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Uniform Sobolev estimates on compact manifolds involving singular potentials

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Abstract. We obtain generalizations of the uniform Sobolev inequalities of Kenig, Ruiz and the fourth author (1986) for Euclidean spaces and Dos Santos Ferreira, Kenig and Salo (2014) for compact Riemannian manifolds involving critically singular potentials $V \in L^{n/2}$. We also obtain the analogous improved quasimode estimates of the first, third and fourth author (2021), Hassell and Tacy (2015), the first and fourth author (2019), and Hickman (2020), as well as analogues of the improved uniform Sobolev estimates of Bourgain, Shao, the fourth author and Yao (2015), and Hickman (2020), involving such potentials. Additionally, on S^n, we obtain sharp uniform Sobolev inequalities involving such potentials for the optimal range of exponents, which extend the results of S. Huang and the fourth author (2014). For general Riemannian manifolds, we improve the earlier results in of the first, third and fourth authors (2021) by obtaining quasimode estimates for a larger (and optimal) range of exponents under the weaker assumption that $V \in L^{n/2}$.

1. Introduction and main results

The main purpose of this paper is to extend the uniform Sobolev inequalities on compact Riemannian manifolds (M, g) of [9], [10] and [24] to include Schrödinger operators,

$$H_V = -\Delta_g + V(x), \tag{1.1}$$

with critically singular potentials V, which are always assumed to be real-valued. For the most part, we shall merely assume that

$$V \in L^{n/2}(M). \tag{1.2}$$

In an earlier work of three of the authors [3], in addition to (1.2), it was assumed that $V \in \mathcal{K}$, where $\mathcal{K}$ is the Kato class (see Section 2). The spaces $L^{n/2}$ and $\mathcal{K}$ have the same scaling properties, and both obey the scaling law of the Laplacian, which accounts for their criticality. As was shown in [3], the condition that V is a Kato potential is necessary to obtain quasimode estimates for $q = \infty$. On the other hand, for the exponents arising

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in uniform Sobolev assumptions we merely need to assume (1.2). There is also recent related work of the second and fourth author [16] and Frank and Sabin [11] involving the Weyl counting problem for Kato potentials. Using the uniform Sobolev estimates that we shall prove, we shall easily be able to obtain L^q quasimode estimates for the optimal range of exponents (1.9), and if we assume, in addition to (1.2), that $V_- = \max\{0, -V\}$ is in the Kato space $\mathcal{K}(M)$, we shall also be able to prove quasimode estimates for larger exponents. In an earlier work, the stronger assumption that $V \in \mathcal{K}(M)$ was used to obtain results for large exponents.

As we shall show in the appendix, if we assume (1.2), then H_V is essentially self-adjoint and bounded from below with discrete spectrum, Spec H_V. After adding a constant to V, we may, without loss of generality assume, as we shall throughout, that

$$0 \in \operatorname{Spec} H_V \quad \text{and} \quad \operatorname{Spec} H_V \subset \mathbb{R}_+ = [0, \infty). \tag{1.3}$$

In order to prove these uniform Sobolev estimates, we shall use the following generalized second resolvent formula, which holds for all $n \geq 3$ if V satisfies (1.2):

$$\begin{aligned}(-\Delta_g + V - \zeta)^{-1} &- (-\Delta_g - \zeta)^{-1} \\ &= -[|V|^{1/2}(-\Delta_g - \overline{\zeta})^{-1}]^* \circ [V^{1/2}(-\Delta_g + V - \zeta)^{-1}], \quad \operatorname{Im}\zeta \neq 0,\end{aligned} \tag{1.4}$$

along with quasimode estimates and uniform Sobolev estimates for the unperturbed operator $H_0 = -\Delta_g$ from [9], [10], [24] and [27]. Here $V^{1/2} = (\operatorname{sgn} V)|V|^{1/2}$, and $[\cdot]$ denotes the (unique) bounded extension to the whole space. The resolvent formula (1.4) also holds for a more general class of potentials; see, e.g., [19] and [21] for more details.

We shall also mention that, for $n \geq 5$, we have the following simpler form of the second resolvent formula:

$$(-\Delta_g + V - \zeta)^{-1} - (-\Delta_g - \zeta)^{-1} = -(-\Delta_g - \zeta)^{-1} V(-\Delta_g + V - \zeta)^{-1}, \tag{1.5}$$

since, as we shall show in the appendix, for these dimensions, the operator domains of $H_V - \zeta$ and $-\Delta_g - \zeta$ coincide if $\operatorname{Im}\zeta \neq 0$.

The universal uniform Sobolev estimates and quasimode estimates that we can obtain are the following.

Theorem 1.1. *Let $n \geq 3$ and suppose that*

$$\min(q, p(q)') \geq \tfrac{2(n+1)}{n-1} \quad \text{and} \quad \tfrac{1}{p(q)} - \tfrac{1}{q} = \tfrac{2}{n}. \tag{1.6}$$

Then if $V \in L^{n/2}(M)$ satisfies (1.3) and $\delta > 0$ is fixed, we have the uniform bounds

$$\|u\|_q \leq C_V \|(H_V - \zeta)u\|_{p(q)} \quad \text{if } \zeta \in \Omega_\delta, \tag{1.7}$$

where

$$\Omega_\delta = \{\zeta \in \mathbb{C} : (\operatorname{Im}\zeta)^2 \geq \delta|\operatorname{Re}\zeta| \text{ if } \operatorname{Re}\zeta \geq 1, \text{ and } \operatorname{dist}(\zeta, \mathbb{R}_+) \geq \delta \text{ if } \operatorname{Re}\zeta < 1\}. \tag{1.8}$$

Also, suppose that

$$2 < q \leq \tfrac{2n}{n-4} \text{ if } n \geq 5 \quad \text{or} \quad 2 < q < \infty \text{ if } n = 3, 4. \tag{1.9}$$

Then if $u \in \mathrm{Dom}(H_V)$*, we have*

$$\|u\|_q \lesssim \lambda^{\sigma(q)-1}\|(H_V - \lambda^2 + i\lambda)u\|_2 \quad \text{if } \lambda \geq 1, \tag{1.10}$$

where

$$\sigma(q) = \begin{cases} n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}, & q \geq \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{q}), & 2 \leq q < \frac{2(n+1)}{n-1}. \end{cases} \tag{1.11}$$

Here, $\mathrm{Dom}(H_V)$ denotes the domain of H_V. Also, r' denotes the conjugate exponent for r, i.e., the one satisfying $1/r + 1/r' = 1$. Additionally, we are using the notation $A \lesssim B$, which means that A is bounded from above by a constant times B. The implicit constant might depend on the parameters involved, such as (M, g), q and V in (1.10).

The range of exponents in (1.6) for the uniform Sobolev estimates (1.7) is more restrictive than the corresponding estimates for $\mathbb{R}^n$ in [20], since we require certain $L^2 \to L^r$ quasimode estimates from [27] for both $r = q$ and $r = p(q)'$, which are only valid when the first part of (1.6) holds. Succinctly put, our proof of (1.7) requires that we use the manifold version of the Stein–Tomas extension theorem [34], which is only valid when this condition holds (see [29] for more details).

The condition in the uniform Sobolev inequalities for $\mathbb{R}^n$ in [20] is to replace (1.6) with the weaker requirement that

$$\min(q, p(q)') > \frac{2n}{n-1} \quad \text{and} \quad \frac{1}{p(q)} - \frac{1}{q} = \frac{2}{n}, \tag{1.12}$$

which was shown to be sharp in [20]. The gap condition in (1.6) and (1.12), that is, $1/p(q) - 1/q = 2/n$, follows from scaling considerations, while the necessity of the first part of (1.12) is related to the fact that the Fourier transform of the surface measure on the sphere in $\mathbb{R}^n$ is not in $L^q(\mathbb{R}^n)$ if $q \leq \frac{2n}{n-1}$.

Even though the range of exponents for the uniform Sobolev estimates above might be non-optimal, the ones in (1.9) for the quasimode estimates (1.10) are best possible. For $n \geq 4$, this is due to a counterexample for the case $V \equiv 0$ in [30] (see also [31]), and for $n = 3$, it follows from a counterexample in Section 1 of [3], involving a nontrivial $L^{n/2}$ potential. It was a bit surprising to us that, even though the range of exponents for the uniform Sobolev estimates (1.7) might be a bit restrictive, we can use them along with their proof to obtain quasimode bounds as in (1.10) for the optimal range of exponents.

In an earlier work [3], bounds of the form (1.10) were only obtained for the smaller range where $q < \frac{2n}{n-3}$. Moreover, the bounds (1.10) also improve the earlier ones, since we are only assuming that $V \in L^{n/2}(M)$ and not that V is a Kato potential, i.e., $V \in \mathcal{K}(M)$.

As we mentioned before, if in addition to (1.2), we also assume that the negative part of V satisfies $V_- \in \mathcal{K}(M)$, then we can also obtain the (modified) quasimode estimates in (1.10) and the related spectral projection estimates for larger exponents. See the end of Section 2.

We would also like to note that, by using the quasimode estimates (1.10) in Theorem 1.1, we can obtain, as a corollary, Sobolev estimates for H_V in higher dimensions, which appear to be new since they only involve the assumption $V \in L^{n/2}(M)$ under which favorable heat kernel estimates need not be valid (see Aizenman and Simon [1] and Simon [25]).

Corollary 1.2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$ and assume that H_V is as above with $V \in L^{n/2}(M)$. Then*

$$\|(H_V+1)^{-\alpha/2} f\|_{L^q(M)} \lesssim \|f\|_{L^p(M)}, \tag{1.13}$$

provided that

$$n(\tfrac{1}{p}-\tfrac{1}{q}) = \alpha \quad \text{and} \quad \tfrac{2n}{n+4} \leq p \leq 2 \leq q \leq \tfrac{2n}{n-4}. \tag{1.14}$$

The proof is simple. Since we are assuming (1.3), we obtain from the spectral theorem and the special case of (1.10) with $\lambda = 1$ that $(H_V+1)^{-1}\colon L^2(M) \to L^{\frac{2n}{n-4}}(M)$, and, by duality, it also maps $L^{\frac{2n}{n+4}}(M) \to L^2(M)$. By applying Stein's interpolation theorem, the spectral theorem and the trivial L^2 bounds, we deduce that $(H_V+1)^{-\alpha/2}\colon L^2(M) \to L^q(M)$ for $2 \leq q \leq \frac{2n}{n-4}$, with $\alpha = n(1/2-1/q)$, and also $(H_V+1)^{-\alpha/2}\colon L^p(M) \to L^2(M)$ for $\frac{2n}{n+4} \leq p \leq 2$, with $\alpha = n(1/p-1/2)$. Since these two facts yield the desired $L^p(M) \to L^q(M)$ bounds for $(H_V+1)^{-\alpha/2}$, the proof is complete.

As in [9], in certain geometries, we can obtain improved uniform Sobolev estimates and quasimode estimates using improved bounds for the unperturbed operator H_0.

First, if we use the improved spectral projection estimates of Hassell and Tacy [12] and two of us [4], we can obtain the following.

Theorem 1.3. *Let $n \geq 3$ and suppose that*

$$\min(q, p(q)') > \tfrac{2(n+1)}{n-1} \quad \text{and} \quad \tfrac{1}{p(q)} - \tfrac{1}{q} = \tfrac{2}{n}. \tag{1.15}$$

Assume also that (M,g) has nonpositive sectional curvatures, $V \in L^{n/2}(M)$ satisfies (1.3) and that $\delta > 0$ is fixed. Then we have

$$\|u\|_q \leq C\|(H_V-\zeta)u\|_{p(q)} \quad \text{if } \zeta \in \Omega_{\varepsilon,\delta}, \tag{1.16}$$

where

$$\begin{aligned}\Omega_{\varepsilon,\delta} = \big\{\zeta : (\operatorname{Im}\zeta)^2 \geq \delta(\varepsilon(\lambda))^2\,|\operatorname{Re}\zeta| \text{ if } \operatorname{Re}\zeta \geq 1,\\ \text{and } \operatorname{dist}(\zeta,\mathbb{R}_+) \geq \delta \text{ if } \operatorname{Re}\zeta < 1\big\},\end{aligned} \tag{1.17}$$

with

$$\varepsilon(\lambda) = (\log(2+\lambda))^{-1}. \tag{1.18}$$

Also, suppose that

$$\tfrac{2(n+1)}{n-1} < q \leq \tfrac{2n}{n-4} \text{ if } n \geq 5 \quad \text{or} \quad \tfrac{2(n+1)}{n-1} < q < \infty \text{ if } n = 3,4. \tag{1.19}$$

Then, if $\sigma(q)$ is as in (1.11) and $u \in \operatorname{Dom}(H_V)$,

$$\|u\|_q \lesssim \big(\sqrt{\varepsilon(\lambda)}\big)^{-1}\lambda^{\sigma(q)-1}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)u\|_2 \quad \text{if } \lambda \geq 1. \tag{1.20}$$

Finally, if $q = q_c = \frac{2(n+1)}{n-1}$, we have for some $\delta_n > 0$ depending on the dimension,

$$\|u\|_{q_c} \lesssim \lambda^{\sigma(q_c)-1}(\varepsilon(\lambda))^{-1+\delta_n}\,\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2 \tag{1.21}$$

The quasimode estimates (1.20) improve those in [3] in several ways. First, as noted before, we are not assuming that V is a Kato potential, only (1.2). Moreover, unlike [3], we also do not have to assume that V has small $L^{n/2}$-norm. We also obtain the bounds in (1.20) for the optimal range of exponents given by (1.9), and the bounds (1.21) for the critical exponent $q = q_c$ are new. We have only stated the bounds of the form (1.21) for $q = q_c$; however, if one interpolates with the trivial L^2 estimate, one sees that bounds of the form (1.21) also hold for all $q \in (2, q_c)$ if δ_n is replaced with the appropriate $\delta_{n,q} > 0$.

As we noted after Theorem 1.1, we also can obtain quasimode bounds for exponents larger than the ones in (1.20) if we assume that $V_- \in \mathcal{K}(M)$, and in this case too, we can drop the smallness assumption that was used in [3].

By results in [31], the bounds in (1.20) are equivalent to the following spectral projection bounds

$$\|\chi^V_{[\lambda,\lambda+(\log\lambda)^{-1}]}\|_{L^2(M)\to L^q(M)} \lesssim (\log\lambda)^{-1/2}\lambda^{\sigma(q)}, \quad \lambda \ge 1, \tag{1.22}$$

for q as in (1.19), where $\chi^V_{[\lambda,\lambda+(\log 2+\lambda)^{-1}]}$ denotes the spectral projection operator which projects onto the part of the spectrum of $\sqrt{H_V}$ in the corresponding shrinking intervals $[\lambda, \lambda + (2 + \log\lambda)^{-1}]$. If, in addition to (1.2), we also assume that V is in the Kato class, then we also have (1.22), as in the case $V \equiv 0$ in Hassell and Tacy [12] for all $p > \frac{2(n+1)}{n-1}$. The bounds in (1.21) extend the log-improvements of two of us [4] to include singular potentials as above. Just as was the case for (1.22), the quasimode estimates in (1.21) yield the equivalent log-improved spectral projection estimates

$$\|\chi^V_{[\lambda,\lambda+(\log(2+\lambda))^{-1}]} f\|_{q_c} \lesssim \lambda^{\sigma(q_c)}(\log(2+\lambda))^{-\delta_n}\|f\|_2. \tag{1.23}$$

Additionally, in Section 5, we shall obtain quasimode estimates of the form (1.20) and (1.21) when $n = 2$; however, as in [3] (which handled small potentials), in this case we shall have to assume that $V \in L^1(M) \cap \mathcal{K}(M)$. We improve the corresponding results in [3], though, by dropping the smallness assumption on V.

As was shown in Hickman [13] in higher dimensions, and in Bourgain, Shao, Sogge and Yao [9] for $n = 3$, one can use the decoupling theorem of Bourgain and Demeter [6] to obtain substantial improvements of (1.22) when $M = \mathbb{T}^n$ is the torus, which correspond to taking $\varepsilon(\lambda) = \lambda^{-1/3+c}$ for all $c > 0$. Using these improved quasimode estimates, we can prove the corresponding stronger version of Theorem 1.3 for tori.

Theorem 1.4. *Let $n \ge 3$ and assume that $p(q)$ and q are as in (1.12). Then, for $V \in L^{n/2}(\mathbb{T}^n)$ satisfying (1.3), and $\delta > 0$ and $c_0 > 0$ fixed, we have*

$$\|u\|_{L^q(\mathbb{T}^n)} \le C\|(H_V - \zeta)u\|_{L^{p(q)}(\mathbb{T}^n)} \quad \text{if } \zeta \in \Omega_{\varepsilon,\delta}, \tag{1.24}$$

where $\Omega_{\varepsilon,\delta}$ is as in (1.17), with

$$\varepsilon(\lambda) = \begin{cases} \lambda^{-\beta_1(n,p(q)')+c_0}, & \frac{2n}{n-1} < q < \frac{2n}{n-2}, \\ \lambda^{-\beta_1(n,q)+c_0}, & \frac{2n}{n-2} \le q < \frac{2n}{n-3}, \end{cases} \tag{1.25}$$

for certain $\beta_1(n,r) > 0$ and $p(q)'$ such that $1/p(q)' + 1/p(q) = 1$. Also, suppose that

$$\varepsilon(\lambda) = \begin{cases} \lambda^{-\beta(n,q)+c_0}, & \frac{2(n+1)}{n-1} < q < \frac{2n}{n-2}, \\ \lambda^{-1/3+c_0}, & \frac{2n}{n-2} \le q \le \frac{2n}{n-4} \text{ if } n \ge 5, \text{ or } \frac{2n}{n-2} \le q < \infty \text{ if } n = 3, 4, \end{cases}$$

where

$$\beta(n,q) = \min\{\beta_1(n, p(q)'), \tfrac{(n-1)^2q-2(n-1)(n+1)}{(n+1)(n-1)q-2(n+1)^2+8}\}.$$

Then we have the analogue of (1.20) *on* $\mathbb{T}^n$ *for* q *satisfying* (1.19),

$$\|u\|_{L^q(\mathbb{T}^n)} \lesssim \big(\sqrt{\varepsilon(\lambda)}\big)^{-1}\lambda^{\sigma(q)-1}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)u\|_{L^2(\mathbb{T}^n)} \quad \text{if } \lambda\ge 1. \tag{1.26}$$

Additionally, for the critical point $q_c = \frac{2(n+1)}{n-1}$, *suppose that* $\varepsilon(\lambda)=\lambda^{-\beta_1(n,p(q_c)')+c_0}$ *which satisfies* (1.25), *or more explicitly*

$$\varepsilon(\lambda)=\lambda^{-1/5+c_0} \ \text{if } n\ge 4 \quad \text{and} \quad \varepsilon(\lambda)=\lambda^{-3/16+c_0} \ \text{if } n=3, \tag{1.27}$$

we have, for $u\in \mathrm{Dom}(H_V)$,

$$\|u\|_{L^{q_c}(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}(\varepsilon(\lambda))^{-\frac{n+3}{2(n+1)}}\lambda^{-\frac{n+3}{2(n+1)}}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_{L^2(\mathbb{T}^n)}, \quad \lambda\ge 1. \tag{1.28}$$

We shall give the explicit definition of $\beta_1(n,q)$ later in (4.59). As we shall see, $\beta_1(n,q)$ is a number that decreases from $1/3$ to 0 when q increases from $\frac{2n}{n-2}$ to $\frac{2n}{n-3}$. Similarly, by an explicit calculation, $\beta(n,q)$ is a number that increases from 0 to $1/3$ when q increases from $\frac{2(n+1)}{n-1}$ to $\frac{2n}{n-2}$, in particular, when $q=\frac{2n}{n-2}$, $\beta_1(n,q)=\beta(n,q)=1/3$. As a result, (1.24) generalizes the uniform resolvent estimates of Hickman [13] to the setting of Schrödinger operators with $V\in L^{n/2}(\mathbb{T}^n)$, which also gives us certain uniform resolvent estimates on the torus for general pairs of exponents (p,q) satisfying (1.12). On the other hand, when $q=\frac{2n}{n-2}$, if we take u in (1.26) to be $\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]}f$, we have

$$\|\chi^V_{[\lambda,\lambda+\rho)]}f\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \le (\rho\lambda)^{1/2}\|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \delta_0>0,\ \rho\ge\lambda^{-1/3+\delta_0},$$

which generalizes the spectral projection estimates in [13] (and [9] for the $n=3$ case) to the setting of Schrödinger operators.

Theorems 1.3 and 1.4 represent an improvement in terms of the $\varepsilon(\lambda)$ defining $\Omega_{\varepsilon,\delta}$ as well as the parameter occurring in the quasimode estimates (1.22) over Theorem 1.1, which corresponds to $\varepsilon(\lambda)\equiv 1$.

For the sphere, no such improvement over the case $\varepsilon(\lambda)\approx 1$ is possible, since one cannot have $\varepsilon(\lambda)\to 0$ as $\lambda\to+\infty$ in this case (see [14] and [26]). Notwithstanding, for S^n, we can get an improvement over Theorems 1.3 and 1.4 for the uniform Sobolev estimates by obtaining bounds for the optimal range of exponents satisfying (1.12). This improvement is possible due to the fact that when $M=S^n$, uniform Sobolev estimates for H_0 are known for this range of exponents (see [14]).

Theorem 1.5. *Consider the standard sphere* S^n *for* $n\ge 3$ *and assume that* $V\in L^{n/2}(S^n)$. *If* (1.12) *is valid, we have*

$$\|u\|_q \le C\|(H_V-\zeta)u\|_{p(q)} \quad \text{if } \zeta\in\Omega_\delta, \tag{1.29}$$

where Ω_δ *is as in* (1.8). *Also, for* q *satisfying* (1.9), *if* $\sigma(q)$ *is as in* (1.11) *and* $u\in \mathrm{Dom}(H_V)$,

$$\|u\|_q \lesssim \lambda^{\sigma(q)-1}\|(H_V-\lambda^2+i\lambda)u\|_2 \quad \text{if } \lambda\ge 1. \tag{1.30}$$

It would be interesting to see if the uniform Sobolev bounds (1.29) are universally true or hold for generic Riemannian manifolds.

The study of Schrödinger operators can be found in a vast amount of papers in the literature, especially in the Euclidean case, see, e.g., [17], [18], [23]. In a companion paper [15], the second and fourth author will obtain related uniform Sobolev estimates for $\mathbb{R}^n$ which improve those in [3] and provide natural generalizations of those in [20].

2. Universal Sobolev inequalities on compact manifolds: Abstract universal bounds

The purpose of this section is to prove simple abstract theorems that will allow us to prove Theorems 1.1–1.5, and to also improve the quasimode estimates of [3] for the operators H_V, provided that we have the analogous improved estimates (quasimode and uniform Sobolev) for the unperturbed operators $H_0 = -\Delta_g$. Throughout this section we shall assume that $n \geq 3$, since we shall be using uniform Sobolev estimates for $-\Delta_g$ which break down in two-dimensions. We shall obtain improved quasimode estimates compared to those in [3] later by adapting the arguments here.

In this section we shall consider a pair of exponents (p, q) which are among those in the sharp range of exponents in the uniform Sobolev estimates in [20] for the Euclidean case, i.e., $1 < p < 2 < q < \infty$, and, moreover,

$$\tag{2.1} \tfrac{1}{p} - \tfrac{1}{q} = \tfrac{2}{n}, \quad \min(q, p') > \tfrac{2n}{n-1}.$$

For later use, observe that if the pair (p, q) is as in (2.1), then so is (q', p'). We also note that if (p, q) is as in (2.1), then $\frac{2n}{n-1} < q < \frac{2n}{n-3}$.

For both of the exponents in (2.1), we shall assume that we have improvements of the classical quasimode estimates of the fourth author [27] of the form

$$\tag{2.2} \|u\|_r \leq C\delta(\lambda, r)\,\lambda^{\sigma(r)-1}\,(\varepsilon(\lambda))^{-1}\,\|(-\Delta_g - \lambda^2 + i\varepsilon(\lambda)\lambda)u\|_2$$

for $r = q, p'$ and $\lambda \geq 1$, where $\sigma(r)$ is as in (1.11). The $\delta(\lambda, r)$ and $\varepsilon(\lambda)$ are assumed to be continuous functions of $\lambda \in [1, \infty)$. In practice they are nonpositive powers of λ or $\log(2 + \lambda)$.

In order to have improvements over the results in [27], for $\varepsilon(\lambda) \equiv 1$, we shall assume that

$$\tag{2.3} \varepsilon(\lambda) \searrow \quad \text{and} \quad \varepsilon(\lambda) \in [1/\lambda, 1], \quad \lambda \geq 1.$$

We make the assumption that $\varepsilon(\lambda) \geq 1/\lambda$, since on compact manifolds it is unreasonable to expect meaningful bounds of the form (2.2) when $\varepsilon(\lambda)$ is smaller than the associated wavelength $1/\lambda$ with λ large. The estimates in [27] and the spectral theorem imply that (2.2) is valid when $\delta(\lambda, r) \equiv 1$, and so we shall also assume that

$$\tag{2.4} (\varepsilon(\lambda))^{1/2} \leq \delta(\lambda, r) \leq 1 \quad \text{and} \quad \delta(\lambda, r) \searrow, \quad \lambda \geq 1.$$

We assume that $\delta(\lambda, r) \geq (\varepsilon(\lambda))^{1/2}$, since, by (5.1.12) and (5.1.13) in [29], (2.2) cannot hold if $(\varepsilon(\lambda))^{1/2}/\delta(\lambda, r) \to \infty$ as $\lambda \to +\infty$.

Note that (1.20) corresponds to the "critical case", where $\delta(\lambda, r) = (\varepsilon(\lambda))^{1/2}$ for $\varepsilon(\lambda)$ as in (1.18) in the case of manifolds of nonpositive curvature, as do the results of [9] for $n = 3$ and [13] for $n \geq 4$ with a more favorable numerology on tori.

Although a bit more cryptic at first, it is also natural to assume that

$$\limsup_{\lambda\to\infty} \lambda^{\sigma(q)+\sigma(p')-2}(\varepsilon(\lambda))^{-2}\,\delta(\lambda, q)\,\delta(\lambda, p') = 0. \tag{2.5}$$

This condition arises naturally in the proofs, and one can check that, for the exponents in (2.1), it holds for the special case where $\varepsilon(\lambda) = \delta(\lambda, q) = \delta(\lambda, p') \equiv 1$, which will be a useful observation when we prove certain estimates on S^n. Also, by the first part of (2.4), we have (2.5) if

$$\limsup_{\lambda\to\infty} \lambda^{\sigma(q)+\sigma(p')-2}\,(\varepsilon(\lambda))^{-2} = 0, \tag{2.6}$$

which is a bit more palatable.

In addition to these quasimode estimates, we shall assume that we have the related uniform Sobolev estimates for the unperturbed operators:

$$\|u\|_q \leq C_{\delta_0}\|(-\Delta_g - \lambda^2 + i\mu\varepsilon(\lambda)\lambda))u\|_p \quad \text{when } \lambda \geq 1 \text{ and } |\mu| \geq \delta_0, \tag{2.7}$$

if $\delta_0 > 0$. Here and in what follows, $\mu \in \mathbb{R}$. Similar to the remark after (2.1), observe that if (p, q) are exponents for which (2.7) is valid, then, by duality, this is also true for the pair (q', p').

The abstract theorem that will allow us to prove Theorems 1.1–1.5 is the following.

Theorem 2.1. *Assume (M, g) is a compact Riemannian manifold of dimension $n \geq 3$. Assume further that (p, q) is a pair of exponents satisfying (2.1). Suppose also that (2.2), (2.5) and (2.7) are valid, with $\varepsilon(\lambda)$ and $\delta(\lambda, r)$, satisfying (2.3) and (2.4), respectively, with $r = p', q$ in the latter. Then, if $V \in L^{n/2}(M)$, we have*

$$\|u\|_q \leq C\|(-\Delta_g + V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)u\|_p \quad \text{if } |\mu| \geq 1 \text{ and } \lambda \geq \Lambda, \tag{2.8}$$

assuming that $\Lambda = \Lambda(M, q, V) \geq 1$ sufficiently large.

The assumption that λ in (2.8) is large arises for technical reasons from the fact that since we only are assuming that $V \in L^{n/2}$, we only know via (A.7) in the appendix that $u \in L^q(M)$ for $q \leq \frac{2n}{n-2}$ if $u \in \mathrm{Dom}(H_V)$. On the other hand, after proving Theorem 2.1, we can use its proof to establish the following much more favorable results.

Corollary 2.2. *Assume the hypotheses in Theorem 2.1. Then, for $u \in \mathrm{Dom}\, H_V$,*

$$\|u\|_r \leq C_{V,r}\,\delta(\lambda, r)\,\lambda^{\sigma(r)-1}(\varepsilon(\lambda))^{-1}\|(-\Delta_g + V - \lambda^2 + i\varepsilon(\lambda)\lambda)u\|_2 \tag{2.9}$$

if $\lambda \geq 1$ and $r = q$ or $r = p'$. Additionally,

$$\|u\|_r \leq C_{\delta,V,r}\|(-\Delta_g + V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda))u\|_s \quad \text{when } \lambda \geq 1 \text{ and } |\mu| \geq \delta_0, \tag{2.10}$$

if $\delta_0 > 0$ and $(r, s) = (q, p)$ or (p', q').

To prove these results we shall appeal to the following simple lemma.

Lemma 2.3. *Assume that $n \geq 3$. Let (p,q) be as in (2.1) and $W \in L^n(M)$. Then, if (2.7) is valid,*

$$\big\|\big[W(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}\big]^*\big\|_{L^{\bar{p}}(M)\to L^q(M)} \leq C_{\delta_0}\|W\|_{L^n(M)}, \tag{2.11}$$

with $1/\bar{p} = 1/p - 1/n$, and, if (2.2) is valid for $r = s'$,

$$\begin{aligned}\|[W(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}]^*\|_{L^{\bar{s}}(M)\to L^2(M)}& \\ \leq C\|W\|_{L^n(M)}\delta(\lambda,s')\lambda^{\sigma(s')-1}(\varepsilon(\lambda))^{-1}&,\end{aligned} \tag{2.12}$$

with $1/\bar{s} = 1/s - 1/n$. Finally, if (2.2) is valid for $r = q$ and if $W \in L^\infty(M)$,

$$\begin{aligned}\|[W(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}]^*\|_{L^2(M)\to L^q(M)}& \\ \leq C\|W\|_{L^\infty(M)}\delta(\lambda,q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}&.\end{aligned} \tag{2.13}$$

Proof. Note that, since we are assuming $n \geq 3$, the operators in (2.11)–(2.13) are bounded on $L^2(M)$, by duality, Hölder's inequality and Sobolev estimates.

Also, by duality, (2.11) is a consequence of the following:

$$\|W(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}h\|_{L^{(\bar{p})'}(M)} \leq C_{\delta_0}\|W\|_{L^n(M)}\|h\|_{L^{q'}(M)}. \tag{2.14}$$

To prove this, we first observe that

$$1/(\bar{p})' = 1 - 1/\bar{p} = 1 - 1/p + 1/n = 1/p' + 1/n.$$

Thus, by Hölder's inequality and the dual version of (2.7), we have

$$\begin{aligned}\|W(-\Delta_g-\lambda^2-i\mu\varepsilon(\lambda)\lambda)^{-1}h\|_{L^{(\bar{p})'}(M)} &\leq \|W\|_{L^n(M)}\|(-\Delta_g-\lambda^2-i\mu\varepsilon(\lambda)\lambda)^{-1}h\|_{L^{p'}(M)} \\ &\leq C_{\delta_0}\|W\|_{L^n(M)}\|h\|_{L^{q'}(M)},\end{aligned}$$

as desired.

This argument also yields (2.12). One obtains the dual version of (2.12) by applying (2.2) and Hölder's inequality.

Similarly, (2.13) is equivalent to

$$\|W(-\Delta_g-\lambda^2-i\mu\varepsilon(\lambda)\lambda)^{-1}h\|_{L^2(M)} \leq C\|W\|_{L^\infty(M)}\delta(\lambda,q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|h\|_{L^{q'}(M)}.$$

This follows immediately from the dual version of (2.2). ∎

Proof of Theorem 2.1. Let us first note that proving (2.8) is equivalent to showing that

$$\|(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1}\|_{L^p\to L^q} \lesssim 1 \quad \text{if } \lambda \geq \Lambda \text{ and } |\mu| \geq 1,$$

with Λ sufficiently large and (p,q) as in (2.1). By duality, it suffices prove this inequality when

$$\tfrac{2n}{n-1} < q \leq \tfrac{2n}{n-2}. \tag{2.15}$$

Thus, our task is to show that

$$\|(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q(M)} \leq C\|f\|_{L^p(M)} \quad \text{if } \lambda \geq \Lambda \text{ and } |\mu| \geq 1, \tag{2.16}$$

with (p,q) satisfying (2.1) and (2.15). As in Theorem 2.1, we are also assuming that (2.2) and (2.7) are valid for this pair of exponents.

We are assuming (2.15), since, by (A.7) in the appendix, we have

$$u \in L^q(M), \quad 2 \le q \le \tfrac{2n}{n-2} \quad \text{if } (H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)u \in L^2.$$

Thus, for q as in (2.15),

$$\|(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\|_{L^q(M)} < \infty \quad \text{if } f \in L^2(M). \tag{2.17}$$

In proving (2.16), since L^2 is dense in L^p, we may and shall assume that $f \in L^2(M)$ to be able to use (2.17) to justify a bootstrapping argument that follows.

The bootstrapping argument shall also exploit the simple fact that if we let

$$V_{\le N}(x) = \begin{cases} V(x) & \text{if } |V(x)| \le N, \\ 0 & \text{otherwise,} \end{cases} \tag{2.18}$$

then, of course,

$$\|V_{\le N}\|_{L^\infty} \le N, \tag{2.19}$$

and, if $V_{>N}(x) = V(x) - V_{\le N}(x)$,

$$\|V_{>N}\|_{L^{n/2}(M)} \le \delta(N), \quad \text{with } \delta(N) \searrow 0 \text{ as } N \to \infty, \tag{2.20}$$

since we are assuming that $V \in L^{n/2}(M)$.

To exploit this, we use the second resolvent formula (1.4) to write

$$\begin{aligned}(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f &= (-\Delta_g - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f \\ &- \big[|V_{>N_1}|^{1/2}(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}\big]^*\big(V^{1/2}\cdot(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\big) \\ &- \big[|V_{\le N_1}|^{1/2}(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}\big]^*\big((V_{>N_2})^{1/2}\cdot(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\big) \\ &- \big[|V_{\le N_1}|^{1/2}(-\Delta_g - \lambda^2 - i\mu\varepsilon(\lambda)\lambda)^{-1}\big]^*\big((V_{\le N_2})^{1/2}\cdot(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\big) \\ &= \mathrm{I} - \mathrm{II} - \mathrm{III} - \mathrm{IV}.\end{aligned} \tag{2.21}$$

Here and for the remainder of the proof of Theorem 2.1, we are assuming that

$$|\mu| \ge 1.$$

We shall not appeal to our assumption that λ is large until the end of the proof.

By the uniform Sobolev estimates (2.7) for the unperturbed operator, we have

$$\|\mathrm{I}\|_q \le C\|f\|_p. \tag{2.22}$$

Also, by (2.11) and Hölder's inequality

$$\begin{aligned}\|\mathrm{II}\|_q &\le C\||V_{>N_1}|^{1/2}\|_{L^n}\,\|V^{1/2}\cdot(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\|_{\overline{p}} \\ &\le C\|V_{>N_1}\|_{L^{n/2}}^{1/2}\cdot\|V\|_{L^{n/2}}^{1/2}\cdot\|(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\|_{L^q},\end{aligned}$$

since $1/\overline{p} = 1/p - 1/n$. By (2.20), we can fix N_1 large enough so that

$$C\|V_{>N_1}\|_{L^{n/2}}^{1/2}\cdot\|V\|_{L^{n/2}}^{1/2} < \tfrac{1}{6},$$

yielding the bounds

$$\|\mathrm{II}\|_q < \tfrac{1}{6}\|(H_V - \lambda^2 + i\mu\varepsilon(\lambda)\lambda)^{-1} f\|_q. \tag{2.23}$$

Similarly,

$$\begin{aligned}\|\mathrm{III}\|_q &\le C\|V_{\le N_1}\|_{L^{n/2}}^{1/2}\,\|(V_{>N_2})^{1/2}\cdot(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{\overline{p}}\\ &\le C\|V\|_{L^{n/2}}^{1/2}\cdot\|V_{>N_2}\|_{L^{n/2}}^{1/2}\cdot\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q}.\end{aligned}$$

By (2.20), we can fix N_2 large enough so that $C\|V\|_{L^{n/2}}^{1/2}\cdot\|V_{>N_2}\|_{L^{n/2}}^{1/2}<1/6$, which implies

$$\|\mathrm{III}\|_q<\tfrac{1}{6}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_q. \tag{2.24}$$

It remains to estimate the norm of IV in (2.21). We first note that, by (2.13),

$$\begin{aligned}\|\mathrm{IV}\|_q &\le CN_1^{1/2}\delta(\lambda,q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\\ &\quad\times\|(V_{\le N_2})^{1/2}\cdot(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_2\\ &\le CN_1^{1/2}N_2^{1/2}\delta(\lambda,q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_2.\end{aligned} \tag{2.25}$$

We can estimate the last factor by appealing to the second resolvent formula one more time. Here there is no need to split the potential, and, instead, we write

$$\begin{aligned}(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f &= (-\Delta_g-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\\ &\quad-[|V|^{1/2}(-\Delta_g-\lambda^2-i\mu\varepsilon(\lambda)\lambda)^{-1}]^*(V^{1/2}\cdot(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f)\\ &= A-B.\end{aligned} \tag{2.26}$$

By the dual version of (2.2) with $r=p'$, we have

$$\|A\|_2\le C\delta(\lambda,p')\lambda^{\sigma(p')-1}(\varepsilon(\lambda))^{-1}\|f\|_p. \tag{2.27}$$

Also, if $1/\overline{p}=1/p-1/n$, then, by (2.12) and Hölder's inequality,

$$\begin{aligned}\|B\|_2 &\le C\delta(\lambda,p')\lambda^{\sigma(p')-1}(\varepsilon(\lambda))^{-1}\|V\|_{L^{n/2}}^{1/2}\,\|V^{1/2}(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{\overline{p}}\\ &\le C\delta(\lambda,p')\lambda^{\sigma(p')-1}(\varepsilon(\lambda))^{-1}\|V\|_{L^{n/2}}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_q.\end{aligned} \tag{2.28}$$

If we combine (2.26), (2.27) and (2.28), and use (2.25), we conclude that

$$\begin{aligned}\|\mathrm{IV}\|_q &\le CN_1^{1/2}N_2^{1/2}\lambda^{\sigma(q)+\sigma(p')-2}(\varepsilon(\lambda))^{-2}\delta(\lambda,q)\delta(\lambda,p')\\ &\quad\times\big(\|f\|_p+\|V\|_{L^{n/2}}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_q\big)\\ &\le C\|f\|_p+\tfrac{1}{6}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_q,\end{aligned} \tag{2.29}$$

by (2.5), if $\lambda\ge\Lambda$, with Λ sufficiently large, since N_1 and N_2 have been fixed.

If we combine (2.22), (2.23), (2.24) and (2.29), we conclude that for $\lambda\ge\Lambda$, we have

$$\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q(M)}\le C\|f\|_{L^p(M)}+\tfrac{1}{2}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q(M)}.$$

By (2.17), this leads to (2.16), since we are assuming, as we may, that $f\in L^2(M)$. ∎

Remark. In dimensions $n \geq 5$, the arguments can be simplified a little bit, since, in these cases, we may appeal to the more straightforward second resolvent formula (1.5) instead of relying on (1.4) (as we must do for $n = 3, 4$). If we do so for $n \geq 5$, then we may replace (2.21) with a simpler variant

$$\begin{aligned}(H_V - \lambda^2 + i\mu\varepsilon(\lambda))^{-1} f &= (-\Delta_g - \lambda^2 + i\mu\varepsilon(\lambda))^{-1} f \\ &\quad - [(-\Delta_g - \lambda^2 + i\mu\varepsilon(\lambda))^{-1}](V_{>N} \cdot (H_V - \lambda^2 + i\mu\varepsilon(\lambda)) f) \\ &\quad - [(-\Delta_g - \lambda^2 + i\mu\varepsilon(\lambda))^{-1}](V_{\leq N} \cdot (H_V - \lambda^2 + i\mu\varepsilon(\lambda)) f).\end{aligned}$$

Then the arguments that were used to control II and III in (2.21) can easily be adapted to control the second and third terms, respectively, in the right-hand side of the above identity. As we alluded to earlier, we need to use the more complicated second resolvent formula (1.4) when $n = 3, 4$, due to the fact that the *form* domains (but not *operator* domains) of H_V and H_0 coincide in this case, while for $n \geq 5$, we may use (1.6), since, in these cases, the operator domains coincide.[1]

Proof of Corollary 2.2. Let us first prove the quasimode estimates (2.9). To be able to use the uniform Sobolev estimates in Theorem 2.1, we shall initially assume that $\lambda \geq \Lambda$, where $\Lambda = \Lambda(M, q, V) \geq 1$ is as in this theorem.

Proving the quasimode estimate is equivalent to showing that for q as in (2.9), we have

$$\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1}\|_{L^2 \to L^q} \leq C\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}, \quad \lambda \geq \Lambda,$$

or, by duality, for $\lambda \geq \Lambda$,

$$\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{L^2(M)} \leq C\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|f\|_{L^{q'}(M)}. \tag{2.30}$$

To prove this we note that (2.2) and duality yield

$$\|(-\Delta_g - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1}\|_{L^{q'} \to L^2} \leq C\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}, \quad \lambda \geq 1, \tag{2.31}$$

while, (2.8) yields

$$\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1}\|_{L^{q'} \to L^{r'}} \leq C, \quad \lambda \geq \Lambda, \tag{2.32}$$

since, as remarked after (2.1), if (p, q) is as in (2.1), then so is (q', p').

If we use the decomposition (2.26) again with $\mu = 1$, then, by (2.31), we can estimate the first term in the right-hand side of this equality as follows:

$$\|A\|_2 \leq C\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|f\|_{L^{q'}(M)}. \tag{2.33}$$

Since $1/q' - 1/p' = 2/n$, by (2.12) and Hölder's inequality, we also obtain

$$\begin{aligned}\|B\|_2 &\leq C\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|V\|_{L^{n/2}}^{1/2}\|V^{1/2} \cdot (H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{\overline{q}'} \\ &\leq C\|V\|_{L^{n/2}}\delta(\lambda, q)\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{p'},\end{aligned}$$

if the pair (q', p') is as in (2.1) and $1/\overline{q}' = 1/q' - 1/n$.

[1] We are grateful to one of the referees for pointing this out to us.

By (2.8),

$$\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{p'}\le C_{p',V}\|f\|_{q'},\quad \lambda\ge\Lambda,$$

and since $V\in L^{n/2}$, we conclude that $\|B\|_2$ is also dominated by the right-hand side of (2.30) for λ as above.

To obtain the quasimode estimate (2.10) in the corollary, we need to see that the bounds in (2.30) are also valid when $1\le\lambda<\Lambda$, with $\Lambda=\Lambda(M,q,V)\ge 1$ being the fixed constant in Theorem 2.1. This just follows from the fact that $\delta(\lambda,q)$ and $\varepsilon(\lambda)$ are assumed to be nonzero and continuous, and also by the spectral theorem,

$$\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^2(M)}\le C\|(H_V-\lambda^2+i\varepsilon(\Lambda)\Lambda)^{-1}f\|_{L^2(M)} \tag{2.34}$$

if $1\le\lambda\le\Lambda$.

Let us finish the proof of the corollary by proving (2.10), which is equivalent to showing that for (p,q) as in (2.1), we have

$$\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_q\le C_{\delta,V,q}\|f\|_p\quad\text{if } \lambda\ge 1 \text{ and } |\mu|\ge\delta. \tag{2.35}$$

As before, we may assume that $q\in(\frac{2n}{n-1},\frac{2n}{n-2}]$ to justify the bootstrap argument.

Since, similar to (2.34), by the spectral theorem, we have

$$\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^2(M)}\le C_{\delta_0}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^2(M)} \tag{2.36}$$

if $|\mu|\ge\delta_0$ and $\lambda\ge 1$. Thus, by (2.9) and duality,

$$\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}\|_{L^p\to L^2}\le C_{\delta_0}\,\delta(\lambda,p')\,\lambda^{\sigma(p')-1}(\varepsilon(\lambda))^{-1} \tag{2.37}$$

if $|\mu|\ge\delta_0$ and $\lambda\ge 1$. while, by (2.2), we have

$$\|(-\Delta_g-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}\|_{L^2\to L^q}\le C_{\delta_0}\,\delta(\lambda,q)\,\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1} \tag{2.38}$$

if $|\mu|\ge\delta_0$ and $\lambda\ge 1$. Also, by (2.7),

$$\|(-\Delta_g-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}\|_{L^p\to L^q}\le C_{\delta_0}\quad\text{if } |\mu|\ge\delta_0 \text{ and } \lambda\ge 1. \tag{2.39}$$

If we then split as in (2.21) and argue as before, we find that (2.39) yields

$$\|\mathrm{I}\|_q\le C_{\delta_0}\|f\|_p \tag{2.40}$$

and

$$\|\mathrm{II}\|_q+\|\mathrm{III}\|_q\le\tfrac12\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q}, \tag{2.41}$$

if the latter N_1, N_2 are fixed large enough and $|\mu|\ge\delta_0$ and $\lambda\ge 1$.

If we use (2.13) and an earlier argument, we obtain

$$\begin{aligned}\|\mathrm{IV}\|_q&\le C_{\delta_0}N_1^{1/2}N_2^{1/2}\,\delta(\lambda,q)\,\lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1}\|(H_V-\lambda^2+i\mu\varepsilon(\lambda)\lambda)^{-1}f\|_2\\&\le C'_{\delta_0}N_1^{1/2}N_2^{1/2}\,\delta(\lambda,q)\,\lambda^{\sigma(q)-1}\,\delta(\lambda,p')\,\lambda^{\sigma(p')-1}(\varepsilon(\lambda))^{-2}\|f\|_p,\end{aligned}$$

and since we are assuming (2.5), this yields

$$\|\mathrm{IV}\|_q\le C_{\delta_0}\|f\|_p. \tag{2.42}$$

Since (2.39), (2.41) and (2.42) yield (2.35), the proof is complete. ∎

Now we show another abstract theorem that gives us quasimode estimates for larger exponents.

Theorem 2.4. *Assume* (M, g) *is a compact Riemannian manifold of dimension* $n \geq 5$. *Assume further that* (2.9) *holds for some* $\frac{2(n+1)}{n-1} \leq r < \frac{2n}{n-4}$, *with* $\varepsilon(\lambda)$ *and* $\delta(\lambda, r)$ *satisfying* (2.3) *and* (2.4), *respectively. Then, if* $V \in L^{n/2}(M)$, *we have, for* $u \in \mathrm{Dom}(H_V)$,

$$\|u\|_q \leq C_{V,r}\, \delta(\lambda, r)\, \lambda^{\sigma(q)-1} (\varepsilon(\lambda))^{-1} \|(-\Delta_g + V - \lambda^2 + i\varepsilon(\lambda)\lambda)u\|_2 \tag{2.43}$$

if $\lambda \geq 1$, $r < q \leq \frac{2n}{n-4}$. *Similarly, for* $n = 3$ *or* $n = 4$, *assuming that* (2.9) *holds for some* $\frac{2(n+1)}{n-1} \leq r < \infty$, *with* $\varepsilon(\lambda)$ *and* $\delta(\lambda, r)$ *satisfying* (2.3) *and* (2.4), *we have*

$$\|u\|_q \leq C_{V,r}\, \delta(\lambda, r)\, \lambda^{\sigma(q)-1} (\varepsilon(\lambda))^{-1} \|(-\Delta_g + V - \lambda^2 + i\varepsilon(\lambda)\lambda)u\|_2 \tag{2.44}$$

if $\lambda \geq 1$, $r < q < \infty$.

Here compared with the non-perturbed case (2.2), we have $\delta(\lambda, r)$ on the right-hand side of (2.43) and (2.44) instead of $\delta(\lambda, q)$ for larger exponents q. This is because we are using the bound (2.9) for the exponent r in our proof. And as we can see in the first section, except for the case $q_c = \frac{2(n+1)}{n-1}$, for our applications, we have $\delta(\lambda, q) \equiv \sqrt{\varepsilon(\lambda)}$ for all larger exponents in the quasimode estimates.

Proof of Theorem 2.4. Throughout the proof, we shall assume that

$$\frac{2(n+1)}{n-1} \leq r < q \leq \frac{2n}{n-4} \ \text{ if } n \geq 5 \quad \text{or} \quad \frac{2(n+1)}{n-1} \leq r < q < \infty \ \text{ if } n = 3, 4. \tag{2.45}$$

Note that proving (2.43) is equivalent to showing that for q satisfying (2.45),

$$\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_q \leq C_{V,r}\, \delta(\lambda, r)\, \lambda^{\sigma(q)-1} (\varepsilon(\lambda))^{-1} \|f\|_2 \quad \text{if } \lambda \geq 1. \tag{2.46}$$

As before, in order to justify a bootstrapping argument that follows, we shall temporarily assume that for q as in (2.45),

$$\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{L^q(M)} < \infty \quad \text{if } f \in L^2(M). \tag{2.47}$$

We shall give the proof of (2.47) later in Lemma 2.5 by obtaining Sobolev type inequalities for the operator H_V.

Fix a smooth bump function $\beta \in C_0^\infty(1/4, 4)$ with $\beta \equiv 1$ in $(1/2, 2)$, let $P = \sqrt{\Delta_g}$, and write

$$\begin{aligned}(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f &= \beta(P/\lambda)(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f + (1 - \beta(P/\lambda))(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f \\ &= A + B.\end{aligned} \tag{2.48}$$

To deal with the first term, note that, since $\lambda^{-\alpha}\tau^\alpha\beta(\tau/\lambda)$ is a symbol of order 0, by Theorem 4.3.1 in [29], $\lambda^{-\alpha}(-\Delta_g)^{\alpha/2}\beta(P/\lambda)$ is a 0 order pseudo-differential operator, thus

$$\|(-\Delta_g)^{\alpha/2}\beta(P/\lambda)\|_{L^r \to L^r} \lesssim \lambda^\alpha \quad \text{if } 1 < r < \infty. \tag{2.49}$$

So, by the Sobolev estimates (2.49) and (2.9), if $\alpha = n(1/r - 1/q)$, we have

$$\begin{aligned} \|A\|_q &\le \|(\Delta_g)^{\alpha/2}\beta(P/\lambda)(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_r \\ &\le \lambda^{n(1/r-1/q)}\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_r \\ &\le C_{V,r}\,\delta(\lambda, r)\lambda^{n(1/r-1/q)}\lambda^{\sigma(r)-1}\|f\|_2. \end{aligned} \tag{2.50}$$

Since $n(1/r - 1/q) + \sigma(r) = \sigma(q)$, the first term is dominated by the right-hand side of (2.46).

To bound the second term, we shall use the second resolvent formula (1.4) to write

$$\begin{aligned} &(1-\beta(P/\lambda))(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f \\ &\quad = (1-\beta(P/\lambda))(-\Delta_g - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f \\ &- (1-\beta(P/\lambda))[|V_{>N}|^{1/2}(-\Delta_g - \lambda^2 - i\varepsilon(\lambda)\lambda)^{-1}]^*(V^{1/2}\cdot(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f) \\ &- (1-\beta(P/\lambda))[|V_{\le N}|^{1/2}(-\Delta_g - \lambda^2 - i\varepsilon(\lambda)\lambda)^{-1}]^*(V^{1/2}\cdot(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f) \\ &\quad = \mathrm{I} - \mathrm{II} - \mathrm{III}. \end{aligned} \tag{2.51}$$

Since the function $1 - \beta(\tau/\lambda)$ vanishes in a dyadic neighborhood of λ, it is easy to see that

$$(1-\beta(\tau/\lambda))(\tau^2 - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1}(\tau^2 + \lambda^2)$$

is a symbol of order zero and, again by Theorem 4.3.1 in [29],

$$(1-\beta(P/\lambda))(-\Delta_g - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1}(-\Delta_g + \lambda^2)$$

is a 0 order pseudo-differential operator, thus

$$\|(1-\beta(P/\lambda))(-\Delta_g - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_r \lesssim \|(-\Delta_g + \lambda^2)^{-1} f\|_r \tag{2.52}$$

if $1 < r < \infty$. So, by (2.52), Sobolev estimates, the proof of (2.11) and the fact that

$$\begin{aligned} &(1-\beta(P/\lambda))[|V_{>N}|^{1/2}(-\Delta_g - \lambda^2 - i\varepsilon(\lambda)\lambda)^{-1}]^* \\ &\quad = [|V_{>N}|^{1/2}(1-\beta(P/\lambda))(-\Delta_g - \lambda^2 - i\varepsilon(\lambda)\lambda)^{-1}]^*, \end{aligned} \tag{2.53}$$

we have, for q satisfying (2.45),

$$\begin{aligned} \|\mathrm{II}\|_q &\le C\|V_{>N}\|_{L^{n/2}}^{1/2}\|V^{1/2}\cdot(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{\bar{p}} \\ &\le C\|V_{>N}\|_{L^{n/2}}^{1/2}\|V\|_{L^{n/2}}^{1/2}\cdot\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_{L^q}, \end{aligned} \tag{2.54}$$

where $1/\bar{p} - 1/q = 1/n$. By (2.20), we can fix N large enough so that

$$C\|V_{>N}\|_{L^{n/2}}^{1/2}\|V\|_{L^{n/2}}^{1/2} < \tfrac{1}{4},$$

yielding the bounds

$$\|\mathrm{II}\|_q \le \tfrac{1}{4}\|(H_V - \lambda^2 + i\varepsilon(\lambda)\lambda)^{-1} f\|_q. \tag{2.55}$$

To bound the third term, note that since $(\frac{\lambda^2}{\tau^2+\lambda^2})^{1/2}$ is a symbol of order 0, by Theorem 4.3.1 in [29],

$$(-\Delta_g/\lambda^2+1)^{-1/2}$$

is a 0 order pseudo-differential operator, thus if $1/\overline{p}=1/q-1/n$, then, by Sobolev estimates,

$$\|(-\Delta_g+\lambda^2)^{-1}f\|_q \le C\|(-\Delta_g+\lambda^2)^{-1/2}f\|_{\overline{p}} \le C\lambda^{-1}\|f\|_{\overline{p}}. \tag{2.56}$$

Thus, (2.52) and (2.56) and our earlier arguments (i.e., the proof of Lemma 2.3) yield

$$\begin{aligned}\|\mathrm{III}\|_q &\le C\lambda^{-1}N^{1/2}\|V^{1/2}\cdot(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{\overline{p}}\\ &\le C\lambda^{-1}N^{1/2}\|V\|_{L^{n/2}}^{1/2}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_q.\end{aligned} \tag{2.57}$$

If we choose Λ such that $C\Lambda^{-1}N^{1/2}\|V\|_{L^{n/2}}^{1/2}=1/4$, we conclude that

$$\|\mathrm{III}\|_q \le \tfrac{1}{4}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_q \quad \text{if } \lambda\ge\Lambda. \tag{2.58}$$

Also note that for q satisfying (2.45), we have $1/2-1/q\le 2/n$. By Sobolev estimates, if $\alpha=n(1/2-1/q)$,

$$\begin{aligned}&\|(1-\beta(P/\lambda))(-\Delta_g-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_q\\ &\quad\le \|(-\Delta_g)^{\alpha/2}(1-\beta(P/\lambda))(-\Delta_g-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_2.\end{aligned} \tag{2.59}$$

Since the symbol of the operator on the right-hand side of (2.59) satisfies

$$\tau^\alpha(1-\beta(\tau/\lambda))(\tau^2-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}\le\lambda^{\alpha-2}, \tag{2.60}$$

a combination of (2.59) and (2.60) yields the bounds

$$\|\mathrm{I}\|_q\le\lambda^{n(1/2-1/q)-2}\|f\|_2, \tag{2.61}$$

which is better than the right-hand side of (2.43) and (2.44), due to the condition on $\varepsilon(\lambda)$ and $\delta(\lambda,r)$.

If we combine (2.50), (2.55), (2.58) and (2.61), we conclude that for $\lambda\ge\Lambda$, we have

$$\begin{aligned}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q(M)} &\le C_{V,r}\,\delta(\lambda,r)\,\lambda^{n(1/r-1/q)}\lambda^{\sigma(r)-1}\|f\|_{L^2(M)}\\ &\quad+\tfrac{1}{2}\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^q(M)}.\end{aligned} \tag{2.62}$$

By (2.47), this leads to (2.43) and (2.44) for $\lambda\ge\Lambda$ since we are assuming that $f\in L^2(M)$. On the other hand, by (2.34), the quasimode estimates for $1\le\lambda<\Lambda$ follow as a corollary of the special case when $\lambda=\Lambda$.

To finish the proof of Theorem 2.4, we shall need the following lemma, which gives us (2.47).

Lemma 2.5. *Assume (M,g) is a compact Riemannian manifold of dimension $n\ge3$. If $V\in L^{n/2}(M)$, there exists a constant $N_0>1$ large enough such that*

$$\|u\|_q\le\|(-\Delta_g+V+N_0)u\|_{p(q)} \quad \text{if } \tfrac{1}{p(q)}-\tfrac{1}{q}=\tfrac{2}{n} \text{ and } \tfrac{n}{n-2}<q<\infty. \tag{2.63}$$

The condition on q in (2.63) is necessary, since we do not have the corresponding Sobolev inequalities even for the non-perturbed operator at the two endpoints $p = 1$ or $q = \infty$. Also observe that for q satisfying (2.45), we have $p(q) \le 2$. Thus, by the above inequality, we have $\|u\|_{L^q(M)} < \infty$ for q satisfying (2.45) if $u \in \mathrm{Dom}(H_V)$, which implies (2.47).

To prove (2.63), note that it is equivalent to showing that

$$\|(H_V + N_0)^{-1} f\|_{L^q(M)} \le C\|f\|_{L^p(M)} \quad \text{if } f \in L^2(M), \tag{2.64}$$

with (p, q) as in (2.63). By duality, it suffices prove this inequality when

$$\tfrac{n}{n-2} < q \le \tfrac{2n}{n-2}. \tag{2.65}$$

We are assuming (2.65), since by (A.7) in the appendix, we have

$$u \in L^q(M), \quad 2 \le q \le \tfrac{2n}{n-2}, \text{ if } u \in \mathrm{Dom}(H_V).$$

Thus, for q as in (2.65),

$$\|(H_V + N_0)^{-1} f\|_{L^q(M)} < \infty \quad \text{if } f \in L^2(M). \tag{2.66}$$

As before, in proving (2.64), since L^2 is dense in L^p, we shall assume that $f \in L^2(M)$ to be able to use (2.17) to justify a bootstrapping argument that follows.

We shall use the second resolvent formula (1.4) to write

$$\begin{aligned}(H_V + N_0)^{-1} f &= (-\Delta_g + N_0)^{-1} f \\ &\quad - [|V_{>N}|^{1/2}(-\Delta_g + N_0)^{-1}]^* (V^{1/2} \cdot (H_V + N_0)^{-1} f) \\ &\quad - [|V_{\le N}|^{1/2}(-\Delta_g + N_0)^{-1}]^* (V^{1/2} \cdot (H_V + N_0)^{-1} f) \\ &= \mathrm{I} - \mathrm{II} - \mathrm{III}.\end{aligned} \tag{2.67}$$

By the Sobolev estimates for the unperturbed operator, we have

$$\|\mathrm{I}\|_q \le C\|f\|_p, \tag{2.68}$$

where the constant C does not depend on N_0. Similarly, our earlier arguments yield

$$\begin{aligned}\|\mathrm{II}\|_q &\le C\|V_{>N}\|_{L^{n/2}}^{1/2} \|V^{1/2} \cdot (H_V + N_0)^{-1} f\|_{\overline{p}} \\ &\le C\|V_{>N}\|_{L^{n/2}}^{1/2} \|V\|_{L^{n/2}}^{1/2} \cdot \|(H_V + N_0)^{-1} f\|_{L^q},\end{aligned}$$

using Hölder's inequality and the fact that $1/\overline{p} = 1/q + 1/n$ in the last step. By (2.20), we can fix N large enough so that $C\|V_{>N}\|_{L^{n/2}}^{1/2} \|V\|_{L^{n/2}}^{1/2} < 1/4$, yielding the bounds

$$\|\mathrm{II}\|_q < \tfrac{1}{4}\|(H_V + N_0)^{-1} f\|_q. \tag{2.69}$$

To bound the third term, note that since $(\frac{N_0}{\tau^2 + N_0})^{1/2}$ is a symbol of order 0, by Theorem 4.3.1 in [29],

$$(-\Delta_g/N_0 + 1)^{-1/2}$$

is a 0 order pseudo-differential operator, thus

$$\|(-\Delta_g+N_0)^{-1}f\|_q \le C\|(-\Delta_g+N_0)^{-1/2}f\|_{\overline{p}} \le CN_0^{-1/2}\|f\|_{\overline{p}}, \tag{2.70}$$

using Sobolev estimates and the fact that $1/\overline{p}=1/q+1/n$ in the first inequality. Thus,

$$\begin{aligned}\|\mathrm{III}\|_q &\le CN_0^{-1/2}N^{1/2}\|V^{1/2}\cdot(H_V+N_0)^{-1}f\|_{\overline{p}}\\ &\le CN_0^{-1/2}N^{1/2}\|V\|_{L^{n/2}}^{1/2}\|(H_V+N_0)^{-1}f\|_q.\end{aligned} \tag{2.71}$$

If we choose N_0 such that $CN_0^{-1/2}N^{1/2}\|V\|_{L^{n/2}}^{1/2}<1/4$, (2.68), (2.69) and (2.71) imply

$$\|(H_V+N_0)^{-1}f\|_{L^q(M)} \le C\|f\|_{L^p(M)}+\tfrac{1}{2}\|(H_V+N_0)^{-1}f\|_{L^q(M)}.$$

By (2.66), this leads to (2.63), and the proof is complete. ■

Let us next show how Theorem 1.1 is also a corollary of Theorem 2.1 and Theorem 2.4.

Proof of Theorem 1.1. We shall use Theorem 2.1 with

$$\delta(\lambda,r)=\varepsilon(\lambda)\equiv 1,\quad \lambda\ge 1, \tag{2.72}$$

and $r=q$ and $r=p=p(q)'$ satisfying (1.6).

Then, by the spectral projection estimates of the fourth author [27], we have the quasi-mode estimates (2.2) for the unperturbed operators $H_0=-\Delta_g$. The uniform Sobolev estimates (2.8) are due to Dos Santos Ferreira, Kenig and Salo [10]. Also, it is a simple exercise, using (1.11), to check that for (p,q) as above, we have $\sigma(q)+\sigma(p')-2<0$, and so (2.5) is also trivially valid.

Thus, by inequality (2.9) in Corollary 2.2, and Theorem 2.4, we have (1.10) for $q\in[\frac{2(n+1)}{n-1},\frac{2n}{n-4}]$ if $n\ge 5$, and $q\in[\frac{2(n+1)}{n-1},\infty)$ if $n=3$ or 4. If we use the bound for $q=\frac{2(n+1)}{n-1}$ along with Hölder's inequality and the trivial quasimode estimate for $q=2$ (which follows from the spectral theorem), we also see that (1.10) is valid for $2<q<\frac{2(n+1)}{n-1}$.

The other inequality in Corollary 2.2, (2.10), also trivially implies the uniform Sobolev estimates (1.7) in the region where $\mathrm{Re}\,\zeta\ge 1$. Since the bounds for $\{\zeta\in\Omega_\delta:\mathrm{Re}\,\zeta<1\}$ are valid for the unperturbed operators $H_0=-\Delta_g$ by [10], we can use the quasimode estimates (1.10) for $\lambda=1$ and the proof that (2.8) implies (2.10) to see that the uniform Sobolev bounds in Theorem 1.1 in the region $\mathrm{Re}\,\zeta<1$ are also valid, which finishes the proof. ■

Next, let us also see how we can use Theorem 2.1 and Theorem 2.4 to prove Theorem 1.5, which says that when (M,g) is the standard sphere, we can improve Theorem 1.1 by obtaining the inequalities for a larger range of exponents when $V\in L^{n/2}(S^n)$.

Proof of Theorem 1.5. It is easy to modify the proof of Theorem 1.1 to obtain the uniform Sobolev estimates for S^n, which involve the improved range of exponents in (1.12). As in the preceding proof, we shall use Theorem 2.1 with $\delta(\lambda,r)=\varepsilon(\lambda)\equiv 1$ when $\lambda\ge 1$. Here $r=q$ and $r=p=p(q)'$ are assumed to be as in (1.12). A simple calculation using (1.11) then shows that we have $\sigma(q) + \sigma(p') - 2 \in [-1, -1 + 1/2n]$ and so (2.5) is trivially valid. As a result, for $q < \frac{2n}{n-3}$, we would have the bounds in (1.29) and (1.30) when $\operatorname{Re}\zeta$ and λ are larger than one, respectively, if we had the quasimode estimates (2.2) and the uniform Sobolev estimates (2.8) for the unperturbed operators H_0, for $\varepsilon(\lambda)$ and $\delta(\lambda, r)$ as above and exponents satisfying (1.12). The quasimode estimates are due to Sogge [26] (see also [14]), and the uniform Sobolev estimates are due to S. Huang and Sogge [14].

Since the remaining larger exponents q in (1.30) follows from the case $q < \frac{2n}{n-3}$ and Theorem 2.4, and the cases where $\zeta \in \Omega_\delta$ has $\operatorname{Re}\zeta < 1$ or $\lambda \geq 1$ in (1.29) follow from our earlier arguments, the proof is complete. ∎

Spectral projection estimates for larger exponents

Let us conclude this section by briefly reviewing how if, in addition to assuming (1.2) (i.e., $V \in L^{n/2}$), we assume that $V_- = \max\{0, -V\} \in \mathcal{K}(M)$, then we can obtain spectral projection and quasimode estimates for exponents, which are larger than those in Theorems 1.1–1.5 or Corollary 2.2.

Recall that V is in the Kato class $\mathcal{K}(M)$ if

$$\lim_{r \searrow 0} \sup_x \int_{B_r(x)} h_n(d_g(x,y)) |V(y)| \, dy = 0, \tag{2.73}$$

where

$$h_n(r) = \begin{cases} \log(2 + r^{-1}) & \text{if } n = 2, \\ r^{2-n} & \text{if } n \geq 3. \end{cases}$$

Here $d_g(x,y)$ is the geodesic distance between x and y in M and $B_r(x)$ denotes the geodesic ball of radius r about x.

Let us first show that we can use estimates like (2.9) to obtain certain spectral projection estimates. Specifically, if

$$\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]} = \mathbb{1}_{[\lambda,\lambda+\varepsilon(\lambda)]}(\sqrt{H_V}) \tag{2.74}$$

is the projection onto the part of the spectrum of $\sqrt{H_V}$ in the interval $[\lambda, \lambda + \varepsilon(\lambda)]$, then, by the spectral theorem, (2.9) implies that

$$\|\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]} f\|_r \leq C_V \, \delta(\lambda, r) \lambda^{\sigma(r)} \|f\|_2, \quad \lambda \geq 1. \tag{2.75}$$

To see this one takes u in (2.9) to be $\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]} f$ and then uses the spectral theorem to see that that for this choice of u the right-hand side of (2.9) is dominated by the right-hand side of (2.75).

Next, recall that if $V_- \in \mathcal{K}(M)$, then we have favorable heat kernel bounds (see [33]), and, consequently, if $\beta \in C_0^\infty((1/2, 1))$ is a nonnegative function with integral one and if

$$\tilde{\beta}_\lambda(\tau) = \int_0^\infty e^{-t\tau} \lambda^2 \beta(\lambda^2 t) \, dt, \quad \tau \geq 0, \, \lambda \geq 1,$$

we have

$$\|\tilde{\beta}_\lambda(H_V)\|_{L^r \to L^q} \lesssim \lambda^{n(1/r - 1/q)} \quad \text{if } 2 \leq r \leq q \leq \infty. \tag{2.76}$$

For details, see Section 6 of [3].[2] Arguing as in [3] it is a simple matter to use the spectral theorem and (2.76) to see that if (2.75) is valid, then we have

$$\|\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]} f\|_q \lesssim \delta(\lambda,r)\lambda^{\sigma(r)+n(1/r-1/q)}\|f\|_2, \quad \lambda \geq 1, \text{ if } q \in (r,\infty], \tag{2.77}$$

when $V_- \in \mathcal{K}(M)$.

Based on this and the aforementioned relationships between spectral projection estimates and quasimode estimates, if $V \in L^{n/2}(M)$ and $V_- \in \mathcal{K}(M)$, by Theorem 1.1, for all (M,g), we can also obtain (2.75) with $\varepsilon(\lambda) = \delta(\lambda,r) \equiv 1$ when $r > \frac{2n}{n-4}$ if $n \geq 5$, or $r = \infty$ if $n = 3, 4$, since $\sigma(r) + n(1/r - 1/q) = \sigma(q)$ if $\frac{2(n+1)}{n-1} \leq r < q \leq \infty$. Thus, for such exponents, we recover the universal bounds in [3], while for smaller ones, Theorem 1.1 is stronger since it only requires $V \in L^{n/2}(M)$.

In the case of the standard sphere S^n, if $V \in L^{n/2}(M)$ and $V_- \in \mathcal{K}(M)$, we can similarly obtain (2.75), with $\varepsilon(\lambda) = \delta(\lambda,r) \equiv 1$ for $r = \infty$ when $n = 3, 4$, and $r > \frac{2n}{n-4}$ when $n \geq 5$.

We note that Theorem 1.1 says that when $n = 3$ or $n = 4$, we have (2.75) with $\varepsilon(\lambda) = \delta(\lambda,r) \equiv 1$ for all $2 < r < \infty$. As noted in [3], such spectral projection estimates can break down for $r = \infty$ on S^n in all dimensions if one merely assumes $V \in L^{n/2}(S^n)$, and there is related recent results for general manifolds in Frank and Sabin [11].

We have focused here on variants of the spectral projection estimates for larger exponents than the ones in Theorems 1.1 and 1.5. As we shall see in the next two sections, there are similar results corresponding to Theorems 1.3 and 1.4.

3. Improved bounds for manifolds of nonpositive curvature

The main purpose of this section is to prove Theorem 1.3. Consequently, we shall assume throughout this section that $n \geq 3$ and that (M,g) is an n-dimensional manifold whose sectional curvatures are nonpositive. In Section 5 we shall prove that the quasimode estimates in Theorem 1.3 are valid in the two-dimensional case if, in addition to (1.2), we assume that V is a Kato potential.

By Corollary 2.2 and Theorem 2.4, we would have Theorem 1.3 if we knew that for exponents (p,q) satisfying

$$\min(p',q) > \tfrac{2(n+1)}{n-1} \quad \text{and} \quad \tfrac{1}{p} - \tfrac{1}{q} = \tfrac{2}{n}, \tag{3.1}$$

we had the classical quasimode estimates

$$\|u\|_r \lesssim \lambda^{\sigma(r)-1}(\varepsilon(\lambda))^{-1/2}\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2\|_2 \quad \text{for } r = q, p' \text{ and } \lambda \geq 1, \tag{3.2}$$

as well as

$$\|u\|_q \lesssim \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u\|_p, \quad \lambda \geq 1, \tag{3.3}$$

[2] In [3], this inequality was only proved under the stronger assumption that $V \in \mathcal{K}$; however, since the proof only relied on the heat kernel estimates of Sturm [33], which are valid when $V_- \in \mathcal{K}$, it also yields (2.76).

where here and throughout this section, we shall take

$$\varepsilon(\lambda) = (\log(2+\lambda))^{-1}. \tag{3.4}$$

Even though we have replaced $\lambda^2 + i\varepsilon(\lambda)\lambda$ by $(\lambda + i\varepsilon(\lambda))^2$ here to simplify some calculations to follow, (3.2) and (3.3) are equivalent to (2.2) and (2.7), respectively, with $\delta(\lambda, r) = \sqrt{\varepsilon(\lambda)}$ as in (3.4) in the former.

Even though the first inequality is a consequence of spectral projection estimates in Hassell and Tacy [12] following earlier results of Bérard [2], and even though the resolvent estimates are in [9] and [24], let us sketch their proofs since we shall need to adapt them in order to show that we also get improved quasimode estimates for $q = q_c = \frac{2(n+1)}{n-1}$, which is missing in (3.2). We cannot appeal to Corollary 2.2 to obtain these estimates since it is not known whether the uniform Sobolev estimates (3.3) are valid when $q = q_c$. The quasimode estimates for this exponent are analogous involving $L^{n/2}$ potentials of those in [4], which treated the case $V \equiv 0$.

Let us start with the sketch of (3.2). Since both $r = p'$ and $r = q$ in (3.2) are smaller than $\frac{2n}{n-4}$ when $n \geq 4$, by the discussion at the end of the last section, it is simple to see that (3.2) is equivalent to the spectral projection estimates for the unperturbed operator $H_0 = -\Delta_g$:

$$\|\chi_{[\lambda,\lambda+\varepsilon(\lambda)]} f\|_r \lesssim \sqrt{\varepsilon(\lambda)}\, \lambda^{\sigma(r)} \|f\|_2, \quad \lambda \geq 1,\ r > \tfrac{2(n+1)}{n-1}, \tag{3.5}$$

with r as in (3.2) (see [31]). We shall actually indicate why this inequality is valid for all $r > \frac{2(n+1)}{n-1}$. Here $\chi_{[\lambda,\lambda+\varepsilon(\lambda)]}$ is the operator projecting onto the part of the spectrum of $\sqrt{-\Delta_g}$ in the shrinking intervals $[\lambda, \lambda + \varepsilon(\lambda)]$.

To establish this, fix a real-valued function $a \in \mathcal{S}(\mathbb{R})$ satisfying

$$\operatorname{supp} \hat{a} \subset (-\delta_0, \delta_0) \quad \text{and} \quad a(t) \geq 1,\ t \in [-1, 1], \tag{3.6}$$

where $\delta_0 > 0$ will be specified later on. We then claim that (3.5) would be a consequence of the following:

$$\big\|a\big((\varepsilon(\lambda))^{-1}(P-\lambda)\big)h\big\|_r \lesssim \sqrt{\varepsilon(\lambda)}\, \lambda^{\sigma(r)} \|h\|_2, \quad \lambda \geq 1,\ r > \tfrac{2(n+1)}{n-1}, \tag{3.7}$$

if $P = \sqrt{-\Delta_g}$. To verify this claim, one just takes h to be $\tilde{\chi}_{[\lambda,\lambda+\varepsilon(\lambda)]} f$, where

$$\tilde{\chi}_{[\lambda,\lambda+\varepsilon(\lambda)]}(\tau) = \mathbb{1}_{[\lambda,\lambda+\varepsilon(\lambda)]}(\tau) \cdot \big(a\big((\varepsilon(\lambda))^{-1}(\lambda - \tau)\big)\big)^{-1}.$$

Since this function has sup-norm smaller than one and since $a((\varepsilon(\lambda))^{-1}(P-\lambda))h = \chi_{[\lambda,\lambda+\varepsilon(\lambda)]} f$, one obtains (3.5) from (3.7) and the spectral theorem.

We next observe that, by duality, (3.7) is equivalent to the statement that

$$\big\|a\big((\varepsilon(\lambda))^{-1}(P-\lambda)\big)h\big\|_{L^{r'}(M)\to L^2(M)} \lesssim \sqrt{\varepsilon(\lambda)}\, \lambda^{\sigma(r)}, \quad \lambda \geq 1, \text{ if } r > \tfrac{2(n+1)}{n-1}.$$

By a routine TT^* argument, this is equivalent to the following:

$$\big\|b\big((\varepsilon(\lambda))^{-1}(P-\lambda)\big)\big\|_{L^{r'}(M)\to L^r(M)} \lesssim \varepsilon(\lambda)\, \lambda^{2\sigma(r)}, \quad \lambda \geq 1, \text{ if } r > \tfrac{2(n+1)}{n-1}, \tag{3.8}$$

and $b(\tau) = (a(\tau))^2$.

Next, since, by the first part of (3.6), $\hat{b}$ is supported in $(-2\delta_0, 2\delta_0)$, it follows, from Fourier's inversion theorem, Euler's formula and the first part of (3.6), that

$$b\big((\varepsilon(\lambda))^{-1}(P-\lambda)\big)h = \frac{\varepsilon(\lambda)}{\pi}\int_{-T}^{T} \hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}(\cos tP)h\,dt \\ + b\big((\varepsilon(\lambda))^{-1}(P+\lambda)\big)h, \quad \text{where } T = 2\delta_0\cdot(\varepsilon(\lambda))^{-1}. \tag{3.9}$$

Since $\lambda \geq 1$ and $P \geq 0$, using crude eigenfunction bounds, one obtains

$$\big\|b\big((\varepsilon(\lambda))^{-1}(P+\lambda)\big)\big\|_{L^1(M)\to L^\infty(M)} = O(\lambda^{-N}), \quad \lambda\geq 1,\ N=1,2,3,\dots,$$

and consequently we would have (3.8) if we could show that for small enough fixed $\delta_0>0$, we have

$$\Big\|\int_{-T}^{T}\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}\cos tP\,dt\Big\|_{L^{r'}(M)\to L^r(M)} \lesssim \lambda^{2\sigma(r)}, \quad \lambda\geq 1,\ \text{if } r>\tfrac{2(n+1)}{n-1}, \tag{3.10}$$

and $T = 2\delta_0\cdot(\varepsilon(\lambda))^{-1}$.

Next, let us fix $\eta\in C_0^\infty(\mathbb{R})$ satisfying

$$\eta(t)=1,\ \ t\in(-1/2,1/2), \quad \text{and} \quad \operatorname{supp}\eta\subset(-1,1). \tag{3.11}$$

Then it follows from the universal spectral projection estimates of [27] that

$$\Big\|\int \eta(t)\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}(\cos tP)f\,dt\Big\|_r \lesssim \lambda^{2\sigma(r)}\|f\|_{r'}, \quad \lambda\geq 1, \tag{3.12}$$

for all $r>2$. Consequently, we would have (3.10) if we could show that when δ_0, as in (3.6) and (3.10), is sufficiently small, we have

$$\Big\|\int (1-\eta(t))\,\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}\cos tP\,dt\Big\|_{L^{r'}(M)\to L^r(M)} \lesssim \lambda^{2\sigma(r)}, \quad \lambda\geq 1, \tag{3.13}$$

if $r>\frac{2(n+1)}{n-1}$.

Since the function

$$\tau\to\Psi_\lambda(\tau)=\int(1-\eta(t))\,\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}\cos t\tau\,dt$$

clearly satisfies

$$|\Psi_\lambda(\tau)|\lesssim(\varepsilon(\lambda))^{-1},$$

it follows from the spectral theorem that

$$\Big\|\int(1-\eta(t))\,\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}\cos tP\,dt\Big\|_{L^2(M)\to L^2(M)} \lesssim (\varepsilon(\lambda))^{-1}=\log(2+\lambda). \tag{3.14}$$

We claim that if we also had for some $c_0<\infty$,

$$\Big\|\int(1-\eta(t))\,\hat{b}(\varepsilon(\lambda)t)e^{-it\lambda}\cos tP\,dt\Big\|_{L^1(M)\to L^\infty(M)} \lesssim \lambda^{\frac{n-1}{2}}e^{c_0T}\lesssim\lambda^{\frac{n-1}{2}}\lambda^{c_0\delta_0}, \tag{3.15}$$

then for δ_0 small enough depending on r, we would have (3.12). This just follows from a simple interpolation argument and the observation that if $\theta=2/r$, then $(1-\theta)\cdot\frac{n-1}{2}<2\sigma(r)$, provided that $r>\frac{2(n+1)}{n-1}$.

One can prove (3.15) using the Hadamard parametrix after lifting the calculation to the universal cover of (M, g) as in Bérard [2] and Hassell and Tacy [12] (see also [28]). This completes the proof of (3.8) and hence that of (3.2).

The proof of (3.3) is similar. As in Section 2 of [9], we shall use the formula

$$(3.16)\qquad (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} f = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty e^{i\lambda t} e^{-\varepsilon(\lambda)t} (\cos tP) f \, dt.$$

If η is as in (3.11), we shall write

$$(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} = T_\lambda^0 + T_\lambda^1 + R_\lambda,$$

where if $T = 2\delta_0 \cdot (\varepsilon(\lambda))^{-1}$ is as in (3.9),

$$(3.17)\qquad T_\lambda^0 = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \eta(t)\eta(t/T) e^{i\lambda t} e^{-\varepsilon(\lambda)t} \cos tP \, dt$$

is a local operator, while

$$(3.18)\qquad T_\lambda^1 = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty (1 - \eta(t))\eta(t/T) e^{i\lambda t} e^{-\varepsilon(\lambda)t} \cos tP \, dt,$$

and

$$(3.19)\qquad R_\lambda = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty (1 - \eta(t/T)) e^{i\lambda t} e^{-\varepsilon(\lambda)t} \cos tP \, dt.$$

To prove (3.3), by duality, it suffices to handle the case where $q \in (\frac{2(n+1)}{n-1}, \frac{2n}{n-2}]$, in which case the estimate is equivalent to the statement that

$$(3.20)\qquad \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1}\|_{L^{p(q)}(M) \to L^q(M)} = O(1), \quad \lambda \ge 1,$$

if $q \in (\frac{2(n+1)}{n-1}, \frac{2n}{n-2}]$ and $1/p(q) - 1/q = 2/n$. In view of the above decomposition, this would follow from

$$(3.21)\qquad \|S_\lambda\|_{L^{p(q)}(M) \to L^q(M)} = O(1) \quad \text{if } S_\lambda = T_\lambda^0,\ T_\lambda^1 \text{ or } R_\lambda.$$

As observed in [24], the bounds for R_λ are an immediate consequence of (3.5) and a simple orthogonality argument, after observing that $\sigma(q) + \sigma((p(q))') = 1$ if $(p(q), q)$ are as in (3.20) and

$$\tau \to m_\lambda(\tau) = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty (1 - \eta(t/T)) e^{i\lambda t} e^{-\varepsilon(\lambda)t} \cos t\tau \, dt$$

satisfies

$$(3.22)\qquad |m_\lambda(\tau)| \lesssim (\varepsilon(\lambda))^{-1} (1 + (\varepsilon(\lambda))^{-1} |\lambda - \tau|)^{-N} \quad \text{for all } N, \text{ if } \tau \ge 0,\ \lambda \ge 1,$$

assuming, as above, that $T = 2\delta_0 \cdot (\varepsilon(\lambda))^{-1}$.

The local operator T_λ^0 was estimated in [10] and later in [9] (see also [27]), where it was shown that this operator enjoys the bounds in (3.21) even for the larger range of exponents, where $q > \frac{2n}{n-1}$. One proves this result using stationary phase and Stein's oscillatory integral theorem in [32]. For this step, it is convenient to assume, as we may, that the injectivity radius of (M, g) is ten or more.

Based on this, only one estimate in (3.21) remains. We just need to handle T_λ^1, i.e., if $T = 2\delta_0 \cdot \log(2+\lambda)$ with δ_0 small enough,

$$\lambda^{-1}\Big\| \int_0^\infty (1-\eta(t))\,\eta(t/T)\,e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt\Big\|_{L^{p(q)}(M)\to L^q(M)} = O(1).$$

Since $q \le (p(q))'$ if $(p(q), q)$ are is in (3.20) or (3.19), by Hölder's inequality, this would follow from

$$\Big\| \int_0^\infty (1-\eta(t))\eta(t/T)e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt\Big\|_{L^r(M)\to L^{r'}(M)} = O(\lambda) \tag{3.23}$$

if $r' < \frac{2n(n+1)}{n^2-n-4}$, assuming that $\delta_0 > 0$ is small. Here, we use the fact that $(p(q))' < \frac{2n(n+1)}{n^2-n-4}$ (see (3.1)).

One can repeat the proof of (3.14) to see that

$$\begin{aligned}\Big\| \int_0^\infty (1-\eta(t))\,\eta(t/T)\,e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt\Big\|_{L^2(M)\to L^2(M)} &= O(T)\\ &= O(\log(2+\lambda)).\end{aligned} \tag{3.24}$$

Also, by using the Hadamard parametrix and arguing as in [2], one can adapt the proof of (3.15) to see that

$$\Big\| \int_0^\infty (1-\eta(t))\,\eta(t/T)\,e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt\,\Big\|_{L^1(M)\to L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}\lambda^{c_0\delta_0}) \tag{3.25}$$

if $\delta_0 > 0$ is small. Since

$$\tfrac{2n(n+1)}{n^2-n-4} < \tfrac{2(n-1)}{n-3},$$

we have

$$\tfrac{n-1}{2}\cdot(1-\theta) < 1 \quad \text{if } \theta = \tfrac{2}{r'} \text{ and } r' < \tfrac{2(n-1)}{n-3},$$

and we obtain (3.21) via interpolation if $\delta_0 = \delta_0(r')$ is small enough.

This completes our proof of Theorem 1.3 except for the quasimode estimates (1.21) for the critical exponent $q = q_c = \frac{2(n+1)}{n-1}$, which we shall handle in the next subsection.

Improved quasimode bounds for the critical exponent

As we noted before, we cannot appeal to Corollary 2.2 to obtain improved quasimode estimates for the critical exponent $q_c = \frac{2(n+1)}{n-1}$ on manifolds of nonpositive curvature, since we do not have the uniform Sobolev estimates (3.3) when $q = q_c$. Despite this, we can use the above arguments to obtain (1.21), which extends the critical quasimode estimates of two of us [4] for the case $V \equiv 0$ to include singular potentials when $n \ge 3$. In a later section we shall prove analogous estimates for the two-dimensional case.

To prove the quasimode estimates in (1.21), we shall of course use the fact that, by [4], we have (1.21) when $V \equiv 0$, which is equivalent to the following:

$$\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1}\|_{L^2(M)\to L^{q_c}(M)} \lesssim \lambda^{\sigma(q_c)-1}(\varepsilon(\lambda))^{-1+\delta_n}, \tag{3.26}$$

as well as the following bounds for the spectral projection operators associated to $H_0 = -\Delta_g$:

$$\|\chi_{[\lambda,\lambda+\varepsilon(\lambda)]}\|_{L^2(M)\to L^{q_c}(M)} \lesssim \lambda^{\sigma(q_c)}(\varepsilon(\lambda))^{\delta_n}. \tag{3.27}$$

To proceed, just as before we shall write

$$(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} = T_\lambda + R_\lambda, \quad \text{where } T_\lambda = T_0 + T_\lambda^1, \tag{3.28}$$

with T_λ^0, T_λ^1 and R_λ as in (3.17), (3.18) and (3.19), respectively.

Since $R_\lambda = m_\lambda(\sqrt{H_0})$, with $m_\lambda(\tau)$ as in (3.22), one can use (3.27) and a simple orthogonality argument to see that

$$\|R_\lambda\|_{L^2(M)\to L^{q_c}(M)} \lesssim (\varepsilon(\lambda))^{-1+\delta_n}\,\lambda^{\sigma(q_c)-1}, \tag{3.29}$$

and also

$$\|R_\lambda \circ (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)\|_{L^2(M)\to L^{q_c}(M)} \lesssim (\varepsilon(\lambda))^{-1+\delta_n}\,\lambda^{\sigma(q_c)-1}\cdot(\lambda\varepsilon(\lambda)). \tag{3.30}$$

If we set $T_\lambda = T_\lambda^0 + T_\lambda^1$ as above, then, since $T_\lambda = (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} - R_\lambda$, we trivially obtain from (3.26) and (3.29) the bound

$$\|T_\lambda\|_{L^2(M)\to L^{q_c}(M)} \lesssim (\varepsilon(\lambda))^{-1+\delta_n}\,\lambda^{\sigma(q_c)-1}. \tag{3.31}$$

We noted before that

$$\|T_\lambda^0\|_{L^{p(q_c)}(M)\to L^{q_c}(M)} = O(1) \quad \text{if } \tfrac{1}{p(q_c)} - \tfrac{1}{q_c} = \tfrac{2}{n}.$$

Additionally, by our earlier argument, if the $\delta_0 > 0$ used to define T_λ^1 is small enough, we also have

$$\|T_\lambda^1\|_{L^{p(q_c)}(M)\to L^{q_c}(M)} = O(1), \tag{3.32}$$

by Hölder's inequality, as $q_c < (p(q_c))'$ and $(p(q_c))' < \frac{2(n-1)}{n-3}$.

If we combine the last two estimates we conclude that

$$\|T_\lambda\|_{L^{p(q_c)}(M)\to L^{q_c}(M)} = O(1). \tag{3.33}$$

To use these bounds write

$$\begin{aligned} u &= (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} \circ (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u \\ &= T_\lambda(-\Delta_g + V - (\lambda + i\varepsilon(\lambda))^2)u + T_\lambda(V_{\le N}\cdot u) + T_\lambda(V_{>N}\cdot u) \\ &\quad + R_\lambda(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{3.34}$$

with $V_{\le N}$ and $V_{>N}$ as in (2.18).

By (3.31),

$$\|\mathrm{I}\|_{q_c} \lesssim (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \tag{3.35}$$

and, by (3.30), we similarly obtain

$$\begin{aligned} \|\mathrm{IV}\|_{q_c} &\lesssim (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \cdot (\lambda\varepsilon(\lambda))\|u\|_2, \\ &\lesssim (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \end{aligned} \tag{3.36}$$

using the spectral theorem in the last inequality.

If we use (2.20) along with Hölder's inequality, and (3.33) along with the arguments from Section 2, we conclude that we can fix N large enough so that

$$\|\mathrm{III}\|_{q_c} \le \tfrac{1}{2}\|u\|_{q_c}. \tag{3.37}$$

Also, (3.31) and (2.19) yield, for this fixed N,

$$\begin{aligned} \|\mathrm{II}\|_{q_c} &\le C_N (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \|u\|_2, \\ &\lesssim (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \end{aligned} \tag{3.38}$$

using the spectral theorem and the fact that $\varepsilon(\lambda)\cdot\lambda \ge 1$ if $\lambda \ge 1$.

Combining (3.35), (3.36), (3.37) and (3.38) yields

$$\|u\|_{q_c} \lesssim (\varepsilon(\lambda))^{-1+\delta_n} \lambda^{\sigma(q_c)-1} \|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2,$$

and since this is equivalent to (3.26), the proof of the quasimode estimates for $q = q_c$ in Theorem 1.3 is complete.

4. Improved bounds for tori

In this section we shall prove Theorem 1.4. Let us start by going over the proof of quasimode and uniform Sobolev estimates for the unperturbed operator $H_0 = -\Delta_{\mathbb{T}^n}$, which involve the exponent $q = \frac{2n}{n-2}$:

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \lambda^{-1/2}(\varepsilon(\lambda))^{-1/2}\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^2(\mathbb{T}^n)}, \tag{4.1}$$

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)} \tag{4.2}$$

for $\lambda \ge 1$, with

$$\varepsilon(\lambda) = \lambda^{-1/3+\delta_0} \quad \text{for all } \delta_0 > 0. \tag{4.3}$$

Recall that $\sigma(\frac{2n}{n-2}) = \frac{1}{2}$, and so (4.1) corresponds to (2.2) for $q = \frac{2n}{n-2}$ with the optimal $\delta(\lambda, q) = \sqrt{\varepsilon(\lambda)}$.

Even though these estimates are in [9] for $n = 3$, and in [13] for other dimensions, let us start by reviewing their proofs, since, as in the preceding section, we shall need to modify them to handle the estimates for H_V, especially the ones involving exponents q for which appropriate uniform Sobolev estimates are unavailable, which includes the case $q = q_c$.

The main estimate that is used to prove these two inequalities is a discrete version of the Stein–Tomas restriction theorem:

$$\|\chi_{[\lambda,\lambda+\rho]} f\|_{L^{q_c}(\mathbb{T}^n)} \lesssim (\rho\lambda)^{1/q_c} \lambda^{\varepsilon_0} \|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \varepsilon_0 > 0, \tag{4.4}$$

if $q_c = \frac{2(n+1)}{n-1}$ and $\lambda^{-1} \le \rho \le 1$. Here, χ_I denotes the spectral projection operator associated with the interval I for H_0. Since $\sigma(q_c) = 1/q_c$, this represents a substantial improvement over the unit band ($\rho = 1$) spectral projection estimates of [27]. On the other hand, unlike (4.1), it does not involve $\delta(\rho) = \sqrt{\rho}$. Indeed, no such estimate can be valid for ρ close to the associated wavelength λ^{-1}.

Hickman [13] proved (4.4) using the decoupling estimates of Bourgain and Demeter, see [6]. Specifically, Hickman showed that (4.4) is a consequence of Theorem 2.2 in [6]. Before that, Bourgain, Shao, Sogge and Yao [9] obtained a somewhat weaker form of (4.4) when $n = 3$, in which it was required that $\lambda^{-1/3} \le \rho \le 1$. This paper preceded the decoupling estimates of Bourgain and Demeter, and instead relied on multilinear techniques of Bourgain and Guth [7].

We shall require an equivalent form of (4.4):

$$\|m_{\lambda,\rho}(\sqrt{H_0}) f\|_{L^{q_c}(\mathbb{T}^n)} \lesssim \|m_{\lambda,\rho}\|_\infty \cdot (\rho\lambda)^{2/q_c} \lambda^{\varepsilon_0} \|f\|_{L^{q_c'}(\mathbb{T}^n)} \quad \text{for all } \varepsilon_0 > 0, \tag{4.5}$$

if $\operatorname{supp} m_{\lambda,\rho} \subset [\lambda, \lambda+\rho]$ and $\lambda^{-1} \le \rho \le 1$. After observing that (4.4) and orthogonality imply that $\|m_{\lambda,\rho}\|_{L^2(\mathbb{T}^n)\to L^{q_c}(\mathbb{T}^n)} = O((\rho\lambda)^{1/q_c} \lambda^{\varepsilon_0})$ for all $\varepsilon_0 > 0$, one obtains (4.5) from this and a standard TT^* argument.

Let us now briefly recall the proof of (4.1). As we mentioned earlier, it is equivalent to the statement that

$$\|\chi_{[\lambda,\lambda+\varepsilon(\lambda)]}\|_{L^2(\mathbb{T}^n)\to L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \sqrt{\varepsilon(\lambda)}\,\lambda^{1/2} \quad \varepsilon(\lambda) = \lambda^{-1/3+\delta_0}, \text{ for all } \delta_0 > 0. \tag{4.6}$$

If $a_0 \in \mathcal{S}(\mathbb{R})$ satisfies

$$a_0(0) = 1 \quad \text{and} \quad \operatorname{supp} \hat{a}_0 \subset (-1/2, 1/2), \tag{4.7}$$

then (4.6) is equivalent to the statement that $a_0((\varepsilon(\lambda))^{-1}(\lambda - P))$, $P = \sqrt{H_0}$, maps $L^2(\mathbb{T}^n)$ to $L^{\frac{2n}{n-2}}(\mathbb{T}^n)$ with norm $O(\sqrt{\varepsilon(\lambda)}\,\lambda^{1/2})$, and by a simple TT^* argument this in turn is equivalent to the statement that

$$\|a((\varepsilon(\lambda))^{-1}(\lambda - P)) f\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \varepsilon(\lambda)\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}, \quad \text{with } a(\tau) = (a_0(\tau))^2. \tag{4.8}$$

Next we note that

$$\begin{aligned} a((\varepsilon(\lambda))^{-1}(\lambda - P)) f &= \frac{\varepsilon(\lambda)}{2\pi} \int \hat{a}(\varepsilon(\lambda)t) e^{i\lambda t} e^{-itP} f\, dt \\ &= \frac{\varepsilon(\lambda)}{\pi} \int \hat{a}(\varepsilon(\lambda)t) e^{i\lambda t} (\cos tP) f\, dt + a((\varepsilon(\lambda))^{-1}(\lambda + P)) f. \end{aligned}$$

Since $(\varepsilon(\lambda))^{-1}$, $\lambda \geq 1$ and P is a positive operator, it is a simple matter to use either Sobolev estimates or spectral projection estimates from [27] to see that the operator in the last term in the right-hand side maps $L^2(\mathbb{T}^n)$ to $L^{\frac{2n}{n-2}}(\mathbb{T}^n)$ with norm $O(\lambda^{-N})$ for any N. Thus, we would have (4.8), and consequently (4.1), if we could show that

$$\|Tf\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}, \quad \varepsilon(\lambda) = \lambda^{-1/3+\delta_0}, \tag{4.9}$$

where $Tf = \int \hat{a}(\varepsilon(\lambda)t)e^{i\lambda t}(\cos tP)f\,dt$.

Note that, by (4.7), the integrand vanishes when $|t| > 2(\varepsilon(\lambda))^{-1}$. To exploit this, let us fix a Littlewood–Paley bump function $\beta \in C_0^\infty((1/2, 2))$ satisfying

$$\sum_{j=-\infty}^{\infty} \beta(2^{-j}t) \equiv 1, \quad t > 0, \tag{4.10}$$

and set

$$\beta_0(t) = 1 - \sum_{j=1}^{\infty} \beta(2^{-j}|t|) \in C_0^\infty(\mathbb{R}). \tag{4.11}$$

Using these we can split the operator in (4.9) as

$$Tf = \sum_{j=0}^{\infty} T_j f, \tag{4.12}$$

where

$$\begin{cases} T_0 f = \displaystyle\int \beta_0(t)\hat{a}(\varepsilon(\lambda)t)e^{i\lambda t}(\cos tP)f\,dt, \\ T_j f = \displaystyle\int \beta(2^{-j}|t|)\hat{a}(\varepsilon(\lambda)t)e^{i\lambda t}(\cos tP)f\,dt, \quad j = 1, 2, \ldots. \end{cases} \tag{4.13}$$

Clearly, then (4.9) would be a consequence of the following:

$$\|T_j f\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim 2^{-\delta j}\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}, \quad j = 0, 1, 2, \ldots, \tag{4.14}$$

for some $\delta > 0$ which depends on n and $\delta_0 > 0$ in (4.3).

The bound for $j = 0$ is a simple consequence of the spectral projection estimates of one of us [27]. It is simple to check that the remaining bounds follow, by interpolation, from the following two estimates:

$$\|T_j f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0} \lambda^{\frac{n-1}{n+1}} 2^{\frac{2}{n+1}j} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)} \quad \text{for all } \varepsilon_0 > 0, \tag{4.15}$$

and

$$\|T_j f\|_{L^\infty(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-1}{2}} 2^{\frac{n+1}{2}j} \|f\|_{L^1(\mathbb{T}^n)}. \tag{4.16}$$

Indeed, since $\frac{n-2}{2n} = \theta\frac{n-1}{2(n+1)} + (1-\theta)\frac{1}{\infty}$, with $\theta = \frac{(n+1)(n-2)}{n(n-1)}$, by interpolation, (4.15) and (4.16) yield, for all $\varepsilon_0 > 0$,

$$\|T_j\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n) \to L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim \lambda^{1+\varepsilon_0} \lambda^{-1/n} 2^{3j/n}, \tag{4.17}$$

which implies (4.14), since, by (4.3) and (4.7), $T_j = 0$ for 2^j larger than a fixed constant times $\lambda^{-1/3+\delta_0}$. So, given any fixed δ_0 as in (4.3), we obtain (4.14) with $\delta = \delta_0/n$ if the loss $\varepsilon_0 > 0$ here is small enough.

To finish our proof of (4.1), it remains to prove (4.15) and (4.16).

The first inequality follows from applying (4.5) with $\rho = 2^{-j}$, since

$$\int \beta(2^{-j}|t|)\,\hat{a}(\varepsilon(\lambda)t)\,e^{i\lambda t}\cos(t\tau)\,dt = O(2^j(1+2^j|\lambda-\tau|)^{-N}) \quad \text{for all } N,$$

if $\lambda \geq 1$ and $\tau \geq 0$. Note that the integral in the left-hand side vanishes if 2^j is larger than a fixed multiple of $(\varepsilon(\lambda))^{-1}$.

The remaining inequality, (4.16), amounts to showing that the kernel $K_j(x,y)$ of T_j satisfies

$$K_j(x,y) = O(\lambda^{\frac{n-1}{2}} 2^{\frac{n+1}{2}j}). \tag{4.18}$$

If we relate $\mathbb{T}^n$ to $(-\pi,\pi]^n$ and the wave kernel $\cos tP$ on $\mathbb{T}^n$ to the Euclidean one (see, e.g., [28, Section 3.5]), we can write this kernel as follows:

$$K_j(x,y) = (2\pi)^{-n}\sum_{\ell\in\mathbb{Z}^n}\int_{-\infty}^{\infty}\beta(2^{-j}|t|)\,\hat{a}(\varepsilon(\lambda)t)\,e^{i\lambda t}(\cos t\sqrt{-\Delta_{\mathbb{R}^n}})(x,y+\ell)\,dt, \tag{4.19}$$

with $(\cos t\sqrt{-\Delta_{\mathbb{R}^n}})(x,y+\ell)$ denoting the wave kernel in $\mathbb{R}^n$. If we call the ℓ-th summand above $K_{j,\ell}(x,y)$, then, by using stationary phase and arguing as in [9] or as in Section 3.5 of [28], shows that

$$|K_{j,\ell}(x,y)| \lesssim \lambda^{\frac{n-1}{2}}(1+|x-y-\ell|)^{-\frac{n-1}{2}} \lesssim \lambda^{\frac{n-1}{2}}(1+|\ell|)^{-\frac{n-1}{2}} \tag{4.20}$$

for $x, y \in (-\pi,\pi]^n$. Furthermore, by Huygens' principle, $K_{j,\ell}(x,y) = 0$ when $x, y \in (-\pi,\pi]^n$ and $|\ell|$ is larger than a fixed multiple of 2^j. Therefore, for such x, y, we have

$$|K_j(x,y)| \lesssim \lambda^{\frac{n-1}{2}} \sum_{\{\ell\in\mathbb{Z}^n : |\ell|\leq 2^j\}} (1+|\ell|)^{-\frac{n-1}{2}} \lesssim \lambda^{\frac{n-1}{2}} 2^{\frac{n+1}{2}j}, \tag{4.21}$$

as desired.

Let us now see how we can use this argument to prove the uniform Sobolev estimates (4.2). As was the case in Section 3, we shall make use of the splitting of the resolvent operator $(-\Delta_{\mathbb{T}^n} - (\lambda + i\varepsilon(\lambda))^2)^{-1}$, as in (3.16)–(3.21), where $\varepsilon(\lambda)$ now as in (4.3). In our setting, we may simplify things a bit compared to the argument in Section 3 by taking $T = (\varepsilon(\lambda))^{-1}$, with, as we said, $\varepsilon(\lambda)$ being now as in (4.3). We then would obtain (4.2) if we had (3.21) in the current setting.

The bounds there for R_λ follow from a simple orthogonality argument and (4.6). Also, just as before, the bounds in (3.21) for the local operator are known (see [9], [10]).

To prove the bounds for the remaining operator T_λ^1 in (3.21), we split up the integral dyadically as before, by writing

$$T_\lambda^1 = T_\lambda^{1,0} + \sum_{j=1}^{\infty} T_\lambda^{1,j},$$

where, for $j = 1, 2, \dots,$

$$T_\lambda^{1,j} = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \beta(2^{-j}t)(1-\eta(t))\eta(t/T)e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt, \tag{4.22}$$

and $T_\lambda^{1,0}$ is given by an analogous formula with $\beta(2^{-j}t)$ replaced by $\beta_0(t) \in C_0^\infty(\mathbb{R}^n)$. Since $T_\lambda^{1,0}$ is a local operator which shares the same properties as T_λ^0, we have the analogue of (3.21) with $S_\lambda = T_\lambda^{1,0}$. As a result, we would have the remaining inequality (3.21) with $S_\lambda = T_\lambda^1$, if we could show that when (4.3) is valid, we have, as before, for some $\delta > 0$ depending on δ_0 and n,

$$\|T_\lambda^{1,j}\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)\to L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim 2^{-\delta j}. \tag{4.23}$$

Since $T_\lambda^{1,j} = 0$ when 2^j is larger than a fixed multiple of $(\varepsilon(\lambda))^{-1}$, by the proof of (4.1), we would obtain this estimate via interpolation from the following two estimates:

$$\|T_\lambda^{1,j} f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim \lambda^{-1}\cdot\lambda^{\varepsilon_0}\lambda^{\frac{n-1}{n+1}}2^{\frac{2}{n+1}j}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)} \quad \text{for all } \varepsilon_0 > 0, \tag{4.24}$$

and

$$\|T_\lambda^{1,j} f\|_{L^\infty(\mathbb{T}^n)} \lesssim \lambda^{-1}\cdot\lambda^{\frac{n-1}{2}}2^{\frac{n+1}{2}j}\|f\|_{L^1(\mathbb{T}^n)}. \tag{4.25}$$

Due to the $(\lambda + i\varepsilon(\lambda))^{-1}$ factor in (4.22), one sees from this formula and (4.13) that $T_\lambda^{1,j}$ behaves like $\lambda^{-1}T_j$, and so it is clear that the proof of (4.15) and (4.16) yield (4.24) and (4.25), respectively. This finishes our proofs of (4.1) and (4.2).

Using (4.1) and (4.2) along with Corollary 2.2, we obtain the bounds in Theorem 1.4 involving $q = \frac{2n}{n-2}$.

Quasimode and uniform Sobolev estimates for the critical exponent

Suppose that $q_c = \frac{2(n+1)}{n-1}$, $1/p(q_c) - 1/q_c = 2/n$, and, as in Theorem 1.4, let us assume that for an arbitrary fixed $\delta_0 > 0$,

$$\varepsilon(\lambda) = \lambda^{-1/5+\delta_0} \text{ if } n \geq 4 \quad \text{and} \quad \varepsilon(\lambda) = \lambda^{-3/16+\delta_0} \text{ if } n = 3. \tag{4.26}$$

We then recall that the estimates in Theorem 1.4 for $q = q_c$ say that for $u \in \mathrm{Dom}(H_V)$, we have

$$\|u\|_{L^{q_c}(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}(\varepsilon(\lambda))^{-\frac{n+3}{2(n+1)}}\lambda^{-\frac{n+3}{2(n+1)}}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^2(\mathbb{T}^n)}, \quad \lambda \geq 1, \tag{4.27}$$

as well as

$$\|u\|_{L^{q_c}(\mathbb{T}^n)} \lesssim \|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^{p(q_c)}(\mathbb{T}^n)}, \quad \lambda \geq 1. \tag{4.28}$$

As noted before, the inequality (4.27) is equivalent to the spectral projection estimates

$$\|\chi^V_{[\lambda,\lambda+\rho]} f\|_{L^{q_c}(\mathbb{T}^n)} \lesssim (\rho\lambda)^{1/q_c}\lambda^{\varepsilon_0}\|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \delta_0 > 0, \tag{4.29}$$

where $\rho \in [\lambda^{-1/5+\delta_0}, 1]$ if $n \geq 4$, or $\rho \in [\lambda^{-3/16+\delta_0}, 1]$ if $n = 3$. This is weaker than the $V \equiv 0$ results of Hickman [13], i.e., (4.4). Even though the ρ-intervals in (4.29) do not shrink to $\{1\}$ as $n \to \infty$, it would be interesting to try to improve the range of ρ in this inequality.

Since it is straightforward to check that for $\varepsilon(\lambda)$ satisfying (4.26), (2.6) is valid, by Corollary 2.2, we would have (4.27) and (4.28) if we knew that for such $\varepsilon(\lambda)$, we had the quasimode estimates

$$\|u\|_{q_c} \lesssim \lambda^{\varepsilon_0} (\varepsilon(\lambda))^{-\frac{n+3}{2(n+1)}} \lambda^{-\frac{n+3}{2(n+1)}} \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2 u\|_2 \quad \text{for } \lambda \geq 1, \tag{4.30}$$

and

$$\|u\|_{p(q_c)'} \lesssim (\varepsilon(\lambda))^{-1} \lambda^{\sigma(p(q_c)')-1} \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2 u\|_2 \quad \text{for } \lambda \geq 1, \tag{4.31}$$

as well as

$$\|u\|_{q_c} \lesssim \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2) u\|_{p(q_c)}, \quad \lambda \geq 1. \tag{4.32}$$

Inequality (4.30) follows from Hickman's estimate (4.4) and a simple orthogonality argument. And (4.31) follows from the same argument by using the general spectral projection estimates of the fourth author [27]. As we shall see at the end of this section, we can get a better bound than the right-hand side of (4.31) for $q = p(q_c)'$, but as long as (2.6) is valid for $\varepsilon(\lambda)$ satisfying (4.26), the powers of $\varepsilon(\lambda)$ in inequalities (4.30) and (4.31), which are numbers between $[-1, -1/2]$, are not crucial in the proof of (4.28).

Now let us see how we can modify the proof of (4.2) to obtain (4.32). We shall make use of the splitting of the resolvent operator $(-\Delta_{\mathbb{T}^n} - (\lambda + i\varepsilon(\lambda))^2)^{-1}$ as in (3.16)–(3.21), where $\varepsilon(\lambda)$ now as in (4.26). We then would obtain (4.32) if we had (3.21) in the current setting.

Unlike previous cases, we do not have sharp spectral projection bounds here for the exponent q_c. The operator R_λ will be dealt with differently after we established the desired bounds for T_λ.

As we noted earlier the local operator T_λ^0 always satisfies the desired bounds in the uniform Sobolev estimates regardless of the choice of $\varepsilon(\lambda)$:

$$\|T_\lambda^0\|_{L^{p(q_c)}(\mathbb{T}^n) \to L^{q_c}(\mathbb{T}^n)} = O(1) \quad \text{if } \tfrac{1}{p(q_c)} - \tfrac{1}{q_c} = \tfrac{2}{n}, \text{ i.e., } p(q_c) = \tfrac{2n(n+1)}{n^2+3n+4}.$$

For the operator T_λ^1, just as in the proof of (4.2), we shall need to use the dyadic decomposition

$$T_\lambda^1 = T_\lambda^{1,0} + \sum_{j=1}^{\infty} T_\lambda^{1,j}$$

exactly as before, where for $j = 1, 2, 3, \dots,$ $T^{1,j}$ is given by (4.22) and for $j = 0$, the analogue of this identity with $\beta(2^{-j}t)$ replaced by $\beta_0(t) \in C_0^\infty(\mathbb{R})$. Since the factor $(1 - \eta(t))$ in each of these integrals vanishes near the origin, the quasimode estimates in [27] imply that $\|T_\lambda^{1,0}\|_{L^{p(q_c)}(\mathbb{T}^n) \to L^{q_c}(\mathbb{T}^n)} = O(1)$ or, alternatively, one can use the fact that $T_\lambda^{1,0}$ behaves like T_λ^0 and deduce this from arguments in [9], [10] or [27]. Based on the desired bounds for $j = 0$, we conclude that if we could show that for some $\delta > 0$,

$$\|T_\lambda^{1,j}\|_{L^{p(q_c)}(\mathbb{T}^n) \to L^{q_c}(\mathbb{T}^n)} = O(2^{-j\delta}) \quad \text{if } \tfrac{1}{p(q_c)} - \tfrac{1}{q_c} = \tfrac{2}{n},\ j = 1, 2, 3, \dots, \tag{4.33}$$

then we would obtain $\|T_\lambda\|_{L^{p(q_c)}(\mathbb{T}^n)\to L^{q_c}(\mathbb{T}^n)} = O(1)$. As before, δ here depends on the various parameters in (4.26), and, in order to get the bounds in (4.33), we are lead to assume that $\varepsilon(\lambda)$ is as in (4.26).

In order to prove (4.33), we claim that, by interpolation, it suffices to prove, for all $\varepsilon_0 > 0$, the following three inequalities:

$$\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}\lambda^{-1/n}2^{3j/n} \quad \text{if } q = \tfrac{2n}{n-2}, \tag{4.34}$$

$$\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}2^{j} \quad \text{if } q = \tfrac{2n}{n-3} \text{ for } n \ge 4, \text{ or } q = \infty \text{ for } n = 3, \tag{4.35}$$

$$\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}\lambda^{-1/n}2^{\frac{n^2+2n-2}{n^2}j} \quad \text{if } q = \tfrac{2n^2}{(n-1)(n-2)}, \tag{4.36}$$

where $1/p - 1/q = 2/n$.

To verify this claim, we note that if $p = q_c'$ and $q = p(q_c)' = \frac{2n(n+1)}{n^2-n-4}$, when $n \ge 4$, we have

$$\tfrac{1}{q} = \theta\cdot\tfrac{n-2}{2n} + (1-\theta)\cdot\tfrac{(n-1)(n-2)}{2n^2} \quad \text{if } \theta = \tfrac{n^2-3n-2}{(n+1)(n-2)}.$$

Consequently, by interpolation, (4.34) and (4.36) yield, for any $\varepsilon_0 > 0$,

$$\begin{aligned}\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} &\lesssim \lambda^{\varepsilon_0}(\lambda^{-1/n}2^{3j/n})^{\frac{n^2-3n-2}{(n+1)(n-2)}}\cdot(\lambda^{-1/n}2^{\frac{n^2+2n-2}{n^2}j})^{\frac{2n}{(n+1)(n-2)}} \\ &= \lambda^{\varepsilon_0}\lambda^{-1/n}\cdot 2^{5j/n}.\end{aligned} \tag{4.37}$$

When $n = 3$, the above argument does not work, since $\frac{n^2-3n-2}{(n+1)(n-2)} < 0$ if $n = 3$. Instead, we shall use interpolation between (4.35) and (4.36). More precisely, note that if $p = q_c'$ and $q = p(q_c)' = 12$, we have

$$\tfrac{1}{q} = \tfrac{1}{12} = \theta\cdot\tfrac{1}{9} + (1-\theta)\cdot\tfrac{1}{\infty} \quad \text{if } \theta = \tfrac{3}{4}.$$

By interpolation, (4.35) and (4.36) yield, for any $\varepsilon_0 > 0$,

$$\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}(\lambda^{-1/3}2^{13j/9})^{3/4}\cdot(2^j)^{1/4} = \lambda^{\varepsilon_0}\lambda^{-1/4}\cdot 2^{4j/3}. \tag{4.38}$$

By duality, (4.37) and (4.38) leads to (4.33) if we fix $\delta_0 > 0$ in (4.26) and choose ε_0 here to be sufficiently small, since $T_\lambda^{1,j} = 0$ if 2^j is larger than a fixed constant times $(\varepsilon(\lambda))^{-1}$, which satisfies (4.26).

Now we shall give the proof of (4.34)–(4.36). The first inequality, (4.34), follows from (4.17), since, as noted before, the operator $T_\lambda^{1,j}$ behaves like $\lambda^{-1}T_j$.

To prove the second inequality, first note that if $n \ge 4$, by Theorem 2.7 in [6] and a simple orthogonality argument, we have, for all $\varepsilon_0 > 0$,

$$\|\chi_{[\lambda,\lambda+\rho]}\|_{L^2(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \rho^{1/2}\lambda^{\varepsilon_0}\lambda^{\sigma(q)} \quad \text{if } q = \tfrac{2(n-1)}{n-3},\ \rho\in[\lambda^{-1},1]. \tag{4.39}$$

As a consequence of (4.39), we have, for all $\varepsilon_0 > 0$,

$$\|T_\lambda^{1,j}\|_{L^2(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}\lambda^{\sigma(q)-1}2^{j/2} \quad \text{if } q = \tfrac{2(n-1)}{n-3} \tag{4.40}$$

by an orthogonality argument, since, as noted before, $T_\lambda^{1,j} = \lambda^{-1}m_{\lambda,j}(\sqrt{H_0})$, where $m_{\lambda,j}(\tau) = O(2^j(1+2^j|\tau-\lambda|)^{-N})$ for any N, if $\tau \ge 0$ and $\lambda \ge 1$.

Inequality (4.35) now just follows from (4.40) and (4.25) via interpolation. Indeed, since

$$\tfrac{n-3}{2n} = \theta \cdot \tfrac{n-3}{2(n-1)} + (1-\theta)\cdot \tfrac{1}{\infty}, \quad \tfrac{n+1}{2n} = \theta \cdot \tfrac{1}{2} + (1-\theta)\cdot 1,$$

with $\theta = \frac{n-1}{n}$, we deduce that, for all $\varepsilon_0 > 0$ and θ as above, we have

$$\|T_\lambda^{1,j}\|_{L^{\frac{2n}{n+1}}(\mathbb{T}^n)\to L^{\frac{2n}{n-3}}(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}(\lambda^{-\frac{n-3}{2(n-1)}}2^{j/2})^{\theta}\cdot(\lambda^{\frac{n-3}{2}}2^{\frac{n+1}{2}j})^{1-\theta} = \lambda^{\varepsilon_0}2^j,$$

as desired.

The case $n = 3$ in (4.35) follows from exactly the same argument by using the fact that, for all $\varepsilon_0 > 0$,

$$\|\chi_{[\lambda,\lambda+\rho]}\|_{L^2(\mathbb{T}^3)\to L^\infty(\mathbb{T}^3)} \lesssim \rho^{1/2}\lambda^{\varepsilon_0+1}, \quad \rho \in [\lambda^{-1}, 1]. \tag{4.41}$$

If we take $\rho = \lambda^{-1}$ in the above inequality, (4.41) is equivalent to counting the lattice points on a sphere, which has a general upper bound in any dimensions, i.e., for all $\varepsilon_0 > 0$,

$$\|\chi_{[\lambda,\lambda+\lambda^{-1}]}\|_{L^2(\mathbb{T}^n)\to L^\infty(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-2}{2}+\varepsilon_0}, \quad n \geq 2. \tag{4.42}$$

See, e.g., [8] for a more detailed discussion about inequality (4.42). Inequality (4.41) now follows from (4.42) by a simple orthogonality argument.

The third inequality, (4.36), involves the pair of exponents (p, q) which is the intersection of Stein–Tomas restriction line, where $q = \frac{n+1}{n-1}p'$ and the uniform Sobolev line, where $1/p - 1/q = 2/n$. More precisely, note that, by (4.4), after using the same argument as in the proof of (4.40), we have, for all $\varepsilon_0 > 0$,

$$\|T_\lambda^{1,j}\|_{L^2(\mathbb{T}^n)\to L^{q_c}(\mathbb{T}^n)} \lesssim \lambda^{\varepsilon_0}\lambda^{-1}2^j(\lambda 2^{-j})^{1/q_c} \quad \text{if } q_c = \tfrac{2(n+1)}{n-1}. \tag{4.43}$$

Now (4.36) follows from (4.43) and (4.25) via interpolation. Indeed, since

$$\tfrac{(n-1)(n-2)}{2n^2} = \theta\cdot\tfrac{1}{q_c} + (1-\theta)\cdot\tfrac{1}{\infty}, \quad \tfrac{n^2+n+2}{2n^2} = \theta\cdot\tfrac{1}{2} + (1-\theta)\cdot 1,$$

with $\theta = \frac{(n+1)(n-2)}{n^2}$, and $\frac{n^2+n+2}{2n^2} = \frac{(n-1)(n-2)}{2n^2} + 2/n$, we deduce that for all $\varepsilon_0 > 0$ and θ as above, we have

$$\begin{aligned}\|T_\lambda^{1,j}\|_{L^{p(q_c)}(\mathbb{T}^n)\to L^{(p(q_c))'}(\mathbb{T}^n)} &\lesssim \lambda^{\varepsilon_0}(\lambda^{-\frac{n+3}{2(n+1)}}2^{\frac{n+3}{2(n+1)}j})^{\theta}\cdot(\lambda^{\frac{n-3}{2}}2^{\frac{n+1}{2}j})^{1-\theta}\\ &= \lambda^{\varepsilon_0}\lambda^{-1/n}2^{\frac{n^2+2n-2}{n^2}j},\end{aligned}$$

as desired.

For the remaining operator R_λ, we claim that it has the same mapping properties as the operator $T_\lambda^{1,j}$ where $2^j \approx \varepsilon(\lambda)^{-1}$. Recall that in proving (4.34), (4.35) and (4.36), the only properties we required for the operator $T_\lambda^{1,j}$ are

$$|T_\lambda^{1,j}(\tau)| = O(2^j(1+2^j|\tau-\lambda|)^{-N}) \tag{4.44}$$

and

$$|T_\lambda^{1,j}(x,y)| = O(\lambda^{\frac{n-3}{2}}2^{\frac{n+1}{2}j}). \tag{4.45}$$

Similarly, for the operator R_λ, if $2^j \approx \varepsilon(\lambda)^{-1}$, by (3.22), we have

$$|R_\lambda(\tau)| = O(2^j(1+2^j|\tau-\lambda|)^{-N}). \tag{4.46}$$

For the other kernel bounds (4.45), if we use the dyadic decomposition $R_\lambda = \sum_{k=0}^{\infty} R_\lambda^k$, where

$$R_\lambda^k = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \beta(2^{-k+1}\varepsilon(\lambda)t)(1-\eta(\varepsilon(\lambda))t)\, e^{i\lambda t} e^{-\varepsilon(\lambda)t} \cos tP\, dt, \tag{4.47}$$

and argue as in (4.19)–(4.20) using stationary phase, we have

$$|R_\lambda^k(x,y)| \lesssim \lambda^{\frac{n-3}{2}} 2^{\frac{n+1}{2}k} (\varepsilon(\lambda))^{\frac{n+1}{2}} e^{-2^k}.$$

After summing over k, we conclude that

$$|R_\lambda(x,y)| = O(\lambda^{\frac{n-3}{2}} (\varepsilon(\lambda))^{\frac{n+1}{2}}). \tag{4.48}$$

As a consequence of (4.46) and (4.48), by using the same argument as for the operator $T_\lambda^{1,j}$, we obtain that

$$\|R_\lambda\|_{L^{p(q_c)}(\mathbb{T}^n)\to L^{q_c}(\mathbb{T}^n)} = O(1), \tag{4.49}$$

which completes the proof of (4.32).

Quasimode and uniform Sobolev estimates for general exponents

Now we will see how we can modify the above argument to show that (1.24) and (1.26) hold for general exponents q. We shall first give the proof of (1.24), since essentially it does not require sharp spectral projection bounds. To see this, by Corollary 2.2, we would have (1.24) if we knew that for exponents (p,q) satisfying (1.12), we had the quasimode estimates

$$\|u\|_r \lesssim \lambda^{\sigma(r)-1}(\varepsilon(\lambda))^{-1}\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2 u\|_2 \quad \text{for } r = q, p' \text{ and } \lambda \geq 1, \tag{4.50}$$

as well as

$$\|u\|_q \lesssim \|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u\|_p, \quad \lambda \geq 1, \tag{4.51}$$

where, for all $\delta_0 > 0$,

$$\varepsilon(\lambda) = \begin{cases} \lambda^{-\beta_1(n,p(q)')+\delta_0} & \text{if } \frac{2n}{n-1} < q < \frac{2n}{n-2}, \\ \lambda^{-\beta_1(n,q)+\delta_0} & \text{if } \frac{2n}{n-2} \leq q < \frac{2n}{n-3}. \end{cases} \tag{4.52}$$

We shall give the explicit form of $\beta_1(n,q)$ later in (4.59). Roughly speaking, it is a number that decreases from $1/3$ to 0 as q increases from $\frac{2n}{n-2}$ to $\frac{2n}{n-3}$.

Here (4.50) follows easily from the spectral projection bounds of [27] and a simple orthogonality argument. We shall obtain an improvement over (4.46) at the end of this section by modifying the previous argument that was used to prove (4.1). Right now the bounds in (4.50) is sufficient, since for $\varepsilon(\lambda)$ satisfying (4.52), (2.6) is valid for all exponents (p,q) satisfying (1.12), by (4.50).

To prove (4.51), by duality, it suffices to handle the case where $q \in [\frac{2n}{n-2}, \frac{2n}{n-3})$. As before, we shall split the resolvent operator as

$$(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} = T_\lambda^0 + T_\lambda^1 + R_\lambda.$$

As noted earlier the local operator T_λ^0 always satisfies the desired bounds regardless of the choice of $\varepsilon(\lambda)$. That is,

$$\|T_\lambda^0\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} = O(1) \quad \text{if } \tfrac{1}{p}-\tfrac{1}{q}=\tfrac{2}{n} \text{ and } \tfrac{2n}{n-1}<q<\tfrac{2n}{n-3}, \tag{4.53}$$

see, e.g., [24] and [14] for a proof of the above inequality.

For the operator T_λ, we shall need to use the dyadic decomposition

$$T_\lambda^1 = T_\lambda^{1,0} + \sum_{j=1}^{\infty} T_\lambda^{1,j}$$

exactly as before, where for $j=1,2,3,\dots$, $T^{1,j}$ is given by (4.22) and for $j=0$, the analogue of this identity with $\beta(2^{-j}t)$ replaced by $\beta_0(t)\in C_0^\infty(\mathbb{R})$. The operator $T_\lambda^{1,0}$ behaves like T_λ^0, and it is not hard to see that it satisfies (4.53). Based on the desired bounds for $j=0$, we conclude that if we could show that for some $\delta>0$,

$$\|T_\lambda^{1,j}\|_{L^{p(q)}(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} = O(2^{-j\delta}) \quad \text{if } \tfrac{1}{p}-\tfrac{1}{q}=\tfrac{2}{n},\ j=1,2,3,\dots, \tag{4.54}$$

then we would obtain

$$\|T_\lambda\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} = O(1) \quad \text{as well as} \quad \|R_\lambda\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} = O(1),$$

since, as mentioned before, the operator R_λ behaves like $T_\lambda^{1,j}$.

Given (4.34), (4.35) and (4.36), the above inequality now follows easily from an interpolation argument. First, for $\frac{2n}{n-2}\le q\le \frac{2n^2}{(n-1)(n-2)}$, write

$$\tfrac{1}{q} = \theta_1\cdot\tfrac{n-2}{2n} + (1-\theta_1)\cdot\tfrac{(n-1)(n-2)}{2n^2} \quad \text{if } \theta_1 = \tfrac{2n^2}{n-2}\big(\tfrac{1}{q}-\tfrac{(n-1)(n-2)}{2n^2}\big). \tag{4.55}$$

Consequently, by interpolation, (4.34) and (4.36) yield, for any $\varepsilon_0>0$,

$$\begin{aligned}\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} &\lesssim \lambda^{\varepsilon_0}(\lambda^{-1/n}2^{3j/n})^{\theta_1}\cdot(\lambda^{-1/n}2^{\frac{n^2+2n-2}{n^2}j})^{1-\theta_1}\\ &= \lambda^{\varepsilon_0}\lambda^{-1/n}\cdot 2^{\frac{n^2+2n-2}{n^2}j}2^{-\frac{n^2-n-2}{n^2}\theta_1 j}.\end{aligned} \tag{4.56}$$

Similarly, for $\frac{2n^2}{(n-1)(n-2)}\le q<\frac{2n}{n-3}$, write

$$\tfrac{1}{q} = \theta_2\cdot\tfrac{(n-1)(n-2)}{2n^2} + (1-\theta_2)\cdot\tfrac{n-3}{2n} \quad \text{if } \theta_2 = n^2\big(\tfrac{1}{q}-\tfrac{n-3}{2n}\big). \tag{4.57}$$

By interpolation, (4.35) and (4.36) yield, for any $\varepsilon_0>0$,

$$\begin{aligned}\|T_\lambda^{1,j}\|_{L^p(\mathbb{T}^n)\to L^q(\mathbb{T}^n)} &\lesssim \lambda^{\varepsilon_0}(\lambda^{-1/n}2^{\frac{n^2+2n-2}{n^2}j})^{\theta_2}\cdot(2^j)^{1-\theta_2}\\ &= \lambda^{\varepsilon_0}\lambda^{-\theta_2/n}\cdot 2^j\cdot 2^{\frac{2n-2}{n^2}\theta_2 j}.\end{aligned} \tag{4.58}$$

As a result, given θ_1 and θ_2 as in (4.55) and (4.57), if we define

$$\beta_1(n,q) = \begin{cases} \frac{n}{n^2+2n-2-(n^2-n-2)\theta_1} & \text{if } \frac{2n}{n-2}\le q\le\frac{2n^2}{(n-1)(n-2)},\\ \frac{n\theta_2}{n^2+(2n-2)\theta_2} & \text{if } \frac{2n^2}{(n-1)(n-2)}\le q<\frac{2n}{n-3},\end{cases} \tag{4.59}$$

by (4.56) and (4.58), we obtain (4.54) if we fix $\delta_0>0$ in (4.52) and choose ε_0 above to be sufficiently small, since $T_\lambda^{1,j}=0$ if 2^j is larger than a fixed constant times $(\varepsilon(\lambda))^{-1}$. Thus, the proof of (4.51) is complete.

To conclude, we shall give the proof of (1.26). We shall focus on the case $\frac{2(n+1)}{n-1} < q \le \frac{2n}{n-2}$, since, the estimates for $q > \frac{2n}{n-2}$ follow as a corollary of Theorem 2.4.

To proceed, note that by Corollary 2.2 as well as (4.51), we would have (1.26) if we knew that for exponents (p, q) satisfying

$$\frac{2(n+1)}{n-1} < q \le \frac{2n}{n-2} \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{2}{n}, \tag{4.60}$$

we had the quasimode estimates

$$\|u\|_r \lesssim \lambda^{\sigma(r)-1}(\varepsilon(\lambda))^{-1/2}\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2 u\|_2 \quad \text{for } r = q, p' \text{ and } \lambda \ge 1, \tag{4.61}$$

where we shall take

$$\varepsilon(\lambda) = \lambda^{-\beta_2(n,q)+\delta_0} \quad \text{for all } \delta_0 > 0, \tag{4.62}$$

with

$$\beta_2(n, q) = \frac{(n-1)^2 q - 2(n-1)(n+1)}{(n+1)(n-1)q - 2(n+1)^2 + 8}. \tag{4.63}$$

Actually, given (4.51) and (4.61), in order to apply Corollary 2.2, it suffices to check (2.6) is valid, which is equivalent to $\varepsilon(\lambda) \ge \lambda^{-1/2}$ when $\frac{2(n+1)}{n-1} < q \le \frac{2n}{n-2}$. However, for such exponents q, we have

$$\min(\beta_1(n, q), \beta_2(n, q)) \le \tfrac{1}{3},$$

which implies (2.6). Also, as before, the inequality for $r = p'$ (4.64) is not crucial for our proof. Indeed, simple quasimode estimates as in (4.50) are sufficient for our use.

Note that, compared with (4.50), the power on $\varepsilon(\lambda)$ in (4.61) is sharp, which, as before, is equivalent to the spectral projection estimates

$$\|\chi_{[\lambda,\lambda+\rho]} f\|_{L^q(\mathbb{T}^n)} \lesssim \rho^{1/2}\lambda^{\sigma(q)}\|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \rho \ge \varepsilon(\lambda). \tag{4.64}$$

To prove (4.64), if we repeat the argument in (4.7)–(4.13), by using a TT^* argument, it suffices to prove that for $q > \frac{2(n+1)}{n-1}$ and $\varepsilon(\lambda)$ satisfying (4.62),

$$\|Tf\|_{L^q(\mathbb{T}^n)} \lesssim \lambda^{2\sigma(q)}\|f\|_{L^{q'}(\mathbb{T}^n)}, \tag{4.65}$$

where

$$Tf = \int \hat{a}(\varepsilon(\lambda)t)e^{i\lambda t}(\cos tP) f \, dt, \tag{4.66}$$

with $a \in \mathcal{S}(\mathbb{R})$ defined as in (4.8).

As before we shall split the operator in (4.66) as

$$Tf = \sum_{j=0}^{\infty} T_j f,$$

where the operator T_j is defined as in (4.13).

Clearly, then (4.66) would be a consequence of the following:

$$\|T_j f\|_{L^q(\mathbb{T}^n)} \lesssim 2^{-\delta j}\lambda^{2\sigma(q)}\|f\|_{L^{q'}(\mathbb{T}^n)}, \quad j = 0, 1, 2, \dots, \tag{4.67}$$

for some $\delta > 0$ which depends on n and $\delta_0 > 0$ in (4.3).

The bound for $j = 0$ is a simple consequence of the spectral projection estimates of one of us [27], while the remaining bounds follow by interpolation from (4.15) and (4.16). Indeed, since for any $q > \frac{2(n+1)}{n-1}$, $\frac{1}{q} = \theta \cdot \frac{n-1}{2(n+1)} + (1-\theta) \cdot \frac{1}{\infty}$, with $\theta = \frac{2(n+1)}{(n-1)q}$, (4.15) and (4.16) yield, for all $\varepsilon_0 > 0$,

$$\|T_j\|_{L^q(\mathbb{T}^n) \to L^{q'}(\mathbb{T}^n)} \lesssim \lambda^{2\sigma(q)+\varepsilon_0 - \frac{n-1}{2}\cdot\frac{(n-1)q-2(n+1)}{(n-1)q}} 2^{j\frac{n+1}{2}\cdot\frac{(n-1)q-2(n+1)}{(n-1)q} + j\frac{2}{n+1}\cdot\frac{2(n+1)}{(n-1)q}}. \tag{4.68}$$

As a result, given any fixed δ_0 as in (4.62), we obtain (4.67) if the loss $\varepsilon_0 > 0$ is small enough, since, by (4.3) and (4.7), $T_j = 0$ for 2^j larger than a fixed constant times $\varepsilon(\lambda)^{-1}$, defined as in (4.62).

For later use, note that the above argument works for any $n \geq 2$. When $n = 2$, it gives the following analogue of (4.64):

$$\|\chi_{[\lambda,\lambda+\rho]} f\|_{L^q(\mathbb{T}^n)} \lesssim \rho^{1/2} \lambda^{\sigma(r)} \|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \rho \geq \lambda^{-\frac{q-6}{3q-10}+\delta_0},\ \delta_0 > 0, \tag{4.69}$$

if $q > 6$. In particular, at the point $q = \infty$, we have

$$\|\chi_{[\lambda,\lambda+\rho]} f\|_{L^\infty(\mathbb{T}^n)} \lesssim \rho^{1/2} \lambda^{\sigma(r)} \|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \rho \geq \lambda^{-1/3}, \tag{4.70}$$

by using (4.16) directly without interpolation with (4.15).

Remark. We shall briefly mention that improvements over the inequality (4.64) can be made in several ways. First, if we take $\rho = \lambda^{-1}$ in (4.64), it is conjectured by Bourgain in [5] that for $n \geq 3$,

$$\|\chi_{[\lambda,\lambda+\lambda^{-1}]} f\|_{L^q(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-2}{2}-\frac{n}{q}+\delta_0} \|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } \delta_0 > 0,\ \lambda \geq 1 \text{ and } q \geq \frac{2n}{n-2}. \tag{4.71}$$

As in (4.43), by Theorem 2.7 in [6], (4.71) holds for all $q \geq \frac{2(n-1)}{n-3}$, which is currently the best partial results for this problem. It is interesting and not known to the authors whether one can use (4.71) for $q \geq \frac{2(n-1)}{n-3}$ to improve the range of ρ in the inequality (4.64) when $\frac{2(n+1)}{n-1} < \rho \leq \frac{2n}{n-2}$.

On the other hand, as in [9] and [13], we can slightly improve the kernel bound (4.21), and thus obtain an improvement on the range of $\varepsilon(\lambda)$ in inequalities such as (4.51), (4.61) and (4.64), by exploiting the cancellation between different terms in (4.19), using exponential sum estimates. We omit the details here for simplicity.

5. Improved quasimode estimates when $n = 2$

The purpose of this section is to derive improved quasimode estimates under certain geometric assumptions for $n = 2$. Throughout this section, we shall assume that $V \in \mathcal{K}(M)$ satisfies (2.73), since in two dimensions, $V \in L^1(M)$ cannot ensure that the associated Schrödinger operator is self-adjoint. For a proof of self-adjointness of Schrödinger operators with Kato potentials, see, e.g., [3].

Unlike what was the case for higher dimensions in Theorem 1.1, we cannot improve the universal quasimode bounds in [3] when $n = 2$. We can, however, improve the bounds in Theorem 1.3 in two dimensions by removing the smallness assumption on V that was made in [3], and we can also obtain new bounds for two-dimensional tori.

First, let us see the following analogue of Theorem 1.3.

Theorem 5.1. *Assume that (M, g) is a Riemannian surface of nonpositive curvature and that $V \in \mathcal{K}(M)$. Then, for $q \geq 6$ and*

$$\delta(q) = \begin{cases} 1/72 & \text{if } q = 6, \\ 1/2 & \text{if } q > 6, \end{cases} \tag{5.1}$$

we have, for $u \in \mathrm{Dom}(H_V)$ and $\lambda \geq 1$,

$$\|u\|_q \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1+\delta(q)}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \tag{5.2}$$

where $\varepsilon(\lambda) = (\log(2+\lambda))^{-1}$. Consequently,

$$\|\chi^V_{[\lambda,\lambda+\varepsilon(\lambda)]} f\|_q \lesssim \lambda^{\sigma(q)}(\log(2+\lambda))^{-\delta(q)}\|f\|_2. \tag{5.3}$$

To prove (5.2), as before, we shall use the fact that by [4] and [12], we have (5.2) when $V \equiv 0$, which is equivalent to the following:

$$\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1}\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1+\delta(q)}, \tag{5.4}$$

as well as bounds for the spectral projection operators associated to $H_0 = -\Delta_g$:

$$\|\chi_{[\lambda,\lambda+\varepsilon(\lambda)]}\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)}(\varepsilon(\lambda))^{\delta(q)}. \tag{5.5}$$

The proof of (5.2) is based on the same idea as in the critical exponent case for higher dimensions. And unlike in higher dimensions, where we are able to prove uniform Sobolev estimates for certain range of exponents, the fact that $\delta(q) = 1/2$ for $q > 6$ is not crucial in our proof.

Proof of Theorem 5.1. As in [3], we shall first prove (5.2) for the exponent $q = \infty$, and then use it to obtain (5.2) for $6 \leq q < \infty$.

To proceed, just as before, we shall write

$$(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} = T_\lambda + R_\lambda, \quad \text{where } T_\lambda = T_0 + T^1_\lambda, \tag{5.6}$$

with T^0_λ, T^1_λ and R_λ as in (3.17), (3.18) and (3.19), respectively.

Since $R_\lambda = m_\lambda(\sqrt{H_0})$ with $m_\lambda(\tau)$ as in (3.22), one can use (5.5) and a simple orthogonality argument to see that for all $q \geq 6$,

$$\|R_\lambda\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1+\delta(q)}, \tag{5.7}$$

and also

$$\|R_\lambda \circ (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1+\delta(q)} \cdot (\lambda\,\varepsilon(\lambda)). \tag{5.8}$$

If we set $T_\lambda = T^0_\lambda + T^1_\lambda$ as above, then, since $T_\lambda = (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} - R_\lambda$, we trivially obtain from (5.4) and (5.7) the bound

$$\|T_\lambda\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1+\delta(q)}. \tag{5.9}$$

Note that, by (3.25), if the $\delta_0 > 0$ used to define T^1_λ is small enough, we have

$$\|T^1_\lambda\|_{L^1(M)\to L^\infty(M)} = O(\lambda^{-1/2}\lambda^{c_0\delta_0}) \ll 1. \tag{5.10}$$

Also for the local operator T_λ^0, we have the following kernel estimates:

$$|T_\lambda^0(x,y)| \le \begin{cases} C_0 |\log(\lambda d_g(x,y)/2)| & \text{if } d_g(x,y) \le \lambda^{-1}, \\ C_0 \lambda^{-1/2}(d_g(x,y))^{-1/2} & \text{if } \lambda^{-1} \le d_g(x,y) \le 1, \end{cases} \tag{5.11}$$

which comes from using stationary phase and the formulas

$$S_\lambda^0 = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \eta(\lambda t)\eta(t/T)e^{i\lambda t}e^{-\varepsilon(\lambda)t} \cos tP \, dt \tag{5.12}$$

and

$$S_\lambda^1 = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty (\eta(t) - \eta(\lambda t))\eta(t/T)e^{i\lambda t}e^{-\varepsilon(\lambda)t} \cos tP \, dt, \tag{5.13}$$

separately.

To see this, note that the multiplier associated to the operator S_λ^0 is

$$S_\lambda^0(\tau) = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \eta(\lambda t)\eta(t/T)e^{i\lambda t}e^{-\varepsilon(\lambda)t} \cos t\tau \, dt.$$

Using integration by parts, it is not hard to see that for $j = 0, 1, 2, \dots$,

$$\Big|\frac{d^j}{d\tau^j} S_0(\tau)\Big| \le \begin{cases} C_j \lambda^{-2-j} & \text{if } |\tau| \le \lambda, \\ C_j |\tau|^{-2-j} & \text{if } |\tau| > \lambda. \end{cases} \tag{5.14}$$

Given (5.14), if we argue as in the proof of Theorem 4.3.1 of [29], along with a change of variables, we have $|S_0(x,y)| \le C_0|\log(\lambda d_g(x,y)/2)|\mathbf{1}_{d_g(x,y)<\lambda^{-1}}(x,y)$. The kernel for the operator S_λ^1 is a consequence of stationary phase argument after using Hadamard parametrix, see [9] and [24] for more details.

Since by heat kernel methods, we have $\mathrm{Dom}(H_V) \subset L^\infty(M)$ when $n = 2$, by the very definition of the Kato space, $S_\lambda^0(Vu)$ is given by an absolutely convergent integral. Thus, if $\Lambda = \Lambda(M, V) \ge 1$ is sufficiently large, we have, since $V \in \mathcal{K}$,

$$\|S_\lambda^0(Vu)\|_{L^\infty(M)} \le \tfrac{1}{4}\|u\|_{L^\infty(M)} \quad \text{if } \lambda \ge \Lambda. \tag{5.15}$$

To use these bounds write

$$\begin{aligned} u &= (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} \circ (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u \\ &= T_\lambda(-\Delta_g + V - (\lambda + i\varepsilon(\lambda))^2)u + T_\lambda(V_{\le N} \cdot u) + T_\lambda(V_{>N} \cdot u) \\ &\quad + R_\lambda(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)u \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{5.16}$$

with $V_{\le N}$ and $V_{>N}$ as in (2.18).

By (5.9),

$$\|\mathrm{I}\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \tag{5.17}$$

and, by (5.8), we similarly obtain

$$\begin{aligned} \|\mathrm{IV}\|_\infty &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2} \cdot (\lambda\varepsilon(\lambda))\|u\|_2 \\ &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2, \end{aligned} \tag{5.18}$$

using the spectral theorem in the last inequality.

If we use (5.10), (5.11) and (5.15), along with Hölder's inequality, we conclude that we can fix N large enough so that

$$\|\mathrm{III}\|_\infty \le \tfrac{1}{2}\|u\|_\infty \quad \text{if } \lambda \ge \Lambda. \tag{5.19}$$

Also, (5.9) and (2.19) yield, for this fixed N,

$$\begin{aligned}\|\mathrm{II}\|_\infty &\le C_N(\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|u\|_2\\ &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2,\end{aligned} \tag{5.20}$$

using the spectral theorem and the fact that $\varepsilon(\lambda)\cdot\lambda \ge 1$ if $\lambda \ge 1$.

Combining (5.17), (5.18), (5.19) and (5.20) yields

$$\|u\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2 \quad \text{if } \lambda \ge \Lambda. \tag{5.21}$$

To obtain the quasimode estimate (5.2) for $q=\infty$, we need to see that the bounds in (5.21) are also valid when $1 \le \lambda < \Lambda$. As before this just follows from the fact that

$$\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^2(M)} \le C\|(H_V-\lambda^2+i\varepsilon(\Lambda)\Lambda)^{-1}f\|_{L^2(M)} \quad \text{if } 1\le\lambda\le\Lambda,$$

where C is a constant that depends on Λ.

Now we shall prove (5.2) for $6 \le q < \infty$. We shall focus on the term III, since by (5.7), (5.8) and (5.9), the other three terms are easily bounded by the right-hand side of (5.2). Note that, by (5.11), we have

$$\sup_y\Big(\int_M |T_\lambda^0(x,y)|^q\,dx\Big)^{1/q} \le C\lambda^{-2/q} \quad \text{if } 6\le q<\infty.$$

Whence by Minkowski's integral inequality,

$$\|T_\lambda^0\|_{L^1(M)\to L^q(M)} \le C\lambda^{-2/q}. \tag{5.22}$$

If we combine (5.10) and (5.22), by Hölder's inequality,

$$\|T_\lambda(V_{>N}u)\|_q \le C\lambda^{-2/q}\|V_{>N}u\|_1 \le C\lambda^{-2/q}\|V\|_1\|u\|_\infty.$$

Since we have just proved that

$$\|u\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2,$$

we conclude that the term III is dominated by the right-hand side of (5.2). ∎

We can also obtain the following improved quasimode estimates for the two-dimensional torus.

Theorem 5.2. *Let $\mathbb{T}^2$ denote the two-dimensional torus with flat metric, and assume that $V \in \mathcal{K}(\mathbb{T}^2)$. Then for $q > 6$ and*

$$\varepsilon(\lambda)=\varepsilon(\lambda,q)=\begin{cases}\lambda^{-\frac{q-6}{3q-10}+\delta_0} & \textit{for all } \delta_0>0 \textit{ if } 6<q<\infty,\\ \lambda^{-1/3} & \textit{if } q=\infty,\end{cases} \tag{5.23}$$

we have, for $u \in \mathrm{Dom}(H_V)$ *and* $\lambda \geq 1$,

$$\|u\|_{L^q(\mathbb{T}^2)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1/2}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^2(\mathbb{T}^2)}. \tag{5.24}$$

Similarly, if $\varepsilon(\lambda) \geq \lambda^{-1/5}$, *we have*

$$\|u\|_{L^6(\mathbb{T}^2)} \lesssim \lambda^{\varepsilon_0}(\lambda \cdot \varepsilon(\lambda))^{-5/6}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_{L^2(\mathbb{T}^2)}. \tag{5.25}$$

To prove (5.24), we shall of course use the fact that, by the spectral projection bounds in (4.69) and (4.70), if $\varepsilon(\lambda)$ satisfies (5.23), we have (5.24) when $V \equiv 0$, which is equivalent to the following:

$$\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1}\|_{L^2(M)\to L^q(M)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1/2}. \tag{5.26}$$

Also for the critical point $q = 6$, we shall use

$$\|(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1}\|_{L^2(M)\to L^6(M)} \lesssim \lambda^{\varepsilon_0}(\lambda\varepsilon(\lambda))^{-5/6} \tag{5.27}$$

for all $\varepsilon_0 > 0$, if $\lambda^{-1} \leq \varepsilon(\lambda) \leq 1$, which is a consequence of the spectral projection estimates in (4.4).

Now let us see how we can modify the proof of (5.2) to obtain (5.24) and (5.25). As before, we shall first prove (5.2) for the exponent $q = \infty$, and then use it to obtain similar inequalities for $6 \leq q < \infty$.

To proceed, write

$$(-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} = T_\lambda + R_\lambda, \quad \text{where } T_\lambda = T_0 + T_\lambda^1,$$

with T_λ^0, T_λ^1 and R_λ as in (3.17), (3.18) and (3.19), respectively.

Since $R_\lambda = m_\lambda(\sqrt{H_0})$, with $m_\lambda(\tau)$ as in (3.22), one can use (4.69), (4.70) and a simple orthogonality argument to see that for all $q > 6$,

$$\|R_\lambda\|_{L^2(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1/2} \tag{5.28}$$

and also

$$\|R_\lambda \circ (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)\|_{L^2(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1/2} \cdot (\lambda\varepsilon(\lambda)). \tag{5.29}$$

If we set $T_\lambda = T_\lambda^0 + T_\lambda^1$ as above, then, since $T_\lambda = (-\Delta_g - (\lambda + i\varepsilon(\lambda))^2)^{-1} - R_\lambda$, we trivially obtain from (5.26) and (5.28) the bound

$$\|T_\lambda\|_{L^2(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \lesssim \lambda^{\sigma(q)-1}(\varepsilon(\lambda))^{-1/2} \quad \text{if } q > 6. \tag{5.30}$$

For the operator T_λ^1, we claim that if $\varepsilon(\lambda) \geq \lambda^{-1/3}$ as in (5.23), we have

$$\|T_\lambda^1\|_{L^1(\mathbb{T}^2)\to L^\infty(\mathbb{T}^2)} = O(1). \tag{5.31}$$

To see this, we shall split the integral dyadically as before by writing

$$T_\lambda^1 = T_\lambda^{1,0} + \sum_{j=1}^{\infty} T_\lambda^{1,j},$$

where, for $j = 1, 2, \dots,$

$$T_\lambda^{1,j} = \frac{i}{\lambda + i\varepsilon(\lambda)} \int_0^\infty \beta(2^{-j}t)(1-\eta(t))\,\eta(t/T)\,e^{i\lambda t}e^{-\varepsilon(\lambda)t}\cos tP\,dt,$$

and $T_\lambda^{1,0}$ is given by an analogous formula with $\beta(2^{-j}t)$ replaced by $\beta_0(t) \in C_0^\infty(\mathbb{R}^n)$.

If $j = 0$, by using the spectral projection estimates of [27] and the fact that

$$T_\lambda^{1,0}(\tau) \lesssim \lambda^{-1}(1+|\lambda-\tau|)^{-N} \quad \text{for all } N, \text{ if } \lambda \ge 1 \text{ and } \tau \ge 0,$$

it is not hard to obtain

$$\|T_\lambda^{1,0}\|_{L^1(\mathbb{T}^2)\to L^\infty(\mathbb{T}^2)} = O(1). \tag{5.32}$$

On the other hand, if $j > 0$, by using (4.25) for $n = 2$, we have

$$\|T_\lambda^{1,j} f\|_{L^\infty(\mathbb{T}^2)} \lesssim 2^{3j/2}\lambda^{-1/2}\|f\|_{L^1(\mathbb{T}^2)}, \quad j = 1, 2, \dots. \tag{5.33}$$

Since $T_\lambda^{1,j} = 0$ if 2^j is larger than a fixed constant times $(\varepsilon(\lambda))^{-1}$, after summing over j, if $\varepsilon(\lambda) \ge \lambda^{-1/3}$, we obtain (5.31).

As for the local operator T_λ^0, by repeating the argument in (5.12)–(5.15), we have the following kernel estimates:

$$|T_\lambda^0(x,y)| \le \begin{cases} C_0|\log(\lambda d_g(x,y)/2)| & \text{if } d_g(x,y) \le \lambda^{-1}, \\ C_0\lambda^{-1/2}(d_g(x,y))^{-1/2} & \text{if } \lambda^{-1} \le d_g(x,y) \le 1, \end{cases} \tag{5.34}$$

which is independent of the choice of $\varepsilon(\lambda)$.

To use these bounds write

$$\begin{aligned} u &= (-\Delta_g - (\lambda+i\varepsilon(\lambda))^2)^{-1} \circ (-\Delta_g - (\lambda+i\varepsilon(\lambda))^2)u \\ &= T_\lambda(-\Delta_g + V - (\lambda+i\varepsilon(\lambda))^2)u + T_\lambda(V_{\le N}\cdot u) + T_\lambda(V_{>N}\cdot u) \\ &\quad + R_\lambda(-\Delta_g - (\lambda+i\varepsilon(\lambda))^2)u \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{5.35}$$

with $V_{\le N}$ and $V_{>N}$ as in (2.18).

By (5.30),

$$\|\mathrm{I}\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V - (\lambda+i\varepsilon(\lambda))^2)u\|_2, \tag{5.36}$$

and, by (5.29), we similarly obtain

$$\begin{aligned} \|\mathrm{IV}\|_\infty &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\cdot(\lambda\,\varepsilon(\lambda))\|u\|_2 \\ &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V - (\lambda+i\varepsilon(\lambda))^2)u\|_2, \end{aligned} \tag{5.37}$$

using the spectral theorem in the last inequality.

If we use (5.31), (5.34), and the definition of Kato class, we conclude as before that we can fix N large enough so that

$$\|\mathrm{III}\|_\infty \le \tfrac{1}{2}\|u\|_\infty \quad \text{if } \lambda \ge \Lambda. \tag{5.38}$$

Also, (5.30) and (2.19) yield, for this fixed N,

$$\begin{aligned}\|\mathrm{II}\|_\infty &\le C_N(\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|u\|_2 \\ &\lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2,\end{aligned} \tag{5.39}$$

using the spectral theorem and the fact that $\varepsilon(\lambda)\cdot\lambda \ge 1$ if $\lambda \ge 1$.

Combining (5.36), (5.37), (5.38) and (5.39) yields

$$\|u\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V-(\lambda+i\varepsilon(\lambda))^2)u\|_2 \quad \text{if } \lambda \ge \Lambda. \tag{5.40}$$

To obtain the quasimode estimate (5.24) for $q=\infty$, we need to see that the bounds in (5.21) are also valid when $1 \le \lambda < \Lambda$. As before, this just follows from the fact that

$$\|(H_V-\lambda^2+i\varepsilon(\lambda)\lambda)^{-1}f\|_{L^2(\mathbb{T}^2)} \le C\|(H_V-\lambda^2+i\varepsilon(\Lambda)\Lambda)^{-1}f\|_{L^2(\mathbb{T}^2)} \quad \text{if } 1\le\lambda\le\Lambda,$$

where C is a constant that depend on Λ.

Now we shall prove quasimode estimates for $q<\infty$. First, if $6<q<\infty$, by using (5.28), (5.29) and (5.30), we see that the terms I, II, and IV are bounded by the right-hand side of (5.24). Thus, we only need to focus on the third term III. Note that, by (5.34), we have

$$\sup_y\Big(\int_M |T^0_\lambda(x,y)|^q\,dx\Big)^{1/q} \le C\lambda^{-2/q} \quad \text{if } 6\le q<\infty.$$

Whence, by Minkowski's integral inequality,

$$\|T^0_\lambda\|_{L^1(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \le C\lambda^{-2/q}. \tag{5.41}$$

The $T^{1,0}_\lambda$ operator behaves like the local operator, and we can also use the spectral projection estimates in [27] to get

$$\|T^{1,0}_\lambda\|_{L^1(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \le C\lambda^{-2/q}. \tag{5.42}$$

To obtain the analogue of (5.42) for the operator $T^{1,j}_\lambda$, we shall use interpolation between (5.33) and the following estimates:

$$\|T^{1,j}_\lambda f\|_{L^2(\mathbb{T}^2)} \lesssim 2^{j/2}\lambda^{-1/2}\|f\|_{L^1(\mathbb{T}^2)} \quad j=1,2,\dots, \tag{5.43}$$

which follows from applying a dual version of (4.69) with $\rho=2^{-j}$ as well as the fact that

$$T^{1,j}_\lambda(\tau)=O(2^j(1+2^j|\lambda-\tau|)^{-N}) \quad \text{for all } N, \text{ if } \lambda\ge 1 \text{ and } \tau\ge 0.$$

Since $\frac{1}{q}=\frac{1}{2}\cdot\theta+\frac{1}{\infty}\cdot(1-\theta)$, with $\theta=\frac{2}{q}$, by interpolation between (5.33) and (5.43), we get

$$\|T^{1,j}_\lambda\|_{L^1(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \le C\lambda^{-1/2}2^{j(3/2-2/q)} \quad \text{if } 2<q<\infty. \tag{5.44}$$

After summing over $j\in\mathbb{N}$, with $2^j \lesssim \varepsilon(\lambda)^{-1}$, we conclude that

$$\|T^1_\lambda\|_{L^1(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \le C\lambda^{-1/2}\varepsilon(\lambda)^{2/q-3/2}+C\lambda^{-2/q}. \tag{5.45}$$

Thus, we would have

$$\|T^1_\lambda\|_{L^1(\mathbb{T}^2)\to L^q(\mathbb{T}^2)} \le C\lambda^{-2/q}, \tag{5.46}$$

if we knew $\varepsilon(\lambda) \ge \lambda^{-\frac{q-4}{3q-4}}$. However, since $\frac{q-4}{3q-4} > \frac{q-6}{3q-10}$, this yields (5.46) for all $\varepsilon(\lambda)$ satisfying (5.23), if $6 < q < \infty$.

If we combine (5.41) and (5.46), by Hölder's inequality,

$$\|T_\lambda(V_{>N}u)\|_q \le C\lambda^{-2/q}\|V_{>N}u\|_1 \le C\lambda^{-2/q}\|V\|_1\|u\|_\infty.$$

Since we have just proved that

$$\|u\|_\infty \lesssim (\varepsilon(\lambda))^{-1/2}\lambda^{-1/2}\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2,$$

we conclude that the term III is dominated by the right-hand side of (5.2), which completes the proof of (5.24).

To conclude the section, we shall prove (5.25), by using (5.27), (4.4), and by repeating the arguments above, we can easily see that the terms I, II, and IV are bounded by the right-hand side of (5.25). For the third term III, if we combine (5.41) and (5.45), and use (5.24) for $q = \infty$, as above, we have

$$\begin{aligned}\|T_\lambda(V_{>N}u)\|_6 &\le C(\lambda^{-1/3} + \lambda^{-1/2}\varepsilon(\lambda)^{1/3-3/2})\|V_{>N}u\|_1 \\ &\le C(\lambda^{-1/3} + \lambda^{-1/2}\varepsilon(\lambda)^{-7/6})\|V\|_1\|u\|_\infty \\ &\le \big((\varepsilon(\lambda))^{-1/2}\lambda^{-5/6} + (\varepsilon(\lambda))^{-5/3}\lambda^{-1}\big)\|(H_V - (\lambda + i\varepsilon(\lambda))^2)u\|_2,\end{aligned} \tag{5.47}$$

which is bounded by the right-hand side of (5.25) if $\varepsilon(\lambda) \ge \lambda^{-1/5}$. Thus, the proof of (5.25) is complete.

Appendix: Self-adjointness and limited Sobolev estimates

As we stated before, for brevity, dx denotes the Riemannian volume element for (M, g).

Proposition A.1. *For $n \ge 3$, if $V \in L^{n/2}(M)$, the quadratic form*

$$q_V(u, v) = -\int_M Vu\bar{v}\,dx + \int -\Delta_g u\bar{v}\,dx, \quad u, v \in \mathrm{Dom}(\sqrt{-\Delta_g + 1}), \tag{A.1}$$

is bounded from below and defines a unique semi-bounded self-adjoint operator H_V on L^2. Moreover, $C^\infty(M)$ constitutes a form core for q_V.[3]

Proof. Since $(-\Delta_g + 1)^{1/2}$ is self-adjoint, by perturbation theory (specifically the KLMN theorem, see Theorem X.17 of [22]), it suffices to prove that for any $0 < \varepsilon < 1$, there is a constant $C_\varepsilon < \infty$ such that

$$\int |V||u|^2\,dx \le \varepsilon\|(-\Delta_g + 1)^{1/2}u\|_2^2 + C_\varepsilon\|u\|_2^2, \quad u \in \mathrm{Dom}(\sqrt{H_0}), \tag{A.2}$$

where $H_0 = -\Delta_g + 1$.

[3] Recall that a *form core* for q_V is a subspace S which approximates elements u in the domain of the form, in that there exists a sequence $u_m \in S$ satisfying $\lim_m \|u - u_m\|^2 + q_V(u - u_m, u - u_m) = 0$.

To prove this, for each small $\delta > 0$, choose a maximal δ-separated collection of points $x_j \in M$, $j = 1, \dots, N_\delta$, $N_\delta \approx \delta^{-n}$. Thus, $M = \cup B_j$ if B_j is the δ-ball about x_j, and if B_j^* is the 2δ-ball with the same center,

$$\sum_{j=1}^{N_\delta} \mathbf{1}_{B_j^*}(x) \le C_M, \tag{A.3}$$

where C_M is independent of $\delta \ll 1$ if $\mathbf{1}_{B_j^*}$ denotes the indicator function of B_j^*. Since $V \in L^{n/2}(M)$, for any fixed ε, we can choose $\delta > 0$ small enough so that

$$C_M \Big(C_0 \sup_{x \in M} \|V\|_{L^{n/2}(B(x,2\delta))} \Big) < \varepsilon, \tag{A.4}$$

where C_0 is the constant in (A.5) below.

Now for each B_j, define a smooth bump function ϕ_j, with $\phi_j \equiv 1$ on B_j, and $\phi_j \equiv 0$ outside of B_j^*. Since $M = \cup B_j$, we have

$$\begin{aligned} \int |V|\,|u|^2\,dx &\le \sum_j \int |V|\,|\phi_j u|^2\,dx \\ &\le \Big(\sup_{x\in M} \|V\|_{L^{n/2}(B(x,2\delta))} \Big) \sum_j \|\phi_j u\|^2_{\frac{2n}{n-2}} \\ &\le C_0 \Big(\sup_{x\in M} \|V\|_{L^{n/2}(B(x,2\delta))} \Big) \sum_j \|\nabla(\phi_j u)\|_2^2 \\ &\le C_0 \Big(\sup_{x\in M} \|V\|_{L^{n/2}(B(x,2\delta))} \Big) \sum_j \big(\|\nabla(u)\|^2_{L^2(B_j^*)} + \|(\nabla\phi_j)u)\|_2^2 \big) \\ &\le \varepsilon \|(-\Delta_g + 1)^{1/2} u\|_2^2 + C_\varepsilon \|u\|_2^2, \quad u \in \mathrm{Dom}(\sqrt{H_0}), \end{aligned} \tag{A.5}$$

where $H_0 = -\Delta_g + 1$. Here we have used Sobolev estimates as well as (A.4). ■

If $u \in \mathrm{Dom}(\sqrt{-\Delta_g + 1})$, then $-\Delta_g u$ and Vu are both distributions. If H_V is the self-adjoint operator given by the proposition, then $\mathrm{Dom}(H_V)$ is all such u for which $-\Delta_g u + Vu \in L^2$.

If we take $\varepsilon = 1/2$ in (A.2), we indeed get, for large enough N,

$$\begin{aligned} \|\sqrt{-\Delta_g + 1}\,u\|_2^2 = \int (-\Delta_g + 1) u \overline{u}\,dy &\le 2 \int (-\Delta_g + V + N) u \overline{u}\,dy \\ &= 2 \|\sqrt{H_V + N} u\|_2^2 \quad \text{if } H_V = -\Delta_g + V. \end{aligned} \tag{A.6}$$

Thus, $(-\Delta_g + 1)^{1/2}(H_V + N)^{-1/2}$ and $(H_V + N)^{-1/2}(-\Delta_g + 1)^{1/2}$ are bounded on L^2. Since $(-\Delta_g + 1)^{-1/2}$ is a compact operator on L^2, so must be $(H_V + N)^{-1/2}$. From this, we conclude that the self-adjoint operator H_V has *discrete spectrum*.

A combination of Sobolev estimates for the unperturbed operator and (A.6) also gives us

$$\|u\|_{\frac{2n}{n-2}} \le C \|\sqrt{H_V + N} u\|_2 \quad \text{if } u \in \mathrm{Dom}(H_V). \tag{A.7}$$

Note that in the above inequality (A.5), and thus (A.6), we need the condition $n \geq 3$, because we do not have a suitable Sobolev inequality at $\frac{2n}{n-2}$ when $n = 2$. Additionally, if $n \geq 5$, by an analogous argument as in (A.5), we have, for any $0 < \varepsilon < 1$, that there is a constant $C_\varepsilon < \infty$ such that

$$\int |Vu|^2 \, dx \leq \varepsilon \|(-\Delta_g + 1)u\|_2^2 + C_\varepsilon \|u\|_2^2, \quad u \in \mathrm{Dom}(H_0), \tag{A.8}$$

where $H_0 = -\Delta_g + 1$.

Inequality (A.8) also appears in Theorem X.21 of [22] under a weaker assumption on V. The reason it does not hold when $n = 3, 4$ is that we do not have an appropriate Sobolev inequality at $\frac{2n}{n-4}$ when $n = 3, 4$. As a consequence of (A.8), we have, for large enough N,

$$C_1 \|(-\Delta_g + 1)u\|_2 \leq \|(H_V + N)u\|_2 \leq C_2 \|(-\Delta_g + 1)u\|_2 \tag{A.9}$$

if $H_V = -\Delta_g + V$.

After replacing V by $V + N$ to simplify the notation, we may assume, as we have throughout starting with (1.3), that (A.5) holds with $N = 0$. This just shifts the spectrum and does not change the eigenfunctions. In this case, the spectrum of H_V is positive and its eigenfunctions therefore are distributional solutions of

$$H_V e_\lambda = \lambda^2 e_\lambda \quad \text{for some } \lambda > 0,$$

which means here that λ is the eigenvalue of the "first order" operator $\sqrt{H_V}$, i.e.,

$$P_V e_\lambda = \lambda e_\lambda \quad \text{if } P_V = \sqrt{H_V}. \tag{A.10}$$

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