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Hypercontractivity on the unit circle for ultraspherical measures: linear case

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Abstract. In this paper we extend complex uniform convexity estimates for $\mathbb{C}$ to $\mathbb{R}^n$ and determine best constants. Furthermore, we provide the link to log-Sobolev inequalities and hypercontractivity estimates for ultraspherical measures.

1. Introduction

The starting point of this paper is Bonami's sharp complex convexity estimate (see Chapter III, Theorem 7 of [3])

$$\int_{\mathbb{S}^1} |x + a\zeta| \,\mathrm{dm}(\zeta) \geq \Big(|x|^2 + \frac{1}{2}a^2\Big)^{1/2} \quad \text{for } x \in \mathbb{R}^2,\ a \in [0, \infty), \tag{1.1}$$

where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$ and m denotes the usual Haar measure on $\mathbb{S}^1$, with $\mathrm{m}(\mathbb{S}^1) = 1$. Davis, Garling and Tomczak-Jaegermann, see Proposition 3.1 of [4], presented a proof of (1.1) based on the power series representation of elliptic integrals. We remark that the estimate (1.1) can be seen as a corollary of hypercontractivity on the unit circle for analytic polynomials. Independently, Rothaus [7] and Weissler [8] showed that for any $1 \leq p < q \leq \infty$ and any trigonometric polynomial $f = \sum a_k \zeta^k$, one has that

$$\Big(\int_{\mathbb{S}^1} \Big|\sum a_k r^{|k|} \zeta^k\Big|^q \,\mathrm{dm}(\zeta)\Big)^{1/q} \leq \Big(\int_{\mathbb{S}^1} \Big|\sum a_k \zeta^k\Big|^p \,\mathrm{dm}(\zeta)\Big)^{1/p}$$

holds if and only if $|r| \leq \sqrt{(p-1)/(q-1)}$, $r \in \mathbb{R}$. If f is an analytic polynomial, i.e., $f = \sum_{k\geq 0} a_k \zeta^k$, then using a personal communication by Janson, Weissler (see Corollary 2.1 of [8]) obtains that $\| \sum_{k\geq 0} a_k r^k \zeta^k \|_q \leq \| \sum_{k\geq 0} a_k \zeta^k \|_p$ holds if and only if $|r| \leq \sqrt{p/q}$ for all $0 < q \leq p \leq \infty$. The choice $q = 2$, $a_0 = x, a_1 = 1$ and $a_k = 0$ for all $k \geq 2$ gives

$$\Big(\int_{\mathbb{S}^1} |x + a\zeta|^p \,\mathrm{dm}(\zeta)\Big)^{1/p} \geq \Big(|x|^2 + \frac{p}{2}a^2\Big)^{1/2} \quad \text{for } x \in \mathbb{R}^2,\ a \in [0, \infty), \tag{1.2}$$

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for all $0 < p \leq 2$, and $p/2$ is the best (i.e., largest) real constant satisfying (1.2). Aleksandrov, see Lemma 9.11 of [1], presented an elegant analytic proof of (1.2). The proofs in [8], respectively [1], of (1.2) are complex analytic in nature; they do not seem to work in the *vector-valued* case, i.e., find the largest $C = C(p,n) > 0$ such that

$$\Big(\int_{\mathbb{S}^{n-1}} |x + a\zeta|^p \, \mathrm{d}\sigma\Big)^{1/p} \geq (|x|^2 + Ca^2)^{1/2} \quad \text{for } x \in \mathbb{R}^n,\ a \in [0,\infty), \tag{1.3}$$

where $|\cdot|$ is the n-dimensional Euclidean norm, and m is replaced by σ, the normalized Haar measure on the unit sphere in $\mathbb{R}^n$. Let us also mention that Beckner's hypercontractivity [2] on n-sphere implies the bound $\|1+rH_1(\zeta)\|_{L^q(\mathbb{S}^{n-1},\mathrm{d}\sigma)} \leq \|1+H_1(\zeta)\|_{L^p(\mathbb{S}^{n-1},\mathrm{d}\sigma)}$ for all $r \leq \sqrt{(p-1)/(q-1)}$, where $1 < p \leq q \leq \infty$, and $H_1 \colon \mathbb{S}^{n-1} \to \mathbb{C}$ is any spherical harmonic of degree 1, i.e., $\Delta_{\mathbb{S}^{n-1}} H_1 = -(n-1)H_1$. While Beckner's result pertains to the circle of ideas discussed in the present paper, it does not seem to directly imply our estimate (1.3).

Recently, see [5], we recorded a proof of (1.2), based on Green's identities and subharmonicity estimates, such as

$$\int_{\mathbb{S}^1} |x + a\zeta|^\beta \, \mathrm{dm}(\zeta) \geq \max\{a, |x|\}^\beta, \quad \beta \in \mathbb{R},\ x \in \mathbb{R}^2,\ a \in [0,\infty).$$

In the present paper we obtain the largest C in (1.3) in dimensions $n \geq 3$. The cases $n = 3$ and $n \geq 4$ are treated separately. For $n = 3$, we were able to adjust the argument in [5]. In dimensions four and higher, our proof uses Riesz potential operators on $\mathbb{R}^n$, acting on the surface measure σ.

In Section 3 we exhibit connections between the inequalities (1.3) and advanced techniques based on logarithmic Sobolev inequalities. By change of variables, we reduce the question to the study of hypercontractivity for ultraspherical measures on the unit circle,

$$\mathrm{d}\nu_m(z) = c_m |\sin(\theta)|^m \, \mathrm{d}\theta, \quad z = e^{i\theta} \in \mathbb{S}^1,\ \nu_m(\mathbb{S}^1) = 1,\ m > -1,$$

applied to "linear polynomials" on $\mathbb{S}^1$ given by $f(z) = a + bz$.

For $m = -1$, by definition, we set $\mathrm{d}\nu_{-1}(z) = \frac{1}{2}(\delta_1(z) + \delta_{-1}(z))$. We are interested in real numbers m, p, q, r, with $0 < p \leq q < \infty$ and $r \in \mathbb{R}$, such that

$$\|1 + rbz\|_{L^q(\mathbb{S}^1,\mathrm{d}\nu_m)} \leq \|1 + bz\|_{L^p(\mathbb{S}^1,\mathrm{d}\nu_m)} \quad \text{for all } b \in \mathbb{R}. \tag{1.4}$$

Taking $b \to 0$ in (1.4), one easily obtains a necessary condition on the 4-tuple (m,p,q,r), namely,

$$|r| \leq \sqrt{\frac{p+m}{q+m}}. \tag{1.5}$$

If $m = -1$, then we are in the setting of a celebrated theorem of Bonami [3], also known as Bonami–Beckner–Gross "two-point inequality", which says that (1.5) implies (1.4) when $(m,p,q,r) = (-1,p,q,r)$ and $q \geq p > 1$. A theorem of Weissler [8] shows that (1.5) implies (1.4) when $(m,p,q,r) = (0,p,q,r)$ and $q \geq p > 0$. Inequality (1.3) with the largest C, the main theorem of our paper, in an equivalent way can be restated as (1.5) implies (1.4) when $(m, p, q, r) = (n-2, p, 2, r)$, with $n \geq 2$, $n \in \mathbb{N}$ and $2 \geq p > 0$. In Section 3, using log-Sobolev inequalities for ultraspherical measures, we show that (1.5) implies (1.4) for 4-tuples (m, p, q, r), with $q \geq p \geq 6$ and all $m \geq -1$. Despite of partial progresses, the description of all 4-tuples (m, p, q, r) for which (1.4) holds true remains an open question.

Perhaps an advantage of the reformulation (1.4) over the vector-valued inequality (1.3) is that the estimate of the type (1.4) can be asked for semigroups such that the analytic polynomials P_k, $\deg(P_k) = k$, orthogonal with respect to the measure $d\nu_m$, are eigenfunctions of the generator of the semigroup. Namely, given a sequence $0 = \lambda_0 < \lambda_1 \leq \cdots$ (eigenvalues), $0 < p \leq q < \infty$, find the largest $C > 0$ such that for all $r \in \mathbb{R}$, $|r| \leq C$, we have

$$\Big\| \sum_{k \geq 0} r^{\lambda_k} a_k P_k(z) \Big\|_{L^q(\mathbb{S}^1, d\nu_m)} \leq \Big\| \sum_{k \geq 0} a_k P_k(z) \Big\|_{L^p(\mathbb{S}^1, d\nu_m)}, \tag{1.6}$$

for all $a_k \in \mathbb{C}$, $k \geq 0$. Here we assume that $a_j = 0$ starts for some large $j \geq N$ in order to avoid convergence issues of the infinity series. Our main results only cover the linear case $a_0, a_1 \in \mathbb{R}$, and $a_k = 0$ for all $k \geq 2$, and they do not cover hypercontractivity in such generality as (1.6).

For the reader's convenience, we state explicitly the higher dimensional results that we obtain in this paper using both approaches. In Section 2 we prove

$$\|x + ray\|_{L^q(\mathbb{S}^{n-1}, d\sigma(y))} \leq \|x + ay\|_{L^p(\mathbb{S}^{n-1}, d\sigma(y))} \quad \text{for all } x \in \mathbb{R}^n,\ a \in \mathbb{R}, \tag{1.7}$$

if $q = 2$, $0 < p \leq 2$, $n \geq 2$, $|r| \leq \sqrt{(p+n-2)/n}$, $r \in \mathbb{R}$. In Section 3, in particular, we verify inequality (1.7) if $6 \leq p \leq q$, $n \geq 2$, $|r| \leq \sqrt{(p+n-2)/(q+n-2)}$, $r \in \mathbb{R}$.

2. Main theorem

In this section, $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, σ denotes the normalized Haar measure on $\mathbb{S}^{n-1}$, and $\mathrm{B}_r^n(x)$ denotes the open ball in $\mathbb{R}^n$ with radius $r > 0$, centered at $x \in \mathbb{R}^n$; and we set for convenience $\mathrm{B}_r^n = \mathrm{B}_r^n(0)$. We remark that the notations used in Section 3 will be different from the ones in Section 2.

Theorem 2.1. *Let $n \in \mathbb{N}$, with $n \geq 2$. Let $p \in (0, 2]$ and $\lambda \leq (n+p-2)/n$. Then*

$$\int_{\mathbb{S}^{n-1}} |x - az|^p \, d\sigma(z) \geq (|x|^2 + \lambda a^2)^{p/2} \quad \textit{for } x \in \mathbb{R}^n,\ a \in [0, \infty), \tag{2.1}$$

and $(n+p-2)/n$ is the best (i.e., largest) constant satisfying (2.1).

We start with the elementary observation that $(n+p-2)/n$ is the best (i.e., largest) constant satisfying (2.1). For $x \in \mathbb{R}^2$, with $|x| = 1$, and $a, \lambda \in \mathbb{R}^+$, define

$$I(a) = \int_{\mathbb{S}^{n-1}} |x - az|^p \, d\sigma(z) \quad \text{and} \quad g(a) = (1 + \lambda a^2)^{p/2}. \tag{2.2}$$

Assuming that (2.1) holds true, for $\lambda > 0$, we have

$$I(a) \geq g(a) \quad \text{for } a \geq 0. \tag{2.3}$$

We now show that (2.3) implies that $\lambda \leq (n+p-2)/n$. Clearly, we have that $I(0) = 1$, $g(0) = 1$, $g'(0) = 0$ and $g''(0) = p/\lambda$. Next, since

$$\partial_a |x - az|^p = p|x - az|^{p-2} z \cdot (az - x),$$

we have $I'(0) = 0$. Hence, (2.3) implies that $I''(0) \geq g''(0)$. Calculating further

$$\partial_a^2 |x - az|^p = p(p-2)|x - az|^{p-4}(z \cdot (az - x))^2 + p|x - az|^{p-2}|z|^2,$$

and invoking the integral identity

$$\int_{\mathbb{S}^{n-1}} |(x \cdot z)|^2 \, \mathrm{d}\sigma(z) = \frac{1}{n}$$

gives $I''(0) = p(p-2)/n + p$. Thus, $I''(0) \geq g''(0)$, implies that $\lambda \leq (n+p-2)/n$.

Before turning to the proof of Theorem 2.1, we determine the parameters n and q for which $x \mapsto |x|^q$ is a subharmonic mapping on $\mathbb{R}^n$, and draw consequences (analogous to Jensen's formula in complex analysis).

Lemma 2.1. *Let $n \in \mathbb{N}$ and $q \in \mathbb{R}$. The function $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $x \mapsto |x|^q$, is subharmonic if and only if $q \geq \max\{0, 2-n\}$ or $q \leq \min\{0, 2-n\}$, and then*

$$\int_{\mathbb{S}^{n-1}} |x - az|^q \, \mathrm{d}\sigma(z) \geq \max\{a, |x|\}^q \quad \textit{for } a \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{2.4}$$

Proof. For $i \in \{1, \ldots, n\}$, we have

$$\partial_i f(x) = q x_i |x|^{q-1}, \quad \partial_i^2 f(x) = q|x|^{q-2} + q(q-2)x_i^2 |x|^{q-4},$$

and therefore

$$\Delta f(x) = q(n+q-2)|x|^{q-2}. \tag{2.5}$$

Clearly the sign of the factor $q(n+q-2)$ determines if f is subharmonic or not.

We next turn to verifying that $q(n+q-2) \geq 0$ implies (2.4). If $a < |x|$, the mean value property of subharmonic functions directly yields

$$\int_{\mathbb{S}^{n-1}} |x - az|^q \, \mathrm{d}\sigma(z) \geq |x|^q.$$

To treat the case $a > |x|$, we define $H_a\colon \mathrm{B}_a^n(0) \to \mathbb{R}$ by

$$H_a(x) := \int_{\mathbb{S}^{n-1}} |x - az|^q \, \mathrm{d}\sigma(z).$$

and notice that H_a is subharmonic and rotational invariant, i.e., there exists a function $h_a\colon [0, a) \to \mathbb{R}$ such that

$$H_a(x) = h_a(|x|) \quad \text{for } x \in [0, a).$$

Using subharmonicity and rotational invariance, together with the representation of the Laplace operator in n-dimensional spherical coordinates, we obtain

$$0 \leq \Delta H_a(x) = |x|^{1-n} \partial_r (r \mapsto r^{n-1} \partial h_a(r))(|x|) \quad \text{for } |x| \in (0, a).$$

This yields

$$r^{n-1} \partial h_a(r) \geq 0 \quad \text{for } r \in [0, a)$$

and, consequently,

$$h_a(r) \geq h_a(0) = a^q \quad \text{for } r \in [0, a).$$

Hence, for $a > |x|$, we have $H_a(x) = h_a(|x|) \geq a^q$, and hence

$$\int_{\mathbb{S}^{n-1}} |x - az|^q \, \mathrm{d}\sigma(z) \geq a^q. \qquad \blacksquare$$

Proof. We now prove that (2.1) holds true for $\lambda := (n + p - 2)/n$. Since the case $n = 2$ is already known, we consider $n \geq 3$. An application of the divergence theorem yields that

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |x - az|^p \, \mathrm{d}\sigma(z) \\ = 1 + a^2 p(p + n - 2) \int_0^1 \int_0^t \left(\frac{r}{t}\right)^{n-1} \int_{\mathbb{S}^{n-1}} |x - az|^{p-2} \, \mathrm{d}\sigma(z) \, \mathrm{d}r \, \mathrm{d}t. \end{aligned} \tag{2.6}$$

Indeed, put $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(y) := |x - ay|^p$, and define a vector field X by $X(y) := \nabla f(ty)$. Then $\operatorname{div} X(y) = t \Delta f(ty)$ and, by the divergence theorem,

$$\begin{aligned} \partial_t \int_{\mathbb{S}^{n-1}} f(tz) \, \mathrm{d}\sigma(z) &= \int_{\mathbb{S}^{n-1}} X(z) \cdot z \, \mathrm{d}\sigma(z) \\ &= \frac{1}{n \mathrm{Vol}_n(\mathrm{B}_2^n)} t \int_{\mathrm{B}_2^n} \Delta f(ty) \, \mathrm{d}y \\ &= \frac{1}{n \mathrm{Vol}_n(\mathrm{B}_2^n)} \int_{t\mathrm{B}_2^n} \Delta f(y) \, \mathrm{d}y \\ &= \frac{1}{t^{n-1}} \int_0^t \int_{\mathbb{S}^{n-1}} r^{n-1} \Delta f(rz) \, \mathrm{d}r \, \mathrm{d}\sigma(z). \end{aligned} \tag{2.7}$$

Integrating the identity (2.7) from $t = 0$ to $t = 1$ and invoking (2.5) gives (2.6). Define

$$H(a, x) := \int_{\mathbb{S}^{n-1}} |x - az|^{p-2} \, \mathrm{d}\sigma(z).$$

Then $H(a, \cdot)$ is rotational invariant, i.e., there exists a function $h\colon [0, \infty)^2 \to \mathbb{R}$ such that $H(a, x) = h(a, |x|)$. By (2.6) and re-scaling, we have

$$\int_{\mathbb{S}^{n-1}} |x - az|^p \, \mathrm{d}\sigma(z) = 1 + p(p + n - 2) \int_0^a \int_0^t t^{1-n} u^{n-1} h(u, 1) \, \mathrm{d}u \, \mathrm{d}t. \tag{2.8}$$

The proof of Theorem 2.1 will be obtained by proving suitable lower estimates for the volume integral appearing on the right-hand side of (2.8). We will distinguish the case where $x \mapsto |x|^{p-2}$ is sub-harmonic (corresponding to $n = 3$ and $p \leq 1$), and the case where sub-harmonicity fails (corresponding to $n \geq 4$ or $p > 1$).

2.1. Case $n = 3$ and $p \le 1$

First note that

$$1 + p(p+1)\int_0^a \int_0^t t^{-2}u^2 \max\{1,u\}^{p-2}\,\mathrm{d}u\,\mathrm{d}t = \begin{cases} 1 + \frac{p(p+1)}{6}a^2, & a \in [0,1], \\ a^p + \frac{p(2-p)}{3a} + \frac{(p-1)p}{2}, & a > 1. \end{cases} \tag{2.9}$$

Indeed, (2.9) follows from a direct calculation separating the cases $a \le 1$ and $a > 1$.

For $a \le 1$, we calculate

$$\int_0^a \int_0^t t^{-2}u^2 \max\{1,u\}^{p-2}\,\mathrm{d}u\,\mathrm{d}t = \int_0^a t^{-2}\int_0^t u^2\,\mathrm{d}u\,\mathrm{d}t = \frac{a^2}{6},$$

which yields (2.9) for $a \le 1$.

For $a > 1$, we calculate

$$\begin{aligned}&\int_0^a \int_0^t t^{-2}u^2 \max\{1,u\}^{p-2}\,\mathrm{d}u\,\mathrm{d}t \\ &= \int_0^1 t^{-2}\int_0^t u^2\,\mathrm{d}u\,\mathrm{d}t + \int_1^a t^{-2}\int_0^1 u^2\,\mathrm{d}u\,\mathrm{d}t + \int_1^a t^{-2}\int_1^t u^p\,\mathrm{d}u\,\mathrm{d}t \\ &= \frac{1}{6} - \frac{a^{-1}-1}{3} + \frac{a^p - 1}{(p+1)p} + \frac{a^{-1}-1}{(1+p)},\end{aligned}$$

which yields (2.9) for $a \ge 1$, by arithmetic.

Since $x \mapsto |x|^{p-2}$ is subharmonic, for $n = 3$ and $p \in (0,1]$, Lemma 2.1 yields $h(a,x) \ge \max\{1,a\}^{p-2}$. Applying this estimate to (2.8) and invoking (2.9), we obtain

$$\int_{\mathbb{S}^2} |x - az|^p\,\mathrm{d}\sigma(z) \ge \begin{cases} 1 + \frac{p(p+1)}{6}a^2, & a \le 1, \\ a^p + \frac{p(2-p)}{3a} - \frac{p(1-p)}{2}, & a > 1. \end{cases} \tag{2.10}$$

Defining

$$g(a) := \begin{cases} 1 + \frac{p(p+1)}{6}a^2, & a \in [0,1], \\ a^p + \frac{p(2-p)}{3a} - \frac{p(1-p)}{2}, & a^2 \in (1, \frac{3}{2-p}), \\ a^p, & a^2 \ge \frac{3}{2-p}, \end{cases}$$

it suffices to show

$$\int_{\mathbb{S}^2} |x - az|^p\,\mathrm{d}\sigma(z) \ge g(a) \ge \Big(1 + \frac{p+1}{3}a^2\Big)^{p/2}. \tag{2.11}$$

We first consider $a^2 \ge 3/(2-p)$. In this case, we have

$$\int_{\mathbb{S}^2} |x - az|^p\,\mathrm{d}\sigma(z) \ge a^p \ge \Big(1 + \frac{p+1}{3}a^2\Big)^{p/2}. \tag{2.12}$$

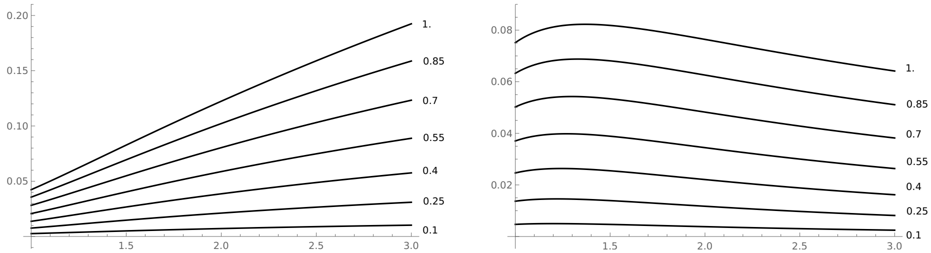

Figure 1. Plots of the functions ϕ and ϕ' for $p \in \{0.1, \dots, 0.85, 1\}$.

Indeed, by Lemma 2.1, $x \mapsto |x|^p$ is subharmonic. Taking into account that $|x| = 1$ and $a^2 > 3/(2-p)$, Lemma 2.1 yields

$$\int_{\mathbb{S}^{n-1}} |x - az|^p \, \mathrm{d}\sigma(z) \geq \max\{a, 1\}^p \geq a^p.$$

To obtain the second estimate in (2.12) note that $a^2 \geq 3/(2-p)$ holds if and only if $a^2 \geq 1 + \frac{p+1}{n} a^2$.

We now turn to the case $a^2 < 3/(2-p)$. By (2.10), in this case, it remains to show the second inequality of (2.12). If moreover $a \in [0, 1]$, this is just Bernoulli's inequality. If finally $a^2 \in (1, 3/(2-p))$, we proceed as follows: For $p \in (0, 1]$, we define

$$\phi(t) := t^{p/2} + \frac{p(2-p)}{3\sqrt{t}} - \frac{p(1-p)}{2} - \Big(1 + \frac{p+1}{3} t\Big)^{p/2}.$$

We show that $\phi(t) \geq 0$ for $t \in (1, 3/(2-p))$. Indeed, since $t < 3/(2-p)$ holds if and only if $t < 1 + \frac{p+1}{3} t$, we get

$$\begin{aligned}
\phi'(t) &= \frac{p}{2} t^{(p-2)/2} - \frac{p(2-p)}{6} t^{-3/2} - \frac{p(p+1)}{6}\Big(1 + \frac{p+1}{3} t\Big)^{(p-2)/2} \\
&\geq \Big(\frac{p}{2} - \frac{p(p+1)}{6}\Big) t^{(p-2)/2} - \frac{p(2-p)}{6} t^{-3/2} \\
&= \frac{p(2-p)}{6} (t^{(p-2)/2} - t^{-3/2}) \geq 0.
\end{aligned}$$

Due to $\phi(1) \geq 0$, this implies $\phi(t) \geq 0$ for $t \in (1, 3/(2-p))$. Summing up for $p \in (0, 1]$ and $t = a^2 \in (1, 3/(2-p))$, we have

$$a^p + \frac{p(2-p)}{3a} - \frac{p(1-p)}{2} \geq \Big(1 + \frac{p+1}{3} a^2\Big)^{p/2}.$$

2.2. Case $n > 3$ or $p > 1$

Since we cannot apply Lemma 2.1, we need another lower bound for $h(a, 1)$. In order to accomplish that, we use the formula

$$r^{-\zeta} = \frac{1}{\Gamma(\zeta/2)} \int_0^\infty t^{-\zeta/2-1} \exp\Big(-\frac{r^2}{t}\Big) \mathrm{d}t = \frac{1}{\Gamma(\zeta/2)} \int_0^\infty t^{\zeta/2-1} \exp(-r^2 t) \, \mathrm{d}t, \tag{2.13}$$

which holds for all $r > 0$ and $\Re\zeta > 0$. Putting $\zeta := 2 - p$, i.e., $p = 2 - \zeta$, we get

$$\begin{aligned} H(a,x) &= \int_{\mathbb{S}^{n-1}} |x - az|^{p-2}\,\mathrm{d}\sigma(z) \\ &= \frac{1}{\Gamma(\zeta/2)} \int_0^\infty \int_{\mathbb{S}^{n-1}} t^{-\zeta/2-1} \exp\Big(-\frac{|x-az|^2}{t}\Big)\,\mathrm{d}\sigma(z)\,\mathrm{d}t \\ &= \frac{1}{\Gamma(\zeta/2)} \int_0^\infty \Big(\int_{\mathbb{S}^{n-1}} \exp\Big(\frac{2ax\cdot z}{t}\Big)\,\mathrm{d}\sigma(z)\Big) t^{-\zeta/2-1} \exp\Big(-\frac{1+a^2}{t}\Big)\,\mathrm{d}t. \end{aligned}$$

We are thus left with finding a good lower bound for

$$\int_{\mathbb{S}^{n-1}} \exp(\lambda x\cdot z)\,\mathrm{d}\sigma(z) = \int_{\mathbb{S}^{n-1}} \cosh(\lambda |x\cdot z|)\,\mathrm{d}\sigma(z),$$

where $\lambda > 0$ and $|x| = 1$. The obvious bound is 1, which eventually turns out not to be sufficient for $p < 4/(n+2)$, so we take the second Taylor approximation, that is, $\cosh s \geq 1 + s^2/2$. By (2.13) and the functional equation of the gamma function, we conclude

$$\begin{aligned} h(a,1) &\geq (1+a^2)^{-\zeta/2} + \frac{2a^2}{n\Gamma(\zeta/2)} \int_0^\infty t^{-\zeta/2-3} \exp\Big(-\frac{1+a^2}{t}\Big)\,\mathrm{d}t \\ &= (1+a^2)^{-\zeta/2} + \frac{2a^2\Gamma(\zeta/2+2)}{n\Gamma(\zeta/2)}(1+a^2)^{-\zeta/2-2} \\ &= (1+a^2)^{p/2-1}\Big(1 + \frac{(4-p)(2-p)}{2n}\,\frac{a^2}{(1+a^2)^2}\Big) =: \psi(a). \end{aligned}$$

According to (2.8), it remains to prove that

$$1 + p(p+n-2)\int_0^a \int_0^t t^{1-n}u^{n-1}\psi(u)\,\mathrm{d}u\,\mathrm{d}t \geq \Big(1 + \frac{p+n-2}{n}\,a^2\Big)^{p/2}.$$

We set $c := (n+p-2)/n$ and show

$$F(a) := 1 + p(p+n-2)\int_0^a \int_0^t t^{1-n}u^{n-1}\psi(u)\,\mathrm{d}u\,\mathrm{d}t - (1+ca^2)^{p/2} \geq 0.$$

Since $F(0) = 0$, this follows from $F' \geq 0$, i.e.,

$$n\int_0^a u^{n-1}\psi(u)\,\mathrm{d}u - a^n(1+ca^2)^{p/2-1} \geq 0,$$

which in turn follows from

$$na^{n-1}\psi(a) - \partial_a(a^n(1+ca^2)^{p/2-1}) \geq 0.$$

Rearranging terms this amounts to

$$1 + \frac{(4-p)(2-p)a^2}{2n(1+a^2)^2} - \Big(\frac{1+a^2}{1+ca^2}\Big)^{1-p/2} + \frac{a^2c(2-p)}{n(1+a^2)}\Big(\frac{1+a^2}{1+ca^2}\Big)^{2-p/2} \geq 0.$$

Put $x := (1+ca^2)/(1+a^2)$; then $x \in (c,1)$ and

$$a^2 = \frac{1-x}{x-c}, \quad 1+a^2 = \frac{1-c}{x-c} \quad \text{and} \quad \frac{a^2}{1+a^2} = \frac{1-x}{1-c}.$$

Thus, we have to show that

$$1 + \frac{(4-p)(2-p)(1-x)(x-c)}{2n(1-c)^2} - x^{p/2-1} + \frac{c(2-p)(1-x)}{n(1-c)} x^{p/2-2} \geq 0,$$

i.e.,

$$x^{2-p/2} \geq \frac{1 + \frac{n(1-c)+c(2-p)}{n(1-c)}(x-1)}{1 + \frac{(4-p)(2-p)(1-x)(x-c)}{2n(1-c)^2}} = \frac{1-c-(1-c^2)(1-x)}{1-c+(2-\frac{p}{2})(1-x)(x-c)}.$$

Considering $n \geq 4$ or $p > 1$, we have $c = 1-(2-p)/n \geq 1/2$. So, eventually it suffices to prove that given $q := 2-p/2 \in [0,1]$, then for all $(x,y) \in [0,1]^2$ satisfying $x \geq y \geq 1/2$, we have

$$x^q(1-y+q(1-x)(x-y)) \geq 1-y-(1-y^2)(1-x). \tag{2.14}$$

The function $q \mapsto x^q(1-y+q(1-x)(x-y))$ is decreasing. Indeed, the derivative of the logarithm with respect to q is

$$\frac{(1-x)(x-y)}{1-y+q(1-x)(x-y)} - \log\frac{1}{x} \leq \frac{(1-x)(x-y)}{1-y} - \log\frac{1}{x} \leq 1-x-\log\frac{1}{x} \leq 0,$$

where we simply used the fact $y \leq x \leq 1$. Thus, we only have to prove, that, assuming $1/2 \leq y \leq x \leq 1$, we have

$$x^2\big(1-y+2(1-x)(x-y)\big) - 1 + y + (1-y^2)(1-x) \geq 0.$$

The left-hand side is a polynomial in x of order 4, which factorizes to

$$(1-x)(x-y)(2x^2+y-1).$$

Due to the conditions on x and y, this is obviously non-negative. ∎

However, the polynomial is negative for $y \leq x < 1/2$, and thus inequality (2.14) does not hold for small values of p and $n \in \{2,3\}$. Hence, the above argument does not apply to dimensions two and three!

3. Hypercontractivity for ultraspherical measures on the unit circle

Here we place the estimates of Theorem 2.1 in a wider framework, provided by logarithmic Sobolev inequalities and hypercontractivity. To this end, we first rewrite it as follows: For $0 < p \leq 2$, $n \geq 2$, $|r| \leq \sqrt{(p+n-2)/n}$, $r \in \mathbb{R}$, we have

$$\|x + ray\|_{L^2(\mathbb{S}^{n-1}, \mathrm{d}\sigma(y))} \leq \|x + ay\|_{L^p(\mathbb{S}^{n-1}, \mathrm{d}\sigma(y))} \quad \text{for all } x \in \mathbb{R}^n,\ a \in \mathbb{R}. \tag{3.1}$$

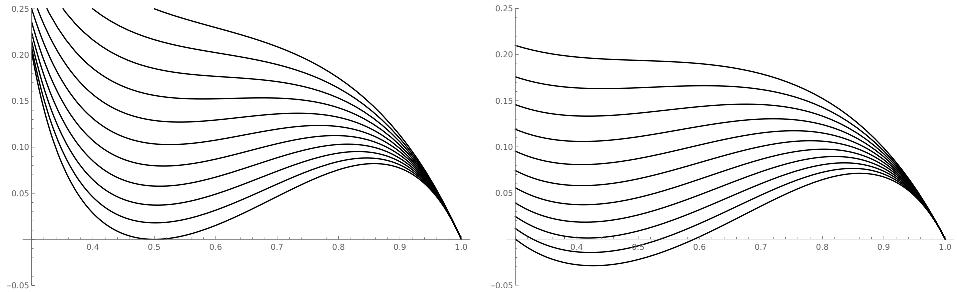

Figure 2. Plots of $x \mapsto x^q - \frac{1-y-(1-y^2)(1-x)}{1-y+q(1-x)(x-y)}$ for $y \in \{0.5, 0.3\}$ and $q \in \{1, 1.1, \ldots, 2\}$.

In this section we consider (3.1) for the range of parameters $n \geq 2$, and $0 < p \leq q < \infty$. We are interested to find the largest possible constant $C = C(n, p, q) > 0$ such that for all $r \in \mathbb{R}$, $|r| \leq C(p, q, r)$, we have

$$\|x + ary\|_{L^q(\mathbb{S}^{n-1}, \mathrm{d}\sigma(y))} \leq \|x + ay\|_{L^p(\mathbb{S}^{n-1}, \mathrm{d}\sigma(y))} \quad \text{for all } x \in \mathbb{R}^n,\ a \in \mathbb{R}. \tag{3.2}$$

First we prove a theorem on the unit circle for ultraspherical measures

$$\mathrm{d}\nu_m(z) = c_m |\sin(\theta)|^m \, \mathrm{d}\theta \quad \text{for all real } m > -1,$$

where $z = e^{i\theta} \in \mathbb{S}^1$, and the scalar $c_m := \frac{\Gamma(m/2+1)}{2\Gamma(1/2)\Gamma(m/2+1/2)}$ is chosen in such a way that $\nu_m(\mathbb{S}^1) = 1$. For $m = -1$, we set $\mathrm{d}\nu_{-1}(z) = \frac{1}{2}(\delta_{-1}(z) + \delta_1(z))$.

Theorem 3.1. *Let $m \geq -1$ and $6 \leq p \leq q$. We have*

$$\|1 + rbz\|_{L^q(\mathbb{S}^1, \mathrm{d}\nu_m)} \leq \|1 + bz\|_{L^p(\mathbb{S}^1, \mathrm{d}\nu_m)} \quad \textit{for all } b \in \mathbb{R} \tag{3.3}$$

if and only if $|r| \leq \sqrt{(p+m)/(q+m)}$.

Let us show that the theorem implies the following corollary.

Corollary 3.1. *For any $6 \leq p \leq q$, all integers $n \geq 2$, and any real $|r| \leq \sqrt{\frac{p+n-2}{q+n-2}}$, inequality (3.2) holds true.*

Indeed, without loss of generality, we can assume $|x| = 1$ in (3.2). Next, for $y = (y_1, \ldots, y_n) \in \mathbb{S}^{n-1}$ and $\lambda = (n-2)/2$, we have

$$\begin{aligned}
\|x + ay\|^p_{L^p(\mathbb{S}^{n-1}, \mathrm{d}\sigma(y))} &= \int_{\mathbb{S}^{n-1}} (1 + 2a\langle x, y\rangle + a^2)^{p/2} \, \mathrm{d}\sigma(y) \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)} \int_{-1}^{1} (1 + 2at + a^2)^{p/2} (1-t^2)^{\lambda-(1/2)} \, \mathrm{d}t \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)} \int_0^{\pi} (1 + 2a\cos(\theta) + a^2)^{p/2} \sin^{2\lambda}(\theta) \, \mathrm{d}\theta \quad (t = \cos(\theta)) \\
&= \int_{\mathbb{S}^1} |1 + az|^p \, \mathrm{d}\nu_{2\lambda}(z) = \|1 + az\|^p_{L^p(\mathbb{S}^1, \mathrm{d}\nu_{n-2})}.
\end{aligned}$$

Similarly, we have $\|x+ray\|_{L^q(\mathbb{S}^{n-1},\mathrm{d}\sigma(y))}=\|1+raz\|_{L^q(\mathbb{S}^1,\mathrm{d}\nu_{n-2})}$. Thus, the inequalities (3.2) and (3.3) are the same with $m=n-2$.

Next we prove Theorem 3.1.

Proof. As the measure $\mathrm{d}\nu_{-1}(z)=\frac{1}{2}(\delta_{-1}(z)+\delta_1(z))$ is the weak* limit of the measures $\mathrm{d}\nu_m(z)$ when $m\to-1$, $m>-1$, without loss of generality, we can assume that $m>-1$ in the theorem.

First we show that the assumption $|r|\le\sqrt{(p+m)/(q+m)}$ is necessary for the hypercontractivity (3.3). Indeed, notice that

$$\int_{\mathbb{S}^1}(\Re(z))^2\,\mathrm{d}\nu_m(z)=c_m\int_0^{2\pi}\cos^2(\theta)|\sin(\theta)|^m\mathrm{d}\theta=1-\frac{c_m}{c_{m+2}}=1-\frac{m+1}{m+2}=\frac{1}{m+2}.$$

Therefore,

$$\begin{aligned}\|1+bz\|_{L^p(\mathbb{S}^1,\mathrm{d}\nu_m)}&=\Big(\int_{\mathbb{S}^1}|1+bz|^p\,\mathrm{d}\nu_m(z)\Big)^{1/p}\\&=\Big(\int_{\mathbb{S}^1}(1+2b\Re(z)+b^2)^{p/2}\,\mathrm{d}\nu_m(z)\Big)^{1/p}\\&=\Big(\int_{\mathbb{S}^1}1+\frac{p}{2}(2b\Re(z)+b^2)+\frac{p}{4}\Big(\frac{p}{2}-1\Big)4b^2(\Re(z))^2+o(b^2)\,\mathrm{d}\nu_m(z)\Big)^{1/p}\\&=\Big(1+\frac{p}{2}b^2+\frac{p(p-2)}{2}b^2\int(\Re(z))^2\,\mathrm{d}\nu_m\Big)^{1/p}\\&=1+\frac{b^2}{2}+\frac{p-2}{2}b^2\frac{1}{m+2}+o(b^2)\\&=1+\frac{b^2}{2}\cdot\frac{m+p}{m+2}+o(b^2).\end{aligned}$$

So the inequality $\|1+rbz\|_{L^q(\mathbb{S}^1,\mathrm{d}\mu_m)}\le\|1+bz\|_{L^p(\mathbb{S}^1,\mathrm{d}\mu_m)}$ implies $r^2\frac{m+q}{m+2}\le\frac{m+p}{m+2}$. Since $p,q>-m$, we obtain $|r|\le\sqrt{(m+p)/(m+q)}$.

Next we show that the necessary condition $|r|\le\sqrt{(p+m)/(q+m)}$ is also sufficient for (3.3). Since $q\ge1$ and $\mathrm{d}\nu_m(z)=\mathrm{d}\nu_m(-z)$, the map $r\mapsto\|1+rbz\|_{L^q(\mathbb{S}^1,\mathrm{d}\nu_m)}$ is even and convex on $\mathbb{R}$, and hence it is nondecreasing on $[0,\infty)$. Thus, it suffices to prove (3.3) in the case when $r=\sqrt{(p+m)/(q+m)}$. Let $m=2\lambda$. After rescaling b as $b\mapsto b/\sqrt{p+m}$, we can rewrite (3.3) as follows:

$$\begin{aligned}&\Big(\int_{-1}^{-1}\Big(1+\frac{2bt}{\sqrt{q+2\lambda}}+\frac{b^2}{q+2\lambda}\Big)^{q/2}\mathrm{d}\mu_\lambda(t)\Big)^{1/q}\\&\qquad\le\Big(\int_{-1}^{-1}\Big(1+\frac{2bt}{\sqrt{p+2\lambda}}+\frac{b^2}{p+2\lambda}\Big)^{p/2}\mathrm{d}\mu_\lambda(t)\Big)^{1/p},\end{aligned}\tag{3.4}$$

where $\mathrm{d}\mu_\lambda(t)=2c_{2\lambda}(1-t^2)^{\lambda-(1/2)}\mathrm{d}t$ is a probability measure on $[-1,1]$. Rescaling b as $b\mapsto b/\sqrt{2}$, we see that inequality (3.4) simply means that the map

$$s\mapsto\Big(\int_{-1}^{1}\Big(1+\frac{2bt}{\sqrt{s+\lambda}}+\frac{b^2}{s+\lambda}\Big)^s\mathrm{d}\mu_\lambda(t)\Big)^{1/s}$$

is nonincreasing on $(3,\infty)$ (here $s=p/2$). If we differentiate in s, then after a certain calculation we see that it suffices to show the following log-Sobolev inequality: Put $f(t)=1+2bt/\sqrt{s+\lambda}+b^2/(s+\lambda)$, then

$$\begin{aligned}\int f^s \ln f^s \, \mathrm{d}\mu_\lambda - \int f^s \, \mathrm{d}\mu_\lambda \ln \int f^s \, \mathrm{d}\mu_\lambda \\ \leq -s^2 \int f^{s-1} \frac{d}{ds} f \, \mathrm{d}\mu_\lambda = s^2 \int f^{s-1} \big(bt(s+\lambda)^{-3/2} + b^2(s+\lambda)^{-2}\big) \, \mathrm{d}\mu_\lambda.\end{aligned}$$

Therefore, if we let $b(s+\lambda)^{-1/2}=\tilde{b}$ and $g(t)=1+2\tilde{b}t+\tilde{b}^2$, then our log-Sobolev inequality rewrites as follows:

$$\int g^s \ln g^s \, \mathrm{d}\mu_\lambda - \int g^s \, \mathrm{d}\mu_\lambda \ln \int g^s \, \mathrm{d}\mu_\lambda \leq \frac{s^2}{s+\lambda} \int g^{s-1} (\tilde{b}t + \tilde{b}^2) \, \mathrm{d}\mu_\lambda. \tag{3.5}$$

The log-Sobolev inequality of Mueller–Weissler, see p. 277 of [6], for $\mathrm{d}\mu_\lambda$ states that

$$\begin{aligned}\int g^s \ln g^s \, \mathrm{d}\mu_\lambda - \int g^s \, \mathrm{d}\mu_\lambda \ln \int g^s \, \mathrm{d}\mu_\lambda \\ \leq \frac{s^2}{2(2\lambda+1)} \frac{2\lambda+1}{2(\lambda+1)} \int (g')^2 g^{s-2} \, \mathrm{d}\mu_{\lambda+1} = \frac{s^2}{4(\lambda+1)} \int (g')^2 g^{s-2} \, \mathrm{d}\mu_{\lambda+1}.\end{aligned} \tag{3.6}$$

Thus, we need to show that

$$\int (g')^2 g^{s-2} \, \mathrm{d}\mu_{\lambda+1} \leq \frac{4(\lambda+1)}{s+\lambda} \int g^{s-1} (\tilde{b}t + \tilde{b}^2) \, \mathrm{d}\mu_\lambda.$$

After an integration by parts, we can rewrite the left-hand side of the last inequality as

$$\frac{4(\lambda+1)}{s-1} \int g^{s-1} t \tilde{b} \, \mathrm{d}\mu_\lambda$$

(here we used the fact that $\frac{c_{2(\lambda+1)}}{c_\lambda} = \frac{\lambda+1}{\lambda+1/2}$). Hence, to prove (3.3) it suffices to show that

$$\frac{1}{s-1} \int g^{s-1} t \, \mathrm{d}\mu_\lambda \leq \frac{1}{s+\lambda} \int g^{s-1} (t+\tilde{b}) \, \mathrm{d}\mu_\lambda. \tag{3.7}$$

We can rewrite (3.7) as

$$\int g^{s-1} \, \mathrm{d}\mu_\lambda \geq \int \frac{t(\lambda+1)}{b(s-1)} g^{s-1} \, \mathrm{d}\mu_\lambda.$$

Integrating the right-hand side by parts, we see that it is enough to show

$$\int (1+2at+a^2)^{s-1} \, \mathrm{d}\mu_\lambda(t) \geq \int (1+2at+a^2)^{s-2} \, \mathrm{d}\mu_{\lambda+1}(t)$$

for all $a=\tilde{b}>0$. We claim that it suffices to consider the case when $a\in(0,1)$. Indeed, otherwise, we can write

$$\begin{aligned}\int (1+2at+a^2)^{s-1}\,\mathrm{d}\mu_\lambda(t) &= a^{2(s-1)}\int (a^{-2}+2a^{-1}t+1)^{s-1}\,\mathrm{d}\mu_\lambda(t)\\ &\ge a^{2(s-1)}\int (a^{-2}+2a^{-1}t+1)^{s-2}\,\mathrm{d}\mu_{\lambda+1}(t)\\ &= a^2\int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_{\lambda+1}(t)\\ &\ge \int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_{\lambda+1}(t).\end{aligned}$$

The inequality $(s-1)/(s-2)\ge 1$ implies

$$\int (1+2at+a^2)^{s-1}\,\mathrm{d}\mu_\lambda(t) \ge \Big(\int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_\lambda(t)\Big)^{(s-1)/(s-2)}.$$

Next, by Jensen's inequality, we have $\int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_\lambda \ge (1+a^2)^{s-2}\ge 1$. Thus,

$$\Big(\int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_\lambda(t)\Big)^{(s-1)/(s-2)} \ge \int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_\lambda(t).$$

So, we need to show that $\int (1+2at+a^2)^{s-2}\,\mathrm{d}\mu_\lambda(t)\ge\int(1+2at+a^2)^{s-2}\,\mathrm{d}\mu_{\lambda+1}(t)$. The inequality trivially holds true if $s=3$. Considering the linear function $F(t)=1+2at+a^2$, it suffices to show that

$$\int_0^\infty r^{s-3}\mu_\lambda(t\in[-1,1]\colon F(t)>r)\,\mathrm{d}r \ge \int_0^\infty r^{s-3}\mu_{\lambda+1}(t\in[-1,1]\colon F(t)>r)\,\mathrm{d}r. \tag{3.8}$$

Consider $h(u)=\mu_\lambda(t\in[-1,1]:t>u)-\mu_{\lambda+1}(t\in[-1,1]:t>u)$. Clearly, $h(-1)=h(0)=h(1)=0$. Also

$$\begin{aligned}h'(u) &= -2c_{2\lambda}(1-u^2)^{\lambda-1/2}+2c_{2\lambda+2}(1-u^2)^{\lambda+1/2}\\ &= 2c_{2\lambda+2}(1-u^2)^{\lambda-1/2}\Big(\frac{1}{2(\lambda+1)}-u^2\Big).\end{aligned}$$

It follows that $h(u)\le 0$ on $[-1,0]$ and $h(u)\ge 0$ on $[0,1]$. Thus, $\varphi(r)=\mu_\lambda(t\in[-1,1]:F(t)>r)-\mu_{\lambda+1}(t\in[-1,1]:F(t)>r)$ changes sign only once, that is, there exists $r_0\in[0,\infty)$ such that $\varphi(r)\le 0$ on $[0,r_0]$ and $\varphi(r)\ge 0$ on $[r_0,\infty)$. If $r_0=0$, then (3.8) trivially holds true. If $r_0>0$, then we have

$$\int_0^\infty \Big(\Big(\frac{r}{r_0}\Big)^{s-3}-1\Big)\varphi(r)\,\mathrm{d}r\ge 0 \tag{3.9}$$

because the integrand has nonnegative sign. Therefore, inequality (3.9), together with $\int_0^\infty \varphi(r)\,\mathrm{d}r=0$, implies $\int_0^\infty r^{s-3}\varphi(r)\,\mathrm{d}r\ge 0$. ■

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