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# A spectral characterization and an approximation scheme for the Hessian eigenvalue

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**Abstract.** We revisit the $k$-Hessian eigenvalue problem on a smooth and bounded $(k-1)$-convex domain in $\mathbb{R}^n$. First, we obtain a spectral characterization of the $k$-Hessian eigenvalue as the infimum of the first eigenvalues of linear second-order elliptic operators whose coefficients belong to the dual of the corresponding Gårding cone. Second, we introduce a non-degenerate inverse iterative scheme to solve the eigenvalue problem for the $k$-Hessian operator. We show that the scheme converges, with a rate, to the $k$-Hessian eigenvalue for all $k$. When $2 \leq k \leq n$, we also prove a local $L^1$ convergence of the Hessian of solutions of the scheme. Hyperbolic polynomials play an important role in our analysis.

## 1. Introduction and statements of the main results

In this paper, we consider the $k$-Hessian counterparts of some results on the Monge–Ampère eigenvalue problem. We begin by recalling these results and the relevant background.

### 1.1. The Monge–Ampère eigenvalue problem

The Monge–Ampère eigenvalue problem, on smooth, bounded and uniformly convex domains $\Omega$ in $\mathbb{R}^n$ $(n \geq 2)$, was first investigated by Lions [17]. He showed that there exist a unique positive constant $\lambda = \lambda(n;\Omega)$ and a unique (up to positive multiplicative constants) nonzero convex function $u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ solving the eigenvalue problem for the Monge–Ampère operator $\det D^2 u$:

$$\det D^2 u = \lambda^n |u|^n \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

The constant $\lambda(n;\Omega)$ is called the Monge–Ampère eigenvalue of $\Omega$. The functions $u$ solving (1.1) are called the Monge–Ampère eigenfunctions. Lions also found a spectral characterization of the Monge–Ampère eigenvalue via the first eigenvalues of linear second-order elliptic operators in non-divergence form.

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Let $V_n = V_n(\Omega)$ be the set of all matrices $A = (a_{ij})_{1\leq i,j\leq n}$, with $a_{ij} \in C(\Omega)$,

$$(a_{ij}) = (a_{ji}) > 0 \quad \text{in } \Omega, \qquad \det(A) \geq \frac{1}{n^n}.$$

For $A \in V_n$, let $\lambda_1^A$ be the first (positive) eigenvalue of the linear second-order operator $-a_{ij}D_{ij}$, with zero Dirichlet boundary condition on $\partial\Omega$, i.e., there exist $v \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, with $v \not\equiv 0$, such that

$$-a_{ij}D_{ij}v = \lambda_1^A v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

The corresponding eigenfunctions $v$, up to multiplicative constants, are positive in $\Omega$ and unique. We refer the readers to the Appendix in [17] for more information about the first eigenvalues for $-a_{ij}D_{ij}$, where $A \in V_n$. Lions [17] showed that

$$\lambda(n;\Omega) = \min_{A\in V_n} \lambda_1^A. \tag{1.2}$$

A variational characterization of $\lambda(n;\Omega)$ was first discovered by Tso [23]. Denote the Rayleigh quotient (for the Monge–Ampère operator) of a nonzero convex function $u$ by

$$R_n(u) = \frac{\int_\Omega |u| \det D^2u\, dx}{\int_\Omega |u|^{n+1}\, dx}.$$

When $u$ is merely a convex function, $\det D^2u\, dx$ is interpreted as the Monge–Ampère measure associated with $u$; see Figalli [8] and Gutiérrez [10]. Tso showed that

$$[\lambda(n;\Omega)]^n = \inf\big\{R_n(u) : u \in C^{0,1}(\overline{\Omega}) \cap C^\infty(\Omega),\\ u \text{ is convex, nonzero in } \Omega,\ u = 0 \text{ on } \partial\Omega\big\}. \tag{1.3}$$

Recently, the author [15] studied the Monge–Ampère eigenvalue problem for general open bounded convex domains and established the singular counterparts of previous results by Lions and Tso. Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$. Define the constant $\lambda = \lambda[n;\Omega]$, via the infimum of the Rayleigh quotient, by

$$(\lambda[n;\Omega])^n = \inf\big\{R_n(u) : u \in C(\overline{\Omega}),\ u \text{ is convex, nonzero in } \Omega,\ u = 0 \text{ on } \partial\Omega\big\}. \tag{1.4}$$

Then, by [15], the infimum in (1.4) is achieved because there exists a nonzero convex eigenfunction $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ solving the Monge–Ampère eigenvalue problem (1.1) with $\lambda = \lambda[n;\Omega]$. When $\Omega$ is a smooth, bounded and uniformly convex domain, the class of competitor functions in the minimization problem (1.4) is larger than that of the minimization problem (1.3); however, it was shown in [15] that $\lambda(n;\Omega) = \lambda[n;\Omega]$.

In [1], Abedin and Kitagawa introduced a numerically appealing inverse iterative scheme

$$\det D^2u_{m+1} = R_n(u_m)|u_m|^n \quad \text{in } \Omega, \qquad u_{m+1} = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

to solve the Monge–Ampère eigenvalue problem (1.1) on a bounded convex domain $\Omega \subset \mathbb{R}^n$. They proved that the scheme (1.5) converges to the Monge–Ampère eigenvalue problem (1.1) for all convex initial data $u_0$ satisfying $R_n(u_0) < \infty$, $u_0 \leq 0$ on $\partial\Omega$, and $\det D^2 u_0 \geq 1$ in $\Omega$. When $m \geq 1$, (1.5) is a degenerate Monge–Ampère equation for $u_{m+1}$ because the right-hand side tends to 0 near the boundary $\partial\Omega$.

In this paper, we prove a spectral characterization of the $k$-Hessian eigenvalue similar to (1.2) (Theorem 1.1), and study a non-degenerate inverse iterative scheme (1.15), similar to (1.5), to solve the $k$-Hessian eigenvalue problem. We will review this problem in Section 1.2. The main results concerning the scheme (1.15) include convergence to the $k$-Hessian eigenvalue (Theorem 1.2) and local $W^{2,1}$ type convergence (Theorem 1.3). The common thread in our investigation is hyperbolic polynomials to be reviewed in Section 2. Our approach, which is based on certain integration by parts inequalities, differs from [1] even in the Monge–Ampère case. As an illustration, for the Monge–Ampère case, our approach gives a sharp reverse Aleksandrov estimate for the Monge–Ampère equation and a convergence rate of $R_n(u_m)$ to $(\lambda(n;\Omega))^n$ in terms of the convergence rate of $u_m$ to a nonzero Monge–Ampère eigenfunction $u_\infty$ (see, Theorem 1.2 (ii), (iii)). This is new compared to currently known iteration schemes for the $p$-Laplace equation [2], [3], [13].

### 1.2. The $k$-Hessian eigenvalue problem

Let $1 \leq k \leq n$ $(n \geq 2)$. Let $\Omega$ be a bounded open and smooth domain in $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$, let $S_k(D^2u)$ denote the $k$-th elementary symmetric function of the eigenvalues $\lambda(D^2u) = (\lambda_1(D^2u), \dots, \lambda_n(D^2u))$ of the Hessian matrix $D^2u$:

$$S_k(D^2u) = \sigma_k(\lambda(D^2u)) := \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}(D^2u)\cdots\lambda_{i_k}(D^2u).$$

For convenience, we denote $\sigma_0(\lambda) = 1$. A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is called $k$-admissible if $\lambda(D^2u) \in \Gamma_k$, where $\Gamma_k$ is an open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for all } j = 1, \dots, k\}. \tag{1.6}$$

We also call $\Gamma_k$ the Gårding cone of the $k$-Hessian operator. *All functions involved in $S_k$ below are assumed to be $k$-admissible.* If $k \geq 2$, we also assume $\partial\Omega$ to be uniformly $(k-1)$-convex, that is, $\sigma_{k-1}(\kappa_1, \dots, \kappa_{n-1}) \geq c_0 > 0$, where $\kappa_1, \dots, \kappa_{n-1}$ are principle curvatures of $\partial\Omega$ relative to the interior normal. Note that $n$-admissible functions are strictly convex, and uniformly $(n-1)$-convex domains are simply uniformly convex domains.

The eigenvalue problem for the $k$-Hessian operator $S_k(D^2u)$ on a bounded, open, smooth and $(k-1)$-convex domain $\Omega$ in $\mathbb{R}^n$, that is,

$$S_k(D^2w) = [\lambda(k;\Omega)]^k |w|^k \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

was first introduced by Wang in [25] (see also [26]) who extended the results of Lions [17] and Tso [23] from the case $k = n$ to the general case $1 \leq k \leq n$. Wang introduced the constant

$$\lambda_1 = \sup\{\lambda > 0 : \text{there is a solution } u_\lambda \in C^2(\overline{\Omega}) \text{ of } (1.9)\}, \tag{1.8}$$

where (1.9) is given by

$$S_k(D^2u) = (1 - \lambda u)^k \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.9}$$

Wang [25] showed that $\lambda_1 \in (0, \infty)$, and that, as $\lambda \to \lambda_1$, $u_\lambda \|u_\lambda\|^{-1}_{L^\infty(\Omega)}$ converges in $C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ to a solution $w \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ of (1.7), with $\lambda(k;\Omega) = \lambda_1$ there.

The eigenvalue problem (1.7) has the following uniqueness property: If $(\bar{\lambda}, \bar{w})$ is a solution of (1.7), where $\bar{\lambda} \geq 0$ and $\bar{w} \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ is $k$-admissible with $w = 0$ on $\partial\Omega$, then $\bar{\lambda} = \lambda(k;\Omega)$ and $\bar{w} = cw$ for some positive constant $c$. The constant $\lambda(k;\Omega)$ is called the $k$-Hessian eigenvalue, and $w$ in (1.7) is called a $k$-Hessian eigenfunction. The scheme (1.8)–(1.9) to compute the $k$-Hessian eigenvalue involves solving the $k$-Hessian equation (1.9) with right-hand side depending on the solution $u$ itself. This equation is more difficult to handle, analytically and numerically, than one with right-hand side depending only on the spatial variables.

Let $R_k(u)$ denote the Rayleigh quotient for the $k$-Hessian operator

$$R_k(u) = \frac{\int_\Omega |u| S_k(D^2u)\, dx}{\int_\Omega |u|^{k+1}\, dx} \tag{1.10}$$

for a $C^2$ function $u$. The requirement that $\|u\|_{L^{k+1}(\Omega)} < \infty$ is implicit in definition (1.10).

Wang [25] also proved the following fundamental property for the variational characterization of $\lambda(k;\Omega)$:

$$[\lambda(k;\Omega)]^k = \inf\big\{R_k(u) : u \in C(\overline{\Omega}) \cap C^2(\Omega), \\ u \text{ is } k\text{-admissible, nonzero in } \Omega,\ u = 0 \text{ on } \partial\Omega\big\}. \tag{1.11}$$

Using (1.11), Liu, Ma and Xu [18] obtained a Brunn–Minkowski inequality for the 2-Hessian eigenvalue in three-dimensional convex domains.

### 1.3. A spectral characterization of the $k$-Hessian eigenvalue

Let $x \cdot y$ denote the standard inner product for $x, y \in \mathbb{R}^n$. Following Kuo–Trudinger [14], let $\Gamma_k^*$ be the dual cone of the Gårding cone $\Gamma_k$, given by

$$\Gamma_k^* = \{\lambda \in \mathbb{R}^n : \lambda \cdot \mu \geq 0 \text{ for all } \mu \in \Gamma_k\}. \tag{1.12}$$

Clearly, $\Gamma_k^* \subset \Gamma_l^*$ for $k \leq l$. For $\lambda \in \Gamma_k^*$, denote

$$\rho_k^*(\lambda) = \inf\left\{\frac{\lambda \cdot \mu}{n} : \mu \in \Gamma_k,\ S_k(\mu) \geq \binom{n}{k}\right\}.$$

Observe that $\Gamma_n^* = \overline{\Gamma}_n$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n^*$, then $\lambda_i \geq 0$ and $\rho_n^*(\lambda) = (\prod_{i=1}^n \lambda_i)^{1/n}$.

For a matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, we write $A \in \Gamma_k^*$ if $\lambda(A) \in \Gamma_k^*$ and define

$$\rho_k^*(A) = \rho_k^*(\lambda(A)).$$

Let $V_k = V_k(\Omega)$ be the following set of positive definite symmetric matrices whose entries are continuous functions on $\Omega$:

$$(1.13)\qquad V_k = \Big\{ A = (a_{ij})_{1\leq i,j\leq n} : (a_{ij}) = (a_{ji}) > 0 \text{ in } \Omega, \\ a_{ij} \in C(\Omega),\ A \in \Gamma_k^*,\ \text{and } \rho_k^*(A) \geq \frac{1}{n}\binom{n}{k}^{1/k} \Big\}.$$

Note that $V_k \subset V_{k+1}$. This follows from the Maclaurin inequalities and the fact that $c(n,k) > c(n,k+1)$ for all $k \leq n-1$, where $c(n,k) := \frac{1}{n}\binom{n}{k}^{1/k}$. Indeed, suppose $A \in V_k$. If $\mu \in \Gamma_{k+1}$, with $S_{k+1}(\mu) \geq \binom{n}{k+1}$, then from the Maclaurin inequality

$$\left(\frac{S_k(\mu)}{\binom{n}{k}}\right)^{1/k} \geq \left(\frac{S_{k+1}(\mu)}{\binom{n}{k+1}}\right)^{1/(k+1)},$$

we find that $\mu \in \Gamma_k$ with $S_k(\mu) \geq \binom{n}{k}$. Hence, $\frac{\lambda(A)\cdot\mu}{n} \geq \rho_k^*(A) \geq c(n,k) > c(n,k+1)$, and therefore $A \in V_{k+1}$. Note that $c(n,k) > c(n,k+1)$ follows from

$$\left[\frac{c(n,k)}{c(n,k+1)}\right]^{k(k+1)} = \binom{n}{k}\left[\binom{n}{k}\binom{n}{k+1}^{-1}\right]^{k} = \binom{n}{k}\frac{(k+1)^k}{(n-k)^k} \\ = \frac{n(n-1)\cdots(n-k+1)}{(n-k)^k}\,\frac{(k+1)^k}{k!} > 1.$$

We now have the following increasing sequence of cones:

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n.$$

Extending Lions' result (1.2) from $k = n$ to all other values of $k$, we have the following theorem.

**Theorem 1.1** (A spectral characterization for the Hessian eigenvalue). *Assume $1 \leq k \leq n$. Let $\Omega$ be a bounded, open, smooth, and uniformly convex domain in $\mathbb{R}^n$. Let $V_k$ be as in (1.13). For $A \in V_k$, let $\lambda_1^A$ be the first positive eigenvalue of the linear second-order operator $-a_{ij}D_{ij}$ with zero Dirichlet boundary condition on $\partial\Omega$. Then*

$$(1.14)\qquad \lambda(k;\Omega) = \min_{A\in V_k} \lambda_1^A.$$

The interest in the above theorem is when $k \geq 2$. When $k = 1$, we have

$$V_1 = \{mI_n : m \geq 1\}$$

where $I_n$ is the identity $n \times n$ matrix and thus the conclusion of Theorem 1.1 is obvious.

### 1.4. A non-degenerate inverse iterative scheme for the $k$-Hessian eigenvalue problem

Inspired by the scheme (1.5), we propose the following non-degenerate inverse iterative scheme to solve the eigenvalue problem (1.7), starting from a $k$-admissible function $u_0 \in C^2(\overline{\Omega})$ with $u_0 \leq 0$ on $\partial\Omega$:

$$(1.15)\qquad S_k(D^2u_{m+1}) = R_k(u_m)|u_m|^k + (m+1)^{-2} \quad \text{in } \Omega, \qquad u_{m+1} = 0 \quad \text{on } \partial\Omega.$$

We add the positive constant $(m+1)^{-2}$, which vanishes in the limit $m \to \infty$, to make the right-hand side of (1.15) strictly positive for each $m$. Thus, for each $m \geq 0$, (1.15) is a non-degenerate $k$-Hessian equation for $u_{m+1}$. See also Remark 1.4. The requirement $u_0 \leq 0$ on $\partial\Omega$ is only used to have $u_0 \leq 0$ in $\Omega$ and thus $R_k(u_0)|u_0|^k = R_k(u_0)(-u_0)^k \in C^2(\overline{\Omega})$.

By a classical result of Caffarelli, Nirenberg and Spruck, Theorem 1 of [4] (see also Theorem 3.4 of [26]), for each $m$, the scheme (1.15) has a unique $k$-admissible solution $u_{m+1} \in C^{3,\alpha}(\overline{\Omega})$ for all $0 < \alpha < 1$. Moreover, $u_m < 0$ in $\Omega$ for all $m \geq 1$. The sequence $(u_m)$ is obtained by repeatedly inverting the $k$-Hessian operator with Dirichlet boundary condition.

In the next theorem, we show that $R(u_m)$ converges to $[\lambda(k;\Omega)]^k$, thus making (1.15) more appealing for numerically computing the $k$-Hessian eigenvalue $\lambda(k;\Omega)$.

**Theorem 1.2** (Convergence to the Hessian eigenvalue of the non-degenerate inverse iterative scheme). *Let $1 \leq k \leq n$, where $n \geq 2$. Let $\Omega$ be a bounded, open, smooth domain in $\mathbb{R}^n$. Assume that $\partial\Omega$ is uniformly $(k-1)$-convex if $k \geq 2$. Consider the reverse iterative scheme* (1.15)*, where $u_0 \in C^2(\overline{\Omega})$, with $u_0 \leq 0$ on $\partial\Omega$, and $u_m$ is $k$-admissible for all $m \geq 0$. Let $w \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ be a nonzero $k$-Hessian eigenfunction as in* (1.7)*. Then the following hold*:

(i) *$R_k(u_m)$ converges to $[\lambda(k;\Omega)]^k$*:

$$\lim_{m\to\infty} R_k(u_m) = [\lambda(k;\Omega)]^k. \tag{1.16}$$

(ii) *There exists a subsequence $u_{m_j}$ that converges weakly in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for all $q < \frac{nk}{n-k}$ to a nonzero function $u_\infty \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^k(\Omega)$. Moreover,*

$$\lim_{j\to\infty} \int_\Omega |u_{m_j}|^k \, dx = \int_\Omega |u_\infty|^k \, dx \tag{1.17}$$

*and, for all $m \geq 1$,*

$$\begin{aligned} R_k^{1/k}(u_m) - \lambda(k;\Omega) &\leq \lambda(k;\Omega) \frac{\int_\Omega (|u_{m+1}| - |u_m|)|w|^k \, dx}{\int_\Omega |u_1||w|^k \, dx} \\ &\leq \lambda(k;\Omega) \frac{\int_\Omega (|u_\infty| - |u_m|)|w|^k \, dx}{\int_\Omega |u_1||w|^k \, dx}. \end{aligned} \tag{1.18}$$

(iii) *When $k = n$, $\{u_m\}$ converges uniformly on $\overline{\Omega}$ to a non-zero Monge–Ampère eigenfunction $u_\infty$ of $\Omega$.*

(iv) *When $k = 1$, $\{u_m\}$ converges in $W^{1,2}_0(\Omega)$ to a non-zero first Laplace eigenfunction $u_\infty$ of $\Omega$.*

We point out that part (iv) of Theorem 1.2 was included for completeness, as it was contained in [2], [3] and [13], when there is no term $(m+1)^{-2}$ on the right-hand side of (1.15).

In the convex case when $k = n$, in view of the work [15], the Monge–Ampère eigenvalue problem (1.1) with $u$ only being convex (so less regular) is now well understood, and this plays a key role in the proof of Theorem 1.2 (iii). The work [15] relies on the regularity theory of weak solutions to the Monge–Ampère equation developed by Caffarelli [5], [6]. To the best of the author's knowledge, for $2 \le k \le n-1$, the $k$-Hessian counterparts of these Monge–Ampère results are still lacking. Thus, showing that $u_\infty$ in Theorem 1.2 (ii) is a $k$-Hessian eigenfunction is still an interesting open problem. One possible alternate route is to upgrade the convergence of $u_m$ to $u_\infty$ in $W^{1,q}_{\mathrm{loc}}(\Omega)$ to that in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $p > k$. So far, we can prove a sort of local $W^{2,1}(\Omega)$ convergence. It is in fact a local $W^{2,1}(\Omega)$ convergence when $k = n$ (see also Theorem 6.2). We have the following theorem.

**Theorem 1.3** (Local $W^{2,1}$ convergence of the non-degenerate inverse iterative scheme). *Assume $2 \le k \le n$. Let $\Omega$ be a bounded, open, smooth, and uniformly $(k-1)$ convex domain in $\mathbb{R}^n$. Let $w \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ be a nonzero $k$-Hessian eigenfunction as in (1.7). Consider the scheme (1.15), where $u_0 \in C^2(\overline{\Omega})$, with $u_0 \le 0$ on $\partial\Omega$, and $u_m$ is $k$-admissible for all $m \ge 0$. Consider a subsequence of $(u_{m_j})$ and its limit $u_\infty$ as in Theorem 1.2 (ii). Let $\lambda_{k,i}(D^2 w, D^2 u_{m+1})$ be defined by*

$$S_k(tD^2 w + D^2 u_{m+1}) = S_k(D^2 w) \prod_{i=1}^{k} (t + \lambda_{k,i}(D^2 w, D^2 u_{m+1})) \quad \text{for all } t \in \mathbb{R}.$$

*When $k = n$, the $\lambda_{k,i}(D^2 w, D^2 u_{m+1})$ are eigenvalues of $D^2 u_{m+1}(D^2 w)^{-1}$. Then*

$$\lambda_{k,i}(D^2 w, D^2 u_{m_j+1}) \to \frac{|u_\infty|}{|w|} \quad \text{locally in } L^1 \text{ when } j \to \infty.$$

*Up to a further extraction of a subsequence, we have the following pointwise convergence:*

$$D^2 u_{m_j+1}(x) \to \frac{|u_\infty(x)|}{|w(x)|} D^2 w(x) \quad \text{for a.e. } x \in \Omega. \tag{1.19}$$

**Remark 1.4.** The conclusions of Theorems 1.2 and 1.3 hold if we replace $(m+1)^{-2}$ in the scheme (1.15) by $a_m > 0$, where $\sum_{m=0}^{\infty} a_m < \infty$. When $k = n$, we can also take $a_m = 0$, and in this case, (i) and (iii) of Theorem 1.2 were obtained in [1] with a different proof.

We now say a few words about the proofs of Theorems 1.2 and 1.3. When $k < n$, the lack of convexity of $k$-admissible functions is the main difficulty in the proof of Theorem 1.2. Our approach is based on the following nonlinear integration by parts inequality for the $k$-Hessian operator.

**Proposition 1.5** (Nonlinear integration by parts inequality for the $k$-Hessian operator). *Let $\Omega$ be a bounded, open, smooth domain in $\mathbb{R}^n$. Assume that $\partial\Omega$ is uniformly $(k-1)$-convex if $k \ge 2$. Then, for $k$-admissible functions $u, v \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ with $u = v = 0$ on $\partial\Omega$, one has*

$$\int_\Omega |v| S_k(D^2 u)\, dx \ge \int_\Omega |u| [S_k(D^2 u)]^{(k-1)/k} [S_k(D^2 v)]^{1/k}\, dx. \tag{1.20}$$

*If $k \ge 2$ and the equality holds in (1.20), then there is a positive, continuous function $\mu$ such that*

$$D^2 u(x) = \mu(x) D^2 v(x) \quad \text{for all } x \in \Omega.$$

The Monge–Ampère case of (1.20), that is, when $k = n$ and $u$ and $v$ are convex, was established in [15] under more relaxed conditions on $u, v$ and $\Omega$.

We will prove Proposition 1.5 and its extensions, using Gårding's inequality [9] for hyperbolic polynomials, of which $\sigma_k$ and $S_k$ (viewed as functions of matrices) are examples.

For the proof of Theorem 1.3, we find that quantitative forms of (1.20) whose defects measure certain closeness of $D^2u$ to $D^2v$ guarantee the interior $W^{2,1}$ convergence of $u_m$ to $u_\infty$. They are proved using quantitative Gårding's inequalities for hyperbolic polynomials; see Lemma 2.4.

**Remark 1.6.** When $k = 1$, (1.20) becomes an equality and it is an integration by parts formula. If we just require that $\lambda(D^2u), \lambda(D^2v) \in \overline{\Gamma}_k$ instead of $\lambda(D^2u), \lambda(D^2v) \in \Gamma_k$, then (1.20) still holds. To see this, take a $k$-admissible function $w \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ with $w = 0$ on $\partial\Omega$. Then we apply the current version of (1.20) to $u + \varepsilon w$ and $v + \varepsilon w$ and then let $\varepsilon \to 0$.

The rest of the paper is organized as follows. In Section 2, we recall some basics of hyperbolic polynomials and Gårding's inequality. In Section 3, we prove Proposition 1.5 and its extensions to other hyperbolic polynomials. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2. The proof of Theorem 1.3 will be given in Section 6.

## 2. Hyperbolic polynomials

In this section, we recall some basics of hyperbolic polynomials and Gårding's inequality. See also Harvey–Lawson [11] for a simple and self-contained account of Gårding's theory of hyperbolic polynomials [9].

Suppose that $p$ is a homogeneous real polynomial of degree $k$ on $\mathbb{R}^N$. Given $a \in \mathbb{R}^N$, we say that $p$ is *$a$-hyperbolic* if $p(a) > 0$, and for each $x \in \mathbb{R}^N$, $p(ta + x)$ can be factored as

$$p(ta + x) = p(a) \prod_{i=1}^{k} (t + \lambda_i(p; a, x)) \quad \text{for all } t \in \mathbb{R},$$

where $\lambda_i(p; a, x)$ $(i = 1, \dots, k)$ are real numbers. The functions $\lambda_i(p; a, x)$ are called the *$a$-eigenvalues of $x$*, and they are well defined up to permutation. In what follows, identities between $\lambda_i(p; \cdot, \cdot)$ are understood modulo the permutation group $\mathcal{S}_k$ of order $k$.

For the reader's convenience, we mention here some examples of $a$-hyperbolic polynomials, mostly taken from [9]. The polynomials $P_k$ in Example 2.5 are most relevant for the results of this paper.

**Example 2.1.** The quadratic polynomial

$$p(x) = x_1^2 - x_2^2 - \cdots - x_N^2, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

is $e_1$-hyperbolic, where $e_1 = (1, 0, \dots, 0)$. The $e_1$-eigenvalues of $x \in \mathbb{R}^N$ are given by

$$\{\lambda_1(p; e_1, x),\ \lambda_2(p; e_1, x)\} = \{x_1 \pm \sqrt{|x|^2 - x_1^2}\}.$$

**Example 2.2.** The polynomial

$$p(x) = \prod_{i=1}^{N} x_i, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N,$$

is $a$-hyperbolic for any $a \in \mathbb{R}^N$ with $p(a) > 0$. The $a$-eigenvalues of $x \in \mathbb{R}^N$ are given by

$$\{\lambda_i(p; a, x) : i = 1, \ldots, N\} = \{x_i/a_i : i = 1, \ldots, N\}.$$

Suppose $p$ is $a$-hyperbolic. Observe from the definition of $a$-eigenvalues of $x$ that

$$\frac{p(x)}{p(a)} = \prod_{i=1}^{k} \lambda_i(p; a, x). \tag{2.1}$$

Denote

$$p'_x(a) = \frac{d}{dt}\Big|_{t=0} p(a + tx). \tag{2.2}$$

Then

$$\frac{p'_x(a)}{p(a)} = \sum_{i=1}^{k} \lambda_i(p; a, x). \tag{2.3}$$

Note that

$$\begin{gathered} \lambda_i(p; a, a) = 1; \quad \lambda_i(p; a, tx) = t\lambda_i(p; a, x) \bmod \mathcal{S}_k, \\ \lambda_i(p; a, ta + x) = t + \lambda_i(p; a, x) \bmod \mathcal{S}_k. \end{gathered} \tag{2.4}$$

If $p$ is be $a$-hyperbolic, then we denote its *edge at $a$* by

$$E_a(p) = \left\{x \in \mathbb{R}^N : \lambda_1(p; a, x) = \cdots = \lambda_k(p; a, x) = 0\right\}.$$

We have

$$\lambda_i(p; a, x) = \mu \text{ for all i } \Leftrightarrow \lambda_i(p; a, x - \mu a) = 0 \text{ for all i } \Leftrightarrow x - \mu a \in E_a(p). \tag{2.5}$$

The Gårding cone of $p$ at $a$ is defined to be

$$\Gamma_a(p) = \{x \in \mathbb{R}^N : \lambda_i(p; a, x) > 0 \text{ for all } i = 1, \ldots, k\}.$$

A fundamental result of Gårding, Theorem 2 of [9], states that if $p$ is $a$-hyperbolic and $b \in \Gamma_a(p)$, then $p$ is $b$-hyperbolic and $\Gamma_b(p) = \Gamma_a(p)$. Therefore, we use $\Gamma(p)$ to denote $\Gamma_a(p)$ whenever $p$ is $a$-hyperbolic. Another fundamental result of Gårding, Theorem 3 of [9], says that the edge $E_a(p)$ of a hyperbolic polynomial $p$ at $a$ is equal to the *linearity $L(p)$ of $p$*, where

$$L(p) = \left\{x \in \mathbb{R}^N : p(tx + y) = p(y) \text{ for all } t \in \mathbb{R} \text{ and } y \in \mathbb{R}^N\right\}.$$

For later reference, we summarize these results in the following theorem.

**Theorem 2.3** (Gårding). *Let $p$ be hyperbolic at $a \in \mathbb{R}^N$. Then the following hold:*

(i) *If $b \in \Gamma_a(p)$, then $p$ is $b$-hyperbolic and $\Gamma_b(p) = \Gamma_a(p)$.*
(ii) $E_a(p) = L(p)$.

From (2.1) and (2.3), we obtain the following quantitative Gårding's inequality.

**Lemma 2.4** (Quantitative Gårding's inequality). *Suppose $p$ is a homogeneous real polynomial of degree $k$ on $\mathbb{R}^N$ and $p$ is $a$-hyperbolic. If $x \in \Gamma(p)$, then*

$$\frac{1}{k}\frac{p'_x(a)}{p(a)} \geq \left(\frac{p(x)}{p(a)}\right)^{1/k} + \frac{1}{k}\sum_{i=1}^{k}\left[\sqrt{\lambda_i(p;a,x)} - \left(\frac{p(x)}{p(a)}\right)^{1/(2k)}\right]^2. \tag{2.6}$$

*In particular, if $k \geq 2$ and $x \in \Gamma(p)$ with*

$$\frac{1}{k}\frac{p'_x(a)}{p(a)} = \left(\frac{p(x)}{p(a)}\right)^{1/k}, \tag{2.7}$$

*then there is a positive constant $\mu$ such that $x - \mu a \in E_a(p)$.*

*Proof.* Without the last nonnegative term, (2.6) is the original Gårding's inequality whose proof uses (2.1), (2.3) and the Cauchy inequality for $k$ positive numbers.

For the full version of (2.6), we use (2.1), (2.3) and the following quantitative version of Cauchy's inequality: If $x_1, \ldots, x_k$ are $k$ ($k \geq 2$) nonnegative numbers, then

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{k} x_i - (x_1\cdots x_k)^{1/k} - \frac{1}{k}\sum_{i=1}^{k}\left(\sqrt{x_i} - (x_1\cdots x_k)^{1/(2k)}\right)^2 \\ = 2(x_1\cdots x_k)^{1/(2k)}\left(\frac{1}{k}\sum_{i=1}^{k}\sqrt{x_i} - (x_1\cdots x_k)^{1/(2k)}\right) \geq 0.\end{aligned} \tag{2.8}$$

Clearly, (2.6) follows from (2.8) applied to $\lambda_i(p;a,x)$. Moreover, if $k \geq 2$ and (2.7) holds, then we must have $\lambda_i(p;a,x) = \cdots = \lambda_k(p;a,x) = \mu$ for some positive constant $\mu$. Hence, the last assertion follows from (2.5). ∎

**Example 2.5.** Let $N = \frac{1}{2}n(n+1)$ and let $A$ be a symmetric $n \times n$ matrix $A = (a_{ij})$. We can view $A$ as a point in $\mathbb{R}^N$. Then $P(A) = \det A$ is $A$-hyperbolic for any positive definite matrix $A$. Let $I_n$ be the identity $n \times n$ matrix. Define $P_k$ by

$$\det(tI_n + A) = P(tI_n + A) = \sum_{k=0}^{n} t^{n-k} P_k(A) \quad \text{for all } t \in \mathbb{R}. \tag{2.9}$$

Then $P_k$ is a homogeneous polynomial of degree $k$ on $\mathbb{R}^N$; moreover, $P_k$ is $I_n$-hyperbolic (see, Example 3 and the discussion at the end of p. 959 in [9]).

From now on, let $P_k$ be as in Example 2.5. From this example, we know that $P_k$ is $I_n$-hyperbolic. Thus, for any symmetric $n \times n$ matrix $A$, we have from the definition of $I_n$-hyperbolicity that the $I_n$-eigenvalues $\lambda_i(P_k; I_n, A)$ are real numbers for all $i = 1, \ldots, k$.

Suppose furthermore that $A$ is a symmetric $n \times n$ matrix with $\lambda(A) \in \Gamma_k$ (as defined in (1.6)). Then, from $\lambda_i(P_k; I_n, A) \in \mathbb{R}$,

$$P_k(tI_n + A) = \sum_{i=0}^{k} \binom{n-i}{k-i} t^{k-i} \sigma_i(\lambda(A)) = P_k(I_n) \prod_{i=1}^{k} (t + \lambda_i(P_k; I_n, A))$$

and $\sigma_i(\lambda(A)) > 0$ for all $i$, we easily find that $\lambda_i(P_k; I_n, A) > 0$ for all $i = 1, \ldots, k$. Hence, $A \in \Gamma(P_k)$, from which we deduce that $P_k$ is $A$-hyperbolic by Theorem 2.3. Recall that we use $\Gamma(P_k)$ to denote the Gårding cone of $P_k$ at $I_n$. Vice versa, if $A \in \Gamma(P_k)$, then by definition, $\lambda_i(P_k; I_n, A) > 0$ for all $i = 1, \ldots, k$, and therefore $\sigma_i(\lambda(A)) > 0$ for all $i = 1, \ldots, k$, which show that $\lambda(A) \in \Gamma_k$. Thus, we have

$$\Gamma(P_k) = \{A \in \mathbb{R}^N : \lambda(A) \in \Gamma_k\}. \tag{2.10}$$

The following lemma shows the triviality of the edge of $P_k$ when $k \geq 2$.

**Lemma 2.6.** *If $k \geq 2$, then*

$$E_{A_0}(P_k) = \{0\} \quad \text{whenever } P_k \text{ is } A_0\text{-hyperbolic}. \tag{2.11}$$

*Proof.* In the proof, we use Theorem 2.3 (ii), which implies that the edge $E_a(p)$ of a hyperbolic polynomial $p$ at $a$ does not depend on $a$. We apply this fact to $p = P_k$ and deduce that if $P_k$ is $A_0$-hyperbolic, then

$$E_{A_0}(P_k) = L(P_k) = E_{I_n}(P_k) = \big\{A \in \mathbb{R}^N : \lambda_1(P_k; I_n, A) = \cdots = \lambda_k(P_k; I_n, A) = 0\big\}.$$

Let $A \in E_{I_n}(P_k)$. Then $\lambda_1(P_k; I_n, A) = \cdots = \lambda_k(P_k; I_n, A) = 0$, so the above expansion of $P_k(tI_n + A)$ shows that $\sigma_i(\lambda(A)) = 0$ for all $i = 1, \ldots, k$. In particular, since $k \geq 2$, we find

$$\sigma_1(\lambda(A)) = \sigma_2(\lambda(A)) = 0.$$

Therefore, the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$ of the symmetric matrix $A$ satisfy

$$\sum_{i=1}^{n} [\lambda_i(A)]^2 = [\sigma_1(\lambda(A))]^2 - 2\sigma_2(\lambda(A)) = 0.$$

It follows that $A$ is the 0 matrix. This shows that $E_{A_0}(P_k) = E_{I_n}(P_k) = \{0\}$, as claimed. ∎

Note that the conclusion of Lemma 2.6 is false for $k = 1$, since

$$E_{A_0}(P_k) = L(P_1) = \{A \in \mathbb{R}^N : P_1(A) = \text{trace}(A) = 0\}.$$

We have the following lemma.

**Lemma 2.7.** *Let $p$ be a homogeneous real polynomial of degree $k$ on $\mathbb{R}^N$. Suppose that $p$ is $a$-hyperbolic with $E_a(p) = \{0\}$. Assume that $\{b^{(m)}\} \subset \mathbb{R}^N$ satisfies $\lambda_i(p; a, b^{(m)}) \to 0$ when $m \to \infty$ for all $i = 1, \ldots, k$. Then $b^{(m)} \to 0$ when $m \to \infty$.*

The lemma is perhaps standard; however, we could not locate a precise reference so we include its proof here. In the proof, we use that $\lambda_i(p; a, x)$, modulo $\mathcal{S}_k$, is continuous in $x$ (see p. 1105 of [11]). This comes from the algebraic fact that roots of a degree $k$ polynomial depend continuously on its coefficients.

*Proof of Lemma 2.7.* We first show that $b^{(m)}$ is bounded. Suppose that $\|b^{(m)}\| = M_m \to \infty$. Consider $\tilde{b}^{(m)} = b^{(m)}/M_n$. Then $\|\tilde{b}^{(m)}\| = 1$, while, modulo $\mathcal{S}_k$,

$$\lambda_i(p; a, \tilde{b}^{(m)}) = \frac{\lambda_i(p; a, b^{(m)})}{M_n} \to 0 \quad \text{for all } i = 1, \dots, k.$$

Up to extracting a subsequence, we have $\tilde{b}^{(m)} \to b$ with $\|b\| = 1$, while $\lambda_i(p; a, \tilde{b}^{(m)}) \to \lambda_i(p; a, b) = 0$ for all $i = 1, \dots, k$. Thus, $b \in E_a(p)$, which shows that $b = 0$, a contradiction.

Next, we show that $b^{(m)}$ converges to 0. We already known that there is $M > 0$ such that $\|b^{(m)}\| \le M$ for all $m$. Suppose there exists $\delta > 0$ such that there is a subsequence, still denoted $b^{(m)}$, satisfying $M \ge \|b^{(m)}\| \ge \delta > 0$. We use compactness as above to get a $b$ with $\|b\| = 1$, while $\lambda_i(p; a, b) = 0$ for all $i = 1, \dots, k$, a contradiction. ■

## 3. Nonlinear integration by parts inequalities

In this section, we prove Proposition 1.5, which is concerned with $P_k$ and its extensions to other hyperbolic polynomials.

*Proof of Proposition 1.5.* Since $u, v \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ are $k$-admissible functions with $u = v = 0$ on $\partial\Omega$, we have $u, v \le 0$ in $\Omega$. We view $S_k$ as a function on $n \times n$ matrices $r = (r_{ij})_{1 \le i,j \le n}$, where

$$S_k(r) = \sigma_k(\lambda(r)).$$

Let

$$S_k^{ij}(D^2u) = \frac{\partial}{\partial r_{ij}} S_k(D^2u).$$

Then, it is well known that (see, for example, [19], [25], [26])

$$S_k(D^2u) = \frac{1}{k} \sum_{i,j=1}^n S_k^{ij}(D^2u) D_{ij}u,$$

and for each $i = 1, \dots, n$, we have the following divergence-free property of the matrix $(S_k^{ij}(D^2u))$:

$$\sum_{j=1}^n D_j S_k^{ij}(D^2u) = 0.$$

Therefore, integrating by parts twice, we get

$$\begin{aligned} (3.1) \quad \int_\Omega |v| S_k(D^2u)\,dx &= \frac{1}{k} \int_\Omega \sum_{i,j=1}^n (-v) S_k^{ij}(D^2u) D_{ij}u\,dx \\ &= \frac{1}{k} \int_\Omega \sum_{i,j=1}^n D_j[v S_k^{ij}(D^2u)] D_i u\,dx = \frac{1}{k} \int_\Omega \sum_{i,j=1}^n D_j v S_k^{ij}(D^2u) D_i u\,dx \\ &= \frac{1}{k} \int_\Omega (-u) \sum_{i,j=1}^n S_k^{ij}(D^2u) D_{ij} v\,dx = \frac{1}{k} \int_\Omega |u| \sum_{i,j=1}^n S_k^{ij}(D^2u) D_{ij} v\,dx. \end{aligned}$$

We need to show that

$$\frac{1}{k}\sum_{i,j=1}^{n} S_k^{ij}(D^2u)D_{ij}v \geq [S_k(D^2u)]^{(k-1)/k}[S_k(D^2v)]^{1/k}. \tag{3.2}$$

Let $P_k$ be as in (2.9). We use the notation $p'_x(a)$ as defined by (2.2). Note that, for $C^2$ functions $u$ and $v$, we have

$$S_k(D^2u) = P_k(D^2u) \quad \text{and} \quad (P_k)'_{D^2v}(D^2u) = \sum_{i,j=1}^{n} S_k^{ij}(D^2u)D_{ij}v.$$

Since $u$ and $v$ are $k$-admissible, we have

$$D^2u, D^2v \in \Gamma(P_k).$$

Thus, by Gårding's inequality (Lemma 2.4),

$$\begin{aligned}\frac{1}{k}\sum_{i,j=1}^{n} S_k^{ij}(D^2u)D_{ij}v &= \frac{1}{k}(P_k)'_{D^2v}(D^2u)\\ &\geq P_k(D^2u)\Big(\frac{P_k(D^2v)}{P_k(D^2u)}\Big)^{1/k} = [P_k(D^2u)]^{(k-1)/k}[P_k(D^2v)]^{1/k}.\end{aligned}$$

Therefore, (3.2) holds and we obtain (1.20).

If $k \geq 2$ and the equality holds in (1.20), then (3.2) must be an equality for almost all $x \in \Omega$. For those $x$, using the last assertion of Lemma 2.4, we can find a positive number $\mu(x)$ such that

$$D^2u(x) - \mu(x)D^2v(x) \in E_{D^2u(x)}(P_k) = \{0\},$$

where we used (2.11) in the last equality. Since $u, v \in C^3(\Omega)$, $\mu$ is a continuous function on $\Omega$ and

$$D^2u(x) = \mu(x)D^2v(x) \quad \text{for all } x \in \Omega.$$

The proof of the proposition is complete. ■

A particular consequence of Proposition 1.5 is the following corollary.

**Corollary 3.1.** *Let $\Omega$ be a bounded, open, smooth, uniformly $(k-1)$-convex (if $k \geq 2$) domain in $\mathbb{R}^n$. Let $w \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ be a $k$-Hessian eigenfunction as in (1.7). Then, for any $k$-admissible function $v \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$, with $v = 0$ on $\partial\Omega$, one has*

$$\lambda(k;\Omega)\int_\Omega |v||w|^k\,dx \geq \int_\Omega |w|^k[S_k(D^2v)]^{1/k}\,dx. \tag{3.3}$$

Corollary 3.1 is sharp since equality holds when $v$ is a $k$-Hessian eigenfunction of $\Omega$. When $k = n$, (3.3) can be viewed as a reverse version of the celebrated Aleksandrov's maximum principle for the Monge–Ampère equation (see Theorem 2.8 of [8] and Theorem 1.4.2 of [10]), which states: If $u \in C(\overline{\Omega})$ is a convex function on an open, bounded and convex domain $\Omega \subset \mathbb{R}^n$, with $u = 0$ on $\partial\Omega$, then

$$|u(x)|^n \leq C(n)(\operatorname{diam}\Omega)^{n-1}\operatorname{dist}(x,\partial\Omega)\int_\Omega \det D^2u\,dx \quad \text{for all } x \in \Omega. \tag{3.4}$$

In fact, the reverse Aleksandrov estimate holds for more relaxed conditions on the domains and convex functions involved.

**Proposition 3.2** (Reverse Aleksandrov estimate). *Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$. Let $\lambda[n;\Omega]$ be the Monge–Ampère eigenvalue of $\Omega$ and let $w$ be a nonzero Monge–Ampère eigenfunction of $\Omega$ (see also (1.1)). Assume that $u \in C^5(\Omega)\cap C(\overline{\Omega})$ is a strictly convex function in $\Omega$, with $u = 0$ on $\Omega$, satisfying*

$$\int_\Omega (\det D^2u)^{1/n}|w|^{n-1}\,dx < \infty.$$

*Then*

$$\lambda[n;\Omega]\int_\Omega |u||w|^n\,dx \geq \int_\Omega (\det D^2u)^{1/n}|w|^n\,dx. \tag{3.5}$$

*Proof of Proposition* 3.2. For the proof, we recall the *nonlinear integration by parts* inequality established in Proposition 1.7 of [15] (see also [16]): If $u, v \in C(\overline{\Omega})\cap C^5(\Omega)$ are strictly convex functions in $\Omega$, with $u = v = 0$ on $\partial\Omega$, and if

$$\int_\Omega (\det D^2u)^{1/n}(\det D^2v)^{(n-1)/n}\,dx < \infty \quad \text{and} \quad \int_\Omega \det D^2v\,dx < \infty,$$

then

$$\int_\Omega |u|\det D^2v\,dx \geq \int_\Omega |v|(\det D^2u)^{1/n}(\det D^2v)^{(n-1)/n}\,dx. \tag{3.6}$$

We apply (3.6) to $u$ and $v = w$. Then, using $\det D^2w = (\lambda[n;\Omega]|w|)^n$, we get

$$\begin{aligned}(\lambda[n;\Omega])^n\int_\Omega |u||w|^n = \int_\Omega |u|\det D^2w\,dx &\geq \int_\Omega |w|(\det D^2u)^{1/n}(\det D^2w)^{(n-1)/n}\,dx\\ &= (\lambda[n;\Omega])^{n-1}\int_\Omega (\det D^2u)^{1/n}|w|^n\,dx.\end{aligned}$$

Dividing the first and last expressions in the above estimates by $(\lambda[n;\Omega])^{n-1}$, we obtain inequality (3.5). ■

**Remark 3.3.** The method of proof of Proposition 1.5 relies on the divergence form structure of the $k$-Hessian operator $S_k(D^2u)$. If we replace $P_k(A)$ in the proof of Proposition 1.5 by other homogeneous, hyperbolic polynomials $P(A)$ of degree $K$, then the conclusion still holds as long as the following conditions are satisfied:

(P1) Let

$$P^{ij}(D^2u) = \frac{\partial}{\partial r_{ij}}P(D^2u).$$

Then

$$P(D^2u) = \frac{1}{K}\sum_{i,j=1}^n P^{ij}(D^2u)D_{ij}u.$$

(P2) For each $u \in C^3(\Omega)$ and $i = 1, \dots, n$, we have the following divergence-free property of the matrix $(P^{ij}(D^2u))$:

$$\sum_{j=1}^{n} D_j P^{ij}(D^2u) = 0.$$

Due to the homogeneity of $P$, property (P1) always holds, in view of Euler's formula. Properties (P1) and (P2) hold for the following hyperbolic polynomials:

$$[P_k(A)]^l, \quad \text{where } l = 1, 2, \dots .$$

Note that

$$K = kl \quad \text{and} \quad \Gamma(P_k) = \Gamma([P_k]^l),$$

so we obtain the following result.

**Proposition 3.4.** *Let $\Omega$ be a bounded, open, smooth domain in $\mathbb{R}^n$. Assume that $\partial\Omega$ is uniformly $(k-1)$-convex if $k \geq 2$. Let $l$ be a positive integer. Then, for $k$-admissible functions $u, v \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$, with $u = v = 0$ on $\partial\Omega$, one has*

$$\int_\Omega |v|[S_k(D^2u)]^l \, dx \geq \int_\Omega |u|[S_k(D^2u)]^{(kl-1)/k}[S_k(D^2v)]^{1/k} \, dx. \tag{3.7}$$

*If $k \geq 2$ and the equality holds in (3.7), then there is a positive, continuous function $\mu$ such that*

$$D^2u(x) = \mu(x) D^2v(x) \quad \textit{for all } x \in \Omega.$$

**Remark 3.5.** If $k \geq 2$, then the quantity $\int_\Omega |v| S_k(D^2u)\, dx$ in Proposition 1.5 is called the non-commutative inner product of two functions $v$ and $u$ on the cone of $k$-admissible functions in [24]. Verbitsky proved in Theorem 3.1 of [24] the following fully nonlinear Schwarz's inequality:

$$\int_\Omega |v| S_k(D^2u)\, dx \leq \Big(\int_\Omega |u| S_k(D^2u)\, dx\Big)^{k/(k+1)} \Big(\int_\Omega |v| S_k(D^2v)\, dx\Big)^{1/(k+1)}, \tag{3.8}$$

which has many applications in the Hessian Sobolev inequalities.

We also note that the proof of (3.8) in [24] also used exactly the properties of $P$ in Remark 3.3. Thus, for $k$-admissible functions $u, v \in C^{1,1}(\overline{\Omega}) \cap C^3(\Omega)$ with $u = v = 0$ on $\partial\Omega$, we also have

$$\int_\Omega |v|[S_k(D^2u)]^l dx \leq \Big(\int_\Omega |u|[S_k(D^2u)]^l dx\Big)^{\frac{kl}{kl+1}} \Big(\int_\Omega |v|[S_k(D^2v)]^l dx\Big)^{\frac{1}{kl+1}}. \tag{3.9}$$

To conclude this section, we note that there are homogeneous hyperbolic polynomials $P$ which do not have property (P2) in Remark 3.3. We may call these *non-divergence form* hyperbolic polynomials. One example is the Monge–Ampère type operator

$$\mathcal{M}_{n-1}(D^2u) := \det((\Delta u) I_n - D^2u), \tag{3.10}$$

which appears in many geometric contexts, both real and complex; see, for example [12], [20], [21] and the references therein. When $n = 3$, we have

$$P(D^2u) := \mathcal{M}_2(D^2u) = \det((\Delta u) I_3 - D^2u) = S_1(D^2u)S_2(D^2u) - S_3(D^2u).$$

For $u(x) = x_1^3 + x_2^2 + x_3^2$, one can check, using the divergence-free property of the matrices $S_k^{ij}$ for $k = 1, 2, 3$, that

$$\sum_{j=1}^{3} D_j P^{1j}(D^2u) = \sum_{j=1}^{3}\Big(S_1^{1j}(D^2u)\frac{\partial}{\partial x_j}S_2(D^2u) + \frac{\partial}{\partial x_j}S_1(D^2u)S_2^{1j}(D^2u)\Big) = 48 \neq 0.$$

## 4. A spectral characterization of the $k$-Hessian eigenvalue via dual Gårding cone

In this section, we prove Theorem 1.1.

Let $\Gamma_k$ and $\Gamma_k^*$ be as in (1.6) and (1.12), respectively. We recall the following result of Kuo and Trudinger, Proposition 2.1 of [14].

**Proposition 4.1.** *For matrices $B = (b_{ij}) \in \Gamma_k$, $A = (a_{ij}) \in \Gamma_k^*$, $k = 1, \ldots, n$, we have*

$$[S_k(B)]^{1/k}\rho_k^*(A) \leq \frac{1}{n}\binom{n}{k}^{1/k}\text{trace}(AB).$$

*Proof of Theorem 1.1.* Let $w \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ be a nonzero $k$-Hessian eigenfunction, so $w$ satisfies (1.7). Then $D^2w \in \Gamma_k$. Let $A = (a_{ij}) \in V_k$. Then

$$\rho_k^*(A) \geq \frac{1}{n}\binom{n}{k}^{1/k}.$$

Applying Proposition 4.1 to $D^2w$ and $A$, we have

$$[S_k(D^2w)]^{1/k} \leq \frac{1}{\rho_k^*(A)}\frac{1}{n}\binom{n}{k}^{1/k}\text{trace}(AD^2w) \leq a_{ij}D_{ij}w. \tag{4.1}$$

Since $[S_k(D^2w)]^{1/k} = \lambda(k;\Omega)|w| = -\lambda(k;\Omega)w$, we obtain

$$a_{ij}D_{ij}w + \lambda(k;\Omega)w \geq 0 \quad \text{in } \Omega.$$

By Proposition A.2 (ii) of [17], we find that

$$\lambda(k;\Omega) \leq \lambda_1^A.$$

Hence,

$$\lambda(k;\Omega) \leq \inf_{A\in V_k}\lambda_1^A.$$

Now, we show that the infimum is achieved. Note that, if $u$ is $k$-admissible, then we have $(S_k^{ij}(D^2u))_{i\leq i,j\leq n} \in \Gamma_k^*$. Moreover, as a consequence of Gårding's inequality (3.2), we find

$$\rho_k^*(S_k^{ij}(D^2u)) = \frac{k}{n}[S_k(D^2u)]^{(k-1)/k}\binom{n}{k}^{1/k}.$$

Observe that

$$\begin{aligned}-\lambda(k;\Omega)w = [S_k(D^2 w)]^{1/k} &= [S_k(D^2 w)]^{-(k-1)/k} S_k(D^2 w)\\ &= \frac{1}{k}[S_k(D^2 w)]^{-(k-1)/k} S_k^{ij}(D^2 w) D_{ij} w.\end{aligned}$$

Thus, $\lambda(k;\Omega)$ is the first eigenvalue of $-a_{ij} D_{ij}$, where

$$(a_{ij})_{1\le i,j\le j} = \Big(\frac{1}{k}[S_k(D^2 w)]^{-(k-1)/k} S_k^{ij}(D^2 w)\Big)_{1\le i,j\le n} \in V_k,$$

with $\rho_k^*((a_{ij})) = \frac{1}{n}\binom{n}{k}^{1/k}$. ∎

**Remark 4.2.** Let $V_k$ be as in (1.13). Observe from (4.1) that for $u$ $k$-admissible, we have

$$[S_k(D^2 u)]^{1/k} = \inf_{A=(a_{ij})\in V_k} a_{ij} D_{ij} u.$$

## 5. Convergence to the $k$-Hessian eigenvalue

In this section, we prove Theorem 1.2.

*Proof of Theorem* 1.2. (i) For $m \ge 0$, multiplying both sides of (1.15) by $|u_{m+1}|$ and then integrating over $\Omega$, we find

$$\begin{aligned}R_k(u_{m+1})\|u_{m+1}\|^{k+1}_{L^{k+1}(\Omega)} &= \int_\Omega |u_{m+1}| S_k(D^2 u_{m+1})\,dx\\ &= R_k(u_m)\int_\Omega |u_m|^k |u_{m+1}|\,dx + \frac{1}{(m+1)^2}\int_\Omega |u_{m+1}|\,dx\\ &\le R_k(u_m)\|u_m\|^k_{L^{k+1}(\Omega)}\|u_{m+1}\|_{L^{k+1}(\Omega)}\\ &\quad + \frac{|\Omega|^{k/(k+1)}}{(m+1)^2}\|u_{m+1}\|_{L^{k+1}(\Omega)}.\end{aligned}$$

It follows that

$$R_k(u_{m+1})\|u_{m+1}\|^k_{L^{k+1}(\Omega)} \le R_k(u_m)\|u_m\|^k_{L^{k+1}(\Omega)} + \frac{|\Omega|^{k/(k+1)}}{(m+1)^2}. \tag{5.1}$$

Therefore, by iterating, we obtain

$$\begin{aligned}R_k(u_{m+1})\|u_{m+1}\|^k_{L^{k+1}(\Omega)} &\le R_k(u_0)\|u_0\|^k_{L^{k+1}(\Omega)} + |\Omega|^{k/(k+1)}\sum_{m=0}^\infty \frac{1}{(m+1)^2}\\ &= R_k(u_0)\|u_0\|^k_{L^{k+1}(\Omega)} + \frac{\pi^2}{6}|\Omega|^{k/(k+1)}.\end{aligned}$$

From (1.11), we know that

$$R_k^{1/k}(u_m) \ge \lambda(k;\Omega) \quad \text{for } m \ge 1. \tag{5.2}$$

Hence, there exists a constant $C_1(k,u_0,\Omega)$ independent of $m$ such that

$$\|u_{m+1}\|_{L^{k+1}(\Omega)} \le C_1(k,u_0,\Omega). \tag{5.3}$$

By the uniqueness (up to positive multiplicative constants) of the $k$-Hessian eigenfunctions, we can assume that $w \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ is a $k$-Hessian eigenfunction with $L^\infty$ norm 1, that is,

$$S_k(D^2 w) = [\lambda(k;\Omega)]^k |w|^k \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \qquad \|w\|_{L^\infty(\Omega)} = 1. \tag{5.4}$$

Then, we use the nonlinear integration by parts inequality (1.20) to get

$$\int_\Omega |u_{m+1}| S_k(D^2 w)\, dx \geq \int_\Omega |w| [S_k(D^2 w)]^{(k-1)/k} [S_k(D^2 u_{m+1})]^{1/k}\, dx. \tag{5.5}$$

Therefore, recalling (5.4), we find, after dividing both sides of the above inequality by $[\lambda(k;\Omega)]^k$, that

$$\begin{aligned} &\int_\Omega |u_{m+1}||w|^k\, dx \\ &\geq \int_\Omega [\lambda(k;\Omega)]^{-1} |w|^k \Big[ R_k(u_m)|u_m|^k + \frac{1}{(m+1)^2} \Big]^{1/k} dx \\ &> \int_\Omega [\lambda(k;\Omega)]^{-1} |w|^k [R_k(u_m)]^{1/k} |u_m|\, dx \\ &= \int_\Omega |u_m||w|^k\, dx + [R_k^{1/k}(u_m) - \lambda(k;\Omega)][\lambda(k;\Omega)]^{-1} \int_\Omega |u_m||w|^k\, dx. \end{aligned} \tag{5.6}$$

Thus, (5.6), together with (5.2), implies that the sequence $\{\int_\Omega |u_m||w|^k\, dx\}_{m=1}^\infty$ is increasing. On the other hand, using (5.3) and (5.4), we find that

$$\int_\Omega |u_m||w|^k\, dx \leq \int_\Omega |u_m|\, dx \leq C_2(k, u_0, \Omega).$$

It follows from $u_1 < 0$ in $\Omega$ that $\int_\Omega |u_m||w|^k\, dx$ converges to a limit

$$\lim_{m\to\infty} \int_\Omega |u_m||w|^k\, dx = L \in (0, \infty). \tag{5.7}$$

For $m \geq 1$, we get from (5.6) that

$$\begin{aligned} R_k^{1/k}(u_m) - \lambda(k;\Omega) &\leq \lambda(k;\Omega) \frac{\int_\Omega (|u_{m+1}| - |u_m|)|w|^k\, dx}{\int_\Omega |u_m||w|^k\, dx} \\ &\leq \lambda(k;\Omega) \frac{\int_\Omega (|u_{m+1}| - |u_m|)|w|^k\, dx}{\int_\Omega |u_1||w|^k\, dx}. \end{aligned} \tag{5.8}$$

Letting $m \to \infty$ in (5.8) and recalling (5.7), we conclude that the whole sequence $R_k(u_m)$ converges to $[\lambda(k;\Omega)]^k$ as asserted in (1.16).

In particular, we have $R_k(u_m) \leq C_3(k, u_0, \Omega)$ and hence, using the Hölder inequality and (5.3),

$$\int_\Omega S_k(D^2 u_{m+1})\, dx = (m+1)^{-2}|\Omega| + R_k(u_m) \int_\Omega |u_m|^k\, dx \leq C_4(k, u_0, \Omega). \tag{5.9}$$

(ii) From (5.3) and the local $W^{1,q}_{\rm loc}(\Omega)$ estimate for the $k$-Hessian equation (see, Theorem 5.1 below) for all $q < \frac{nk}{n-k}$, we have the uniform bound for $u_m$ in $W^{1,q}(V)$ for each $V \subset\subset \Omega$. Thus, there exists a subsequence $(u_{m_j})$ that converges weakly in $W^{1,q}_{\rm loc}(\Omega)$ for all $q < \frac{nk}{n-k}$ to a function $u_\infty \in W^{1,q}_{\rm loc}(\Omega)$. From the compactness of the Sobolev embedding $W^{1,q}$ to $L^k$ on smooth bounded sets for all $q$ sufficiently close to $\frac{nk}{n-k}$, we can also assume that $u_{m_j}$ converges strongly to $u_\infty$ in $L^k_{\rm loc}(\Omega)$. From the second inequality in (5.9) and Fatou's lemma, we have

$$\infty > \liminf_{j\to\infty} \int_\Omega |u_{m_j}|^k\,dx \geq \int_\Omega |u_\infty|^k\,dx. \tag{5.10}$$

On the other hand, using (5.3), we find that for each $V \subset\subset \Omega$,

$$\begin{aligned}\int_\Omega |u_{m_j}|^k\,dx &= \int_{\Omega\setminus V} |u_{m_j}|^k\,dx + \int_V |u_{m_j}|^k\,dx\\ &\leq \|u_{m_j}\|^k_{L^{k+1}(\Omega\setminus V)}|\Omega \setminus V|^{1/(k+1)} + \int_V |u_{m_j}|^k\,dx\\ &\leq C_1^k|\Omega \setminus V|^{1/(k+1)} + \int_V |u_{m_j}|^k\,dx.\end{aligned}$$

Therefore, using the strong convergence of $u_{m_j}$ to $u_\infty$ in $L^k(V)$, we get

$$\begin{aligned}\limsup_{j\to\infty} \int_\Omega |u_{m_j}|^k\,dx &\leq C_1^k|\Omega \setminus V|^{1/(k+1)} + \int_V |u_\infty|^k\,dx\\ &\leq C_1^k|\Omega \setminus V|^{1/(k+1)} + \int_\Omega |u_\infty|^k\,dx.\end{aligned} \tag{5.11}$$

Combining (5.10) with (5.11), we obtain (1.17) as claimed. Clearly, (1.17) and the increasing property of $\int_\Omega |u_m||w|^k\,dx$ implies that $u_\infty$ is nonzero.

Finally, from (5.8), (1.17) and the increasing property of $\{\int_\Omega |u_m||w|^k\,dx\}_{m=1}^\infty$, we obtain (1.18).

(iii) Assume now $k = n$. We show the convergence of $u_m$ to a nontrivial Monge–Ampère eigenfunction $u_\infty$ of $\Omega$. A similar result was proved in [1]. However, our scheme (1.15) and approach are a bit different, so we include the details.

As mentioned in the introduction, we can define the Rayleigh quotient $R_n(u)$ (for the Monge–Ampère operator), as in (1.10), of a nonzero merely convex function $u$, where $\det D^2u\,dx$ is interpreted as the Monge–Ampère measure $Mu$ associated with $u$. It is defined by

$$Mu(E) = |\partial u(E)|, \quad \text{where } \partial u(E) = \bigcup_{x\in E} \partial u(x) \text{ for each Borel set } E \subset \Omega,$$

with

$$\partial u(x) := \{p \in \mathbb{R}^n : u(y) \geq u(x) + p\cdot(y-x) \text{ for all } y \in \Omega\}.$$

In what follows, when $u$ is merely convex, $R_n(u)$ and $\det D^2u$ are understood in the above sense.

Applying the Aleksandrov estimate (3.4) to $u_{m+1}$, where $m \geq 0$, and invoking (5.9), we find

$$\|u_{m+1}\|^n_{L^\infty(\Omega)} \leq C(n,\Omega)\int_\Omega \det D^2 u_{m+1}\, dx \leq C(n,\Omega) C_4(n,u_0,\Omega) \leq C(n,\Omega,u_0).$$

Hence, we obtain the uniform $L^\infty$ bound

$$\|u_m\|_{L^\infty(\Omega)} \leq C(n,\Omega,u_0) < \infty.$$

Again, the Aleksandrov estimate and the convexity of $u_m$ give the uniform $C^{0,1/n}(\overline{\Omega})$ bound for $u_m$:

$$\|u_m\|_{C^{0,1/n}(\overline{\Omega})} \leq C(n,\Omega,u_0) \quad \text{for all } m \geq 1.$$

Therefore, up to extracting a subsequence, we have the uniform convergence

$$u_{m_j} \to u_\infty \not\equiv 0$$

for a convex function $u_\infty \in C(\overline{\Omega})$, with $u_\infty = 0$ on $\partial\Omega$, while we also have the uniform convergence

$$u_{m_j+1} \to w_\infty \not\equiv 0$$

for a convex function $w_\infty \in C(\overline{\Omega})$, with $w_\infty = 0$ on $\partial\Omega$.

Thus, letting $j \to \infty$ in

$$\det D^2 u_{m_j+1} = R_n(u_{m_j})|u_{m_j}|^n + (m_j+1)^{-2},$$

using (1.16) and the weak convergence of the Monge–Ampère measure (see Corollary 2.12 of [8] and Lemma 5.3.1 of [10]), we get

$$\det D^2 w_\infty = (\lambda(n;\Omega)|u_\infty|)^n. \tag{5.12}$$

In view of (5.1), we have

$$R_n(u_{m_j+1})\|u_{m_j+1}\|^n_{L^{n+1}(\Omega)} \leq R_n(u_{m_j})\|u_{m_j}\|^n_{L^{n+1}(\Omega)} + |\Omega|^{n/(n+1)}(m_j+1)^{-2}. \tag{5.13}$$

Letting $j \to \infty$ in (5.13) and recalling (1.16), we first find that

$$\|w_\infty\|_{L^{n+1}(\Omega)} \leq \|u_\infty\|_{L^{n+1}(\Omega)}.$$

In fact, we have the equality. To see this, we use $m_{j+2} \geq m_j + 2$ and iterate (5.1) from $m_j+1$ to $m_{j+2}-1$ to get

$$R_n(u_{m_{j+2}})\|u_{m_{j+2}}\|^n_{L^{n+1}(\Omega)} \leq R_n(u_{m_j+1})\|u_{m_j+1}\|^n_{L^{n+1}(\Omega)} + |\Omega|^{n/(n+1)}\sum_{s=m_j+2}^{m_{j+2}} s^{-2}.$$

Again, letting $j \to \infty$ in the above inequality and recalling (1.16), we obtain

$$\|u_\infty\|_{L^{n+1}(\Omega)} \leq \|w_\infty\|_{L^{n+1}(\Omega)}.$$

In conclusion, we have

$$\|w_\infty\|_{L^{n+1}(\Omega)} = \|u_\infty\|_{L^{n+1}(\Omega)}.$$

However, from (5.12),

$$\begin{aligned} R_n(w_\infty)\|w_\infty\|^{n+1}_{L^{n+1}(\Omega)} &= \int_\Omega |w_\infty| \det D^2 w_\infty \, dx \\ &= [\lambda(n;\Omega)]^n \int_\Omega |u_\infty|^n |w_\infty| \, dx \\ &\le [\lambda(n;\Omega)]^n \|u_\infty\|^n_{L^{n+1}(\Omega)} \|w_\infty\|_{L^{n+1}(\Omega)} \\ &= [\lambda(n;\Omega)]^n \|w_\infty\|^{n+1}_{L^{n+1}(\Omega)}. \end{aligned}$$

Since, by (1.4), $R_n(w_\infty) \ge (\lambda[n;\Omega])^n = [\lambda(n;\Omega)]^n$, we must have $R_n(w_\infty) = [\lambda(n;\Omega)]^n$, and the inequality above must be an equality, but this gives $u_\infty = c w_\infty$ for some constant $c > 0$. Thus, from (5.12), we have

$$\det D^2 w_\infty = c^n [\lambda(n;\Omega)]^n |w_\infty|^n. \tag{5.14}$$

Note that the quantities $\lambda(n;\Omega)$ in (1.3) and $\lambda[n;\Omega]$ in (1.4) are a priori different. In [15], the bracket notation $\lambda[n;\Omega]$ is most relevant for $\Omega$ with corners or flat parts on $\partial\Omega$. However, when $\Omega$ is a smooth, bounded and uniformly convex domain, it was shown in [15] that $\lambda(n;\Omega) = \lambda[n;\Omega]$.

By the uniqueness of the Monge–Ampère eigenfunctions (Theorem 1 of [17] and Theorem 1.1 of [15]), it follows from (5.14) that $c = 1$ and $w_\infty = u_\infty$ is a Monge–Ampère eigenfunction of $\Omega$. From (5.7), we have

$$\int_\Omega |u_\infty||w|^n \, dx = \lim_{m\to\infty} \int_\Omega |u_m||w|^n \, dx = L.$$

With this property and the uniqueness, up to positive multiplicative constants, of the Monge–Ampère eigenfunctions of $\Omega$, we conclude that the limit $u_\infty$ does not depend on the subsequence $u_{m_j}$. This shows that the whole sequence $u_m$ converges to a nonzero Monge–Ampère eigenfunction $u_\infty$ of $\Omega$.

(iv) When $k = 1$, we prove the full convergence in $W^{1,2}_0(\Omega)$ of $u_m$ to a first Laplace eigenfunction $u_\infty$ of $\Omega$. We sketch its proof along the lines of (iii). Note that the Rayleigh quotient for $R_1(u)$ in (1.10) is defined originally for $u \in C^2(\Omega)$. If furthermore, $u \le 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$, then an integration by parts gives

$$R_1(u) = \frac{\int_\Omega |Du|^2 \, dx}{\int_\Omega |u|^2 \, dx},$$

which is the usual Rayleigh quotient for the Laplace operator with $u \in W^{1,2}_0(\Omega)$. In this proof, all functions involved, including $u_m \le 0$, belong to $W^{1,2}_0(\Omega)$ so this is the formula for $R_1(u)$ that we will use.

Recall that the first Laplace eigenvalue of $\Omega$ has the following variational characterization:

$$\lambda(1;\Omega) = \inf\{R_1(u) : u \in W^{1,2}_0(\Omega) \setminus \{0\}\}.$$

Since $u_m \le 0$ in $\Omega$ for all $m$, we can rewrite (1.15) as

$$(5.15)\qquad -\Delta u_{m+1} = R_1(u_m)u_m - (m+1)^{-2} \quad \text{in } \Omega, \qquad u_m = 0 \quad \text{on } \partial\Omega.$$

By (5.3), we have, for all $m \ge 1$,

$$\|u_m\|_{L^2(\Omega)} \le C_1(u_0, \Omega).$$

As observed right before (5.9), we also have $R_1(u_m) \le C_3(u_0, \Omega)$ for all $m \ge 1$. Hence,

$$\int_\Omega (|Du_m|^2 + |u_m|^2)\,dx = [R_1(u_m) + 1]\|u_m\|^2_{L^2(\Omega)} \le C_4(u_0, \Omega).$$

The sequence $\{u_m\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$. Therefore, there is a subsequence $u_{m_j}$ such that $u_{m_j}$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to $u_\infty \in W_0^{1,2}(\Omega)$, where $u_\infty \le 0$. As noticed in (ii), we have $u_\infty \not\equiv 0$.

From (5.15), we deduce that $u_{m_j+1}$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to $w_\infty \in W_0^{1,2}(\Omega)$, where $w_\infty \le 0$ and $w_\infty \not\equiv 0$.

Now, letting $j \to \infty$ in

$$-\Delta u_{m_j+1} = R_1(u_{m_j})u_{m_j} - (m_j+1)^{-2} \quad \text{in } \Omega, \qquad u_{m_j} = 0 \quad \text{on } \partial\Omega,$$

we obtain as in (iii), using (1.16), that

$$(5.16)\qquad -\Delta w_\infty = \lambda(1;\Omega)u_\infty \quad \text{in } \Omega$$

and

$$\|w_\infty\|_{L^2(\Omega)} = \|u_\infty\|_{L^2(\Omega)}.$$

The equation (5.16) is understood in the sense that for all $\varphi \in W_0^{1,2}(\Omega)$, we have

$$\int_\Omega Dw_\infty \cdot D\varphi\,dx = \lambda(1;\Omega)\int_\Omega u_\infty \varphi\,dx.$$

In particular,

$$\int_\Omega |Dw_\infty|^2\,dx = \lambda(1;\Omega)\int_\Omega u_\infty w_\infty\,dx.$$

Applying the Hölder inequality in

$$\begin{aligned} R_1(w_\infty)\|w_\infty\|^2_{L^2(\Omega)} &= \int_\Omega |Dw_\infty|^2\,dx = \lambda(1;\Omega)\int_\Omega u_\infty w_\infty\,dx \\ &\le \lambda(1;\Omega)\|u_\infty\|_{L^2(\Omega)}\|w_\infty\|_{L^2(\Omega)} = \lambda(1;\Omega)\|w_\infty\|^2_{L^2(\Omega)} \end{aligned}$$

together $R_1(w_\infty) \ge \lambda(1;\Omega)$, we find that the above inequality becomes an equality and we obtain a constant $c > 0$ such that $u_\infty = cw_\infty$ and

$$-\Delta w_\infty = c\lambda(1;\Omega)w_\infty \quad \text{in } \Omega.$$

Since $w_\infty \le 0$, $w_\infty \not\equiv 0$, we deduce that $c\lambda(1;\Omega)$ is the first Laplace eigenvalue. Its uniqueness then allows us to conclude that $c = 1$ and $u_\infty = w_\infty$ is a first Laplace eigenfunction of $\Omega$.

Using (5.7) as in (iii), we find that whole sequence $u_m$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to $u_\infty$. This convergence is strong in $W_0^{1,2}(\Omega)$. Indeed, by (5.15), we can write

$$-\Delta(u_{m+1}-u_\infty) = R_1(u_m)(u_m-u_\infty) + [R_1(u_m)-\lambda(1;\Omega)]u_\infty - (m+1)^{-2} \quad \text{in } \Omega.$$

Multiplying both sides by $u_{m+1}-u_\infty$ and integrating by parts, we easily conclude that $\|D(u_{m+1}-u_\infty)\|^2_{L^2(\Omega)} \to 0$. ∎

In the proof of Theorem 1.2 (ii), we use the following estimate due to Trudinger–Wang.

**Theorem 5.1** ([22], Theorem 4.1). *Let $u \in C^2(\Omega)$ be $k$-admissible satisfying $u \le 0$ in $\Omega$. Then for any subdomain $V \subset\subset \Omega$ and all $q < \frac{nk}{n-k}$, we have the estimate*

$$\int_V |Du|^q\,dx \le C(V,\Omega,n,k,q)\Big(\int_\Omega |u|\,dx\Big)^q.$$

**Remark 5.2.** (a) Let $w \in C^\infty(\Omega)\cap C^{1,1}(\overline{\Omega})$ be a nonzero $k$-Hessian eigenfunction as in (1.7). In view of Theorem 1.2 (i), we deduce from (5.5) and (5.6) the following result for the scheme (1.15):

$$\lim_{m\to\infty}\Big[\int_\Omega |u_{m+1}|S_k(D^2w)\,dx - \int_\Omega |w|[S_k(D^2w)]^{(k-1)/k}[S_k(D^2u_{m+1})]^{1/k}\,dx\Big] = 0. \tag{5.17}$$

Indeed, let $b_m$ be the difference in the above bracket. Then $b_m \ge 0$ by (5.5). As in (5.6), we have

$$\begin{aligned}
&[\lambda(k;\Omega)]^{-k} b_m \\
&= \int_\Omega |u_{m+1}||w|^k\,dx - \int_\Omega [\lambda(k;\Omega)]^{-1}|w|^k\big[R_k(u_m)|u_m|^k + (m+1)^{-2}\big]^{1/k}\,dx \\
&< \int_\Omega |u_{m+1}||w|^k\,dx - \int_\Omega [\lambda(k;\Omega)]^{-1}[R_k(u_m)]^{1/k}|u_m||w|^k\,dx \to 0 \text{ when } m\to\infty.
\end{aligned}$$

In the last convergence, we used (5.7) and $[\lambda(k;\Omega)]^{-1}[R_k(u_m)]^{1/k} \to 1$ as given by (1.16).

(b) We can use Lemma 2.2 of [22] to show that the limit function $u_\infty$ in Theorem 1.2 (ii) possesses certain convexity properties, called $k$-convexity in [22].

**Remark 5.3.** Consider the case $2 \le k \le n-1$. As remarked after the statement of Theorem 1.2, showing that $u_\infty$ in Theorem 1.2 (ii) is a $k$-Hessian eigenfunction is an interesting open problem. Moreover, we do not know how to prove the full convergence of $u_m$ to $u_\infty$ in some suitable sense as in the Monge–Ampère case. In the Monge–Ampère case, the uniqueness issue of the Monge–Ampère eigenvalue problem (1.1) with $u$ only being convex is now well understood and this plays a key role in the proof of Theorem 1.2 (iii), as it was used to conclude that $c = 1$, among other results.

For a $k$-convex function $u$, we can define a weak notion of its $k$-Hessian, still denoted by $S_k(D^2u)$, (see [22] for example). Consider the following degenerate $k$-Hessian equation:

$$S_k(D^2w) = \lambda|w|^k \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \tag{5.18}$$

for a nonzero $k$-convex function $w$ and a positive constant $\lambda > 0$. To the best of the author's knowledge, the following questions concerning (5.18) are still open:

(i) Is $w$ smooth in $\Omega$?
(ii) Is $\lambda$ unique?
(iii) Is $w$ unique up to a positive multiplicative constant?

In the Monge–Ampère case, the answer to all these questions is positive, see [15], which relies on the regularity theory of weak solutions to the Monge–Ampère equation developed by Caffarelli [5], [6]. The $k$-Hessian counterparts of these Monge–Ampère results are still lacking.

## 6. $W^{2,1}$ convergence for the non-degenerate inverse iterative scheme

In this section, we prove Theorem 1.3.

*Proof of Theorem* 1.3. Recall that we are considering the case $2 \le k \le n$ for the scheme (1.15). Integrating by parts as in (3.1), we have

$$\int_\Omega |u_{m+1}| S_k(D^2 w)\, dx = \int_\Omega \frac{1}{k} |w| S_k^{ij}(D^2 w) D_{ij} u_{m+1}\, dx. \tag{6.1}$$

Consider the following hyperbolic polynomial as defined in (2.9):

$$p(A) = P_k(A), \quad \text{where } \lambda(A) \in \Gamma_k.$$

Recall the notation $\lambda_i(P_k; A, X)$ in Section 2 and $\Gamma_k$ as in (1.6). To simplify the notation, we denote

$$\lambda_{k,i}(A, X) = \lambda_i(P_k; A, X).$$

When $k = n$, $\lambda_{k,i}(A, X)$ are all eigenvalues of the matrix $XA^{-1}$. Let

$$A = D^2 w, \quad X_m = D^2 u_{m+1}.$$

Then

$$S_k^{ij}(D^2 w) D_{ij} u_{m+1} = p'_{X_m}(A) = \sum_{i=1}^k \lambda_{k,i}(A, X_m) p(A), \quad \frac{p(X_m)}{p(A)} = \prod_{i=1}^k \lambda_{k,i}(A, X_m)$$

and

$$\left[\frac{p(X_m)}{p(A)}\right]^{1/k} p(A) = [p(X_m)]^{1/k} [p(A)]^{(k-1)/k} = [S_k(D^2 w)]^{(k-1)/k} [S_k(D^2 u_{m+1})]^{1/k}.$$

Using (2.6), we find that

$$\begin{aligned} \frac{1}{k} S_k^{ij}(D^2 w) D_{ij} u_{m+1} &= \frac{1}{k} \sum_{i=1}^k \lambda_{k,i}(A, X_m) p(A) = \frac{1}{k} p'_{X_m}(A) p(A) \\ &\ge [S_k(D^2 w)]^{(k-1)/k} [S_k(D^2 u_{m+1})]^{1/k} \\ &\quad + \frac{1}{k} \sum_{i=1}^k \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[\frac{p(X_m)}{p(A)}\right]^{1/(2k)} \right)^2 p(A). \end{aligned} \tag{6.2}$$

By combining (6.1), (6.2) and (5.17), we deduce that

$$\int_\Omega |w| \sum_{i=1}^{k} \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 p(A)\, dx \to 0 \quad \text{when } m \to \infty. \tag{6.3}$$

From the uniform $L^1(\Omega)$ bound for $u_m$ which can be derived from (5.3), and (6.1), we get

$$\int_\Omega \sum_{i=1}^{k} \lambda_{k,i}(A, X_m) p(A) |w|\, dx \le C(k, u_0, \Omega).$$

Since $|w| \ge c(V) > 0$ for each $V \subset\subset \Omega$, and $p(A) = S_k(D^2 w) = [\lambda(k;\Omega)]^k |w|^k$, we obtain that

$$\int_V \sum_{i=1}^{k} \lambda_{k,i}(A, X_m) dx \le C(V) \quad \text{for each } V \subset\subset \Omega. \tag{6.4}$$

Thus, (6.3) and (6.4) imply the following convergence:

$$\lambda_{k,i}(A, X_m) - \left[ \frac{p(X_m)}{p(A)} \right]^{1/k} \to 0 \quad \text{locally in } L^1 \text{ when } m \to \infty. \tag{6.5}$$

To see this, let $V \subset\subset \Omega$ be a non-empty open set. Then

$$[p(A)]^{1/k} = \lambda(k;\Omega)|w| \ge \lambda(k;\Omega) c(V) = c_5(k, \Omega, V) > 0.$$

Thus, (6.3) implies that

$$\int_\Omega \sum_{i=1}^{k} \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 dx \to 0 \quad \text{when } m \to \infty. \tag{6.6}$$

By the Hölder inequality and (5.9), we find

$$\begin{aligned} \int_V \left[ \frac{p(X_m)}{p(A)} \right]^{1/k} dx &\le \frac{1}{c_5(k, \Omega, V)} \int_V [S_k(D^2 u_{m+1})]^{1/k}\, dx \\ &\le \frac{|V|^{(k-1)/k}}{c_5(k, \Omega, V)} \left( \int_V S_k(D^2 u_{m+1}) dx \right)^{1/k} \le C_6(k, u_0, \Omega, V). \end{aligned} \tag{6.7}$$

Again, by the Hölder inequality, we have

$$\begin{aligned} &\left( \int_V \left| \lambda_{k,i}(A, X_m) - \left[ \frac{p(X_m)}{p(A)} \right]^{1/k} \right| dx \right)^2 \\ &\le \int_V \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 dx \int_V \left( \sqrt{\lambda_{k,i}(A, X_m)} + \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 dx \\ &\le 2 \int_V \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 dx \int_V \left( \lambda_{k,i}(A, X_m) + \left[ \frac{p(X_m)}{p(A)} \right]^{1/k} \right) dx \\ &\le 2(C(V) + C_6(k, u_0, \Omega, V)) \int_V \left( \sqrt{\lambda_{k,i}(A, X_m)} - \left[ \frac{p(X_m)}{p(A)} \right]^{1/(2k)} \right)^2 dx. \end{aligned}$$

In the last estimate, we used (6.4) and (6.7). Now, letting $m \to \infty$ in the above inequality and recalling (6.6), we obtain (6.5), as claimed.

Recall from parts (i) and (ii) of Theorem 1.2 that, when $j \to \infty$,

$$\frac{p(X_{m_j})}{p(A)} = \frac{S_k(D^2 u_{m_j+1})}{S_k(D^2 w)} = \frac{R_k(u_{m_j})|u_{m_j}|^k + (m_j+1)^{-2}}{[\lambda(k;\Omega)]^k |w|^k} \tag{6.8}$$
$$\to \frac{[\lambda(k;\Omega)]^k |u_\infty|^k}{[\lambda(k;\Omega)]^k |w|^k} = \frac{|u_\infty|^k}{|w|^k} \quad \text{locally in } L^1.$$

In the above local $L^1$ convergence, as in (6.5), we also use that $|w|$ has a positive lower bound on each compact subset of $\Omega$.

It follows from (6.5) and (6.8) that

$$\lambda_{k,i}(D^2 w, D^2 u_{m_j+1}) = \lambda_{k,i}(A, X_{m_j}) \to \frac{|u_\infty|}{|w|} \quad \text{locally in } L^1 \text{ when } j \to \infty.$$

Finally, we prove the pointwise convergence (1.19). The above local $L^1$ convergence shows that, up to extracting a subsequence, still denoted $(u_{m_j})$, we have

$$\lambda_{k,i}(A(x), X_{m_j}(x)) \to \frac{|u_\infty(x)|}{|w(x)|} \quad \text{for a.e. } x \in \Omega, \text{ for all } i = 1, \ldots, k.$$

Thus, we deduce from (2.4) that

$$\lambda_{k,i}\Big(A(x), X_{m_j}(x) - \frac{|u_\infty(x)|}{|w(x)|} A(x)\Big) \to 0 \quad \text{for a.e. } x \in \Omega, \text{ for all } i = 1, \ldots, k. \tag{6.9}$$

Since $k \geq 2$, we know from Lemma 2.6 that $E_{A(x)}(P_k) = \{0\}$ for all $x \in \Omega$. It follows from (6.9) and Lemma 2.7 that

$$D^2 u_{m_j+1}(x) - \frac{|u_\infty(x)|}{|w(x)|} D^2 w(x) \to 0 \quad \text{for a.e. } x \in \Omega.$$

Therefore, we have (1.19) and the proof of our theorem is complete. ∎

**Remark 6.1.** The $L^1$ convergence in Theorem 1.3 uses the fact that when $k \geq 2$, the second sum in (2.6) gives nontrivial information. When $k = 1$, this sum is 0; however, by Theorem 1.2 (iv), we have $u_\infty = cw$ for some constant $c > 0$, and the following full convergence:

$$\lambda_{1,1}(A, X_m) = \frac{\Delta u_{m+1}}{\Delta w} = \frac{R_1(u_m)u_m - (m+1)^{-2}}{\lambda(1;\Omega) w} \to \frac{u_\infty}{w} = c \quad \text{locally in } W^{1,2}.$$

Now, consider the scheme (1.15) with $k = n$. Then $\lambda_{n,1}(A, X_m), \ldots, \lambda_{n,n}(A, X_m)$ in Theorem 1.3 are eigenvalues of $D^2 u_{m+1}(D^2 w)^{-1}$. From Theorem 1.2 (iii), we know that $u_m$ converges uniformly to a nonzero Monge–Ampère eigenfunction $u_\infty$, which is a positive multiple of $w$. Without loss of generality, we can assume that $u_\infty = w$. Thus, Theorem 1.3 shows that

$$D^2 u_{m+1}(D^2 w)^{-1} \to I_n \quad \text{locally in } L^1(\Omega)$$

and hence $D^2 u_{m+1} \to D^2 w$ locally in $L^1(\Omega)$. Thus, a rigidity form of Proposition 1.5, that is (2.6) of Lemma 2.4, improves the uniform convergence of $u_m$ to $w$ to an interior

$W^{2,1}$ convergence. Note that this $W^{2,1}$ convergence also follows from the general result in De Philippis–Figalli, Theorem 1.1 of [7], but our proof here is different and it also works for the $k$-Hessian eigenvalue problem. We state this convergence in the following theorem.

**Theorem 6.2.** *Let $\Omega$ be a bounded, open, smooth and uniformly convex domain in $\mathbb{R}^n$. Let $k = n$ and let $w$ be a nonzero Monge–Ampère eigenfunction of $\Omega$ to which the solution $u_m$ of* (1.15) *converges uniformly. Then $D^2 u_m$ converges locally in $L^1$ to $D^2 w$ in $\Omega$.*

**Remark 6.3.** Hidden in the variational characterizations (1.3) and (1.11) of the Monge–Ampère and $k$-Hessian eigenvalues via the Rayleigh quotients defined in (1.10) is the divergence form of the $k$-Hessian operators. For $k = 1$, the frequently used Rayleigh quotient is

$$Ra(u) = \frac{\int_\Omega |Du|^2\,dx}{\int_\Omega |u|^2\,dx}.$$

When $u \in C^2(\overline{\Omega})$ with $u \le 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$, $Ra(u)$ is equal to $R_1(u)$ (defined in (1.10)) due to a simple integration by parts. Thus, the divergence form of $S_1(D^2u) = \Delta u$ is used here. For non-divergence form operators such as $\mathcal{M}_{n-1}(D^2u)$ in (3.10), we do not expect their first eigenvalues (if any) to have a variational characterization as the $k$-Hessian eigenvalues. However, we expect the spectral characterizations of the $k$-Hessian eigenvalues in Theorem 1.1 to have counterparts in purely non-divergence form operators generated by hyperbolic polynomials.

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