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# Cones, rectifiability, and singular integral operators

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**Abstract.** Let  $\mu$  be a Radon measure on  $\mathbb{R}^d$ . We define and study conical energies  $\mathcal{E}_{\mu,p}(x, V, \alpha)$ , which quantify the portion of  $\mu$  lying in the cone with vertex  $x \in \mathbb{R}^d$ , direction  $V \in G(d, d-n)$ , and aperture  $\alpha \in (0, 1)$ . We use these energies to characterize rectifiability and the big pieces of Lipschitz graphs property. Furthermore, if we assume that  $\mu$  has polynomial growth, we give a sufficient condition for  $L^2(\mu)$ -boundedness of singular integral operators with smooth odd kernels of convolution type.

## 1. Introduction

Let  $m < d$  be positive integers. Given an  $m$ -plane  $V \in G(d, m)$ , a point  $x \in \mathbb{R}^d$ , and  $\alpha \in (0, 1)$ , we define

$$K(x, V, \alpha) = \{y \in \mathbb{R}^d : \text{dist}(y, V + x) < \alpha|x - y|\}.$$

That is,  $K(x, V, \alpha)$  is an open cone centered at  $x$ , with direction  $V$ , and aperture  $\alpha$ .

Let  $0 < n < d$ . It is well known that if a set  $E \subset \mathbb{R}^d$  satisfies, for some  $V \in G(d, d-n)$ ,  $\alpha \in (0, 1)$ , the condition

$$(1.1) \quad x \in E \implies E \cap K(x, V, \alpha) = \emptyset,$$

then  $E$  is contained in some  $n$ -dimensional Lipschitz graph  $\Gamma$ , and  $\text{Lip}(\Gamma) \leq 1/\alpha$ , see, e.g., the proof of Lemma 15.13 in [42].

To what extent can we weaken the condition (1.1) and still get meaningful information about the geometry of  $E$ ? It depends on what we mean by “meaningful information”, naturally. One could ask for the rectifiability of  $E$ , or if  $E$  contains big pieces of Lipschitz graphs, or whether nice singular integral operators are bounded on  $L^2(E)$ . The aim of this paper is to answer these three questions.

### 1.1. Rectifiability

A measurable set  $E \subset \mathbb{R}^d$  is  $n$ -rectifiable if there exists a countable number of Lipschitz maps  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^d$  such that

$$\mathcal{H}^n\left(E \setminus \bigcup_i f_i(\mathbb{R}^n)\right) = 0,$$

where  $\mathcal{H}^n$  denotes the  $n$ -dimensional Hausdorff measure. More generally, a Radon measure  $\mu$  is said to be  $n$ -rectifiable if  $\mu \ll \mathcal{H}^n$  and there exists an  $n$ -rectifiable set  $E \subset \mathbb{R}^d$  such that  $\mu(\mathbb{R}^d \setminus E) = 0$ .

A measure-theoretic analogue of (1.1), well-suited to the study of rectifiability, is that of an *approximate tangent plane*. We recall the definition below.

For  $r > 0$ , we define the truncated cone

$$K(x, V, \alpha, r) = K(x, V, \alpha) \cap B(x, r),$$

and for  $0 < r < R$ , we define the doubly truncated cone

$$K(x, V, \alpha, r, R) = K(x, V, \alpha, R) \setminus K(x, V, \alpha, r).$$

Given a Radon measure  $\mu$  on  $\mathbb{R}^d$  and  $x \in \text{supp } \mu$ , the lower and upper densities of  $\mu$  at  $x$  are defined as

$$\Theta_*^n(\mu, x) = \liminf_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n} \quad \text{and} \quad \Theta^{n,*}(\mu, x) = \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^n}.$$

Recall that if  $\mu$  is  $n$ -rectifiable, then  $0 < \Theta_*^n(\mu, x) = \Theta^{n,*}(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \text{supp } \mu$ . In that case we set  $\Theta^n(\mu, x) := \Theta_*^n(\mu, x) = \Theta^{n,*}(\mu, x)$ .

**Definition 1.1.** We say that an  $n$ -plane  $W \in G(d, n)$  is an *approximate tangent plane* to a Radon measure  $\mu$  at  $x \in \text{supp } \mu$  if  $\Theta^{n,*}(\mu, x) > 0$ , and for every  $\alpha \in (0, 1)$ ,

$$(1.2) \quad \lim_{r \rightarrow 0} \frac{\mu(K(x, W^\perp, \alpha, r))}{r^n} = 0.$$

The following classical characterization of rectifiable measures holds.

**Theorem 1.2** (Theorem 9.1 of [27]). *Let  $\mu$  be a finite Radon measure on  $\mathbb{R}^d$  satisfying  $0 < \Theta^{n,*}(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Then the following are equivalent:*

- (a)  $\mu$  is  $n$ -rectifiable,
- (b) for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exists a unique approximate tangent plane to  $\mu$  at  $x$ ,
- (c) for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exist  $W_x \in G(d, n)$  and  $\alpha_x \in (0, 1)$  such that

$$(1.3) \quad \limsup_{r \rightarrow 0} \frac{\mu(K(x, W_x^\perp, \alpha_x, r))}{r^n} < (\alpha_x)^n \varepsilon(n) \Theta^{n,*}(\mu, x),$$

where  $\varepsilon(n)$  is a small dimensional constant.

The results we prove in this paper are of similar nature. More precisely, we introduce and study *conical energies*.

**Definition 1.3.** Suppose  $\mu$  is a Radon measure on  $\mathbb{R}^d$ , and  $x \in \text{supp } \mu$ . Let  $V \in G(d, d - n)$ ,  $\alpha \in (0, 1)$ ,  $1 \leq p < \infty$  and  $R > 0$ . We define the  $(V, \alpha, p)$ -conical energy of  $\mu$  at  $x$  up to scale  $R$  as

$$\mathcal{E}_{\mu,p}(x, V, \alpha, R) = \int_0^R \left( \frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

For  $E \subset \mathbb{R}^d$ , we set also  $\mathcal{E}_{E,p}(x, V, \alpha, R) = \mathcal{E}_{\mathcal{H}^n|_E,p}(x, V, \alpha, R)$ .

Note that the definition above depends on the dimension parameter  $n$ , so it would be more precise to say that  $\mathcal{E}_{\mu,p}(x, V, \alpha, R)$  is the  $n$ -dimensional  $(V, \alpha, p)$ -conical energy. For the sake of brevity, throughout the paper we will consider  $n$  to be fixed, and we will usually not point out this dependence. The same applies to other definitions.

We are ready to state our first result.

**Theorem 1.4.** Let  $1 \leq p < \infty$ . Suppose that  $\mu$  is a Radon measure on  $\mathbb{R}^d$  satisfying  $\Theta^{n,*}(\mu, x) > 0$  and  $\Theta_*^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Assume that for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exist some  $V_x \in G(d, d - n)$  and  $\alpha_x \in (0, 1)$  such that

$$(1.4) \quad \mathcal{E}_{\mu,p}(x, V_x, \alpha_x, 1) < \infty.$$

Then  $\mu$  is  $n$ -rectifiable.

Conversely, if  $\mu$  is  $n$ -rectifiable, then for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exists  $V_x \in G(d, d - n)$  such that for all  $\alpha \in (0, 1)$ , we have

$$(1.5) \quad \mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) < \infty.$$

**Remark 1.5.** The “necessary” part of Theorem 1.4 improves on Theorem 1.2 in the following way. Existence of approximate tangents means that the conical density simply converges to 0, while (1.5) means that the conical density satisfies a Dini-type condition, and converges to 0 rather fast.

**Remark 1.6.** Concerning the “sufficient” part of Theorem 1.4, clearly, condition (1.3) is weaker than (1.4). However, Theorem 1.4 has the following advantage over Theorem 1.2: we only require  $\Theta^{n,*}(\mu, x) > 0$  and  $\Theta_*^n(\mu, x) < \infty$  for our criterion to hold. In particular, we do not assume  $\mu \ll \mathcal{H}^n$ . It is not clear to the author how to show a criterion involving (1.3) or (1.2) without assuming a priori  $\mu \ll \mathcal{H}^n$ .

**Question 1.7.** Suppose  $\mu$  is a Radon measure on  $\mathbb{R}^d$  satisfying  $\Theta^{n,*}(\mu, x) > 0$  and  $\Theta_*^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Assume that for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exists an approximate tangent plane to  $\mu$  at  $x$ . Does this imply that  $\mu$  is  $n$ -rectifiable?

Let us mention that in recent years many similar characterizations of rectifiable measures have been obtained. By “similar” we mean the pointwise finiteness of a square function involving some flatness quantifying coefficients. The most famous coefficients of this type are  $\beta$  numbers, first introduced in [33] and further developed by David and Semmes [21], [22]. A necessary condition for rectifiability that uses  $\beta_p$  numbers was shown in [59], see Theorem 9.3 for the precise statement. Its sufficiency (under various assumptions on densities of the measure) was proved in [4], [9], [25], [51]. Measures carried by rectifiable curves are studied using  $\beta$  numbers in [3], [8], [9], [10], [38], [39], [45]; see also the survey [6].

Finiteness of a square function involving  $\alpha$  coefficients (defined in [56]) is shown to be necessary for rectifiability in [59]. The opposite implication is studied in [2], [5], [49]. In [15], [17], rectifiable measures were characterized using  $\alpha_2$  numbers, first defined in [57]. Square functions involving centers of mass are studied in [44] and [63]. Finally, [60], [62] are devoted to a square function involving  $\Delta$  numbers, where

$$\Delta_\mu(x, r) = \left| \frac{\mu(B(x, r))}{r^n} - \frac{\mu(B(x, 2r))}{(2r)^n} \right|.$$

For related characterizations of rectifiable measures in terms of tangent measures, see Chapter 16 of [42] and Section 5 of [53]. For a study of tangent points of Jordan curves in terms of  $\beta$  numbers, see [11], and for a generalization of this result for lower content regular sets of arbitrary dimension, see [64].

The behavior of conical densities on purely unrectifiable sets is studied in [14] and in Section 5 of [35]. In [14], [36], [37], [41], the relation between conical densities for higher dimensional sets and their porosity is investigated.

Higher order rectifiability in terms of approximate differentiability of sets is studied in [54]. In [24], Del Nin and Idu characterize  $C^{1,\alpha}$  rectifiable sets using approximate tangents parabolooids, essentially obtaining a  $C^{1,\alpha}$  counterpart of Theorem 1.2. See also [28] and [29] for related results.

We would also like to mention recent results of Badger and Naples that nicely complement Theorem 1.4. In Theorem D of [45], Naples showed that a modified version of (1.2) can be used to characterize pointwise doubling measures *carried by Lipschitz graphs*, that is, measures vanishing outside of a countable union of  $n$ -dimensional Lipschitz graphs. In an even more recent paper [7], Badger and Naples completely describe measures carried by  $n$ -dimensional Lipschitz graphs on  $\mathbb{R}^d$ . They use a Dini condition imposed on the so-called *conical defect*, and their condition is closely related to (1.4). Note the absence of densities in the assumptions (and conclusion) of their results. If one adds an assumption  $\Theta_*^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , then it actually follows from [7] that  $\mu$ -a.e. finiteness of their conical Dini function implies that  $\mu$  is  $n$ -rectifiable. We would like to stress, however, that neither Theorem 1.4 implies the results from [7], nor the other way around.

### 1.2. Big pieces of Lipschitz graphs

Before stating our next theorem, we need to recall some definitions.

**Definition 1.8.** We say that  $E \subset \mathbb{R}^d$  is *n-Ahlfors–David regular* (abbreviated as  $n$ -ADR) if there exist constants  $C_0, C_1 > 0$  such that for all  $x \in E$  and  $0 < r < \text{diam}(E)$ ,

$$C_0 r^n \leq \mathcal{H}^n(E \cap B(x, r)) \leq C_1 r^n.$$

The constants  $C_0, C_1$  will be referred to as the ADR constants of  $E$ .

**Definition 1.9.** We say that an  $n$ -ADR set  $E \subset \mathbb{R}^d$  has *big pieces of Lipschitz graphs* (BPLG) if there exist constants  $\kappa, L > 0$ , such that the following holds.

For all balls  $B$  centered at  $E$ ,  $0 < r(B) < \text{diam}(E)$ , there exists a Lipschitz graph  $\Gamma_B$ , with  $\text{Lip}(\Gamma_B) \leq L$ , such that

$$\mathcal{H}^n(E \cap B \cap \Gamma_B) \geq \kappa r(B)^n.$$

Sets with BPLG were studied, e.g., in [18], [22], [23] as one of the possible quantitative counterparts of rectifiability. Let us point out that the class of sets with BPLG is strictly smaller than the class of uniformly rectifiable sets, introduced in the seminal work of David and Semmes [21], [22]. An example of a uniformly rectifiable set that does not contain BPLG is due to Hrycak, although he never wrote it down, see the appendix of [1].

While there are available many characterizations of uniformly rectifiable sets, the sets containing BPLG are not as well understood. David and Semmes showed in [23] that a set contains BPLG if and only if it has *big projections* and satisfies the *weak geometric lemma*. We refer the reader to [23] or Section I.1.5 of [22] for details.

Very recently, Orponen characterized the BPLG property in terms of the *big projections in plenty of directions* property [50], answering an old question of David and Semmes. A little before that, Martikainen and Orponen [40] characterized sets with BPLG in terms of  $L^2$  norms of their projections. Interestingly, the authors use the information about projections of an  $n$ -ADR set  $E$  to draw conclusions about intersections with cones of some subset  $E' \subset E$  with  $\mathcal{H}^n(E') \approx \mathcal{H}^n(E)$ . This in turn allows them to find a Lipschitz graph intersecting an ample portion of  $E'$ . We will use some of their techniques to prove a characterization of sets containing BPLG in terms of the following property.

**Definition 1.10.** Let  $1 \leq p < \infty$ . We say that a measure  $\mu$  has *big pieces of bounded energy for  $p$* , abbreviated as BPBE( $p$ ), if there exist constants  $\alpha, \kappa, M_0 > 0$  such that the following holds.

For all balls  $B$  centered at  $\text{supp } \mu$ ,  $0 < r(B) < \text{diam}(\text{supp } \mu)$ , there exist a set  $G_B \subset B$ , with  $\mu(G_B) \geq \kappa\mu(B)$ , and a direction  $V_B \in G(d, d - n)$  such that for all  $x \in G_B$ ,

$$(1.6) \quad \mathcal{E}_{\mu,p}(x, V_B, \alpha, r(B)) = \int_0^{r(B)} \left( \frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \leq M_0.$$

**Theorem 1.11.** Let  $1 \leq p < \infty$ . Suppose  $E \subset \mathbb{R}^d$  is  $n$ -ADR. Then  $E$  has BPLG if and only if  $\mathcal{H}^n|_E$  has BPBE( $p$ ).

**Remark 1.12.** In particular, for  $n$ -ADR sets, the condition BPBE( $p$ ) is equivalent to BPBE( $q$ ) for all  $1 \leq p, q < \infty$ .

**Remark 1.13.** In fact, one can show that an a priori slightly weaker condition than BPBE is already sufficient for BPLG. To be more precise, in (1.6) replace  $K(x, V_B, \alpha, r)$  with  $K(x, V_B, \alpha, r) \cap G_B$ , so that we get

$$(1.7) \quad \int_0^{r(B)} \left( \frac{\mathcal{H}^n(K(x, V_B, \alpha, r) \cap E \cap G_B)}{r^n} \right)^p \frac{dr}{r} \leq M_0.$$

We show that this “weak” BPBE is sufficient for BPLG in Proposition 10.1. It is obvious that (1.7) is also necessary for BPLG: if  $E$  contains BPLG, then choosing  $G_B = \Gamma_B$  as in Definition 1.9, one can pick the corresponding  $V_B$  and  $\alpha$  so that  $K(x, V_B, \alpha, r) \cap \Gamma_B = \emptyset$ .

It is tempting to consider also the following definition.

**Definition 1.14.** Let  $1 \leq p < \infty$ . We say that a measure  $\mu$  has *bounded mean energy* (BME) for  $p$  if there exist constants  $\alpha, M_0 > 0$ , and for every  $x \in \text{supp } \mu$ , there exists a direction  $V_x \in G(d, d - n)$  such that the following holds.

For all balls  $B$  centered at  $\text{supp } \mu$ ,  $0 < r(B) < \text{diam}(\text{supp } \mu)$ , we have

$$\int_B \mathcal{E}_{\mu,p}(x, V_x, \alpha, r(B)) d\mu(x) = \int_B \int_0^{r(B)} \left( \frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} d\mu(x) \leq M_0 \mu(B).$$

In other words, we require  $\mu(K(x, V_x, \alpha, r))^p r^{-np} \frac{dr}{r} d\mu(x)$  to be a Carleson measure. This condition looks quite natural due to many similar characterizations of uniform rectifiability, e.g., the geometric lemma of [21], [22] or the results from [56], [57].

It is easy to see, using the compactness of  $G(d, d - n)$  and Chebyshev’s inequality, that BME for  $p$  implies BPBE( $p$ ). However, the reverse implication does not hold. In [16], we give an example of a set containing BPLG that does not satisfy BME. The problem is the following. In the definition above, the plane  $V_x$  is fixed for every  $x \in \text{supp } \mu$  once and for all, and we do not allow it to change between different scales. This is too rigid.

**Question 1.15.** Can one modify the definition of BME, allowing the planes  $V_x$  to depend on the scale  $r$ , so that the modified BME could be used to characterize BPLG, or uniform rectifiability?

It seems likely that every uniformly rectifiable measure would satisfy such relaxed BME (the idea would be similar to what is done in Section 9: use the  $\beta$ -numbers characterization of UR to get an upper bound for  $\beta$ -numbers, and then estimate the measure of cones from above by the  $\beta$ -numbers). It is less clear whether this relaxed BME would imply uniform rectifiability. Perhaps additional control for the oscillation of  $V_{x,r}$  would be needed.

### 1.3. Boundedness of SIOs

We will be concerned with singular integral operators of convolution type, with odd  $C^2$  kernels  $k: \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$  satisfying, for some constant  $C_k > 0$ ,

$$(1.8) \quad |\nabla^j k(x)| \leq \frac{C_k}{|x|^{n+j}} \quad \text{for } x \neq 0 \text{ and } j \in \{0, 1, 2\}.$$

We will denote the class of all such kernels by  $\mathcal{K}^n(\mathbb{R}^d)$ . Note that these kernels are particularly nice examples of Calderón–Zygmund kernels (see p.48 of [58] for definition), which will let us use many tools from the Calderón–Zygmund theory. Since the measures we work with may be non-doubling, our main reference will be Chapter 2 of [58]. For the more classical theory, we refer the reader to Chapter 5 of [31], Chapter 4 of [32].

**Definition 1.16.** Given a kernel  $k \in \mathcal{K}^n(\mathbb{R}^d)$ , a constant  $\varepsilon > 0$ , and a (possibly complex) Radon measure  $\nu$ , we set

$$T_\varepsilon \nu(x) = \int_{|x-y|>\varepsilon} k(y-x) d\nu(y), \quad x \in \mathbb{R}^d.$$

For a fixed positive Radon measure  $\mu$  and all functions  $f \in L^1_{\text{loc}}(\mu)$ , we define

$$T_{\mu,\varepsilon} f(x) = T_\varepsilon(f\mu)(x).$$

We say that  $T_\mu$  is bounded in  $L^2(\mu)$  if all  $T_{\mu,\varepsilon}$  are bounded in  $L^2(\mu)$ , uniformly in  $\varepsilon > 0$ . Let  $M(\mathbb{R}^d)$  denote the space of all finite real Borel measures on  $\mathbb{R}^d$ . When endowed with the total variation norm  $\|\cdot\|_{\text{TV}}$ , this is a Banach space. We say that  $T$  is bounded from  $M(\mathbb{R}^d)$  to  $L^{1,\infty}(\mu)$  if there exists a constant  $C$  such that for all  $\nu \in M(\mathbb{R}^d)$  and all  $\lambda > 0$ ,

$$\mu(\{x \in \mathbb{R}^d : |T_\varepsilon \nu(x)| > \lambda\}) \leq \frac{C \|\nu\|_{\text{TV}}}{\lambda},$$

uniformly in  $\varepsilon > 0$ .

The main motivation for developing the theory of quantitative rectifiability was finding necessary and/or sufficient conditions for boundedness of singular integral operators. David and Semmes showed in [21] that, for an  $n$ -ADR set, the  $L^2$  boundedness of all singular integral operators with smooth and odd kernels is equivalent to uniform rectifiability. The famous David–Semmes problem asks whether the  $L^2$  boundedness of a single SIO, the Riesz transform, is already sufficient for uniform rectifiability. It was shown that the answer is affirmative for  $n = 1$  in [43], for  $n = d - 1$  in [46], and the problem is open for other  $n$ .

In the non-ADR setting less is known. A necessary condition for the boundedness of SIOs in  $L^2(\mu)$ , where  $\mu$  is Radon and non-atomic, is the *polynomial growth condition*:

$$(1.9) \quad \mu(B(x, r)) \leq C_1 r^n \quad \text{for all } x \in \text{supp } \mu, r > 0,$$

see Proposition 1.4 in Part III of [19]. Eiderman, Nazarov and Volberg showed in [26] that if  $\mu$  is a measure on  $\mathbb{R}^2$ ,  $\mathcal{H}^1(\text{supp } \mu) < \infty$ , and  $\mu$  has vanishing lower 1-density, then the Riesz transform is unbounded. Their result was generalized to SIOs associated to gradients of single layer potentials in [13]. Nazarov, Tolsa and Volberg proved in [47] that if  $E \subset \mathbb{R}^{n+1}$  satisfies  $\mathcal{H}^n(E) < \infty$  and the  $n$ -dimensional Riesz transform is bounded in  $L^2(\mathcal{H}^n|_E)$ , then  $E$  is  $n$ -rectifiable. That the same is true for gradients of single layer potentials was shown by Prat, Puliatti and Tolsa in [52].

Concerning sufficient conditions for boundedness of SIOs, in [4] Azzam and Tolsa estimated the Cauchy transform of a measure using its  $\beta$  numbers. Their method was further developed by Girela-Sarrión [30]. He gives a sufficient condition for boundedness of singular integral operators with kernels in  $\mathcal{K}^n(\mathbb{R}^d)$  in terms of  $\beta$  numbers. We use the main lemma from [30] to prove the following criterion involving 2-conical energy.

**Theorem 1.17.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^d$  satisfying the polynomial growth condition (1.9). Suppose that  $\mu$  has BPBE(2). Then all singular integral operators  $T_\mu$  with kernels  $k \in \mathcal{K}^n(\mathbb{R}^d)$  are bounded in  $L^2(\mu)$ , with norm depending only on BPBE constants, the polynomial growth constant  $C_1$ , and the constant  $C_k$  from (1.8).*

**Remark 1.18.** A similar result, with BPBE(2) condition replaced by BPBE(1) condition, has already been shown in Theorem 10.2 of [12]. It is easy to see that for measures satisfying the polynomial growth (1.9), we have

$$\mathcal{E}_{\mu,2}(x, V, \alpha, R) \leq C_1 \mathcal{E}_{\mu,1}(x, V, \alpha, R),$$

and so BPBE(2) is a weaker assumption than BPBE(1). Moreover, in [16] we show that the measure constructed in [34] does not satisfy BPBE(1), but it trivially satisfies BPBE(2). Hence, Theorem 1.17 really does improve on Theorem 10.2 of [12].

**Remark 1.19.** Recall that for  $n$ -ADR sets, the condition  $\text{BPBE}(p)$  was equivalent to BPLG, regardless of  $p$ . By the remark above, it is clear that if we replace the  $n$ -ADR condition with polynomial growth (i.e., if we drop the lower regularity assumption), then the condition  $\text{BPBE}(p)$  is no longer independent of  $p$ . In general we only have one implication: for  $1 \leq p < q < \infty$ ,

$$\text{BPBE}(p) \implies \text{BPBE}(q).$$

**Remark 1.20.** Theorem 1.17 is sharp in the following sense. If one tried to weaken the assumption  $\text{BPBE}(2)$  to  $\text{BPBE}(p)$  for some  $p > 2$ , then the theorem would no longer hold. The reason is that for any  $p > 2$ , one may construct a Cantor-like probability measure  $\mu$ , say on a unit square in  $\mathbb{R}^2$ , that has linear growth and such that for all  $x \in \text{supp } \mu$ ,

$$\int_0^1 \left( \frac{\mu(B(x, r))}{r} \right)^p \frac{dr}{r} \lesssim 1$$

(that is, a much stronger version of  $\text{BPBE}(p)$  holds), but nevertheless, the Cauchy transform is not bounded on  $L^2(\mu)$ . See Chapter 4.7 of [58].

Sadly, the implication of Theorem 1.17 cannot be reversed. Let  $E \subset \mathbb{R}^2$  be the previously mentioned example of a 1-ADR uniformly rectifiable set that does not contain BPLG. In particular, by Theorem 1.11,  $E$  does not satisfy  $\text{BPBE}(p)$  for any  $p$ . Nevertheless, by the results of David and Semmes [21], all nice singular integral operators are bounded on  $L^2(E)$ .

**1.4. Cones and projections**

Let us note that Theorem 10.2 of [12] was merely a tool to prove the main result of [12]: a lower bound on analytic capacity involving  $L^2$  norms of projections. Chang and Tolsa proved also an interesting inequality showing the connection between 1-conical energy and  $L^2$  norms of projections. We introduce additional notation before stating their result.

**Definition 1.21.** Suppose  $V \in G(d, d - n)$ ,  $\alpha \in (0, 1)$ , and  $1 \leq p < \infty$ . Let  $B$  be a ball. The  $(V, \alpha, p)$ -conical energy of  $\mu$  in  $B$  is

$$\mathcal{E}_{\mu,p}(B, V, \alpha) = \int_B \int_0^{r(B)} \left( \frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} d\mu(x).$$

We define also

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) = \int_{\mathbb{R}^d} \int_0^\infty \left( \frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} d\mu(x).$$

We will often suppress the arguments  $V, \alpha$ , and write simply  $\mathcal{E}_{\mu,p}(B), \mathcal{E}_{\mu,p}(\mathbb{R}^d)$ .

**Remark 1.22.** For  $p = 1$ , we have

$$\begin{aligned} (1.10) \quad \int_0^\infty \frac{\mu(K(x, V, \alpha, r))}{r^n} \frac{dr}{r} &= \int_{K(x, V, \alpha)} \int_{|x-y|}^\infty \frac{dr}{r^{n+1}} d\mu(y) \\ &= n^{-1} \int_{K(x, V, \alpha)} \frac{1}{|x-y|^n} d\mu(y), \end{aligned}$$



and so

$$(1.11) \quad \mathcal{E}_{\mu,1}(\mathbb{R}^d, V, \alpha) = n^{-1} \int_{\mathbb{R}^d} \int_{K(x,V,\alpha)} \frac{1}{|x-y|^n} d\mu(y) d\mu(x).$$

In their paper, Chang and Tolsa were working with the expression from the right-hand side above.

Given  $V \in G(d, m)$ , we will denote by  $\pi_V: \mathbb{R}^d \rightarrow V$  the orthogonal projection onto  $V$ , and by  $\pi_V^\perp: \mathbb{R}^d \rightarrow V^\perp$ , the orthogonal projection onto  $V^\perp$ . We endow  $G(d, m)$  with the natural probability measure  $\gamma_{d,m}$ , see Chapter 3 of [42], and with a metric  $d(V, W) = \|\pi_V - \pi_W\|_{\text{op}}$ , where  $\|\cdot\|_{\text{op}}$  is the operator norm. We write  $\pi_V \mu$  to denote the image measure of  $\mu$  by the projection  $\pi_V$ . If  $\pi_V \mu \ll \mathcal{H}^n|_V$ , then we identify  $\pi_V \mu$  with its density with respect to  $\mathcal{H}^n|_V$ , and  $\|\pi_V \mu\|_{L^2(V)}$  denotes the  $L^2$  norm of this density. Otherwise, we set  $\|\pi_V \mu\|_{L^2(V)} = \infty$ .

**Proposition 1.23** (Corollary 3.11 of [12]). *Let  $V_0 \in G(d, n)$  and  $\alpha > 0$ . Then, there exist constants  $\lambda, C > 1$  such that for any finite Borel measure  $\mu$  in  $\mathbb{R}^d$  (see (1.11)),*

$$\begin{aligned} \mathcal{E}_{\mu,1}(\mathbb{R}^d, V_0^\perp, \alpha) &\approx \int_{\mathbb{R}^d} \int_{K(x,V_0^\perp,\alpha)} \frac{1}{|x-y|^n} d\mu(y) d\mu(x) \\ &\leq C \int_{B(V_0,\lambda\alpha)} \|\pi_V \mu\|_{L^2(V)}^2 d\gamma_{d,n}(V). \end{aligned}$$

Let us note that a variant of this estimate was also proved in [40], for a measure of the form  $\mu = \mathcal{H}^n|_E$ , with  $E$  a suitable set.

The inequality converse to that of Proposition 1.23 in general is not true, but it is not far off. Additional assumptions on  $\mu$  are necessary, and one has to add another term to the left-hand side. See Remark 3.12, Appendix A, of [12].

In the light of results mentioned above, as well as the characterization of sets with BPLG from [40], the connection between  $L^2$  norms of projections and cones is quite striking. Note that the proof of the Besicovitch–Federer projection theorem also involves careful analysis of measure in cones, see Chapter 18 of [42]. Exploring further the relationship between cones and projections would be very interesting.

**Question 1.24.** Is it possible to obtain an inequality similar to that of Proposition 1.23, but with  $\mathcal{E}_{\mu,2}$  on the left-hand side, and some quantity involving  $\pi_V \mu$  on the right-hand side?

### 1.5. Organization of the article

In Section 2 we introduce additional notation and recall the properties of the David–Mattila lattice  $\mathcal{D}_\mu$ . In Section 3 we state our main lemma, a corona decomposition-like result. Roughly speaking, it says that if a measure  $\mu$  has polynomial growth, and for some  $V \in G(d, d-n)$ ,  $\alpha \in (0, 1)$ , we have  $\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty$ , then we can decompose  $\mathcal{D}_\mu$  into a family of trees such that:

- for every tree,  $\mu$  is “well-behaved” at the scales and locations of the tree,
- we have a good control on the number of trees (see (3.2)).

We prove the main lemma in Sections 4–6. Let us point out that in the case  $p = 1$ , an analogous corona decomposition was already shown in Lemma 5.1 of [12]. Our proof follows the same general strategy, but some key estimates had to be done differently (most notably the estimates in Section 5).

In Section 7 we show how to use the main lemma and results from [30] to get Theorem 1.17. Sections 8 and 9 are dedicated to the proof of Theorem 1.4. The “sufficient part” follows from our main lemma, while the “necessary part” is deduced from the corresponding  $\beta_2$  result of Tolsa [59]. Finally, we prove Theorem 1.11 in Sections 10 and 11. To show the “sufficient part”, we use the results from [40], whereas the “necessary part” follows from a simple geometric argument.

## 2. Preliminaries

### 2.1. Additional notation

We will write  $A \lesssim B$  if there exists some constant  $C$  such that  $A \leq CB$ .  $A \approx B$  means that  $A \lesssim B \lesssim A$ . If the constant  $C$  depends on some parameter  $t$ , we will write  $A \lesssim_t B$ . We usually omit the dependence on  $n$  and  $d$ .

$B(x, r)$  stands for the open ball  $\{y \in \mathbb{R}^d : |y - x| < r\}$ . On the other hand, if  $B$  is a ball, then  $r(B)$  denotes its radius.

A characteristic function of a set  $E \subset \mathbb{R}^d$  will be denoted by  $\mathbb{1}_E$ .

Given a Radon measure  $\mu$  and a ball  $B = B(x, r)$ , we set

$$\Theta_\mu(B) = \Theta_\mu(x, r) = \frac{\mu(B)}{r^n}.$$

If  $T$  is a singular integral operator as in Definition 1.16, then the associated maximal operator  $T_*$  is defined as

$$T_*v(x) = \sup_{\varepsilon > 0} |T_\varepsilon v(x)| \quad \text{for } v \in M(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Given an  $n$ -plane  $L$ ,  $\pi_L$  will denote the orthogonal projection onto  $L$ , and  $\pi_L^\perp$  will denote the orthogonal projection onto  $L^\perp$ .

Given two bounded sets  $E, F \subset \mathbb{R}^d$ ,  $\text{dist}_H(E, F)$  will stand for the Hausdorff distance between  $E$  and  $F$ .

### 2.2. David–Mattila lattice

In the proof of Theorem 1.17 we will use the lattice of “dyadic cubes” constructed by David and Mattila [20]. Their construction depends on parameters  $C_0 > 1$  and  $A_0 > 5000C_0$ . The parameters can be chosen in such a way that the following lemmas hold.

**Lemma 2.1** (Theorem 3.2, Lemma 5.28 of [20]). *Let  $\mu$  be a Radon measure on  $\mathbb{R}^d$ ,  $E = \text{supp } \mu$ . There exists a sequence of partitions of  $E$  into Borel subsets  $Q \in \mathcal{D}_{\mu,k}$ ,  $k \geq 0$ , with the following properties:*

- (a) *For each integer  $k \geq 0$ ,  $E$  is the disjoint union of the “cubes”  $Q \in \mathcal{D}_{\mu,k}$ , and if  $k < l$ ,  $Q \in \mathcal{D}_{\mu,l}$ , and  $R \in \mathcal{D}_{\mu,k}$ , then either  $Q \cap R = \emptyset$  or else  $R \subset Q$ .*

- (b) *The general position of the cubes  $Q$  can be described as follows. For each  $k \geq 0$  and each cube  $Q \in \mathcal{D}_{\mu,k}$ , there exists a ball  $B(Q) = B(x_Q, r(Q))$  such that*

$$x_Q \in Q, \quad A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k},$$

$$E \cap B(Q) \subset Q \subset E \cap 28B(Q) = E \cap B(x_Q, 28r(Q)),$$

and the balls  $5B(Q)$ ,  $Q \in \mathcal{D}_{\mu,k}$ , are disjoint.

- (c) *Denote by  $\mathcal{D}_{\mu}^{\text{db}}$  the family of doubling cubes, i.e.,  $Q \in \mathcal{D}_{\mu} = \bigcup_{k \geq 0} \mathcal{D}_{\mu,k}$  satisfying*

$$(2.1) \quad \mu(100B(Q)) \leq C_0 \mu(B(Q)).$$

*Then, for any  $R \in \mathcal{D}_{\mu}$ , there exists a family  $\{Q_i\}_{i \in I} \subset \mathcal{D}_{\mu}^{\text{db}}$  such that  $Q_i \subset R$  and  $\mu(R \setminus \bigcup_i Q_i) = 0$ .*

For any  $Q \in \mathcal{D}_{\mu}$ , we denote by  $\mathcal{D}_{\mu}(Q)$  the family of  $P \in \mathcal{D}_{\mu}$  such that  $P \subset Q$ . Given  $Q \in \mathcal{D}_{\mu,k}$ , we set  $J(Q) = k$  and  $\ell(Q) = 56C_0 A_0^{-k}$ . Note that  $r(Q) \approx \ell(Q)$ .

We define  $B_Q = 28B(Q) = B(x_Q, 28r(Q))$ , so that

$$E \cap \frac{1}{28}B_Q \subset Q \subset B_Q.$$

Note that if  $Q \subset P$ , then  $B_Q \subset B_P$ .

**Lemma 2.2** (Lemma 2.4 of [4]). *Suppose the cubes  $Q \in \mathcal{D}_{\mu}$ ,  $R \in \mathcal{D}_{\mu}$ ,  $Q \subset R$ , are such that all the intermediate cubes  $Q \subsetneq S \subsetneq R$  are non-doubling, i.e.,  $S \notin \mathcal{D}_{\mu}^{\text{db}}$ . Then*

$$(2.2) \quad \Theta_{\mu}(100B(Q)) \leq (C_0 A_0)^d A_0^{-9d(J(Q)-J(R)-1)} \Theta_{\mu}(100B(R))$$

and

$$\sum_{S \in \mathcal{D}_{\mu}: Q \subset S \subset R} \Theta_{\mu}(100B(S)) \lesssim \Theta_{\mu}(100B(R)).$$

Let us remark that the constant  $9d$  in the exponent of (2.2) could be replaced by any other positive constant, if  $C_0$  and  $A_0$  are chosen suitably, see equation (5.30) of [20].

**Lemma 2.3** (Lemma 4.5 of [12]). *Let  $R \in \mathcal{D}_{\mu}^{\text{db}}$ . Then there exists another doubling cube  $Q \subsetneq R$ ,  $Q \in \mathcal{D}_{\mu}^{\text{db}}$ , such that*

$$\mu(Q) \approx \mu(R) \quad \text{and} \quad \ell(Q) \approx \ell(R).$$

From now on we will treat  $C_0$  and  $A_0$  as absolute constants, and we will not track the dependence on them in our estimates.

### 3. Main lemma

In order to formulate our main lemma we need to introduce some vocabulary.

Let  $\mu$  be a compactly supported Radon measure with polynomial growth (1.9). Suppose  $\mathcal{D}_{\mu}$  is the associated David–Mattila lattice, and assume that

$$R_0 = \text{supp } \mu \in \mathcal{D}_{\mu}$$

is the biggest cube.

Given a family of cubes  $\text{Top} \subset \mathcal{D}_\mu^{\text{db}}$  satisfying  $R_0 \in \text{Top}$ , we define the following families associated to each  $R \in \text{Top}$ :

- $\text{Next}(R)$  is the family of maximal cubes  $Q \in \text{Top}$  strictly contained in  $R$ ,
- $\text{Tr}(R)$  is the family of cubes  $Q \in \mathcal{D}_\mu$  contained in  $R$ , but *not* contained in any  $P \in \text{Next}(R)$ .

Clearly,  $\mathcal{D}_\mu = \bigcup_{R \in \text{Top}} \text{Tr}(R)$ . Define

$$\text{Good}(R) = R \setminus \bigcup_{Q \in \text{Next}(R)} Q.$$

**Lemma 3.1** (Main lemma). *Let  $\mu$  be a compactly supported Radon measure on  $\mathbb{R}^d$ . Suppose there exists  $r_0 > 0$  such that for all  $x \in \text{supp } \mu$ ,  $0 < r \leq r_0$ , we have*

$$(3.1) \quad \mu(B(x, r)) \leq C_1 r^n.$$

Assume further that for some  $V \in G(d, d - n)$ ,  $\alpha \in (0, 1)$ , and  $1 \leq p < \infty$ , we have  $\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty$ . Then there exists a family of cubes  $\text{Top} \subset \mathcal{D}_\mu^{\text{db}}$  and a corresponding family of Lipschitz graphs  $\{\Gamma_R\}_{R \in \text{Top}}$  satisfying:

- (i) the Lipschitz constants of  $\Gamma_R$  are uniformly bounded by a constant depending on  $\alpha$ ,
- (ii)  $\mu$ -almost all  $\text{Good}(R)$  is contained in  $\Gamma_R$ ,
- (iii) for all  $Q \in \text{Tr}(R)$ , we have  $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_R)$ .

Moreover, the following packing condition holds:

$$(3.2) \quad \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim_\alpha (C_1)^p \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha).$$

The implicit constant does not depend on  $r_0$ .

We prove the lemma above in Sections 4–6. From this point on, until the end of Section 6, we assume that  $\mu$  is a compactly supported Radon measure satisfying the growth condition (3.1), and that there exist  $V \in G(d, d - n)$ ,  $\alpha \in (0, 1)$ ,  $1 \leq p < \infty$ , such that

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty.$$

For simplicity, in our notation we will suppress the parameters  $V$  and  $\alpha$ . That is, we will write  $\mathcal{E}_{\mu,p}(\mathbb{R}^d) = \mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha)$ , as well as  $K = K(0, V, \alpha)$ ,  $K(x) = K(x, V, \alpha)$ , and  $K(x, r) = K(x, V, \alpha, r)$ . Finally, given  $0 < r < R$ , set

$$K(x, r, R) = K(x, R) \setminus K(x, r).$$

**Parameters**

In the proof of Lemma 3.1 we will use a number of parameters. To make it easier to keep track of what depends on what, and at which point the parameters get fixed, we list them below. Recall that “ $C_1 = C_1(C_2)$ ” means that “the value of  $C_1$  depends on the value of  $C_2$ .”

- $A = A(p) > 1$  is the “HD” constant; it is fixed in Lemma 6.1.

- $\tau = \tau(\alpha, t)$  is the “LD” constant; it is fixed in (5.1).
- $M = M(\alpha) > 1$  is the “key estimate” constant; it is chosen in Lemma 4.3.
- $\eta = \eta(M, t) \in (0, 1)$  is the constant from the definition of  $\mathcal{E}_{\mu,p}(Q)$  in (4.1); it is fixed in the proof of Lemma 5.4.
- $t = t(M, \alpha) > M$  is the “ $t$ -neighbor” constant, see Section 4.3. It is fixed just below (5.6), but depends also on Lemma 4.5 and Lemma 4.7.
- $\Lambda = \Lambda(M) > 2M$  is the constant from Lemma 4.8.
- $\varepsilon = \varepsilon(\tau, \alpha, \eta) \in (0, 1)$  is the “BCE” constant; it is fixed in Lemma 5.4.

### 4. Construction of a Lipschitz graph $\Gamma_R$

Suppose  $R \in \mathcal{D}_\mu^{\text{db}}$ . In this section we will construct a corresponding tree of cubes  $\text{Tree}(R)$ , and a Lipschitz graph  $\Gamma_R$  that “approximates  $\mu$  at scales and locations from  $\text{Tree}(R)$ ”; see Lemma 4.8.

#### 4.1. Stopping cubes

Consider constants  $A \gg 1$ ,  $0 < \varepsilon \ll \tau \ll 1$ , and  $0 < \eta \ll 1$ , which will be fixed later on. Given  $Q \in \mathcal{D}_\mu$ , we set

$$(4.1) \quad \mathcal{E}_{\mu,p}(Q) = \frac{1}{\mu(Q)} \int_{2B_Q} \int_{\eta r(Q)}^{\eta^{-1}r(Q)} \left( \frac{\mu(K(x,r))}{r^n} \right)^p \frac{dr}{r} d\mu(x).$$

For any  $R \in \mathcal{D}_\mu^{\text{db}}$ , we define the following families of cubes:

- $\text{BCE}_0(R)$ , the family of big conical energy cubes, consisting of  $Q \in \mathcal{D}_\mu(R)$  such that

$$\sum_{Q \subset P \subset R} \mathcal{E}_{\mu,p}(P) > \varepsilon \Theta_\mu(2B_R)^p.$$

- $\text{HD}_0(R)$ , the high density family, consisting of  $Q \in \mathcal{D}_\mu^{\text{db}}(R) \setminus \text{BCE}_0(R)$  such that

$$\Theta_\mu(2B_Q) > A \Theta_\mu(2B_R).$$

- $\text{LD}_0(R)$ , the low density family, consisting of  $Q \in \mathcal{D}_\mu(R) \setminus \text{BCE}_0(R)$  such that

$$\Theta_\mu(2B_Q) < \tau \Theta_\mu(2B_R).$$

We denote by  $\text{Stop}(R)$  the family of maximal (hence, disjoint) cubes from  $\text{BCE}_0(R) \cup \text{HD}_0(R) \cup \text{LD}_0(R)$ , and we set  $\text{BCE}(R) = \text{BCE}_0(R) \cap \text{Stop}(R)$ ,  $\text{HD}(R) = \text{HD}_0(R) \cap \text{Stop}(R)$ ,  $\text{LD}(R) = \text{LD}_0(R) \cap \text{Stop}(R)$ .

Note that the cubes in  $\text{HD}(R)$  are doubling (by the definition), while the cubes from  $\text{LD}(R)$  and  $\text{BCE}(R)$  may be non-doubling.

We define  $\text{Tree}(R)$  as the family of cubes from  $\mathcal{D}_\mu(R)$  which are not strictly contained in any cube from  $\text{Stop}(R)$  (in particular,  $\text{Stop}(R) \subset \text{Tree}(R)$ ). Note that it may happen that  $R \in \text{BCE}(R)$ , in which case  $\text{Tree}(R) = \{R\}$ .

Basic properties of cubes in  $\text{Tree}(R)$  are collected in the lemma below.

**Lemma 4.1.** *Suppose  $Q \in \text{Tree}(R)$ . Then*

$$(4.2) \quad \Theta_\mu(2B_Q) \lesssim A\Theta_\mu(2B_R).$$

Moreover, for  $Q \in \text{Tree}(R) \setminus \text{Stop}(R)$ ,

$$(4.3) \quad \tau \Theta_\mu(2B_R) \leq \Theta_\mu(2B_Q),$$

$$(4.4) \quad \sum_{Q \subset P \subset R} \varepsilon_{\mu,p}(P) \leq \varepsilon \Theta_\mu(2B_R)^p.$$

Finally, for every  $Q \in \text{Tree}(R)$ , there exists a doubling cube  $P(Q) \in \text{Tree}(R) \cap \mathcal{D}_\mu^{\text{db}}$  such that  $Q \subset P(Q)$  and  $\ell(P(Q)) \lesssim_{A,\tau} \ell(Q)$ . If  $R \notin \text{Stop}(R)$ , we have  $P(Q) \in \text{Tree}(R) \cap \mathcal{D}_\mu^{\text{db}} \setminus \text{Stop}(R)$ .

*Proof.* First, note that if  $R \in \text{Stop}(R)$ , then  $\text{Tree}(R) = \{R\}$  and the lemma above is trivial. Assume that  $R \notin \text{Stop}(R)$ .

Inequalities (4.3) and (4.4) are obvious by the definition  $\text{LD}(R)$  and  $\text{BCE}(R)$ .

Concerning (4.2), note that for  $Q \in \text{Tree}(R) \cap \mathcal{D}_\mu^{\text{db}} \setminus \text{Stop}(R)$ , we have  $\Theta_\mu(2B_Q) \leq A\Theta_\mu(2B_R)$ , from the high density stopping condition. In general, given  $Q \in \text{Tree}(R)$ , let  $P(Q)$  be the smallest doubling cube containing  $Q$ , other than  $Q$ . Since  $R \in \mathcal{D}_\mu^{\text{db}}$  and  $R \notin \text{Stop}(Q)$ , we certainly have  $P(Q) \in \text{Tree}(R) \cap \mathcal{D}_\mu^{\text{db}} \setminus \text{Stop}(R)$ , and therefore  $\Theta_\mu(2B_{P(Q)}) \leq A\Theta_\mu(2B_R)$ .

Denote by  $P_1, P_2, \dots, P_k$  all the intermediate cubes, so that  $Q \subset P_1 \subset P_2 \subset \dots \subset P_k \subset P(Q)$ . Since  $P_j$  are non-doubling, we have, by Lemma 2.2,

$$\begin{aligned} \Theta_\mu(2B_Q) &\lesssim \Theta_\mu(2B_{P_1}) \lesssim \Theta_\mu(100B(P_1)) \leq (C_0A_0)^d A_0^{-9d(k-1)} \Theta_\mu(100B(P(Q))) \\ &\lesssim \Theta_\mu(2B_{P(Q)}) \leq A\Theta_\mu(2B_R), \end{aligned}$$

which proves (4.2).

Finally, to see that  $\ell(P(Q)) \lesssim_{A,\tau} \ell(Q)$ , note that  $P_1 \in \text{Tree}(R) \setminus \text{Stop}(R)$ , and so  $\tau \Theta_\mu(2B_R) \leq \Theta_\mu(2B_{P_1})$ . On the other hand, a minor modification of the computation above shows that

$$\Theta_\mu(2B_{P_1}) \lesssim_{C_0,A_0} A_0^{-9d(k-1)} A\Theta_\mu(2B_R).$$

It follows that  $k \lesssim_{A,\tau} 1$ . ■

The following estimate of the measure of cubes in  $\text{BCE}(R)$  will be used later on in the proof of the packing estimate (3.2).

**Lemma 4.2.** *We have*

$$(4.5) \quad \sum_{Q \in \text{BCE}(R)} \mu(Q) \leq \frac{1}{\varepsilon \Theta_\mu(2B_R)^p} \sum_{P \in \text{Tree}(R)} \varepsilon_{\mu,p}(P) \mu(P).$$

*Proof.* We use the fact that for  $Q \in \text{BCE}(R)$ , we have

$$\sum_{Q \subset P \subset R} \varepsilon_{\mu,p}(P) > \varepsilon \Theta_\mu(2B_R)^p$$

to conclude that

$$\begin{aligned} \Theta_\mu(2B_R)^p \sum_{Q \in \text{BCE}(R)} \mu(Q) &\leq \frac{1}{\varepsilon} \sum_{Q \in \text{BCE}(R)} \mu(Q) \sum_{\substack{P \in \mathcal{D}_\mu \\ Q \subset P \subset R}} \mathcal{E}_{\mu,p}(P) \\ &= \frac{1}{\varepsilon} \sum_{P \in \text{Tree}(R)} \mathcal{E}_{\mu,p}(P) \sum_{\substack{Q \in \text{BCE}(R) \\ Q \subset P}} \mu(Q) \leq \frac{1}{\varepsilon} \sum_{P \in \text{Tree}(R)} \mathcal{E}_{\mu,p}(P) \mu(P). \quad \blacksquare \end{aligned}$$

### 4.2. Key estimate

We introduce some additional notation. Given  $x \in \mathbb{R}^d$  and  $\lambda > 0$ , set

$$K^\lambda(x) = K(x, V, \lambda\alpha).$$

For  $Q \in \mathcal{D}_\mu$ , we denote

$$K_Q^\lambda = \bigcup_{x \in Q} K^\lambda(x).$$

If  $\lambda = 1$ , we will write  $K_Q$  instead of  $K_Q^1$ .

**Lemma 4.3.** *There exists a constant  $M = M(\alpha) > 1$  such that if  $Q \in \text{Tree}(R)$  and  $P \in \mathcal{D}_\mu(R)$  satisfy*

$$P \cap K_Q^{1/2} \setminus MB_Q \neq \emptyset \quad \text{and} \quad \text{dist}(Q, P) \geq Mr(P),$$

then  $P \notin \text{Tree}(R)$ .

*Proof.* Taking  $M = M(\alpha) > 1$  big enough, we can choose cubes  $P', Q' \in \mathcal{D}_\mu(R)$  such that

- $P \subsetneq P' \subset R$ ,  $P' \subset K_Q^{3/4}$ , and  $\ell(P') \approx \text{dist}(P', Q)$ ,
- $Q \subsetneq Q' \subset R$ ,  $\ell(Q') \approx M^{-1}\ell(P')$ , and  $\text{dist}(P', Q') \approx \ell(P')$ .

Moreover, if  $M$  is taken big enough, we have, for all  $x \in 2B_{Q'}$ ,

$$2B_{P'} \subset K(x).$$

Thus, if  $\eta$  is taken small enough (say,  $\eta \ll M^{-1}$ ), we have

$$\begin{aligned} (4.6) \quad \left(\frac{\mu(2B_{P'})}{\ell(P')^n}\right)^p \mu(2B_{Q'}) &\lesssim_\eta \int_{2B_{Q'}} \int_{\eta r(Q')}^{\eta^{-1}r(Q')} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x) \\ &= \mathcal{E}_{\mu,p}(Q') \mu(Q'). \end{aligned}$$

Since  $Q \in \text{Tree}(R)$  and  $Q \subsetneq Q'$ , we have  $Q' \in \text{Tree}(R) \setminus \text{Stop}(R)$ , and so (from (4.6) and (4.4))

$$\Theta_\mu(2B_{P'})^p \approx \left(\frac{\mu(2B_{P'})}{\ell(P')^n}\right)^p \lesssim_\eta \frac{\mu(Q')}{\mu(2B_{Q'})} \mathcal{E}_{\mu,p}(Q') \leq \mathcal{E}_{\mu,p}(Q') \leq \varepsilon \Theta_\mu(2B_R)^p.$$

It follows that, for  $\varepsilon$  small enough,  $P' \in \text{LD}_0(R)$ . Since  $P \subsetneq P'$ , we get that  $P \notin \text{Tree}(R)$ . The proof is complete. ■

We set

$$(4.7) \quad G_R = R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \quad \text{and} \quad \tilde{G}_R = \bigcap_{k=1}^{\infty} \bigcup_{\substack{Q \in \text{Tree}(R) \\ r(Q) \leq A_0^{-k}}} 2MB_Q.$$

Note that  $G_R \subset \tilde{G}_R$ .

**Lemma 4.4.** *For all  $x, y \in \tilde{G}_R$ , we have  $y \notin K^{1/2}(x)$ . Thus,  $\tilde{G}_R$  is contained in an  $n$ -dimensional Lipschitz graph with Lipschitz constant depending only on  $\alpha$ .*

*Proof.* Proof by contradiction. Suppose that  $x, y \in \tilde{G}_R$  and  $x - y \in K^{1/2}$ . Let  $Q, P \in \text{Tree}(R)$  be such that  $x \in 2MB_Q, y \in 2MB_P$ , with side length so small that  $P \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$  and  $\text{dist}(Q, P) \geq Mr(P)$  (note that this can be done because  $K^{1/2}$  is an open cone, and so  $x' - y' \in K^{1/2}$  also for  $x' \in B(x, \varepsilon')$  and  $y' \in B(y, \varepsilon')$ , assuming  $\varepsilon' > 0$  small enough). It follows by Lemma 4.3 that  $P \notin \text{Tree}(R)$ , and so we reach a contradiction. ■

### 4.3. Construction of $\Gamma_R$

The Lipschitz graph from Lemma 4.4 can be thought of as a first approximation of  $\Gamma_R$ . It contains the “good set”  $\tilde{G}_R$ , but we would also like for  $\Gamma_R$  to lie close to cubes from  $\text{Tree}(R)$ . In this subsection we show how to do it.

Given  $t > 1$ , we say that cubes  $Q, P \in \mathcal{D}_\mu$  are  $t$ -neighbors if they satisfy

$$(4.8) \quad t^{-1}r(Q) \leq r(P) \leq tr(Q)$$

and

$$(4.9) \quad \text{dist}(Q, P) \leq t(r(Q) + r(P)).$$

If at least one of the conditions above does not hold, we say that  $Q$  and  $P$  are  $t$ -separated. We will also say that a family of cubes is  $t$ -separated if the cubes from that family are pairwise  $t$ -separated.

Consider a big constant  $t = t(M, \alpha) > M$  which will be fixed later on. We denote by  $\text{Sep}(R)$  a maximal  $t$ -separated subfamily of  $\text{Stop}(R)$  (it exists by Zorn’s lemma). Clearly, for every  $Q \in \text{Stop}(R)$ , there exists some  $P \in \text{Sep}(R)$  which is a  $t$ -neighbor of  $Q$ .

Furthermore, we define  $\text{Sep}^*(R)$  as the family of all cubes  $Q \in \text{Sep}(R)$  satisfying the following two conditions:

$$(4.10) \quad 2MB_Q \cap \tilde{G}_R = \emptyset,$$

and for all  $P \in \text{Sep}(R), P \neq Q$ , we have

$$(4.11) \quad 2MB_P \not\subset 2MB_Q.$$

**Lemma 4.5.** *Suppose  $t = t(M)$  is big enough. Then, for all  $Q, P \in \text{Sep}^*(R), Q \neq P$ , we have  $Q \not\subset 1.5MB_P$ .*

*Proof.* Suppose  $Q \in \text{Sep}^*(R)$  and  $Q \subset 1.5MB_P$ . We will show that  $P \notin \text{Sep}^*(R)$ .



Firstly, if  $r(Q) > t^{-1}r(P)$ , then  $Q \subset 1.5MB_P$  implies that  $Q$  and  $P$  are  $t$ -neighbors (for  $t$  big enough), and so  $P \notin \text{Sep}^*(R)$ . On the other hand, if  $r(Q) \leq t^{-1}r(P)$ , then (if  $t$  is big enough)  $Q \subset 1.5MB_P$  implies  $2MB_Q \subset 2MB_P$ , contradicting (4.11). ■

**Lemma 4.6.** *For every  $Q \in \text{Sep}(R)$  at least one of the following is true:*

- (a)  $2MB_Q \cap \tilde{G}_R \neq \emptyset$ ,
- (b) *there exists  $P \in \text{Sep}^*(R)$  such that  $2MB_P \subset 2MB_Q$ .*

*Proof.* If  $Q \in \text{Sep}^*(R)$ , then of course (b) holds (with  $P = Q$ ). Suppose that  $Q \notin \text{Sep}^*(R)$ , and that (a) does not hold (i.e.,  $2MB_Q \cap \tilde{G}_R = \emptyset$ ). We will find  $P \in \text{Sep}^*(R)$  such that  $2MB_P \subset 2MB_Q$ .

Since  $Q \notin \text{Sep}^*(R)$  and (4.10) holds, condition (4.11) must be false. Thus, we get a cube  $Q_1 \in \text{Sep}(R)$  such that  $2MB_{Q_1} \subset 2MB_Q$ . If  $Q_1 \in \text{Sep}^*(R)$ , we get (b) with  $P = Q_1$ . Otherwise, we continue as follows.

Reasoning as before,  $Q_1 \in \text{Sep}(R) \setminus \text{Sep}^*(R)$  and  $2MB_{Q_1} \cap \tilde{G}_R = \emptyset$  ensures that there exists a cube  $Q_2 \in \text{Sep}(R)$  such that  $2MB_{Q_2} \subset 2MB_{Q_1}$ . Iterating this process, we get a (perhaps infinite) sequence of cubes  $Q_0 := Q, Q_1, Q_2, \dots$  satisfying  $2MB_{Q_{j+1}} \subset 2MB_{Q_j}$ .

If the algorithm never stops, then  $\bigcap_{j=0}^\infty 2MB_{Q_j} \neq \emptyset$ . But, by the definition (4.7) of  $\tilde{G}_R$ , we have  $\bigcap_{j=0}^\infty 2MB_{Q_j} \subset \tilde{G}_R$ , and so we get a contradiction with  $2MB_Q \cap \tilde{G}_R = \emptyset$ . Thus, the algorithm stops at some cube  $Q_m$ , which means that  $Q_m \in \text{Sep}^*(R)$ . Setting  $P = Q_m$  finishes the proof. ■

**Lemma 4.7.** *Suppose  $t = t(M)$  is big enough. Then*

- (a) *for all  $Q, P \in \text{Sep}^*(R)$ ,  $Q \neq P$ , we have*

$$(4.12) \quad Q \cap K_P^{1/2} = P \cap K_Q^{1/2} = \emptyset,$$

- (b) *for all  $x \in \tilde{G}_R$  and for all  $Q \in \text{Sep}^*(R)$ , we have*

$$x \notin K_Q^{1/2} \quad \text{and} \quad Q \cap K^{1/2}(x) = \emptyset.$$

*Proof of (a).* Proof by contradiction. Suppose  $Q \cap K_P^{1/2} \neq \emptyset$  (which by symmetry of cones implies  $P \cap K_Q^{1/2} \neq \emptyset$ ). Without loss of generality, assume that  $r(Q) \leq r(P)$ . Since  $Q$  and  $P$  are  $t$ -separated, at least one of the conditions (4.8), (4.9) fails, i.e.,

$$r(Q) \leq t^{-1}r(P) \quad \text{or} \quad \text{dist}(Q, P) > t(r(Q) + r(P)).$$

We know by Lemma 4.5 that  $Q \not\subset 1.5MB_P$ . It is easy to see that in either of the cases considered above, this implies  $Q \cap 1.2MB_P = \emptyset$ . It follows that  $Q \cap (K_P^{1/2} \setminus MB_P) \neq \emptyset$  and  $r(Q) \leq r(P) \leq M^{-1} \text{dist}(Q, P)$ . Hence, we can use Lemma 4.3 to conclude that  $Q \notin \text{Tree}(R)$ . This contradicts  $Q \in \text{Sep}^*(R)$ . ■

*Proof of (b).* Proof by contradiction. Suppose  $x \in K_Q^{1/2}$ . We have  $x \notin 2MB_Q$  by (4.10). Since  $x \in \tilde{G}_R$ , we can find an arbitrarily small cube  $P \in \text{Tree}(R)$  such that  $x \in 2MB_P$ . Taking  $r(P)$  small enough, we have  $r(P) \leq M^{-1} \text{dist}(Q, P)$  and  $P \cap K_Q^{1/2} \setminus MB_Q \neq \emptyset$  (because  $x \in K^{1/2}(x') \setminus 2MB_Q$  for some  $x' \in Q$ , and  $K^{1/2}(x')$  is an open set). Lemma 4.3 yields  $P \notin \text{Tree}$ , a contradiction. ■

**Lemma 4.8.** *There exists a Lipschitz graph  $\Gamma_R$ , with Lipschitz constant depending only on  $\alpha$ , such that*

$$\tilde{G}_R \subset \Gamma_R.$$

Moreover, there exists a big constant  $\Lambda = \Lambda(M, t) > 1$  such that for every  $Q \in \text{Tree}(R)$ , we have

$$(4.13) \quad \Lambda B_Q \cap \Gamma_R \neq \emptyset.$$

*Proof.* Recall that for each cube  $Q \in \mathcal{D}_\mu$  we have a ‘‘center’’ denoted by  $x_Q \in Q$ . Set  $F = \{x_Q : Q \in \text{Sep}^*(R)\} \cup \tilde{G}_R$ . It follows, by Lemma 4.4 and Lemma 4.7, that for any  $x, y \in F$ , we have  $x - y \notin K^{1/2}$ . Thus, there exists a Lipschitz graph  $\Gamma_R$ , with slope depending only on  $\alpha$ , such that  $F \subset \Gamma_R$ .

Concerning the second statement, it is clearly true for  $Q \in \text{Sep}^*(R)$  (even with  $\Lambda = 1$ ). For  $Q \in \text{Sep}(R)$ , we have, by Lemma 4.6, that either  $2MB_Q \cap \tilde{G}_R \neq \emptyset$  or there exists  $P \in \text{Sep}^*(R)$  with  $2MB_P \subset 2MB_Q$ . Thus, (4.13) holds if  $\Lambda \geq 2M$ .

If  $Q \in \text{Stop}(R)$ , there exists some  $P \in \text{Sep}(R)$ , which is a  $t$ -neighbor of  $Q$ , so that for some  $\Lambda = \Lambda(t, M) > 1$ , we have  $\Lambda B_Q \supset 2MB_P$ , and  $2MB_P$  intersects  $\Gamma_R$ . Finally, for a general  $Q \in \text{Tree}(R)$ , either  $Q$  contains some cube from  $\text{Stop}(R)$ , or  $Q \subset \tilde{G}_R$ . In any case,  $\Lambda B_Q \cap \Gamma_R \neq \emptyset$ . ■

**Remark 4.9.** Note that while for a general cube  $Q \in \text{Tree}(R)$ , we only have  $\Lambda B_Q \cap \Gamma_R \neq \emptyset$ , we have a better estimate for the root  $R$ :

$$(4.14) \quad B_R \cap \Gamma_R \neq \emptyset.$$

Indeed, (4.14) is clear if the set  $\tilde{G}_R$  is non-empty. If  $\tilde{G}_R = \emptyset$ , then  $\text{Sep}^*(R) \neq \emptyset$ , so that for some  $P \in \text{Sep}^*(R)$ , we have  $x_P \in \Gamma_R \cap B_R$ .

### 5. Small measure of cubes from LD(R)

In the proof of the packing estimate (3.2), it will be crucial to have a bound on the measure of low density cubes.

**Lemma 5.1.** *We have*

$$\sum_{Q \in \text{LD}(R)} \mu(Q) \lesssim_{t,\alpha} \tau \mu(R).$$

In particular, for  $\tau$  small enough, we have

$$(5.1) \quad \sum_{Q \in \text{LD}(R)} \mu(Q) \leq \tau^{1/2} \mu(R).$$

We begin by defining some auxiliary subfamilies of LD(R).

**Lemma 5.2.** *There exists a  $t$ -separated family  $\text{LD}_{\text{Sep}}(R) \subset \text{LD}(R)$  such that*

$$\sum_{Q \in \text{LD}(R)} \mu(Q) \lesssim_t \sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q).$$

*Proof.* We construct the family  $LD_{\text{Sep}}(R)$  in the following way. Define  $LD_1(R)$  as a maximal  $t$ -separated subfamily of  $LD(R)$ . Next, define  $LD_2(R)$  as a maximal  $t$ -separated subfamily of  $LD(R) \setminus LD_1(R)$ . In general, having defined  $LD_j(R)$ , we define  $LD_{j+1}(R)$  to be a maximal  $t$ -separated subfamily of  $LD(R) \setminus (LD_1(R) \cup \dots \cup LD_j(R))$ .

We claim that there is only a bounded number of non-empty families  $LD_j(R)$ , with the bound depending on  $t$ . Indeed, if  $Q \in LD_j(R)$ , then  $Q$  has at least one  $t$ -neighbor in each family  $LD_k(R)$ ,  $k \leq j$ . It follows easily from the definition of  $t$ -neighbors that the number of  $t$ -neighbors of any given cube is bounded by a constant  $C(t)$ . Hence,  $j \leq C(t)$ .

Set  $LD_{\text{Sep}}(R)$  to be the family  $LD_j(R)$  maximizing  $\sum_{Q \in LD_j(R)} \mu(Q)$ . Then

$$\sum_{Q \in LD(R)} \mu(Q) \leq C(t) \sum_{Q \in LD_{\text{Sep}}(R)} \mu(Q). \quad \blacksquare$$

We define also a family  $LD_{\text{Sep}}^*(R) \subset LD_{\text{Sep}}(R)$  in the following way: we remove from  $LD_{\text{Sep}}(R)$  all the cubes  $P$  for which there exists some  $Q \in LD_{\text{Sep}}(R)$  such that

$$(5.2) \quad 1.1B_Q \cap 1.1B_P \neq \emptyset \quad \text{and} \quad r(Q) < r(P).$$

**Lemma 5.3.** *For each  $Q \in LD_{\text{Sep}}(R)$  at least one of the following is true:*

- (a)  $1.2B_Q \cap \tilde{G}_R \neq \emptyset$ ,
- (b) *there exists some  $P \in LD_{\text{Sep}}^*(R)$  such that  $1.2B_P \subset 1.2B_Q$ .*

*Proof.* Suppose  $Q \in LD_{\text{Sep}}(R)$ , and that (a) does not hold. We will find  $P$  such that (b) is satisfied.

If  $Q \notin LD_{\text{Sep}}^*(R)$ , then there exists some cube  $Q_1 \in LD_{\text{Sep}}(R)$  such that

$$1.1B_Q \cap 1.1B_{Q_1} \neq \emptyset \quad \text{and} \quad r(Q_1) < r(Q).$$

Since  $Q$  and  $Q_1$  are  $t$ -separated, and (4.9) holds, it follows that  $tr(Q_1) < r(Q)$ . Thus,  $Q_1$  is tiny compared to  $Q$ , and we have  $1.2B_{Q_1} \subset 1.2B_Q$ . If  $Q_1 \in LD_{\text{Sep}}^*(R)$ , we set  $P = Q_1$  and we are done. Otherwise, we iterate as in Lemma 4.6 (with  $2M$  replaced by 1.2) to find a finite sequence  $Q_1, Q_2, \dots, Q_m$  satisfying  $1.2B_{Q_{j+1}} \subset 1.2B_{Q_j}$ , and such that  $Q_m \in LD_{\text{Sep}}^*(R)$ . ■

**Lemma 5.4.** *For each  $Q \in LD_{\text{Sep}}^*(R)$ , we have*

$$(5.3) \quad \mu\left(Q \cap \bigcup_{P \in LD_{\text{Sep}}^*(R)} (K_P^{1/2} \setminus MB_P)\right) \lesssim_{\tau, \alpha, \eta} \varepsilon \mu(Q).$$

*In particular, if  $\varepsilon$  is small enough, then for each  $Q \in LD_{\text{Sep}}^*(R)$ , we can choose a point*

$$(5.4) \quad w_Q \in Q \setminus \bigcup_{P \in LD_{\text{Sep}}^*(R)} (K_P^{1/2} \setminus MB_P).$$

*Proof.* Suppose  $Q \in LD_{\text{Sep}}^*(R)$  and that we have  $Q \cap K_P^{1/2} \setminus MB_P \neq \emptyset$  for some  $P \in LD_{\text{Sep}}^*(R)$ . Note that if we had  $Mr(Q) \leq \text{dist}(Q, P)$ , then the assumptions of Lemma 4.3 would be satisfied, and we would arrive at  $Q \notin \text{Tree}(R)$ , a contradiction. Thus,

$$(5.5) \quad \text{dist}(Q, P) \leq Mr(Q) < tr(Q).$$

It follows that (4.9) – one of the  $t$ -neighborhood conditions – is satisfied. Since  $Q$  and  $P$  are  $t$ -separated, we necessarily have  $tr(Q) \leq r(P)$  or  $tr(P) \leq r(Q)$ .

If we had  $tr(Q) \leq r(P)$ , then (5.5) implies  $\text{dist}(Q, P) \leq r(P)$ . Hence,  $1.1B_Q \cap 1.1B_P \neq \emptyset$ . But this cannot be true, by the definition of  $\text{LD}_{\text{Sep}}^*(R)$ . It follows that

$$(5.6) \quad tr(P) \leq r(Q).$$

Let  $S \supset P$  be the biggest ancestor of  $P$  satisfying  $r(S) \leq \delta r(Q)$  for some small constant  $\delta = \delta(\alpha)$ , which will be fixed in a few lines. If  $t$  is big enough, then  $S \neq P$ . Thus,  $r(S) \approx_\delta r(Q)$ , and  $S \in \text{Tree}(R) \setminus \text{Stop}(R)$ . Recall that by the definition of  $\text{LD}_{\text{Sep}}^*(R)$ , we have  $1.1B_Q \cap 1.1B_P = \emptyset$ . It follows that if  $\delta < 0.001$ , then  $4B_S \cap 1.05B_Q = \emptyset$ . Now, using this separation, it is not difficult to check that for  $\delta = \delta(\alpha)$  small enough, for any  $x \in K_P^{1/2} \cap Q$ , we have

$$2B_S \subset K(x).$$

Observe also that, due to (5.5) and the fact that  $r(S) \leq \delta r(Q)$ , we have

$$2B_S \subset B(x, r) \quad \text{for } r \in \left(\frac{\eta^{-1}}{2} r(Q), \eta^{-1} r(Q)\right),$$

provided that  $\eta$  is small enough (say,  $\eta^{-1} \gg t$ ). Putting together the two estimates above, we get that

$$\mu(2B_S) \leq \mu(K(x, r))$$

for any  $x \in K_P^{1/2} \cap Q \supset Q \cap K_P^{1/2} \setminus MB_P$  and all  $r \in (\eta^{-1}r(Q)/2, \eta^{-1}r(Q))$ .

Integrating the above over all  $x \in A$ , where  $A \subset Q \cap K_P^{1/2} \setminus MB_P$  is an arbitrary measurable subset, yields (see (4.3))

$$(5.7) \quad \begin{aligned} \mu(A)\Theta_\mu(2B_R)^p &\leq \tau^{-1} \mu(A)\Theta_\mu(2B_S)^p \approx_{\tau, \alpha} \mu(A) \left(\frac{\mu(2B_S)}{r(Q)^n}\right)^p \\ &\lesssim_\eta \int_A \int_{\eta r(Q)}^{\eta^{-1}r(Q)} \left(\frac{\mu(K(x, r))}{r^n}\right)^p \frac{dr}{r} d\mu(x). \end{aligned}$$

Now, let  $P_i$  be some ordering of cubes  $P \in \text{LD}_{\text{Sep}}^*(R)$  satisfying  $Q \cap K_P^{1/2} \setminus MB_P \neq \emptyset$ . We define  $A_1 = Q \cap K_{P_1}^{1/2} \setminus MB_{P_1}$ , and for  $i > 1$ ,

$$A_i = Q \cap K_{P_i}^{1/2} \setminus \left(MB_{P_i} \cup \bigcup_{j=1}^{i-1} A_j\right).$$

Observe that  $A_i$  are pairwise disjoint and their union is  $Q \cap \bigcup_{P \in \text{LD}_{\text{Sep}}^*(R)} (K_P^{1/2} \setminus MB_P)$ . Thus, from (5.7), we have

$$\begin{aligned} \mu\left(Q \cap \bigcup_{P \in \text{LD}_{\text{Sep}}^*(R)} K_P^{1/2} \setminus MB_P\right)\Theta_\mu(2B_R)^p &= \sum_i \mu(A_i)\Theta_\mu(2B_R)^p \\ &\lesssim_{\tau, \alpha, \eta} \int_{\bigcup_i A_i} \int_{\eta r(Q)}^{\eta^{-1}r(Q)} \left(\frac{\mu(K(x, r))}{r^n}\right)^p \frac{dr}{r} d\mu(x) \leq \mathcal{E}_{\mu, p}(Q) \mu(Q). \end{aligned}$$

Note that since  $Q \notin \text{BCE}(R)$ , we have  $\mathcal{E}_{\mu, p}(Q) \mu(Q) \leq \varepsilon \Theta_\mu(2B_R)^p \mu(Q)$ . So the estimate (5.3) holds. ■

**Lemma 5.5.** *There exists an  $n$ -dimensional Lipschitz graph  $\Gamma_{LD}$  passing through all the points  $w_P$ ,  $P \in LD_{Sep}^*(R)$ . The Lipschitz constant of  $\Gamma_{LD}$  depends only on  $\alpha$ .*

*Proof.* It suffices to show that for any  $Q, P \in LD_{Sep}^*(R)$ ,  $Q \neq P$ , we have

$$(5.8) \quad w_Q - w_P \notin K^{1/2}.$$

Without loss of generality, assume  $r(P) \leq r(Q)$ . By (5.4), we have

$$w_Q \notin K_P^{1/2} \setminus MB_P.$$

In particular,

$$w_Q \notin K^{1/2}(w_P) \setminus MB_P.$$

So, to prove (5.8), it is enough to show that

$$(5.9) \quad w_Q \notin MB_P.$$

Assume the contrary, i.e.,  $w_Q \in MB_P$ . Then

$$\text{dist}(Q, P) \leq CMr(P) \leq t(r(Q) + r(P)).$$

That is, (4.9) holds. But  $Q$  and  $P$  are  $t$ -separated, and so (4.8) must fail. Hence,

$$r(P) \leq t^{-1}r(Q).$$

$Q$  and  $P$  belong to  $LD_{Sep}^*(R)$ , so by (5.2) we have  $1.1B_Q \cap 1.1B_P = \emptyset$ . Thus,

$$\text{dist}(w_Q, B_P) \geq 0.1r(B_Q) \geq Ctr(B_P) > Mr(B_P).$$

So (5.9) holds. ■

We can finally finish the proof of Lemma 5.1.

*Proof of Lemma 5.1.* By Lemma 5.2, it suffices to estimate the measure of cubes from  $LD_{Sep}(R)$ . Let  $\mathcal{G}$  denote an arbitrary finite subfamily of  $LD_{Sep}(R)$ . We use the covering lemma (Theorem 9.31 of [58]) to choose a subfamily  $\mathcal{F} \subset \mathcal{G}$  such that

$$\bigcup_{Q \in \mathcal{G}} 1.5B_Q \subset \bigcup_{Q \in \mathcal{F}} 2B_Q,$$

and the balls  $\{1.5B_Q\}_{Q \in \mathcal{F}}$  are of bounded superposition.

The above and the LD stopping condition give

$$(5.10) \quad \sum_{Q \in \mathcal{G}} \mu(Q) \leq \sum_{Q \in \mathcal{F}} \mu(2B_Q) \lesssim \tau^{\Theta_\mu}(2B_R) \sum_{Q \in \mathcal{F}} r(B_Q)^n.$$

Now, it follows from Lemma 5.3 and Lemma 5.5 that for each  $Q \in \mathcal{G} \subset LD_{Sep}(R)$ , there exists either  $w_Q \in \Gamma_{LD} \cap 1.2B_Q$  or  $x \in \tilde{G}_R \cap 1.2B_Q \subset \Gamma_R \cap 1.2B_Q$ . Hence,

$$\mathcal{H}^n(1.5B_Q \cap (\Gamma_{LD} \cup \Gamma_R)) \approx_\alpha r(B_Q)^n.$$

Now, using the bounded superposition property of  $\mathcal{F}$ , we get

$$\begin{aligned} \sum_{Q \in \mathcal{F}} r(B_Q)^n &\approx_\alpha \sum_{Q \in \mathcal{F}} \mathcal{H}^n(1.5B_Q \cap (\Gamma_{LD} \cup \Gamma_R)) \lesssim \mathcal{H}^n\left(\bigcup_{Q \in \mathcal{F}} 1.5B_Q \cap (\Gamma_{LD} \cup \Gamma_R)\right) \\ &\leq \mathcal{H}^n(2B_R \cap (\Gamma_{LD} \cup \Gamma_R)) \approx_\alpha r(R)^n \approx \mu(2B_R)\Theta_\mu(2B_R)^{-1} \approx \mu(R)\Theta_\mu(2B_R)^{-1}, \end{aligned}$$

since  $R \in \mathcal{D}_\mu^{\text{db}}$ . Together with (5.10), this gives

$$\sum_{Q \in \mathcal{G}} \mu(Q) \lesssim_\alpha \tau\mu(R).$$

Since  $\mathcal{G}$  was an arbitrary finite subfamily of  $\text{LD}_{\text{Sep}}(R)$ , we finally arrive at

$$\sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q) \lesssim_\alpha \tau\mu(R). \quad \blacksquare$$

## 6. Top cubes and packing estimate

### 6.1. Definition of Top

In order to define the Top family, we need to introduce some additional notation. Given  $Q \in \mathcal{D}_\mu$ , let  $\mathcal{MD}(Q)$  denote the family of maximal cubes from  $\mathcal{D}_\mu^{\text{db}}(Q) \setminus \{Q\}$ . It follows from Lemma 2.1 (c) that the cubes from  $\mathcal{MD}(Q)$  cover  $\mu$ -almost all of  $Q$ .

Given  $R \in \mathcal{D}_\mu^{\text{db}}$ , set

$$\text{Next}(R) = \bigcup_{Q \in \text{Stop}(R)} \mathcal{MD}(Q).$$

Since we always have  $\mathcal{MD}(Q) \neq \{Q\}$ , it is clear that  $\text{Next}(R) \neq \{R\}$ .

Observe that if  $P \in \text{Next}(R)$ , then, by Lemma 4.1 and Lemma 2.2, we have for all intermediate cubes  $S \in \mathcal{D}_\mu$ ,  $P \subset S \subset R$ ,

$$(6.1) \quad \Theta_\mu(2B_S) \lesssim_A \Theta_\mu(2B_R).$$

We are finally ready to define Top. It is defined inductively as  $\text{Top} = \bigcup_{k \geq 0} \text{Top}_k$ . First, set

$$\text{Top}_0 = \{R_0\},$$

where  $R_0$  was defined as  $\text{supp } \mu$ . Having defined  $\text{Top}_k$ , we set

$$\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R).$$

Note that for each  $k \geq 0$ , the cubes from  $\text{Top}_k$  are pairwise disjoint.

### 6.2. Definition of ID

We distinguish a special type of Top cubes. We say that  $R \in \text{Top}$  is increasing density,  $R \in \text{ID}$ , if

$$\mu\left(\bigcup_{Q \in \text{HD}(R)} Q\right) \geq \frac{1}{2}\mu(R).$$

**Lemma 6.1.** *If  $A$  is big enough, then for all  $R \in \text{ID}$ ,*

$$(6.2) \quad \Theta_\mu(2B_R)^p \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q)^p \mu(Q).$$

*Proof.* The definition of ID and the HD stopping condition imply that for any  $R \in \text{ID}$ ,

$$\Theta_\mu(2B_R)^p \mu(R) \leq 2\Theta_\mu(2B_R)^p \sum_{Q \in \text{HD}(R)} \mu(Q) \leq 2A^{-p} \sum_{Q \in \text{HD}(R)} \Theta_\mu(2B_Q)^p \mu(Q).$$

Note that all  $Q \in \text{HD}(R)$  are doubling, and so, by Lemma 2.3,

$$\Theta_\mu(2B_Q)^p \mu(Q) \lesssim \sum_{P \in \mathcal{MD}(Q)} \Theta_\mu(2B_P)^p \mu(P) = \sum_{\substack{P \in \text{Next}(R) \\ P \subset Q}} \Theta_\mu(2B_P)^p \mu(P).$$

If  $A$  is taken big enough, then the estimates above yield (6.2). ■

### 6.3. Packing condition

We will now establish the packing condition (3.2). For  $S \in \text{Top}$ , set  $\text{Top}(S) = \text{Top} \cap \mathcal{D}_\mu(S)$  and  $\text{Top}_j(S) = \text{Top}_j \cap \mathcal{D}_\mu(S)$ . For  $k \geq 0$ , we also define

$$\text{Top}_0^k(S) = \bigcup_{0 \leq j \leq k} \text{Top}_j(S) \quad \text{and} \quad \text{ID}_0^k(S) = \text{ID} \cap \text{Top}_0^k(S).$$

Recall that  $\mu$  satisfies the following polynomial growth condition: there exist  $C_1 > 0$  and  $r_0 > 0$  such that for all  $x \in \text{supp } \mu$ ,  $0 < r \leq r_0$ , we have

$$(6.3) \quad \mu(B(x, r)) \leq C_1 r^n.$$

**Lemma 6.2.** *For all  $S \in \text{Top}$ , we have*

$$(6.4) \quad \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \mu(R) \lesssim_{\varepsilon, \eta, \tau} (C_1)^p \mu(S) + \int_{2B_S} \int_0^{\eta^{-1}C_0 r(S)} \left( \frac{\mu(K(x, r))}{r^n} \right)^p \frac{dr}{r} d\mu(x).$$

*The implicit constant does not depend on  $r_0$ .*

*Proof.* First, we deal with ID cubes. Note that, from (6.2),

$$\begin{aligned} \sum_{R \in \text{ID}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) &\leq \frac{1}{2} \sum_{R \in \text{ID}_0^k(S)} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q)^p \mu(Q) \\ &\leq \frac{1}{2} \sum_{Q \in \text{Top}_0^{k+1}(S)} \Theta_\mu(2B_Q)^p \mu(Q), \end{aligned}$$

where the last inequality follows from the fact that  $\bigcup_{R \in \text{Top}_0^k} \text{Next}(R) = \text{Top}_0^{k+1}$ . Now, observe that for  $Q \in \text{Top}_{k+1}$ , we have  $r(Q) \leq C_0 A_0^{-k} r(R_0)$ , and so if  $k$  is big enough, then  $r(2B_Q) \leq r_0$ . Thus, by (6.3),

$$(6.5) \quad \Theta_\mu(2B_Q) \leq C_1.$$

Hence,

$$(6.6) \quad \begin{aligned} \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) &= \sum_{R \in \text{Top}_0^k(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) + \sum_{R \in \text{ID}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) \\ &\leq \sum_{R \in \text{Top}_0^k(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) + \frac{1}{2} \sum_{R \in \text{Top}_0^{k+1}(S)} \Theta_\mu(2B_R)^p \mu(R) \\ &\leq \sum_{R \in \text{Top}_0^k(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) + \frac{1}{2} \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) + \frac{(C_1)^p}{2} \mu(S). \end{aligned}$$

Note that for small cubes  $Q \in \text{Top}_0^k(S)$  (i.e., satisfying  $r(2B_Q) \leq r_0$ ), we have (6.5), while for big cubes, the trivial estimate  $\Theta_\mu(2B_Q) \leq \mu(2B_S)r_0^{-n}$  holds. It follows that

$$\sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) \leq (k + 1)((C_1)^p + \mu(2B_S)^p r_0^{-np}) \mu(S) < \infty,$$

and so we may deduce from (6.6) that

$$\sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R)^p \mu(R) \leq 2 \sum_{R \in \text{Top}_0^k(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) + (C_1)^p \mu(S).$$

Letting  $k \rightarrow \infty$ , we arrive at

$$(6.7) \quad \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \mu(R) \leq 2 \sum_{R \in \text{Top}(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) + (C_1)^p \mu(S).$$

Now, we need to estimate the sum from the right-hand side. By the definition of ID, we have, for all  $R \in \text{Top}(S) \setminus \text{ID}$ ,

$$\mu\left(R \setminus \bigcup_{Q \in \text{HD}(R)} Q\right) \geq \frac{1}{2} \mu(R),$$

and so by Lemma 2.1 (c), we get

$$\begin{aligned} \mu(R) &\leq 2\mu\left(R \setminus \bigcup_{Q \in \text{Stop}(R)} Q\right) + 2\mu\left(\bigcup_{Q \in \text{Stop}(R) \setminus \text{HD}(R)} Q\right) \\ &= 2\mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) + 2 \sum_{Q \in \text{LD}(R)} \mu(Q) + 2 \sum_{Q \in \text{BCE}(R)} \mu(Q). \end{aligned}$$

The measure of low density cubes is small due to (5.1), and so for  $\tau$  small enough, we have

$$\mu(R) \leq 3\mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) + 3 \sum_{Q \in \text{BCE}(R)} \mu(Q).$$



Thus,

$$(6.8) \quad \sum_{R \in \text{Top}(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \mu(R) \leq 3 \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) + 3 \sum_{R \in \text{Top}(S) \setminus \text{ID}} \Theta_\mu(2B_R)^p \sum_{Q \in \text{BCE}(R)} \mu(Q).$$

Concerning the first sum, notice that if  $\mu(R \setminus \bigcup_{Q \in \text{Next}(R)} Q) > 0$ , then we have arbitrarily small cubes  $P$  belonging to  $\text{Tree}(R)$ . In particular, by (4.3) and (6.3), we have  $\Theta_\mu(2B_R) \leq \tau^{-1} \Theta_\mu(2B_P) \leq \tau^{-1} C_1$ , taking  $P \in \text{Tree}(R) \setminus \text{Stop}(R)$  small enough. Recall also that for  $R \in \text{Top}(S)$ , the sets  $R \setminus \bigcup_{Q \in \text{Next}(R)} Q$  are pairwise disjoint. Hence,

$$(6.9) \quad \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \mu\left(R \setminus \bigcup_{Q \in \text{Next}(R)} Q\right) \leq (\tau^{-1} C_1)^p \mu(S).$$

To estimate the second sum, from (6.8), we apply (4.5) to get

$$\begin{aligned} \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \sum_{Q \in \text{BCE}(R)} \mu(Q) &\leq \frac{1}{\varepsilon} \sum_{R \in \text{Top}(S)} \sum_{P \in \text{Tree}(R)} \mathcal{E}_{\mu,p}(P) \mu(P) \\ &\leq \frac{1}{\varepsilon} \sum_{P \in \mathcal{D}_\mu(S)} \mathcal{E}_{\mu,p}(P) \mu(P). \end{aligned}$$

By the definition of  $\mathcal{E}_{\mu,p}(P)$ , and the bounded intersection property of the balls  $2B_P$  for cubes  $P$  of the same generation, we have

$$\begin{aligned} \sum_{P \in \mathcal{D}_\mu(S)} \mathcal{E}_{\mu,p}(P) \mu(P) &= \sum_k \sum_{P \in \mathcal{D}_{\mu,k}(S)} \int_{2B_P} \int_{\eta r(P)}^{\eta^{-1} r(P)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} \\ &\lesssim \sum_k \int_{2B_S} \int_{\eta A_0^{-k}}^{\eta^{-1} C_0 A_0^{-k}} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} \\ &\lesssim_\eta \int_{2B_S} \int_0^{\eta^{-1} C_0 r(S)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x). \end{aligned}$$

Consequently,

$$\sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R)^p \sum_{Q \in \text{BCE}(R)} \mu(Q) \lesssim_{\varepsilon,\eta} \int_{2B_S} \int_0^{\eta^{-1} C_0 r(S)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x).$$

Together with (6.7), (6.8) and (6.9), this gives (6.4). ■

Let us put together all the ingredients of the proof of the main lemma.

*Proof of Lemma 3.1.* Let  $\text{Top} \subset \mathcal{D}_\mu^{\text{db}}$  be as above, and let  $\{\Gamma_R\}_{R \in \text{Top}}$  be as in Lemma 4.8. Then, properties (i) and (ii) are ensured by Lemma 4.8. Property (iii) follows from (6.1). We get the packing estimate (3.2) from (6.4), by taking  $S = R_0$ . ■

### 7. Application to singular integral operators

To prove Theorem 1.17, we will use geometric characterizations of boundedness of operators from  $\mathcal{K}^n(\mathbb{R}^d)$  shown in Sections 4, 5, 9 of [30]. For  $n = 1, d = 2$ , a variant of this characterization valid for the Cauchy transform was already proved in [55].

For  $Q, S \in \mathcal{D}_\mu, Q \subset S$ , we set

$$\delta_\mu(Q, S) = \int_{2B_S \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y).$$

The notation  $\text{Good}(R), \text{Tr}(R), \text{Next}(R)$ , used below, was introduced in Section 3.

**Lemma 7.1** ([30]). *Let  $\mu$  be a compactly supported Radon measure on  $\mathbb{R}^d$  satisfying the growth condition (1.9). Assume there exist a family of cubes  $\text{Top} \subset \mathcal{D}_\mu^{\text{db}}$  and a corresponding family of Lipschitz graphs  $\{\Gamma_R\}_{R \in \text{Top}}$  satisfying:*

- (i) *Lipschitz constants of  $\Gamma_R$  are uniformly bounded by some absolute constant,*
- (ii)  *$\mu$ -almost all  $\text{Good}(R)$  is contained in  $\Gamma_R$ ,*
- (iii) *for all  $Q \in \text{Tr}(R)$ , we have  $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_R)$ .*
- (iv) *for all  $Q \in \text{Next}(R)$ , there exists  $S \in \mathcal{D}_\mu, Q \subset S$ , such that  $\delta_\mu(Q, S) \lesssim \Theta_\mu(2B_R)$  and  $2B_S \cap \Gamma_R \neq \emptyset$ .*

Then, for every singular integral operator  $T$  with kernel  $k \in \mathcal{K}^n(\mathbb{R}^d)$ , we have

$$\sup_{\varepsilon > 0} \|T_\varepsilon \mu\|_{L^2(\mu)}^2 \lesssim \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^2 \mu(R),$$

with the implicit constant depending on  $C_1$  and the constant  $C_k$  from (1.8).

The result above is not explicitly stated in [30], but it is essentially Lemma 1, Section 5, of [30]. The ‘‘corona decomposition’’ assumptions of Lemma 7.1 come from Lemma D of [30], which is treated there as a black-box. The proof of Lemma 1 of [30] is concluded in Section 9 of [30], and it is evident from its last line that we may replace the  $\beta$ -number right-hand side of Lemma 1 of [30] by the sum-over-Top-cubes right-hand side of Lemma 7.1.

We are going to use Lemma 3.1 together with Lemma 4.8 and Lemma 7.1 to get the following.

**Lemma 7.2.** *Let  $\mu$  be a compactly supported Radon measure on  $\mathbb{R}^d$  satisfying the growth condition (1.9). Assume further that for some  $V \in G(d, d - n), \alpha \in (0, 1)$ , we have*

$$\mathcal{E}_{\mu,2}(\mathbb{R}^d, V, \alpha) < \infty.$$

Then, for every singular integral operator  $T$  with kernel  $k \in \mathcal{K}^n(\mathbb{R}^d)$ , we have

$$(7.1) \quad \sup_{\varepsilon > 0} \|T_\varepsilon \mu\|_{L^2(\mu)}^2 \lesssim \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,2}(\mathbb{R}^d, V, \alpha),$$

with the implicit constant depending on  $C_1, \alpha$  and the constant  $C_k$  from (1.8).

*Proof.* Using Lemma 3.1 (with  $p = 2$ ), it is clear that the assumptions (i)–(iii) of Lemma 7.1 are satisfied. We still have to check if (iv) holds. Once we do that, the packing estimate (3.2) together with Lemma 7.1 will ensure that (7.1) holds.

Suppose  $R \in \text{Top}$ ,  $Q \in \text{Next}(R)$ . We are looking for  $S \in \mathcal{D}_\mu$  such that  $\delta_\mu(Q, S) \lesssim \Theta_\mu(2B_R)$  and  $2B_S \cap \Gamma_R \neq \emptyset$ . Let  $P \in \text{Stop}(R)$  be such that  $Q \subset P$ . By Lemma 4.8, we have some constant  $\Lambda$  such that

$$\Lambda B_P \cap \Gamma_R \neq \emptyset.$$

Together with (4.14), this implies that there exists  $S \in \text{Tree}(R)$  such that  $P \subset S$ ,  $r(S) \approx_\Lambda r(P)$ , and

$$2B_S \cap \Gamma_R \neq \emptyset.$$

We split

$$\delta_\mu(Q, S) = \int_{2B_S \setminus 2B_P} \frac{1}{|y - x_Q|^n} d\mu(y) + \int_{2B_P \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y).$$

Concerning the first integral, for  $y \in 2B_S \setminus 2B_P$ , we have  $|y - x_Q| \approx r(S) \approx_\Lambda r(P)$ , and so (from (4.2))

$$\int_{2B_S \setminus 2B_P} \frac{1}{|y - x_Q|^n} d\mu(y) \lesssim \Theta_\mu(2B_S) \lesssim_A \Theta_\mu(2B_R).$$

To deal with the second integral, observe that there are no doubling cubes between  $Q$  and  $P$ . Then it follows from Lemma 2.2 that

$$\int_{2B_P \setminus 2B_Q} \frac{1}{|y - x_Q|^n} d\mu(y) \lesssim \Theta_\mu(100B(P)).$$

If  $P = R$ , then  $P$  is doubling and we have  $\Theta_\mu(100B(P)) \lesssim \Theta_\mu(2B_R)$ . Otherwise, the parent of  $P$ , denoted by  $P'$ , belongs to  $\text{Tree}(R) \setminus \text{Stop}(R)$ . Since  $100B(P) \subset 2B_{P'}$ , we get (from (4.2))

$$\Theta_\mu(100B(P)) \lesssim \Theta_\mu(2B_{P'}) \lesssim_A \Theta_\mu(2B_R).$$

Either way, we get that  $\delta_\mu(Q, S) \lesssim_A \Theta_\mu(2B_R)$ , and so the assumption (iv) of Lemma 7.1 is satisfied. ■

Lemma 7.2 allows us to use the non-homogeneous  $T1$  theorem of Nazarov, Treil and Volberg [48] to prove a version of Theorem 1.17 in the case of a fixed direction  $V$ , i.e., if for all  $x \in \text{supp } \mu$ , we have  $V_x \equiv V$ .

**Lemma 7.3.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^d$  satisfying the polynomial growth condition (1.9). Suppose that there exist  $M_0 > 1$ ,  $\alpha \in (0, 1)$ ,  $V \in G(d, d - n)$  such that for every ball  $B$ , we have*

$$(7.2) \quad \mathcal{E}_{\mu,2}(B, V, \alpha) \leq M_0 \mu(B).$$

*Then all singular integral operators  $T_\mu$  with kernels in  $\mathcal{K}^d(\mathbb{R}^d)$  are bounded in  $L^2(\mu)$ . The bound on the operator norm of  $T_\mu$  depends only on  $C_1, \alpha, M_0$ , and the constant  $C_k$  from (1.8).*

*Proof.* We apply Lemma 7.2 to  $\mu|_B$ , where  $B$  is an arbitrary ball, and get that

$$\sup_{\varepsilon>0} \|T_\varepsilon(\mu|_B)\|_{L^2(\mu|_B)}^2 \lesssim_{C_1,\alpha,C_k} \mu(B) + \mathcal{E}_{\mu|_B,2}(\mathbb{R}^d, V, \alpha).$$

It is easy to see that, using the assumptions (1.9) and (7.2), we have

$$\mathcal{E}_{\mu|_B,2}(\mathbb{R}^d, V, \alpha) \lesssim \mathcal{E}_{\mu,2}(B, V, \alpha) + C_1^2 \mu(B) \leq (M_0 + C_1^2) \mu(B).$$

Hence,

$$(7.3) \quad \sup_{\varepsilon>0} \|T_\varepsilon(\mu|_B)\|_{L^2(\mu|_B)}^2 \lesssim_{C_1,\alpha,C_k,M_0} \mu(B).$$

The  $L^2$  boundedness of  $T_\mu$  follows by the non-homogeneous  $T1$  theorem from [48]. The condition (7.3) is slightly weaker than the original assumption in [48], but this is not a problem, see the discussion in Section 3.7.2 of [58]. ■

We are ready to finish the proof of Theorem 1.17.

*Proof of Theorem 1.17.* Let  $B$  be an arbitrary ball intersecting  $\text{supp } \mu$ . Recall that, by the definition of BPBE(2), there exist  $M_0 > 1, \kappa > 0, V_B \in G(d, d - n)$ , and  $G_B \subset B$  such that  $\mu(G_B) \geq \kappa \mu(B)$ , and for all  $x \in G_B$ ,

$$\int_0^{r(B)} \left( \frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \leq M_0.$$

By the polynomial growth condition (1.9), we also have

$$\int_{r(B)}^\infty \left( \frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \leq \int_{r(B)}^\infty \frac{\mu(B)^2}{r^{2n+1}} dr \lesssim \frac{\mu(B)^2}{r(B)^{2n}} \leq C_1^2.$$

Hence, for all  $x \in G_B$ ,

$$\int_0^\infty \left( \frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \lesssim_{C_1,M_0} 1.$$

Set  $\nu = \mu|_{G_B}$ . The estimate above implies that for all balls  $B' \subset \mathbb{R}^d$ , we have

$$\mathcal{E}_{\nu,2}(B', V_B, \alpha) = \int_{B'} \int_0^{r(B')} \left( \frac{\nu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} d\nu(x) \lesssim_{C_1,M_0} \nu(B').$$

Clearly,  $\nu = \mu|_{G_B}$  has polynomial growth, and so we may apply Lemma 7.3 to conclude that all singular integral operators  $T_\nu$  with kernels in  $\mathcal{K}^n(\mathbb{R}^d)$  are bounded in  $L^2(\nu)$ . Thus, the corresponding maximal operators  $T_*$  are bounded from  $M(\mathbb{R}^d)$  to  $L^{1,\infty}(\nu)$ , see Theorem 2.21 of [58].

Recall that for all balls  $B$ , we have  $\mu(G_B) \approx_\kappa \mu(B)$ . For any fixed  $T$ , the operator norm of  $T_{\mu|_{G_B,\varepsilon}}: L^2(\mu|_{G_B}) \rightarrow L^2(\mu|_{G_B})$  is bounded uniformly in  $B$  and  $\varepsilon$ , and so the same is true for the operator norm of  $T_*: M(\mathbb{R}^d) \rightarrow L^{1,\infty}(\mu|_{G_B})$ . Hence, we may use the good lambda method, Theorem 2.22 of [58], to conclude that  $T_\mu$  is bounded in  $L^2(\mu)$ . ■

### 8. Sufficient condition for rectifiability

The aim of this section is to prove the following sufficient condition for rectifiability.

**Proposition 8.1.** *Suppose  $\mu$  is a Radon measure on  $\mathbb{R}^d$  satisfying  $\Theta^{n,*}(\mu, x) > 0$  and  $\Theta_*^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Assume further that for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exist some  $V_x \in G(d, d - n)$  and  $\alpha_x \in (0, 1)$  such that*

$$(8.1) \quad \int_0^1 \left( \frac{\mu(K(x, V_x, \alpha_x, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Then  $\mu$  is  $n$ -rectifiable.

We reduce the proposition above to the following lemma.

**Lemma 8.2.** *Suppose  $\mu$  is a Radon measure on  $B(0, 1) \subset \mathbb{R}^d$ , and assume that there exists a constant  $C_* > 0$  such that  $\Theta_*^n(\mu, x) \leq C_*$  and  $\Theta^{n,*}(\mu, x) > 0$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Assume further that there exist  $M_0 > 0$ ,  $V \in G(d, d - n)$  and  $\alpha \in (0, 1)$  such that for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ ,*

$$(8.2) \quad \int_0^1 \left( \frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \leq M_0.$$

Then  $\mu$  is  $n$ -rectifiable.

*Proof of Proposition 8.1 using Lemma 8.2.* To show that  $\mu$  is rectifiable, it suffices to prove that for any bounded  $E \subset \text{supp } \mu$  of positive measure, there exists  $F \subset E$ ,  $\mu(F) > 0$ , such that  $\mu|_F$  is rectifiable. Given any such  $E$  we may rescale it and translate it, so, without loss of generality,  $E \subset B(0, 1)$ .

Since  $0 < \Theta^{n,*}(\mu, x)$  and  $\Theta_*^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in E$ , choosing  $C_* > 1$  big enough, we get that the set

$$(8.3) \quad E' = \{x \in E : \Theta^{n,*}(\mu, x) > 0, \Theta_*^n(\mu, x) \leq C_*\}$$

has positive  $\mu$ -measure.

Let  $\{V_k\}_{k \in \mathbb{N}}$  be a countable and dense subset of  $G(d, d - n)$ . It is clear that for any  $\alpha \in (0, 1)$ ,  $V \in G(d, d - n)$ , there exists  $k \in \mathbb{N}$  such that  $K(0, V_k, k^{-1}) \subset K(0, V, \alpha)$ . Set

$$E_k = \left\{ x \in \mathbb{R}^d : \int_0^1 \left( \frac{\mu(K(x, V_k, k^{-1}, r))}{r^n} \right)^p \frac{dr}{r} \leq k \right\}.$$

It is a simple exercise to check that for each  $k \in \mathbb{N}$ , the set  $E_k$  is Borel. Moreover, it follows from (8.1) that  $\mu(\mathbb{R}^d \setminus \bigcup_k E_k) = 0$ . Pick any  $k \in \mathbb{N}$  with  $\mu(E' \cap E_k) > 0$  and set  $F = E' \cap E_k$ . Using the Lebesgue differentiation theorem and (8.3), it is easy to see that for  $\mu$ -a.e.  $x \in F$ , we have  $\Theta^{n,*}(\mu|_F, x) = \Theta^{n,*}(\mu, x) > 0$  and  $\Theta_*^n(\mu|_F, x) = \Theta_*^n(\mu, x) \leq C_*$ . Hence,  $\mu|_F$  satisfies the assumptions of Lemma 8.2, and so it is  $n$ -rectifiable. ■

#### 8.1. Proof of Lemma 8.2 for $\mu \ll \mathcal{H}^n$

First, we will prove Lemma 8.2 under the additional assumption  $\Theta^{n,*}(\mu, x) < \infty$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$  (which is equivalent to  $\mu \ll \mathcal{H}^n$ ).

Using similar tricks as in the proof of Proposition 8.1, it is easy to see that we may actually replace  $\Theta^{n,*}(\mu, x) < \infty$  by a stronger condition: without loss of generality, we can assume that there exist  $C_1 > 0$  and  $r_0 > 0$  such that for all  $x \in \text{supp } \mu$  and all  $0 < r \leq r_0$ , we have

$$(8.4) \quad \mu(B(x, r)) \leq C_1 r^n.$$

Then the assumptions of Lemma 3.1 are satisfied, and we get a family of cubes  $\text{Top} \subset \mathcal{D}_\mu^{\text{db}}$  and an associated family of Lipschitz graphs  $\Gamma_R, R \in \text{Top}$ . The cubes from  $\text{Top}$  satisfy the packing condition

$$\sum_{R \in \text{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim \mu(\mathbb{R}^d) + \varepsilon_{\mu,p}(\mathbb{R}^d, V, \alpha) \leq (1 + M_0) \mu(B(0, 1)).$$

It follows that for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , we have

$$\sum_{R \in \text{Top}: R \ni x} \Theta_\mu(2B_R)^p < \infty.$$

Fix some  $x$  for which the above holds. Denote by  $R_0 \supset R_1 \supset \dots$  the sequence of cubes from  $\text{Top}$  containing  $x$ . We claim that for  $\mu$ -a.e.  $x$ , this sequence is finite.

Indeed, if the sequence is infinite, we have  $\Theta_\mu(2B_{R_i}) \rightarrow 0$ . On the other hand, let  $i \geq 0$  and  $r(R_{i+1}) \leq r \leq r(R_i)$ . Since  $R_{i+1} \in \text{Next}(R_i)$ , we get, from (6.1),

$$\Theta_\mu(x, r) \lesssim_A \Theta_\mu(2B_{R_i}).$$

In consequence,

$$\Theta^{n,*}(\mu, x) \lesssim_A \limsup_{i \rightarrow \infty} \Theta_\mu(2B_{R_i}) = 0,$$

which may happen only on a set of  $\mu$ -measure 0 because  $\Theta^{n,*}(\mu, x) > 0$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ .

Hence, for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , the sequence  $\{R_i\}$  is finite. This means that if  $R_k$  denotes the smallest  $\text{Top}$  cube containing  $x$ , then  $x \in \text{Good}(R_k)$ . It follows that

$$\mu\left(\mathbb{R}^d \setminus \bigcup_{R \in \text{Top}} \text{Good}(R)\right) = 0.$$

By Lemma 3.1 (ii), we have  $\mu(\text{Good}(R_k) \setminus \Gamma_{R_k}) = 0$ . Hence,

$$\mu\left(\mathbb{R}^d \setminus \bigcup_{R \in \text{Top}} \Gamma_R\right) = 0,$$

and so  $\mu$  is  $n$ -rectifiable.

### 8.2. Proof of Lemma 8.2 in full generality

Thanks to the partial result from the preceding subsection, it is clear that to prove Lemma 8.2 in full generality, it suffices to show that for  $\mu$  satisfying the assumptions of Lemma 8.2, we have

$$M_n \mu(x) = \sup_{r>0} \frac{\mu(B(x, r))}{r^n} < \infty \quad \text{for } \mu\text{-a.e. } x \in B(0, 1).$$

To do that, we will use techniques from Section 5 of [61].

**Lemma 8.3** (Lemma 5.1 of [61]). *Let  $C > 2$ . Suppose that  $\mu$  is a Radon measure on  $\mathbb{R}^d$ , and that  $\Theta_*^n(\mu, x) \leq C_*$  for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ . Then, for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exists a sequence of radii  $r_k \rightarrow 0$  such that*

$$\mu(B(x, Cr_k)) \leq 2C^d \mu(B(x, r_k)) \leq 20C_* C^{n+d} r_k^n.$$

Let  $\lambda < 1/2$  be a small constant depending on  $\alpha$ , to be chosen later. By the lemma above (used with  $C = \lambda^{-1}$ ) and Vitali’s covering theorem (see Theorem 2.8 of [42]), there exists a family of pairwise disjoint closed balls  $B_i, i \in I$ , centered at  $x_i \in \text{supp } \mu \subset B(0, 1)$ , which cover  $\mu$ -almost all of  $B(0, 1)$ , and which satisfy

$$\mu(B_i) \leq 2\lambda^{-d} \mu(\lambda B_i) \leq 20C_* \lambda^{-d} r(B_i)^n,$$

and

$$r(B_i) \leq \rho$$

for some arbitrary fixed  $\rho > 0$ . We may assume that (8.2) holds for all the centers  $x_i$ . Choose  $I_0 \subset I$  a finite subfamily such that

$$\mu\left(B(0, 1) \setminus \bigcup_{i \in I_0} B_i\right) \leq \varepsilon \mu(B(0, 1)),$$

where  $\varepsilon > 0$  is some small constant. Clearly,  $I_0 = I_0(\rho, \varepsilon)$ .

For each  $i \in I_0$ , we consider an  $n$ -dimensional disk  $D_i$ , centered at  $x_i$ , parallel to  $V^\perp \in G(d, n)$ , with radius  $\lambda r(B_i)$ . We define an approximating measure

$$\nu = \sum_{i \in I_0} \frac{\mu(B_i)}{\mathcal{H}^n(D_i)} \mathcal{H}^n|_{D_i}.$$

Note that

$$(8.5) \quad \nu(D_i) = \mu(B_i) \approx_\lambda \mu(\lambda B_i) \lesssim_\lambda C_* r(B_i)^n.$$

Moreover, since  $I_0$  is a finite family, the definition of  $\nu$  and (8.5) imply that  $\nu$  satisfies the polynomial growth condition (3.1) with  $r_0 = \min_{i \in I_0} r(B_i)/2$  and  $C_1 = C(\lambda)C_*$ , i.e., for  $0 < r < r_0$  and  $x \in \text{supp } \nu$ ,

$$(8.6) \quad \nu(B(x, r)) \leq C(\lambda)C_* r^n.$$

**Lemma 8.4.** *For  $\lambda = \lambda(\alpha) < 1/2$  small enough, we have*

$$\mathcal{E}_{\nu, p}(\mathbb{R}^d, V, \frac{1}{2}\alpha) \lesssim_{\lambda, p} (M_0 + \mu(B(0, 1))^p) \mu(B(0, 1)).$$

*The implicit constant does not depend on  $\rho, \varepsilon$ .*

*Proof.* Let  $i \in I_0$  and  $x \in D_i$ . We will estimate the  $\nu$ -measure of  $K(x, V, \frac{1}{2}\alpha, r)$ .

First, note that  $\nu(K(x, V, \frac{1}{2}\alpha, r)) = \nu(K(x, V, \frac{1}{2}\alpha, r) \setminus B_i)$ . Indeed,  $B_i \cap \text{supp } \nu = D_i$ , and  $D_i \cap K(x, V, \frac{1}{2}\alpha) = \emptyset$  because  $D_i$  is parallel to  $V^\perp$ . Thus,  $\nu(K(x, V, \frac{1}{2}\alpha, r) \cap B_i) = 0$ . It follows immediately that for  $r \leq (1 - \lambda)r(B_i)$ , we have  $\nu(K(x, V, \frac{1}{2}\alpha, r)) = 0$ .

Concerning  $r > (1 - \lambda)r(B_i)$ , if  $\lambda = \lambda(\alpha)$  is small enough, then

$$K(x, V, \frac{1}{2}\alpha, r) \setminus B_i \subset K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i$$

because  $x \in \lambda B_i$ . Thus, it suffices to estimate  $v(K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i)$ .

Suppose  $r > (1 - \lambda)r(B_i)$  and  $j \in I_0$  is such that  $D_j \cap K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i \neq \emptyset$ . Since  $B_i$  and  $B_j$  are disjoint, we have

$$r(B_j) + r(B_i) + \text{dist}(B_i, B_j) \leq 3r \quad \text{and} \quad \text{dist}(D_i, D_j) \geq \frac{r(B_i)}{2} + \frac{r(B_j)}{2}.$$

It follows easily that, for  $\lambda = \lambda(\alpha)$  small enough, we get  $\lambda B_j \subset K(x_i, V, \alpha, 4r)$ . Thus, using (8.5),

$$\begin{aligned} v(K(x_i, V, \frac{3}{4}\alpha, 2r)) &= v(K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i) \leq \sum_{j \in I_0: \lambda B_j \subset K(x_i, V, \alpha, 4r)} v(D_j) \\ &\approx_\lambda \sum_{j \in I_0: \lambda B_j \subset K(x_i, V, \alpha, 4r)} \mu(\lambda B_j) \leq \mu(K(x_i, V, \alpha, 4r)). \end{aligned}$$

Hence, thanks to (8.2),

$$\int_0^{1/4} \left( \frac{v(K(x_i, V, \frac{3}{4}\alpha, 2r))}{r^n} \right)^p \frac{dr}{r} \lesssim_\lambda \int_0^1 \left( \frac{\mu(K(x_i, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \leq M_0.$$

This gives

$$\begin{aligned} \int_{D_i} \int_0^\infty \left( \frac{v(K(x, V, \frac{1}{2}\alpha, r))}{r^n} \right)^p \frac{dr}{r} dv(x) &\leq \int_{D_i} \int_0^\infty \left( \frac{v(K(x_i, V, \frac{3}{4}\alpha, r))}{r^n} \right)^p \frac{dr}{r} dv(x) \\ &\leq C(\lambda)M_0v(D_i) + \int_{D_i} \int_{1/4}^\infty \left( \frac{v(\mathbb{R}^d)}{r^n} \right)^p \frac{dr}{r} dv(x) \\ &\lesssim_{\lambda,p} M_0v(D_i) + v(\mathbb{R}^d)^p v(D_i) \leq M_0\mu(B_i) + \mu(B(0, 1))^p \mu(B_i). \end{aligned}$$

Summing over  $i \in I_0$  yields

$$\mathcal{E}_{v,p}(\mathbb{R}^d, V, \frac{1}{2}\alpha) \lesssim_{\lambda,p} (M_0 + \mu(B(0, 1))^p)\mu(B(0, 1)). \quad \blacksquare$$

**Lemma 8.5.** *For  $\lambda = \lambda(\alpha) < 1/2$  small enough, we have*

$$\int M_n v(x)^p dv(x) \lesssim_{\alpha,\lambda,p} ((C_*)^p + M_0 + \mu(B(0, 1))^p)\mu(B(0, 1)).$$

The constants on the right-hand side do not depend on  $\rho, \varepsilon$ .

*Proof.* By (8.6) and Lemma 8.4, we may use Lemma 3.1 to get a family of cubes  $\text{Top}_\rho$ , satisfying properties (i)–(iii) of Lemma 3.1, and such that

$$\begin{aligned} (8.7) \quad \sum_{R \in \text{Top}_\rho} \Theta_v(2B_R)^p v(R) &\lesssim_{\alpha,\lambda} (C_*)^p v(\mathbb{R}^d) + C(p)(M_0 + \mu(B(0, 1))^p)\mu(B(0, 1)) \\ &\lesssim_{\alpha,\lambda,p} ((C_*)^p + M_0 + \mu(B(0, 1))^p)\mu(B(0, 1)). \end{aligned}$$



Now, property (iii) of Lemma 3.1 lets us estimate  $M_n v(x)$ . Indeed, suppose  $x \in \text{supp } v$ , and let  $r_1 > 0$  be such that

$$M_n v(x) \leq 2 \frac{v(B(x, r_1))}{r_1^n}.$$

Since  $\text{supp } v \subset B(0, 2)$ , we have  $r_1 \leq 4$ . Let  $Q \in \mathcal{D}_v$  be the smallest cube satisfying  $x \in Q$  and  $B(x, r_1) \cap \text{supp } v \subset 2B_Q$  (such a cube exists because the largest cube  $Q_0 := \text{supp } v$  clearly satisfies  $\text{supp } v \subset 2B_{Q_0}$ ). Let  $R \in \text{Top}_v$  be the top cube such that  $Q \in \text{Tr}(R)$ . Clearly,  $\ell(Q) \approx r_1$ . By Lemma 3.1 (iii), we have

$$\frac{v(B(x, r_1))}{r_1^n} \lesssim \Theta_v(2B_Q) \lesssim \Theta_v(2B_R).$$

Thus,  $M_n v(x)^p \lesssim \sum_{R \in \text{Top}_v} \mathbb{1}_R(x) \Theta_v(2B_R)^p$ . Integrating with respect to  $v$  and applying (8.7) yields the desired estimate. ■

**Lemma 8.6.** *We have*

$$\int M_n \mu(x)^p d\mu(x) \lesssim_{\alpha, \lambda, p} ((C_*)^p + M_0 + \mu(B(0, 1))^p) \mu(B(0, 1)).$$

In particular,  $M_n \mu(x) < \infty$  for  $\mu$ -a.e.  $x \in B(0, 1)$ .

*Proof.* Denote

$$M_{n, \rho} \mu(x) = \sup_{r \geq \rho} \frac{\mu(B(x, r))}{r^n}.$$

Recall that  $I_0 = I_0(\rho, \varepsilon)$  and set

$$E_{\varepsilon, \rho} = \text{supp } \mu \cap \bigcup_{i \in I_0} B_i.$$

We claim that

$$(8.8) \quad \int_{E_{\varepsilon, \rho}} M_{n, \rho}(\mathbb{1}_{E_{\varepsilon, \rho}} \mu)(x)^p d\mu(x) \lesssim \int M_{n, \rho} v(x)^p dv(x).$$

Indeed, let  $x, x' \in B_j, j \in I_0$ , and  $r \geq \rho$ . Then, using repeatedly the fact that  $r(B_i) \leq \rho \leq r$  for  $i \in I_0$ ,

$$\begin{aligned} \mu(B(x, r) \cap E_{\varepsilon, \rho}) &\leq \mu(B(x', 3r) \cap E_{\varepsilon, \rho}) \leq \sum_{i \in I_0: B_i \cap B(x', 3r) \neq \emptyset} \mu(B_i) \\ &= \sum_{i \in I_0: B_i \cap B(x', 3r) \neq \emptyset} v(D_i) \leq v(B(x', 5r)). \end{aligned}$$

Hence, for all  $x \in B_j, j \in I_0$ ,

$$M_{n, \rho}(\mathbb{1}_{E_{\varepsilon, \rho}} \mu)(x) \leq 5^n \inf_{x' \in B_j} M_{n, \rho} v(x').$$

Integrating both sides of the inequality with respect to  $\mu$  in  $E_{\varepsilon, \rho}$  yields (8.8).

Lemma 8.5 and (8.8) give

$$\int_{E_{\varepsilon,\rho}} M_{n,\rho}(\mathbb{1}_{E_{\varepsilon,\rho}}\mu)(x)^p d\mu(x) \leq C(\alpha, \lambda, p)((C_*)^p + M_0 + \mu(B(0, 1))^p)\mu(B(0, 1)) =: K,$$

where  $K$  is independent of  $\rho$  and  $\varepsilon$ .

Set  $\varepsilon_k = 2^{-k}$ . Observe that, for a fixed  $\rho > 0$ , we have  $\mu(\mathbb{R}^d \setminus \liminf_k E_{\varepsilon_k,\rho}) = 0$ , where

$$\liminf_k E_{\varepsilon_k,\rho} = \bigcup_{j=1}^{\infty} G_j \quad \text{and} \quad G_j = \bigcap_{k=j}^{\infty} E_{\varepsilon_k,\rho}.$$

The inclusion  $G_j \subset E_{\varepsilon_j,\rho}$  gives

$$\int_{G_j} M_{n,\rho}(\mathbb{1}_{G_j}\mu)(x)^p d\mu(x) \leq \int_{E_{\varepsilon_j,\rho}} M_{n,\rho}(\mathbb{1}_{E_{\varepsilon_j,\rho}}\mu)(x)^p d\mu(x) \leq K.$$

Since the sequence of sets  $G_j$  is increasing, we easily get that for  $\mu$ -a.e.  $x \in B(0, 1)$ ,

$$\mathbb{1}_{G_j}(x) M_{n,\rho}(\mathbb{1}_{G_j}\mu)(x) \xrightarrow{j \rightarrow \infty} M_{n,\rho}\mu(x),$$

and the convergence is monotone. Hence, by the monotone convergence theorem,

$$\int M_{n,\rho}\mu(x)^p d\mu(x) \leq K.$$

The estimate is uniform in  $\rho$ , and so once again monotone convergence gives

$$\int M_n\mu(x)^p d\mu(x) \leq K. \quad \blacksquare$$

Taking into account Lemma 8.6 and Section 8.1, the proof of Lemma 8.2 is finished.

### 9. Necessary condition for rectifiability

In this section we will prove the following.

**Proposition 9.1.** *Suppose  $\mu$  is an  $n$ -rectifiable measure on  $\mathbb{R}^d$ , and  $1 \leq p < \infty$ . Then, for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , there exists  $V_x \in G(d, d - n)$  such that for any  $\alpha \in (0, 1)$ , we have*

$$\int_0^1 \left( \frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

First, we recall the definition of  $\beta_2$  numbers, as defined by David and Semmes [21].

**Definition 9.2.** Given a Radon measure  $\mu$ ,  $x \in \text{supp } \mu$ ,  $r > 0$ , and an  $n$ -plane  $L$ , define

$$\beta_{\mu,2}(x, r) = \inf_L \left( \frac{1}{r^n} \int_{B(x,r)} \left( \frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y) \right)^{1/2},$$

where the infimum is taken over all  $n$ -planes intersecting  $B(x, r)$ .

Tolsa [59] showed the following necessary condition for rectifiability in terms of  $\beta_2$  numbers.

**Theorem 9.3** ([59]). *Suppose  $\mu$  is an  $n$ -rectifiable measure on  $\mathbb{R}^d$ . Then, for  $\mu$ -a.e.  $x \in \mathbb{R}^d$ , we have*

$$(9.1) \quad \int_0^1 \beta_{\mu,2}(x, r)^2 \frac{dr}{r} < \infty.$$

**Remark 9.4.** When showing that rectifiable sets have approximate tangents almost everywhere, one uses the so-called *linear approximation* properties, see Theorems 15.11 and 15.19 of [42]. The theorem of Tolsa improves on the linear approximation property and allows us to improve on the classical approximate tangent plane result.

For a fixed  $n$ -rectifiable measure  $\mu$ , let  $L_{x,r}$  denote a plane minimizing  $\beta_{\mu,2}(x, r)$  (it may be non-unique, in which case we simply choose one of the minimizers).

Recall that in Definition 1.1 we defined the approximate tangent to  $\mu$  to be an  $n$ -plane  $W'_x \in G(d, n)$ . Let  $W_x := x + W'_x$ , whenever the approximate tangent exists and is unique (that is, for  $\mu$ -a.e.  $x$ , by Theorem 1.2). In order to apply Tolsa’s result in our setting, we need the following intuitively clear result.

**Lemma 9.5.** *Let  $\mu$  be a rectifiable measure. Then, for  $\mu$ -a.e.  $x \in \text{supp } \mu$ , we have*

$$(9.2) \quad \frac{\text{dist}_H(L_{x,r} \cap B(x, r), W_x \cap B(x, r))}{r} \xrightarrow{r \rightarrow 0} 0.$$

A relatively simple (although lengthy) proof can be found in Appendix A.

Before proving Proposition 9.1, we need one more lemma. Recall that if  $\alpha > 0$ ,  $W$  is an  $n$ -plane, and  $0 < r < R$ , then  $K(x, W^\perp, \alpha, r, R) = K(x, W^\perp, \alpha, R) \setminus K(x, W^\perp, \alpha, r)$ .

**Lemma 9.6.** *Let  $\alpha, \varepsilon \in (0, 1)$  be some constants satisfying  $\eta := 1 - \alpha - 3\varepsilon > 0$ . Let  $x \in \mathbb{R}^d$ ,  $r > 0$ , and suppose that  $W$  and  $L$  are  $n$ -planes satisfying  $x \in W$  and*

$$(9.3) \quad \text{dist}_H(L \cap B(x, r), W \cap B(x, r)) \leq \varepsilon r.$$

Then

$$K(x, W^\perp, \alpha, r, 2r) \subset B(x, 2r) \setminus B_{\eta r}(L).$$

*Proof.* Suppose  $y \in K(x, W^\perp, \alpha, r, 2r)$ , so that  $r < |x - y| < 2r$  and  $|x - \pi_W(y)| < \alpha|x - y|$ . We need to show that  $\text{dist}(y, L) > \eta r$ .

Set  $y' = \pi_L(y)$ ,  $x' = \pi_L(x)$ . Then, using (9.3),

$$\begin{aligned} \text{dist}(y, L) &= |y - y'| \geq |x - y| - |x' - y'| - |x - x'| \\ &= |x - y| - |x' - y'| - \text{dist}(x, L) \geq |x - y| - |x' - y'| - \varepsilon r. \end{aligned}$$

Let  $\tilde{\pi}_W$  and  $\tilde{\pi}_L$  denote the orthogonal projections onto the  $n$ -planes parallel to  $W$  and  $L$  passing through the origin. It follows from (9.3) that  $\|\tilde{\pi}_W - \tilde{\pi}_L\|_{\text{op}} \leq \varepsilon$ . Thus,

$$|x' - y'| = |\tilde{\pi}_L(x - y)| \leq |\tilde{\pi}_W(x - y)| + \|\tilde{\pi}_W - \tilde{\pi}_L\|_{\text{op}} |x - y| \leq |\tilde{\pi}_W(x - y)| + 2\varepsilon r.$$

Hence, using the fact that  $|\tilde{\pi}_W(x - y)| = |x - \pi_W(y)| < \alpha|x - y|$ , we get, from the two estimates above,

$$\text{dist}(y, L) \geq |x - y| - |\tilde{\pi}_W(x - y)| - 3\varepsilon r \geq (1 - \alpha)|x - y| - 3\varepsilon r \geq (1 - \alpha - 3\varepsilon)r = \eta r.$$

The proof is complete. ■

*Proof of Proposition 9.1.* Let  $\mu$  be  $n$ -rectifiable. For  $r > 0$  and  $x \in \text{supp } \mu$ , let  $L_{x,r}$  be the  $n$ -plane minimizing  $\beta_{\mu,2}(x, r)$ . We know that for  $\mu$ -a.e.  $x \in \text{supp } \mu$ , we have (9.1) and (9.2) (in particular, the approximate tangent plane  $W_x$  exists). Fix such  $x$ . Set  $V_x = W_x^\perp$ , let  $\alpha \in (0, 1)$  be arbitrary, and for  $0 < r < R$ , set  $K(r) = K(x, V_x, \alpha, r)$ ,  $K(r, R) = K(x, V_x, \alpha, r, R)$ . We will show that

$$(9.4) \quad \int_0^1 \left( \frac{\mu(K(r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Let  $\varepsilon > 0$  be a constant so small that  $\eta := 1 - \alpha - 3\varepsilon > 0$ . Use Lemma 9.5 to find  $r_0 > 0$  such that for  $0 < r \leq r_0$ , we have

$$\text{dist}_H(L_{x,r} \cap B(x, r), W_x \cap B(x, r)) \leq \varepsilon r.$$

Then it follows from Lemma 9.6 that for all  $0 < r \leq r_0$ ,

$$K(r, 2r) \subset B(x, 2r) \setminus B_{\eta r}(L_{x,r}).$$

Note that, by Chebyshev’s inequality,

$$\mu(B(x, 2r) \setminus B_{\eta r}(L_{x,r})) \leq \eta^{-2} \int_{B(x,2r)} \left( \frac{\text{dist}(y, L_{x,r})}{r} \right)^2 d\mu(y) = \eta^{-2} (2r)^n \beta_{\mu,2}(x, 2r)^2.$$

Hence, for  $0 < r \leq r_0$ , we have

$$\frac{\mu(K(r, 2r))}{r^n} \lesssim_{\eta} \beta_{\mu,2}(x, 2r)^2,$$

and so, from (9.1),

$$(9.5) \quad \int_0^{r_0} \frac{\mu(K(r, 2r))}{r^n} \frac{dr}{r} \lesssim_{\eta} \int_0^{2r_0} \beta_{\mu,2}(x, r)^2 \frac{dr}{r} < \infty.$$

Now, observe that for any integer  $N > 0$ ,

$$\begin{aligned} \int_{2^{-N}r_0}^{r_0} \frac{\mu(K(r))}{r^n} \frac{dr}{r} &\lesssim (r_0)^{-n} \sum_{k=0}^N \mu(K(2^{-k}r_0))2^{kn} \\ &\leq 2^n (r_0)^{-n} \sum_{k=0}^N \mu(K(2^{-k}r_0))2^{kn} - (r_0)^{-n} \sum_{k=0}^N \mu(K(2^{-k}r_0))2^{kn} \\ &= (r_0)^{-n} \sum_{k=0}^N \mu(K(2^{-k}r_0))2^{(k+1)n} - (r_0)^{-n} \sum_{k=0}^N \mu(K(2^{-k}r_0))2^{kn} \end{aligned}$$

$$\begin{aligned}
 &= (r_0)^{-n} \sum_{k=1}^{N+1} (\mu(K(2^{-k+1}r_0)) - \mu(K(2^{-k}r_0)))2^{kn} + \frac{\mu(K(2^{-(N+1)}r_0))}{(2^{-(N+1)}r_0)^n} - \frac{\mu(K(r_0))}{r_0^n} \\
 &\lesssim \int_0^{r_0/2} \frac{\mu(K(r, 2r))}{r^n} \frac{dr}{r} + \Theta_\mu(x, 2^{-(N+1)}r_0) + 0.
 \end{aligned}$$

Letting  $N \rightarrow \infty$ , we get from the above and (9.5) that

$$\int_0^{r_0} \frac{\mu(K(r))}{r^n} \frac{dr}{r} \lesssim_\eta \int_0^{2r_0} \beta_{\mu,2}(x, r)^2 \frac{dr}{r} + \Theta^{n,*}(\mu, x) < \infty$$

for  $\mu$ -a.e.  $x \in \text{supp } \mu$ , where we also used the fact that  $\Theta^{n,*}(\mu, x) < \infty$   $\mu$ -almost everywhere (because  $\mu$  is  $n$ -rectifiable). The integral  $\int_{2r_0}^1 \frac{\mu(K(r))}{r^n} \frac{dr}{r}$  is obviously finite, and so we get that

$$\int_0^1 \frac{\mu(K(r))}{r^n} \frac{dr}{r} < \infty,$$

which is precisely (9.4) with  $p = 1$ . To get the same result with  $p > 1$ , note that since  $\Theta^{n,*}(\mu, x) < \infty$  for  $\mu$ -a.e.  $x$ , we have

$$\begin{aligned}
 \int_0^1 \left(\frac{\mu(K(r))}{r^n}\right)^p \frac{dr}{r} &\leq \int_0^1 \frac{\mu(K(r))}{r^n} \Theta_\mu(x, r)^{p-1} \frac{dr}{r} \\
 &\leq \sup_{0 < r < 1} \Theta_\mu(x, r)^{p-1} \int_0^1 \frac{\mu(K(r))}{r^n} \frac{dr}{r} < \infty. \quad \blacksquare
 \end{aligned}$$

### 10. Sufficient condition for BPLG

In this section we prove the ‘‘sufficient part’’ of Theorem 1.11. After a suitable translation and rescaling, it suffices to show the following.

**Proposition 10.1.** *Suppose  $p \geq 1$ ,  $E \subset \mathbb{R}^d$  is  $n$ -AD-regular, and  $0 \in E$ . Let  $\alpha > 0$ ,  $M_0 > 1$ ,  $\kappa > 0$ , and assume that there exist  $F \subset E \cap B(0, 1)$  and  $V \in G(d, d - n)$  such that  $\mathcal{H}^n(F) \geq \kappa$ , and for all  $x \in F$ ,*

$$(10.1) \quad \int_0^1 \left(\frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap F)}{r^n}\right)^p \frac{dr}{r} \leq M_0.$$

*Then there exists a Lipschitz graph  $\Gamma$ , with Lipschitz constant depending on  $\alpha, n, d$ , such that*

$$(10.2) \quad \mathcal{H}^n(F \cap \Gamma) \gtrsim 1,$$

*with the implicit constant depending on  $\kappa, p, M_0, \alpha, n, d$ , and the AD-regularity constants of  $E$ .*

To prove the above we will use techniques developed in [40]. Fix  $V \in G(d, d - n)$ . Let  $\theta > 0$  and  $M \in \{0, 1, 2, \dots\}$ . In the language of Martikainen and Orponen, a set  $E \subset \mathbb{R}^d$  has the  $n$ -dimensional  $(\theta, M)$ -property if for all  $x \in E$ ,

$$\#\{j \in \mathbb{Z} : K(x, V, \theta, 2^{-j}, 2^{-j+1}) \cap E \neq \emptyset\} \leq M.$$

It is easy to see that if  $E$  has the  $n$ -dimensional  $(\theta, 0)$ -property, then  $E$  is contained in a Lipschitz graph with Lipschitz constant bounded by  $1/\theta$ , see Remark 1.11 of [40].

The main proposition of [40] reads as follows.

**Proposition 10.2** (Proposition 1.12 of [40]). *Assume that  $E$  is  $n$ -AD-regular, and assume that  $F_1 \subset E \cap B(0, 1)$  is an  $\mathcal{H}^n$ -measurable subset with  $\mathcal{H}^n(F_1) \approx_C 1$ . Suppose further that  $F_1$  satisfies the  $n$ -dimensional  $(\theta, M)$ -property for some  $\theta > 0, M \geq 0$ . Then there exists an  $\mathcal{H}^n$ -measurable subset  $F_2 \subset F_1$ , with  $\mathcal{H}^n(F_2) \approx_{C, \theta, M} 1$ , which satisfies the  $(\theta/b, 0)$ -property. Here  $b \geq 1$  is a constant depending only on  $d$ .*

**Remark 10.3.** It follows immediately from the proposition above that if we construct  $F_1 \subset E \cap B(0, 1)$  with  $\mathcal{H}^n(F_1) \approx \kappa$  satisfying the  $n$ -dimensional  $(\alpha/2, M)$ -property, then we will get a Lipschitz graph  $\Gamma$  such that (10.2) holds. Hence, we will be done with the proof of Proposition 10.1.

To construct  $F_1$ , we will use another lemma from [40].

**Lemma 10.4** (Lemma 2.1 of [40]). *Let  $E$  be an  $n$ -AD-regular set with  $\mathcal{H}^n(E) \geq C > 0$ , let  $F \subset E \cap B(0, 1)$  be an  $\mathcal{H}^n$ -measurable subset, and let*

$$F_\varepsilon = \{x \in F : \mathcal{H}^n(F \cap B(x, r_x)) \leq \varepsilon r_x^n \text{ for some radius } 0 < r_x \leq 1\}.$$

Then  $\mathcal{H}^n(F_\varepsilon) \lesssim \varepsilon$  with the bound depending only on  $C$  and the AD-regularity constant of  $E$ .

Note that the set  $F \setminus F_\varepsilon$  does not have to be AD-regular. Nevertheless, we gain some extra regularity that will prove useful.

Now, let  $E$  and  $F \subset E \cap B(0, 1)$  be as in the assumptions of Proposition 10.1. We apply Lemma 10.4 to conclude that for some  $\varepsilon$ , depending on  $\kappa$  and the AD-regularity constant of  $E$ , we have

$$\mathcal{H}^n(F \setminus F_\varepsilon) \geq \frac{\kappa}{2}.$$

Set  $F_1 = F \setminus F_\varepsilon$ .

**Lemma 10.5.** *There exists  $M = M(M_0, \varepsilon, \alpha, n)$  such that  $F_1$  satisfies the  $n$ -dimensional  $(\alpha/2, M)$ -property.*

*Proof.* Denote by  $F_{\text{Bad}} \subset F_1$  the set of  $x \in F_1$  such that

$$(10.3) \quad \#\{j \in \mathbb{Z} : K(x, V, \alpha/2, 2^{-j}, 2^{-j+1}) \cap F_1 \neq \emptyset\} > M.$$

We will show that, if  $M$  is chosen big enough, the set  $F_{\text{Bad}}$  is empty.

Let  $x \in F_{\text{Bad}}$  and  $j \in \mathbb{Z}$  be such that there exists  $x_j \in K(x, V, \alpha/2, 2^{-j}, 2^{-j+1}) \cap F_1$ . It is easy to see that for some  $\lambda = \lambda(\alpha)$ , independent of  $j$ , we have

$$B(x_j, \lambda 2^{-j}) \subset K(x, V, \alpha, 2^{-j-1}, 2^{-j+2}).$$

Since  $x_j \in F_1 = F \setminus F_\varepsilon$ , it follows that

$$\mathcal{H}^n(F \cap B(x_j, \lambda 2^{-j})) > \varepsilon (\lambda 2^{-j})^n.$$

The two observations above give

$$\frac{\mathcal{H}^n(F \cap K(x, V, \alpha, 2^{-j+2}))}{(2^{-j+2})^n} \geq \frac{\mathcal{H}^n(F \cap K(x, V, \alpha, 2^{-j-1}, 2^{-j+2}))}{(2^{-j+2})^n} \gtrsim_{\alpha, \lambda} \varepsilon.$$

By (10.3), there are more than  $M$  different scales (i.e.,  $j$ 's) for which the above holds. Thus, for  $x \in F_{\text{Bad}}$ , we have

$$\int_0^1 \left( \frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap F)}{r^n} \right)^p \frac{dr}{r} \gtrsim_{\alpha, \lambda} M \varepsilon^p.$$

Taking  $M = M(M_0, \varepsilon, \alpha, n, p)$  big enough, we get a contradiction with (10.1). Thus,  $F_{\text{Bad}}$  is empty. Now, it follows trivially by the definition of  $F_{\text{Bad}}$  that  $F_1$  satisfies the  $n$ -dimensional  $(\alpha/2, M)$ -property. ■

By Remark 10.3, this finishes the proof of Proposition 10.1.

### 11. Necessary condition for BPLG

In this section we prove the “necessary part” of Theorem 1.11. After rescaling, translating, and using the BPLG property, it is clear that it suffices to show the following.

**Proposition 11.1.** *Suppose  $E \subset \mathbb{R}^d$  is  $n$ -AD-regular and  $0 \in E$ . Let  $p \geq 1$ . Assume there exists a Lipschitz graph  $\Gamma$  such that  $\mathcal{H}^n(\Gamma \cap E \cap B(0, 1)) \geq \kappa$ . Then there exists  $\alpha = \alpha(\text{Lip}(\Gamma)) > 0$ ,  $V \in G(d, d - n)$  and a set  $F \subset \Gamma \cap E \cap B(0, 1)$  such that  $\mathcal{H}^n(F) \gtrsim \kappa$ , and for  $x \in F$ ,*

$$(11.1) \quad \int_0^1 \left( \frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap E)}{r^n} \right)^p \frac{dr}{r} \leq M_0,$$

where  $M_0 > 1$  is a constant depending on  $p, \text{Lip}(\Gamma), \kappa$  and the AD-regularity constant of  $E$ .

We begin by fixing some additional notation. Set  $\mu = \mathcal{H}^n|_E$ . We will denote the AD-regularity constant of  $E$  by  $C_0$ , so that for every  $x \in E, 0 < r < \text{diam}(E)$ ,

$$C_0^{-1}r^n \leq \mu(B(x, r)) \leq C_0r^n.$$

**Remark 11.2.** Since we assume that  $E$  is AD-regular, the exponent  $p$  in (11.1) does not really matter. For any  $p > 1$ , we have

$$\left( \frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap E)}{r^n} \right)^p \leq C_0^{p-1} \frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap E)}{r^n},$$

and so it is enough to prove (11.1) for  $p = 1$ .

Set  $L = \text{Lip}(\Gamma)$ . Let  $V \in G(d, d - n)$  be such that  $\Gamma$  is an  $L$ -Lipschitz graph over  $V^\perp$ , and let  $\theta = \theta(L) > 0$  be such that

$$K(x, V, \theta) \cap \Gamma = \emptyset \quad \text{for all } x \in \Gamma.$$

Set  $\alpha = \min(\frac{\theta}{2}, 0.1, \frac{1}{4L})$ .

For every  $x \in E \cap B(0, 1) \setminus \Gamma$ , consider the ball  $B_x = B(x, 0.01 \operatorname{dist}(x, \Gamma))$ . We use the  $5r$ -covering lemma to choose a countable subfamily of pairwise disjoint balls  $B_j = B(x_j, r_j)$ ,  $r_j = 0.01 \operatorname{dist}(x_j, \Gamma)$ ,  $j \in \mathbb{Z}$ , such that

$$E \cap B(0, 1) \setminus \Gamma \subset \bigcup_{j \in \mathbb{Z}} 5B_j.$$

Observe that

$$(11.2) \quad \sum_{j \in \mathbb{Z}} r_j^n \leq C_0 \sum_{j \in \mathbb{Z}} \mu(B_j) = C_0 \mu\left(\bigcup_{j \in \mathbb{Z}} B_j\right) \leq C_0 \mu(B(0, 2)) \lesssim C_0^2.$$

For each  $j \in \mathbb{Z}$ , set

$$K_j = \bigcup_{y \in 5B_j} K(y, V, \alpha), \quad K_j(r) = \bigcup_{y \in 5B_j} K(y, V, \alpha, r).$$

**Lemma 11.3.** *For each  $j \in \mathbb{Z}$ , we have*

$$(11.3) \quad \mathcal{H}^n(K_j \cap \Gamma) \lesssim_L r_j^n.$$

Moreover,

$$(11.4) \quad K_j(r) \cap \Gamma = \emptyset \quad \text{for } r < r_j.$$

*Proof.* Equation (11.4) is easy to prove: observe that for  $r < r_j$ , we have  $K_j(r) \subset 6B_j$ , and so for  $y \in K_j(r)$ ,

$$\operatorname{dist}(y, \Gamma) \geq \operatorname{dist}(x_j, \Gamma) - 6r_j = (1 - 0.06) \operatorname{dist}(x_j, \Gamma) > 0.$$

Concerning (11.3), we claim that since  $\Gamma = \operatorname{graph}(F)$  for some  $L$ -Lipschitz function  $F: V^\perp \rightarrow V$ , and since  $\alpha$  is sufficiently small, for all  $x \in \mathbb{R}^d$ , we have

$$(11.5) \quad K(x, V, \alpha) \cap \Gamma \subset B(x, C \operatorname{dist}(x, \Gamma)),$$

where  $C = C(L) > 1$ . Indeed, if  $\operatorname{dist}(x, \Gamma) = 0$ , then  $K(x, V, \alpha) \cap \Gamma = \emptyset$  and there is nothing to prove. Suppose  $\operatorname{dist}(x, \Gamma) > 0$ ,  $y \in K(x, V, \alpha) \cap \Gamma$ , and let  $z \in \Gamma$  be the image of  $x$  under the projection onto  $\Gamma$  orthogonal to  $V^\perp$ , i.e.,  $z = \pi_{V^\perp}^\perp(x) + F(\pi_{V^\perp}^\perp(x))$ .

Observe that, since  $\Gamma$  is a Lipschitz graph,

$$|x - z| \lesssim_L \operatorname{dist}(x, \Gamma),$$

and also  $\pi_{V^\perp}^\perp(x) = \pi_{V^\perp}^\perp(z)$ . By the definition of a cone,  $y \in K(x, V, \alpha)$  gives

$$|\pi_{V^\perp}^\perp(z - y)| = |\pi_{V^\perp}^\perp(x - y)| < \alpha|x - y|.$$

On the other hand,  $y \in \Gamma$  and the above imply

$$|\pi_V(z - y)| \leq L|\pi_{V^\perp}^\perp(z - y)| < L\alpha|x - y|.$$



The three estimates above yield

$$\begin{aligned} |x - y| &\leq |x - z| + |z - y| \leq C(L) \operatorname{dist}(x, \Gamma) + |\pi_V^\perp(z - y)| + |\pi_V(z - y)| \\ &\leq C(L) \operatorname{dist}(x, \Gamma) + \alpha|x - y| + L\alpha|x - y| \leq C(L) \operatorname{dist}(x, \Gamma) + \frac{1}{2}|x - y|. \end{aligned}$$

Hence,  $|x - y| \lesssim_L \operatorname{dist}(x, \Gamma)$  and (11.5) follows.

Now, going back to (11.3), note that for  $y \in 5B_j$ , we have  $\operatorname{dist}(y, \Gamma) \approx r_j$ , so that  $K(y, V, \alpha) \cap \Gamma \subset B(y, Cr_j)$  for some  $C = C(L)$ . Moreover,  $B(y, Cr_j) \subset B(x_j, 10Cr_j)$ . Therefore,  $K_j \cap \Gamma \subset B(x_j, 10Cr_j) \cap \Gamma$ , and (11.3) easily follows. ■

*Proof of Proposition 11.1.* Let  $x \in \Gamma \cap B(0, 1)$  and  $0 < r < 1$ . Since  $\{5B_j\}_{j \in \mathbb{Z}}$  cover  $E \cap B(0, 1) \setminus \Gamma$ , and  $K(x, V, \alpha, r) \cap \Gamma = \emptyset$ , we have

$$\mu(K(x, V, \alpha, r)) \leq \sum_{j \in \mathbb{Z}: 5B_j \cap K(x, V, \alpha, r) \neq \emptyset} \mu(5B_j) \lesssim C_0 \sum_{j \in \mathbb{Z}: 5B_j \cap K(x, V, \alpha, r) \neq \emptyset} r_j^n.$$

Notice that  $5B_j \cap K(x, V, \alpha, r) \neq \emptyset$  if and only if  $x \in K_j(r)$ . Hence, using the above and Lemma 11.3 yields

$$\begin{aligned} &\int_{\Gamma \cap B(0, 1)} \int_0^1 \frac{\mu(K(x, V, \alpha, r))}{r^n} \frac{dr}{r} d\mathcal{H}^n(x) \\ &\lesssim_{C_0} \int_{\Gamma \cap B(0, 1)} \int_0^1 \frac{1}{r^n} \sum_{j \in \mathbb{Z}} r_j^n \mathbb{1}_{K_j(r)}(x) \frac{dr}{r} d\mathcal{H}^n(x) \\ &= \sum_{j \in \mathbb{Z}} r_j^n \int_{\Gamma \cap B(0, 1)} \int_0^1 \frac{1}{r^n} \mathbb{1}_{K_j(r)}(x) \frac{dr}{r} d\mathcal{H}^n(x) \\ &\leq \sum_{j \in \mathbb{Z}} r_j^n \int_{K_j \cap \Gamma} \int_{r_j}^1 \frac{1}{r^n} \frac{dr}{r} d\mathcal{H}^n(x) \lesssim \sum_{j \in \mathbb{Z}} r_j^n \int_{K_j \cap \Gamma} r_j^{-n} d\mathcal{H}^n(x) \lesssim_L \sum_{j \in \mathbb{Z}} r_j^n \lesssim_{C_0} 1, \end{aligned}$$

see (11.2)–(11.4). We know that  $\mathcal{H}^n(\Gamma \cap B(0, 1) \cap E) \geq \kappa$ , and so we can use Chebyshev’s inequality to conclude that there exist  $M_0 = M_0(L, C_0, \kappa) > 1$  and  $F \subset \Gamma \cap B(0, 1) \cap E$ , with  $\mathcal{H}^n(F) \geq \kappa/2$ , such that for all  $x \in F$ ,

$$\int_0^1 \frac{\mu(K(x, V, \alpha, r))}{r^n} \frac{dr}{r} \leq M_0. \quad \blacksquare$$

### A. Proof of Lemma 9.5

For reader’s convenience, we restate Lemma 9.5 below.

**Lemma A.1.** *Let  $\mu$  be a  $n$ -rectifiable measure. For  $x \in \operatorname{supp} \mu$  and  $r > 0$ , let  $L_{x,r}$  denote a minimizing plane for  $\beta_{\mu,2}(x, r)$ , let  $W'_x$  be the approximate tangent plane to  $\mu$  at  $x$ , whenever it exists, and let  $W_x = W'_x + x$ . Then for  $\mu$ -a.e.  $x \in \operatorname{supp} \mu$ , we have*

$$(A.1) \quad \frac{\operatorname{dist}_H(L_{x,r} \cap B(x, r), W_x \cap B(x, r))}{r} \xrightarrow{r \rightarrow 0} 0.$$

*Proof.* Recall that since  $\mu$  is  $n$ -rectifiable, the density  $\Theta^n(\mu, x)$  exists and satisfies  $0 < \Theta^n(\mu, x) < \infty$  for  $\mu$ -a.e.  $x$ . Let  $M \geq 100$  be some big constant. Define

$$E_M := \{x \in \text{supp } \mu : M^{-1} \leq \Theta^n(\mu, x) \leq M\}.$$

Note that  $\mu(\mathbb{R}^d \setminus \bigcup_{M \geq 100} E_M) = 0$ , and so it suffices to show that for all  $M \geq 100$ , (A.1) holds for  $\mu$ -a.e.  $x \in E_M$ . Fix some big  $M$ , and set  $\nu = \mu|_{E_M}$ . It is well known that

$$(A.2) \quad M^{-1} \leq \Theta^n(\nu, x) = \Theta^n(\mu, x) \leq M \quad \text{for } \nu\text{-a.e. } x \in \text{supp } \nu,$$

which can be shown, e.g., using Corollary 6.3 of [42] in conjunction with the Lebesgue differentiation theorem. For  $\nu$ -a.e.  $x$ , the plane  $W_x$  is well defined by Theorem 1.2, and also by Theorem 9.3,

$$(A.3) \quad \int_0^1 \beta_{\mu,2}(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

Fix  $x \in E_M$  such that (A.3) and (A.2) hold, and such that  $W_x$  is well defined. Once we show that (A.1) holds at  $x$ , the proof will be finished. From now on we will suppress the subscript  $x$ , so that  $L_r =: L_{x,r}$ ,  $W =: W_x$ . By applying an appropriate translation, we may assume that  $x = 0$ .

Given some small  $r > 0$ , let  $A_r(y) = y/r$ , so that  $A_r(B(0, r)) = B(0, 1)$ . Set  $L'_r = A_r(L_r)$ . It is easy to see that (A.1) is equivalent to showing

$$\text{dist}_H(L'_r \cap B(0, 1), W \cap B(0, 1)) \xrightarrow{r \rightarrow 0} 0.$$

We will prove that the convergence above holds by contradiction. Suppose it is not true, so that there exist  $\varepsilon > 0$  and a sequence  $r_k \rightarrow 0$  such that for all  $k$ , we have

$$(A.4) \quad \text{dist}_H(L'_{r_k} \cap B(0, 1), W \cap B(0, 1)) \geq \varepsilon.$$

Let  $\eta > 0$  be some tiny constant. Observe that, by (A.3), for  $k \geq k_0(\eta, M)$  large enough, we have

$$(A.5) \quad \beta_{\mu,2}(0, r_k)^2 \leq \frac{\eta^3}{M}.$$

Indeed, otherwise one could use the simple fact that  $\beta_{\mu,2}(0, r) \lesssim \beta_{\mu,2}(0, 2r)$  to conclude that  $\int_0^1 \beta_{\mu,2}(0, r)^2 dr/r = \infty$ . Moreover, let us remark that for every  $0 < \delta < 1/2$ , if  $k = k(\delta)$  is large enough, then we have  $L'_{r_k} \cap B(0, \delta) \neq \emptyset$ . This can be shown easily using the fact that  $\Theta^n(\mu, x) \geq M^{-1}$ , that  $L_{r_k}$  are minimizers of  $\beta_{\mu,2}(0, r_k)$ , and the fact that  $\beta_{\mu,2}(0, r_k) \rightarrow 0$ . We leave checking the details to the reader.

Now, we use the fact that for  $k$  large enough  $L'_{r_k} \cap B(0, \delta) \neq \emptyset$  and the compactness properties of the Hausdorff distance to conclude that there exists some subsequence (again denoted by  $r_k$ ) such that  $L'_{r_k} \cap \overline{B(0, 1)}$  converges in Hausdorff distance to a compact set of the form  $V \cap \overline{B(0, 1)}$ , where  $V$  is an  $n$ -plane intersecting  $B(0, \delta)$ . Since  $\delta > 0$  can be chosen arbitrarily small, we get that  $V$  passes through 0. Note that, by (A.4),

$$(A.6) \quad \text{dist}_H(V \cap B(0, 1), W \cap B(0, 1)) \geq \varepsilon.$$

Let  $B_{\eta r_k}(V)$  denote the  $\eta r_k$ -neighborhood of  $V$ . We will show now that a large portion of measure  $\nu$  in  $B(0, r_k)$  is concentrated at the intersection of  $B_{\eta r_k}(V)$  and  $B_{\eta r_k}(W)$ .

Since  $V$  passes through 0, for every  $r > 0$ , we have  $A_r^{-1}(V) = V$ . Thus,

$$(A.7) \quad \frac{\text{dist}_H(L_{r_k} \cap B(0, r_k), V \cap B(0, r_k))}{r_k} \xrightarrow{k \rightarrow \infty} 0.$$

Note that, for  $k$  big enough,

$$\begin{aligned} \frac{1}{\nu(B(0, r_k))} \int_{B(0, r_k)} \left(\frac{\text{dist}(y, V)}{r_k}\right)^2 d\nu(y) &\leq \frac{1}{\nu(B(0, r_k))} \int_{B(0, r_k)} \left(\frac{\text{dist}(y, L_{r_k})}{r_k}\right)^2 d\nu(y) \\ &\quad + \left(\frac{\text{dist}_H(L_{r_k} \cap B(0, 2r_k), V \cap B(0, 2r_k))}{r_k}\right)^2 \\ &\leq \frac{r_k^n}{\nu(B(0, r_k))} \beta_{\mu, 2}(0, r_k)^2 + \eta^3 \\ &\leq 2M\beta_{\mu, 2}(0, r_k)^2 + \eta^3 \leq 3\eta^3. \end{aligned}$$

see (A.2), (A.5), (A.7). It follows from Chebyshev’s inequality and the estimate above that

$$\nu(B(0, r_k) \setminus B_{\eta r_k}(V)) \leq \eta^{-2} \int_{B(0, r_k)} \left(\frac{\text{dist}(y, V)}{r_k}\right)^2 d\nu(y) \leq 3\eta\nu(B(0, r)).$$

Hence,

$$\nu(B(0, r_k) \cap B_{\eta r_k}(V)) \geq (1 - 3\eta r_k)\nu(B(0, r_k)).$$

On the other hand, by the definition of the approximate tangent plane  $W$  and (A.2), for any  $0 < \alpha < 1$ , we have

$$\begin{aligned} \nu(K(0, W, \alpha, r_k)) &= \nu(B(0, r_k)) - \nu(K(0, W^\perp, \sqrt{1 - \alpha^2}, r_k)) \\ &\geq \nu(B(0, r_k)) - \frac{\eta}{2M} r_k^n \geq (1 - \eta)\nu(B(0, r_k)), \end{aligned}$$

if  $k$  is large enough (depending on  $\alpha, \eta$  and  $M$ ). Note that  $K(0, W, \alpha, r_k) \subset B_{\alpha r_k}(W) \cap B(0, r_k)$ . Thus, choosing  $\alpha = \eta$ , if we define

$$S = S(k, \eta) = B(0, r_k) \cap B_{\eta r_k}(V) \cap B_{\eta r_k}(W),$$

then by the two previous estimates, we have

$$(A.8) \quad \nu(S) \geq (1 - 4\eta)\nu(B(0, r_k)) \geq \frac{1}{2M} r_k^n,$$

where in the second inequality we used (A.2).

We will show that if  $\eta$  is chosen small enough (depending on  $\varepsilon$ , the constant from (A.6)), then the estimate above leads to a contradiction. Roughly speaking, (A.8) means that a lot of measure is concentrated in the intersection of  $B_{\eta r_k}(V)$  and  $B_{\eta r_k}(W)$ , but since  $V$  and  $W$  are somewhat well-separated by (A.6), this intersection behaves approximately like an  $(n - 1)$ -dimensional set.

Let us start by exploiting (A.6). By the definition of Hausdorff distance and the fact that  $V$  and  $W$  are  $n$ -planes, it follows from basic linear algebra that there exists some  $w \in W^\perp$  with  $|w| = 1$  and  $|\pi_V(w)| \geq \varepsilon$ . Let  $v_1 = \pi_V(w)/|\pi_V(w)|$ , and let  $V_0 \subset V$  be the orthogonal complement of  $\text{span}(v_1)$  in  $V$ .

We define  $T = T(k, \eta)$  to be a tube-like set defined as

$$T = T(k, \eta) = \{z \in \mathbb{R}^d : |z \cdot v_1| \leq 2\eta\varepsilon^{-1}r_k, |\pi_{V_0}(z)| \leq r_k, |\pi_{V^\perp}^\perp(z)| \leq \eta r_k\}.$$

We claim that  $S(k, \eta) \subset T(k, \eta)$ . Indeed, let  $z \in S$ . The estimate  $|\pi_{V_0}(z)| \leq r_k$  is trivial since  $S \subset B(0, r_k)$ . The estimate  $|\pi_{V^\perp}^\perp(z)| \leq \eta r_k$  follows from the fact that  $z \in B_{\eta r_k}(V)$ . Concerning  $|z \cdot v_1|$ , note that since  $z \in B_{\eta r_k}(W)$  and  $w \in W^\perp$ , we have  $|z \cdot w| \leq \eta r_k$ . We can use our choice of  $w$  and  $v_1 = \pi_V(w)/|\pi_V(w)|$  to get

$$\begin{aligned} \eta r_k &\geq |z \cdot w| = |z \cdot \pi_V(w) + z \cdot \pi_{V^\perp}^\perp(w)| \\ &\geq |z \cdot \pi_V(w)| - |z \cdot \pi_{V^\perp}^\perp(w)| = |z \cdot v_1| |\pi_V(w)| - |\pi_{V^\perp}^\perp(z) \cdot \pi_{V^\perp}^\perp(w)| \\ &\geq |z \cdot v_1| \varepsilon - |\pi_{V^\perp}^\perp(z)| |\pi_{V^\perp}^\perp(w)| \geq |z \cdot v_1| \varepsilon - \eta r_k, \end{aligned}$$

where in the last inequality we used again  $z \in B_{\eta r_k}(V)$ . Thus, we have  $|z \cdot v_1| \leq 2\eta\varepsilon^{-1}r_k$ , and the proof of  $S(k, \eta) \subset T(k, \eta)$  is finished.

Choose  $\eta = \gamma\varepsilon$  for some tiny  $\gamma = \gamma(M) > 0$ , and let  $k$  be large enough for (A.8) to hold. It follows from the definition of  $T$  that we can cover  $T$  with a family of balls  $\{B_i\}_{i \in I}$  such that  $r(B_i) = \eta r_k$  and  $\#I \lesssim \varepsilon^{-1}\eta^{-(n-1)}$ . It is well known that (A.2) implies that for all  $y \in \mathbb{R}^d$  and  $r > 0$ , we have  $\nu(B(y, r)) \leq Mr^n$ . In particular, for each  $i \in I$  we have  $\nu(B_i) \leq M(\eta r_k)^n$ . Thus, from (A.8),

$$\begin{aligned} \frac{1}{2M}r_k^n &\leq \nu(S) \leq \nu(T) \leq \sum_{i \in I} \nu(B_i) \\ &\leq \#IM(\eta r_k)^n \lesssim \varepsilon^{-1}\eta^{-(n-1)}M(\eta r_k)^n = \varepsilon^{-1}\eta M r_k^n. \end{aligned}$$

That is,

$$M^{-2} \lesssim \varepsilon^{-1}\eta = \gamma.$$

This is a contradiction for  $\gamma = \gamma(M)$  small enough. Hence, (A.4) is false, and so (A.1) holds for  $\mu$ -a.e.  $x \in E_M$ . Taking  $M \rightarrow \infty$  finishes the proof. ■

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