Rev. Mat. Iberoam. 38 (2022), no. 5, 1529–1551
DOI 10.4171/RMI/1302

Oscillating spectral multipliers on groups of Heisenberg type

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Abstract. We establish endpoint estimates for a class of oscillating spectral multipliers on Lie groups of Heisenberg type. The analysis follows an earlier argument due to the second and fourth author [Springer INdAM Ser., vol. 45 (2021)], but requires the detailed analysis of the wave equation on these groups due to Müller and Seeger [Anal. PDE 8 (2015)].

We highlight and develop the connection between sharp bounds for oscillating spectral multipliers and the problem of determining the minimal amount of smoothness required for Mihlin–Hörmander multipliers, a problem that has been solved for groups of Heisenberg type but remains open for other groups.

1. Introduction

Let G be a connected Lie group and let $X_1, X_2, \dots, X_n$ be left invariant vector fields on G which satisfy Hörmander's condition; that is, they generate, together with the iterative commutators, the tangent space of G at every point (a special case is a Lie group of Heisenberg type; see the next section for precise definitions). The sublaplacian $\mathcal{L} = -\sum_{j=1}^n X_j^2$ is then a nonnegative, second order hypoelliptic operator which is essentially self-adjoint on $L^2(\mu)$, where μ is the right Haar measure on G. Hence $\sqrt{\mathcal{L}}$ admits a spectral resolution $\{E_\lambda\}$; if m is a bounded, Borel measurable function on $[0, \infty)$, then the spectral theorem implies that

$$m(\sqrt{\mathcal{L}}\,) = \int_0^\infty m(\lambda)\, dE_\lambda$$

is a bounded operator on $L^2(G)$.

In this paper we will consider a general framework of spectral multipliers (introduced by Miyachi in the Euclidean setting, when $G = \mathbb{R}^d$; see [15] and [16]) which contains oscillating examples of the form

$$m_{\theta,\beta}(\lambda) = \frac{e^{i\lambda^\theta}}{\lambda^{\theta\beta/2}}\, \chi(\lambda) \tag{1.1}$$

2020 Mathematics Subject Classification: 42B15, 42B20, 43A22.
Keywords: Heisenberg type groups, spectral multipliers, sublaplacians, Mihlin–Hörmander multipliers, Hardy spaces.

for any $\theta, \beta \geq 0$. Here $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi(\lambda) \equiv 0$ for $\lambda \leq 1$ and $\chi(\lambda) \equiv 1$ when λ is large. Our interest will be in establishing endpoint bounds for the corresponding spectral operators. When $G = \mathbb{R}^d$ and $0 < \theta < 1$, endpoint bounds were established by C. Fefferman and Stein [22] and [5], where they introduced the tool of establishing Hardy space bounds from which one can interpolate to obtain the L^p endpoint estimate when $d|1/p - 1/2| = \beta/2$.

A consequence of our analysis is the following result.

Theorem 1.1. *Let G be a Lie group of Heisenberg type and $1 < p$. For all $\theta \geq 0$ with $\theta \neq 1$, $m_{\theta,\beta}(\sqrt{\mathcal{L}})$ is bounded on $L^p(G)$ if and only if $\beta/2 \geq d\,|1/p - 1/2|$. Furthermore, $m_{1,\beta}(\sqrt{\mathcal{L}})$ is bounded on $L^p(G)$ if and only if $\beta/2 \geq (d-1)|1/p - 1/2|$. Here d denotes the topological dimension of G.*

The second part is a statement about the wave operator on groups of Heisenberg type and is not new. The sufficiency is established by Müller and Seeger [17] (see earlier work of Müller and Stein in [19]), and the necessity follows from recent work in [20] which is valid on stratified groups of arbitrary step. The sufficiency of the first part is only new at the endpoint $\beta/2 = d|1/p - 1/2|$. The open range $\beta/2 > d|1/p - 1/2|$ follows from work on a closely related problem concerning Mihlin–Hörmander spectral multipliers.

The analysis in [20] and [12] shows that for the oscillating spectral multipliers $m_{2,\beta}(\lambda) = e^{i\lambda^2}\lambda^{-\beta}\chi(\lambda)$, the operator $m_{2,\beta}(\sqrt{\mathcal{L}})$ is unbounded on $L^p(G)$ when $\beta/2 < d|1/p - 1/2|$; this being the case of the Schrödinger group and is valid for sublaplacians $\mathcal{L}$ on any stratified Lie group G. This gives the necessity of the first statement in Theorem 1.1.

Theorem 1.1 is a consequence of a more general theorem. See Section 3.

An interesting, ongoing problem is to determine the minimal amount of smoothness for spectral multipliers m which guarantees that $m(\sqrt{\mathcal{L}})$ is bounded on all $L^p, 1 < p < \infty$ (and weak-type (1, 1), bounded on H^1, etc.). This problem has been studied extensively in the context of Mihlin–Hörmander multipliers where a scale-invariant Sobolev condition of the form

$$\|m\|_{L^q_{s,\mathrm{sloc}}} := \sup_{t \geq 0} \|m(t\cdot)\chi\|_{L^q_s(\mathbb{R}_+)} < \infty \tag{1.2}$$

is imposed for an appropriate $q \in [1, \infty]$ and $s \in [0, \infty)$; here[1] $\chi \in C^\infty_c(0, \infty)$ is any nontrivial smooth cut-off function and L^q_s is the L^q Sobolev space of order s. Following Martini and Müller [12] (see also [20]), we denote the Mihlin–Hörmander threshold $\mathfrak{s}(\mathcal{L})$ as the infimum in $s \in [0, \infty)$, where

$$\forall p \in (1, \infty),\ \exists C = C_{p,s} < \infty \quad \text{such that} \quad \|m(\sqrt{\mathcal{L}})\|_{p \to p} \leq C\|m\|_{L^2_{s,\mathrm{sloc}}} \tag{1.3}$$

holds for all bounded, Borel measurable m.

When $G = \mathbb{R}^d$ and $\mathcal{L} = -\Delta$ is the usual Laplacian, then it is well known that $\mathfrak{s}(-\Delta) = d/2$. A fundamental result of Christ [3] and Mauceri–Meda [13] shows that on any stratified Lie group G, $\mathfrak{s}(\mathcal{L}) \leq Q/2$, where Q is the *homogeneous dimension* of G. Shortly

[1] We will use χ to denote smooth cut-off functions in different situations (as in (1.1)). The context should be clear.

afterwards, Hebisch [6] and Müller and Stein [18] observed that when G is a group of Heisenberg type, then $\mathfrak{s}(\mathcal{L}) \le d/2$, where d is the topological dimension[2] of G. A number of results in this direction have been obtained since then, and we now know that $d/2 \le \mathfrak{s}(\mathcal{L}) < Q/2$ for any 2-step stratified group G, [12], and that the equality $\mathfrak{s}(\mathcal{L}) = d/2$ holds in a number of cases; see [10], [11] and [9].

Let G be a Lie group of Heisenberg type (in particular a 2-step stratified group) so that $\mathfrak{s}(\mathcal{L}) = d/2$. Since $\|m\|_{L^2_{s,\mathrm{sloc}}}$ is smaller than $\|m\|_{L^\infty_{s,\mathrm{sloc}}}$ and $\|m\|_{L^\infty_{0,\mathrm{sloc}}} \sim \|m\|_{L^\infty}$, we can interpolate the estimates in (1.3) with trivial L^2 bounds to conclude that

$$\|m(\sqrt{\mathcal{L}})\|_{p\to p} \le C_{p,s}\, \|m\|_{L^\infty_{s,\mathrm{sloc}}} \tag{1.4}$$

holds for all $p \in (1,\infty)$ and $s > d|1/p - 1/2|$. In particular, since the oscillating multiplier $m_{\theta,\beta}(\lambda)$ in (1.1) satisfies $\|m_{\theta,\beta}\|_{L^\infty_{\beta/2,\mathrm{sloc}}} < \infty$, we see that $m_{\theta,\beta}(\sqrt{\mathcal{L}})$ is bounded on $L^p(G)$ if $d|1/p - 1/2| < \beta/2$. Hence it is the endpoint L^p bound when $\beta/2 = d|1/p - 1/2|$ which interests us in this paper.

The estimates (1.4) motivate the interest in another threshold exponent $\mathfrak{s}_-(\mathcal{L})$, also introduced in [12], defined as the infimum in $s \in [0,\infty)$, where

$$\forall p \in (1,\infty),\ \exists\, C = C_{p,s} < \infty \quad \text{such that} \quad \|m(\sqrt{\mathcal{L}})\|_{p\to p} \le C\, \|m\|_{L^\infty_{s,\mathrm{sloc}}} \tag{1.5}$$

holds for all bounded, Borel measurable m. By interpolation, $\mathfrak{s}_-(\mathcal{L}) \le \mathfrak{s}(\mathcal{L})$, and so the exponent $\mathfrak{s}_-(\mathcal{L})$ can be used to provide lower bounds for $\mathfrak{s}(\mathcal{L})$.

In [20], the lower bound $d/2 \le \mathfrak{s}_-(\mathcal{L})$ was established in great generality, including sublaplacians $\mathcal{L}$ on any stratified Lie group of arbitrary step. In fact, in [20] it was shown that if (1.4) holds for some $1 \le p$ and $s \ge 0$ and all bounded Borel measurable m, then necessarily $s \ge d|1/p - 1/2|$. This implies that $d/2 \le \mathfrak{s}_-(\mathcal{L})$ $(\le \mathfrak{s}(\mathcal{L}))$ in this case. The same conclusions were obtained in [12] in less generality; for sublaplacians on any step 2 stratified group, with less robust methods.

Yet another threshold exponent $\mathfrak{s}_+(\mathcal{L})$ was introduced in [12], which will be useful for us. It is defined as the infimum in $s \in [0,\infty)$, where

$$\forall \text{compact } K \subset \mathbb{R},\ \exists C_{K,s} < \infty \quad \text{such that} \quad \|F(\sqrt{\mathcal{L}})\|_{1\to 1} \le C_{K,s} \|F\|_{L^2_s} \tag{1.6}$$

holds for all Borel measurable F with $\mathrm{supp}(F) \subset K$. The estimate (1.6) is the key estimate behind establishing the endpoint bound $\|m(\sqrt{\mathcal{L}})\|_{L^1\to L^{1,\infty}} \le C\|m\|_{L^2_{s,\mathrm{sloc}}}$ via the Calderón–Zygmund theory. Hence $\mathfrak{s}(\mathcal{L}) \le \mathfrak{s}_+(\mathcal{L})$; see [8]. In [12], the inequalities $d/2 \le \mathfrak{s}_-(\mathcal{L}) \le \mathfrak{s}(\mathcal{L}) \le \mathfrak{s}_+(\mathcal{L}) < Q/2$ were established for any 2-step stratified Lie group although in many cases, including the case of groups of Heisenberg type, we have $d/2 = \mathfrak{s}_-(\mathcal{L}) = \mathfrak{s}(\mathcal{L}) = \mathfrak{s}_+(\mathcal{L})$.

To see the relevance of (1.6), consider the natural decomposition

$$m(\lambda) = \sum_{j\in\mathbb{Z}} m(\lambda)\, \phi(2^{-j}\lambda) =: \sum_{j\in\mathbb{Z}} m_j(\lambda) \tag{1.7}$$

[2] For any stratified Lie group G, the topological dimension is always strictly smaller than the homogeneous dimension, except when G is Euclidean.

of a general spectral multiplier m, where $\phi \in C_c^\infty(0,\infty)$ satisfies $\sum_{j\in\mathbb{Z}} \phi(2^{-j}\lambda) \equiv 1$. Setting $m^j(\lambda) := m(2^j\lambda)\phi(\lambda)$, we see that for our oscillating multipliers $m_{\theta,\beta}$, we have the uniform bound

$$\|m^j_{\theta,\beta}\|_{L^2_s} \le C\, 2^{|j|\theta(s-\beta/2)}, \quad \forall s \ge 0. \tag{1.8}$$

Therefore if $\beta > 2\mathfrak{s}_+(\mathcal{L})$, we can find an s with $\mathfrak{s}_+(\mathcal{L}) < s < \beta/2$ and hence

$$\|m_{\theta,\beta}(\sqrt{\mathcal{L}})\|_{1\to 1} \le \sum_{j\in\mathbb{Z}} \|m^j_{\theta,\beta}(\sqrt{\mathcal{L}})\|_{1\to 1} \le C_s \sum_{j\in\mathbb{Z}} \|m^j\|_{L^2_s} \le C_s \sum_{j\in\mathbb{Z}} 2^{-|j|\theta(\beta/2-s)},$$

showing that $m_{\theta,\beta}(\sqrt{\mathcal{L}})$ is bounded on $L^1(G)$ when $\beta > 2\mathfrak{s}_+(\mathcal{L})$. Now when $\beta < 2s_+$, where $s_+ = \mathfrak{s}_+(\mathcal{L})$, we can embed $m_{\theta,\beta}$ into the analytic family of multipliers $m_z(\lambda) = \lambda^{\theta/2(\beta-(2s_++\delta)z)} m_{\theta,\beta}(\lambda)$ and use Stein's analytic interpolation theorem to conclude the following.

Lemma 1.2. *If $\beta < 2\mathfrak{s}_+(\mathcal{L})$, then the operator $m_{\theta,\beta}(\sqrt{\mathcal{L}})$ is bounded on $L^p(G)$ whenever $2\mathfrak{s}_+(\mathcal{L})|1/p-1/2| < \beta/2$.*

In particular when $\mathfrak{s}_+(\mathcal{L}) = d/2$, this gives us an alternative proof of the L^p boundedness for $m_{\theta,\beta}(\sqrt{\mathcal{L}})$ in the open range $d|1/p-1/2| < \beta/2$.

From the work of Fefferman and Stein [5] and [22] mentioned earlier in the Euclidean setting, we see that a natural way to establish the endpoint L^p bound when $d|1/p - 1/2| = \beta/2$ is to embed the oscillating multiplier $m_{\theta,\beta}$ with $\beta < d$ into the analytic family $\mathfrak{m}_z(\lambda) = \lambda^{\theta/2(\beta-dz)} m_{\theta,\beta}$ of multipliers so that $\mathfrak{m}_{\beta/d} = m_{\theta,\beta}$, and prove some Hardy space estimate $H^1 \to L^1$ for $\mathfrak{m}_{1+iy}$ with polynomial bounds in $|y|$. Since $\mathfrak{m}_{iy} \in L^\infty$, uniformly in y, the multiplier operator is uniformly bounded on L^2 and so Stein's analytic interpolation theorem can then be invoked to show $\mathfrak{m}_{\beta/d} = m_{\theta,\beta}$ is bounded on L^p for $d|1/p-1/2| = \beta/2$. This will be the procedure we will follow.

In the Euclidean setting, a different type of endpoint results for the Miyachi class of multipliers were established by Baernstein and Sawyer [1] (H^p, with $p < 1$), Seeger [21] (L^p, with $1 < p \le 2$) and Beltran and Bennett [2] (weighted estimates). It would interesting to explore to what extent these results can be extended to the Lie group setting.

Notation. We use the notation $A \lesssim B$ between two positive quantities A and B to denote $A \le CB$ for some constant C. We sometimes use the notation $A \lesssim_k B$ to emphasize that the implicit constant depends on the parameter k. We sometimes use $A = O(B)$ to denote the inequality $A \lesssim B$. Furthermore, we use $A \ll B$ to denote $A \le \delta B$ for a sufficiently small constant $\delta > 0$ whose smallness will depend on the context.

Outline of paper. In the next section we review the definition of Lie groups of Heisenberg type and recall the key results from [17], where Müller and Seeger give a detailed analysis of the wave equation in this setting, including the introduction of a local, isotropic Hardy space $h^1_{\text{iso}}(G)$ which is compatible with the underlying group structure. In Section 3 we develop a general framework of spectral multipliers which include both Mihlin–Hörmander multipliers as well as the oscillating examples (1.1). We formulate the main estimate in Theorem 1.1 in this more general framework. In Section 4 we give the proof of this main estimate on $h^1_{\mathrm{iso}}(G)$, up to the final step which requires a fine decomposition of the wave operator in [17]. We describe this decomposition in Section 5. In Section 6, we provide the final step in the the proof.

2. Groups of Heisenberg type: the work of Müller–Seeger [17]

Let G be a connected Lie group and let $\mathfrak{g}$ denote the associated Lie algebra. We say that G is a Lie group of Heisenberg type if its Lie algebra $\mathfrak{g}$ is a Lie algebra of Heisenberg type. A Lie algebra of Heisenberg type is a 2-step stratified Lie algebra, i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subspaces of dimensions d_1 and d_2 satisfying $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_2 \subseteq \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$. Moreover, the definition requires that, endowing $\mathfrak{g}$ with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal subspaces, the unique skew-symmetric endomorphisms J_μ on $\mathfrak{g}_1$, with $\mu \in \mathfrak{g}_2^* \setminus \{0\}$, defined by

$$\langle J_\mu(V), W \rangle = \mu([V, W]) \quad \text{for all pairs } V, W \in \mathfrak{g}_1$$

satisfy $J_\mu^2 = -|\mu|^2 I$. In particular, this implies that $\dim \mathfrak{g}_1 = d_1$ is even.

We fix an orthonormal basis $X_1, \ldots, X_{d_1}$ of $\mathfrak{g}_1$ and an orthonormal basis $U_1, \ldots, U_{d_2}$ of $\mathfrak{g}_2$. We identify the dual spaces $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ via the inner product.

From now on, G will denote a Lie group of Heisenberg type (and $\mathfrak{g}$ will denote a Lie algebra of Heisenberg type).

The operators $\mathcal{L}, -iU_1, \ldots, -iU_{d_2}$ form a set of positive strongly commuting self-adjoint operators and admit a joint spectral resolution. However we will only need to consider operators of the form $\phi(\mathcal{L}, |U|)$, where $U := (-iU_1, \ldots, -iU_{d_2})$.

We will identify G with its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \simeq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ via the exponential map and we write points in G as (x, u), where $x \in \mathfrak{g}_1 \simeq \mathbb{R}^{d_1}$ and $u \in \mathfrak{g}_2 \simeq \mathbb{R}^{d_2}$. The topological dimension of G is $d = d_1 + d_2$ and the homogeneous dimension is $Q = d_1 + 2d_2$. We can write the group law on G as

$$(x, u) \cdot (x', u') = (x + x', u + u' + \tfrac{1}{2}\langle Jx, x' \rangle),$$

where $x \in \mathfrak{g}_1$ and $u \in \mathfrak{g}_2$ and $\langle Jx, x' \rangle$ denotes a vector in $\mathfrak{g}_2$ with components $\langle J_{U_i} x, x' \rangle$.

Consider the positive sublaplacian

$$\mathcal{L} = -(X_1^2 + \cdots + X_{d_1}^2)$$

with spectral resolution $\sqrt{\mathcal{L}} = \int_0^\infty \lambda \, dE_\lambda$. Then

$$m(\sqrt{\mathcal{L}}) = \int_0^\infty m(\lambda)\, dE_\lambda \tag{2.1}$$

defines a spectral multiplier operator which is bounded on $L^2(G)$ precisely when m is a bounded, Borel measurable function on $\mathbb{R}_+ = [0, \infty)$.

Abusing notation, we will also denote by $m(\sqrt{\mathcal{L}})$ the convolution kernel of the operator $m(\sqrt{\mathcal{L}})$.

The main result of Müller and Seeger in [17] states that the wave operator $m_{1,\beta}(\sqrt{\mathcal{L}}) = e^{i\sqrt{\mathcal{L}}}(1+\sqrt{\mathcal{L}})^{-\beta/2}$ on a group G of Heisenberg type is bounded on $L^p(G)$ when $\beta/2 = (d-1)|1/p - 1/2|$. Recall that we included this in the statement of Theorem 1.1.

Hence solutions

$$u(\cdot,\tau) = \cos(\tau\sqrt{\mathcal{L}})\, f + \frac{\sin(\tau\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}\, g$$

of the Cauchy problem

$$(\partial_\tau^2 + \mathcal{L})u = 0, \quad u|_{\tau=0} = f, \quad \partial_\tau u|_{\tau=0} = g$$

satisfy the Sobolev inequality

$$\|u(\cdot,\tau)\|_p \le C\left[\|(1+\tau^2\mathcal{L})^{\gamma/2} f\|_p + \|\tau(1+\tau^2\mathcal{L})^{\gamma/2-1} g\|_p\right], \tag{2.2}$$

where $\gamma = (d-1)|1/p - 1/2|$.

The proof in [17] involves a detailed analysis of the singularities of the wave kernel on groups of Heisenberg type and an appropriate corresponding Littlewood–Paley type decomposition of the wave operator. Their analysis also gives sharp L^1 estimates for wave operators whose symbols are supported in a dyadic interval.

Theorem 2.1 ([17]). *Let $\chi \in C_c^\infty$ be supported in $(1/2, 2)$. Then the L^1 operator norm of $\chi(\lambda^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ has the following bound:*

$$\|\chi(\lambda^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}\|_{L^1(G)\to L^1(G)} \le C\,(1+|\lambda|)^{(d-1)/2}. \tag{2.3}$$

Such L^1 estimates immediately show that $\mathfrak{s}_+(\mathcal{L}) \le d/2$ on groups of Heisenberg type.

Proposition 2.1. *For all F compactly supported in $\mathbb{R}^+$, and for all $s > d/2$, we can find a constant $C = C_s$ such that*

$$\int_G |F(\sqrt{\mathcal{L}})(x)|\,dx \le C_s\,\|F\|_{L^2_s} \tag{2.4}$$

holds. Here we are employing our convention that $F(\sqrt{\mathcal{L}})(x)$ also denotes the convolution kernel of the operator $F(\sqrt{\mathcal{L}})$.

Proof. By the Fourier inversion formula, we write

$$F(t) = \int \widehat{F}(\tau)e^{it\tau}\,d\tau.$$

Let $\chi \in C_c^\infty(\mathbb{R})$ such that $F(t) = \chi(t)F(t)$ and hence

$$F(\sqrt{\mathcal{L}}) = \int \widehat{F}(\tau)\,\chi(\sqrt{\mathcal{L}})\,e^{i\tau\sqrt{\mathcal{L}}}\,d\tau. \tag{2.5}$$

By Theorem 2.1, we see that the operator $\chi(\sqrt{\mathcal{L}})\,e^{i\tau\sqrt{\mathcal{L}}}$ is bounded on $L^1(G)$ with operator norm $O((1+|\tau|)^{(d-1)/2})$. Hence by Cauchy–Schwarz,

$$\|F(\sqrt{\mathcal{L}})\|_{L^1(G)} \le \int |\widehat{F}(\tau)|(1+|\tau|)^{(d-1)/2}\,d\tau \lesssim_s \|F\|_{L^2_s}$$

for any $s > d/2$, establishing (2.4). ∎

As we already mentioned, the estimate (2.4) implies (via Calderón–Zygmund techniques; see [8]) that

$$\|m(\sqrt{\mathcal{L}})\|_{L^1\to L^{1,\infty}} \quad \text{and} \quad \|m(\sqrt{\mathcal{L}})\|_{L^p\to L^p} \le C\,\|m\|_{L^2_{s,\mathrm{sloc}}} \tag{2.6}$$

for any $s > d/2$ and all $1 < p < \infty$.

In [17], Müller and Seeger give an alternative proof of (2.6) which directly uses the L^1 operator norm bound of $\chi(\lambda^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ in (2.3), together with a pointwise bound on the convolution kernel of $\chi(\lambda^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ which is easily derived from the fundamental finite propagation speed property of solutions of the wave equation in this context (see [14]).

In fact the real part of the convolution kernel of $\chi(\lambda^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ can be written as $\varphi_\lambda * \mathcal{P}$, where φ_λ and $\mathcal{P}$ are the convolution kernels of $\chi(\lambda^{-1}\sqrt{\mathcal{L}})$ and $\cos(\sqrt{\mathcal{L}})$, respectively. Here $\varphi_\lambda(x,u) = \lambda^Q\varphi(\delta_\lambda(x,u))$ is a dilate of a Schwartz function φ and the finite speed property implies that $\mathcal{P}$ is a compactly supported distribution of finite order. Hence there exists an $M \ge 1$ such that the bound $|\varphi_\lambda * P(x,u)| \lesssim_N \lambda^M \|\delta_\lambda(x,u)\|_E^{-N}$ holds for large λ and any $N \ge 1$ (here $\|\cdot\|_E$ denotes the Euclidean norm on G). A similar bound holds for the imaginary part (see [17], Proposition 8.8, for further details).

We record this bound in the following proposition.

Proposition 2.2. *For $\lambda \ge 1$ and $\chi \in \mathcal{S}(\mathbb{R})$, we have the following bound for the convolution kernel of the operator $\chi(\lambda^{-1}\sqrt{\mathcal{L}})\,e^{i\sqrt{\mathcal{L}}}$: there exists an $M \ge 1$ such that*

$$\big|\chi(\lambda^{-1}\sqrt{\mathcal{L}})\,e^{i\sqrt{\mathcal{L}}}(x,u)\big| \le C_N\,\lambda^M(\lambda|x| + \lambda^2|u|)^{-N} \tag{2.7}$$

holds for any $N \ge 1$. Here C_N depends only on N and a suitable Schwartz norm of χ.

Müller and Seeger use (2.3) and (2.7) to give a short proof of (2.6). Their argument also shows that

$$\|m(\sqrt{\mathcal{L}})\|_{H^1(G)\to L^1(G)} \le C\,\|m\|_{L^2_{s,\mathrm{sloc}}} \tag{2.8}$$

for any $s > d/2$. Here $H^1(G)$ denotes the Hardy space on G defined with respect to the nonisotropic automorphic dilations

$$\delta_r(x,u) := (rx, r^2u), \quad r > 0, \tag{2.9}$$

together with the Korányi balls

$$B_r(x,u) := \{(y,v) \in G : \|(y,v)^{-1}\cdot(x,u)\|_K < r\}, \tag{2.10}$$

where $\|(x,u)\|_K := (|x|^4 + |4u|^2)^{1/4}$ defines the Korányi norm on G.

As outlined at the end of the Introduction, to prove the endpoint L^p bound for the oscillating spectral multipliers $m_{\theta,\beta}$ it suffices to embed $m_{\theta,\beta}$ into an analytic family $\mathfrak{m}_z$ and establish an appropriate Hardy space bound $H^1 \to L^1$ for $\mathfrak{m}_{1+iy}$ corresponding to $m_{\theta,d}$. It is natural to try to do this with the Hardy space $H^1(G)$ described above, defined with the nonisotropic automorphic dilations (2.9). However we are unable to do this; the geometry of the Korányi balls (2.10) does not seem appropriate for our problem. Instead we need a Hardy space that is defined using isotropic dilations $r(x,u) := (rx, ru)$

but which is also compatible with the Heisenberg group structure. For similar reasons, this was also the case for Müller and Seeger in their proof of (2.2), where they introduced a local isotropic Hardy space $h^1_{\mathrm{iso}}(G)$ and proved that $m_{1,d-1} = e^{i\sqrt{\mathcal{L}}}(1+\sqrt{\mathcal{L}})^{-(d-1)/2}$ is bounded from $h^1_{\mathrm{iso}}(G)$ to $L^1(G)$. Our main effort will be to establish that for $\theta \neq 1$, $m_{\theta,d}$ is bounded from $h^1_{\mathrm{iso}}(G)$ to $L^1(G)$.

A local, isotropic Hardy space

In [17] a local, isotropic Hardy space $h^1_{\mathrm{iso}}(G)$ was introduced in their study of the wave equation on Lie groups G of Heisenberg type. The classical Hardy space $H^1(G)$ is defined with respect to the homogeneous balls

$$B_r(x,u) = \{(y,v) \in G : \|(y,v)^{-1} \cdot (x,u)\|_K \leq r\}$$

so that $|B_r(x,u)| = c r^{d_1+2d_2} = c r^Q$. The space $h^1_{\mathrm{iso}}(G)$ is defined with respect to *isotropic balls* skewed by the Heisenberg group translation

$$B_r^E(x,u) := \{(y,v) \in G : \|(y,v)^{-1} \cdot (x,u)\|_E \leq r\},$$

where $\|(x,u)\|_E := |x| + |u|$ is comparable to the classical Euclidean norm on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Hence $|B_r^E(x,u)| = c r^d = c r^{d_1+d_2}$.

For $0 < r \leq 1$, we define a (P,r) atom as a function b supported in the isotropic Heisenberg ball $B_r^E(P)$ with radius r and centre P, such that $\|b\|_2 \leq r^{-d/2}$, and further, in the case where $r \leq 1/2$, we require the cancellation $\int b = 0$. A function f belongs to $h^1_{\mathrm{iso}}(G)$ if $f = \sum c_\nu b_\nu$, where b_ν is a (P_ν, r_ν) atom for some centre P_ν and radius $r_\nu \leq 1$ with $\sum |c_\nu| < \infty$. The norm of $f \in h^1_{\mathrm{iso}}(G)$ is defined as

$$\|f\|_{h^1_{\mathrm{iso}}(G)} := \inf \sum_\nu |c_\nu|,$$

where the infimum is taken over all representations $f = \sum_\nu c_\nu b_\nu$ where each b_ν is an atom. The space $h^1_{\mathrm{iso}}(G)$ is a closed subspace of $L^1(G)$ and for every $1 < p < 2$, $L^p(G)$ is a complex interpolation space for the couple $(h^1_{\mathrm{iso}}(G), L^2(G))$. See [17], where these facts are established.

3. A more general framework

For the oscillating spectral multipliers $m_{\theta,\beta}$ in (1.1), large $\lambda \gg 1$ are the important spectral frequencies and so, on the convolution kernel side, small balls are the important ones. Hence the local space $h^1_{\mathrm{iso}}(G)$ is the relevant one for these multipliers.

We now describe a more general framework of spectral multipliers (introduced by Miyachi [15] and [16] in the Euclidean setting) which include both the class of Mihlin–Hörmander multipliers as well as the oscillating examples $m_{\theta,\beta}$. We introduce a general class of spectral multipliers which satisfy some scale-invariant conditions (as in (1.2)) and which depend on an oscillation parameter $\theta \geq 0$, a decay parameter $\beta \geq 0$ and a smoothness parameter $s \geq 0$. We introduce the following class $M_{\theta,\beta,s}$ of spectral multipliers. Fix a nontrivial cut-off $\chi \in C_c^\infty(0,\infty)$. When $0 \le t \le 1$, we impose the standard uniform L^2 Sobolev norm condition

$$\sup_{0\le t\le 1} \|m(t\cdot)\chi\|_{L_s^2(\mathbb{R}_+)} < \infty, \tag{3.1}$$

and for $1 \le t$, we impose the conditions

$$\sup_{1\le t} t^{\theta\beta/2}\|m(t\cdot)\chi\|_{L^\infty(\mathbb{R}_+)} < \infty \quad \text{and} \quad \sup_{1\le t} t^{-\theta(2s-\beta)/2}\|m(t\cdot)\chi\|_{L_s^2(\mathbb{R}_+)} < \infty. \tag{3.2}$$

The conditions (3.1) and (3.2) do not depend on the choice of χ. For $m \in M_{\theta,\beta,s}$, we define $C_m^{\theta,\beta,s}$ to be the maximum of the quantities appearing in (3.2).

When $\theta = 0$, these conditions reduce to the condition $\sup_{t\ge 0}\|m(t\cdot)\chi\|_{L_s^2} < \infty$ and if this holds for some $s > d/2$, the fundamental work of Hebisch [6] and Müller–Stein [18] show that the spectral multiplier operator is bounded on all $L^p(G)$, $1 < p < \infty$.

The examples $m_{\theta,\beta}(\lambda) = e^{i\lambda^\theta}\lambda^{-\theta\beta/2}\chi(\lambda)$ when $\theta > 0$ from (1.1) satisfy the conditions (3.1) and (3.2). Note that (3.2) expresses a growth in the L_s^2 Sobolev norm of $m(t\cdot)\chi$ (when $s > \beta/2$) and a decay in the L^∞ norm of $m(t\cdot)\chi$. If the L^2 Sobolev condition in (3.2) is satisfied for some $s > 0$, then by complex interpolation, it is also satisfied for all $0 \le s' \le s$ with $C_m^{\theta,\beta,s'} \lesssim C_m^{\theta,\beta,s}$ since the $s' = 0$ case $\|m(t\cdot)\chi\|_{L^2} \lesssim t^{-\theta\beta/2}$ is implied by the L^∞ condition.

Define

$$\mathcal{M}_\beta := \bigcup_{\theta\ge 0,\theta\ne 1,s>d/2} M_{\theta,\beta,s}. \tag{3.3}$$

This puts us in the position to employ analytic interpolation arguments to deduce that $m \in \mathcal{M}_\beta$ is an $L^p(G)$ multiplier in the range $d|1/p - 1/2| \le \beta/2$ from a Hardy space bound for multiplier operators associated to $m \in \mathcal{M}_d$. Furthermore, from the invariance of $\mathcal{M}_d$ under multiplication by λ^{iy} for any real y (with resulting polynomial in y bounds in (3.1) and (3.2)), it suffices to show $m(\sqrt{\mathcal{L}})\colon h^1_{\text{iso}}(G) \to L^1(G)$ for $m \in \mathcal{M}_d$.

The main result here is to establish the endpoint bound, that a strong-type $L^p(G)$ estimate holds for $m(\sqrt{\mathcal{L}})$ with $m \in \mathcal{M}_\beta$ and $\beta < d$, when $\beta/2 = d|1/p - 1/2|$.

Theorem 3.1. *If $\beta < d$, then for any $m \in \mathcal{M}_\beta$, $m(\sqrt{L})$ is bounded on all $L^p(G)$ with $\beta/2 \ge d|1/p - 1/2|$.*

We will use an analytic interpolation argument to establish Theorem 3.1. First we we smoothly decompose the multiplier

$$m(t) = m_{\text{small}}(t) + m_{\text{large}}(t) \tag{3.4}$$

into high and low frequency parts, where $m_{\text{small}}(t) = m(t)$ for small $0 < t \lesssim 1$ and $m_{\text{large}}(t) = m(t)$ for large $t \gg 1$. We note that m_{small} is a Mihlin–Hörmander multiplier and we appeal to established results (see for instance [6]).

Therefore it suffices to treat the operator $m_{\text{large}}(\sqrt{\mathcal{L}})$. The key estimate for these operators is contained in the following theorem.

Theorem 3.2. *Suppose that $m \in M_{\theta,d,s}$ for some $\theta > 0$, $\theta \neq 1$, and some $s > d/2$. Let m_{large} be the large frequency part of m as described in (3.4). Then*

$$\|m_{\text{large}}(\sqrt{\mathcal{L}})f\|_{L^1(G)} \lesssim C_m^{\theta,d,s}\,\|f\|_{h^1_{\text{iso}}(G)},$$

where we recall $C_m^{\theta,d,s}$ is the maximum of the quantities appearing in (3.2).

To see how to complete the proof of Theorem 3.1 from Theorem 3.2, let $T(\sqrt{\mathcal{L}}) = m_{\text{large}}(\sqrt{\mathcal{L}})$ be the operator associated to the large frequencies of an m in the statement of Theorem 3.1. It suffices to establish an $L^p(G)$ bound at the endpoint $\beta/2 = d|1/p - 1/2|$ for $T(\sqrt{\mathcal{L}})$ when $m \in \mathcal{M}_\beta$. In particular, $m \in M_{\theta,\beta,s}$ for some $\theta \geq 0$, $\theta \neq 1$, and some $s > d/2$. It suffices to consider the case $\theta > 0$ since the case $\theta = 0$ corresponds to Mihlin–Hörmander multipliers, which have been successfully treated in the setting of groups of Heisenberg type; see [6] or [18].

For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \in [0,1]$, consider

$$\mathfrak{m}^z(\lambda) := m_{\text{large}}(\lambda)\,\lambda^{\theta/2(\beta - dz)}$$

and denote by $T_z(\sqrt{\mathcal{L}})$ the associated spectral multiplier operator. Note that $T_\sigma = T$, where $\sigma = \beta/d$ and $1/p = \sigma + (1-\sigma)/2$ when $1 < p < 2$ and $\beta/2 = d|1/p - 1/2|$. If $\mathrm{Re}(z) = 0$, then $\|\mathfrak{m}^{iy}\|_{L^\infty} \leq C_m^{\theta,\beta,s}$ since $m \in M_{\theta,\beta,s}$ and hence $T_{iy}(\sqrt{\mathcal{L}})$ is bounded on $L^2(G)$, uniformly in $y \in \mathbb{R}$. When $z = 1 + iy$, we have $\mathfrak{m}^{1+iy} \in M_{\theta,d,s}$, with

$$C_{\mathfrak{m}^{1+iy}}^{\theta,d,s} \lesssim |y|^s\,C_m^{\theta,\beta,s},$$

implying that $T_{1+iy}(\sqrt{\mathcal{L}})$ maps $h^1_{\text{iso}}(G)$ into $L^1(G)$ with polynomial bounds in y by Theorem 3.2. Hence $T(\sqrt{\mathcal{L}})$ is bounded on $L^p(G)$ by Stein's analytic interpolation theorem.

In proving Theorem 3.2, we will not explicitly state the constants $C_m^{\theta,d,s}$ when applying the condition (3.2), but the dependence will be clear whenever this condition is applied.

We follow an argument developed in [4] which establishes a Hardy space $H^1(G)$ bound for oscillating multipliers on general stratified groups. Our original hope was to adapt this argument, using only the bounds (2.3) and (2.7) from [17]. However, at the final step of the argument, we need to employ the full analysis of the wave operator as detailed in [17].

4. The first part of the proof of Theorem 3.2

We fix a spectral multiplier $m \in M_{\theta,d,s}$ for some $\theta \neq 1$ with $\theta > 0$ and some $s > d/2$, and consider the large frequency part m_{large} as described in (3.4).

We fix an atom a_B supported in an isotropic, Heisenberg ball $B = B_r^E(P)$ with $r \leq 1$ and we want to prove

$$\int_G |T(\sqrt{\mathcal{L}})a_B(x)|\,dx \lesssim 1, \tag{4.1}$$

where $T(\sqrt{\mathcal{L}}) = m_{\text{large}}(\sqrt{\mathcal{L}})$.

We decompose $m_{\mathrm{large}} = \sum_{j>0} m_j$, where $m_j(t) = m(t)\phi(2^{-j}t)$ for an appropriate smooth ϕ supported in $[1/2, 2]$. Hence $T(\sqrt{\mathcal{L}}) = \sum_{j>0} m_j(\sqrt{\mathcal{L}})$ and only the conditions for m in (3.2) are relevant.

Without loss of generality, we may assume that the ball B is centered at the origin, $P = 0$. Let $L \leq 0$ be such that $2^{L-1} < r \leq 2^L$. The L^2 boundedness of $T(\sqrt{\mathcal{L}})$ implies that, for any fixed $C > 0$,

$$\int_{\|(x,u)\|_E \leq C2^L} |T(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du \lesssim 1$$

via the Cauchy–Schwarz inequality, and so it suffices to show that

$$\int_{\|(x,u)\|_E \gg 2^L} |T(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du \lesssim 1. \tag{4.2}$$

We bound the integral in (4.2) by

$$\mathcal{I} := \int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du.$$

Writing $m^j(\lambda) := m_j(2^j\lambda) = m(2^j\lambda)\phi(\lambda)$, we have $m_j(\sqrt{\mathcal{L}}) = m^j(2^{-j}\sqrt{\mathcal{L}})$. Now, using (2.5), we write

$$m^j(\sqrt{\mathcal{L}}) = \int \widehat{m}^j(\tau)\,\chi(\sqrt{\mathcal{L}})\,e^{i\tau\sqrt{\mathcal{L}}}\,d\tau,$$

where $\chi \equiv 1$ on the support of ϕ so that

$$m_j(\sqrt{\mathcal{L}}) = \int \widehat{m}^j(\tau)\,\chi(2^{-j}\sqrt{\mathcal{L}})\,e^{i2^{-j}\tau\sqrt{\mathcal{L}}}\,d\tau. \tag{4.3}$$

We remark that in the τ integral above, we may assume $|\tau| \gg 1$ since

$$\int_{|\tau|\lesssim 1} |\widehat{m}^j(\tau)|\,\|\chi(2^{-j}\sqrt{\mathcal{L}})e^{i2^{-j}\tau\sqrt{\mathcal{L}}}\|_{L^1\to L^1}\,d\tau \lesssim \int_{|\tau|\lesssim 1} |\widehat{m}^j(\tau)|\,d\tau \lesssim 2^{-j\theta d/2}$$

by the uniform boundedness of $\|\chi(2^{-j}\sqrt{\mathcal{L}})e^{i2^{-j}\tau\sqrt{\mathcal{L}}}\|_{L^1\to L^1}$ for small $|\tau|$ (by dilation-invariance of the L^1 operator norm and an application of Hulanicki's theorem [7]), and the L^∞ assumption on m^j. This is summable over $j > 0$ since $\theta > 0$.

Henceforth we shall assume τ is large in the integral (4.3) representing $m_j(\sqrt{\mathcal{L}})$.

We split the integral $\mathcal{I} = \mathrm{I} + \mathrm{II}$ into two parts, where

$$\begin{aligned}\mathrm{I} + \mathrm{II} := &\sum_{j\in\mathcal{A}} \int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du \\ &+ \sum_{j\in\mathcal{B}} \int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du,\end{aligned}$$

where $\mathcal{A} = \{j > 0 :\ j(1-\theta) + L \leq 0\}$ and $\mathcal{B} = \{j > 0 :\ j(1-\theta) + L > 0\}$. Note that the set $\mathcal{B}$ is empty for $\theta > 1$.

For $j \in \mathcal{B}$,

$$\int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}}) a_B(x,u)|\, dx\, du \le \int_G |a_B(y,v)| \Big[\int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}})((y,v)^{-1} \cdot (x,u))|\, dx\, du \Big]\, dy\, dv,$$

and when $\|(x,u)\|_E \gg 2^L$, we have $\|(y,v)^{-1} \cdot (x,u)\|_E \ge 2^L$ whenever $\|(y,v)\|_E \le 2^L$. This follows easily from the group law

$$(y,v)^{-1} \cdot (x,u) = (x-y, u-v+\tfrac{1}{2}\langle Jx, y\rangle),$$

noting that $L \le 0$. Hence

$$\int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}}) a_B(x,u)|\, dx\, du \le \int_{\|(x,u)\|_E \ge 2^L} |m_j(\sqrt{\mathcal{L}})(x,u)|\, dx\, du,$$

where we used the fact that for atoms $\|a_B\|_{L^1(G)} \lesssim 1$.

Let us denote by $K_\tau(x,u)$ the convolution kernel of $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$. Then for $g = 2^j/\tau$, we have that $(K_\tau(x,u))_g := g^Q K_\tau(\delta_g(x,u))$ is the convolution kernel associated to $\chi(2^{-j}\sqrt{\mathcal{L}})e^{i2^{-j}\tau\sqrt{\mathcal{L}}}$. Hence by (4.3),

$$\int_{\|(x,u)\|_E \ge 2^L} |m_j(\sqrt{\mathcal{L}})(x,u)|\, dx\, du \le \int |\widehat{m}^j(\tau)| \Big[\int_{\|(x,u)\|_E \ge 2^L} |(K_\tau)_g(x,u)|\, dx\, du \Big] d\tau.$$

But $\|\delta_{g^{-1}}(x,u)\|_E \ge 2^L$ implies $\|\delta_\tau(x,u)\|_E \ge 2^{j+L}$ since $j > 0$. Hence we have

$$\int_{\|(x,u)\|_E \ge 2^L} |(K_\tau)_g(x,u)|\, dx\, du \le \int_{\|\delta_\tau(x,u)\|_E \ge 2^{j+L}} |K_\tau(x,u)|\, dx\, du.$$

Recall the pointwise estimate (2.7) for $K_\tau(x,u)$ (valid for large τ):

$$|K_\tau(x,u)| \lesssim_N |\tau|^M \, \|\delta_\tau(x,u)\|_E^{-N}$$

for some $M \ge 1$ and any $N \ge 1$. Hence if $|\tau| \le 2^{j+L}$, choosing N large enough we have

$$\int_{\|\delta_\tau(x,u)\|_E \ge 2^{j+L}} |K_\tau(x,u)|\, dx du \lesssim_N 2^{-N(j+L)}$$

and therefore

$$\int_{|\tau| \le 2^{j+L}} |\widehat{m}^j(\tau)| \Big[\int_{\|(x,u)\|_E \ge 2^L} |(K_\tau)_g(x,u)| dx du \Big] d\tau \lesssim_N 2^{-N(j+L)} \int_{|\tau| \le 2^{j+L}} |\widehat{m}^j(\tau)|\, d\tau$$

which by Cauchy–Schwarz and (3.2) is at most

$$2^{-(N-1/2)(j+L)} \|\widehat{m}^j\|_{L^2} \lesssim 2^{-(N-1/2)(j+L)} 2^{-j\theta d/2} \le 2^{-(N-1/2)(j+L)}.$$

Note that $j + L > 0$ for $j \in \mathcal{B}$, when $\mathcal{B}$ is non empty. Indeed, in this case we must have $0 < \theta < 1$, which implies $j + L > j(1-\theta) + L > 0$.

To treat the remaining part

$$\int_{|\tau|\geq 2^{j+L}} |\widehat{m}^j(\tau)|\Big[\int_{\|(x,u)\|_E\geq 2^L} |(K_\tau)_g(x,u)|\,dx\,du\Big]\,d\tau,$$

we promote the integration over $\|(x,u)\|_E \geq 2^L$ to all of G so that the inner integral is at most

$$\begin{aligned}\int_{|\tau|\geq 2^{j+L}} &|\widehat{m}^j(\tau)|\Big[\int_G |(K_\tau)_g(x,u)|\,dx\,du\Big]\,d\tau\\ &= \int_{|\tau|\geq 2^{j+L}} |\widehat{m}^j(\tau)|\Big[\int_G |K_\tau(x,u)|\,dx\,du\Big]\,d\tau \lesssim \int_{|\tau|\geq 2^{j+L}} |\widehat{m}^j(\tau)|\,|\tau|^{(d-1)/2}\\ &\leq 2^{-(\sigma-1/2)(j+L)}\,\|m^j\|_{L^2_{\sigma+(d-1)/2}} \lesssim 2^{-(\sigma-1/2)(j(1-\theta)+L)}\end{aligned}$$

for some $\sigma > 1/2$. The first inequality uses (2.3) in Theorem 2.1 followed by the Cauchy–Schwarz inequality and our hypothesis (3.2) on the L^2 Sobolev norms of m^j. Therefore,

$$\int_{\|(x,u)\|_E\geq 2^L} |m_j(\sqrt{\mathcal{L}})(x,u)|\,dx\,du \lesssim 2^{-(\sigma-1/2)(j(1-\theta)+L)}$$

and so

$$\int_{\|(x,u)\|_E\gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du \lesssim 2^{-(\sigma-1/2)(j(1-\theta)+L)},$$

showing that we can sum over all $j \in \mathcal{B}$ (since $\theta \neq 1$) and uniformly bound II.

For I, we split $\mathcal{A} = \mathcal{A}_- \cup \mathcal{A}_+$ further such that $\mathcal{A}_- = \{j \in \mathcal{A} : j + L \leq 0\}$ and $\mathcal{A}_+ = \{j \in \mathcal{A} : j + L > 0\}$. This splits $\mathrm{I} = \mathrm{I}_- + \mathrm{I}_+$ accordingly.

For the sum over $j \in \mathcal{A}_+$, we split each integral in I_+ into two parts,

$$\int_{\|(x,u)\|_E\gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du = S_{\Lambda,j} + L_{\Lambda,j}$$

for some positive $\Lambda > 0$, where

$$S_{\Lambda,j} := \int_{2^L\ll\|(x,u)\|_E\leq 2^{L+\Lambda}} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du$$

and $L_{\Lambda,j}$ is defined similarly but with the integration taken over (x,u) such that $2^{L+\Lambda} \leq \|(x,u)\|_E$.

For $S_{\Lambda,j}$ we use Cauchy–Schwarz, the L^2 bound $\|a_B\|_2 \leq 2^{-Ld/2}$ on our atom and our L^∞ hypothesis on m_j to see that

$$S_{\Lambda,j} \leq 2^{(L+\Lambda)d/2}\,\|m_j(\sqrt{\mathcal{L}})a_B\|_{L^2} \lesssim 2^{(L+\Lambda)d/2}\,2^{-j\theta d/2}\,\|a_B\|_{L^2} \leq 2^{\Lambda d/2}\,2^{-j\theta d/2}.$$

Next we will show that

$$L_{\Lambda,j} \lesssim_\sigma 2^{(\sigma-1/2)j\theta}\,2^{-(\sigma-1/2)(j+L+\Lambda)} \tag{4.4}$$

for some $\sigma > 1/2$. Choosing Λ such that

$$2^{\Lambda} = 2^{j\theta}\, 2^{-\frac{\sigma-1/2}{\sigma-1/2+d/2}(j+L)}$$

optimizes the bound for the sum $S_{\Lambda,j} + L_{\Lambda,j}$.

With this choice of Λ (which is > 0 since $j \in \mathcal{A}$) we have $S_{\Lambda,j} + L_{\Lambda,j} \lesssim 2^{-\epsilon(j+L)}$, where

$$\epsilon = \frac{d}{2} \cdot \frac{\sigma - 1/2}{\sigma - 1/2 + d/2} > 0,$$

and this shows that

$$\mathrm{I}_{+} = \sum_{j \in A_{+}} \left[S_{\Lambda,j} + L_{\Lambda,j} \right] \lesssim 1$$

is uniformly bounded. We now turn to establish (4.4).

Proceeding as for terms $j \in \mathcal{B}$, and using the formula (4.3), we see that

$$L_{\Lambda,j} \le \int |\widehat{m}^{j}(\tau)| \Big[\int_{2^{L+\Lambda} \le \|(x,u)\|_E} |(K_\tau)_g(x,u)|\, dx\, du \Big]\, d\tau =: L'_{\Lambda,j},$$

where we recall that $g = 2^j/\tau$, $(K_\tau)_g(x,u) = g^Q K_\tau(\delta_g(x,u))$ and $K_\tau(x,u)$ is the convolution kernel of $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$. We split $L'_{\Lambda,j} = \mathrm{III} + \mathrm{IV}$, where

$$\mathrm{III} := \int_{|\tau| \le 2^{j+L+\Lambda}} |\widehat{m}^{j}(\tau)| \Big[\int_{2^{L+\Lambda} \le \|(x,u)\|_E} |(K_\tau)_g(x,u)|\, dx\, du \Big]\, d\tau$$

and IV is defined in the same way except the integration in τ is over $|\tau| \ge 2^{j+L+\Lambda}$.

Again since $\|\delta_{g^{-1}}(x,u)\|_E \ge 2^{L+\Lambda}$ implies $\|\delta_\tau(x,u)\|_E \ge 2^{j+L+\Lambda}$ since $j > 0$, we have

$$\int_{\|(x,u)\|_E \ge 2^{L+\Lambda}} |(K_\tau)_g(x,u)|\, dx\, du \le \int_{\|\delta_\tau(x,u)\|_E \ge 2^{j+L+\Lambda}} |K_\tau(x,u)|\, dx\, du$$

and so

$$\mathrm{III} \le \int_{|\tau| \le 2^{j+L+\Lambda}} |\widehat{m}^{j}(\tau)| \Big[\int_{2^{j+L+\Lambda} \le \|\delta_\tau(x,u)\|_E} |K_\tau(x,u)|\, dx\, du \Big]\, d\tau,$$

which by (2.7) (recall we have reduced to large τ) implies that

$$\mathrm{III} \lesssim_N 2^{-N(j+L+\Lambda)} \int_{|\tau| \le 2^{j+L+\Lambda}} |\widehat{m}^{j}(\tau)|\, d\tau \lesssim 2^{-(N-1/2)(j+L+\Lambda)}\, 2^{-j\theta d/2}$$

for any $N \ge 1$. For IV, since τ is large we can use (2.3), Cauchy–Schwarz and our L^2 Sobolev condition on m^j to see that

$$\mathrm{IV} \lesssim \int_{2^{j+L+\Lambda} \le |\tau|} |\widehat{m}^{j}(\tau)|\, |\tau|^{(d-1)/2}\, d\tau \lesssim_\sigma 2^{-(\sigma-1/2)(j+L+\Lambda)}\, 2^{(\sigma-1/2)j\theta}$$

for some $\sigma > 1/2$. Hence $L_{\Lambda,j} \le \mathrm{III} + \mathrm{IV} \lesssim 2^{(\sigma-1/2)j\theta}\, 2^{-(\sigma-1/2)(j+L+\Lambda)}$ holds for some $\sigma > 1/2$, establishing (4.4).

Finally, we turn to the sum over $j \in \mathcal{A}_-$, where $j + L \leq 0$, and it is here that we would like to use the cancellation of the atom a_B. We are allowed to use the cancellation since if there are j's such that $j + L \leq 0$, we must have $L \leq -1$, so that the atom is supported in a ball of radius $r \leq 1/2$. We have

$$m_j(\sqrt{\mathcal{L}})a_B(x,u) = \int \big[m_j(\sqrt{\mathcal{L}})((y,v)^{-1}\cdot(x,u)) - m_j(\sqrt{\mathcal{L}})(x,u)\big]a_B(y,v)\,dy\,dv,$$

and through basic estimates we can bound the difference by $\|\delta_{2^j}(y,v)\|_K = 2^j\|(y,v)\|_K$. If this Korányi norm could be controlled by the Euclidean norm, then since $\|(y,v)\|_E \leq 2^L$ for $(y,v) \in B$, we would gain a factor of 2^{j+L} which is summable for $j \in \mathcal{A}_-$. It is here where we see the two incompatible geometries (coming from the Korányi norm on the one hand, and the Euclidean norm on the other) creating an obstacle.

To overcome this, we will need to employ a refined decomposition in [17] of the operator $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ appearing in the formula (4.3):

$$m_j(\sqrt{\mathcal{L}}) = \int \widehat{m}^j(\tau)\,\chi(2^{-j}\sqrt{\mathcal{L}})\,e^{i2^{-j}\tau\sqrt{\mathcal{L}}}\,d\tau.$$

In the next section we will describe this decomposition, but now we make a couple preliminary remarks. First recall that we may assume τ is large in the above integral.

Second, we can easily handle any error terms $E_{\tau,j}(\mathcal{L})$ arising in our decomposition of $\chi(2^{-j}\sqrt{\mathcal{L}})e^{i2^{-j}\tau\sqrt{\mathcal{L}}}$ which have a uniform L^1 operator norm of $O(|\tau|^{-2})$ since

$$\int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\|E_{\tau,j}\mathcal{L}\|_{L^1\to L^1}\,d\tau \lesssim \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\,|\tau|^{-2}d\tau \lesssim 2^{-j\theta d/2}, \tag{4.5}$$

which again is summable for $j > 0$.

5. A refined decomposition of $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$

In this section we will recall a decomposition of the operator $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$ from [17] which we will need to carry out the final step in the argument. For the reader's convenience, we give an outline of how this decomposition is derived, but refer to [17] for the precise details of each step.

The decomposition is based on the following subordination formula which relates the wave operator $e^{i\sqrt{\mathcal{L}}}$ to the Schrödinger group $\{e^{it\mathcal{L}}\}$. See Proposition 4.1 in [17].

Lemma 5.1. *Let $\chi_1 \in C_c^\infty$ be equal to 1 on the support of χ. Then there exist smooth functions $a_\tau(s)$ and $\rho_\tau(s)$ supported for $s \sim 1$ such that*

$$\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}} = \chi_1(\tau^{-2}\mathcal{L})\sqrt{\tau}\int e^{i\frac{\tau}{4s}}\,a_\tau(s)\,e^{is\mathcal{L}/\tau}ds + \rho_\tau(\tau^{-2}\mathcal{L}). \tag{5.1}$$

Furthermore, the $L^p(G)$ operator norm of $\rho_\tau(\tau^{-2}\mathcal{L})$ is $O(|\tau|^{-N})$ for every $N \geq 1$.

Hence by (4.5), the contribution corresponding to the term $\rho_\tau(\tau^{-2}\mathcal{L})$ can be easily treated, being uniformly summable in $j > 0$.

The formula (5.1) is an immediate consequence of the spectral resolution (2.1) for $\sqrt{\mathcal{L}}$ and the stationary phase formula

$$\chi(\sqrt{x})e^{i\lambda\sqrt{x}} = \chi_1(x)\sqrt{\lambda}\int e^{i\frac{\lambda}{4s}}\,a_\lambda(s)\,e^{i\lambda s x}\,ds + \rho_\lambda(x), \tag{5.2}$$

together with the change of variables $x \to x/\lambda^2$. The phase $\Phi(s) = \lambda x s + \lambda/4s$ in the oscillatory integral in (5.2) has a unique nondegenerate critical point at $s_* = 1/2\sqrt{x}$, which is comparable to 1 due to the support of χ and χ_1. Also $\Phi(s_*) = \lambda\sqrt{x}$ and $\Phi''(s_*) = 4\lambda x^{3/2}$. Hence the stationary phase formula says

$$\chi_1(x)\sqrt{\lambda}\int e^{i\Phi(s)}a_\lambda(s)\,ds = c\,\chi_1(x)\,\frac{e^{i\Phi(s_*)}}{\sqrt{\Phi''(s_*)}} + \text{error} = \tilde{\chi}(\sqrt{x})e^{i\lambda\sqrt{x}} + \text{error},$$

but of course a careful analysis of this principle with uniform control of the various terms, especially the error term, is required to make this precise and useful. See [17] for details.

Lemma 5.1 allows us to understand the wave operator $e^{i\sqrt{\mathcal{L}}}$ via the Schrödinger group $\{e^{it\mathcal{L}}\}$, where explicit formulae are well known. For instance, the convolution kernel S_t for $e^{it\mathcal{L}}$ is given by

$$S_t(x,u) = \int_{\mathbb{R}^{d_2}}\Big(\frac{|\mu|}{2\sin(2\pi t|\mu|)}\Big)^{d_1/2} e^{-i|x|^2\frac{\pi}{2}|\mu|\cot(2\pi t|\mu|)}\,e^{2\pi i u\cdot\mu}\,d\mu. \tag{5.3}$$

From (5.1), we see that $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}} = \chi_1(\tau^{-2}\mathcal{L})n_\tau(\mathcal{L}) + \rho_\tau(\tau^{-2}\mathcal{L})$, where

$$n_\tau(\mathcal{L}) = \sqrt{\tau}\int_{\mathbb{R}} e^{i\frac{\tau}{4s}}\,a_\tau(s)\,e^{is\mathcal{L}/\tau}\,ds$$

and so by (5.3), the convolution kernel of $n_\tau(\mathcal{L})$ is given by the formula

$$n_\tau(x,u) = \sqrt{\tau}\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}} e^{i\frac{\tau}{4s}}\,a_\tau(s)\Big(\frac{|\mu|}{2\sin(2\pi s|\mu|/\tau)}\Big)^{d_1/2} \times e^{-i|x|^2\frac{\pi}{2}|\mu|\cot(2\pi s|\mu|/\tau)}\,e^{2\pi i u\cdot\mu}\,ds\,d\mu.$$

As the phase above exhibits periodic singularities (where $\sin(2\pi s|\mu|/\tau) = 0$), it is natural to introduce an equally spaced decomposition in the central Fourier variables $|\mu|$: that is, with respect to the spectrum of the operator $|U|$ (recall that $U := (-iU_1,\dots,-iU_{d_2})$).

We fix an $\eta_0 \in C_c^\infty(\mathbb{R})$ supported in a small neighborhood of the origin and such that $\sum_{k\in\mathbb{Z}}\eta_0(t-k\pi) \equiv 1$. We decompose $n_\tau(\mathcal{L}) = n_\tau^0(\mathcal{L},|U|) + n_\tau^1(\mathcal{L},|U|)$, where

$$n_\tau^0(\mathcal{L},|U|) = \sqrt{\tau}\int_{\mathbb{R}} e^{i\frac{\tau}{4s}}\,a_\tau(s)\,\eta_0\Big(\frac{s}{\tau}|U|\Big)\,e^{is\mathcal{L}/\tau}\,ds$$

and $n_\tau^1(\mathcal{L},|U|) = \sum_{k\geq 1} n_{\tau,k}(\mathcal{L},|U|)$, where

$$n_{\tau,k}(\mathcal{L},|U|) = \sqrt{\tau}\int_{\mathbb{R}} e^{i\frac{\tau}{4s}}\,a_\tau(s)\,\eta_0\Big(\frac{s}{\tau}|U| - k\pi\Big)\,e^{is\mathcal{L}/\tau}\,ds.$$

The refined L^1 estimates established in [17] which prove (2.3) in Theorem 2.1 are the following.

Proposition 5.1. *We have*

(5.4) $$\|n^0_\tau(\mathcal{L}, |U|)\|_{L^1\to L^1} \lesssim (1+|\tau|)^{(d-1)/2}$$

and for each $k \geq 1$,

(5.5) $$\|n_{\tau,k}(\mathcal{L}, |U|)\|_{L^1\to L^1} \lesssim k^{-(d_1+1)/2}\,(1+|\tau|)^{(d-1)/2}.$$

Since $d_1 \geq 2$, the sum over $k \geq 1$ converges and so Theorem 2.1 is an immediate consequence of Proposition 5.1.

Denote by $\mathcal{V}_\tau$ and $\mathcal{W}_\tau$ the operators $\chi_1(\tau^{-2}\mathcal{L})n^0_\tau(\mathcal{L}, |U|)$ and $\chi_1(\tau^{-2}\mathcal{L})n^1_\tau(\mathcal{L}, |U|)$, respectively, so that

(5.6) $$\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}} = \mathcal{V}_\tau + \mathcal{W}_\tau + E_\tau,$$

where the operator E_τ is negligible; the L^1 operator norm is $O(|\tau|^{-N})$ for any $N \geq 1$ and so by (4.5) gives rise to a bound of the form $2^{-j\theta d/2}$.

Finally, we need to decompose $\mathcal{W}_\tau = \sum_{n\geq 0} \mathcal{W}_{\tau,n}$, which will localize $|U|/\tau$ (or the central Fourier variables $|u|/\tau$). Fix two smooth functions ζ_0 and ζ_1, with ζ_0 supported in $(-1, 1)$ and ζ_1 supported in $\pm(1/2, 2)$ so that $\zeta_0(t) + \sum_{n\geq 1} \zeta_1(2^{-n}t) \equiv 1$. Define

$$\mathcal{W}_{\tau,0} := \zeta_0(\tau^{-1}|U|)\mathcal{W}_\tau, \quad \text{and for } n \geq 1, \quad \mathcal{W}_{\tau,n} := \zeta_1(\tau^{-1}2^{-n}|U|)\mathcal{W}_\tau$$

so that $\mathcal{W}_\tau = \sum_{n\geq 0} \mathcal{W}_{\tau,n}$ holds. This finer decomposition will allow us to identify when we will be able to implement the argument outlined at the end of previous section which uses the cancellation of the atom a_B.

6. The final step in the proof of Theorem 3.2

Recall that the last step in the proof of Theorem 3.2 is to uniformly bound

(6.1) $$\mathrm{I}_- := \sum_{j>0:j+L\leq 0} \int_{\|(x,u)\|_E \gg 2^L} |m_j(\sqrt{\mathcal{L}})a_B(x,u)|\,dx\,du =: \sum_{j+L\leq 0} \mathrm{I}_j.$$

From the previous section, we see that the operator $\chi(2^{-j}\sqrt{\mathcal{L}})e^{i2^{-j}\tau\sqrt{\mathcal{L}}}$ appearing in the formula (4.3),

$$m_j(\sqrt{\mathcal{L}}) = \int \widehat{m}^j(\tau)\,\chi(2^{-j}\sqrt{\mathcal{L}})\,e^{i2^{-j}\tau\sqrt{\mathcal{L}}}\,d\tau,$$

can be written as $(\mathcal{V}_\tau)_g + (\mathcal{W}_\tau)_g + (E_\tau)_g$, where the L^1 operator norm of $(E_\tau)_g$ is uniformly $O(|\tau|^{-N})$ for any $N \geq 1$, $g = 2^j/\tau$ and the convolution kernel of $(\mathcal{W}_\tau)_g$, say, is given by the L^1 invariant dilate $(\mathcal{W}_\tau)_g(x,u) = g^Q\,\mathcal{W}_\tau(\delta_g(x,u))$.

From (4.5), we have the bound $\mathrm{I}_j \leq \mathrm{I}^1_j + \mathrm{I}^2_j + O(2^{-j\theta d/2})$, where

(6.2) $$\mathrm{I}^1_j = \int_{\|(x,u)\|_E\gg 2^L} \left| \int_{|\tau|\gg 1} \widehat{m}^j(\tau)(\mathcal{V}_\tau)_g\,a_B(x,u)\,d\tau \right| dx\,du$$

and

$$\mathrm{I}_j^2 = \int_{\|(x,u)\|_E \gg 2^L} \left| \int_{|\tau|\gg 1} \widehat{m}^j(\tau)(\mathcal{W}_\tau)_g\, a_B(x,u)\, d\tau \right| dx\, du.$$

We first treat I_j^2, noting

$$\mathrm{I}_j^2 \le \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\, \|(\mathcal{W}_\tau)_g a_B\|_{L^1(G)}\, d\tau \le \sum_{n\ge 0} \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\, \|(\mathcal{W}_{\tau,n})_g a_B\|_{L^1(G)}\, d\tau,$$

where we employed the decomposition $\mathcal{W}_\tau = \sum_{n\ge 0} \mathcal{W}_{\tau,n}$ from the end of the previous section.

From (5.5) in Proposition 5.1, we see that $\|n_\tau^1\|_{L^1\to L^1} \lesssim (1+|\tau|)^{(d-1)/2}$ and hence $\|\mathcal{W}_{\tau,0}\|_{L^1\to L^1} \lesssim (1+|\tau|)^{(d-1)/2}$. Also, we can write

$$\mathcal{W}_{\tau,n} = \zeta_1(\tau^{-1}2^{-n}|U|)\mathcal{W}_\tau = \sum_{k\ge n+C} \zeta_1(\tau^{-1}2^{-n}|U|)\, n_{\tau,k},$$

where the constant C depends only on support properties of η_0 and a_τ, thus the estimate $\|\mathcal{W}_{\tau,n}\|_{L^1\to L^1} \lesssim 2^{-n(d_1-1)/2}(1+|\tau|)^{(d-1)/2}$ follows again from (5.5). Hence

$$\|\mathcal{W}_{\tau,n}\|_{L^1\to L^1} \lesssim 2^{-n(d_1-1)/2}(1+|\tau|)^{(d-1)/2} \tag{6.3}$$

holds for all $n \ge 0$.

We now split the sum over $n \ge 0$; for large n, we will use the exponential decay in n in the estimate (6.3) and here we will not need the cancellation of the atom a_B. For small n, we will use the cancellation of the atom, but the incompatibility of the two geometries will diminish the favorable factor 2^{j+L} coming from the difference with an exponential growth in n which nonetheless will be admissible if n is small enough.

Recall that when $j \in \mathcal{A}_-$, we have $2^{j+L} \le 1$ and we split the sum in n, writing $\mathrm{I}_j^2 \le \mathrm{I}_{j,1}^2 + \mathrm{I}_{j,2}^2$, where

$$\mathrm{I}_{j,1}^2 := \sum_{n\in\mathcal{N}_1} \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\|(\mathcal{W}_{\tau,n})_g a_B\|_{L^1(G)}\, d\tau$$

and $\mathcal{N}_1 = \{n \ge 0 : 2^{-\epsilon(j+L)} \le 2^n\}$ for some fixed small $\epsilon > 0$. We define $\mathrm{I}_{j,2}^2$ similarly, where now the sum over n lies in $\mathcal{N}_2 = \{n \ge 0 : 2^n \le 2^{-\epsilon(j+L)}\}$.

For $\mathrm{I}_{j,1}^2$ we use the bound in (6.3) to conclude that

$$\mathrm{I}_{j,1}^2 \lesssim \sum_{n\in\mathcal{N}_1} 2^{-n(d_1-1)/2} \int |\widehat{m}^j(\tau)||\tau|^{(d-1)/2}\, d\tau \lesssim 2^{\delta(j+L)} \int |\widehat{m}^j(\tau)||\tau|^{(d-1)/2}\, d\tau$$

for some $\delta > 0$. But

$$\int |\widehat{m}^j(\tau)||\tau|^{(d-1)/2}\, d\tau = \int_{|\tau|\le 2^{j\theta}} |\widehat{m}^j(\tau)||\tau|^{(d-1)/2}\, d\tau + \int_{|\tau|\ge 2^{j\theta}} |\widehat{m}^j(\tau)||\tau|^{(d-1)/2}\, d\tau$$

and the first integral is uniformly bounded in j by an application of Cauchy–Schwarz and our L^∞ condition on m^j. The second integral is also uniformly bounded in j by another application of Cauchy–Schwarz and our L^2 Sobolev condition on m^j. Hence $\mathrm{I}_{j,1}^2 \lesssim 2^{\delta(j+L)}$, which is uniformly summable over j with $j + L \le 0$.

To treat $\mathrm{I}_{j,2}^2$, we make the understanding that $\zeta = \zeta_0$ when $n = 0$ and $\zeta = \zeta_1$ when $n \geq 1$. Hence

$$(\mathcal{W}_{\tau,n})_g(\mathcal{L}) = n_\tau^1(\tau^2 2^{-2j}\mathcal{L}, \tau 2^{-j}|U|)\, \chi_1(2^{-2j}\mathcal{L})\, \zeta(2^{-j-n}|U|)$$

and so $(\mathcal{W}_{\tau,n})_g(\mathcal{L})a_B(x,u) = (\mathcal{W}_{\tau,n})_g(\mathcal{L})(a_B * H_{j,n})(x,u)$, where $H_{j,n}$ is defined as

$$f * H_{j,n} := \chi_1'(2^{-2j}\mathcal{L})\, \zeta'(2^{-j-n}|U|)\, f,$$

where χ_1' and ζ' are smooth cut-off functions which are identically equal to 1 on the supports of χ_1 and ζ, respectively. Hence $\chi_1 = \chi_1\chi_1'$ and $\zeta = \zeta\zeta'$.

Therefore,

$$\mathrm{I}_{j,2}^2 \leq \sum_{n\in\mathcal{N}_2} \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\, \|\mathcal{W}_{\tau,n}\|_{L^1\to L^1} \|a_B * H_{j,n}\|_{L^1(G)}\, d\tau.$$

We note that $H_{j,n}$ is a δ_{2^j} dilate of the convolution of the Schwartz function h_1 (coming from $\chi_1'(\mathcal{L})$) and the 2^n dilate of the convolution kernel of $\zeta'(|U|)$, which has the form $\delta \otimes h_2$, where δ is the Dirac measure on $\mathbb{R}^{d_1}$ and h_2 is a Schwartz function on $\mathbb{R}^{d_2}$. Hence

$$H_{j,n}(x,u) = \int 2^{j(d_1+2d_2)}\, h_1(2^j x, 2^{2j} w)\, 2^{d_1(j+n)}\, h_2(2^j(u-w))\, dw \tag{6.4}$$

and in particular, we see that $\|H_{j,n}\|_{L^1} \lesssim 1$.

We now use the cancellation of the atom a_B. We have

$$\begin{aligned} a_B * H_{j,n}(x,u) &= \int a_B(y,v)\big[H_{j,n}((y,v)^{-1}\cdot(x,u)) - H_{j,n}(x,u)\big]\, dy\, dv \\ &= -\int a_B(y,v)\Big(\int_0^1 \big\langle y, \nabla_x H_{j,n}(x-sy, u-sv+\tfrac{1}{2}\langle Jx,y\rangle)\big\rangle \\ &\qquad + \big\langle v+\tfrac{1}{2}\langle Jx,y\rangle, \nabla_u H_{j,n}(x-sy, u-sv+\tfrac{1}{2}\langle Jx,y\rangle)\big\rangle\, ds\Big)\, dy\, dv. \end{aligned}$$

Using $\langle Jx,y\rangle = \langle J(x-sy),y\rangle$, we see that

$$\|a_B * H_{j,n}\|_{L^1(G)} \lesssim 2^L\left[\|\nabla_x H_{j,n}\|_{L^1(G)} + \|\nabla_u H_{j,n}\|_{L^1(G)} + \||x|\nabla_u H_{j,n}\|_{L^1(G)}\right]$$

and so (6.4) implies $\|a_B * H_{j,n}\|_{L^1(G)} \lesssim 2^n 2^{j+L}$.

We should mention that the bound $\|a_B * H_{j,n}\|_{L^1} \lesssim \min(1, 2^n 2^{j+L})$ was used in [17] (see Lemma 9.1 there). We have reproduced a sketch the proof here for the convenience of the reader.

The bound $\|a_B * H_{j,n}\|_{L^1(G)} \lesssim 2^n 2^{j+L}$, together with $\|\mathcal{W}_{\tau,n}\|_{L^1\to L^1} \lesssim |\tau|^{(d-1)/2}$, shows

$$\mathrm{I}_{j,2}^2 \lesssim \sum_{n\in\mathcal{N}_2} 2^n 2^{j+L} \int |\widehat{m}^j(\tau)|\,|\tau|^{(d-1)/2}\, d\tau \lesssim 2^{\delta(j+L)} \int |\widehat{m}^j(\tau)|\,|\tau|^{(d-1)/2}\, d\tau$$

for some $\delta > 0$. Since we have seen that the above integral is uniformly bounded in j, we see that

$$|\mathrm{I}_j^2| \lesssim 2^{\delta(j+L)} \quad \text{for some } \delta > 0, \tag{6.5}$$

which is uniformly summable over j with $j + L \leq 0$. This proves that $\sum_{j:j+L\leq 0} \mathrm{I}_j^2 \lesssim 1$.

It remains to treat the terms I_j^1 in (6.2). Recall that $\mathcal{V}_\tau = \chi_1(\tau^{-2}\mathcal{L})n_\tau^0(\mathcal{L},|U|)$, and the convolution kernel of $n_\tau^0(\mathcal{L},|U|)$ is given by the formula

$$K_\tau^0(x,u) = \sqrt{\tau}\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}} e^{i\frac{\tau}{4s}}\, a_\tau(s)\,\eta_0(s|\mu|/\tau)\Big(\frac{|\mu|}{2\sin(2\pi s|\mu|/\tau)}\Big)^{d_1/2} \times e^{-i|x|^2\frac{\pi}{2}|\mu|\cot(2\pi s|\mu|/\tau)}\, e^{2\pi i u\cdot\mu}\, ds\, d\mu.$$

A straightforward analysis of the phase in the above oscillatory integral representation of the kernel K_τ^0 shows a couple of regions in (x,u) space where the phase is nondegenerate, and an integration by parts argument shows the following rapid decay estimates (see Lemma 6.2 in [17] for details).

Proposition 6.1. *For every $N \ge 1$, we have*

$$|K_\tau^0(x,u)| \lesssim_N (1+|\tau|)^{(Q+1)/2-N}(|x|^2+|u|)^{-N}, \quad \textit{when } |x|^2+|u| \ge 2,$$

and

$$|K_\tau^0(x,u)| \lesssim_N (1+|\tau|)^{(Q+1)/2-N}(1+|u|)^{-N}, \quad \textit{when } |x|^2 \le 1/20.$$

Now let $\phi \in C_c^\infty(\mathbb{R}^{d_1+d_2})$ be such that $\phi(x,u) = 1$ when $|x|^2+|u| \le 2$, and $\phi(x,u) = 0$ when $|x|^2+|u| \ge 3$. Also fix $\psi \in C_c^\infty(\mathbb{R}^{d_1})$ such that $\psi(x) = 1$ when $|x|^2 \le 1/40$ and $\psi(x) = 0$ when $|x|^2 \ge 1/20$. Decompose $K_\tau^0 = K_\tau^{0,1} + K_\tau^{0,2}$, where

$$K_\tau^{0,1}(x,u) := K_\tau^0(x,u)\,\phi(x,u)(1-\psi(x))$$

so that by Proposition 6.1, $\|K_\tau^{0,2}\|_{L^1} \lesssim |\tau|^{-N}$ for any $N \ge 1$. Furthermore, from Hulanicki's result in [7],

$$\|\mathcal{V}_\tau - n_\tau^0\|_{L^1\to L^1} = \|\chi_1(\tau^{-2}\mathcal{L})n_\tau^0 - n_\tau^0\|_{L^1\to L^1} \lesssim (1+|\tau|)^{-N}$$

for every $N \ge 1$ (see Lemma 6.1 in [17]). Hence if $\mathcal{K}_\tau^{0,1}$ denotes the operator of convolution with $K_\tau^{0,1}$, then $\|\mathcal{V}_\tau - \mathcal{K}_\tau^{0,1}\|_{L^1\to L^1} \lesssim (1+|\tau|)^{-N}$ and so if

$$\mathrm{II}_j^1 := \int_{\|(x,u)\|_E \gg 2^L}\Big|\int_{|\tau|\gg 1}\widehat{m}^j(\tau)a_B * (K_\tau^{0,1})_g(x,u)\,d\tau\Big|\,dx\,du,$$

then

$$\mathrm{I}_j^1 \lesssim \mathrm{II}_j^1 + \int_{|\tau|\gg 1}|\widehat{m}^j(\tau)|\|(\mathcal{V}_\tau - \mathcal{K}_\tau^{0,1})_g\|_{L^1\to L^1}\,d\tau = \mathrm{II}_j^1 + O(2^{-j\theta d/2}). \tag{6.6}$$

For a fixed τ and (x,u) let us examine the convolution

$$a_B * (K_\tau^{0,1})_g(x,u) = \int g^Q\, K_\tau^0(\delta_g((y,v)^{-1}\cdot(x,u)))\,\phi(\delta_g((y,v)^{-1}\cdot(x,u))) \times (1-\psi(2^j/\tau(x-y))\,a_B(y,v)\,dy\,dv.$$

Note that $|x-y| \sim |\tau|/2^j$, but since $2^{j+L} \le 1 \ll |\tau|$, we have $|y| \le 2^L \ll |\tau|/2^j$ and so $|x| \sim |\tau|/2^j$ or $|\tau| \sim 2^j|x|$. Therefore,

$$\mathrm{II}_j^1 = \int_{\|(x,u)\|_E \gg 2^L}\Big|\int_{|\tau|\gg 1,\ |\tau|\sim 2^j|x|}\widehat{m}^j(\tau)a_B * (K_\tau^{0,1})_g(x,u)\,d\tau\Big|\,dx\,du.$$

Now we work backwards, recalling the operator decomposition $n_\tau^0 = \mathcal{K}_\tau^{0,1} + \mathcal{K}_0^{0,2}$, so that

$$\mathcal{V}_\tau = \mathcal{K}_\tau^{0,1} + [\mathcal{V}_\tau - n_\tau^0] + \mathcal{K}_\tau^{0,2}.$$

Furthermore,

$$\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}} = \mathcal{V}_\tau + \mathcal{W}_\tau + E_\tau = \mathcal{K}_\tau^{0,1} + \mathcal{W}_\tau + [\mathcal{V}_\tau - n_\tau^0] + \mathcal{K}_\tau^{0,2} + E_\tau.$$

The terms $[\mathcal{V}_\tau - n_\tau^0]$, $\mathcal{K}_\tau^{0,2}$ and E_τ are negligible, each with an L^1 operator norm of $O(|\tau|^{-2})$ which gives rise to the admissible bound $2^{-j\theta d/2}$ by (4.5). Hence, using the estimates in (6.5), we see that

$$|\mathrm{II}_j^1| \lesssim \int_{\|(x,u)\|_E \gg 2^L} \left| \int_{|\tau|\sim 2^j|x|} \widehat{m}^j(\tau) a_B * (K_\tau)_g(x,u)\, d\tau \right| dx\, du + 2^{\delta(j+L)} + 2^{-j\theta d/2},$$

where K_τ denotes (as before) the convolution kernel of $\chi(\tau^{-1}\sqrt{\mathcal{L}})e^{i\sqrt{\mathcal{L}}}$.

Therefore, setting

$$\mathrm{III}_j^1 := \int_{\|(x,u)\|_E \gg 2^L} \left| \int_{|\tau|\gg 1, |\tau|\sim 2^j|x|} \widehat{m}^j(\tau) a_B * (K_\tau)_g(x,u)\, d\tau \right| dx\, du,$$

matters are reduced to showing $\sum_{j+L\le 0} \mathrm{III}_j^1 \lesssim 1$. We have

$$\mathrm{III}_j^1 \le \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)| \int |a_B(y,v)| \times \left[\int_{|x|\sim|\tau|/2^j} g^Q |K_\tau(\delta_g((y,v)^{-1}\cdot(x,u)))|\, dx\, du \right] dy\, dv\, d\tau.$$

Recall that

$$(y,v)^{-1}\cdot(x,u) = (x-y, u-v+\tfrac{1}{2}\langle Jx, y\rangle).$$

In the (x,u) integral above, make the change of variables $u' = u - v + \frac{1}{2}\langle Jx, y\rangle$, followed by $x' = x - y$, noting that

$$|x-y| \sim |\tau|/2^j \;\Leftrightarrow\; |x| \sim |\tau|/2^j \;\text{ since }\; |y| \le 2^L \ll |\tau|/2^j,$$

to conclude

$$\mathrm{III}_j^1 \le \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)| \int_{|x|\sim|\tau|/2^j} g^Q |K_\tau(\delta_g(x,u))|\, dx\, du\, d\tau \le \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)| \int_{|x|\sim 1} |K_\tau(x,u)|\, dx\, du\, d\tau.$$

By (2.7), we have the pointwise estimate $|K_\tau(x,u)| \lesssim |\tau|^M(|\tau x| + |\tau^2 u|)^{-N}$ for some $M \ge 1$ and every $N \ge 1$ whenever τ is large. Therefore,

$$\int_{|x|\sim 1} |K_\tau(x,u)|\, dx\, du \lesssim \frac{1}{|\tau|^{N-M}} \int_{\mathbb{R}^{d_2}} \frac{1}{(1+|\tau u|)^N}\, du = c|\tau|^{-(N+d_2-M)}$$

and so

$$\mathrm{III}_j^1 \lesssim \int_{|\tau|\gg 1} |\widehat{m}^j(\tau)|\,|\tau|^{-2}\,d\tau \lesssim 2^{-j\theta d/2},$$

which is summable for $j > 0$ with $j + L \le 0$.

This completes the proof of Theorem 3.2.

Acknowledgements. The authors would like to thank the referee for helpful suggestions.

Funding. The first and the second author were partially supported by GNAMPA (Project 2020 "Alla frontiera tra l'analisi complessa in più variabili e l'analisi armonica"). The third author is supported by a UK EPSRC scholarship at the Maxwell Institute Graduate School.

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Received December 17, 2020. Published online August 3, 2021.

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