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On a problem by Nathan Jacobson

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Abstract. We prove a coordinatization theorem for unital alternative algebras containing 2×2 matrix algebra with the same identity element 1. This solves an old problem announced by Nathan Jacobson on the description of alternative algebras containing a generalized quaternion algebra $\mathbb{H}$ with the same 1, for the case when the algebra $\mathbb{H}$ is split. In particular, this is the case when the basic field is finite or algebraically closed.

1. Introduction

The classical Wedderburn coordinatization theorem says that if a unital associative algebra A contains a matrix algebra $M_n(F)$ with the same identity element, then it is itself a matrix algebra, $A \cong M_n(D)$, "coordinated" by D. Generalizations and analogues of this theorem were proved for various classes of algebras and superalgebras [2, 4, 6–10, 12, 13]. The common content of all these results is that if an algebra (or superalgebra) contains a certain subalgebra (matrix algebra, octonions, Albert algebra) with the same identity element, then the algebra itself has the same structure, but not over the basic field rather over a certain algebra that "coordinatizes" it. The coordinatization theorems play an important role in structure theories, especially in classification theorems, and also in representation theory, since quite often an algebra A coordinated by D is Morita equivalent to D, though they could belong to different classes (for instance, Jordan algebras are coordinated by associative and alternative algebras).

In this paper we consider alternative algebras. Recall that an algebra $\mathcal{A}$ is called *alternative* if it satisfies the following identities:

$$x^2 y = x(xy), \quad (xy)y = xy^2, \tag{1.1}$$

for all $x, y \in \mathcal{A}$. All associative algebras are clearly alternative. A classical example of a non-associative alternative algebra is the Cayley (or generalized octonion) algebra $\mathbb{O}$ (see [3, 5, 13, 16]). Kaplansky [4] proved an analogue of Wedderburn's theorem for alternative algebras containing the Cayley algebra. He showed that if $\mathcal{A}$ is an alternative algebra

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with identity element 1 which contains a subalgebra $\mathcal{B}$ isomorphic to a Cayley algebra and if 1 is contained in $\mathcal{B}$, then $\mathcal{A}$ is isomorphic to the Kronecker product $\mathcal{B} \otimes T$, where T is the center of $\mathcal{A}$. Jacobson gave a new proof of Kaplansky's result, using his classification of irreducible alternative bimodules, and in addition proved an analogue of this theorem for Jordan algebras [2], where the role of Cayley algebra is played by the Albert algebra, the exceptional simple Jordan algebra of dimension 27. These results have important applications in the theory of representations of alternative and Jordan algebras [1, 3].

The Wedderburn coordinatization theorem in the case $n \geq 3$ admits a generalization for alternative algebras, since every alternative algebra $\mathcal{A}$ which contains a subalgebra $M_n(F)$ ($n \geq 3$) with the same identity element is associative (see Corollary 11 in Chapter 2 of [13]). The result is not true for $n = 2$: the split Cayley algebra and its 6-dimensional subalgebra are counterexamples. The problem of description of alternative algebras containing $M_2(F)$ or, more generally, a generalized quaternion algebra $\mathbb{H}$ with the same identity element, was posed by Jacobson [2]. In this paper, we solve this problem for the split case $\mathbb{H} \cong M_2(F)$, without any restriction on the dimension and characteristic of the base field F.

Our $M_2(F)$-coordinatization involves two ingredients: an alternative $M_2(F)$-algebra $\mathcal{A}$ is "coordinated" by an associative algebra $\mathcal{B}$ and by a commutative $\mathcal{B}$-bimodule V (that is, V is annihilated by any commutator of elements of $\mathcal{B}$), on which a skew-symmetric mapping is defined with values in the center of $\mathcal{B}$, satisfying the Plücker relations. More exactly, $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$, with a properly defined multiplication. The details are given in the main Theorem 5.1.

The paper is organized as follows. In Section 2 we give definitions and some known results on alternative algebras and bimodules. In Section 3 we prove that a unital alternative algebra $\mathcal{A}$ containing the generalized quaternion algebra $\mathbb{H}$ with the same unit admits a Z_2-grading $\mathcal{A} = \mathcal{A}_a \oplus \mathcal{A}_c$ with associative 0-component $\mathcal{A}_a$. In the next section, we determine multiplication in the 1-component $\mathcal{A}_c$. In Section 5 we prove the main theorem on $M_2(F)$-coordinatization of alternative algebras. Sections 6 and 7 are devoted to examples and open questions.

Throughout this paper, the ground field F is of arbitrary characteristic.

2. Definitions and known results

Let $\mathcal{A}$ be an arbitrary algebra. Denote by $(x, y, z) = (xy)z - x(yz)$ the *associator* of the elements $x, y, z \in \mathcal{A}$, and by $[x, y] = xy - yx$ the *commutator* of the elements $x, y \in \mathcal{A}$. For subsets $\mathcal{B}, \mathcal{C}, \mathcal{D}$ of $\mathcal{A}$, we denote by $(\mathcal{B}, \mathcal{C}, \mathcal{D})$ the *associator space* generated by all the associators (b, c, d), $b \in \mathcal{B}$, $c \in \mathcal{C}$, $d \in \mathcal{D}$. The *associative center* $N(\mathcal{A})$, the *commutative center* $K(\mathcal{A})$ and the *center* $Z(A)$ are respectively defined as follows:

$$N(\mathcal{A}) = \{a \in \mathcal{A} | (a, \mathcal{A}, \mathcal{A}) = (\mathcal{A}, a, \mathcal{A}) = (\mathcal{A}, \mathcal{A}, a) = 0\},$$
$$K(\mathcal{A}) = \{a \in \mathcal{A} | [a, \mathcal{A}] = 0\},$$
$$Z(\mathcal{A}) = N(\mathcal{A}) \cap K(\mathcal{A}).$$

In terms of associators, identities (1.1) defining alternative algebras can be written as

$$(x,x,y)=0, \quad (x,y,y)=0.$$

The first of them is called the *left alternative identity* and the second one, the *right alternative identity*.

Linearizing the left and right alternative identities, we obtain

$$(x,z,y)+(z,x,y)=0, \quad (x,y,z)+(x,z,y)=0,$$

which show that in an alternative algebra the associator is an antisymmetric function of its arguments. Also, these identities can be written as

$$(x\circ z)y-x(zy)-z(xy)=0, \quad (xy)z+(xz)y-x(y\circ z)=0, \tag{2.1}$$

where $a\circ b=ab+ba$.

Throughout the article we will make use of some identities that are valid in any alternative algebra and will be mentioned at the time be required.

2.1. Alternative bimodules

Let $\mathcal{A}$ be an alternative algebra over F and let V be a bimodule over $\mathcal{A}$, this is, V is a vector space over F equipped with the applications $\mathcal{A}\otimes V\longrightarrow V$, $a\otimes v\longmapsto av$, and $V\otimes\mathcal{A}\longrightarrow V$, $v\otimes a\longmapsto va$, for $a\in\mathcal{A}$, $v\in V$. Define on the vector space $E=\mathcal{A}\oplus V$ a binary operation $\cdot : E\times E\longrightarrow E$ by

$$(a+v)\cdot(b+w)=ab+(av+wb),$$

where $a,b\in\mathcal{A}$, $v,w\in V$. Then E with the operation (product) $\cdot$ becomes an algebra, the *split null extension* of $\mathcal{A}$ by bimodule V, where $\mathcal{A}$ is a subalgebra and V is an ideal such that $V^2=0$. Now, V is called an *alternative bimodule* over $\mathcal{A}$ if E is an alternative algebra with respect to $\cdot$.

Due to identities (1.1), a bimodule V over $\mathcal{A}$ is an alternative bimodule if and only if the following relationships are satisfied:

$$(a,a,v)=0,(a,v,b)+(v,a,b)=0,$$
$$(v,b,b)=0,(a,v,b)+(a,b,v)=0,$$

for all $a,b\in\mathcal{A}$, $v\in V$.

Let $\mathcal{A}$ be a composition algebra (see [3,5,13,16]). Recall that $\mathcal{A}$ is a unital alternative algebra, it has an involution $a\mapsto a^*$ such that the *trace* $t(a)=a+a^*$ and the *norm* $n(a)=aa^*$ lie in F.

An alternative bimodule V over a composition algebra $\mathcal{A}$ is called a *Cayley bimodule* if it satisfies the relation

$$av=va^*, \tag{2.2}$$

where $a\in\mathcal{A}$, $v\in V$, and $a\to a^*$ is the canonical involution in $\mathcal{A}$.

Typical examples of composition algebras are the algebras of (generalized) quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ with symplectic involutions. Recall that $\mathbb{O} = \mathbb{H} \oplus v\mathbb{H}$, with the product defined by

$$a \cdot b = ab, \quad a \cdot vb = v(a^*b), \quad vb \cdot a = v(ab), \quad va \cdot vb = (ba^*)v^2, \tag{2.3}$$

where $a, b \in \mathbb{H},\ 0 \neq v^2 \in F,\ a \mapsto a^*$ is the symplectic involution in $\mathbb{H}$.

The subspace $v\mathbb{H} \subset \mathbb{O}$ is invariant under multiplication by elements of $\mathbb{H}$, and it gives an example of a *Cayley bimodule* over $\mathbb{H}$. If $\mathbb{H}$ is a division algebra, then $v\mathbb{H}$ is irreducible, otherwise $\mathbb{H} \cong M_2(F)$ and

$$v\mathbb{H} = \langle ve_{22}, -ve_{12}\rangle \oplus \langle -ve_{21}, ve_{11}\rangle,$$

where the $M_2(F)$-bimodules $\langle ve_{22},\ -ve_{12}\rangle$ and $\langle -ve_{21}, ve_{11}\rangle$ are both isomorphic to the 2-dimensional Cayley bimodule $\mathrm{Cay} = F \cdot m_1 + F \cdot m_2$, with the action of $M_2(F)$ given by

$$e_{ij} \cdot m_k = \delta_{ik} m_j, \quad m \cdot a = a^* \cdot m, \tag{2.4}$$

where $a \in M_2(F),\ m \in \mathrm{Cay},\ i, j, k \in \{1, 2\}$ and $a \mapsto a^*$ is the symplectic involution in $M_2(F)$.

We will denote the Cayley bimodule $v\mathbb{H}$ for division $\mathbb{H}$ as $\mathrm{Cay}\,\mathbb{H}$, and the regular (associative) $\mathbb{H}$-bimodule by Reg .

3. $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_a + \mathcal{A}_c$

The statement of the next result follows from Lemma 11 in [15] and its proof.

Proposition 3.1 (Lemma 11 in [15]). *Let $\mathcal{A}$ be a unitary alternative algebra over the field F which contains a composition subalgebra $\mathcal{C}$ with the same identity element. Suppose that a subspace V of $\mathcal{A}$ is $\mathcal{C}$-invariant and satisfies* (2.2)*. Then, the following identities are valid for any $a, b \in \mathcal{C}$, $r \in \mathcal{A}$, $u, v \in V$:*

$$(ab)v = b(av), \quad v(ab) = (vb)a, \tag{3.1}$$

$$a(ur) = u(a^*r), \quad (ru)a = (ra^*)u, \tag{3.2}$$

$$a(uv) = u(va), \quad (uv)a = (au)v, \tag{3.3}$$

$$(u, v, a) = [uv, a]. \tag{3.4}$$

It is important to know the structure of unitary alternative $\mathbb{H}$-bimodules. Their structure is given by the following result.

Proposition 3.2 (Lemma 12 in [15]). *Every unitary alternative $\mathbb{H}$-bimodule V is completely reducible and admits a decomposition $V = V_a \oplus V_c$, where V_a is an associative $\mathbb{H}$-bimodule and V_c is a Cayley bimodule over $\mathbb{H}$; furthermore, the subbimodule V_c coincides with the associator subspace $(V, \mathbb{H}, \mathbb{H})$. Every irreducible component of the subbimodule V_a is isomorphic to the regular $\mathbb{H}$-bimodule* Reg*, and every irreducible component of the subbimodule V_c is isomorphic to* $\mathrm{Cay}\,\mathbb{H}$ *if $\mathbb{H}$ is a division algebra, and to* Cay *if $\mathbb{H} \cong M_2(F)$.*

Let $\mathcal{A}$ be an alternative algebra such that $\mathcal{A}$ contains $\mathbb{H}$ with the same identity element, so we can consider $\mathcal{A}$ as a unitary alternative $\mathbb{H}$-bimodule. Then, by Proposition 3.2, $\mathcal{A}$ is completely reducible and admits the decomposition

$$\mathcal{A} = \mathcal{A}_a \oplus \mathcal{A}_c,$$

where $\mathcal{A}_a$ is a unitary associative $\mathbb{H}$-bimodule and $\mathcal{A}_c$ is a unitary Cayley $\mathbb{H}$-bimodule.

Denote $Z_a = \{u \in \mathcal{A}_a | [u, \mathbb{H}] = 0\}$. Since $\mathcal{A}_a$ is isomorphic to a direct sum of bimodules Reg, we have $\mathcal{A}_a = \oplus_i \mathrm{Reg}_i$, $\mathrm{Reg}_i \cong \mathrm{Reg}$ for all i. This implies that $\mathcal{A}_a$ contains a set of elements $\{u_i\}$ (the images of 1 under the isomorphisms with Reg) such that $\mathrm{Reg}_i = u_i\mathbb{H}$ with $u_i \in Z_a$, and each element of $\mathcal{A}_a$ can be written in only one way in the form $\sum u_i a_i$, $a_i \in \mathbb{H}$. Now, of course, $Z_a \neq 0$ and $\mathcal{A}_a = Z_a\mathbb{H}$.

Also, by Proposition 3.2, the bimodule $\mathcal{A}_c$ coincides with $(\mathcal{A}, \mathbb{H}, \mathbb{H})$ and is completely reducible; this is, $\mathcal{A}_c = \oplus_j \widetilde{\mathrm{Cay}}_j$, where $\widetilde{\mathrm{Cay}}$ is equal to $\mathrm{Cay}\,\mathbb{H}$ or to Cay. Therefore,

$$\mathcal{A} = (\oplus_i \mathrm{Reg}_i) \oplus (\oplus_j \widetilde{\mathrm{Cay}}_j).$$

The statements and demonstrations of Lemmas 3.3 and 3.4 are similar to Lemmas 3.1 and 3.2 of [7] given there for superbimodules over the superalgebra $B(4, 2) = \mathbb{H} + \mathrm{Cay}$.

Lemma 3.3. *Let $\mathcal{A} = \mathcal{A}_a \oplus \mathcal{A}_c$ be the decomposition of $\mathcal{A}$ from above. Then for any $m, n \in \mathcal{A}_c$, $a \in \mathbb{H}$,*

$$(mn)a = (am)n, \quad a(mn) = m(na), \tag{3.5}$$

and for any $u \in \mathcal{A}_a$, $m \in \mathcal{A}_c$, $a, b \in \mathbb{H}$,

$$(um)a = (ua^*)m \tag{3.6}$$

$$a(mu) = m(a^*u), \tag{3.7}$$

$$((um)a)b = (um)(ba), \tag{3.8}$$

$$b(a(mu)) = (ab)(mu), \tag{3.9}$$

$$(um, a, b) = (um)[b, a], \tag{3.10}$$

$$(b, a, mu) = [b, a](mu). \tag{3.11}$$

Proof. First, observe that (3.5) follows from (3.1), and that (3.6), (3.7) follow from (3.2). Now let $u \in \mathcal{A}_a$, $m \in \mathcal{A}_c$, $a, b \in \mathbb{H}$. Then by (3.6),

$$(um)a.b = (ua^*.m)b = (ua^*.b^*)m = (u.(ba)^*)m = (um)(ba),$$

which proves (3.8). Similarly, by (3.7), we get (3.9). Finally, by (3.8) and (3.9) we have

$$(um, a, b) = ((um)a)b - (um)(ab) = (um)(ba) - (um)(ab) = (um)[b, a],$$
$$(b, a, mu) = (ba)(mu) - b(a(mu)) = (ba)(mu) - (ab)(mu) = [b, a](mu),$$

which proves (3.10) and (3.11). ∎

Lemma 3.4. *The products $\mathcal{A}_a\mathcal{A}_a$, $\mathcal{A}_a\mathcal{A}_c$, $\mathcal{A}_c\mathcal{A}_a$ and $\mathcal{A}_c\mathcal{A}_c$ are $\mathbb{H}$-invariant subspaces. Moreover, $\mathcal{A}_a\mathcal{A}_c + \mathcal{A}_c\mathcal{A}_a \subseteq \mathcal{A}_c$ and $\mathcal{A}_c\mathcal{A}_c \subseteq \mathcal{A}_a$.*

Proof. Since $\mathcal{A}_a$ and $\mathcal{A}_c$ are $\mathbb{H}$-invariant, in order to prove the first part of the lemma it suffices to show that the product of any $\mathbb{H}$-invariant subspaces U and W is again $\mathbb{H}$-invariant.

We have, by the linearized identity of the right alternativity (2.1),

$$(UW)\mathbb{H} \subseteq U(W \circ \mathbb{H}) + (U\mathbb{H})W \subseteq UW,$$

and similarly $\mathbb{H}(UW) \subseteq UW$.

Now, let us demonstrate that $\mathcal{A}_a\mathcal{A}_c + \mathcal{A}_c\mathcal{A}_a \subseteq \mathcal{A}_c$. Recall that by Proposition 3.2, $\mathcal{A}_c = (\mathcal{A}, \mathbb{H}, \mathbb{H})$. Choose $a, b \in \mathbb{H}$ such that $0 \neq [a, b]^2 \in F$; then by (3.11),

$$\mathcal{A}_c\mathcal{A}_a = [a, b]^2(\mathcal{A}_c\mathcal{A}_a) \subseteq [a, b](\mathcal{A}_c\mathcal{A}_a) = (a, b, \mathcal{A}_c\mathcal{A}_a) \subseteq (\mathbb{H}, \mathbb{H}, \mathcal{A}) = \mathcal{A}_c,$$

and similarly $\mathcal{A}_a\mathcal{A}_c \subseteq \mathcal{A}_c$. Finally, for any $m, n \in \mathcal{A}_c$ and $a \in \mathbb{H}$, we have by (3.5) and (3.1),

$$((mn)a)b = ((am)n)b = (b(am))n = ((ab)m)n = (mn)(ab),$$

which proves $\mathcal{A}_c\mathcal{A}_c \subseteq \mathcal{A}_a$. ∎

Lemma 3.5. *$\mathcal{A}_a$ is an associative subalgebra of $\mathcal{A}$.*

Proof. Recall the following identities, valid in every alternative algebra (see [15, 16]):

$$(xy)(zx) = x(yz)x, \tag{3.12}$$
$$[x, yz] = [x, y]z + y[x, z] - 3(x, y, z), \tag{3.13}$$
$$(xy, z, t) = x(y, z, t) + (x, z, t)y - (x, y, [z, t]), \tag{3.14}$$
$$2[(x, y, z), t] = ([x, y], z, t) + ([y, z], x, t) + ([z, x], y, t), \tag{3.15}$$
$$[x, y](x, y, z) = (x, y, (x, y, z)) = -(x, y, z)[x, y], \tag{3.16}$$
$$\begin{aligned}((z, w, t), x, y) = {}& ((z, x, y), w, t) + (z, (w, x, y), t)\\ &+ (z, w, (t, x, y)) - [w, (z, t, [x, y])] + ([z, t], w, [x, y]).\end{aligned} \tag{3.17}$$

Let us fix arbitrary elements $u, v, w \in Z_a$ and $a, b, c \in \mathbb{H}$. Then by (3.15),

$$([a, b], u, v) = 2[(a, b, u), v] - ([b, u], a, v) - ([u, a], b, v) = 0.$$

So by (3.14),

$$(uv, a, b) = u(v, a, b) + (u, a, b)v - (u, v, [a, b]) = -([a, b], u, v) = 0,$$

which implies $(Z_aZ_a, \mathbb{H}, \mathbb{H}) = 0$. By linearization of (3.16), choosing $a, b \in \mathbb{H}$ such that $[a, b]^2 = \alpha \in F$, $\alpha \neq 0$, we have for any $x \in \mathcal{A}_a$,

$$\begin{aligned}[a, b](u, x, c) &= -[u, b](a, x, c) - [a, x](u, b, c) - [u, x](a, b, c) + (a, b, (u, x, c))\\ &\quad + (u, b, (a, x, c)) + (a, x, (u, b, c)) + (u, x, (a, b, c))\\ &= (a, b, (u, x, c))\\ &\overset{(3.17)}{=} ((u, a, b), x, c) + (u, (x, a, b), c) + (u, x, (c, a, b))\\ &\quad - [x, (u, c, [a, b])] + ([u, c], x, [a, b]) = 0.\end{aligned}$$

So $\alpha(u,x,c) = [a,b]^2(u,x,c) = [a,b]([a,b](u,x,c)) = 0$, thus $(u,x,c) = 0$, which implies $(Z_a, \mathcal{A}_a, \mathbb{H}) = 0$. In particular, $(Z_a, Z_a, \mathbb{H}) = 0$. Then, by (3.13),

$$[a, uv] = [a,u]v + u[a,v] - 3(a,u,v) = 0,$$

and so $[\mathbb{H}, Z_a Z_a] = 0$. Therefore $Z_a Z_a \subseteq Z_a$.

By linearization of (3.16), we have

$$\begin{aligned}[a,b](u,v,w) = -[a,v](u,b,w) - [u,b](a,v,w) - [u,v](a,b,w) + (a,b,(u,v,w)) \\ + (a,v,(u,b,w)) + (u,b,(a,v,w)) + (u,v,(a,b,w)) = 0.\end{aligned}$$

Choose again $a, b \in \mathbb{H}$ such that $[a,b]^2 = \alpha \in F, \alpha \neq 0$. Then

$$\alpha(u,v,w) = [a,b]^2(u,v,w) = [a,b]([a,b](u,v,w)) = 0,$$

and so $(u,v,w) = 0$. Thus Z_a is an associative algebra.

Consequently, by linearization of the central Moufang identity (3.12) and using the fact that $\mathcal{A}_a$ is an associative $\mathbb{H}$-bimodule, we have

$$\begin{aligned}(ua)(vb) &= -(ba)(vu) + (u(av))b + (b(av))u = -(ba)(vu) + (u(va))b + ((ba)v)u \\ &= -(ba)(vu) + ((uv)a)b + (ba)(vu) = (uv)(ab).\end{aligned}$$

Therefore $\mathcal{A}_a \mathcal{A}_a \subseteq \mathcal{A}_a$, that is, $\mathcal{A}_a$ is a subalgebra of $\mathcal{A}$.

Recalling that $(Z_a, \mathcal{A}_a, \mathbb{H}) = 0$, then for all $x, y \in \mathcal{A}_a$ we have

$$(ua, x, y) \overset{(3.14)}{=} u(a,x,y) + (u,x,y)a - (u,a,[x,y]) = u(a,x,y) + (u,x,y)a.$$

Thus, using the last equality several times and the fact that Z_a is associative, we have

$$\begin{aligned}(ua, vb, wc) &= u(a, vb, wc) + (u, vb, wc)a \\ &= -u(v(b,a,wc) + (v,a,wc)b) - (v(b,u,wc) + (v,u,wc)b)a \\ &= -((w(c,v,u) + (w,v,u)c)b)a = 0.\end{aligned}$$

Therefore, $\mathcal{A}_a$ is an associative subalgebra of $\mathcal{A}$. ■

It follows immediately from Lemmas 3.4 and 3.5 the following result.

Corollary 3.6. *$\mathcal{A} = \mathcal{A}_a \oplus \mathcal{A}_c$ is a $\mathbb{Z}_2$-graded algebra, where $\mathcal{A}_a$ is the even part and $\mathcal{A}_c$ is the odd part of the $\mathbb{Z}_2$-grading of $\mathcal{A}$.*

In what follows we will use in a permanent way the fact that $\mathcal{A}$ is a $\mathbb{Z}_2$-graded alternative algebra. Thus, we have $(\mathcal{A}_c, \mathbb{H}, \mathcal{A}_c) \subseteq (\mathcal{A}_c, \mathcal{A}_a, \mathcal{A}_c) \subseteq \mathcal{A}_a$, and $[Z_a, \mathcal{A}_c] \subseteq \mathcal{A}_c$.

Lemma 3.7. $[Z_a, \mathcal{A}_c] = (Z_a, \mathcal{A}, \mathcal{A}) = 0$.

Proof. Let us fix arbitrary elements $u, v, w \in Z_a, m, n \in \mathcal{A}_c$ and $a, b, c \in \mathbb{H}$. In the proof of the previous lemma we have shown that $(Z_a, \mathcal{A}_a, \mathbb{H}) = 0$. So, let us generalize the previous equality and show first

$$(Z_a, \mathcal{A}, \mathbb{H}) = 0. \tag{3.18}$$

By the fact that $\mathcal{A}_c$ is a Cayley $\mathbb{H}$-bimodule, we have

$$(a,u,m) = (au)m - a(um) = (au)m - (um)a^* = (au)m - (ua)m = [a,u]m = 0,$$

which proves $(\mathbb{H}, Z_a, \mathcal{A}_c) = 0$. Thus,

$$(Z_a, \mathcal{A}, \mathbb{H}) \subseteq (Z_a, \mathcal{A}_a, \mathbb{H}) + (Z_a, \mathcal{A}_c, \mathbb{H}) = 0,$$

which proves (3.18).

In addition, consider the identity

$$([x,y],y,z) = [y,(x,y,z)] \tag{3.19}$$

which is valid in every alternative algebra. Using its linearization, we obtain

$$([u,m],a,b) = -([u,a],m,b) + [m,(u,a,b)] + [a,(u,m,b)] = 0.$$

Thus $([Z_a, \mathcal{A}_c], \mathbb{H}, \mathbb{H}) = 0$, that is, $[Z_a, \mathcal{A}_c] \subseteq \mathcal{A}_a$. Therefore,

$$[Z_a, \mathcal{A}_c] \subseteq \mathcal{A}_a \cap \mathcal{A}_c = 0,$$

which implies $[Z_a, \mathcal{A}_c] = 0$.

By linearization of (3.16), by (3.17), choosing $a, b \in \mathbb{H}$ such that $0 \neq [a,b]^2 = \alpha \in F$, we have

$$\begin{aligned} [a,b](u,m,n) &= -[u,b](a,m,n) - [a,m](u,b,n) - [u,m](a,b,n) + (a,b,(u,m,n)) \\ &\quad + (u,b,(a,m,n)) + (a,m,(u,b,n)) + (u,m,(a,b,n)) \\ &= (u,m,(a,b,n)) \\ &= ((u,m,a),b,n) - ((u,b,n),m,a) - (u,(m,b,n),a) \\ &\quad + [m,(u,a,[b,n])] - ([u,a],m,[b,n]) = 0, \end{aligned}$$

hence $\alpha(u,m,n) = [a,b]^2(u,m,n) = [a,b]([a,b](u,m,n)) = 0$ and $(u,m,n) = 0$; therefore, $(Z_a, \mathcal{A}_c, \mathcal{A}_c) = 0$. Also

$$\begin{aligned} [a,b](u,n,v) &= -[u,b](a,n,v) - [a,n](u,b,v) - [u,n](a,b,v) + (a,b,(u,n,v)) \\ &\quad + (u,b,(a,n,v)) + (a,n,(u,b,v)) + (u,n,(a,b,v)) \\ &= (a,b,(u,n,v)) = ((u,n,v),a,b) \\ &\overset{(3.17)}{=} -(u,(a,n,v),b) - (u,a,(b,n,v)) + ((u,a,b),n,v) \\ &\quad + [a,(u,b,[n,v])] - ([u,b],a,[n,v]) = 0, \end{aligned}$$

so $\alpha(u,n,v) = [a,b]^2(u,n,v) = [a,b]([a,b](u,n,v)) = 0$ and $(u,n,v) = 0$; thus, $(Z_a, \mathcal{A}_c, Z_a) = 0$. Then by (3.14) and (3.18),

$$(ua,v,m) = u(a,v,m) + (u,v,m)a - (u,a,[v,m]) = 0;$$

so $(Z_a\mathbb{H}, Z_a, \mathcal{A}_c) = 0$. Therefore $(\mathcal{A}_a, Z_a, \mathcal{A}_c) = 0$, and we have

$$(Z_a, \mathcal{A}, \mathcal{A}) \subseteq (Z_a, \mathcal{A}_a, \mathcal{A}_a) + (Z_a, \mathcal{A}_c, \mathcal{A}_a) + (Z_a, \mathcal{A}_c, \mathcal{A}_c) = 0,$$

so $Z_a \subseteq N(\mathcal{A})$. ∎

Corollary 3.8. $\mathcal{A}_a = Z_a \otimes_F \mathbb{H}$.

Proof. As $\mathcal{A}_a = \sum \oplus u_i \mathbb{H}$, every element of $\mathcal{A}_a$ can be written uniquely in the form $\sum u_i a_i$, with $a_i \in \mathbb{H}$. We know that $\mathcal{A}_a$ is associative. On the other hand, let $x = \sum u_i a_i \in Z_a$; then $ax = xa$ for all $a \in \mathbb{H}$ by $[Z_a, \mathbb{H}] = 0$. Therefore, we have

$$\sum u_i a a_i = \sum u_i a_i a;$$

so, $aa_i = a_i a$. But as $\mathbb{H}$ is central, we have $a_i = \alpha_i 1$, $\alpha_i \in F$. Then $Z_a = \sum F u_i$ and $\mathcal{A}_a = Z_a \otimes_F \mathbb{H}$. ■

Lemma 3.9. $[Z_a, Z_a]\mathcal{A}_c = \mathcal{A}_c[Z_a, Z_a] = 0$.

Proof. In the proof of Lemma 3.7 we have obtained $[Z_a, \mathcal{A}_c] = 0$. Thus, by (3.13) and again by Lemma 3.7,

$$[Z_a, Z_a]\mathcal{A}_c \subseteq [Z_a, Z_a\mathcal{A}_c] - Z_a[Z_a, \mathcal{A}_c] + 3(Z_a, Z_a, \mathcal{A}_c) = 0,$$

and similarly $\mathcal{A}_c[Z_a, Z_a] = 0$. ■

Remark 3.10. Note that in general Z_a is not commutative. For example, if $\mathcal{A} = M_n(\mathbb{H})$, then $Z_a \cong M_n(F)$. If $\mathcal{A}$ is prime and nonassociative, then by the Corollary to Theorem 8.11 in [16], $N(\mathcal{A}) = Z(\mathcal{A})$, hence $Z_a \subseteq Z(\mathcal{A})$ is commutative. In fact, in this case A is a Cayley–Dickson ring (see [16]).

4. Multiplication in $\mathcal{A}_c$

In the previous section we described, in particular, the structure of the associative part $\mathcal{A}_a$. This section is devoted to description of the multiplication in the Cayley part $\mathcal{A}_c$. Here and below we will assume that the quaternion algebra $\mathbb{H}$ is split, that is, $\mathbb{H} \cong M_2(F)$.

We have already mentioned that the Cayley $\mathbb{H}$-bimodule $\mathcal{A}_c$ is completely reducible and is a direct sum of bimodules isomorphic to the Cayley bimodule $\mathrm{Cay} = F \cdot m_1 + F \cdot m_2$ from (2.4). Denote by $V(1)$ and $V(2)$ the subspaces of $\mathcal{A}_c$ spanned by the elements of type m_1 and m_2, respectively; then the mappings

$$\pi_{12} : V(1) \to V(2), \quad v \mapsto e_{12} \cdot v,$$
$$\pi_{21} : V(2) \to V(1), \quad v \mapsto e_{21} \cdot v$$

are mutually inverse and establish isomorphisms between $V(1)$ and $V(2)$. Clearly, $\mathcal{A}_c = V(1) \oplus V(2)$. Let $V = V(1)$. For any $v \in V$, we denote $v(1) = v,\ v(2) = \pi_{12}(v)$. Then $\mathrm{Cay}(v) = F \cdot v(1) + F \cdot v(2) \cong \mathrm{Cay}$.

Proposition 4.1. *For any $u, v \in V$ we have*

$$\mathrm{Cay}(u) \cdot \mathrm{Cay}(v) = \langle u, v\rangle\, \mathbb{H},$$

where $\langle , \rangle \colon V \times V \to Z(\mathcal{A})$ is a skew-symmetric bilinear mapping. In particular, $\mathrm{Cay}(v)^2 = 0$ *for any $v \in V$.*

Proof. We have, by the identities of right and left alternativity,

$$(u(1)v(1))e_{11} = -(u(1)e_{11})v(1) + u(1)(v(1) \circ e_{11}) = u(1)v(1),$$
$$e_{11}(u(1)v(1)) = -u(1)(e_{11}v(1)) + (e_{11} \circ u(1))v(1) = -u(1)v(1) + u(1)v(1) = 0,$$

which shows that $u(1)v(1) \in e_{22}\mathcal{A}_a e_{11} = Z_a e_{21}$, hence $u(1)v(1) = ze_{21}$ for some $z \in Z_a$. Furthermore, we have

$$ze_{22} = (ze_{21})e_{12} = (u(1)v(1))e_{12} = -(u(1)e_{12})v(1) + u(1)(v(1) \circ e_{12}) = u(2)v(1),$$
$$ze_{11} = e_{12}(ze_{21}) = e_{12}(u(1)v(1)) = -u(1)(e_{12}v(1)) + (e_{12} \circ u(1))v(1) = -u(1)v(2),$$
$$ze_{12} = e_{12}(ze_{22}) = e_{12}(u(2)v(1)) = -u(2)(e_{12}v(1)) + (e_{12} \circ u(2))v(1) = -u(2)v(2),$$

which proves that $\mathrm{Cay}(u) \cdot \mathrm{Cay}(v) = z\,\mathbb{H}$.

Since $z \in Z_a$, we have $[z, \mathbb{H}] = [z, \mathcal{A}_c] = 0$. Hence in order to prove that $z \in Z(\mathcal{A})$, it remains to show that $[z, Z_a] = 0$. Note that $z = z(e_{11} + e_{22}) = u(2)v(1) - u(1)v(2) \in \mathcal{A}_c^2$. Therefore,

$$[z, Z_a] \subset [\mathcal{A}_c^2, Z_a] \subseteq \mathcal{A}_c[\mathcal{A}_c, Z_a] + [\mathcal{A}_c, Z_a]\mathcal{A}_c + 3(\mathcal{A}_c, \mathcal{A}_c, Z_a) = 0.$$

Finally, denote $z = z(u, v)$ and consider

$$e_{11}(u(1) \circ v(1)) = (e_{11}u(1))v(1) + (e_{11}v(1))u(1) = u(1) \circ v(1).$$

On the other hand, $e_{11}(u(1) \circ v(1)) = e_{11}((z(u, v) + z(v, u))e_{21}) = 0$. Hence $u(1) \circ v(1) = 0$ and $z(u, v) = -z(v, u)$. Denote $\langle u, v\rangle = z(u, v)$; then we have, as above,

$$\langle u, v\rangle = z(u, v) = u(2)v(1) - u(1)v(2), \tag{4.1}$$

which proves that $\langle u, v\rangle$ is a bilinear function of u, v. ∎

Lemma 4.2. *For any $u, v, w, t \in V$ the following identities hold*:

$$\langle u, v\rangle w + \langle v, w\rangle u + \langle w, u\rangle v = 0, \tag{4.2}$$
$$\langle u, v\rangle\langle w, t\rangle + \langle v, w\rangle\langle u, t\rangle + \langle w, u\rangle\langle v, t\rangle = 0. \tag{4.3}$$

Proof. Recall that in the proof of Proposition 4.1 we obtained the equalities

$$u(1)v(1) = \langle u, v\rangle\, e_{21}, \tag{4.4}$$
$$u(1)v(2) = -\langle u, v\rangle\, e_{11}, \tag{4.5}$$
$$u(2)v(1) = \langle u, v\rangle\, e_{22}, \tag{4.6}$$
$$u(2)v(2) = -\langle u, v\rangle\, e_{12}. \tag{4.7}$$

Therefore, using the fact that $\langle u, v\rangle \in Z(A)$, by the linearized right alternative identity we have

$$\begin{aligned} 0 &= (u(1), v(1), w(2)) + (u(1), w(2), v(1)) \\ &= \langle u, v\rangle e_{21}w(2) + u(1)\langle v, w\rangle e_{11} - \langle u, w\rangle e_{11}v(1) - u(1)\langle w, v\rangle e_{22} \\ &= \langle u, v\rangle w + 0 - \langle u, w\rangle v - \langle w, v\rangle u = \langle u, v\rangle w + \langle w, u\rangle v + \langle v, w\rangle u, \end{aligned}$$

which proves (4.2). Multiplying (4.2) by the element $t \in V$, we get (4.3). ∎

Corollary 4.3. *Let $\{v_i \mid i \in I\}$ be a basis of the space V and let $u_{ij} = \langle v_i, v_j\rangle \in Z(\mathcal{A})$. Then the elements u_{ij} satisfy the Plücker relations*

$$u_{ij} = -u_{ji}, \quad u_{ij}u_{kl} + u_{ik}u_{lj} + u_{il}u_{jk} = 0. \tag{4.8}$$

An example of a family of elements $u_{ij} = -u_{ji}$ satisfying relations (4.8) may be obtained by taking in an associative commutative algebra K elements $a_1, \dots, a_n$ and setting $u_{ij} = a_i - a_j$.

Another example is the coordinate algebra of the Grassmanian $G_{2,n}$ (see, for example, p. 42 in [14]).

Lemma 4.4. *Consider the algebra of polynomials $F[x_1, \dots, x_n; y_1, \dots, y_n]$, and let $\alpha_{ij} = \det\left[\begin{smallmatrix} x_i & y_i \\ x_j & y_j \end{smallmatrix}\right] \in F[x_1, \dots, x_n; y_1, \dots, y_n]$. Then the elements $\alpha_{ij} = -\alpha_{ji}$ satisfy relations (4.8). Moreover, the algebra $F[\alpha_{ij} \mid 1 \le i < j \le n]$ is a free algebra modulo relations (4.8).*

Proof. Firstly, one can easily check that the elements α_{ij} satisfy relations (4.8). Furthermore, it follows from the relation

$$\alpha_{12}\alpha_{ij} + \alpha_{1i}\alpha_{j2} + \alpha_{1j}\alpha_{2i} = 0$$

that α_{ij} for $i, j > 2$ lies in the algebra $F[\alpha_{1i}, \alpha_{1j}, \alpha_{2i}, \alpha_{2j}, \alpha_{12}^{-1}] \subset F(x_1, \dots, x_n; y_1, \dots, y_n)$. Therefore,

$$F[\alpha_{ij} \mid 1 < i \le n,\ 2 < j \le n] \subseteq F[\alpha_{12}, \dots, \alpha_{1n}; \alpha_{23}, \dots, \alpha_{2n}, \alpha_{12}^{-1}]. \tag{4.9}$$

Observe that $y_2 = \frac{1}{x_1}\alpha_{12} + \frac{x_2 y_1}{x_1}$, hence $F(x_1, x_2, y_1, y_2) = F(x_1, x_2, y_1, \alpha_{12})$.

Similarly, resolving with respect to $x_n,\ y_n$ the system

$$\alpha_{1n} = x_1 y_n - y_1 x_n,$$
$$\alpha_{2n} = x_2 y_n - y_2 x_n,$$

we get

$$x_n = \frac{x_2\alpha_{1n} - x_1\alpha_{2n}}{\alpha_{12}} \quad \text{and} \quad y_n = \frac{y_2\alpha_{1n} - y_1\alpha_{2n}}{\alpha_{12}};$$

hence

$$x_n,\ y_n \in F(\alpha_{1n},\ \alpha_{2n},\ x_1,\ x_2,\ y_1,\ y_2) = F(\alpha_{1n},\ \alpha_{2n},\ x_1,\ x_2,\ y_1,\ \alpha_{12}).$$

Therefore,

$$F(x_1, \dots, x_n, y_1, \dots, y_n) = F(x_1, x_2, y_1, \alpha_{12}, \dots, \alpha_{1n}, \alpha_{23}, \dots, \alpha_{2n}),$$

and $\operatorname{tr.deg} F(x_1, x_2, y_1, \alpha_{12}, \dots, \alpha_{1n}, \alpha_{23}, \dots, \alpha_{2n}) = 2n$, which means that the elements $\alpha_{12}, \dots, \alpha_{1n}, \alpha_{23}, \dots, \alpha_{2n}$ are algebraically independent.

Now, let $F[u_{ij}]$ be a free algebra modulo relations (4.8). Consider the epimorphism $\pi\colon F[u_{ij}] \longrightarrow F[\alpha_{ij}];\ u_{ij} \longmapsto \alpha_{ij}$. We will prove that $\ker\pi = 0$. Let $f(u_{12}, \dots, u_{(n-1)n}) \in \ker\pi$, that is, $f(\alpha_{12}, \dots, \alpha_{(n-1)n}) = 0$. Inclusions (4.9) follow from relations (4.8), hence they are valid in the algebra $F[u_{ij}]$ as well. Therefore, there exists k such that

$$u_{12}^k f(u_{12}, \dots, u_{(n-1)n}) = g(u_{12}, \dots, u_{2n})$$

for some $g(u_{12}, \dots, u_{2n}) \in F[u_{12}, \dots, u_{2n}]$. Clearly, $g(\alpha_{12}, \dots, \alpha_{2n}) = 0$. Since the elements $\alpha_{12}, \dots, \alpha_{2n}$ are algebraically independent, we have $g = 0$. But the algebra $F[u_{ij}]$ is a domain (see, for example, Chapter 8 of [11]), therefore $f = 0$, proving the lemma. ∎

Recall that by Corollary 3.8, $\mathcal{A}_a \cong M_2(Z_a)$, hence $\mathcal{A} = M_2(Z_a) \oplus V(1) \oplus V(2)$.

Proposition 4.5. *Let $X, Y \in \mathcal{A}$, $X = X_a + x(1) + y(2)$, $Y = Y_a + z(1) + t(2)$, where $X_a = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $Y_a = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$, $a, b, c, d, e, f, g, h \in Z_a$, $x, y, z, t \in V$. Then the product XY is given by*

$$XY = X_aY_a + \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} + (az + ct + hx - gy)(1) + (bz + dt - fx + ey)(2).$$

Proof. The proof follows from identities (2.4), (4.4)–(4.7), and Lemma 3.7. ∎

We can make the formula defining the product in $\mathcal{A}$ more transparent by using the following notation: for $u, v \in V$, we denote

$$(u, v) = u(1) + v(2).$$

With this notation, using usual matrix multiplication and the fact that $[Z_a, V_c] = 0$, we have for $X = X_a + (x, y)$, $Y = Y_a + (z, t)$,

$$XY = X_aY_a + \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} + (z, t)X_a + (x, y)(Y_a)^*, \tag{4.10}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

In the next section we will prove that Proposition 4.5 in fact describes all unital alternative extensions of the algebra $M_2(F)$.

5. The main theorem

Let $\mathcal{B}$ be an associative unital algebra and let V be a left $\mathcal{B}$-module such that $[\mathcal{B}, \mathcal{B}]$ annihilates V. Clearly, in this case V has a structure of a commutative $\mathcal{B}$-bimodule with $v \cdot b = b \cdot v$, $v \in V$, $b \in \mathcal{B}$. Assume that there exists a $\mathcal{B}$-bilinear skew-symmetric mapping $\langle , \rangle : V \times V \to \mathcal{B}$ such that $\langle V, V\rangle \subseteq Z(\mathcal{B})$ and formula (4.2) holds for any $u, v, w \in V$.

Let $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$, where $V^2 = \{(u, v) \mid u, v \in V\} \cong V \oplus V$. Let $X, Y \in \mathcal{A}$, $X = X_a + (x, y)$, $Y = Y_a + (z, t)$, where X_a, $Y_a \in M_2(\mathcal{B})$ and $(x, y), (z, t) \in V^2$. Define a product in $\mathcal{A}$ by formula (4.10):

$$XY = X_aY_a + \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} + (z, t)X_a + (x, y)(Y_a)^*.$$

Theorem 5.1. *The algebra $\mathcal{A}$ with the product defined above is an alternative unital algebra containing $M_2(F)$ with the same unit. Conversely, every unital alternative algebra that contains the matrix algebra $M_2(F)$ with the same unit has this form.*

Proof. The second part of the theorem follows from Proposition 4.5 with $B = Z_a$. Let us now prove that $\mathcal{A}$ is alternative.

Let us first prove that V^2 is a right alternative bimodule over $M_2(\mathcal{B})$. Let $A, B \in M_2(\mathcal{B})$, $(x, y) \in V^2$. One can easily check that $(x, y)((AB)^* - B^*A^*) = 0$ (since $V[\mathcal{B}, \mathcal{B}] = 0$). Therefore,

$$((x, y), A, A) = ((x, y)A^*)A^* - (x, y)(A^2)^* = 0.$$

Furthermore,

$$\begin{aligned}(A, (x, y), B) + (A, B, (x, y)) &= ((x, y)A)B^* - ((x, y)B^*)A + (x, y)(AB) - ((x, y)B)A \\ &= (x, y)(AB^* - B^*A + AB - BA) = (x, y)[A, B + B^*] = 0\end{aligned}$$

since $B + B^* = \operatorname{tr}(B)$ commutes with A on V^2. Therefore, V^2 is a right alternative bimodule over $M_2(\mathcal{B})$. Now let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{B}$, and consider

$$\begin{aligned}(A, (x, y), (x, y)) &= ((x, y)A) \cdot (x, y) - \begin{pmatrix} -\langle x, y\rangle & 0 \\ 0 & \langle y, x\rangle \end{pmatrix} A \\ &= (xa + yc, xb + yd) \cdot (x, y) + \langle x, y\rangle A \\ &= \begin{pmatrix} -\langle xa + yc, y\rangle & -\langle xb + yd, y\rangle \\ \langle xa + yc, x\rangle & \langle xb + yd, x\rangle \end{pmatrix} + \langle x, y\rangle A \\ &= \begin{pmatrix} -\langle xa, y\rangle & -\langle xb, y\rangle \\ \langle yc, x\rangle & \langle yd, x\rangle \end{pmatrix} + \langle x, y\rangle A = -\langle x, y\rangle A + \langle x, y\rangle A = 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}&((x, y), A, (u, v)) + ((x, y), (u, v), A) \\ &= ((x, y)A^*) \cdot (u, v) - (x, y) \cdot ((u, v)A) + ((x, y) \cdot (u, v))A - (x, y) \cdot ((u, v)A^*) \\ &= (xd - yc, -xb + ya) \cdot (u, v) - (x, y) \cdot (ua + vc, ub + vd) \\ &\quad + \begin{pmatrix} -\langle x, v\rangle & -\langle y, v\rangle \\ \langle x, u\rangle & \langle y, u\rangle \end{pmatrix} A - (x, y) \cdot (ud - vc, -ub + va) \\ &= \begin{pmatrix} -\langle xd - yc, v\rangle & -\langle -xb + ya, v\rangle \\ \langle xd - yc, u\rangle & \langle -xb + ya, u\rangle \end{pmatrix} - \begin{pmatrix} -\langle x, ub + vd\rangle & -\langle y, ub + vd\rangle \\ \langle x, ua + vc\rangle & \langle y, ua + vc\rangle \end{pmatrix} \\ &\quad + \begin{pmatrix} -\langle x, v\rangle a - \langle y, v\rangle c & -\langle x, v\rangle b - \langle y, v\rangle d \\ \langle x, u\rangle a + \langle y, u\rangle c & \langle x, u\rangle b + \langle y, u\rangle d \end{pmatrix} - \begin{pmatrix} -\langle x, -ub + va\rangle & -\langle y, -ub + va\rangle \\ \langle x, ud - vc\rangle & \langle y, ud - vc\rangle \end{pmatrix} \\ &= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.\end{aligned}$$

We have

$$\begin{aligned}X_{11} &= -\langle xd - yc, v\rangle + \langle x, ub + vd\rangle - \langle x, v\rangle a - \langle y, v\rangle c + \langle x, -ub + va\rangle \\ &= -\langle x, v\rangle d + \langle y, v\rangle c + \langle x, u\rangle b + \langle x, v\rangle d - \langle x, v\rangle a - \langle y, v\rangle c - \langle x, u\rangle b + \langle x, v\rangle a \\ &= 0,\end{aligned}$$

and similarly,

$$\begin{aligned}X_{12} &= -\langle -xb + ya, v\rangle + \langle y, ub + vd\rangle - \langle x, v\rangle b - \langle y, v\rangle d + \langle y, -ub + va\rangle\\ &= \langle x, v\rangle b - \langle y, v\rangle a + \langle y, u\rangle b + \langle y, v\rangle d - \langle x, v\rangle b - \langle y, v\rangle d - \langle y, u\rangle b + \langle y, v\rangle a\\ &= 0,\\ X_{21} &= \langle xd - yc, u\rangle - \langle x, ua + vc\rangle + \langle x, u\rangle a + \langle y, u\rangle c - \langle x, ud - vc\rangle\\ &= \langle x, u\rangle d - \langle y, u\rangle c - \langle x, u\rangle a - \langle x, v\rangle c + \langle x, u\rangle a + \langle y, u\rangle c - \langle x, u\rangle d + \langle x, v\rangle c\\ &= 0,\\ X_{22} &= \langle -xb + ya, u\rangle - \langle y, ua + vc\rangle + \langle x, u\rangle b + \langle y, u\rangle d - \langle y, ud - vc\rangle\\ &= \langle x, u\rangle d - \langle y, u\rangle c - \langle x, u\rangle a - \langle x, v\rangle c + \langle x, u\rangle a + \langle y, u\rangle c - \langle x, u\rangle d + \langle x, v\rangle c\\ &= 0.\end{aligned}$$

Finally,

$$\begin{aligned}((x, y), (z, t), (z, t)) &= \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} \cdot (z, t) - (x, y) \cdot \begin{pmatrix} -\langle z, t\rangle & 0 \\ 0 & \langle t, z\rangle \end{pmatrix}\\ &= (z, t) \begin{pmatrix} -\langle x, t\rangle & -\langle y, t\rangle \\ \langle x, z\rangle & \langle y, z\rangle \end{pmatrix} + \langle z, t\rangle (x, y)\\ &= (-\langle x, t\rangle z + \langle x, z\rangle t + \langle z, t\rangle x, -\langle y, t\rangle z + \langle y, z\rangle t + \langle z, t\rangle y)\\ &= (\langle t, x\rangle z + \langle x, z\rangle t + \langle z, t\rangle x, \langle t, y\rangle z + \langle y, z\rangle t + \langle z, t\rangle y) \overset{(4.2)}{=} (0, 0).\end{aligned}$$

Therefore, $\mathcal{A}$ is right alternative. Similarly, one can prove that $\mathcal{A}$ is left alternative. ■

6. Examples

6.1. Algebra of octonions

Let $\mathcal{B}$ be a unital associative commutative algebra and let $\mathcal{A} = \mathbb{O}(\mathcal{B})$ be the split octonion algebra over $\mathcal{B}$. In this case, $\mathcal{A} = M_2(\mathcal{B}) \oplus vM_2(\mathcal{B})$ with $v^2 = 1$, $\mathcal{A}_a = M_2(\mathcal{B})$, $\mathcal{A}_c = vM_2(\mathcal{B})$, $Z_a = Z(\mathcal{A}) = \mathcal{B}$.

Take $V = \mathcal{B}^2 = \{(a, b) \mid a, b \in \mathcal{B}\}$, $(a, b)(1) = v\left(\begin{smallmatrix} 0 & 0 \\ -b & a \end{smallmatrix}\right)$, $(a, b)(2) = v\left(\begin{smallmatrix} b & -a \\ 0 & 0 \end{smallmatrix}\right)$. Then we have $\mathcal{A} = M_2(\mathcal{B}) \oplus V(1) \oplus V(2)$, with $\langle (a, b), (c, d)\rangle = -\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

In fact, by (4.1),

$$\begin{aligned}\langle (a, b), (c, d)\rangle &= (a, b)(2) \cdot (c, d)(1) - (a, b)(1) \cdot (c, d)(2)\\ &= v\begin{pmatrix} b & -a \\ 0 & 0 \end{pmatrix} \cdot v\begin{pmatrix} 0 & 0 \\ -d & c \end{pmatrix} - v\begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix} \cdot v\begin{pmatrix} d & -c \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} 0 & 0 \\ -d & c \end{pmatrix} \cdot \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} - \begin{pmatrix} d & -c \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\end{aligned}$$

Now for any $u = (a, b),\ v = (c, d),\ w = (e, f) \in V$ we have

$$\begin{aligned}\langle u, v\rangle w &+ \langle v, w\rangle u + \langle w, u\rangle v\\ &= -\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}(e, f) - \det\begin{pmatrix} c & d \\ e & f\end{pmatrix}(a, b) - \det\begin{pmatrix} e & f \\ a & b\end{pmatrix}(c, d)\\ &= -(ad - bc)(e, f) - (cf - de)(a, b) - (eb - fa)(c, d) = (0, 0),\end{aligned}$$

hence $\mathbb{O}(\mathcal{B})$ satisfies (4.2).

The following proposition gives conditions under which the algebra $\mathcal{A}$ from Theorem 5.1 is isomorphic to the octonion algebra $\mathbb{O}(\mathcal{B})$.

Proposition 6.1. *The unital algebra $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$ from Theorem 5.1 is isomorphic to the octonion algebra $\mathbb{O}(\mathcal{B})$ if and only if there exist $x, y \in V$ such that $\langle x, y\rangle = 1$.*

Proof. We have already checked that the algebra $\mathbb{O}(\mathcal{B})$ has the form $M_2(\mathcal{B}) \oplus V^2$. It suffices to note that $\langle x, y\rangle = 1$ for $x = (1, 0),\ y = (0, -1) \in V$.

Let now $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$ be such that there exist $x, y \in V$ with $\langle x, y\rangle = 1$. Observe first that for any $u, v \in V,\ a, b \in \mathcal{B}$ the following equality holds:

$$[a, b]\langle u, v\rangle = 0. \tag{6.1}$$

In fact, we have

$$ab\langle u, v\rangle = a\langle bu, v\rangle = \langle bu, av\rangle = b\langle u, av\rangle = ba\langle u, v\rangle.$$

For any $a, b \in \mathcal{B}$ we now have $0 = [a, b]\langle x, y\rangle = [a, b]$, hence $\mathcal{B}$ is commutative. Consider $\mathcal{C} = M_2(F) + \mathrm{Cay}(x) + \mathrm{Cay}(y)$. It follows from Proposition 4.1 and its proof that $\mathcal{C}$ is a subalgebra of $\mathcal{A}$ isomorphic to the split octonion algebra $\mathbb{O}(F)$. Therefore, by the Kaplansky–Jacobson theorem, $\mathcal{A} \cong \mathbb{O}(A)$ for some commutative associative algebra A. It follows from (4.2) that $V = \mathcal{B} \cdot \mathrm{Cay}(x) + \mathcal{B} \cdot \mathrm{Cay}(y)$ and $A = \mathcal{B}$. ∎

6.2. Algebras obtained by (commutative) Cayley–Dickson process

Note that if the mapping $\langle ,\rangle \colon V^2 \to Z(\mathcal{B})$ is trivial, then the algebra $\mathcal{A}$ is just a split null extension of the algebra $M_2(\mathcal{B})$ by a bimodule V^2. In this case, V may be an arbitrary associative $\mathcal{B}$-module (annihilated by $[\mathcal{B}, \mathcal{B}]$ if $\mathcal{B}$ is not commutative). For instance, when $\mathcal{B} = F$ and $V = F$ we get in this way the algebra $\mathcal{A} = M_2(F) \oplus \mathrm{Cay}$.

If the mapping $\langle ,\rangle \colon V^2 \to Z(\mathcal{B})$ is not trivial, then by (4.2) the rank of V as a $\mathcal{B}$-module is less than 3. Observe that the left side of (4.2) is $\mathcal{B}$-multilinear and skew-symmetric on u, v, w. Therefore, it holds when $\Lambda^3(V_{\mathcal{B}}) = 0$. In particular it holds if the rank of V is less or equal to 2. If $V \subseteq \mathcal{B} \cdot x$ then the mapping $\langle ,\rangle$ is trivial by skew-symmetry. Let us consider now the case when V is a 2-generated $\mathcal{B}$-module.

Let A be an associative commutative algebra and $\alpha \in A$. Denote by $\mathrm{CD}(M_2(A), \alpha)$ the algebra $M_2(A) \oplus vM_2(A)$ with a product defined by the following analogue of (2.3):

$$a \cdot b = ab, \quad a \cdot vb = v(a^*b), \quad vb \cdot a = v(ab), \quad va \cdot vb = \alpha(ba^*), \tag{6.2}$$

where $a, b \in M_2(A),\ a \mapsto a^*$ is the symplectic involution in $M_2(A)$.

The algebra $\mathrm{CD}(M_2(A),\alpha)$ is an alternative algebra containing $M_2(A)$ with the same unit. We will call it the *algebra obtained from $M_2(A)$ by the Cayley–Dickson process with parameter* α. The algebra $\mathrm{CD}(M_2(A),\alpha)$ is an octonion algebra if and only if the parameter α is invertible in A.

Theorem 6.2. *Let $\mathcal{B}$ be a unital commutative algebra, let $V=\mathcal{B}^2$, and let $\langle,\rangle\colon V^2\to\mathcal{B}$ be a skew-symmetric $\mathcal{B}$-bilinear mapping. Then the algebra $\mathcal{A}=M_2(\mathcal{B})\oplus V^2$ is isomorphic to an algebra* $\mathrm{CD}(M_2(\mathcal{B}),\alpha)$*, where $\alpha=-\langle(1,0),(0,1)\rangle$. Conversely, every algebra* $\mathrm{CD}(M_2(A),\alpha)$ *has this form.*

Proof. Let $\mathcal{A}=\mathrm{CD}(M_2(A),\alpha)$. Take $V=A^2=\{(a,b)\,|\,a,b\in A\}$, $(a,b)(1)=v\left(\begin{smallmatrix}0&0\\-b&a\end{smallmatrix}\right)$, and $(a,b)(2)=v\left(\begin{smallmatrix}b&-a\\0&0\end{smallmatrix}\right)\in vM_2(A)$. Then we have, as before, $\mathcal{A}=M_2(A)\oplus V(1)\oplus V(2)$, with $\langle(a,b),(c,d)\rangle=-\alpha\det\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. In particular, $\langle(1,0),(0,1)\rangle=-\alpha$.

Conversely, let $\mathcal{A}=M_2(\mathcal{B})\oplus V^2$, where $V\cong\mathcal{B}^2$ and $\langle(1,0),(0,1)\rangle=-\alpha$. Define the mapping $\varphi\colon V^2=V(1)\oplus V(2)\to vM_2(\mathcal{B})\subset\mathrm{CD}(M_2(\mathcal{B}),\alpha)$ by sending, for any $a,b\in\mathcal{B}$,

$$(a,b)(1)\mapsto v\begin{pmatrix}0&0\\-b&a\end{pmatrix},\quad (a,b)(2)\mapsto v\begin{pmatrix}b&-a\\0&0\end{pmatrix}.$$

It is easy to see that φ is an isomorphism of alternative $M_2(\mathcal{B})$-bimodules. Furthermore, let $x=(a,b),\ y=(c,d)\in V=\mathcal{B}^2$, then we have

$$\begin{aligned}\langle x,y\rangle&=\langle(a,b),(c,d)\rangle=\langle a(1,0)+b(0,1),c(1,0)+d(0,1)\rangle\\&=(ad-bc)\langle(1,0),(0,1)\rangle=-\alpha(ad-bc).\end{aligned}$$

Let $z=(e,f),\ t=(g,h)\in V$; then we have by (4.10),

$$(x,y)(z,t)=\begin{pmatrix}-\langle x,t\rangle&-\langle y,t\rangle\\\langle x,z\rangle&\langle y,z\rangle\end{pmatrix}=-\alpha\begin{pmatrix}-ah+bg&-ch+dg\\af-be&cf-de\end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\varphi(x,y)\cdot\varphi(z,t)&=v\begin{pmatrix}d&-c\\-b&a\end{pmatrix}\cdot v\begin{pmatrix}h&-g\\-f&e\end{pmatrix}=\alpha\begin{pmatrix}h&-g\\-f&e\end{pmatrix}\cdot\begin{pmatrix}a&c\\b&d\end{pmatrix}\\&=\alpha\begin{pmatrix}ah-bg&ch-dg\\-af+be&-cf+ed\end{pmatrix}.\end{aligned}$$

Therefore, the mapping

$$\mathrm{id}+\varphi:\mathcal{A}=M_2(\mathcal{B})\oplus V^2\to\mathrm{CD}(M_2(\mathcal{B}),\alpha)=M_2(\mathcal{B})\oplus vM_2(\mathcal{B})$$

is an isomorphism. ■

6.3. Algebras obtained by noncommutative Cayley–Dickson process

Let us now generalize the Cayley–Dickson process for non-commutative coefficient algebras. Let A be a unital associative algebra, not necessarily commutative, and let $\alpha\in A$ be such that $\alpha A\subseteq Z(A)$. Denote $\mathrm{NCD}(M_2(A),\alpha)=M_2(A)\oplus vM_2(\bar{A})$, where $\bar{A}=A/[A,A]A$, and define a product in it by setting

$$a\cdot b=ab,\quad a\cdot v\bar{b}=v(\bar{a}^*\bar{b}),\quad v\bar{b}\cdot a=v(\bar{a}\bar{b}),\quad v\bar{a}\cdot v\bar{b}=\alpha(b_1a_1^*),\tag{6.3}$$

where $a, b \in M_2(A)$, $\bar{a}, \bar{b}$ are their images in $M_2(\bar{A})$, $\bar{a} \mapsto \bar{a}^*$ is the symplectic involution in $M_2(\bar{A})$, and $a_1^*, b_1 \in M_2(A)$ are some pre-images of $\bar{a}^*, \bar{b}$ under the canonical epimorphism $M_2(A) \to M_2(\bar{A})$. Observe that the last product in (6.3) is defined correctly since $\alpha[A, A] = 0$.

Proposition 6.3. *The algebra* $\mathrm{NCD}(M_2(A), \alpha)$ *is a unital alternative algebra that contains* $M_2(A)$ *with the same unit.*

Proof. Denote $I = [A, A]A$. Then $M_2(I)$ is an ideal of $\mathrm{NCD}(M_2(A), \alpha)$ which annihilates $vM_2(\bar{A})$ and is annihilated by α; moreover, $\mathrm{NCD}(M_2(A), \alpha)/I \cong \mathrm{CD}(M_2(\bar{A}), \alpha)$. Therefore, the $M_2(A)$-bimodule $vM_2(\bar{A})$ is in fact an $M_2(\bar{A})$-bimodule, and since the algebra $\mathrm{CD}(M_2(\bar{A}), \alpha)$ is alternative, $vM_2(\bar{A})$ is an alternative $M_2(A)$-bimodule. In this way, it suffices to check the alternativity identities only when we have at least two arguments belonging to $vM_2(\bar{A})$.

For any $a, b \in M_2(A)$, we have

$$(a, v\bar{b}, v\bar{b}) = (v(\bar{a}^*\bar{b})) \cdot v\bar{b} - a(\alpha b b_1^*) = \alpha\, b(b_1^* a) - \alpha\, a(b b_1^*),$$

where $\overline{b_1^*} = \bar{b}^*$. Consider

$$\overline{b(b_1^* a) - a(b b_1^*)} = \bar{b}(\bar{b}^*\bar{a}) - \bar{a}(\bar{b}\bar{b}^*) = (\det \bar{b})\bar{a} - \bar{a}(\det \bar{b}) = \bar{0}.$$

Thus $b(b_1^* a) - a(b b_1^*) \in M_2(I)$ and $\alpha(b(b_1^* a) - a(b b_1^*)) = 0$.

Furthermore,

$$(v\bar{a}, v\bar{b}, v\bar{b}) = (\alpha b a_1^*) \cdot v\bar{b} - v\bar{a} \cdot (\alpha b b_1^*) = \alpha v((\bar{a}\bar{b}^*)\bar{b} - (\bar{b}\bar{b}^*)\bar{a}) = 0.$$

Finally, consider, for $c \in M_2(A)$,

$$\begin{aligned}(v\bar{a}, v\bar{b}, c) + (v\bar{a}, c, v\bar{b}) &= \alpha\, b a_1^* \cdot c - v\bar{a} \cdot v(\bar{c}\bar{b}) + v(\bar{c}\bar{a}) \cdot v\bar{b} - v\bar{a} \cdot v(\bar{c}^*\bar{b}) \\ &= \alpha\, (b a_1^* \cdot c - cb \cdot a_1^* + b \cdot a_1^* c_1^* - c_1^* b \cdot a_1^*).\end{aligned}$$

We have

$$\begin{aligned}\overline{b a_1^* \cdot c - cb \cdot a_1^* + b \cdot a_1^* c_1^* - c_1^* b \cdot a_1^*} &= \bar{b}\bar{a}^* \cdot \bar{c} - \bar{c}\bar{b} \cdot \bar{a}^* + \bar{b} \cdot \bar{a}^*\bar{c}^* - \bar{c}^*\bar{b} \cdot \bar{a}^* \\ &= \bar{b}\bar{a}^*\, t(\bar{c}) - t(\bar{c})\, \bar{b}\bar{a}^* = \bar{0}.\end{aligned}$$

Hence $b a_1^* \cdot c - cb \cdot a_1^* + b \cdot a_1^* c_1^* - c_1^* b \cdot a_1^* \in M_2(I)$ and $\alpha\, (b a_1^* \cdot c - cb \cdot a_1^* + b \cdot a_1^* c_1^* - c_1^* b \cdot a_1^*) = 0$.

We have proved that the algebra $\mathrm{NCD}(M_2(A), \alpha)$ is right alternative. Similarly, one can check that it is left alternative. ∎

Now we can generalize Theorem 6.2 to the case when $\mathcal{B}$ is not commutative.

Theorem 6.4. *Let* $\mathcal{B}$ *be a unital associative algebra,* $\overline{\mathcal{B}} = \mathcal{B}/[\mathcal{B}, \mathcal{B}]\mathcal{B}$, $V = \overline{\mathcal{B}}^2$ *and let* $\langle , \rangle : V^2 \to \mathcal{B}$ *be a skew-symmetric* $\mathcal{B}$*-bilinear mapping. Then the algebra* $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$ *is isomorphic to the algebra* $\mathrm{NCD}(M_2(\mathcal{B}), \alpha)$*, where* $\alpha = -\langle (1, 0), (0, 1) \rangle$. *Conversely, every algebra* $\mathrm{NCD}(M_2(A), \alpha)$ *has this form.*

Proof. Let first $\mathcal{A} = \mathrm{NCD}(M_2(A), \alpha)$. Denote $\bar{A} = A/[A, A]A$ and take $V = \bar{A}^2 = \{(\bar{a}, \bar{b}) \mid a, b \in A\}$, $(\bar{a}, \bar{b})(1) = v\left(\begin{smallmatrix} 0 & 0 \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$, $(\bar{a}, \bar{b})(2) = v\left(\begin{smallmatrix} \bar{b} & -\bar{a} \\ 0 & 0 \end{smallmatrix}\right) \in vM_2(\bar{A})$; then we have, as before, $\mathcal{A} = M_2(A) \oplus V(1) \oplus V(2)$, with $\langle(\bar{a}, \bar{b}), (\bar{c}, \bar{d})\rangle = -\alpha(ad - bc)$. In particular, $\langle(\bar{1}, \bar{0}), (\bar{0}, \bar{1})\rangle = -\alpha$.

Conversely, let $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$, where $V \cong \overline{\mathcal{B}}^2$, $\overline{\mathcal{B}} = \mathcal{B}/[\mathcal{B}, \mathcal{B}]\mathcal{B}$ and $\langle(\bar{1}, \bar{0}), (\bar{0}, \bar{1})\rangle = -\alpha$. Define the mapping $\varphi\colon V^2 = V(1) \oplus V(2) \to vM_2(\overline{\mathcal{B}}) \subset \mathrm{NCD}(M_2(\mathcal{B}), \alpha)$ by sending, for any $a, b \in \mathcal{B}$,

$$(\bar{a}, \bar{b})(1) \mapsto v\begin{pmatrix} 0 & 0 \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad (\bar{a}, \bar{b})(2) \mapsto v\begin{pmatrix} \bar{b} & -\bar{a} \\ 0 & 0 \end{pmatrix}.$$

Then, as in the proof of Theorem 6.2, one can easily see that the mapping

$$\mathrm{id} + \varphi : \mathcal{A} = M_2(\mathcal{B}) \oplus V^2 \to \mathrm{NCD}(M_2(\mathcal{B}), \alpha) = M_2(\mathcal{B}) \oplus vM_2(\overline{\mathcal{B}})$$

is an isomorphism. ∎

Algebras of type $\mathrm{CD}(M_2(A), \alpha)$ can be constructed for any commutative algebra A and any $\alpha \in A$. For algebras of type $\mathrm{NCD}(M_2(A), \alpha)$, one have to check the condition $[\alpha A, A] = 0$. For instance, one can take $A = F\langle x, y \mid y[x, y] = [x, y]y = 0\rangle$, with $\alpha = y^2$.

6.4. The case when V is not 2-generated

In all the examples considered above, the $\mathcal{B}$-module V was 2-generated. Here we will give an example where V is 3-generated.

Let $\mathcal{B}$ be a commutative unital algebra, let $a, b, c \in \mathcal{B}$, and let $V = \mathcal{B}^3/I$, where $I = \mathcal{B} \cdot (a, b, c)$. Denote $e_1 = (1, 0, 0) + I$, $e_2 = (0, 1, 0) + I$ and $e_3 = (0, 0, 1) + I$. Then we have $V = \mathcal{B} \cdot e_1 + \mathcal{B} \cdot e_2 + \mathcal{B} \cdot e_3$, where $a \cdot e_1 + b \cdot e_2 + c \cdot e_3 = 0$. Define a $\mathcal{B}$-bilinear skew-symmetric mapping $\langle , \rangle\colon V \times V \to \mathcal{B}$ by setting

$$\langle e_1, e_2\rangle = c, \quad \langle e_2, e_3\rangle = a, \quad \langle e_3, e_1\rangle = b.$$

One can easily check that the mapping $\langle , \rangle$ is defined correctly. Moreover, we have

$$\langle e_1, e_2\rangle e_3 + \langle e_2, e_3\rangle e_1 + \langle e_3, e_1\rangle e_2 = c \cdot e_3 + a \cdot e_1 + b \cdot e_2 = 0,$$

that is, identity (4.2) is true for $u = e_1$, $v = e_2$, $w = e_3$. Since the left side of (4.2) is skew-symmetric and multilinear on u, v, w, it follows that (4.2) is valid in V. By Theorem 5.1, the algebra $\mathcal{A} = M_2(\mathcal{B}) \oplus V^2$ is a unital alternative algebra containing $M_2(\mathcal{B})$ as a unital subalgebra.

Observe that taking here $a = b = 0$, we get the algebra $\mathrm{CD}(\mathcal{B}, c)$ from Theorem 6.2.

Moreover, following the scheme from the previous section, this construction can be extended for noncommutative algebras $\mathcal{B}$. One has only to choose the elements $a, b, c \in \mathcal{B}$ such that $a\mathcal{B} + b\mathcal{B} + c\mathcal{B} \subset Z(\mathcal{B})$.

7. Open questions

(1) The first natural question which we leave open is the case when the algebra $\mathbb{H}$ is not split, that is, when $\mathbb{H}$ is a division algebra. This case is more complicated since while $\mathrm{Cay}_i \cdot \mathrm{Cay}_j = \alpha_{ij}\,\mathbb{H}$, the product $\mathrm{Reg}_{\,i}\,\mathbb{H} \cdot \mathrm{Reg}_{\,j}\,\mathbb{H} = \sum_{k=1}^{4} \alpha_{ij}^{k}\,\mathbb{H}$ for some $\alpha_{ij}^{k} \in Z(\mathcal{A})$, and instead of Plücker relations (4.8), the elements α_{ij}^{k} satisfy more complicated system of relations. We plan to consider this case in a forthcoming paper.

(2) An interesting question is to study the alternative algebras that contain $\mathbb{H}$ (or $\mathbb{H}$-*algebras*) from a categorical point of view. Clearly, the class of $\mathbb{H}$-algebras form a category, with morphisms being the homomorphisms acting identically on $\mathbb{H}$. Given an $\mathbb{H}$-bimodule V, the free $\mathbb{H}$-algebra over V or *tensor algebra* $\mathbb{H}[V]$ *of the bimodule* V plays a role of a free object in this category. When $V = V_a$ is associative, $V = \oplus_{i=1}^{m}\mathrm{Reg}_{\,i}\,\mathbb{H}$ and the algebra $\mathbb{H}[V]$ is associative and is isomorphic to $\mathbb{H} \otimes F\langle x_1, \dots, x_m\rangle$, where $F\langle x_1, \dots, x_m\rangle$ is the free associative algebra on m generators.

When $V = V_c$ is a Cayley $\mathbb{H}$-bimodule, $V = \oplus_{i=1}^{m}\mathrm{Cay}_i$, the situation is not so clear even in the split case. For $m = 1$, $\mathbb{H}[V] = \mathbb{H} \oplus \mathrm{Cay}$ with $\mathrm{Cay}^2 = 0$ is just a well-known 6-dimensional subalgebra of a split Cayley–Dickson algebra; for $n = 2$, we have $\mathbb{H}[V] \cong \mathrm{CD}(M_2(F[\alpha_{12}]), \alpha_{12})$, but for $n \geq 3$, the structure of the algebra $\mathbb{H}[V]$ is not known.

The situation is even more complicated for the mixed case, when $V = V_a \oplus V_c$ with $V_a,\ V_c \neq 0$, again except some trivial cases.

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