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Mean convex properly embedded $[\varphi, \vec{e}_3]$-minimal surfaces in $\mathbb{R}^3$

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Abstract. We establish curvature estimates and a convexity result for mean convex properly embedded $[\varphi, \vec{e}_3]$-minimal surfaces in $\mathbb{R}^3$, i.e., φ-minimal surfaces when φ depends only on the third coordinate of $\mathbb{R}^3$. Led by the works on curvature estimates for surfaces in 3-manifolds, due to White for minimal surfaces, to Rosenberg, Souam and Toubiana for stable CMC surfaces, and to Spruck and Xiao for stable translating solitons in $\mathbb{R}^3$, we use a compactness argument to provide curvature estimates for a family of mean convex $[\varphi, \vec{e}_3]$-minimal surfaces in $\mathbb{R}^3$. We apply this result to generalize the convexity property of Spruck and Xiao for translating solitons. More precisely, we characterize the convexity of a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$ with non-positive mean curvature when the growth at infinity of φ is at most quadratic.

1. Introduction

In 1983, Schoen [17] obtained an estimate for the length of the second fundamental form $\mathcal{S}$ of a stable minimal surface Σ in a 3-manifold. In particular, in $\mathbb{R}^3$, he proved the existence of a constant C such that

$$|\mathcal{S}(p)| \leq \frac{C}{d_\Sigma(p, \partial\Sigma)}, \quad p \in \Sigma,$$

where d_Σ stands for the intrinsic distance of Σ. Later, in 2010, Rosenberg, Souam and Toubiana [15] obtained an estimate for the length of the second fundamental form, depending on the distance to the boundary, for any stable H-surface Σ in a complete Riemannian 3-manifold of bounded sectional curvature $|\mathbb{K}| \leq \beta < +\infty$. More precisely, they proved the existence of a constant $C > 0$ such that

$$|\mathcal{S}(p)| \leq \frac{C}{\min\left\{d_\Sigma(p, \partial\Sigma), \pi/(2\sqrt{\beta})\right\}}, \quad p \in \Sigma.$$

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More recently, in 2016, White [20] obtained an estimate for the length of the second fundamental form for minimal surfaces with finite total absolute curvature less than 4π in 3-manifolds, depending on the distance to the boundary, the sectional curvature and the gradient of the sectional curvature of the ambient space.

In this paper we are interested in the so-called φ-minimal surfaces in a domain Ω of $\mathbb{R}^3$, which arise as critical points of the weighted area functional

$$\mathcal{A}^{\varphi}(\Sigma) = \int_{\Sigma} e^{\varphi}\, d\Sigma, \tag{1.1}$$

where $d\Sigma$ is the area element of Σ and φ is a smooth function defined in Ω.

When φ only depends on the third coordinate in $\mathbb{R}^3$, Σ will be called a $[\varphi, \vec{e}_3]$-*minimal surface* and then, the Euler–Lagrange equation for (1.1) is given in terms of the mean curvature vector $\mathbf{H}$ of Σ as follows:

$$\mathbf{H} = \dot{\varphi}\, \vec{e}_3^{\perp},$$

where $(\dot{\ })$ denotes the derivative with respect to the third coordinate, and $\perp$ is the projection to the normal bundle of Σ.

Classical minimal surfaces are obtained for $\dot{\varphi} \equiv 0$. When $\dot{\varphi}$ is a non-null constant, the correspondent class gives the translating solitons for the mean curvature flow, that is, surfaces Σ such that $t \mapsto \Sigma + t\vec{e}_3$ is a solution of the mean curvature flow.

Following the Colding and Minicozzi method [4, 5], Spruck and Xiao [18] have also obtained area and curvature bounds for complete mean convex translating solitons in $\mathbb{R}^3$. As an application, and using the Omori–Yau maximum principle (see, for example, [1]), they have proved one of the fundamental results in the recent development of translating solitons theory conjectured by Wang in [19].

Theorem 1.1 (Theorem 1.1 in [18]). *Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton with non-negative mean curvature. Then Σ is convex.*

Here we extend the results of [18] to mean convex $[\varphi, \vec{e}_3]$-minimal surfaces. In our case, by mean convex surfaces we will refer to those surfaces with $H \leq 0$ everywhere. More precisely, we will consider mean convex $[\varphi, \vec{e}_3]$-minimal surfaces with empty boundary in $\mathbb{R}^3_\alpha = \{p \in \mathbb{R}^3 \mid \langle p, \vec{e}_3\rangle > \alpha\}$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$\dot{\varphi} > 0, \quad \ddot{\varphi} \geq 0 \quad \text{on }]\alpha, \infty[. \tag{1.2}$$

We prove that mean convex $[\varphi, \vec{e}_3]$-minimal surfaces are stable (Proposition 4.4), and we show area estimates (Theorem 4.10) when

$$\Gamma := \sup_{]\alpha, +\infty[} (2\ddot{\varphi} - \dot{\varphi}^2) < +\infty. \tag{1.3}$$

To obtain curvature bounds, we need a good control at infinity of the function φ. To be more precise, we are going to consider that $z \mapsto \dot{\varphi}(z)/z$ is analytic at $+\infty$; i.e., $\dot{\varphi}$ has the following series expansion at $+\infty$:

$$\dot{\varphi}(u) = \Lambda\, u + \beta + \sum_{i=1}^{\infty} \frac{c_i}{u^i}, \quad \text{for } u \text{ large enough}, \tag{1.4}$$

with $\Lambda \geq 0$ and $\beta > 0$ if $\Lambda = 0$.

It is worth to note that condition (1.4) implies (1.3). Apart of a natural extension of the best known examples, conditions (1.2) and (1.4) are interesting because under these assumptions it is possible to know explicitly the asymptotic behaviour of rotational and translational invariant examples (see [13]).

The main results obtained in this paper can be summarized in the following two theorems.

Theorem A. *Let Σ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3_\alpha$ with non-positive mean curvature, locally bounded genus and $\varphi: \mathbb{R} \to \mathbb{R}$ satisfying* (1.2) *and* (1.4). *Then $|\mathcal{S}|/\dot{\varphi}$ is bounded on Σ. In particular, if $\Lambda = 0$, $|\mathcal{S}|$ is bounded, and if $\Lambda \neq 0$, $|\mathcal{S}|$ may go to infinity but with at most a linear growth in height.*

Theorem B. *Let Σ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3_\alpha$ with non-positive mean curvature, locally bounded genus and $\varphi: \mathbb{R} \to \mathbb{R}$ satisfying* (1.2), (1.4) *and $\dddot{\varphi} \leq 0$ on $]\alpha, +\infty[$. Then Σ is convex if and only if the function ΛK is bounded from below, where K is the Gauss curvature.*

The paper is organized as follows. In Sections 2 and 3, we show some facts about the geometry of the Ilmanen space, introduced in [12], and give some notations and fundamental equations of $[\varphi, \vec{e}_3]$-minimal surfaces. Following a similar approach as in [18] and using a compactness result of White (Theorem 2.1 in [21]), we obtain a blow-up theorem for $[\varphi, \vec{e}_3]$-minimal which allow us to prove Theorem A. This is needed for the proof of Theorem B in Section 5, which is based on a generalized Omori–Yau maximum principle (see Theorem 3.2 in [1]).

2. Geometry of the Ilmanen space

Let $\varphi: I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function defined in an open interval I of $\mathbb{R}$. Following [12], consider the Ilmanen space Ω^φ as the Riemannian manifold $\Omega = \mathbb{R}^2 \times I$ with the Euclidean conformal metric $\langle \cdot, \cdot \rangle^\varphi$ defined at any point $p = (x_1, x_2, x_3) \in \Omega$ by

$$\langle \cdot, \cdot \rangle^\varphi_p = e^{\varphi(x_3)} \langle \cdot, \cdot \rangle_p .$$

Denote by D and R (respectively, D^φ and R^φ) the Levi-Civita connection and the curvature tensor of the Euclidean space $\mathbb{R}^3$ (respectively, of the Ilmanen space Ω^φ). Then, for any orthonormal frame $\{e_i\}_{i=1,2,3}$ of $\mathbb{R}^3$, we obtain

$$D^\varphi_X Y = D_X Y + \frac{1}{2} \dot{\varphi} \left(\langle X, e_3 \rangle Y + \langle Y, e_3 \rangle X - \langle X, Y \rangle e_3 \right), \tag{2.1}$$

and

$$\begin{aligned} R^\varphi(X, Y, Y, X) = &-\frac{e^\varphi}{4} |X|^2 \left((2\ddot{\varphi} - \dot{\varphi}^2) \langle Y, e_3 \rangle^2 + |Y|^2 \dot{\varphi}^2 \right) \\ &+ \frac{e^\varphi}{4} \langle X, Y \rangle \left((2\ddot{\varphi} - \dot{\varphi}^2) \langle Y, e_3 \rangle \langle X, e_3 \rangle + \langle X, Y \rangle \dot{\varphi}^2 \right) \\ &+ \frac{e^\varphi}{4} \langle X, Y \rangle \left((2\ddot{\varphi} - \dot{\varphi}^2) \langle Y, e_3 \rangle \langle X, e_3 \rangle \right) - \frac{e^\varphi}{4} |Y|^2 \left((2\ddot{\varphi} - \dot{\varphi}^2) \langle X, e_3 \rangle^2 \right) \end{aligned} \tag{2.2}$$

for any tangent vector fields $X, Y \in T\Omega$; here $(\dot{\ })$ denotes the derivative with respect to x_3.

From (2.1) and (2.2), we also have the following.

Lemma 2.1. *Consider the orthonormal frame of Ω^φ given by $\{e_i^\varphi = e^{-\varphi/2} e_i\}$. Then,*

$$\langle D^\varphi_{e_i^\varphi} e_j^\varphi, e_k^\varphi\rangle^\varphi = \frac{1}{2}\, e^{-\varphi/2}\,\dot\varphi\,(\delta_{3j}\,\delta_{ik} - \delta_{ij}\,\delta_{3k}),$$
$$\mathbb{K}^\varphi(e_i^\varphi, e_j^\varphi) = \frac{1}{4}\, e^{-\varphi}\left((\dot\varphi^2 - 2\ddot\varphi)\,\delta_{i3} - \dot\varphi^2\right) \quad \textit{for } i \neq j,$$
$$\overline{\nabla}^\varphi \mathbb{K}^\varphi(e_i^\varphi, e_j^\varphi) = \frac{1}{4}\left(\dot\varphi^3 - (\dot\varphi^3 - 2\dot\varphi\,\ddot\varphi)\,\delta_{i3} + 2(\dot\varphi\,\ddot\varphi - \dddot\varphi)\,\delta_{i3} - 2\dot\varphi\,\ddot\varphi\right) e_3,$$

where $\overline{\nabla}^\varphi$ and $\mathbb{K}^\varphi$ are, respectively, the usual gradient operator and the sectional curvature of Ω^φ.

Definition 2.2. We say that the Ilmanen space Ω^φ has *bounded geometry* if the sectional curvature $\mathbb{K}^\varphi$ is bounded and the injectivity radius is bounded from below.

From Lemma 2.1 and the work of Cheeger, Gromov and Taylor [2], we can prove:

Proposition 2.3. *The following statements hold.*

(1) *If φ is a positive smooth function outside of a compact set, then the Ilmanen space is complete.*

(2) *If φ is a smooth function such that $e^{-\varphi}\max\{\dot\varphi^2, \ddot\varphi\}$ is bounded outside a compact set, then the Ilmanen space has bounded geometry.*

Remark 2.4. Throughout this paper, we will consider Σ as a connected and orientable surface with empty boundary in $\Omega \subseteq \mathbb{R}^3$.

Denote by N and $\mathcal{S}$ the Gauss map and the second fundamental form of Σ in $\mathbb{R}^3$, respectively. Then the shape operators $\mathbf{S}^\varphi$ and $\mathbf{S}$ of Σ in Ω^φ and $\mathbb{R}^3$, respectively, satisfy

$$-\mathbf{S}^\varphi_p u = D^\varphi_u N^\varphi = e^{-\varphi/2}\Big(-\mathbf{S}_p u + \frac{1}{2}\,\dot\varphi\langle N(p), e_3\rangle\, u\Big)$$

for any point $p \in \Sigma$ and any vector $u \in T_p\Sigma$, where $N^\varphi = e^{-\varphi/2}N$ is the Gauss map of Σ in the Ilmanen space. The above relation gives the following.

Proposition 2.5. *There holds that*

$$\mathcal{S}^\varphi_p(u, v) = e^{\varphi/2}\left(\mathcal{S}_p(u, v) + \frac{1}{2}\,\dot\varphi\langle N(p), e_3\rangle\langle u, v\rangle\right),$$
$$k_i^\varphi(p) = e^{-\varphi/2}\left(k_i(p) + \frac{1}{2}\,\dot\varphi\langle N(p), e_3\rangle\right)$$

for any $u, v \in T_p\Sigma$, where $\mathcal{S}^\varphi$ and k_i^φ (respectively, $\mathcal{S}$ and k_i) are the second fundamental form and the principal curvatures of Σ in Ω^φ (respectively, in $\mathbb{R}^3$).

In particular, the corresponding mean curvatures satisfy

$$H^\varphi = e^{-\varphi/2}(H + \dot\varphi\langle N, e_3\rangle).$$

3. Short background of $[\varphi, \vec{e}_3]$-minimal surfaces

In what follows, ∇, Δ and ∇^2 will denote, respectively, the gradient, the Laplacian and the Hessian operators on Σ associated to the induced metric from $\mathbb{R}^3$.

Definition 3.1. An orientable immersion Σ in $\mathbb{R}^3$ is called $[\varphi, \vec{e}_3]$-minimal if and only if the mean curvature H satisfies that $H = -\dot{\varphi}\,\langle \vec{e}_3, N\rangle$.

A $[\varphi, \vec{e}_3]$-minimal Σ in $\mathbb{R}^3$ can be viewed either as a critical point of the weighted area functional

$$\mathcal{A}^{\varphi}(\Sigma) := \int_{\Sigma} e^{\varphi}\, dA_{\Sigma}, \tag{3.1}$$

where dA_{Σ} is the volume element of Σ, or as a minimal surface in the Ilmanen space Ω^{φ}. From this property of minimality, a tangency principle can be applied and any two different φ-minimal surfaces cannot "touch" each other at one interior or boundary point (see Theorem 1 and Theorem 1a in [6]).

Let Σ be a $[\varphi, \vec{e}_3]$-minimal surface and denote by

$$\mu := \langle p, \vec{e}_3\rangle, \quad \eta := \langle N(p), \vec{e}_3\rangle, \quad p \in \Sigma,$$

their height and angle function, respectively. The following list of fundamental equations that will appear throughout this paper were proved in Section 2 of [13].

Lemma 3.2. *The following relations hold.*

(1) $\nabla\mu = \vec{e}_3^{\top}, \quad \langle\nabla\eta, \cdot\rangle = \mathcal{S}(\nabla\mu, \cdot)$,
(2) $\dot{\varphi}^2 = \dot{\varphi}^2|\nabla\mu|^2 + H^2$,
(3) $\dot{\varphi}\,\nabla^2\mu = H\mathcal{S}$,
(4) $\nabla^2\eta = \nabla_{\nabla\mu}\mathcal{S} - \eta\,\mathcal{S}^{[2]}$,
(5) $\Delta\mu = \dot{\varphi}(1 - |\nabla\mu|^2)$,
(6) $\Delta N + \dot{\varphi}\,\nabla\eta + \ddot{\varphi}\,\eta\,\nabla\mu + |\mathcal{S}|^2 N = 0$,
(7) $\nabla^2 H = -\eta\,\nabla^2\dot{\varphi} - \nabla_{\nabla\varphi}\mathcal{S} - H\mathcal{S}^{[2]} + \mathcal{B}$,
(8) $\Delta\mathcal{S} + \nabla_{\nabla\varphi}\mathcal{S} + \eta\,\nabla^2\dot{\varphi} + |\mathcal{S}|^2\mathcal{S} - \mathcal{B} = 0$,

where $\vec{e}_3^{\top}$ denotes the tangent projection of $\vec{e}_3$ in $T\Sigma$, ∇_X is the induced Levi-Civita connection, and $\mathcal{S}^{[2]}$ and $\mathcal{B}$ are the symmetric 2-tensors given by the following expressions:

$$\mathcal{S}^{[2]}(X, Y) = \sum_k \mathcal{S}(X, E_k)\,\mathcal{S}(E_k, Y),$$
$$\mathcal{B}(X, Y) = \langle\nabla\dot{\varphi}, X\rangle\,\mathcal{S}(\nabla\mu, Y) + \langle\nabla\dot{\varphi}, Y\rangle\,\mathcal{S}(\nabla\mu, X),$$

for any vector fields $X, Y \in T\Sigma$ and any orthonormal frame $\{E_i\}$ of $T\Sigma$.

From the strong maximum principle, equation (6) in Lemma 3.2 and Definition 3.1, the following result holds.

Theorem 3.3. *Let $\varphi\colon]a,b[\to \mathbb{R}$ be a strictly increasing function satisfying*

$$\ddot{\varphi} + \lambda\dot{\varphi}^2 \geq 0, \quad \text{for some } \lambda > 0, \tag{3.2}$$

and let Σ be a $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^2\times]a,b[$ with $H \leq 0$. If H vanishes anywhere, then H vanishes everywhere and Σ lies in a vertical plane.

Using the Hamilton principle (see [16], Section 2), we also can prove the following.

Theorem 3.4. *Let $\varphi\colon]a,b[\to \mathbb{R}$ be a strictly increasing function satisfying $\ddot{\varphi} \leq 0$, and let Σ be a locally convex $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^2\times]a,b[$. If the Gauss curvature K vanishes anywhere, then K vanishes everywhere.*

4. Stability of $[\varphi, \vec{e}_3]$-minimal surfaces

In this section, we will study the stability of $[\varphi, \vec{e}_3]$-minimal surfaces, where *stable* means stability as minimal surface in the Ilmanen space, i.e., its weighted area functional $\mathcal{A}^\varphi$ is locally minimal.

Proposition 4.1 (see Appendix of [3]). *Let X be a compactly supported variational vector field on the normal bundle of Σ, and let F_t be the normal variation associated to X. If Σ is an oriented $[\varphi, \vec{e}_3]$-minimal surface, then the second derivative of the weighted area functional $\mathcal{A}^\varphi$ is given by*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}^\varphi(F_t(\Sigma)) = \mathcal{Q}_\varphi(u,u), \quad \text{for any } u \in \mathcal{C}_0^\infty(\Sigma),$$

where $\mathcal{Q}_\varphi$ is the symmetric bilinear

$$\mathcal{Q}_\varphi(f,g) = \int_\Sigma e^\varphi \left(\langle \nabla f, \nabla g\rangle - (|\mathcal{S}|^2 - \ddot{\varphi}\eta^2) fg\right) d\Sigma. \tag{4.1}$$

Definition 4.2. We say that an oriented $[\varphi, \vec{e}_3]$-minimal surface Σ without boundary is stable if and only if, for any compactly supported smooth function u, it holds that

$$\mathcal{Q}_\varphi(u,u) = -\int_\Sigma u\mathcal{L}_\varphi(u)\, e^\varphi\, d\Sigma \geq 0, \tag{4.2}$$

where $\mathcal{L}_\varphi$ is a gradient Schrödinger operator defined on $\mathcal{C}^2(\Sigma)$ by

$$\mathcal{L}_\varphi(\cdot) := \Delta^\varphi(\cdot) + (|\mathcal{S}|^2 - \ddot{\varphi}\eta^2)(\cdot) \tag{4.3}$$

and Δ^φ is the drift Laplacian given by $\Delta^\varphi(\cdot) = \Delta(\cdot) + \langle\nabla\varphi, \nabla(\cdot)\rangle$.

Remark 4.3. The existence of stable surfaces is not guaranteed for any function φ. Cheng, Mejia and Zhou [3] proved that if Ω^φ is complete and $\ddot{\varphi} \leq -\varepsilon < 0$ for some positive constant ε, then there are not stable surfaces without boundary and with finite weighted area.

Proposition 4.4. *Let $\varphi\colon]\alpha, +\infty[\to \mathbb{R}$ be a regular function satisfying* (1.2) *and let Σ be an oriented $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^3$ with $H \leq 0$. Then, Σ is stable.*

Proof. From Theorem 3.3, we can assume that $H < 0$ everywhere; otherwise, Σ is a vertical plane and, as we are going to see in Corollary 4.6, Σ will be stable.

Suppose $H < 0$ and consider $w = \log(\eta)$. Then, by equation (6) of Lemma 3.2, we get that

$$\Delta w + \langle \nabla\varphi, \nabla w\rangle = -\frac{|\nabla\eta|^2}{\eta^2} - |\mathcal{S}|^2 - \ddot{\varphi}|\nabla\mu|^2. \tag{4.4}$$

Now, fix a compact domain $\mathcal{K}$ on Σ and consider u as an arbitrary function $\mathcal{C}^2(\Sigma)$ with compact support inside $\mathcal{K}$. By applying the divergence theorem to the expression $\operatorname{div}(e^{\varphi}\, u^2\, \nabla w)$, we have

$$\int_\Sigma e^{\varphi}\, u^2 (\Delta w + \langle \nabla\varphi, \nabla w\rangle)\, d\Sigma = -2\int_\Sigma e^{\varphi}\, u\, \langle \nabla u, \nabla w\rangle\, d\Sigma. \tag{4.5}$$

Now, from (4.4), (4.5) and (4.1), we obtain

$$\mathcal{Q}(u,u) = \int_\Sigma e^{\varphi}\left(\left|\nabla u - \frac{u}{\eta}\nabla\eta\right|^2 + \ddot{\varphi}u^2\right) d\Sigma \geq 0,$$

which concludes the proof. ∎

Fischer-Colbrie and Schoen [8] gave a condition on the first eigenvalue $\lambda_1(\mathcal{L}_\varphi)$ of $\mathcal{L}_\varphi$ which characterizes the stability of minimal surfaces in 3-manifolds. Using this characterization, we have:

Proposition 4.5. *Let Σ be a complete oriented $[\varphi, \vec{e}_3]$-immersion in $\mathbb{R}^3$. The following statements are equivalent.*

(1) *Σ is stable.*

(2) *The first eigenvalue $\lambda_1(\mathcal{L}_\varphi)(\mathcal{K}) < 0$ on any compact domain $\mathcal{K} \subset \Sigma$.*

(3) *There exists a positive function $u \in \mathcal{C}^2(\Sigma)$ such that $\mathcal{L}_\varphi(u) = 0$.*

As consequence of Proposition 4.5, we have the following corollary.

Corollary 4.6. *Let Σ be a complete oriented $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$. Then,*

(1) *if Σ is a graph with respect to a Killing vector V lying in the orthogonal complement of $\vec{e}_3$, then Σ is stable for any smooth function φ;*

(2) *if φ is an increasing convex smooth function and Σ is a graph with respect to $\vec{e}_3$, then Σ is stable.*

Proof. Consider the following smooth function $\nu = \langle V, N\rangle$. By assumption, ν is a positive function on Σ, and from equation (6) of Lemma 3.2, we get that

$$\Delta\nu = -\dot{\varphi}\langle V, \nabla\eta\rangle - \ddot{\varphi}\,\eta\langle V, \nabla\mu\rangle - |\mathcal{S}|^2\nu. \tag{4.6}$$

On the other hand, by equation (1) of Lemma 3.2, the following relations hold:

$$\begin{aligned}\langle \nabla\varphi, \nabla\nu\rangle &= -\dot{\varphi}\langle \mathbf{S}(V, \nabla\mu), N\rangle = \dot{\varphi}\,\mathcal{S}(\nabla\mu, N) = \dot{\varphi}\langle \nabla\eta, V\rangle,\\ \langle V, \nabla\mu\rangle &= \langle V, \vec{e}_3 - \eta N\rangle = -\eta\nu.\end{aligned}$$

From the above expressions and (4.6), we have $\mathcal{L}_\varphi(u) = 0$ and the first statement holds. The second assertion is a consequence of Proposition 4.4. ∎

Remark 4.7. Some results about stable $[\varphi, \vec{e}_3]$-minimal surface with $\ddot{\varphi} < 0$ can be found in [7].

Finally, from Theorem 3 in [3] and Corollary 4.6, we can also prove the following non-existence result.

Theorem 4.8. *Let V be a Killing vector field in the orthogonal complement of $\vec{e}_3$. If φ is a smooth function such that $\ddot{\varphi} \leq -\varepsilon < 0$, for some $\varepsilon > 0$, and the Ilmanen space is complete, then there are not $[\varphi, \vec{e}_3]$-minimal graphs with respect to V with finite weighted area.*

4.1. Intrinsic area estimates

To prove intrinsic area bounds, we will follow the same method as in [18].

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying (1.2) and (1.3), and let Σ be a $[\varphi, \vec{e}_3]$-immersion in $\mathbb{R}^3_\alpha$ with $H \leq 0$. Consider an intrinsic ball $\mathcal{D}_\rho(p)$ in Σ of radius ρ and centered at p.

Lemma 4.9. *If $\rho\dot{\varphi}(\rho + \mu(p)) < \sqrt{2}\pi$, then $\mathcal{D}_\rho(p)$ is disjoint from the conjugate locus of p and*

$$\int_\Sigma |\mathcal{S}|^2 u^2 \, d\Sigma \leq e^{2\rho\dot{\varphi}(\rho+\mu(p))} \int_\Sigma (|\nabla u|^2 + \ddot{\varphi}\, \eta^2 u^2) \, d\Sigma, \tag{4.7}$$

for any $u \in \mathcal{H}^2_0(\mathcal{D}_\rho(p))$.

Proof. As $|\nabla \mu|^2 \leq 1$, it is clear that for any $q \in \mathcal{D}_\rho(p)$, $\mu(p) - \rho \leq \mu(q) \leq \mu(p) + \rho$. Hence

$$\varphi(\mu(p) - \rho) \leq \varphi(\mu(q)) \leq \varphi(\rho + \mu(p)), \quad q \in \mathcal{D}_\rho(p),$$

and we have the following control of the curvature:

$$2K \leq H^2 \leq \dot{\varphi}^2(\mu(q)) \leq \dot{\varphi}^2(\rho + \mu(p)) \quad \text{on } \mathcal{D}_\rho(p).$$

Consequently, the first statement follows from the Rauch comparison theorem. Finally, inequality (4.7) follows from the above inequalities, Proposition 4.4 and the stability inequality (4.2). ∎

Theorem 4.10 (Boundness of area). *Let Σ be a $[\varphi, \vec{e}_3]$- immersion in $\mathbb{R}^3_\alpha$ with $H \leq 0$ and φ satisfying (1.2) and (1.3). If $2\rho\dot{\varphi}(\rho + \mu(p)) < \log(2)$ and $\sqrt{|\Gamma|}\,\rho < 1$, then the geodesic disk $\mathcal{D}_\rho(p)$ of radius ρ centered at p is disjoint from the cut locus of p and*

$$\mathcal{A}(\mathcal{D}_\rho(p)) < 4\pi\rho^2, \tag{4.8}$$

where $\mathcal{A}(\cdot)$ is the intrinsic area of Σ in $\mathbb{R}^3$.

Proof. First, we prove the inequality (4.8). Since $|\mathcal{S}|^2 = H^2 - 2K$, from (1.2), (1.3) and Lemma 4.9, we get that for any $u \in \mathcal{H}^2_0(\mathcal{D}_\rho(p))$,

$$\begin{aligned} -2\int_\Sigma K u^2 \, d\Sigma &\leq e^{2\rho\dot{\varphi}(\rho+\mu(p))} \int_\Sigma (|\nabla u|^2 + \ddot{\varphi}\, \eta^2 u^2) \, d\Sigma - \int_\Sigma \dot{\varphi}^2 \eta^2 u^2 \, d\Sigma \\ &\leq 2\int_\Sigma |\nabla u|^2 \, d\Sigma + \Gamma \int_\Sigma \eta^2 u^2 \, d\Sigma. \end{aligned} \tag{4.9}$$

Moreover, by Gauss–Bonnet, the variation of the length $l(s)$ of $\partial\mathcal{D}_s(\rho)$ is given by

$$l'(s) = \int_{\partial\mathcal{D}_s(p)} k_g \, d\sigma = 2\pi - \int_{\mathcal{D}_s(p)} K \, d\Sigma = 2\pi - K(s), \tag{4.10}$$

where k_g is the geodesic curvature of $\partial\mathcal{D}_s(p)$. If u is a radial function satisfying $u' \leq 0$ and $u(\rho) = 0$, the coarea formula gives

$$\int_{\mathcal{D}_s(p)} K\, u^2 \, d\Sigma = \int_0^\rho u^2(s) \int_{\partial\mathcal{D}_s(p)} K \, d\sigma \, ds = \int_0^\rho u^2(s) K'(s) \, ds,$$
$$\int_{\mathcal{D}_s(p)} |\nabla u|^2 d\Sigma = \int_0^\rho \int_{\partial\mathcal{D}_s(p)} |\nabla u|^2 d\sigma \, ds = \int_0^\rho (u'(s))^2 \, l(s) \, ds.$$

In particular, by taking $u(s) = 1 - cs/\rho$, applying integration by parts and using (4.10) and the above expressions, we have

$$\begin{aligned} -4\pi + 4\,\frac{\mathcal{A}(\mathcal{D}_\rho(p))}{\rho^2} &= -\frac{4}{\rho} \int_0^\rho (2\pi - l'(s)) \Big(1 - \frac{s}{\rho}\Big) \, ds \\ &\leq 2\,\frac{\mathcal{A}(\mathcal{D}_\rho(p))}{\rho^2} + \Gamma \int_\Sigma \eta^2 u^2 \, d\Sigma. \end{aligned} \tag{4.11}$$

If $\Gamma \leq 0$, then the inequality (4.8) trivially holds. If $\Gamma > 0$, using that $\sqrt{\Gamma}\rho < 1$ and (4.11) we get

$$\mathcal{A}(\mathcal{D}_\rho(p)) \leq \frac{4\pi}{2 - \Gamma\rho^2}\, \rho^2 < 4\pi\rho^2.$$

Now we will see that $\mathcal{D}_\rho(p)$ is disjoint from the cut locus of p. Otherwise, there exists $q \in \partial\mathcal{D}_{r_0}(p)$ that lies in the cut locus of p, where $r_0 = \mathrm{Inj}(\Sigma)(p) \leq \rho$. Since $\rho\dot{\varphi}(\rho + \mu(p)) < \sqrt{2}\pi$, from Lemma 4.9 and a Klingenberg-type argument (see for example [14], Chapter 5), there exist two geodesics from p to q which bound a smooth domain $D \subset \mathcal{D}_{r_0}(p)$ with a possible corner at p. By the Gauss–Bonnet Theorem,

$$2\pi = 2\pi - \int_{\partial D} k_g \, d\sigma = \int_D K \, d\Sigma \leq \frac{1}{2}\, \dot{\varphi}^2(r_0 + \mu(p))\, \mathcal{A}(D).$$

Hence,

$$\mathcal{A}(\mathcal{D}_{r_0}(p)) \geq \mathcal{A}(D) \geq \frac{4\pi}{\dot{\varphi}^2(r_0 + \mu(p))}\cdot$$

From the area estimate (4.8) for $\rho = r_0$ and the fact that $\rho\dot{\varphi}(\rho + \mu(p)) < \log(2)/2$, we get that

$$4\pi > 4\pi\, r_0^2\, \dot{\varphi}^2(r_0 + \mu(p)) \geq 4\pi,$$

which is a contradiction. ∎

4.2. Blow-up and curvature estimate

For later use, we will need the following compactness result, which is a consequence of Theorem 2.1 in [21]:

Theorem 4.11. *Let Ω be an open subset of $\mathbb{R}^3$. Let $\{\varphi_n\}$ be a sequence of smooth functions on Ω converging smoothly to φ_∞. Let Σ_n be a sequence of properly embedded minimal surfaces in the corresponding Ilmanen space Ω^{φ_n}. Suppose also that the area and the genus of Σ_n are bounded uniformly on compact subsets of Ω. Then, the total curvatures of Σ_n are also uniformly bounded on compact subsets of Ω and, after passing to a subsequence, Σ_n converge to a smooth properly embedded minimal Σ_∞ in Ω^{φ_∞}. The convergence is smooth away from a discrete set $\mathcal{C}$, and for each connected component Σ_∞^0 of Σ_∞, either*

(1) *the convergence to Σ_∞^0 is smooth everywhere with multiplicity* 1*, or*

(2) *the convergence to Σ_∞^0 is smooth with some multiplicity grater than* 1 *away from $\Sigma_\infty \cap \mathcal{C}$. In this case, if Σ_∞ is two-sided, the it must be stable.*

If the total curvatures of Σ_n are bounded by β, the set $\mathcal{C}$ has at most $\beta/4\pi$ points.

By using the same method as in [20], we prove the following.

Lemma 4.12 (Monoticity formula). *Let Σ be a $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^3_\alpha$ with φ satisfying (1.2). Fix any point $q \in \Sigma$ and consider the Euclidean ball $B(q,r)$ of radius r centered at q. Denote by $\Sigma_r = \Sigma \cap B(q,r)$ and by $\partial\Sigma_r = \Sigma \cap \partial B(q,r)$ and define $A(r) = \mathcal{A}(\Sigma \cap B(q,r))$ and $L(r) = \text{length}(\Sigma \cap \partial B(q,r))$. If there exists $\varepsilon > 0$ such that $0 \le \varphi(\varepsilon) < 1$, then the function*

$$\Theta_\Sigma(r) = \frac{\varphi(r)A(r)}{4\pi r^2}$$

is increasing in r over the interval $]0,\varepsilon[$.

Proof. If we take on Σ the vector field $X(p) = p - q$, $p \in \Sigma$, then, from the divergence theorem, we get that

$$\begin{aligned} 2A(r) &= \int_{\Sigma_r} \operatorname{div}(X)\, d\Sigma_r = \int_{\partial\Sigma_r} \langle X, \nu\rangle\, d\sigma - \int_{\Sigma_r} H\langle X, N\rangle\, d\Sigma_r \\ &= \int_{\partial\Sigma_r} \langle X, \nu\rangle\, d\sigma_r + \int_{\Sigma_r} \dot\varphi\,\eta\,\langle X, N\rangle\, d\Sigma_r, \end{aligned} \tag{4.12}$$

where ν is the conormal vector over $\partial\Sigma_r$, $d\Sigma_r$ is the volume element of Σ induced by the Euclidean metric, and $d\sigma$ is the length element of $\partial\Sigma_r$. From hypothesis, we have that $0 \le \varphi(r) \le 1$ for any $r < \varepsilon$. Moreover, as in the proof of Theorem 3 in [20],

$$L(r) \le A'(r) \quad \text{for any } r,$$

and joining both inequalities to the expression (4.12), we have

$$0 \le rA'(r) + r\dot\varphi(r)A(r) - 2A(r). \tag{4.13}$$

Finally, multiplying by $r^{-3}\varphi(r)$ in (4.13), we get

$$\begin{aligned} 0 &\le r^{-2}\varphi(r)\,A'(r) + r^{-2}\dot\varphi(r)\varphi(r)\,A(r) - 2r^{-3}\varphi(r)\,A(r) \\ &\le r^{-2}\varphi(r)\,A'(r) + r^{-2}\dot\varphi(r)\,A(r) - 2r^{-3}\varphi(r)\,A(r) = (r^{-2}\varphi(r)A(r))', \end{aligned} \tag{4.14}$$

which concludes the proof. ∎

Theorem 4.13 (Blow-up). *Let Σ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3_\alpha$ with $H \leq 0$, locally bounded genus, and with φ satisfying (1.2) and (1.4). Consider any sequence $\{\lambda_n\} \to +\infty$ and suppose that there exists a sequence $\{p_n\}$ in Σ such that $\{\dot{\varphi}(\mu(p_n))/\lambda_n\} \to C$ for some constant $C \geq 0$. Then, after passing to a subsequence, $\Sigma_n = \lambda_n(\Sigma - p_n)$ converge smoothly to*

(i) *a plane when $C = 0$,*

(ii) *one of the following translating solitons when $C > 0$:*

  (a) *a vertical plane,*

  (b) *a grim reaper surface,*

  (c) *a titled grim reaper surface,*

  (d) *a bowl soliton,*

  (e) *a Δ-Wing translating soliton.*

Proof. Consider the sequence of properly embedded surfaces $\Sigma_n = \lambda_n(\Sigma - p_n)$ in $\mathbb{R}^3$. Each Σ_n is a minimal surface in the Ilmanen space Ω^{φ_n}, where $\Omega = \mathbb{R}^3$ and

$$\varphi_n(x_3) = \varphi\Big(\frac{x_3}{\lambda_n} + \mu(p_n)\Big) - \varphi(\mu(p_n)).$$

It is clear form our assumption that

$$\varphi_n \to \varphi_\infty, \quad \text{with} \quad \varphi_\infty(x_3) = C\,x_3. \tag{4.15}$$

For any compact set $\mathcal{K}$ in Ω, we can consider r large enough such that $\mathcal{K}$ is contained in the Euclidean ball $B(0,r)$ of radius r centered at the origin. Then, for any $\epsilon_0 > 0$ and n large enough, it follows from (4.15) that

$$\begin{aligned}\mathcal{A}^{\varphi_n}(\Sigma_n \cap \mathcal{K}) &\leq \int_{\Sigma_n \cap B(0,r)} e^{\varphi_n}\, d\Sigma_n = \int_{\Sigma_n \cap B(0,r)} e^{Cq+\epsilon_0}\, d\Sigma_n \\ &\leq \lambda_n^2 \int_{\Sigma \cap B(p_n, r/\lambda_n)} e^{Cr+\epsilon_0}\, d\Sigma = e^{Cr+\epsilon_0}\lambda_n^2 \mathcal{A}(\Sigma \cap B(p_n, r/\lambda_n)).\end{aligned}$$

As φ can be choose up to a constant, we can assume that there exists $\varepsilon > 0$ such that $0 < \varphi(\varepsilon) < 1$. Since $r/\lambda_n \to 0$, it follows Lemma 4.12 that there must be n_0 such that $r/\lambda_n \leq \varepsilon$ and $\mathcal{O}_\Sigma(r/\lambda_n) \leq \mathcal{O}_\Sigma(\varepsilon)$ for any $n \geq n_0$. Thus,

$$\mathcal{A}(\Sigma \cap B(p_n, r/\lambda_n)) \leq \frac{\varphi(\varepsilon)}{\varphi(r/\lambda_n)} \Big(\frac{r}{\lambda_n}\Big)^2 \frac{\mathcal{A}(\Sigma \cap B(p_n, \varepsilon))}{\varepsilon^2}.$$

Joining both inequalities we have that, for n large enough,

$$\mathcal{A}^{\varphi_n}(\Sigma_n \cap \mathcal{K}) \leq \frac{e^{Cr+\epsilon_0}\varphi(\varepsilon)}{\varphi(r/\lambda_n)}\, r^2 \,\frac{\mathcal{A}(\Sigma \cap B(p_n, \varepsilon))}{\varepsilon^2} \leq 4\pi \,\frac{e^{Cr+\epsilon_0}\varphi(\varepsilon)}{\varphi(r/\lambda_n)}\, r^2.$$

As $\lambda_n \to +\infty$, there exists a positive constant Θ, depending only of φ , such that

$$\mathcal{A}^{\varphi_n}(\Sigma_n \cap \mathcal{K}) \leq \Theta\, \pi e^{Cr} r^2.$$

Consequently, Σ_n have uniformly bounded area on compact subsets of $\mathbb{R}^3$. From Theorem 4.11, the sequence Σ_n converges to a properly embedded $[\varphi_\infty, \vec{e}_3]$-minimal surface Σ_∞ in $\mathbb{R}^3$. Since each Σ_n is stable, Σ_∞ must be a plane in $\mathbb{R}^3$ if $C = 0$ (see [8]). If $C > 0$, then Σ_∞ is a mean convex properly embedded translating soliton in $\mathbb{R}^3$, and from the results in [10] and [18], Σ_∞ must be either a vertical plane, a grim reaper surface, a titled grim reaper surface, a bowl soliton or a Δ-Wing translating soliton.

Finally, we prove that the convergence is smooth everywhere. Otherwise, there exists $q_n \in \Sigma_n$ such that $q_n \to q \in \Sigma_\infty$ and $T_{q_n}\Sigma_n$ does not converge to $T_q\Sigma_\infty$. Arguing as in Theorem 2.4 of [21], there exists $\hat{\lambda}_n \to \infty$ such that, after passing a subsequence, the surfaces $\Sigma'_n = \hat{\lambda}_n(\Sigma_n - q_n)$ converge smoothly and with multiplicity 1 to a complete, smooth, properly embedded, non-flat minimal surface Σ'_∞ in $\mathbb{R}^3$ of finite total curvature. But, as each Σ'_n is stable we have that Σ'_∞ must be a plane, see Corollary 4 in [8], and this is a contradiction. ∎

Now, by combining the methods of Rosenberg, Souam and Toubiana [15], and Spruck and Xiao [18], we will prove Theorem A.

Proof of Theorem A. Suppose that there exists a sequence of points $\{p_n\}$ in Σ such that

$$\lambda_n = |\mathcal{S}|(p_n) \to +\infty, \qquad \lim_{n\to+\infty} \frac{\lambda_n}{\dot{\varphi}(\mu(p_n))} = +\infty.$$

Then, for a subsequence of $\{p_n\}$ we have $\dot{\varphi}(\mu(p_n))/\lambda_n \mapsto 0$, and from Theorem 4.13, the sequence $\Sigma_n = \lambda_n(\Sigma - p_n)$ converges smoothly to a plane Σ_∞ in $\mathbb{R}^3$. Since $|\mathcal{S}_n(p_n)| = 1$ for each n, we also have $|\mathcal{S}_\infty(0)| = 1$, which is a contradiction. ∎

The following results are consequences of Lemma 2.1 and the results in [15, 20].

Theorem 4.14. *Let φ be a smooth function such that*

$$\frac{1}{2}\, e^{-\varphi}(|\max\{\dot{\varphi}^2, \ddot{\varphi}\}|) + |\max\{\dot{\varphi}^3, 2\dot{\varphi}\ddot{\varphi}, \dddot{\varphi}\}| \geq \rho,$$

for some constant $\rho > 0$, and let Σ be a minimal surface (possibly with boundary) in the Ilmanen space with total absolute curvature at most $\lambda < 4\pi$. Then there exists a constant C depending of λ such that

$$|\mathcal{S}^\varphi| \min\{d_\varphi(p, \partial\Sigma), \mathcal{R}\} \leq C \quad \textit{for any } p \in \Sigma,$$

where

$$\mathcal{R} = (\sup|\mathbb{K}^\varphi| + \sup|\overline{\nabla}^\varphi \mathbb{K}^\varphi|^{1/2})^{-1}.$$

Theorem 4.15. *Let φ be a smooth function such that the Ilmanen space is a complete Riemannian manifold with bounded geometry whose sectional curvature $|\mathbb{K}^\varphi| \leq A$ for some constant $A > 0$. For any stable minimal immersion Σ in the Ilmanen space (possibly with boundary), there exists a constant C such that*

$$|\mathcal{S}^\varphi| \min\{d_\varphi(p, \partial\Sigma), \pi/2\sqrt{A}\} \leq C.$$

5. A Spruck–Xiao's type theorem

Using a delicate maximum principle argument, Spruck and Xiao [18] proved that any complete translating soliton in $\mathbb{R}^3$ with $H \leq 0$ is convex. A slightly simplified proof of this result is presented by Hoffman, Ilmanen, Martín and White in [11]. In this section, we consider the same problem for properly embedded $[\varphi, \vec{e}_3]$-minimal surfaces in $\mathbb{R}^3$ with $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying (1.2), (1.4) and $\ddot{\varphi} \leq 0$ on $]\alpha, +\infty[$.

We start with some results we will use.

Theorem 5.1 (Generalized Omori–Yau maximum principle for Δ^ψ, Theorem 3.2 in [1]). *Let Σ be a surface in $\mathbb{R}^3$ and let Δ^ψ be the drift Laplacian operator associated to $\psi \in \mathcal{C}^2(\Sigma)$. Let $\gamma \in \mathcal{C}^2(\Sigma)$ be such that*

$$\gamma(p) \to +\infty \quad \textit{as } p \to \infty, \tag{5.1}$$

$$\Delta^\psi \gamma \leq C \quad \textit{outside a compact subset of } \Sigma, \tag{5.2}$$

$$|\nabla \gamma| \leq C \quad \textit{outside a compact subset of } \Sigma, \tag{5.3}$$

for some constant $C > 0$. If $\nu \in \mathcal{C}^2(\Sigma)$ and $\nu^\star = \sup_\Sigma \nu < +\infty$, then there exists a sequence of points $\{p_n\} \subset \Sigma$ satisfying, for each $n \in \mathbb{N}$,

$$\text{(i)} \quad \nu(p_n) > \nu^\star - \frac{1}{n}, \qquad \text{(ii)} \quad \Delta^\psi \nu(p_n) < \frac{1}{n}, \qquad \text{(iii)} \quad |\nabla \nu(p_n)| < \frac{1}{n}\cdot$$

Lemma 5.2. *Let k_i be the principal curvatures of an immersion Σ in $\mathbb{R}^3$ and let $\mathcal{U}$ be the set of totally umbilical points of Σ. If $\{v_i\}$ is an orthonormal frame of principal directions in $T\Sigma$, then the following statements hold.*

(1) $\nabla_{v_i} v_i = \alpha_i v_j, \ \ \nabla_{v_j} v_i = \alpha_j v_j$ *with* $\alpha_i = -\alpha_j$.

(2) *The coefficients α_i are determined by the formula*

$$\alpha_i = \frac{h_{12,i}}{k_1 - k_2} \quad \textit{in } \Sigma - \mathcal{U}, \ \textit{where } h_{ij,k} = (\nabla_{v_k} \mathcal{S})(v_i, v_j).$$

Proof. The first item is trivially obtained by differentiating $\langle v_i, v_j \rangle = \delta_{ij}$ On the other hand, differentiating $\mathcal{S}(v_1, v_2) = 0$ and using the first item, we get that

$$0 = (\nabla_{v_i} \mathcal{S})(v_1, v_2) + \mathcal{S}(\nabla_i v_1, v_2) + \mathcal{S}(\nabla_{v_i} v_2, v_1) = h_{12,i} + \alpha_i (k_2 - k_1). \qquad \blacksquare$$

Lemma 5.3. *If Σ is a $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^3$, then*

$$\Delta^\varphi k_i = -|\mathcal{S}|^2 k_i - \eta \nabla^2 \dot{\varphi}(v_i, v_i) + \mathcal{B}(v_i, v_i) + 2(-1)^{i+1} \frac{Q^2}{k_1 - k_2} \quad \textit{in } \Sigma - \mathcal{U},$$

where $\mathcal{B}$ is the bilinear form defined in Lemma 3.2 and

$$Q^2 = h_{12,1}^2 + h_{12,2}^2 = h_{11,2}^2 + h_{22,1}^2.$$

Proof. We only prove the formula for the first principal curvature k_1, because the reasoning for k_2 is the same. Fix any point $p \in \Sigma - \mathcal{U}$ and consider a geodesic frame $\{u_1, u_2\}$

of $T_p\Sigma$. Then,

$$\Delta k_1 = \sum_{i=1}^{2}\langle \nabla_{u_i}\nabla k_1, u_i\rangle = \sum_{i=1}^{2}\langle \nabla_{u_i}\nabla \mathcal{S}(v_1,v_1), u_i\rangle. \tag{5.4}$$

From item (1) of Lemma 5.2, $\mathcal{S}(\nabla_{u_i}v_1, v_1) = 0$, and we have

$$\nabla \mathcal{S}(v_1,v_1) = \sum_{i=1}^{2}((\nabla_{u_i}\mathcal{S})(v_1,v_1))u_i. \tag{5.5}$$

By using (5.5) and (5.4), we prove that

$$\Delta k_1 = \sum_{i=1}^{2}\langle \nabla_{u_i}(\nabla_{u_i}\mathcal{S})(v_1,v_1)u_i, u_i\rangle = (\Delta\mathcal{S})(v_1,v_1) + 2\,\frac{Q^2}{k_1-k_2}, \tag{5.6}$$

and the lemma follows from item (8) of Lemma 3.2. ∎

Lemma 5.4. *Let Σ be a $[\varphi,\vec{e}_3]$-minimal immersion in $\mathbb{R}^3_\alpha$ with $k_1<0$, $H=k_1+k_2<0$. If for any positive smooth function $\psi\colon\Sigma\to]0,+\infty[$ we consider the operator*

$$\mathcal{J}^\psi := \Delta^{\varphi+2\log\psi},$$

then on $\Sigma\backslash\mathcal{U}$ we have

$$\mathcal{J}^\eta\Big(\frac{k_2}{\eta}\Big) = -\ddot\varphi\langle\nabla\mu, v_2\rangle^2 + \ddot\varphi\Big(\frac{k_2}{\eta}\Big)(1+2\langle\nabla\mu,v_2\rangle^2) - \frac{2}{\eta}\,\frac{Q^2}{k_1-k_2}, \tag{5.7}$$

$$\begin{aligned}\mathcal{J}^{-k_1}\Big(\frac{\eta}{k_1}\Big) &= \ddot\varphi\langle\nabla\mu,v_1\rangle^2\Big(\frac{\eta}{k_1}\Big)^2 - \ddot\varphi\Big(\frac{\eta}{k_1}\Big)(1+2\langle\nabla\mu,v_1\rangle^2)\\ &\quad -2\Big(\frac{\eta}{k_1}\Big)\frac{Q^2}{k_1(k_1-k_2)}.\end{aligned} \tag{5.8}$$

In particular, if $\ddot\varphi\le 0$ on $]\alpha,+\infty[$ and φ satisfies (1.2), then

$$\mathcal{J}^\eta\Big(\frac{k_2}{\eta}\Big)\ge 0 \quad \textit{on } \{p\in\Sigma : k_2(p)>0\}. \tag{5.9}$$

Proof. It is not difficult to see that

$$\mathcal{J}^\eta\Big(\frac{k_2}{\eta}\Big) = \frac{\eta\,\Delta^\varphi k_2 - k_2\,\Delta^\varphi\eta}{\eta^2} \quad\text{and}\quad \mathcal{J}^{-k_1}\Big(\frac{\eta}{k_1}\Big) = \frac{k_1\,\Delta^\varphi\eta - \eta\,\Delta^\varphi k_1}{k_1^2}. \tag{5.10}$$

Moreover, from Lemmas 3.2 and 5.3, we get that

$$\begin{aligned}\eta\Delta^\varphi k_i &= -|\mathcal{S}|^2k_i\eta - \eta^2(\ddot\varphi\langle\nabla\mu,v_i\rangle^2 - \ddot\varphi\,\eta k_i) + 2\ddot\varphi k_i\langle\nabla\mu,v_i\rangle^2\\ &\quad -2(-1)^{i+1}\eta\,\frac{Q^2}{k_1-k_2},\end{aligned} \tag{5.11}$$

$$k_2\,\Delta^\varphi\eta = -\ddot\varphi\,\eta k_2|\nabla\mu|^2 - |\mathcal{S}|^2k_2\eta, \tag{5.12}$$

$$k_1\,\Delta^\varphi\eta = -\ddot\varphi\,\eta k_1|\nabla\mu|^2 - |\mathcal{S}|^2k_1\eta, \tag{5.13}$$

and we may conclude from (5.10), (5.11), (5.12) and (5.13) by a straightforward computation. ∎

Lemma 5.5. *Let Σ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface without boundary in $\mathbb{R}^3_\alpha$ with $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying (1.2) and (1.4). Then, Σ is complete and the generalized Omori–Yau maximum principle can be applied to Δ^φ.*

Proof. Consider the function $\gamma\colon \Sigma \to \mathbb{R}$ given by $\gamma(p) = 2\log|p|$. Then, as Σ is properly embedded and φ satisfies (1.2) and (1.4), we have

$$\gamma(p) \to +\infty \quad \text{as } p \to \infty, \tag{5.14}$$

$$|\nabla\gamma(p)| = 2\frac{|p^\top|}{|p|^2} \leq 2, \quad |p| \gg 0, \tag{5.15}$$

$$\Delta^\varphi\gamma(p) = -4\frac{|p^\top|^2}{|p|^4} + \frac{2\mu(p)\dot{\varphi}(\mu(p)) + 4}{|p|^2} \leq 2A + 1, \quad |p| \gg 0, \tag{5.16}$$

and from Theorem 5.1 we can apply the generalized Omori–Yau maximum principle to Δ^φ.

By taking γ along any divergent geodesic, it is clear from (5.14) and (5.15) that any properly embedded surface in $\mathbb{R}^3$ is complete. ■

5.1. Proof of Theorem B

Let Σ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3_\alpha$ with $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying (1.2), (1.4) and $\dddot{\varphi} \leq 0$ on $]\alpha, +\infty[$. Then from Theorem 3.3, we can assume that $\eta > 0$ everywhere. Take $k_1 < 0$, $k_1 \leq k_2$, $H = k_1 + k_2 < 0$.

We only need to prove that Σ is convex provided ΛK is bounded from below since the converse is trivial. For proving that, we will argue by contradiction and suppose there exists a point $p_0 \in \Sigma$ such that $K(p_0) < 0$. Then,

$$0 < \vartheta := \sup_\Sigma \frac{k_2}{\eta} = \sup_{\Omega^+} \frac{k_2}{\eta}, \tag{5.17}$$

where $\Omega^+ = \{p \in \Sigma \mid k_2(p) > 0\}$.

Claim 5.6. *The supremum ϑ is not attained.*

Proof. Suppose it is attained at a point p. Then from (5.9) and the strong maximum principle, see Theorem 3.5 in [9], k_2/η is constant on Σ and $Q \equiv 0$. Thus, from Lemma 5.2, $\{v_1, v_2\}$ is parallel and then $k_1 k_2 \equiv 0$, getting a contradiction with (5.17). ■

Claim 5.7. *If $\{p_n\} \subset \Omega^+$ is a sequence of points such that $\frac{k_2}{\eta}(p_n) \to \vartheta$, then after passing to a subsequence, $\mu(p_n) \to +\infty$ and*

(1) *if $\Lambda = 0$ and $\frac{\eta}{k_1}(p_n) \to 0$, then $\eta(p_n) \to 0$,*
(2) *if $\Lambda \neq 0$, then $\eta(p_n) \to 0$ and $\frac{\eta}{k_1}(p_n) \to 0$.*

Proof. From (1.4), the function $2\ddot{\varphi} - \dot{\varphi}^2$ is upper bounded on $]\alpha, +\infty[$ and we can apply Theorem 4.10, getting that the sequence $\Sigma_n = \Sigma - p_n$ has area uniformly bounded on compact subsets of $\mathbb{R}^3_\alpha$.

Each Σ_n is a $[\varphi_n, \vec{e}_3]$-minimal surface with

$$\varphi_n(u) = \varphi(u + \mu(p_n)) - \varphi(\mu(p_n)), \quad \text{for each } n \in \mathbb{N}. \tag{5.18}$$

If $\sup_n\{\mu(p_n)\} < +\infty$ then, by taking an accumulation point μ_∞ of $\{\mu(p_n)\}$ and applying the compactness Theorem 4.11, we get that, after passing to a subsequence, $\mu(p_n) \to \mu_\infty \in \mathbb{R}$, the Σ_n converge to a properly embedded $[\varphi_\infty, \vec{e}_3]$-minimal surface Σ_∞ with $H \leq 0$, where $\varphi_\infty(u) := \varphi(u + \mu_\infty) - \varphi(\mu_\infty)$. From Theorem A, the length of second fundamental form of Σ_n is bounded, therefore the convergence must be smooth at the origin and so the function k_2/η reaches its supremum at the origin. This is a contradiction with Claim 5.6.

If $\Lambda = 0$ and $\frac{\eta}{k_1}(p_n) \to 0$, we consider $\Sigma_n' = \lambda_n(\Sigma - p_n)$, where $\lambda_n = -\frac{k_1}{\eta}(p_n)$. Then, from (1.2) and (1.4) and after passing to a subsequence, we get that

$$\eta(p_n) \to \eta_\infty \quad \text{and} \quad \frac{\dot\varphi(\mu(p_n))}{\lambda_n} = 1 + \frac{k_2}{k_1}(p_n) \to 0.$$

Applying Theorem 4.13, the Σ_n' converge smoothly to a plane Σ_∞, with principal curvatures at the origin given by

$$k_1 = -\eta_\infty \quad \text{and} \quad k_2 = \eta_\infty,$$

which implies that $\eta_\infty = 0$.

If $\Lambda \neq 0$, since $k_1/\eta + k_2/\eta = -\dot\varphi$, we have that $\frac{\eta}{k_1}(p_n) \to 0$. Let us suppose by contradiction that $\eta(p_n) \to \eta_\infty \neq 0$. Then $k_1(p_n) \to -\infty$ and $k_2(p_n) \to \vartheta$, getting to a contradiction with the hypothesis that ΛK is bounded from below. ∎

We will distinguish two cases: $\dot\varphi$ bounded ($\Lambda = 0$) and $\dot\varphi$ unbounded ($\Lambda \neq 0$):

- **Case $\boldsymbol{\Lambda = 0}$.** In this case, from (1.4) we have that on $]\alpha, +\infty[$,

$$0 < \dot\varphi < \sup_{]\alpha,+\infty[} \dot\varphi = \beta. \tag{5.19}$$

Claim 5.8. *The case $\vartheta = +\infty$ is not possible.*

Proof. Assume there exists a sequence of points $\{p_n\}$ such that $\frac{k_2}{\eta}(p_n) \to +\infty$. Using that

$$\left(\frac{k_1}{k_2}\right) + 1 = -\dot\varphi\left(\frac{\eta}{k_2}\right) \quad \text{and} \quad \frac{k_1 + k_2}{\eta} = -\dot\varphi, \tag{5.20}$$

we get $(k_1/k_2)(p_n) \to -1$ and $(\eta/k_1)(p_n) \to 0$. In particular,

$$\tau = \sup_\Sigma \frac{\eta}{k_1} = 0. \tag{5.21}$$

and τ is not attained at a interior point. Now we may apply the generalized Omori–Yau maximum principle for Δ^φ and conclude that there exists a sequence of points, $\{q_n\} \subset \Sigma$, $|q_n| \to +\infty$, such that

$$\frac{\eta}{k_1}(q_n) \to 0, \quad \left|\nabla\left(\frac{\eta}{k_1}\right)\right|(q_n) \to 0 \quad \text{and} \quad \Delta^\varphi\left(\frac{\eta}{k_1}\right)(q_n) \leq 0. \tag{5.22}$$

Consequently, it follows from (5.20), and (5.22) that $(k_2/\eta)(q_n) \to +\infty$ and so, for n large enough, $\{q_n\} \subset \Omega^+$. In particular, there exists $n_0 \in \mathbb{N}$ such that (5.7) and (5.8) hold for $n \geq n_0$. For the rest of the proof of Theorem B, any statement that some quantity tends to a limit refers only to the quantity at the corresponding points.

Now, from Claim 5.7, after passing to subsequence, $\mu \to +\infty$, $\eta \to 0$ and $k_2/k_1 = -\dot{\varphi}\,\eta/k_1 - 1 \to -1$. Thus, from Lemma 3.2, we have

$$\left|\frac{\nabla\eta}{k_1}\right|^2 = \langle\nabla\mu, v_1\rangle^2 + \left(\frac{k_2}{k_1}\right)^2 \langle\nabla\mu, v_2\rangle^2 \to 1, \tag{5.23}$$

and by (5.22) and (5.23),

$$\frac{\nabla\eta}{k_1} \to \mathcal{X}, \qquad \frac{\eta}{k_1}\frac{\nabla k_1}{k_1} \to \mathcal{X}, \quad \mathcal{X} \neq 0. \tag{5.24}$$

Since

$$\frac{\eta}{k_1}\frac{\nabla H}{k_1} = \frac{\eta}{k_1}\frac{\nabla k_1}{k_1} + \frac{\eta}{k_1}\frac{\nabla k_2}{k_1} = -\frac{\eta^2}{k_1}\frac{\nabla\dot{\varphi}}{k_1} - \frac{\eta}{k_1}\frac{\dot{\varphi}\nabla\eta}{k_1}, \tag{5.25}$$

it follows from Lemma 3.2, (5.24) and (5.25) that

$$\frac{\eta}{k_1}\frac{h_{11,2}}{k_1} \to \langle\mathcal{X}, v_2\rangle, \qquad \frac{\eta}{k_1}\frac{h_{22,1}}{k_1} \to -\langle\mathcal{X}, v_1\rangle.$$

In particular,

$$\frac{\eta^2}{k_1^4} Q^2 \to |\mathcal{X}| = 1. \tag{5.26}$$

Multiplying by (η/k_1) in (5.8), we obtain

$$\begin{aligned} &\left(\frac{\eta}{k_1}\right)\Delta^{\varphi}\left(\frac{\eta}{k_1}\right) + 2\left(\frac{\eta}{k_1}\right)\left\langle\nabla\left(\frac{\eta}{k_1}\right), \frac{\nabla k_1}{k_1}\right\rangle \\ &\quad = \dddot{\varphi}\langle\nabla\mu, v_1\rangle^2\left(\frac{\eta}{k_1}\right)^3 - \ddot{\varphi}\left(\frac{\eta}{k_1}\right)^2(1 + 2\langle\nabla\mu, v_1\rangle^2) - 2k_1\left(\frac{\eta^2}{k_1^4}\right)\frac{Q^2}{k_1 - k_2}. \end{aligned} \tag{5.27}$$

Using that $k_2/k_1 \to -1$, (1.2), (1.4), (5.22) and (5.26), we can take limit when $n \to +\infty$ in the above equality to get $0 \leq -1$, a contradiction. ∎

Claim 5.9. *If $\{p_n\} \subset \Sigma$ is a sequence of points such that $k_2/\eta \to \vartheta < +\infty$, then after passing to a subsequence,*

$$\mu \to +\infty, \quad \eta \to 0 \quad \textit{and} \quad \frac{k_1}{k_2} \to -\frac{\beta}{\vartheta} - 1.$$

Proof. By taking $\Sigma_n = \Sigma - p_n$, we can argue as in the first part of Claim 5.7 to prove that after passing to a subsequence, $\mu \to +\infty$. Then, from (5.18),

$$\varphi_n \to \varphi_\infty, \quad \text{with } \varphi_\infty(u) = \beta u,$$

and using again the compactness Theorem 4.11, after passing to a subsequence, we have that the sequence Σ_n converges to a properly embedded translating soliton Σ_∞ containing the origin with $H \leq 0$. But from Theorem A, the length of the second fundamental form of Σ_n is bounded, so the convergence is smooth and we conclude that if Σ_∞ is not a vertical plane, k_2/η attains it supremum value at the origin of Σ_∞, which contradicts Claim 5.6.

Thus, $\eta \to 0$ and

$$\frac{k_1}{k_2} = \frac{H}{k_2} - 1 = -\dot{\varphi}\frac{\eta}{k_2} - 1 \to \frac{\beta}{\vartheta} - 1. \qquad \blacksquare$$

Claim 5.10. *The case* $0 < \vartheta < +\infty$ *is not possible.*

Proof. If

$$0 < \vartheta = \sup_{\Sigma} \frac{k_2}{\eta} = \sup_{\Omega^+} \frac{k_2}{\eta} < \infty,$$

them from Lemma 5.5, Theorem 5.1 and Claim 5.6, there exists a sequence of points $\{p_n\} \subset \Omega^+$, $|p_n| \to +\infty$, such that

$$\left(\frac{k_2}{\eta}\right) \to \vartheta, \quad \left|\nabla\left(\frac{k_2}{\eta}\right)\right| \to 0 \quad \text{and} \quad \Delta^{\varphi}\left(\frac{k_2}{\eta}\right)(p_n) \leq 0. \tag{5.28}$$

From Claim 5.9 we get

$$\nabla\mu \to \vec{e}_3, \quad \mu \to +\infty \quad \text{and} \quad \frac{k_1}{k_2} \to -\frac{\beta}{\vartheta} - 1,$$

and from Lemma 3.2,

$$\left|\frac{k_2}{\eta}\frac{\nabla\eta}{\eta}\right|^2 = \frac{k_2^4}{\eta^4}\left(\frac{k_1^2}{k_2^2}\langle\nabla\mu, v_1\rangle^2 + \langle\nabla\mu, v_2\rangle^2\right) \to C \neq 0,$$

where C is a constant such that $C \in [\vartheta^4, 2\vartheta^4 + \vartheta^2(\beta^2 + 2\beta\vartheta)[$. Then, by (5.28),

$$\frac{\nabla k_2}{\eta} \to \mathcal{Y} \quad \text{and} \quad \frac{k_2}{\eta}\frac{\nabla\eta}{\eta} \to \mathcal{Y}, \quad \mathcal{Y} \neq 0. \tag{5.29}$$

Arguing as in Claim 5.9 we can prove that

$$\frac{h_{11,2}}{\eta} \to -\vartheta^2\left(\frac{\beta}{\vartheta} + 1\right)\langle\vec{e}_3, v_2\rangle \quad \text{and} \quad \frac{h_{22,1}}{\eta} \to -\vartheta^2\left(\frac{\beta}{\vartheta} + 1\right)\langle\vec{e}_3, v_1\rangle, \tag{5.30}$$

and then,

$$\frac{Q^2}{\eta^2} = \frac{h_{11,2}^2 + h_{22,1}^2}{\eta^2} \to \vartheta^4\left(\frac{\beta}{\vartheta} + 1\right)^2. \tag{5.31}$$

Multiplying by (k_2/η) in (5.7), we obtain

$$\begin{aligned} &\frac{k_2}{\eta}\Delta^{\varphi}\left(\frac{k_2}{\eta}\right) + 2\frac{k_2}{\eta}\left\langle\nabla\left(\frac{k_2}{\eta}\right), \frac{\nabla\eta}{\eta}\right\rangle \\ &= -\dddot{\varphi}\frac{k_2}{\eta}\langle\nabla\mu, v_2\rangle^2 + \ddot{\varphi}\left(\frac{k_2}{\eta}\right)^2(1 + 2\langle\nabla\mu, v_2\rangle^2) - 2\left(\frac{Q^2}{\eta^2}\right)\frac{k_2}{k_1 - k_2}. \end{aligned} \tag{5.32}$$

Using that $k_1/k_2 \to -\beta/\vartheta - 1$, (1.2), (1.4), (5.28) and (5.31), we can take limit when $n \to +\infty$ in the above equality to get

$$0 \geq 2\frac{\vartheta^4(\beta/\vartheta + 1)^2}{\beta/\vartheta + 2} > 0,$$

which is contradiction. $\blacksquare$

• **Case $\Lambda \neq 0$**. As the supremum of k_2/η is not attained on Σ, we can take any divergent sequence of points $\{p_n\} \subset \Omega^+$ such that $k_2/\eta \to \vartheta$.

Claim 5.11. *If $\Lambda \neq 0$ and $\{p_n\} \subset \Sigma$ is a sequence of points such that $k_2/\eta \to \vartheta < +\infty$, then, after passing to a subsequence,*

$$\frac{\eta}{k_1} \to 0, \quad \mu \to +\infty, \quad \eta \to 0 \quad \textit{and} \quad \frac{k_2}{k_1} \to 0.$$

Proof. By taking $\Sigma_n = \Sigma - p_n$, we can argue as in the first part of Claim 5.7 to prove that after passing to a subsequence, $\mu \to +\infty$. Since $(k_1 + k_2)/\eta = -\dot{\varphi}$, we have that $\eta/k_1 \to 0$ and Claim 5.7 gives that, after passing to subsequence, $\eta \to 0$. Finally,

$$\frac{k_1}{k_2} = \frac{H}{k_2} - 1 = -\dot{\varphi}\,\frac{\eta}{k_2} \to -\infty. \qquad \blacksquare$$

Claim 5.12. *The case $0 < \vartheta < +\infty$ is not possible.*

Proof. From Theorem 5.1 and Claim 5.6, there exists a sequence of points $\{q_n\} \subset \Omega^+$, $|q_n| \to +\infty$, such that

$$\frac{k_2}{\eta}(q_n) \to \vartheta, \quad \left|\nabla\left(\frac{k_2}{\eta}\right)\right|(q_n) \to 0 \quad \text{and} \quad \Delta^{\varphi}\left(\frac{k_2}{\eta}\right)(q_n) \leq 0. \tag{5.33}$$

By an straightforward computation we obtain

$$\begin{aligned}
\frac{\eta^2}{k_1 k_2^2}\,\nabla k_2 &= \frac{\eta^3}{k_1 k_2^2}\,\nabla\left(\frac{k_2}{\eta}\right) + \frac{\eta}{k_1 k_2}\,\nabla\eta,\\
\frac{\eta^2}{k_1 k_2^2}\,\nabla k_1 &= -\frac{\eta^3}{k_1 k_2^2}\,\nabla\left(\frac{k_2}{\eta}\right) + \frac{\eta}{k_2^2}\,\nabla\eta - \frac{\eta^3\ddot{\varphi}}{k_1 k_2}\,\nabla\mu,
\end{aligned}$$

and using Claim 5.11 and (5.33),

$$\frac{\eta^2}{k_1 k_2^2}\,h_{22,1} \to \frac{1}{\vartheta}\,\langle \vec{e}_3, v_1\rangle \quad \text{and} \quad \frac{\eta^2}{k_1 k_2^2}\,h_{11,2} \to \frac{1}{\vartheta}\,\langle \vec{e}_3, v_2\rangle,$$

which gives

$$\frac{\eta^4}{k_1^2 k_2^4}\,Q^2 = \frac{\eta^4}{k_1^2 k_2^4}\,(h_{11,2}^2 + h_{22,1}^2) \to \frac{1}{\vartheta^2} > 0. \tag{5.34}$$

As the equation (5.7) holds on Ω^+, multiplying by $\frac{\eta^3}{k_1 k_2^2}$ we get that

$$\begin{aligned}
&\frac{\eta^3}{k_1 k_2^2}\,\Delta^{\varphi}\left(\frac{k_2}{\eta}\right) + 2\,\frac{\eta^3}{k_1 k_2^2}\left\langle \nabla\left(\frac{k_2}{\eta}\right), \frac{\nabla\eta}{\eta}\right\rangle\\
&= -\ddot{\varphi}\,\frac{\eta^3}{k_1 k_2^2}\langle\nabla\mu, v_2\rangle^2 + \ddot{\varphi}\,\frac{\eta^3}{k_1 k_2^2}\left(\frac{k_2}{\eta}\right)(1 + 2\langle\nabla\mu, v_2\rangle^2) - 2\,\frac{\eta^3}{k_1 k_2^2}\,\frac{1}{\eta}\,\frac{Q^2}{k_1 - k_2}\cdot \qquad \blacksquare
\end{aligned}$$

Claim 5.13. *The case $\vartheta = +\infty$ is not possible.*

Proof. Assume by contradiction that $\vartheta = +\infty$. Let $g\colon \mathbb{R} \to]-1,1[$ be a bounded smooth function satisfying

$$\dot{g} \geq 0 \quad \text{on } \mathbb{R}, \quad g(x) = 1 - \frac{1}{x} \quad \text{on } [1, +\infty[. \tag{5.35}$$

Let $h\colon \Sigma \to \mathbb{R}$ be the function $h(p) = g\big(\frac{k_2}{\eta}(p)\big)$. Using (5.7), a straightforward computation provides

$$\begin{aligned} \Delta^{\varphi} h + 2\Big\langle \nabla h, \frac{\nabla \eta}{\eta}\Big\rangle &= \ddot{g}\left|\nabla\Big(\frac{k_2}{\eta}\Big)\right|^2 - \dot{g}\,\ddot{\varphi}\langle \nabla\mu, v_2\rangle^2 \\ &\quad + \dot{g}\,\ddot{\varphi}\Big(\frac{k_2}{\eta}\Big)(1 + 2\langle\nabla\mu, v_2\rangle^2) - 2\,\frac{\dot{g}}{\eta}\,\frac{Q^2}{k_1 - k_2}. \end{aligned} \tag{5.36}$$

Since $\vartheta = +\infty$, it is clear that

$$\sup_{\Sigma}\{h\} = 1, \tag{5.37}$$

and it is not attained on Σ. Now, from Lemma 5.5 we can apply the Theorem 5.1 so that there exists a divergent sequence $\{q_n\}$ such that

$$h \to 1, \quad |\nabla h| \to 0 \quad \text{and} \quad \Delta^{\varphi} h(q_n) \leq 0. \tag{5.38}$$

Thus, $k_2/\eta \to +\infty$, $\eta/k_1 \to 0$ and, from Claim 5.7, after passing to a subsequence we have also that $\mu \to +\infty$ and $\eta \to 0$. Now, we can argue as in Claim 5.12 to get that

$$\frac{\eta}{k_1 k_2} h_{22,1} \to \langle \vec{e}_3, v_1\rangle \quad \text{and} \quad \frac{\eta}{k_1 k_2} h_{11,2} \to \langle \vec{e}_3, v_2\rangle,$$

which gives

$$\frac{\eta^2}{k_1^2 k_2^2}\, Q^2 = \frac{\eta^2}{k_1^2 k_2^2}\,(h_{11,2}^2 + h_{22,1}^2) \to 1 > 0. \tag{5.39}$$

Using that $k_2/k_1 \in]-1, 0[$ in Ω^+ and that for n large enough,

$$\dot{g}\Big(\frac{k_2}{\eta}(p_n)\Big) = \frac{\eta^2}{k_2^2}(p_n) \quad \text{and} \quad \ddot{g}\Big(\frac{k_2}{\eta}(p_n)\Big) = -2\,\frac{\eta^3}{k_2^3}(p_n),$$

if we multiply by ηk_1 in the expression (5.36) take limit when $n \to +\infty$, we get

$$0 \leq -\frac{2}{1 - C} < 0,$$

where $k_2/k_1 \to C \in [-1, 0]$, which is a contradiction. ∎

From the above claims, the only possibility is that $\vartheta \leq 0$, which concludes the proof of Theorem B. ∎

From Theorem 4.13, Theorem B and arguing as in Corollary 2.3 of [10], we may obtain the following.

Corollary 5.14. *Let Σ be as in Theorem B with ΛK bounded from below. If $\{p_n\}$ is any divergent sequence in Σ and $\{\lambda_n\} \to +\infty$ any sequence such that $\{\dot{\varphi}(\mu(p_n))/\lambda_n\} \to C$ for some constant $C > 0$, then $\Sigma_n = \lambda_n(\Sigma - p_n)$ converge smoothly (after passing to a subsequence) to a vertical plane, a grim reaper surface, or a titled grim reaper surface.*

Moreover, from Theorem 3.4 and Theorem B, we have:

Corollary 5.15. *Let Σ be as in Theorem B with ΛK bounded from below. If K vanishes anywhere, then Σ has vanishing curvature.*

Some interesting questions. We conclude this paper with two questions related to our Theorem B. The first one is whether an entire $[\varphi, \vec{e}_3]$-minimal vertical graph in $\mathbb{R}^3$ with φ satisfying (1.2) and (1.4) is convex. The second one is whether an entire $[\varphi, \vec{e}_3]$-minimal vertical graph in $\mathbb{R}^3$ with $H(p) \to 0$ as $|p| \to \infty$ and φ satisfying (1.2) and (1.4) is rotationally symmetric. We expect affirmative answers to both questions.

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