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Generalized Gaussian bounds for discrete convolution powers

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Abstract. We prove a uniform generalized Gaussian bound for the powers of a discrete convolution operator in one space dimension. Our bound is derived under the assumption that the Fourier transform of the coefficients of the convolution operator is a trigonometric rational function, which generalizes previous results that were restricted to trigonometric polynomials. We also allow the modulus of the Fourier transform to attain its maximum at finitely many points over a period.

1. Introduction and main result

For $1 \leq q < +\infty$, we let $\ell^q(\mathbb{Z};\mathbb{C})$ denote the Banach space of complex-valued sequences indexed by $\mathbb{Z}$ and such that the ℓ^q norm, defined for $u\colon \mathbb{Z} \to \mathbb{C}$ by

$$\|u\|_{\ell^q} := \Big(\sum_{j \in \mathbb{Z}} |u_j|^q \Big)^{1/q},$$

is finite, and we let $\ell^\infty(\mathbb{Z};\mathbb{C})$ denote the Banach space of bounded complex-valued sequences indexed by $\mathbb{Z}$ and equipped with the norm

$$\|u\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |u_j|.$$

Throughout this article, we use the notations

$$\begin{aligned} \mathcal{U} &:= \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}, \quad \overline{\mathcal{U}} := \mathcal{U} \cup \mathbb{S}^1, \\ \mathbb{D} &:= \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}, \quad \overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{S}^1, \\ \mathbb{S}^1 &:= \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}. \end{aligned}$$

If w is a complex number and ρ a positive real number, the notation $B_\rho(w)$ stands for the open ball in $\mathbb{C}$ centered at w and with radius ρ, that is, $B_\rho(w) := \{z \in \mathbb{C} \mid |z - w| < \rho\}$.

The notation $\sigma(T)$ stands for the spectrum of a bounded operator T acting on a Banach space E. We also use the notation $\| \cdot \|_{E \to E}$ for the operator norm on a Banach space E.

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1.1. A reminder on Laurent operators

Let us recall a few facts about Laurent operators on $\ell^q(\mathbb{Z};\mathbb{C})$. If $a \in \ell^1(\mathbb{Z};\mathbb{C})$, we let L_a denote the so-called Laurent (or convolution) operator associated with the sequence a, see [23, 38], which is defined by

$$L_a\colon\ (u_j)_{j\in\mathbb{Z}} \mapsto \Big(\sum_{\ell\in\mathbb{Z}} a_\ell\, u_{j-\ell}\Big)_{j\in\mathbb{Z}} = a \star u, \tag{1.1}$$

whenever the defining formula (1.1) for the sequence $L_a u$ makes sense. Here and below, $\star$ always stands for the convolution product of two sequences indexed by $\mathbb{Z}$. In particular, Young's inequality shows that L_a acts boundedly on $\ell^q(\mathbb{Z};\mathbb{C})$ for any $q \in [1,+\infty]$:

$$\|L_a u\|_{\ell^q} \le \|a\|_{\ell^1}\, \|u\|_{\ell^q} \quad \forall u \in \ell^q(\mathbb{Z};\mathbb{C}).$$

The spectrum of L_a is also well understood since the celebrated Wiener–Levy theorem, see [22], characterizes the invertible elements of $\ell^1(\mathbb{Z};\mathbb{C})$ for the convolution product (and we have the morphism property $L_a \circ L_b = L_{a\star b}$). Namely, the spectrum of L_a as an operator acting on $\ell^q(\mathbb{Z};\mathbb{C})$ does not depend on q and is nothing but the image of the Fourier transform of the sequence a:

$$\sigma(L_a) = \Big\{\sum_{\ell\in\mathbb{Z}} a_\ell\, \mathrm{e}^{\mathbf{i}\ell\xi} \;\big|\; \xi \in \mathbb{R}\Big\}.$$

Since a belongs to $\ell^1(\mathbb{Z};\mathbb{C})$, its Fourier transform is continuous on $\mathbb{R}$. It actually belongs to the so-called Wiener algebra.

Following, among other works, [9, 27, 28, 36], we are interested here in giving uniform pointwise bounds for the n-th iterated convolution product $a \star \cdots \star a = a^{\star n}$ as the number n gets large. We use the convention $a^{\star 1} := a$ and $a^{\star n} := a^{\star(n-1)} \star a$ for $n \ge 2$. Note that, by the morphism property $L_a \circ L_b = L_{a\star b}$, we have $(L_a)^n = L_{a^{\star n}}$ for any $n \in \mathbb{N}$. Beyond their own analytical interest, sharp bounds for the coefficients $(a^{\star n})_j$ or the precise description of their asymptotic behavior are useful in probability theory and in numerical analysis. In probability theory, the coefficients a_ℓ correspond to the probability $\mathbb{P}(X = \ell)$, where X is a random variable with values in $\mathbb{Z}$. Considering the random walk

$$Y_n := X_1 + \cdots + X_n,$$

where the X_m's are identically distributed, independent and follow the same law as X, we have

$$(a^{\star n})_j = \mathbb{P}(Y_n = j),$$

for any $n \in \mathbb{N}^*$ and $j \in \mathbb{Z}$. In this context, asymptotic expansions for $(a^{\star n})_j$ are referred to as *local limit theorems* and may be found in Chapter VII of [25]. These expansions involve the Gaussian function and Hermite polynomials. In numerical analysis, the study of the iterates $(L_a)^n$, $n \in \mathbb{N}$, arises when one discretizes an evolutionary linear partial differential equation (set on the real line $\mathbb{R}$) by means of a finite difference scheme. The transport equation or the heat equation are typical examples. We refer, for instance, to [14, 29]. From Young's inequality,

$$\|(L_a)^n u\|_{\ell^\infty} = \|L_{a^{\star n}} u\|_{\ell^\infty} \le \|a^{\star n}\|_{\ell^1}\, \|u\|_{\ell^\infty},$$

one observes that boundedness of the sequence $(\|a^{\star n}\|_{\ell^1})_{n\in\mathbb{N}}$ is a sufficient condition[1] for what is, in this context, referred to as *stability in the maximum norm*, that is,

$$\sup_{n\in\mathbb{N}} \|(L_a)^n\|_{\ell^\infty\to\ell^\infty} < +\infty.$$

The fundamental result in [36] characterizes, under suitable assumptions, the elements $a \in \ell^1(\mathbb{Z};\mathbb{C})$ such that the geometric sequence $(a^{\star n})_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z};\mathbb{C})$, see also [7, 9, 16, 28] and references therein for further developments. The sufficient part of the characterization in [36] (see also [7, 32]) is performed by deriving a suitable "algebraic" pointwise bound for the coefficient $(a^{\star n})_j$ (see Lemma 2.4 of [36]). This bound is obtained by integrating by parts the Fourier transform of $a^{\star n}$, and this manipulation requires the Fourier transform of a to be $\mathscr{C}^2$. For the derivation of these algebraic bounds, the support of the sequence a may be arbitrary. Refining and optimizing this approach, the algebraic bound in [36] was turned in [9] into a generalized Gaussian bound, for finitely supported sequences a, thanks to a suitable contour deformation. The contours chosen in [9] can go arbitrarily far away from the real line (where the Fourier transform of a is defined at first), which is the reason why the authors in [9] assume a to have finite support, so that its Fourier transform extends to a holomorphic function on the whole complex plane. The necessity of the conditions of Theorem 1 of [36] for a to be power bounded in $\ell^1(\mathbb{Z};\mathbb{C})$ is proved under the assumption that a is finitely supported. This assumption on the support was removed in [16].

Our goal in this article is to extend the results in [9] in two directions: We first wish to consider sequences a with *infinite* support, since such sequences arise when one considers *implicit* discretizations of partial differential equations. We also wish to relax the assumption made in [9] that the modulus of the Fourier transform of a attains its maximum at only one point over each period (say, at 0, in the interval $[-\pi,\pi]$). When the modulus of the Fourier transform attains its maximum at more than one point over a period, the arguments in either [36] or [9] rely on ($\mathscr{C}^\infty$) partitions of unity, which destroy the holomorphy of the Fourier transform. This is the reason why the bounds obtained in [9] in that situation are only of "sub-Gaussian" type (compare, for instance, Theorem 3.1 of [9] with Theorem 3.5 of [9]). Actually, the results in [9] were first refined and extended in [27], where local limit theorems are proved for complex-valued sequences a, and then further extended in [28] to deal with *multidimensional* situations. Another extension that is achieved in [28] is the proof of generalized Gaussian bounds for finitely supported sequences a whose modulus of the Fourier transforms attains its maximum at several points (under a technical assumption that is discussed below).

Our approach in this article is quite different from the one in [7, 9, 28, 36], where the coefficient $(a^{\star n})_j$ is represented by an integral involving the Fourier transform of a (to the n-th power). Here we rather follow an approach which, in the partial differential equations community, is commonly referred to as "spatial dynamics", and which amounts to representing $(a^{\star n})_j$ in terms of the resolvent of the operator L_a. The link between the two comes from the so-called functional calculus, see Chapter VII of [2], which expresses

[1] It is actually a necessary and sufficient condition, see [36].

the *temporal* Green function (here the coefficient $(a^{\star n})_j$) in terms of the *spatial* Green function, which is the solution to the resolvent equation

$$(zI - L_a)u = \boldsymbol{\delta}, \quad z \notin \sigma(L_a),$$

where $\boldsymbol{\delta}$ stands for the "discrete" Dirac mass ($\boldsymbol{\delta}_j = 1$ if $j = 0$, and 0 otherwise). A detailed analysis of the spatial Green function with sharp holomorphic extensions and bounds is provided in Section 2 of the present article under conditions that are similar to but, to some extent, less restrictive than those in [9, 28]. Our main technical assumption is the fact that we consider nonzero *drift* velocities, which enables us to pass smoothly (that is, holomorphically) from "temporal" to "spatial" representations. Some cases with vanishing *group velocities* (in the terminology of [37]) are discussed at the end of this article.

Once we have sharp holomorphic extensions and bounds for the spatial Green function, our final argument relies on a suitable choice of contours in the defining expression of the temporal Green function. The choice of contours can be interpreted as an application of the saddle point method [6]. A fundamental contribution in this direction is [40] (for the stability analysis of viscous shock profiles) and we also refer to [11] for an application of this method to the stability analysis of *discrete* shock profiles. As a matter of fact, our motivations for deriving generalized Gaussian bounds in the broadest possible context stems from the stability analysis of discrete shock profiles but also from the theory of numerical boundary conditions for hyperbolic equations. An application of the techniques developed in this article to finite rank perturbations of Toeplitz operators (on $\ell^2(\mathbb{N};\mathbb{C})$ rather than $\ell^2(\mathbb{Z};\mathbb{C})$) is given in [4]. Discrete shock profiles will be considered in a forthcoming work. We now make several assumptions and state our main result.

1.2. Assumptions and main result

This work is much inspired by the theory of partial differential equations and its numerical approximations. Hence, instead of sticking to the convolution operators L_a of the introduction, we shall rather use operators of the form

$$S_b\colon\ (u_j)_{j\in\mathbb{Z}} \mapsto \Big(\sum_{\ell\in\mathbb{Z}} b_\ell\, u_{j+\ell}\Big)_{j\in\mathbb{Z}},$$

with $b \in \ell^1(\mathbb{Z};\mathbb{C})$. One of the simplest such operators is the so-called shift operator $\mathbf{S}$ defined by

$$\mathbf{S}\colon\ (u_j)_{j\in\mathbb{Z}} \mapsto (u_{j+1})_{j\in\mathbb{Z}}.$$

The two definitions of operators L_a and S_b are closely related. Namely, given $b\in\ell^1(\mathbb{Z};\mathbb{C})$, we have $S_b = L_a$, where the sequence $a \in \ell^1(\mathbb{Z};\mathbb{C})$ is defined by $a_\ell := b_{-\ell}$ for all $\ell \in \mathbb{Z}$. Our convention, which is different from [9,27,28], is the reason for the minus sign in (1.8) in the term $-\mathbf{i}\alpha_k\xi$ (compare, for instance, with equation (3.3) of [9]).

We thus consider from now on two "convolution" operators Q_0 and Q_1 on $\mathbb{Z}$ with *finite support*:

$$(Q_\sigma u)_j := \sum_{\ell=-r}^{p} a_{\ell,\sigma}\, u_{j+\ell} \quad \forall\sigma = 0, 1,\ \forall j \in \mathbb{Z}, \tag{1.2}$$

where $r, p \in \mathbb{N}$ and the $a_{\ell,\sigma}$'s are complex numbers.[2] In what follows, we always write

$$Q_0 = L_{\phi_0} \quad \text{and} \quad Q_1 = L_{\phi_1}, \tag{1.3}$$

where ϕ_0 and ϕ_1 are *finitely supported* elements of $\ell^1(\mathbb{Z};\mathbb{C})$. Both operators Q_0 and Q_1 act boundedly on any $\ell^q(\mathbb{Z};\mathbb{C})$, $1 \leq q \leq +\infty$. Our main focus below is on the three cases $q = 1$, $q = 2$ and $q = +\infty$. The integers r, p in (1.2) define the common *stencil* of the operators Q_0, Q_1. They are fixed by enforcing the conditions

$$|a_{-r,1}| + |a_{-r,0}| > 0, \quad |a_{p,1}| + |a_{p,0}| > 0.$$

Our first assumption is the following.

Assumption 1.1. The operator Q_1 is an isomorphism on $\ell^2(\mathbb{Z};\mathbb{C})$, that is,

$$\widehat{Q}_1(\kappa) := \sum_{\ell=-r}^{p} a_{\ell,1}\,\kappa^\ell \neq 0 \quad \forall \kappa \in \mathbb{S}^1, \tag{1.4}$$

and it satisfies furthermore the index condition

$$\frac{1}{2\mathbf{i}\pi} \int_{\mathbb{S}^1} \frac{\widehat{Q}_1'(\kappa)}{\widehat{Q}_1(\kappa)}\, \mathrm{d}\kappa = 0. \tag{1.5}$$

The function $\widehat{Q}_1$ in (1.4) is referred to below as the *symbol* of the convolution operator Q_1. We can similarly define the symbol $\widehat{Q}_0$ associated with Q_0:

$$\widehat{Q}_0(\kappa) := \sum_{\ell=-r}^{p} a_{\ell,0}\,\kappa^\ell \quad \forall \kappa \in \mathbb{S}^1.$$

Recalling the definition (1.3) of the sequence ϕ_1, the condition (1.4) implies that ϕ_1 is invertible in $\ell^1(\mathbb{Z};\mathbb{C})$ (thanks to the Wiener–Levy theorem [22]). We are then interested in the operator $\mathcal{L} := Q_1^{-1} Q_0$ and more specifically in its powers $\mathcal{L}^n$ as n becomes large. Since ϕ_1 is invertible in $\ell^1(\mathbb{Z};\mathbb{C})$, we can write $\mathcal{L} = L_\phi$ with $\phi := \phi_1^{-1} \star \phi_0$. Since we are interested in $\mathcal{L} = Q_1^{-1} Q_0$, we can always multiply Q_0 and Q_1 by the same nonzero complex number, which does not modify $\mathcal{L}$. In view of Assumption 1.1, we thus always assume $\widehat{Q}_1(1) = 1$ from now on.

We briefly discuss the support of the sequence ϕ in order to compare our framework with that of [9] or [28]. The generalized Gaussian bounds in [9] or [28] are obtained by assuming that ϕ has finite support (which makes its Fourier transform an entire function). In our case, two situations occur:

- The support of ϕ_1 is a singleton, that is, $Q_1 = a_{\underline{\ell},1}\mathbf{S}^{\underline{\ell}}$ for some integer $\underline{\ell}$ between $-r$ and p (recall the notation $\mathbf{S}$ for the shift operator). Because of the condition (1.5), we have $\underline{\ell} = 0$, and so Q_1 is a nonzero multiple of the identity, and our normalization convention $\widehat{Q}_1(1) = 1$ makes Q_1 be the identity. In that case, $\phi = \phi_0$ is finitely supported. In numerical analysis, this situation corresponds to *explicit schemes*. In probability theory, this situation corresponds to random walks with finite range.

[2] When discretizing partial differential equations with real coefficients, these numbers are real.

- The support of ϕ_1 contains at least two elements. Then the inverse ϕ_1^{-1} of ϕ_1 for the convolution product has an infinite support. Apart from "trivial" cases where a factorization is possible, $\phi = \phi_1^{-1} \star \phi_0$ will also have infinite support. An example of this situation is provided in Section 4. In numerical analysis, this situation corresponds to *implicit schemes*.

As can be expected, a crucial role is played below by the symbol of $\mathscr{L}$, which is defined by

$$F(\kappa) := \frac{\widehat{Q}_0(\kappa)}{\widehat{Q}_1(\kappa)} \quad \forall \kappa \in \mathbb{S}^1. \tag{1.6}$$

The main difference between [9] or [28] and the present work is that we allow $F(\exp(\mathbf{i}\xi))$ to be a trigonometric rational function of ξ rather than just a trigonometric polynomial in ξ. (Other results, such as local limit theorems, are derived in [28] under the assumption that F is of class $\mathscr{C}^\infty$ on $\mathbb{S}^1$, but we focus here on the derivation of generalized Gaussian bounds.) In other words, we deal here with the class of sequences in $\ell^1(\mathbb{Z};\mathbb{C})$ whose Fourier transforms are trigonometric rational functions. The following assumption on F is inspired by the fundamental contribution [36]. The link between our Assumption 1.2 below and the classification obtained in [36], p. 280, in the case of trigonometric polynomials is discussed in Appendix A at the end of this article (see Lemma A.1).

Assumption 1.2. The function F defined in (1.6) satisfies $\max_{\kappa\in\mathbb{S}^1} |F(\kappa)| = 1$. Furthermore, there exists a finite set of points $\{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}$, $K \geq 1$, in $\mathbb{S}^1$ such that

$$|F(\kappa)| < 1 \quad \forall \kappa \in \mathbb{S}^1 \setminus \{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}, \tag{1.7}$$

and for all index $k = 1, \dots, K$, $F(\underline{\kappa}_k)$ belongs to $\mathbb{S}^1$. Moreover, for any $k = 1, \dots, K$, there exist a nonzero real number α_k, an even integer $2\mu_k \geq 2$ and a complex number β_k with positive real part such that

$$\frac{F(\underline{\kappa}_k \mathrm{e}^{\mathbf{i}\xi})}{F(\underline{\kappa}_k)} = \exp\big(-\mathbf{i}\alpha_k \xi - \beta_k \xi^{2\mu_k} + O(\xi^{2\mu_k+1})\big), \tag{1.8}$$

as ξ tends to 0.

As in [9,28,36], the maximum of F on the unit circle $\mathbb{S}^1$ is normalized to be 1. Thanks to Beurling's result (see p. 428 of [30]):

$$\lim_{n\to\infty} \|a^{\star n}\|_{\ell^1}^{1/n} = \max_{\theta\in\mathbb{R}} \Big| \sum_{\ell\in\mathbb{Z}} a_\ell \, \mathrm{e}^{\mathbf{i}\ell\theta} \Big| \quad \forall a \in \ell^1(\mathbb{Z};\mathbb{C}),$$

the case where the maximum equals 1 is the limit case where the question of stability is not straightforward. In view of the result of Lemma A.1 in Appendix A, we just wish to exclude in Assumption 1.2 the case where F has constant modulus on $\mathbb{S}^1$, and we then only consider the so-called points of type γ in the terminology of [36]. The number α_k in (1.8) is necessarily real since $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$. The fact that all real numbers α_k are nonzero is a major assumption that we make. It is fundamental below in the description of the so-called spatial Green function. Examples of operators Q_0, Q_1

for which Assumptions 1.1 and 1.2 are satisfied are provided in Section 4 at the end of this article.

Since $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$, the operator $\mathcal{L}$ is a contraction on $\ell^2(\mathbb{Z};\mathbb{C})$, that is,

$$\|\mathcal{L}u\|_{\ell^2} \leq \|u\|_{\ell^2} \quad \forall u \in \ell^2(\mathbb{Z};\mathbb{C}),$$

since the ℓ^2 norms on both sides can be computed by the Parseval–Bessel identity. Of course, this implies that every power of $\mathcal{L}$ is also a contraction on $\ell^2(\mathbb{Z};\mathbb{C})$. In the field of numerical analysis, this property is referred to as ℓ^2-stability,[3] or strong stability [34,35], for the "numerical scheme"

$$\begin{cases} Q_1 u^{n+1} = Q_0 u^n, & n \in \mathbb{N}, \\ u^0 \in \ell^2(\mathbb{Z}). \end{cases}$$

Let us now define the quantities

$$\mathbb{A}_\ell(z) := z a_{\ell,1} - a_{\ell,0} \quad \forall z \in \mathbb{C},\ \forall \ell = -r,\dots,p. \tag{1.9}$$

The following assumption already appears in several works devoted to the stability analysis of *numerical boundary conditions* for discretized hyperbolic equations, see, e.g., [3,12,15,19,24] and references therein. Not only does it determine the minimal integers r and p in (1.2) (by prohibiting to add artificial zero coefficients), but it is also crucially used below to analyze the so-called resolvent equation (2.1). It might be relaxed though, but a more elaborate analysis would be required.

Assumption 1.3. The functions $\mathbb{A}_{-r}$ and $\mathbb{A}_p$ defined in (1.9) do not vanish on $\overline{\mathcal{U}}$.

For instance, if $a_{-r,1}$ and $a_{p,1}$ are nonzero, Assumption 1.3 means that $a_{-r,0}/a_{-r,1}$ and $a_{p,0}/a_{p,1}$ belong to $\mathbb{D}$. If $a_{-r,1}$ and $a_{p,1}$ are both zero, then $a_{-r,0}$ and $a_{p,0}$ should both be nonzero.

Thanks to Assumption 1.3, we can define the following companion matrix:

$$\mathbb{M}(z) := \begin{bmatrix} -\frac{\mathbb{A}_{p-1}(z)}{\mathbb{A}_p(z)} & \dots & \dots & -\frac{\mathbb{A}_{-r}(z)}{\mathbb{A}_p(z)} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}), \tag{1.10}$$

which is holomorphic on the set $\{z \in \mathbb{C} \mid |z| > \exp(-\underline{\eta})\}$ for some parameter $\underline{\eta} > 0$ which only depends on the location of the root of $\mathbb{A}_p$ (if it exists). A crucial observation is that the upper right coefficient of $\mathbb{M}(z)$ is always nonzero, because of Assumption 1.3 and up to restricting $\underline{\eta}$, so the matrix $\mathbb{M}(z)$ is invertible for all relevant values of z. We shall repeatedly use the inverse matrix $\mathbb{M}(z)^{-1}$ in what follows.

The analysis in this article heavily relies on a precise description of the spectrum of $\mathbb{M}(z)$ as z runs through $\overline{\mathcal{U}}$ (and even sometimes slightly through $\mathbb{D}$). This description is given in Lemma 2.1 below, and uses the following two assumptions.

[3] For scalar problems, this is even equivalent to the so-called von Neumann stability condition [14,29].

Assumption 1.4. Either Q_1 is the identity, or $a_{-r,1}$ and $a_{p,1}$ are nonzero.

If Q_1 is the identity, Assumption 1.1 is trivially satisfied since $\widehat{Q}_1 \equiv 1$. In the other case, the complex numbers $a_{-r,1}$ and $a_{p,1}$ are both nonzero. In that case, $\widehat{Q}_1$ is a meromorphic function on $\mathbb{C}$ with a single pole (that is located at 0) of order r. (When r equals zero, there is no pole.) By the residue theorem [31], the index condition (1.5) is equivalent to $\widehat{Q}_1$ having r zeros (counted with multiplicity) in $\mathbb{D} \setminus \{0\}$. Because $\kappa^r \widehat{Q}_1(\kappa)$ is a polynomial of degree $p+r$, $\widehat{Q}_1$ then has p zeros in $\mathcal{U}$.

Assumption 1.5. For all index $k = 1, \dots, K$, let us define $\underline{z}_k := F(\underline{\kappa}_k) \in \mathbb{S}^1$. Then for any $k = 1, \dots, K$, the set

$$\mathcal{I}_k := \left\{ \nu \in \{1, \dots, K\} \mid \underline{z}_\nu = \underline{z}_k \right\} \tag{1.11}$$

has either one or two elements.[4] Furthermore, in case it has two elements, which we denote $\nu_{k,1}, \nu_{k,2}$, then $\alpha_{\nu_{k,1}} \alpha_{\nu_{k,2}} < 0$. (Let us recall that the drift parameters α_k are given in Assumption 1.2.)

From now on, we always make Assumptions 1.1, 1.2, 1.3, 1.4 and 1.5. Our main result is a partial extension of Theorem 3.1 of [9] and Theorem 1.8 of [28]. It gives a uniform, generalized Gaussian bound for the convolution coefficients of the powers $\mathcal{L}^n$. A precise statement is the following.

Theorem 1.6. *Let the operators Q_0, Q_1 in (1.2) satisfy Assumptions 1.1, 1.2, 1.3, 1.4 and 1.5 and the normalization condition $\widehat{Q}_1(1) = 1$. According to the above two cases in Assumption 1.4, we have*:

Explicit case. *If Q_1 is the identity, then there exist two constants $C > 0$ and $c > 0$ such that the operator $\mathcal{L} = Q_0$ satisfies the uniform generalized Gaussian bound*

$$|(\mathcal{L}^n \boldsymbol{\delta})_j| \le C \sum_{k=1}^{K} \frac{1}{n^{1/(2\mu_k)}} \exp\left(-c\left(\frac{|j-\alpha_k n|}{n^{1/(2\mu_k)}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right) \quad \forall n \in \mathbb{N}^*, \ \forall j \in \mathbb{Z}, \tag{1.12}$$

where $\boldsymbol{\delta}$ denotes the discrete Dirac mass defined by $\boldsymbol{\delta}_j = 1$ if $j = 0$ and $\boldsymbol{\delta}_j = 0$ otherwise.

Implicit case. *If Q_1 is not the identity, then there exist constants $C > 0$, $L > 0$ and $c > 0$ such that the operator $\mathcal{L} = Q_1^{-1} Q_0$ satisfies the bounds, for all $n \in \mathbb{N}^*$,*

$$|(\mathcal{L}^n \boldsymbol{\delta})_j| \le C \sum_{k=1}^{K} \frac{1}{n^{1/(2\mu_k)}} \exp\left(-c\left(\frac{|j-\alpha_k n|}{n^{1/(2\mu_k)}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right) \quad \forall |j| \le Ln, \tag{1.13a}$$

$$\left|(\mathcal{L}^n \boldsymbol{\delta})_j\right| \le C \exp(-cn - c|j|) \quad \forall |j| > Ln, \tag{1.13b}$$

A comparison between Theorem 1.6 and the analogous results in [9, 28, 36] is provided in the next subsection. Otherwise, the rest of this article is organized as follows. In Section 2 we prove sharp bounds on the so-called spatial Green function. This is where Assumptions 1.3, 1.4 and 1.5 are used. Then we use these preliminary bounds in Section 3 to obtain the uniform bounds (1.12) and (1.13) for what we call the temporal Green function. Examples and possible extensions are given in Section 4. The proofs of some intermediate and related results are gathered in Appendix A.

[4] Note that $\mathcal{I}_k$ always contains $\{k\}$.

1.3. What is new and what is not?

Let us first observe that the sequence $\mathscr{G} := \mathscr{L}\,\delta$ corresponds to the Laurent series expansion of F near $\mathbb{S}^1$:

$$F(\kappa) = \sum_{j\in\mathbb{Z}} \mathscr{G}_j\,\kappa^j \quad \forall \kappa \in \mathbb{S}^1.$$

In particular, if $\mathscr{G}$ satisfies a generalized Gaussian bound of the form

$$|\mathscr{G}_j| \leq C\exp(-c|j|^s) \quad \forall j \in \mathbb{Z},$$

for some positive constants C and c and some exponent $s > 1$, then F extends to a holomorphic function on $\mathbb{C}\setminus\{0\}$. In our framework, we have $F(\kappa) = \widehat{Q}_0(\kappa)/\widehat{Q}_1(\kappa)$. Assuming that $\widehat{Q}_0(\kappa)$ and $\widehat{Q}_1(\kappa)$ have no common factor, the only possible case where F extends to a holomorphic function on $\mathbb{C}\setminus\{0\}$ is when $\widehat{Q}_1$ does not vanish on $\mathbb{C}\setminus\{0\}$. Because of the form of $\widehat{Q}_1$ and the index condition (1.5), the only situation in which F extends to a holomorphic function on $\mathbb{C}\setminus\{0\}$ is when Q_1 is the identity.[5] This argument explains why, in (1.13), the bound for $(\mathscr{L}^n\delta)_j$ "degenerates" to $\exp(-c|j|)$ for any fixed n (e.g., $n = 1$) and for large j's.

We now compare Theorem 1.6 with Theorem 3.1 of [9] and Theorem 1.8 of [28], which, to our knowledge, are the two prior references on generalized Gaussian bounds for convolution powers of complex sequences. In Theorem 3.1 of [9], the authors consider (in our notation) the *explicit* case (Q_1 is the identity) with $K = 1$, but they make no assumption on the drift parameter α_1 (while we assume $\alpha_1 \neq 0$ in Theorem 1.6). When specifying Theorem 1.8 of [28] to one space dimension, the result in Theorem 1.8 of [28] covers the *explicit* case (Q_1 is the identity) with $K \geq 1$, but it is then further assumed that

$$\alpha_1 = \cdots = \alpha_K, \quad \mu_1 = \cdots = \mu_K, \quad \beta_1 = \cdots = \beta_K.$$

As in Theorem 3.1 of [9], it is not assumed in Theorem 1.8 of [28] that the common value of the α_k's should be nonzero.

In our opinion, the main novelty here consists in considering sequences a such that the α_k's, μ_k's and β_k's are, to some extent, "arbitrary". This is not entirely true since we assume that the $\underline{z}_k$'s are not "too much" equal (Assumption 1.5) and we further assume that all α_k's are nonzero. We believe that this restriction on the α_k's in Theorem 1.6 is purely technical. For instance, in Corollary 4.2, we give an example of a general situation in the explicit case where some α_k can be zero and where the result of Theorem 1.6 can be used to obtain the same bound as in (1.12). We also note that the theory of numerical boundary conditions in [15] (see [3] for a thorough exposition) covers cases with vanishing group velocities (by making use in several occurrences of Puiseux expansions). In the continuous setting, reaction diffusion equations are another occurrence where spatial dynamics and pointwise Green function bounds cover some problems with zero drift velocities, see, e.g., [10]. We thus hope that we shall be able to fully remove the restriction $\alpha_k \neq 0$ as well as Assumption 1.5 in the future.

Our second improvement is to consider sequences in $\ell^1(\mathbb{Z};\mathbb{C})$ whose Fourier transforms are trigonometric rational functions. This extension is relevant for implicit numerical schemes. Possible extensions of our work are listed in Section 4.

[5] We assume, of course, that Q_0 and Q_1 are irreducible.

2. The spatial Green function

The spectrum of $\mathscr{L}$ as an operator on $\ell^2(\mathbb{Z};\mathbb{C})$ is the parametrized curve $F(\mathbb{S}^1)$. We know from Assumption 1.2 that this curve touches the unit circle $\mathbb{S}^1$ at the points $\underline{z}_k$, $k = 1, \dots, K$, and that it is located inside the open unit disk $\mathbb{D}$ otherwise. Hence the resolvent set of $\mathscr{L}$ contains at least $\overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. For such values of z, we can thus define the sequence $G(z) \in \ell^2(\mathbb{Z};\mathbb{C})$ (the capital G letter stands for Green, as in Green's function) by the formula

$$(zI - \mathscr{L})\, G(z) = \boldsymbol{\delta}, \tag{2.1}$$

where we recall that $\boldsymbol{\delta}$ stands for the Dirac mass ($\boldsymbol{\delta}_j = 1$ if $j = 0$ and $\boldsymbol{\delta}_j = 0$ if $j \in \mathbb{Z} \setminus \{0\}$).

From the definition $\mathscr{L} = Q_1^{-1} Q_0$, equation (2.1) can be equivalently rewritten as

$$(zQ_1 - Q_0)\, G(z) = Q_1 \boldsymbol{\delta},$$

and the definitions (1.2), (1.9) give the final form

$$\sum_{\ell=-r}^{p} \mathbb{A}_\ell(z)\, G_{j+\ell}(z) = (Q_1 \boldsymbol{\delta})_j \quad \forall j \in \mathbb{Z}, \tag{2.2}$$

together with the integrability conditions at infinity, $G(z) \in \ell^2(\mathbb{Z};\mathbb{C})$.

2.1. Spectral properties

We introduce the augmented vectors

$$W_j(z) := \begin{bmatrix} G_{j+p-1}(z) \\ \vdots \\ G_{j-r}(z) \end{bmatrix} \in \mathbb{C}^{p+r} \quad \forall j \in \mathbb{Z}, \qquad \mathbf{e} := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{p+r},$$

and rewrite equivalently (2.2) as

$$W_{j+1}(z) - \mathbb{M}(z) W_j(z) = \frac{(Q_1 \boldsymbol{\delta})_j}{\mathbb{A}_p(z)}\, \mathbf{e} \quad \forall j \in \mathbb{Z}. \tag{2.3}$$

The construction and analysis of the solution to the recurrence relation (2.3) relies on the following spectral splitting lemma, which is originally due to Kreiss [19] in the context of finite difference approximations.

Lemma 2.1 (Spectral splitting). *Let $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$ and let the matrix $\mathbb{M}(z)$ be defined as in (1.10). Then $\mathbb{M}(z)$ has:*

- *no eigenvalue on $\mathbb{S}^1$,*
- *r eigenvalues in $\mathbb{D} \setminus \{0\}$,*
- *p eigenvalues in $\mathcal{U}$ (eigenvalues are counted with multiplicity).*

Let now $k \in \{1, \dots, K\}$ be such that the set $\mathcal{I}_k$ in (1.11) is the singleton $\{k\}$. Then if $\alpha_k > 0$, the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, and it has $r - 1$ eigenvalues in $\mathbb{D}$ and p eigenvalues in $\mathcal{U}$. If $\alpha_k < 0$, the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, and it has r eigenvalues in $\mathbb{D}$ and $p - 1$ eigenvalues in $\mathcal{U}$.

Eventually, let now $k \in \{1, \dots, K\}$ be such that the set $\mathcal{I}_k$ in (1.11) has two elements $\nu_{k,1}, \nu_{k,2}$. Then the matrix $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_{\nu_{k,1}}$ and $\underline{\kappa}_{\nu_{k,2}}$ as simple eigenvalues on $\mathbb{S}^1$, and it has $r - 1$ eigenvalues in $\mathbb{D}$ and $p - 1$ eigenvalues in $\mathcal{U}$.

The arguments are basically the same as in [19] but we give them here for the sake of completeness.

Proof of Lemma 2.1. We first recall that the matrix $\mathbb{M}(z)$ is given by (1.10) and that it is invertible for all z satisfying $|z| > \exp(-\eta)$ (thanks to Assumption 1.3). Hence 0 will never be an eigenvalue of $\mathbb{M}(z)$ for the relevant values of z. Let us then observe that $\kappa \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathbb{M}(z)$ for $z \in \overline{\mathcal{U}}$ if and only if z and κ satisfy the so-called dispersion relation

$$\sum_{\ell=-r}^{p} \mathbb{A}_\ell(z)\kappa^\ell = 0,$$

and the definition (1.9) of the functions $\mathbb{A}_\ell$ yields the equivalent form

$$\widehat{Q}_1(\kappa)z = \widehat{Q}_0(\kappa). \tag{2.4}$$

In particular, for any z in the connected set $\overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$, $\mathbb{M}(z)$ has no eigenvalue on the unit circle $\mathbb{S}^1$, for otherwise we would have $z = F(\kappa)$ for some $\kappa \in \mathbb{S}^1$ and $z \notin \{\underline{z}_1, \dots, \underline{z}_K\}$, which is precluded by Assumption 1.2. To obtain the first statement of Lemma 2.1, it thus remains to count the number of eigenvalues of $\mathbb{M}(z)$ in $\mathbb{D} \setminus \{0\}$ (we shall call such eigenvalues the *stable* ones). By the connectedness of $\overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$, the number of stable eigenvalues of $\mathbb{M}(z)$ does not depend on $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. In order to compute the precise number of such eigenvalues, we shall let z tend to infinity and determine the asymptotic behavior of these eigenvalues. This asymptotic behavior differs completely between the explicit and implicit cases (though the number of stable eigenvalues will be the same in both cases), which is the reason why we now deal with those two cases separately.

The explicit case ($Q_1 = I$). The dispersion relation (2.4) then reduces to

$$z = \sum_{\ell=-r}^{p} a_{\ell,0}\,\kappa^\ell. \tag{2.5}$$

If $r = 0$, then there are no eigenvalues in $\mathbb{D} \setminus \{0\}$ for any z, otherwise there would be at least one eigenvalue in $\mathbb{D} \setminus \{0\}$ for all $z \in \mathcal{U}$ and the triangle inequality in (2.5) would imply

$$|z| \leq \sum_{\ell=0}^{p} |a_{\ell,0}|.$$

which is impossible because $|z|$ can be arbitrarily large. The result is thus proved in the case $r = 0$, so we assume $r \geq 1$ from now on (Assumption 1.3 then yields $a_{-r,0} \neq 0$). Following [19] (see also [3] for the complete details), the number of eigenvalues of $\mathbb{M}(z)$ in

$\mathbb{D} \setminus \{0\}$ is computed by letting z tend to infinity, in which case all such (stable) eigenvalues of $\mathbb{M}(z)$ collapse to zero. Indeed, an eigenvalue of $\mathbb{M}(z)$ in $\mathbb{D} \setminus \{0\}$ cannot remain uniformly away from the origin, otherwise the right-hand side of (2.5) would remain bounded while the left-hand side tends to infinity.

The final argument is the following (see Theorem 4.2.1 of [17] for a general statement). For any $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$, the eigenvalues of $\mathbb{M}(z)$ are those $\kappa \neq 0$ such that

$$\kappa^r = \frac{1}{z} \sum_{\ell=-r}^{p} a_{\ell,0} \, \kappa^{r+\ell},$$

which is just an equivalent way of writing (2.5). Hence, for large z, the small eigenvalues of $\mathbb{M}(z)$ behave at the leading order like the roots of the reduced equation

$$\kappa^r = \frac{a_{-r,0}}{z},$$

and there are exactly r distinct roots close to 0 of that equation. Hence $\mathbb{M}(z)$ has r eigenvalues in $\mathbb{D} \setminus \{0\}$ for any $z \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$.

The implicit case ($Q_1 \neq I$). We then know that $a_{-r,1} \neq 0$ and $a_{p,1} \neq 0$. Moreover, the function $\widehat{Q}_1$ satisfies the index condition (1.5). By the residue theorem [31], this means that $\widehat{Q}_1$ has as many poles as roots in $\mathbb{D}$, and since it only has a pole of order r at 0, we can conclude that $\widehat{Q}_1$ has r roots in $\mathbb{D} \setminus \{0\}$. Since $\kappa^r \, \widehat{Q}_1(\kappa)$ is a polynomial of degree $p + r$, we also conclude that $\widehat{Q}_1$ has p roots in $\mathcal{U}$, as already explained in the introduction.

From the definition (1.9), we compute:

$$\lim_{z \to \infty} \mathbb{M}(z) = \begin{bmatrix} -\frac{a_{p-1,1}}{a_{p,1}} & \cdots & \cdots & -\frac{a_{-r,1}}{a_{p,1}} \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where the eigenvalues of that (invertible) matrix are exactly those $\kappa \neq 0$ that satisfy $\widehat{Q}_1(\kappa) = 0$. Hence, for any sufficiently large z, $\mathbb{M}(z)$ has r eigenvalues in $\mathbb{D} \setminus \{0\}$ and p eigenvalues in $\mathcal{U}$ (which are close to the roots of $\widehat{Q}_1$). This completes the proof of the first statement in Lemma 2.1. It now remains to examine the situation at the points $\underline{z}_k$, $k = 1, \dots, K$. The arguments below are the same for the explicit and implicit cases so we stop distinguishing between the two from now on. We thus consider a point $\underline{z}_k$ for $1 \leq k \leq K$.

Case I. We assume that the index $k \in \{1, \dots, K\}$ is such that the set $\mathcal{I}_k$ in (1.11) is the singleton $\{k\}$, and we assume, for now, $\alpha_k > 0$ in (1.8). Since the eigenvalues of $\mathbb{M}(\underline{z}_k)$ are the roots of the dispersion relation

$$\widehat{Q}_1(\kappa)\underline{z}_k = \widehat{Q}_0(\kappa),$$

we first observe that the only eigenvalue of $\mathbb{M}(\underline{z}_k)$ on $\mathbb{S}^1$ is $\underline{\kappa}_k$, and we are now going to show that this eigenvalue is algebraically (and therefore geometrically) simple. The relation (1.8) gives

$$F'(\underline{\kappa}_k) = -\frac{\underline{z}_k \, \alpha_k}{\underline{\kappa}_k} \neq 0.$$

Moreover, the characteristic polynomial of $\mathbb{M}(z)$ at κ equals $z - F(\kappa)$ up to a nonvanishing holomorphic function of (z, κ) close to $(\underline{z}_k, \underline{\kappa}_k)$. This means that $\underline{\kappa}_k$ is an algebraically simple eigenvalue of $\mathbb{M}(\underline{z}_k)$ and can therefore be extended holomorphically with respect to z in a sufficiently small neighborhood of $\underline{z}_k$. We let $\kappa_k(z)$ denote this holomorphic extension, which satisfies $z = F(\kappa_k(z))$ for any z close to $\underline{z}_k$. Performing a Taylor expansion, we compute

$$\kappa_k(\underline{z}_k(1+\epsilon)) = \underline{\kappa}_k\Big(1 - \frac{\epsilon}{\alpha_k}\Big) + O(\epsilon^2).$$

In particular, $\kappa_k(\underline{z}_k(1+\epsilon))$ belongs to $\mathbb{D}$ for $\epsilon > 0$ small enough.

To conclude, we observe that the $p + r - 1$ eigenvalues of $\mathbb{M}(\underline{z}_k)$ which differ from $\underline{\kappa}_k$ lie in $\mathbb{D} \cup \mathcal{U}$. Those eigenvalues remain in $\mathbb{D} \cup \mathcal{U}$ as $\underline{z}_k$ is perturbed into $\underline{z}_k(1+\epsilon)$ for a sufficiently small $\epsilon > 0$. Using the previous step of the analysis, we know that $\mathbb{M}(\underline{z}_k(1+\epsilon))$ has r eigenvalues in $\mathbb{D}$ and p eigenvalues in $\mathcal{U}$, so the reader will easily get convinced that the only possible situation for the location of the eigenvalues of $\mathbb{M}(\underline{z}_k)$ is the one stated in Lemma 2.1.

Cases II and III. It remains to deal with the case where $\mathcal{I}_k$ is the singleton $\{k\}$ and $\alpha_k < 0$ (case II), and the final case where $\mathcal{I}_k$ has two elements (case III). The argument for case II is the same as for case I, except that now the Taylor expansion of κ_k shows that $\kappa_k(\underline{z}_k(1+\epsilon))$ belongs to $\mathcal{U}$ for $\epsilon > 0$ small enough. The remaining details for that case are easily filled in. For case III, $\mathbb{M}(\underline{z}_k)$ has two eigenvalues on $\mathbb{S}^1$, which are, in our usual notation, $\underline{\kappa}_{\nu_{k,1}}$ and $\underline{\kappa}_{\nu_{k,2}}$. The same argument as in case I or case II shows that one of these eigenvalues moves into $\mathbb{D}$ as z is perturbed from $\underline{z}_k$ to $\underline{z}_k(1+\epsilon)$, and the other eigenvalue moves into $\mathcal{U}$. This situation thus mixes cases I and II. The conclusion follows and the proof of Lemma 2.1 is now complete. ∎

2.2. Estimates for the spatial Green function

This section is devoted to the analysis of the solution to the recurrence relation (2.3), which we recall is an equivalent formulation of (2.2). More precisely, our aim is to derive pointwise estimates on the spatial Green function $G_j(z)$. We will divide the analysis depending on the position of z in the complex plane. Away from the tangency points $\underline{z}_1, \dots, \underline{z}_K$, we expect to obtain a uniform exponential decay, while near the tangency points, only some kind of local boundedness is expected. We will rely on the spectral splitting given by Lemma 2.1 to compute pointwise estimates of the augmented vector $W_j(z)$.

We start with the estimates away from the tangency points.

Lemma 2.2 (Bounds away from the tangency points). *Let $\underline{z} \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$. Then there exists an open ball $B_\delta(\underline{z})$, $\delta > 0$, centered at $\underline{z}$, and there exist two constants $C > 0$, $c > 0$ such that*

$$|G_j(z)| \le C \exp(-c|j|) \quad \forall z \in B_\delta(\underline{z}),\ \forall j \in \mathbb{Z}.$$

Proof. We first introduce some notation. Let $\underline{z} \in \overline{\mathcal{U}} \setminus \{\underline{z}_1, \dots, \underline{z}_K\}$ be fixed. We know that $\mathbb{M}(z)$ in (1.10) is well defined and holomorphic in a sufficiently small neighborhood of $\underline{z}$ (including in the case where $\underline{z}$ belongs to $\mathbb{S}^1$). Moreover, because of the spectral splitting shown in Lemma 2.1, the matrix $\mathbb{M}(z)$ has no eigenvalue on $\mathbb{S}^1$ for z close to $\underline{z}$, and it has r, resp. p, eigenvalues in $\mathbb{D}$, resp. $\mathcal{U}$, for z close to $\underline{z}$. Consequently, for z close to $\underline{z}$, the so-called stable subspace, which is spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues in $\mathbb{D}$, has constant dimension r. Similarly, the unstable subspace, which is spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues in $\mathcal{U}$, has constant dimension p. We let $\mathbb{E}^s(z)$, resp. $\mathbb{E}^u(z)$, denote the stable, resp. unstable, subspace of $\mathbb{M}(z)$ for z close to $\underline{z}$. We have the decomposition

$$\mathbb{C}^{p+r} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z) \quad \forall z \in B_\delta(\underline{z}),$$

for some sufficiently small radius $\delta > 0$. The associated projectors are denoted $\pi^s(z)$ and $\pi^u(z)$. The dynamics of (2.3) is therefore of hyperbolic type for any $z \in B_\delta(\underline{z})$.

The projectors $\pi^s(z)$ and $\pi^u(z)$ are given by contour integrals. For instance, we have:

$$\pi^s(z) = \frac{1}{2\mathbf{i}\pi} \int_\gamma (wI - \mathbb{M}(z))^{-1} \, \mathrm{d}w,$$

where γ is a contour that encloses the stable eigenvalues (those in $\mathbb{D}$) of $\mathbb{M}(z)$ (for instance, $\mathbb{S}^1$ is such a contour). A similar formula holds for $\pi^u(z)$ with a contour that encloses the unstable eigenvalues. This formula shows that $\pi^s(z)$ depends holomorphically on z in the ball $B_\delta(\underline{z})$ and, consequently, the stable and unstable subspaces $\mathbb{E}^s(z)$ and $\mathbb{E}^u(z)$ depend holomorphically on z.[6]

Up to restricting δ, any complex number z in the open ball $B_\delta(\underline{z})$ lies in the resolvent set of the operator $\mathscr{L}$, hence there exists a unique solution $(W_j(z))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{p+r})$ to (2.3). Since the dynamics of the iteration (2.3) for such z enjoys a hyperbolic dichotomy, the solution to (2.3) is given by integrating either from j to $+\infty$, or from $-\infty$ to $j-1$, depending on whether we compute the unstable or stable components of the vector $W_j(z)$. This leads to the expression

$$\pi^u(z) W_j(z) = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell \geq 0} (Q_1 \boldsymbol{\delta})_{j+\ell} \, \mathbb{M}(z)^{-1-\ell} \, \pi^u(z) \mathbf{e} \quad \forall j \in \mathbb{Z}, \tag{2.6}$$

for the unstable components, and to the expression

$$\pi^s(z) W_j(z) = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^{j-1} (Q_1 \boldsymbol{\delta})_\ell \, \mathbb{M}(z)^{j-1-\ell} \, \pi^s(z) \mathbf{e} \quad \forall j \in \mathbb{Z}, \tag{2.7}$$

for the stable components.

At this stage, we observe that the sequence $Q_1 \boldsymbol{\delta}$ only has finitely many nonzero coefficients, which are given by

$$(Q_1 \boldsymbol{\delta})_j = \begin{cases} a_{-j,1} & \text{if } j \in \{-p, \dots, r\}, \\ 0 & \text{otherwise,} \end{cases} \quad \forall j \in \mathbb{Z}.$$

[6] Following the analysis of spectral projectors in [18], we shall say that a vector space $\mathbb{E}(z) \subset \mathbb{C}^N$ that depends on a complex variable z, for z in an open set $\mathcal{O} \subset \mathbb{C}$, and that has constant dimension n, depends holomorphically on z if, for any $\underline{z} \in \mathcal{O}$, there exists a neighborhood $\mathcal{V}$ of $\underline{z}$ in $\mathcal{O}$ and a basis $e_1(z), \dots, e_n(z)$ of $\mathbb{E}(z)$ that depends holomorphically on z in $\mathcal{V}$. This amounts to saying that the vector bundle defined by $\mathbb{E}$ over $\mathcal{O}$ is holomorphic. A typical example is the case $\mathbb{E}(z) = P(z)\mathbb{C}^N$, where $P(z)$ is a projector on $\mathbb{C}^N$ that depends holomorphically on z in an open set $\mathcal{O}$.

Hence we see from (2.6) that $\pi^u(z)W_j(z)$ vanishes for $j \geq r+1$, and we see from (2.7) that $\pi^s(z)W_j(z)$ vanishes for $j \leq -p$. For $j \leq r$, we get

$$\pi^u(z)W_j(z) = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell=\max(-p-j,0)}^{r-j} a_{-j-\ell,1}\, \mathbb{M}(z)^{-1-\ell} \pi^u(z)\mathbf{e},$$

and since the sequence $(\mathbb{M}(z)^{-\ell}\pi^u(z))_{\ell\geq 1}$ is exponentially decreasing, uniformly with respect to $z \in B_\delta(\underline{z})$, we get the uniform bound[7]

$$(2.8) \qquad |\pi^u(z)W_j(z)| \leq \begin{cases} 0 & \text{if } j \geq r+1, \\ C\exp(-c|j|) & \text{if } j \leq r, \end{cases} \qquad \forall z \in B_\delta(\underline{z}),\ \forall j \in \mathbb{Z}.$$

Similar arguments, using the exponential decay of the sequence $(\mathbb{M}(z)^{\ell}\pi^s(z))_{\ell\geq 1}$, yield the bound

$$(2.9) \qquad |\pi^s(z)W_j(z)| \leq \begin{cases} 0 & \text{if } j \leq -p-1, \\ C\exp(-c|j|) & \text{if } j \geq -p, \end{cases} \qquad \forall z \in B_\delta(\underline{z}),\ \forall j \geq 1.$$

Adding (2.8) and (2.9) gives the claim of Lemma 2.2, since the spatial Green function $G_j(z)$ is just one coordinate of the vector $W_j(z) \in \mathbb{C}^{p+r}$. ∎

We are now going to examine the behavior of the spatial Green function $G(z)$ close to any of the points $\underline{z}_k$, $k = 1, \ldots, K$, where the spectrum of $\mathcal{L}$ is tangent to the unit circle. Let us first recall that the exterior $\mathcal{U}$ of the unit disk belongs to the resolvent set of $\mathcal{L}$, hence the spatial Green function $G(z)$ is well defined in the “half-ball” $B_\delta(\underline{z}_k) \cap \mathcal{U}$ for any radius $\delta > 0$. Our goal below is to extend holomorphically $G(z)$ to a whole neighborhood of $\underline{z}_k$ for each k, which amounts to passing through the (essential) spectrum of $\mathcal{L}$. Our results are the following two lemmas.

Lemma 2.3 (Bounds close to the tangency points – Cases I and II). *Let $k \in \{1, \ldots, K\}$ be such that the set $\mathcal{I}_k$ in (1.11) is the singleton $\{k\}$. Then there exists an open ball $B_\epsilon(\underline{z}_k)$ and there exist two constants $C > 0$ and $c > 0$ such that, for any integer $j \in \mathbb{Z}$, the component $G_j(z)$ defined on $B_\epsilon(\underline{z}_k) \cap \mathcal{U}$ extends holomorphically to the whole ball $B_\epsilon(\underline{z}_k)$ with respect to z, and the holomorphic extension satisfies the bounds*

$$(\text{Case I}) \quad |G_j(z)| \leq \begin{cases} C\exp(-c|j|) & \text{if } j \leq 0, \\ C|\kappa_k(z)|^j & \text{if } j \geq 1, \end{cases} \quad \text{if } \alpha_k > 0,\ \forall z \in B_\epsilon(\underline{z}_k),\ \forall j \in \mathbb{Z},$$

$$(\text{Case II}) \quad |G_j(z)| \leq \begin{cases} C|\kappa_k(z)|^{|j|} & \text{if } j \leq 0, \\ C\exp(-cj) & \text{if } j \geq 1, \end{cases} \quad \text{if } \alpha_k < 0,\ \forall z \in B_\epsilon(\underline{z}_k),\ \forall j \in \mathbb{Z},$$

where, in either case, $\kappa_k(z)$ denotes the (unique) holomorphic eigenvalue of $\mathbb{M}(z)$ defined close to $\underline{z}_k$, which satisfies $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$.

[7] Here we also use Assumption 1.3 to get a uniform local bound for $\mathbb{A}_p(z)^{-1}$, including in the case $\underline{z} \in \mathbb{S}^1$, for which z can come inside the unit disk.

Lemma 2.4 (Bounds close to the tangency points – Case III). *Let now $k \in \{1, \dots, K\}$ be such that the set $\mathcal{I}_k$ in (1.11) has two elements $\{\nu_{k,1}, \nu_{k,2}\}$ which are fixed by the convention $\alpha_{\nu_{k,1}} < 0 < \alpha_{\nu_{k,2}}$. Then there exists an open ball $B_\epsilon(\underline{z}_k)$ centered at $\underline{z}_k$ and there exists a constant $C > 0$ such that, for any integer $j \in \mathbb{Z}$, the component $G_j(z)$ defined on $B_\epsilon(\underline{z}_k) \cap \mathcal{U}$ extends holomorphically to the whole ball $B_\epsilon(\underline{z}_k)$ with respect to z, and the holomorphic extension satisfies the bound*

$$\text{(Case III)} \qquad |G_j(z)| \le \begin{cases} C\,|\kappa_{\nu_{k,1}}(z)|^{|j|} & \text{if } j \le 0, \\ C\,|\kappa_{\nu_{k,2}}(z)|^{j} & \text{if } j \ge 1, \end{cases} \qquad \forall z \in B_\epsilon(\underline{z}_k),\ \forall j \in \mathbb{Z},$$

where $\kappa_{\nu_{k,1}}(z)$, resp. $\kappa_{\nu_{k,2}}(z)$, denotes the (unique) holomorphic eigenvalue of $\mathbb{M}(z)$, defined close to $\underline{z}_k$, which satisfies $\kappa_{\nu_{k,1}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,1}}$, resp. $\kappa_{\nu_{k,2}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,2}}$.

The proofs of Lemma 2.3 and Lemma 2.4 are mostly identical so we just give the proof of Lemma 2.3 and indicate the minor refinements for the proof of Lemma 2.4.

Proof of Lemma 2.3. Most ingredients of the proof are similar to what we have already done in the proof of Lemma 2.2. We assume from now on $\alpha_k > 0$, the case $\alpha_k < 0$ being left to the interested reader. We just need to slightly adapt the notation used in the proof of Lemma 2.2, since the hyperbolic dichotomy of $\mathbb{M}(z)$ does not hold any longer in a whole neighborhood of $\underline{z}_k$. Since $\underline{\kappa}_k$ is a simple eigenvalue of $\mathbb{M}(\underline{z}_k)$, we can extend it holomorphically to a simple eigenvalue $\kappa_k(z)$ of $\mathbb{M}(z)$ in a neighborhood of $\underline{z}_k$. This eigenvalue is associated with the eigenvector

$$E_k(z) := \begin{bmatrix} \kappa_k(z)^{p+r-1} \\ \vdots \\ \kappa_k(z) \\ 1 \end{bmatrix} \in \mathbb{C}^{p+r},$$

which also depends holomorphically on z in a neighborhood of $\underline{z}_k$. The vector $E_k(z)$ contributes to the stable subspace of $\mathbb{M}(z)$ for $z \in \mathcal{U}$ close to $\underline{z}_k$ but the situation is unclear as z goes inside $\mathbb{D}$ (it actually depends on the position of z with respect to the spectrum of $\mathcal{L}$). The remaining $p + r - 1$ eigenvalues of $\mathbb{M}(z)$ enjoy the now familiar hyperbolic dichotomy, uniformly with respect to z close to $\underline{z}_k$. We let below $\mathbb{E}^{ss}(z)$, resp. $\mathbb{E}^u(z)$, denote the strongly stable, resp. unstable, subspace of $\mathbb{M}(z)$ associated with those eigenvalues that remain uniformly inside $\mathbb{D}$, resp. $\mathcal{U}$, as z belongs to a neighborhood of $\underline{z}_k$. In particular, $\mathbb{E}^{ss}(z)$, resp. $\mathbb{E}^u(z)$, has dimension $r - 1$, resp. p, thanks to Lemma 2.1, and we have the decomposition

$$\text{(2.10)} \qquad \mathbb{C}^{p+r} = \mathbb{E}^{ss}(z) \oplus \text{Span}\, E_k(z) \oplus \mathbb{E}^u(z) \quad \forall z \in B_\epsilon(\underline{z}_k),$$

for a sufficiently small radius $\epsilon > 0$. We let below $\pi^{ss}(z)$, $\pi_k(z)$ and $\pi^u(z)$ denote the holomorphic projectors associated with the decomposition (2.10).

We first consider a point $z \in B_\epsilon(\underline{z}_k) \cap \mathcal{U}$ so that the decomposition (2.10) holds and the Green function $G(z) \in \ell^2(\mathbb{Z};\mathbb{C})$ is well defined as the only solution to (2.2). We use the equivalent formulation (2.3) and derive the following expressions that are entirely similar to those found in the proof of Lemma 2.2, for all $j \in \mathbb{Z}$:

$$\pi^u(z) W_j(z) = -\frac{1}{\mathbb{A}_p(z)} \sum_{\ell \geq 0} (Q_1 \boldsymbol{\delta})_{j+\ell} \, \mathbb{M}(z)^{-1-\ell} \pi^u(z) \mathbf{e}, \tag{2.11a}$$

$$\pi^{ss}(z) W_j(z) = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^{j-1} (Q_1 \boldsymbol{\delta})_{\ell} \, \mathbb{M}(z)^{j-1-\ell} \pi^{ss}(z) \mathbf{e}, \tag{2.11b}$$

$$\pi_k(z) W_j(z) = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-\infty}^{j-1} (Q_1 \boldsymbol{\delta})_{\ell} \, \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e}. \tag{2.11c}$$

The strongly stable $(\pi^{ss}(z) W_j(z))$ and unstable $(\pi^u(z) W_j(z))$ components obviously extend holomorphically to the whole neighborhood $B_\epsilon(\underline{z}_k)$ of $\underline{z}_k$, since the projectors $\pi^{ss}(z)$ and $\pi^u(z)$ depend holomorphically on z on that set, and the sums on the right-hand side of (2.11a) and (2.11b) are, at most, finite. Furthermore, by using the same type of bounds as in the proof of Lemma 2.2, we obtain

$$|\pi^u(z) W_j(z) + \pi^{ss}(z) W_j(z)| \leq C \exp(-c|j|) \quad \forall z \in B_\epsilon(\underline{z}_k), \ \forall j \in \mathbb{Z}, \tag{2.12}$$

for appropriate constants $c, C > 0$. We now focus on the vector $\pi_k(z) W_j(z)$ in (2.11c), which is aligned with the eigenvector $E_k(z)$. We see from (2.11c) that $\pi_k(z) W_j(z)$ vanishes for $j \leq -p$. For j in the finite set $\{-p+1, \dots, r\}$, we have

$$\pi_k(z) W_j(z) = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-p}^{j-1} a_{-\ell,1} \, \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e},$$

and for $j \geq r+1$, we have

$$\pi_k(z) W_j(z) = \frac{1}{\mathbb{A}_p(z)} \sum_{\ell=-p}^{r} a_{-\ell,1} \, \kappa_k(z)^{j-1-\ell} \pi_k(z) \mathbf{e}.$$

In either case, we see that the component $\pi_k(z) W_j(z)$ extends holomorphically to the whole neighborhood $B_\epsilon(\underline{z}_k)$ of $\underline{z}_k$, and we have a bound of the form[8]

$$|\pi_k(z) W_j(z)| \leq \begin{cases} 0 & \text{if } j \leq -p, \\ C|\kappa_k(z)|^j & \text{if } j \geq 1-p, \end{cases} \quad \forall z \in B_\epsilon(\underline{z}_k), \ \forall j \in \mathbb{Z}. \tag{2.13}$$

In order to conclude, we can always assume that the ball $B_\epsilon(\underline{z}_k)$ is so small that the modulus $|\kappa_k(z)|$ belongs to the interval $[\exp(-c), \exp c]$ (it equals 1 at $\underline{z}_k$).[9] It then remains to add the bounds in (2.12) and (2.13) and to compare which is the largest. This completes the proof of Lemma 2.3 in the case $\alpha_k > 0$. The remaining case II ($\alpha_k < 0$) is handled similarly except that now the eigenvector $E_k(z)$ contributes to the unstable subspace of $\mathbb{M}(z)$ for $|z| > 1$. The minor modifications are left to the reader. ∎

[8] Here we use again that $\mathbb{A}_p(z)$ does not vanish in the ball $B_\epsilon(\underline{z}_k)$ up to restricting the radius ϵ.

[9] The constant c here refers to the same one as in (2.12).

Proof of Lemma 2.4. The proof of Lemma 2.4 is a mixture between cases I and II in which now $\mathbb{M}(z)$ has two (holomorphic) eigenvalues whose modulus equals 1 at $\underline{z}_k$. One contributes to the stable subspace of $\mathbb{M}(z)$ and the other one contributes to the unstable subspace of $\mathbb{M}(z)$ for $|z| > 1$. The same ingredients as in the proof of Lemma 2.3 can then be applied with minor modifications. ∎

Let us remark that all the claims in Lemma 2.2, Lemma 2.3 and Lemma 2.4 do not distinguish between the explicit and implicit case since they only rely on Lemma 2.1. In the implicit case, for the forthcoming estimates of the temporal Green function of Section 3, we will also need to obtain bounds of the spatial Green function $G(z)$ for large values of z. These bounds are provided by the following result.

Lemma 2.5 (Bounds at infinity – Implicit case). *If Q_1 is not the identity, then there exist a radius $R \geq 2$ and two constants $C > 0$, $c > 0$ such that*

$$|G_j(z)| \leq C \exp(-c|j|) \quad \forall z \notin B_R(0),\ \forall j \in \mathbb{Z}.$$

Proof. The proof is basically the same as that of Lemma 2.2. Indeed, we recall that in the implicit case, the matrix $\mathbb{M}(z)$ has a limit at infinity, given by

$$\begin{bmatrix} -\frac{a_{p-1,1}}{a_{p,1}} & \dots & \dots & -\frac{a_{-r,1}}{a_{p,1}} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and this matrix has a hyperbolic dichotomy because of Assumption 1.1. We can thus apply the same argument as in the proof of Lemma 2.2 for z in a neighborhood of infinity. In particular, we can use the fact that the sequences $(\mathbb{M}(z)^{-\ell}\pi^u(z))_{\ell\geq 1}$ and $(\mathbb{M}(z)^{\ell}\pi^s(z))_{\ell\geq 1}$ are exponentially decreasing, uniformly with respect to z in a neighborhood $B_R(0)^c$ of infinity. The conclusion of Lemma 2.5 follows. ∎

Let us observe that we can extend holomorphically each scalar component $G_j(z)$ to a neighborhood $B_\epsilon(\underline{z}_k)$ of $\underline{z}_k$, but that does not necessarily mean that the extended sequence $G(z)$ lies in $\ell^2(\mathbb{Z};\mathbb{C})$. For instance, in case I of Lemma 2.3, the eigenvalue $\kappa_k(z)$ contributes to the stable subspace of $\mathbb{M}(z)$ for $|z| > 1$ but it starts contributing to the unstable subspace of $\mathbb{M}(z)$ as z crosses the spectrum of $\mathcal{L}$ (which coincides with the curve $F(\mathbb{S}^1)$). Hence the holomorphic extension $G(z)$ ceases to be in $\ell^2(\mathbb{Z};\mathbb{C})$ as z crosses the spectrum of $\mathcal{L}$ for it then has an exponentially growing mode in j at $+\infty$.

We finally end this section with the following corollary, which is a direct consequence of Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 above, by applying a standard compactness argument. We refer to Figure 1 for a geometrical representation of this result.

Corollary 2.6. *There exists $\epsilon_\star > 0$ such that for each $\epsilon \in (0,\epsilon_\star)$, there exists $\delta_\epsilon > 0$ such that the Green function $G(z)$, defined initially as the unique solution $G(z) \in \ell^2(\mathbb{Z};\mathbb{C})$ to (2.2) for each z in the resolvent set of $\mathcal{L}$, has a unique holomorphic extension (also denoted $G(z)$) to the set*

$$\mathcal{S} := \left\{\zeta \in \mathbb{C} \mid \mathrm{e}^{-\delta_\epsilon} < |\zeta| \leq \mathrm{e}^{\pi}\right\} \cup \bigcup_{k=1}^{K} B_\epsilon(\underline{z}_k).$$

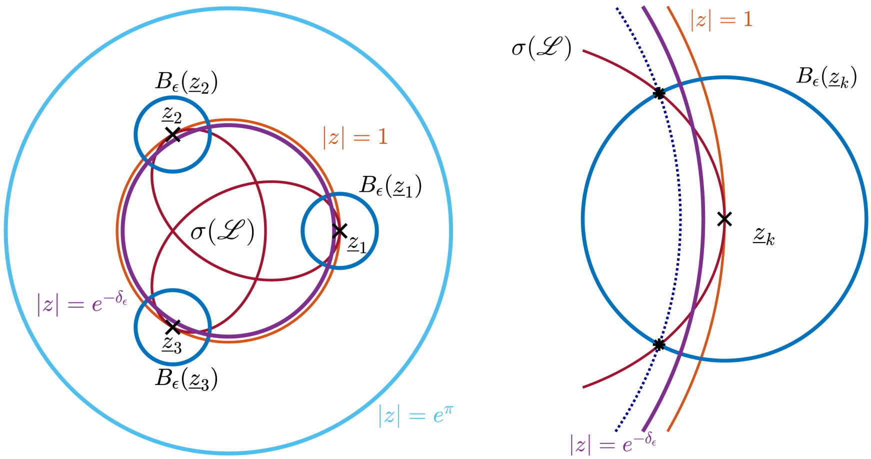

Figure 1. Geometrical illustrations of the set $\mathcal{S}$ given in Corollary 2.6 in the case $K = 3$. The curve of essential spectrum $\sigma(\mathscr{L})$ (dark red curve) is tangent to the unit circle precisely at $\underline{z}_k$ for $k = 1, 2, 3$ (black crosses) and otherwise strictly contained in the unit disk $\mathbb{D}$. For $\epsilon \in (0, \epsilon_\star)$, each ball $B_\epsilon(\underline{z}_k)$ intersects $\sigma(\mathscr{L})$ at two points (black stars in the right panel) and the dashed dark blue line represents the circle passing through the point with largest modulus. We then fix $\delta_\epsilon > 0$ such that the circle $\{z \mid |z| = \mathrm{e}^{-\delta_\epsilon}\}$ (magenta curve) is contained in between the unit circle and the circle passing through the point with largest modulus.

Moreover, there are constants $C > 0$ and $c > 0$ such that:

- *whenever $z \in \mathcal{S} \setminus \bigcup_{k=1}^{K} B_\epsilon(\underline{z}_k)$,*

$$|G_j(z)| \leq C \exp(-c|j|) \quad \forall j \in \mathbb{Z}, \tag{2.14}$$

- *whenever $z \in B_\epsilon(\underline{z}_k)$ for $k = 1, \dots, K$, the Green function $G(z)$ satisfies one of the bounds in Lemma 2.3 or 2.4 depending on the cardinal of $\mathcal{I}_k$ and the sign of α_k.*

Furthermore, in the implicit case, the above uniform exponential bound (2.14) extends to all $|z| \geq \mathrm{e}^{\pi}$.

3. The temporal Green function

The starting point of the analysis is to use the inverse Laplace transform formula to express the so-called Green function $\mathscr{G}^n := \mathscr{L}^n \boldsymbol{\delta}$ as the following contour integral:

$$\mathscr{G}_j^n = (\mathscr{L}^n \boldsymbol{\delta})_j = \frac{1}{2\mathbf{i}\pi} \int_{\widetilde{\Gamma}} z^n G_j(z)\, \mathrm{d}z \quad \forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \tag{3.1}$$

where $\widetilde{\Gamma}$ is a closed curve in the complex plane surrounding the unit disk $\mathbb{D}$ and lying in the resolvent set of $\mathscr{L}$. The idea is to deform $\widetilde{\Gamma}$ in order to obtain sharp pointwise estimates on the temporal Green function using our pointwise estimates on the spatial Green function given in Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 above. To do so, we first change variable in (3.1), by setting $z = \exp(\tau)$, so that we get

$$\mathscr{G}_j^n = \frac{1}{2\mathbf{i}\pi} \int_\Gamma e^{n\tau} \mathbf{G}_j(\tau) \, d\tau, \tag{3.2}$$

where, without loss of generality, $\Gamma = \{s + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ for some $s > 0$ (and actually *any* $s > 0$ thanks to Cauchy's formula), and $\mathbf{G}_j(\tau)$ is given by

$$\mathbf{G}_j(\tau) := G_j(e^\tau) e^\tau \quad \forall j \in \mathbb{Z}.$$

The remaining of this section is devoted to the proof of Theorem 1.6. For the sake of clarity, we first treat the explicit case with $K = 1$, and then deal with the implicit case still with $K = 1$. And finally, we explain how the results generalize to $K > 1$ in both cases. (Let us recall that K denotes the number of tangency points of the spectrum of $\mathscr{L}$ within the unit circle $\mathbb{S}^1$.)

In the explicit case (when Q_1 is the identity), the analysis below uses the fact that each velocity α_k lies in the open interval $(-p, r)$. This fact is stated in the following lemma, which can be seen as a variation of the so-called Courant–Friedrichs–Lewy condition [5] and/or the Bernstein inequality for trigonometric polynomials. A proof of Lemma 3.1 is provided in Appendix A.

Lemma 3.1. *Under Assumptions 1.1 and 1.2, if Q_1 is the identity, then*

$$-p < \alpha_k < r \quad \forall k = 1, \dots, K,$$

where the α_k's are the drift velocities arising in (1.8).

We now deal with the proof of Theorem 1.6.

3.1. The explicit case with $K = 1$

We first remark that, since $\mathscr{L} = Q_0$ is a convolution operator with finite stencil, for each $n \geq 1$, we have

$$\mathscr{G}_j^n = 0 \quad \text{for } j > rn \text{ or } j < -pn.$$

As a consequence, throughout this section, we assume that j and n satisfy

$$-pn \leq j \leq rn, \quad n \geq 1.$$

We also assume without loss of generality that $\underline{\kappa}_1 = \underline{z}_1 = 1$ together with $\alpha_1 > 0$ (the case $\alpha_1 < 0$ being handled similarly). In that case, we have from (1.8) that

$$F(e^{\mathbf{i}\xi}) = \exp\big(-\mathbf{i}\alpha\xi - \beta\xi^{2\mu} + O(\xi^{2\mu+1})\big) \quad \text{as } \xi \to 0, \tag{3.3}$$

where we dropped the index 1 to simplify our notations. Now, using Lemma 2.3, bounds close to the tangency point $z = 1$ for $G_j(z)$ translate into bounds near the origin $\tau = 0$ for $\mathbf{G}_j(\tau)$. More precisely, we have the following lemma which combines Lemma 2.3 and Corollary 2.6.

Lemma 3.2. *There exist $\epsilon_* > 0$ and two constants $0 < \beta_* < \mathrm{Re}(\beta) < \beta^*$ such that for each $\epsilon \in (0, \epsilon_*)$ there exist some width $\eta_\epsilon > 0$ together with two constants, still denoted $C > 0$, $c > 0$, such that, for any integer $j \in \mathbb{Z}$, the component $\mathbf{G}_j(\tau)$ extends holomorphically on $B_\epsilon(0)$ with bounds*

$$|\mathbf{G}_j(\tau)| \leq \begin{cases} C\exp(-c|j|) & \text{if } j \leq 0, \\ C\exp(j\,\mathrm{Re}(\varpi(\tau))) & \text{if } j \geq 1, \end{cases} \quad \forall \tau \in B_\epsilon(0),\ \forall j \in \mathbb{Z}, \tag{3.4}$$

where ϖ is holomorphic on $B_\epsilon(0)$ and has the Taylor expansion

$$\varpi(\tau) = -\frac{1}{\alpha}\tau + (-1)^{\mu+1}\frac{\beta}{\alpha^{2\mu+1}}\tau^{2\mu} + O(|\tau|^{2\mu+1}) \quad \forall \tau \in B_\epsilon(0), \tag{3.5}$$

together with

$$\mathrm{Re}(\varpi(\tau)) \leq -\frac{1}{\alpha}\mathrm{Re}(\tau) + \frac{\beta^*}{\alpha^{2\mu+1}}\mathrm{Re}(\tau)^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}}\mathrm{Im}(\tau)^{2\mu} \quad \forall \tau \in B_\epsilon(0). \tag{3.6}$$

Furthermore, we have

$$|\mathbf{G}_j(\tau)| \leq C\exp(-c|j|) \quad \forall \tau \in \Omega_\epsilon := \{-\eta_\epsilon < \mathrm{Re}(\tau) \leq \pi\} \setminus B_\epsilon(0),\ \forall j \in \mathbb{Z}. \tag{3.7}$$

Proof. The first part of the proof simply relies on writing $\kappa(z) = \exp(\omega(z))$ near $z = 1$ and using $z = \exp(\tau)$, so that after identification, we have $\varpi(\tau) = \omega(\exp(\tau))$. Indeed, the function κ is holomorphic in the ball $B_{\epsilon_0}(1)$ for some $\epsilon_0 > 0$. Upon reducing the size of ϵ_0, we can define a holomorphic function $\omega: B_{\epsilon_0}(1) \to \mathbb{C}$ such that $\kappa(z) = \exp(\omega(z))$ for each $z \in B_{\epsilon_0}(1)$, and $\omega(1) = 0$. We now let $\epsilon_* > 0$ small enough be such that for each $\epsilon \in (0, \epsilon_*)$ and $\tau \in B_\epsilon(0)$, we have $\exp(\tau) \in B_{\epsilon_0}(1)$. We can now define $\varpi: B_\epsilon(0) \to \mathbb{C}$ as $\varpi(\tau) := \omega(\exp(\tau))$, which is holomorphic in $B_\epsilon(0)$ by construction. Finally, we remark that $\mathbf{G}_j(\tau)$ extends holomorphically on $B_\epsilon(0)$ for any $j \in \mathbb{Z}$, since $G_j(z)$ extends holomorphically on $B_{\epsilon_0}(1)$.

Next, we explain how to use the expansion (3.3) to obtain the desired Taylor expansion (3.5) for $\varpi(\tau)$ near $\tau = 0$. We first remark that for each $\epsilon \in (0, \epsilon_*)$ and $\tau \in B_\epsilon(0)$, we have the identity

$$\mathrm{e}^\tau = F(\kappa(\mathrm{e}^\tau)) = F(\mathrm{e}^{\varpi(\tau)}).$$

As a consequence, we get the expansion

$$\tau = -\alpha\varpi(\tau) + \beta(-1)^{\mu+1}\varpi(\tau)^{2\mu} + O(|\varpi(\tau)|^{2\mu+1}),$$

as $\tau \to 0$. Since ϖ is holomorphic in $B_\epsilon(0)$ with $\varpi(0) = 0$, we can use the above equality to obtain, by identification, each term of its Taylor expansion and recover (3.5). Note that we can always reduce the size of ϵ_* so that the expansion is valid for each $\epsilon \in (0, \epsilon_*)$ and τ in $B_\epsilon(0)$.

To complete the proof, we now prove the existence of two positive real numbers β_* and β^* verifying $0 < \beta_* < \mathrm{Re}(\beta) < \beta^*$ such that inequality (3.6) holds true in $B_\epsilon(0)$ for each $\epsilon \in (0, \epsilon_*)$. First we compute

$$(-1)^{\mu+1}\mathrm{Re}(\beta\tau^{2\mu}) = (-1)^{\mu+1}\mathrm{Re}(\beta)\mathrm{Re}(\tau^{2\mu}) - (-1)^{\mu+1}\mathrm{Im}(\beta)\mathrm{Im}(\tau^{2\mu}),$$

i.e.,

$$\begin{aligned}(-1)^{\mu+1}\operatorname{Re}(\beta\tau^{2\mu}) = &-\operatorname{Re}(\beta)\operatorname{Im}(\tau)^{2\mu} - (-1)^{\mu}\operatorname{Re}(\beta)\operatorname{Re}(\tau)^{2\mu}\\ &- \operatorname{Re}(\beta)\sum_{m=1}^{\mu-1}(-1)^m\binom{2\mu}{2m}\operatorname{Re}(\tau)^{2m}\operatorname{Im}(\tau)^{2(\mu-m)}\\ &- \operatorname{Im}(\beta)\sum_{m=0}^{\mu-1}(-1)^{m+1}\binom{2\mu}{2m+1}\operatorname{Re}(\tau)^{2m+1}\operatorname{Im}(\tau)^{2(\mu-m)-1}.\end{aligned}$$

Next, using Young's inequality with some $\delta > 0$, we get

$$\operatorname{Re}(\tau)^k\operatorname{Im}(\tau)^{2\mu-k} \leq \frac{k}{2\mu\delta^{2\mu/k}}\operatorname{Re}(\tau)^{2\mu} + \frac{2\mu-k}{2\mu}\,\delta^{\frac{2\mu}{2\mu-k}}\operatorname{Im}(\tau)^{2\mu}, \quad k = 1,\dots,2\mu-1.$$

And, we also note that the remainder term can be bounded as

$$O(|\tau|^{2\mu+1}) \leq C\epsilon_*(|\operatorname{Re}(\tau)|^{2\mu} + |\operatorname{Im}(\tau)|^{2\mu}), \quad \tau \in B_\epsilon(0),$$

for each $\epsilon \in (0,\epsilon_*)$ and for some constant $C > 0$ independent of ϵ. We finally remark that $\delta > 0$ can be taken arbitrarily small and that the leading order term in $\operatorname{Im}(\tau)^{2\mu}$ comes with a negative sign, since we assume $\operatorname{Re}(\beta) > 0$. As a consequence, upon eventually reducing the size of ϵ_*, we can find $0 < \beta_* < \operatorname{Re}(\beta) < \beta^*$ (depending only on ϵ_*) such that inequality (3.6) holds true in $B_\epsilon(0)$ for each $\epsilon \in (0,\epsilon_*)$. ∎

Using Lemma 3.2, we readily see that when $-np \leq j \leq 0$, our estimates (3.4)–(3.7) from Lemma 3.2 can be combined to

$$|\mathbf{G}_j(\tau)| \leq C\mathrm{e}^{-c|j|} \quad \forall \tau \in \Omega_\epsilon \cup B_\epsilon(0),\ \forall j \leq 0,$$

from which we automatically obtain the following estimate, using the contour:

$$\Gamma = \left\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi,\pi]\right\} \subset \Omega_\epsilon \cup B_\epsilon(0)$$

in (3.2) for any $0 < \eta < \eta_\epsilon$. Modifying the contour in (3.2) is legitimate thanks to Cauchy's formula and also because the integrals on the segments $\{-\upsilon \pm \mathbf{i}\pi \mid \upsilon \in [-\eta,\pi]\}$ compensate one another.

Lemma 3.3. *For each $\epsilon \in (0,\epsilon_*)$ there exists constants $C > 0$ and $c > 0$ such that for all $-np \leq j \leq 0$ with $n \geq 1$,*

$$|\mathscr{G}_j^n| \leq C\mathrm{e}^{-n\eta-c|j|},$$

for any $\eta \in (0,\eta_\epsilon)$ with $\eta_\epsilon > 0$ the width given in Lemma 3.2.

From now on, we assume that $1 \leq j \leq nr$. It turns out that we will need again to divide the analysis into two parts. We will consider first the medium range where $1 \leq j \leq n\delta$, with $\delta := \alpha/2$. In this case we can prove the following lemma.

Lemma 3.4. *For each $\epsilon \in (0, \min(\epsilon_*, (\frac{\alpha^{2\mu}}{2\beta^*})^{\frac{1}{2\mu-1}}))$, there exists a constant $C > 0$ such that for $n \geq 1$ and $1 \leq j \leq n\delta$, the following estimate holds:*

$$|\mathscr{G}_j^n| \leq C\mathrm{e}^{-n\eta/4},$$

for each $\eta \in (0,\eta_\epsilon)$ with $\eta_\epsilon > 0$ the width given in Lemma 3.2.

Proof. For each $\epsilon \in (0, \min(\epsilon_*, (\frac{\alpha^{2\mu}}{2\beta^*})^{\frac{1}{2\mu-1}}))$ and for $\eta \in (0, \eta_\epsilon)$, with $\eta_\epsilon > 0$ given in Lemma 3.2, we use again the segment $\Gamma = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\} \subset \Omega_\epsilon \cup B_\epsilon(0)$ in (3.2). We denote by Γ^{in} and Γ^{out} the portions of the segment $\text{Re}(\tau) = -\eta$ which lie inside $B_\epsilon(0)$ and outside $B_\epsilon(0)$ with $|\text{Im}(\tau)| \leq \pi$, respectively. Standard computations (using Lemma 3.2) lead to

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma^{\text{out}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \leq C\mathrm{e}^{-n\eta-cj}$$

and

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma^{\text{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \leq C\mathrm{e}^{-n\eta}\int_{\Gamma^{\text{in}}} \mathrm{e}^{j\,\text{Re}(\varpi(\tau))}\,\frac{|\mathrm{d}\tau|}{2\pi}.$$

Next, we recall the estimate (3.6) on $\text{Re}(\varpi(\tau))$ from Lemma 3.2, that is,

$$\text{Re}(\varpi(\tau)) \leq -\frac{1}{\alpha}\,\text{Re}(\tau) + \frac{\beta^*}{\alpha^{2\mu+1}}\,\text{Re}(\tau)^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}}\,\text{Im}(\tau)^{2\mu} \quad \forall \tau \in B_\epsilon(0).$$

As a consequence, for all $\tau \in \Gamma^{\text{in}} \subset B_\epsilon(0)$, we have

$$\text{Re}(\varpi(\tau)) \leq \frac{\eta}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}}\,\eta^{2\mu}.$$

Here, we crucially used the fact that the term in $\text{Im}(\tau)^{2\mu}$ in the estimate for $\text{Re}(\varpi(\tau))$ comes with a negative sign. Summarizing, we have obtained that

$$-n\eta + j\,\text{Re}(\varpi(\tau)) \leq n\eta\Big(-\frac{1}{2} + \frac{\beta^*}{2\alpha^{2\mu}}\,\eta^{2\mu-1}\Big)$$

for each $\tau \in \Gamma^{\text{in}}$ and $1 \leq j \leq n\alpha/2$. Finally, since $0 < \eta < \epsilon < (\frac{\alpha^{2\mu}}{2\beta^*})^{\frac{1}{2\mu-1}}$, we have that $\frac{\beta^*}{2\alpha^{2\mu}}\,\eta^{2\mu-1} < 1/4$, and we get the final estimate

$$\mathrm{e}^{-n\eta}\int_{\Gamma^{\text{in}}} \mathrm{e}^{j\,\text{Re}(\varpi(\tau))}\,\frac{|\mathrm{d}\tau|}{2\pi} \leq \mathrm{e}^{-n\eta/4}.$$

This concludes the proof of the lemma. ∎

We now turn to the last case where $n \geq 1$ and $n\delta \leq j \leq nr$ (recall $\delta = \alpha/2$ and $\alpha < r$). Our generalized Gaussian estimates will precisely come from this part of the analysis. In order to proceed, we follow the strategy developed in [40] in the fully continuous case (see also [11] in a fully discrete case that corresponds to $\mu = 1$ in our notation), and introduce a family of parametrized curves given by

$$\Gamma_p := \Big\{\text{Re}(\tau) - \frac{\beta^*}{\alpha^{2\mu}}\,\text{Re}(\tau)^{2\mu} + \frac{\beta_*}{\alpha^{2\mu}}\,\text{Im}(\tau)^{2\mu} = \Psi(\tau_p) \;\Big|\; -\eta \leq \text{Re}(\tau) \leq \tau_p\Big\}, \tag{3.8}$$

with $\Psi(\tau_p) := \tau_p - \frac{\beta^*}{\alpha^{2\mu}}\tau_p^{2\mu}$. Note that the curves Γ_p intersect the real axis at τ_p. We now explain how we choose $\eta > 0$ and $\tau_p > -\eta$ in the above definition of Γ_p.

First, for each $\epsilon \in (0, \epsilon_*)$, we fix $\eta \in (0, \eta_\epsilon)$ with $\eta_\epsilon > 0$ given in Lemma 3.2 such that the curve Γ_p with $\tau_p = 0$ intersects $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ inside the open ball $B_\epsilon(0)$. Then we let $\epsilon_0 \in (0, \epsilon)$ which is uniquely defined as the value of τ_p for which Γ_p with

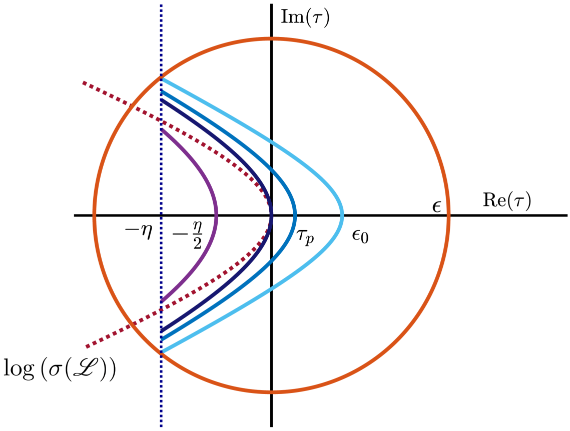

Figure 2. Illustration of the geometry of the family of parametrized curved Γ_p within the ball $B_\epsilon(0)$ for different values of $\tau_p \in [-\eta/2, \epsilon_0]$. The extremal curves are given when $\tau_p = -\eta/2$ to the left (magenta curve) and when $\tau_p = \epsilon_0$ to the right (light blue curve), where $0 < \epsilon_0 < \epsilon$ is precisely defined so that Γ_p with $\tau_p = \epsilon_0$ intersects the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ on the boundary of $B_\epsilon(0)$. The dashed dark red curve represents the logarithm of the spectrum $\sigma(\mathscr{L})$. Note that with our careful choice of parametrization, we have that Γ_p with $\tau_p = 0$ (dark blue curve) lies to the right of the spectral curve with tangency at the origin.

$\tau_p = \epsilon_0$ intersects the segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ precisely on the boundary[10] of $B_\epsilon(0)$ with η fixed previously. And finally, the specific value of τ_p is fixed depending on the ratio ζ/γ as follows:

$$\tau_p := \begin{cases} \rho\left(\frac{\zeta}{\gamma}\right) & \text{if } -\frac{\eta}{2} \leq \rho\left(\frac{\zeta}{\gamma}\right) \leq \epsilon_0, \\ \epsilon_0 & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \epsilon_0, \\ -\frac{\eta}{2} & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}. \end{cases}$$

We refer to Figure 2 for an illustration of the geometry of Γ_p for different values of τ_p. There only remains to define ζ, γ and the function ρ. We let

$$\zeta := \frac{j - n\alpha}{2\mu n} \quad \text{and} \quad \gamma := \frac{j}{n} \frac{\beta^*}{\alpha^{2\mu}} > 0,$$

and $\rho(\zeta/\gamma)$ is the unique real root to the equation

$$-\zeta + \gamma x^{2\mu-1} = 0,$$

that is,

$$\rho\left(\frac{\zeta}{\gamma}\right) := \operatorname{sgn}\left(\frac{\zeta}{\gamma}\right)\left(\frac{|\zeta|}{\gamma}\right)^{\frac{1}{2\mu-1}}.$$

[10] This is possible because the curves Γ_p are symmetric with respect to the real axis.

The motivation for introducing such quantities comes from the estimate (3.6) from Lemma 3.2. More precisely, for all $\tau \in \Gamma_p \subset B_\epsilon(0)$, we have

$$\begin{aligned} j\,\mathrm{Re}(\varpi(\tau)) &\le j\Big(-\frac{1}{\alpha}\,\mathrm{Re}(\tau) + \frac{\beta^*}{\alpha^{2\mu+1}}\,\mathrm{Re}(\tau)^{2\mu} - \frac{\beta_*}{\alpha^{2\mu+1}}\,\mathrm{Im}(\tau)^{2\mu}\Big) \\ &= j\Big(-\frac{\tau_p}{\alpha} + \frac{\beta^*}{\alpha^{2\mu+1}}\,\tau_p^{2\mu}\Big) = -n\tau_p + \frac{n}{\alpha}(-2\mu\zeta\tau_p + \gamma\tau_p^{2\mu}), \end{aligned}$$

and our careful choice of τ_p will always allow us to handle the terms inside the final parenthesis.

We remark that $-\frac{\alpha}{4\mu} \le \zeta \le \frac{r-\alpha}{2\mu}$, and our generalized Gaussian estimates will come from those values of $\zeta \approx 0$. Before proceeding with the analysis, we note that for all $\tau \in \Gamma_p$, we have

$$\mathrm{Re}(\tau) \le \tau_p - c_*\,\mathrm{Im}(\tau)^{2\mu}, \tag{3.9}$$

for some constant $c_* > 0$. Indeed, we remark that the function Ψ, defined as $\Psi(t) = t - \frac{\beta^*}{\alpha^{2\mu}}\,t^{2\mu}$, satisfies $\Psi'(0) = 1$, and for each $t \in [-\eta, \epsilon]$, one has $\Psi'(t) \le c_0$ for some $c_0 > 0$. As a consequence, for each $\tau \in \Gamma_p$, one has

$$-\frac{\beta_*}{\alpha^{2\mu}}\,\mathrm{Im}(\tau)^{2\mu} = \Psi(\mathrm{Re}(\tau)) - \Psi(\tau_p) = -\int_{\mathrm{Re}(\tau)}^{\tau_p} \Psi'(t)\,\mathrm{d}t \ge c_0(\mathrm{Re}(\tau) - \tau_p),$$

which gives the desired estimate (3.9) with $c_* = \frac{\beta_*}{c_0\alpha^{2\mu}}$. Furthermore, a straightforward application of the implicit function theorem gives the following result on the parametrization of the curves Γ_p that we will be using in our estimates below.

Lemma 3.5. *There exist $\epsilon_{**} \in (0, \epsilon_*)$, an analytic function $\Phi\colon (-\epsilon_{**}, \epsilon_{**}) \times (-\epsilon_{**}, \epsilon_{**}) \to \mathbb{R}$ and some constant $C > 0$ such that for any $\epsilon \in (0, \epsilon_{**})$ and $\tau_p \in (-\epsilon, \epsilon)$, the curve Γ_p can be parametrized as*

$$\Gamma_p = \big\{\tau \in B_\epsilon(0) \mid \mathrm{Re}(\tau) = \Phi(\mathrm{Im}(\tau), \tau_p)\big\},$$

with

$$\mathrm{Re}(\tau) = \tau_p - \frac{\beta^*}{\alpha^{2\mu}}\,\mathrm{Im}(\tau)^{2\mu} + O(|\mathrm{Im}(\tau)|^{2\mu+1} + |\tau_p|^{2\mu+1}),$$

together with

$$\Big|\frac{\partial\Phi(\mathrm{Im}(\tau), \tau_p)}{\partial\,\mathrm{Im}(\tau)}\Big| \le C \quad \text{for each } |\mathrm{Im}(\tau)| \le \epsilon \text{ and } |\tau_p| \le \epsilon.$$

Finally, in what follows, we will use the notation $f \lesssim g$ whenever $f \le Cg$ for some constant $C > 0$ independent of j and n.

We first treat the case $-\eta/2 \le \rho(\zeta/\gamma) \le \epsilon_0$. For all $\tau \in \Gamma_p \subset B_\epsilon(0)$, we obtain

$$\begin{aligned} n\,\mathrm{Re}(\tau) + j\,\mathrm{Re}(\varpi(\tau)) &\le n(\mathrm{Re}(\tau) - \tau_p) + \frac{n}{\alpha}(-2\mu\zeta\tau_p + \gamma\tau_p^{2\mu}) \\ &\le -nc_*\,\mathrm{Im}(\tau)^{2\mu} + \frac{n}{\alpha}(-2\mu\zeta\tau_p + \gamma\tau_p^{2\mu}). \end{aligned}$$

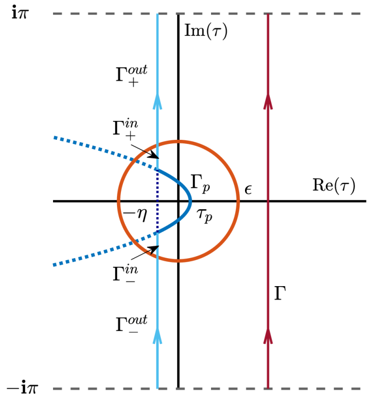

Figure 3. Illustration of the contour used in the case $-\eta/2 \leq \rho(\zeta/\gamma) \leq \epsilon_0$ when $n\delta \leq j \leq nr$. The contour is composed of $\Gamma_-^{\text{out}} \cup \Gamma_-^{\text{in}} \cup \Gamma_p \cup \Gamma_+^{\text{in}} \cup \Gamma_+^{\text{out}}$. The contours $\Gamma_\pm^{\text{in}}$ and $\Gamma_\pm^{\text{out}}$ are the portions of the segment $\mathrm{Re}(\tau) = -\eta$ which lie inside $B_\epsilon(0)$ and outside $B_\epsilon(0)$ with $|\mathrm{Im}(\tau)| \leq \pi$, respectively, while Γ_p is defined in (3.8) and intersects the real axis at τ_p.

For the second term, we will use the specific form of $\tau_p = \rho(\zeta/\gamma) = \mathrm{sgn}(\zeta)(|\zeta|/\gamma)^{\frac{1}{2\mu-1}}$ to get that

$$-2\mu\zeta\tau_p + \gamma\tau_p^{2\mu} = -\gamma(2\mu-1)\left(\frac{|\zeta|}{\gamma}\right)^{\frac{2\mu}{2\mu-1}} < 0.$$

As a consequence, we can derive the following bound:

$$\begin{aligned}\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| &\lesssim \int_{\Gamma_p} \mathrm{e}^{n\,\mathrm{Re}(\tau)+j\,\mathrm{Re}(\varpi(\tau))}|\,\mathrm{d}\tau| \\ &\lesssim \mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{|\zeta|}{\gamma})^{\frac{2\mu}{2\mu-1}}}\int_{\Gamma_p}\mathrm{e}^{-nc_*\,\mathrm{Im}(\tau)^{2\mu}}|\,\mathrm{d}\tau| \\ &\lesssim \frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{|\zeta|}{\gamma})^{\frac{2\mu}{2\mu-1}}}}{n^{1/(2\mu)}}.\end{aligned}$$

In the last inequality, assuming that $\epsilon \in (0, \epsilon_{**})$, we have used Lemma 3.5 to get

$$\int_{\Gamma_p}\mathrm{e}^{-nc_*\,\mathrm{Im}(\tau)^{2\mu}}|\,\mathrm{d}\tau| \lesssim \int_{-\ell_*}^{\ell_*}\mathrm{e}^{-nc_* x^{2\mu}}\,\mathrm{d}x \lesssim \frac{1}{n^{1/(2\mu)}},$$

where $\ell_* \in (0, \epsilon)$ is defined as $\ell_* := (\frac{\alpha^{2\mu}}{\beta^*}(\Psi(\tau_p) - \Psi(-\eta)))^{1/(2\mu)}$, which is the positive root of

$$-\eta - \frac{\beta^*}{\alpha^{2\mu}}\eta^{2\mu} + \frac{\beta_*}{\alpha^{2\mu}}\ell_*^{2\mu} = \Psi(\tau_p) = \tau_p - \frac{\beta^*}{\alpha^{2\mu}}\tau_p^{2\mu}.$$

Next we denote by $\Gamma_\pm^{\text{in}}$ and $\Gamma_\pm^{\text{out}}$ the portions of the segment $\mathrm{Re}(\tau) = -\eta$ which lie inside $B_\epsilon(0)$ and outside $B_\epsilon(0)$ with $|\mathrm{Im}(\tau)| \leq \pi$, respectively. We refer to Figure 3 for an illustration. Usual computations lead to

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\rm out}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \leq C\mathrm{e}^{-n\eta-cj}.$$

For all $\tau\in\Gamma_\pm^{\rm in}$, we use that $\operatorname{Im}(\tau)^2\geq \operatorname{Im}(\tau_*)^2$, where $\tau_*=-\eta+\mathbf{i}\ell_*$, with ℓ_* defined by

$$\ell_*=\Big(\frac{\alpha^{2\mu}}{\beta^*}(\Psi(\tau_p)-\Psi(-\eta))\Big)^{1/(2\mu)}.$$

In other words, the point $\tau_*=-\eta+\mathbf{i}\ell_*$ lies at the intersection of Γ_p and the segment $\{-\eta+\mathbf{i}\ell\mid \ell\in[-\pi,\pi]\}$, with $\tau_*\in B_\epsilon(0)$. As a consequence, for all $\tau\in\Gamma_\pm^{\rm in}$, we have

$$\begin{aligned}\operatorname{Re}(\varpi(\tau)) &\leq -\frac{1}{\alpha}\operatorname{Re}(\tau)+\frac{\beta^*}{\alpha^{2\mu+1}}\operatorname{Re}(\tau)^{2\mu}-\frac{\beta_*}{\alpha^{2\mu+1}}\operatorname{Im}(\tau)^{2\mu}\\ &= -\frac{\tau_p}{\alpha}+\frac{\beta^*}{\alpha^{2\mu+1}}\tau_p^{2\mu}-\frac{\beta_*}{\alpha^{2\mu+1}}\underbrace{(\operatorname{Im}(\tau)^{2\mu}-\ell_*^{2\mu})}_{\geq 0}\\ &\leq -\frac{\tau_p}{\alpha}+\frac{\beta^*}{\alpha^{2\mu+1}}\tau_p^{2\mu}.\end{aligned}$$

Thus, we have

$$\begin{aligned}n\operatorname{Re}(\tau)+j\operatorname{Re}(\varpi(\tau)) &\leq -n\eta+j\Big(-\frac{\tau_p}{\alpha}+\frac{\beta^*}{\alpha^{2\mu+1}}\tau_p^{2\mu}\Big)\\ &= \frac{n}{\alpha}\Big[-\eta\alpha+\frac{j}{n}\Big(-\tau_p+\frac{\beta^*}{\alpha^{2\mu}}\tau_p^{2\mu}\Big)\Big]\\ &\leq \frac{n}{\alpha}\big[-(\eta+\tau_p)\alpha-2\mu\zeta\tau_p+\gamma\tau_p^{2\mu}\big]\\ &= \frac{n}{\alpha}\Big[-(\eta+\tau_p)\alpha-(2\mu-1)\gamma\Big(\frac{|\zeta|}{\gamma}\Big)^{\frac{2\mu}{2\mu-1}}\Big]\end{aligned}$$

for all $\tau\in\Gamma_\pm^{\rm in}$. Finally, as $-\eta/2\leq\rho(\zeta/\gamma)=\tau_p$, we have $\eta+\tau_p\geq\eta/2$, we obtain an estimate of the form

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\rm in}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \leq C\mathrm{e}^{-n\eta/2-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{|\zeta|}{\gamma})^{\frac{2\mu}{2\mu-1}}}.$$

Summarizing, we have obtained

$$\begin{aligned}|\mathscr{G}_j^n| &\leq \Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big|+\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\rm out}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big|+\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\rm in}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big|\\ &\leq C\Big(\frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{|\zeta|}{\gamma})^{\frac{2\mu}{2\mu-1}}}}{n^{1/(2\mu)}}+\mathrm{e}^{-n\eta-cj}+\mathrm{e}^{-n\eta/2-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{|\zeta|}{\gamma})^{\frac{2\mu}{2\mu-1}}}\Big).\end{aligned}$$

Next we consider the case $\rho(\zeta/\gamma)>\epsilon_0$, for which we choose $\tau_p=\epsilon_0$. The contour Γ is decomposed into $\Gamma_p\cup\Gamma_\pm^{\rm out}$, where $\Gamma_\pm^{\rm out}$ are the portions of the segment $\operatorname{Re}(\tau)=-\eta$

which lie outside $B_\epsilon(0)$ with $|\mathrm{Im}(\tau)| \le \pi$. In that case, we have that for all $\tau \in \Gamma_p$,

$$n\,\mathrm{Re}(\tau) + j\,\mathrm{Re}(\varpi(\tau)) \le -nc_*\,\mathrm{Im}(\tau)^{2\mu} + \frac{n}{\alpha}(-2\mu\zeta\tau_p + \gamma\tau_p^{2\mu})$$
$$= -nc_*\,\mathrm{Im}(\tau)^{2\mu} + \frac{n}{\alpha}(-2\mu\zeta\epsilon_0 + \gamma\epsilon_0^{2\mu}),$$

since $\tau_p = \epsilon_0$ in this case. But as $\rho(\zeta/\gamma) > \epsilon_0$, we get that $\zeta > 0$ and $\zeta > \epsilon_0^{2\mu-1}\gamma$, the last term in the previous inequality is estimated via

$$-2\mu\zeta\epsilon_0 + \gamma\epsilon_0^{2\mu} < -(2\mu-1)\gamma\epsilon_0^{2\mu} < 0.$$

As a consequence, we can derive the following bound:

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \lesssim \frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma\epsilon_0^{2\mu}}}{n^{1/(2\mu)}}.$$

With our careful choice of $\epsilon_0 > 0$, the remaining contribution along segments $\Gamma_\pm^{\mathrm{out}}$ with $\mathrm{Re}(\tau) = -\eta$ can be estimated as usual as

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{out}}} \mathrm{e}^{n\tau}\mathbf{G}_\tau(j)\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-n\eta-cj},$$

since $|\tau| \ge \epsilon$ for $\tau \in \Gamma_\pm^{\mathrm{out}}$. Summarizing, we have obtained

$$|\mathcal{G}_j^n| \le \left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| + \left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{out}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right|$$
$$\le C\left(\frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma\epsilon_0^{2\mu}}}{n^{1/(2\mu)}} + \mathrm{e}^{-n\eta-cj}\right).$$

It remains to handle the last case $\rho(\zeta/\gamma) < -\eta/2$, for which we choose $\tau_p = -\eta/2$, and we readily note that in this setting, $\zeta < 0$. The contour Γ is decomposed into $\Gamma_p \cup \Gamma_\pm^{\mathrm{out}} \cup \Gamma_\pm^{\mathrm{in}}$, where once again $\Gamma_\pm^{\mathrm{in}}$ and $\Gamma_\pm^{\mathrm{out}}$ are the portions of the segment $\mathrm{Re}(\tau) = -\eta$ which lie inside $B_\epsilon(0)$ and outside $B_\epsilon(0)$ with $|\mathrm{Im}(\tau)| \le \pi$, respectively. For all $\tau \in \Gamma_p \subset B_\epsilon(0)$, we find that

$$n\,\mathrm{Re}(\tau) + j\,\mathrm{Re}(\varpi(\tau)) \le -nc_*\,\mathrm{Im}(\tau)^{2\mu} + \frac{n}{\alpha}\left(\mu\zeta\eta + \gamma\left(\frac{\eta}{2}\right)^{2\mu}\right).$$

Using that $\rho(\zeta/\gamma) < -\eta/2$, which is equivalent to $\zeta/\gamma < -(\eta/2)^{2\mu-1}$, we get that

$$\mu\zeta\eta + \gamma\left(\frac{\eta}{2}\right)^{2\mu} < -(2\mu-1)\gamma\left(\frac{\eta}{2}\right)^{2\mu}.$$

As a consequence, we can derive the following bound:

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \lesssim \frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{\eta}{2})^{2\mu}}}{n^{1/(2\mu)}}.$$

As usual, we have that

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{out}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-n\eta-cj}.$$

It only remains to estimate the contribution on $\Gamma_\pm^{\mathrm{in}}$. We proceed as before, and we have that for all $\tau\in\Gamma_\pm^{\mathrm{in}}$,

$$\begin{aligned} n\,\mathrm{Re}(\tau)+j\,\mathrm{Re}(\varpi(\tau)) &\le \frac{n}{\alpha}\big[-(\eta+\tau_p)\alpha-2\mu\zeta\tau_p+\gamma\tau_p^{2\mu}\big]\\ &\le \frac{n}{\alpha}\Big[-\frac{\eta}{2}\alpha-(2\mu-1)\gamma\Big(\frac{\eta}{2}\Big)^{2\mu}\Big]\\ &= -n\Big(\frac{\eta}{2}+\frac{(2\mu-1)\gamma}{\alpha}\Big(\frac{\eta}{2}\Big)^{2\mu}\Big), \end{aligned}$$

and this time, we obtain

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-n(\frac{\eta}{2}+\frac{(2\mu-1)\gamma}{\alpha}(\frac{\eta}{2})^{2\mu})}.$$

In conclusion, we have obtained

$$\begin{aligned} |\mathscr{G}_j^n| &\le \left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_p} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| + \left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{out}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| + \left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\pm^{\mathrm{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right|\\ &\le C\Big(\frac{\mathrm{e}^{-\frac{n}{\alpha}(2\mu-1)\gamma(\frac{\eta}{2})^{2\mu}}}{n^{1/(2\mu)}}+\mathrm{e}^{-n\eta-cj}+\mathrm{e}^{-n(\frac{\eta}{2}+\frac{(2\mu-1)\gamma}{\alpha}(\frac{\eta}{2})^{2\mu})}\Big). \end{aligned}$$

As a summary, gathering the above estimates, we can deduce the following result.

Lemma 3.6. *For each $\epsilon\in(0,\epsilon_{**})$, there exist constants $C>0$ and $c>0$ such that for $n\ge 1$ and $n\delta\le j\le nr$, the following estimate holds:*

$$|\mathscr{G}_j^n| \le \frac{C}{n^{1/(2\mu)}}\exp\Big(-c\Big(\frac{|j-\alpha n|}{n^{1/(2\mu)}}\Big)^{\frac{2\mu}{2\mu-1}}\Big),$$

where $\epsilon_{**}>0$ is given in Lemma 3.5.

Proof. One only needs to check that the purely exponentially decaying in n contributions obtained when $\rho(\zeta/\gamma)>\epsilon_0$ or $\rho(\zeta/\gamma)<-\eta/2$ can be subsumed into generalized Gaussian estimates. For example, in the case $\rho(\zeta/\gamma)>\epsilon_0$, there exists some small constant $c>0$ such that

$$-n\le -c\Big(\frac{|j-\alpha n|}{n^{1/(2\mu)}}\Big)^{\frac{2\mu}{2\mu-1}},$$

as

$$\frac{\beta^*}{2\alpha^{2\mu-1}}\le\gamma\le\frac{\beta^*}{\alpha^{2\mu}}\,r$$

and

$$\frac{j}{n\alpha}-1=\frac{2\mu\zeta}{\alpha}>\frac{2\mu}{\alpha}\,\gamma\epsilon_0^{2\mu-1}\ge\frac{\mu\beta^*}{\alpha^{2\mu}}\,\epsilon_0^{2\mu-1}.$$

All other cases can be dealt with in a similar way. ■

Proof of Theorem 1.6. Combining Lemma 3.3, Lemma 3.4 and Lemma 3.6 proves our main Theorem 1.6 in the explicit case with $K = 1$. Indeed, we first fix the parameter ϵ such that

$$\epsilon \in \Big(0,\ \min\Big(\epsilon_{**}, \Big(\frac{\alpha^{2\mu}}{2\beta^*}\Big)^{\frac{1}{2\mu-1}}\Big)\Big),$$

with $0 < \epsilon_{**} < \epsilon_*$ from Lemma 3.5, and then we fix $\eta \in (0, \eta_\epsilon)$, with $\eta_\epsilon > 0$ given in Lemma 3.2, such that the curve Γ_p with $\tau_p = 0$ intersects $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ inside the open ball $B_\epsilon(0)$. As a consequence, we can apply estimates from Lemma 3.3, Lemma 3.4 and Lemma 3.6 with this specific choice of ϵ and η, which gives the proof since purely exponentially decaying in n bounds obtained in Lemma 3.3 and Lemma 3.4 can be subsumed into generalized Gaussian estimates. ∎

3.2. The implicit case with $K = 1$

The main difference compared with the explicit case is that now it is no longer true that $\mathscr{G}_j^n$ vanishes for $j > nr$ or $j < -pn$. Nevertheless, we observe that the results of Lemma 3.3, Lemma 3.4 and Lemma 3.6 still hold true. Actually, the proofs of Lemma 3.3, Lemma 3.4 and Lemma 3.6 naturally extend to $-nL \leq j \leq 0$ (Lemma 3.3), $1 \leq j \leq n\delta$ (Lemma 3.4, unchanged) and $n\delta \leq j \leq nL$ (Lemma 3.6) for any large constant $L \geq \max(p, r)$ that is fixed a priori. As a consequence, one only needs to consider the case $n \geq 1$ and $|j| > nL$ for some large constant $L > 0$ to be determined. To obtain the desired estimate in that case, we use the bound at infinity obtained in Lemma 2.5. More precisely, there exist $R \geq \pi/2$ and two constants $C > 0$, $\underline{c} > 0$ such that

$$|\mathbf{G}_j(\tau)| \leq C \exp(-\underline{c}|j|) \quad \forall \tau \in \{\zeta \in \mathbb{C} \mid \mathrm{Re}(\zeta) \geq \log R\},\ \forall j \in \mathbb{Z}. \tag{3.10}$$

We then have the following result.

Lemma 3.7. *Let $L \geq \max(p, r) > 0$ be large enough such that $L > 2 \log R/\underline{c}$ with R and $\underline{c}$ as in* (3.10)*. Then there exists $C > 0$ such that for $n \geq 1$ and $|j| > nL$, we have*

$$|\mathscr{G}_j^n| \leq C \exp\Big(-n\frac{\underline{c}L}{4} - \frac{\underline{c}}{2}|j|\Big).$$

Proof. In (3.2), we now use the contour $\Gamma = \{\underline{c}|j|/(2n) + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$. With our choice of L, we have that for all $\tau \in \Gamma$, $\mathrm{Re}(\tau) = \underline{c}|j|/(2n) \geq \underline{c}L/2 > \log R$ and so

$$\left|\frac{1}{2\mathbf{i}\pi}\int_\Gamma \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \leq C\mathrm{e}^{-\frac{\underline{c}}{2}|j|}.$$

Finally, we notice that

$$-\frac{\underline{c}}{2}|j| \leq -n\frac{\underline{c}L}{4} - \frac{\underline{c}}{4}|j| \quad \text{for } |j| > nL.$$

This completes the proof of the lemma. ∎

The proof of Theorem 1.6 in the implicit case (for $K = 1$) then follows from the combination of the slight extensions of Lemma 3.3, Lemma 3.4 and Lemma 3.6 (once the large constant L is fixed as in Lemma 3.7).

3.3. The explicit and implicit cases with $K > 1$

We now briefly explain how to handle the general case with $K > 1$ and refer to Figures 4 and 5 for illustrations. From Assumption 1.2, we have the existence of K tangency points $\underline{\kappa}_k$ with associated nonzero real numbers α_k. We will distinguish two cases:

Case A. All α_k are distinct from one and another.

Case B. There exist two or more α_k which are equal.

We only discuss the explicit case here, as the implicit case does not distinguish between Cases A and B. First, let $\tau_k = \mathbf{i}\theta_k := \log(\underline{z}_k)$ for $\theta_k \in [-\pi, \pi]$ and let $\widetilde{\theta}_k \in]-\pi, \pi]$ be such that $\underline{\kappa}_k = \mathrm{e}^{\mathbf{i}\widetilde{\theta}_k}$ for each $k = 1, \dots, K$. In order to proceed, we need the following lemma which is a direct consequence of Corollary 2.6 and whose proof is identical to Lemma 3.2.

Lemma 3.8. *There exist some $\epsilon_* > 0$ and two constants $0 < \beta_* < \mathrm{Re}(\beta_k) < \beta^*$, for $k = 1, \dots, K$, such that for each $\epsilon \in (0, \epsilon_*)$, there exist some width $\eta_\epsilon > 0$ together with two constants, still denoted $C > 0$, $c > 0$, such that, for any integer $j \in \mathbb{Z}$, the component $\mathbf{G}_j(\tau)$ extends holomorphically on each $B_\epsilon(\tau_k)$ with bounds*:

$$\text{(Case I)} \qquad |\mathbf{G}_j(\tau)| \le \begin{cases} C\exp(-c|j|) & \text{if } j \le 0, \\ C\exp(j\,\mathrm{Re}(\varpi_k(\tau))) & \text{if } j \ge 1, \end{cases} \quad \forall \tau \in B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z},$$

$$\text{(Case II)} \qquad |\mathbf{G}_j(\tau)| \le \begin{cases} C\exp(j\,\mathrm{Re}(\varpi_k(\tau))) & \text{if } j \le 0, \\ C\exp(-c|j|) & \text{if } j \ge 1, \end{cases} \quad \forall \tau \in B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z},$$

$$\text{(Case III)} \qquad |\mathbf{G}_j(\tau)| \le \begin{cases} C\exp(j\,\mathrm{Re}(\varpi_{\nu_{k,1}}(\tau))) & \text{if } j \le 0, \\ C\exp(j\,\mathrm{Re}(\varpi_{\nu k,2}(\tau))) & \text{if } j \ge 1, \end{cases} \quad \forall \tau \in B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z},$$

where each ϖ_k is holomorphic on $B_\epsilon(\tau_k)$ and has, for all $\tau \in B_\epsilon(\tau_k)$, the Taylor expansion

$$\varpi_k(\tau) = \mathbf{i}\widetilde{\theta}_k - \frac{1}{\alpha_k}(\tau - \tau_k) + (-1)^{\mu_k+1}\frac{\beta_k}{\alpha_k^{2\mu_k+1}}(\tau - \tau_k)^{2\mu_k} + O(|\tau - \tau_k|^{2\mu_k+1}),$$

together with

$$\text{(3.11)} \qquad \mathrm{Re}(\varpi_k(\tau)) \le -\frac{\mathrm{Re}(\tau)}{\alpha_k} + \frac{\beta^*}{\alpha_k^{2\mu_k+1}}\mathrm{Re}(\tau)^{2\mu_k} - \frac{\beta_*}{\alpha_k^{2\mu_k+1}}(\mathrm{Im}(\tau) - \theta_k)^{2\mu_k}.$$

Furthermore, we have

$$|\mathbf{G}_j(\tau)| \le C\exp(-c|j|) \quad \forall \tau \in \Omega_\epsilon := \{-\eta_\epsilon < \mathrm{Re}(\tau) \le \pi\} \setminus \bigcup_{k=1}^{K} B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z}.$$

Case A. This is precisely the case depicted in Figure 4 with $K = 3$. Without loss of generality, we label the α_k by increasing order so that

$$-p < \alpha_1 < \cdots < \alpha_k < \cdots < \alpha_K < r.$$

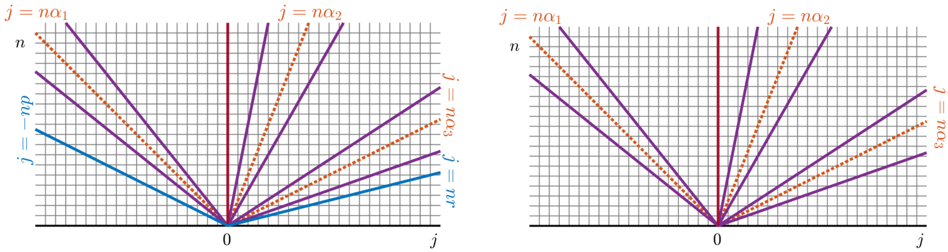

Figure 4. Illustration for the explicit (left) and implicit (right) cases of the different domains in the (j, n) plane where generalized Gaussian estimates are obtained; here $K = 3$ with $\alpha_1 < 0$ and $\alpha_2 \neq \alpha_3 > 0$. Generalized Gaussian estimates are derived near each $j \approx n\alpha_k$, $k = \{1, 2, 3\}$ in the interior of the sectors delimited by the magenta lines. In the explicit case, below the lines $j = -np$ and $j = nr$ (blue), the Green function $\mathscr{G}_j^n$ vanishes.

For each $k = 1, \dots, K$, we define two real numbers $\underline{\delta}_k < \overline{\delta}_k$ such that we have the ordering

$$-p < \underline{\delta}_1 < \alpha_1 < \overline{\delta}_1 < \cdots < \underline{\delta}_k < \alpha_k < \overline{\delta}_k < \cdots < \underline{\delta}_K < \alpha_K < \overline{\delta}_K < r,$$

with $\mathrm{sgn}(\underline{\delta}_k) = \mathrm{sgn}(\overline{\delta}_k) = \mathrm{sgn}(\alpha_k)$. For each $k = 1, \dots, K$, we define the following sectors in the (j, n)-plane:

$$\mathscr{D}_k := \left\{(j, n) \in \mathbb{Z} \times \mathbb{N}^* \mid n\underline{\delta}_k \leq j \leq n\overline{\delta}_k\right\},$$

together with

$$\mathscr{D}_* := \left\{(j, n) \in \mathbb{Z} \times \mathbb{N}^* \mid -np \leq j \leq nr\right\} \setminus \bigcup_{k=1}^{K} \mathscr{D}_k.$$

Our first lemma pertains at obtaining exponential bounds in the region $\mathscr{D}_*$. We introduce two quantities

$$\Lambda_1^* := \min_{k=1,\dots,K} \left(\frac{\alpha_k^{2\mu_k}}{2\beta^*}\right)^{\frac{1}{2\mu_k - 1}} > 0 \quad \text{and} \quad \Lambda_2^* := \min_{k=1,\dots,K} \left(\frac{(\overline{\delta}_k - \alpha_k)\alpha_k^{2\mu_k}}{2r\beta^*}\right)^{\frac{1}{2\mu_k - 1}} > 0.$$

Lemma 3.9. *For each $\epsilon \in (0, \min(\epsilon_*, \Lambda_1^*, \Lambda_2^*))$, there exist $C > 0$ and $\delta > 0$ such that for each $(j, n) \in \mathscr{D}_*$ the following estimate holds*:

$$|\mathscr{G}_j^n| \leq C\,\mathrm{e}^{-n\delta}.$$

Proof. We only sketch the proof as it is almost identical to the proofs of Lemma 3.3 and Lemma 3.4. Let $\epsilon \in (0, \min(\epsilon_*, \Lambda_1^*, \Lambda_2^*))$ and consider $\eta \in (0, \eta_\epsilon)$. We select the contour $\Gamma = \{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ and we denote by Γ_k^{in} the portion of Γ which lie within $B_\epsilon(\tau_k)$ and Γ^{out} the union of the remaining portions. As a consequence, we have $\Gamma = \Gamma_1^{\mathrm{in}} \cup \cdots \cup \Gamma_K^{\mathrm{in}} \cup \Gamma^{\mathrm{out}}$, and we get

$$\left|\frac{1}{2\mathbf{i}\pi} \int_\Gamma \mathrm{e}^{n\tau} \mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \leq \sum_{k=1}^{K} \left|\frac{1}{2\mathbf{i}\pi} \int_{\Gamma_k^{\mathrm{in}}} \mathrm{e}^{n\tau} \mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| + \left|\frac{1}{2\mathbf{i}\pi} \int_{\Gamma^{\mathrm{out}}} \mathrm{e}^{n\tau} \mathbf{G}_j(\tau)\,\mathrm{d}\tau\right|.$$

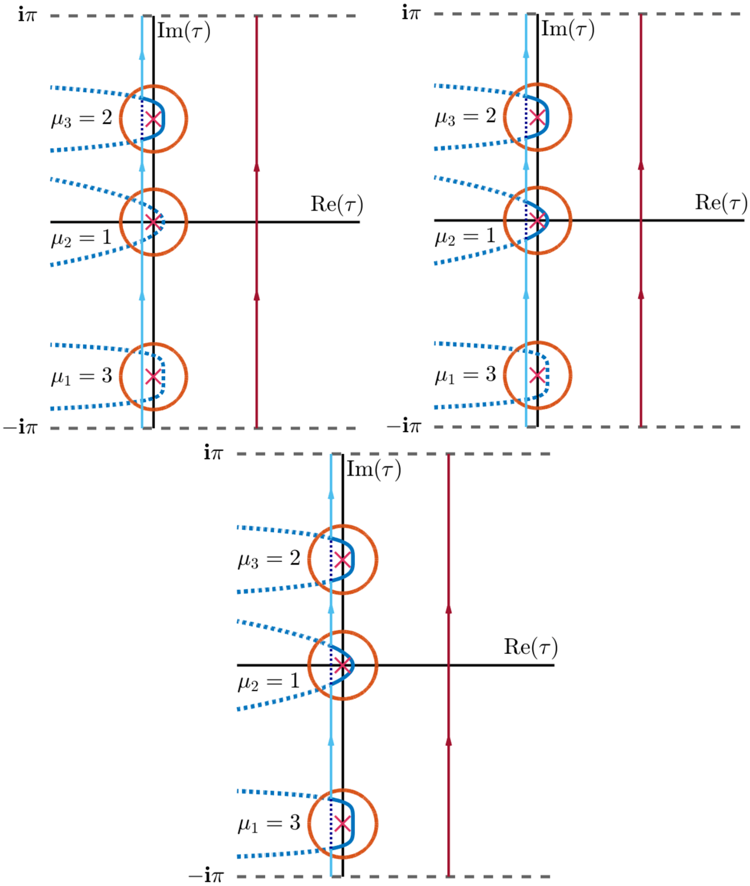

Figure 5. Top left: typical contour used when $(j,n) \in \mathcal{D}_3$ in the case depicted in Figure 4 where all α_k are distinct $\alpha_1 < \alpha_2 < \alpha_3$ (case A). Top right: typical contour used when $(j,n) \in \mathcal{D}_2 = \mathcal{D}_3$ in the case $\alpha_1 < \alpha_2 = \alpha_3$ (case B). Bottom: typical contour used when $(j,n) \in \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3$ in the case $\alpha_1 = \alpha_2 = \alpha_3$ (case B). Here $K = 3$.

Our objective is to bound each above term separately. Along Γ^{out}, we get an estimate of the form

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma^{\mathrm{out}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \leq C\mathrm{e}^{-n\eta-c|j|} \quad \forall (j,n) \in \mathcal{D}_*,$$

as along Γ^{out} the Green function $\mathbf{G}_j(\tau)$ enjoys the pointwise exponential bound from Lemma 3.8.

We now derive pointwise bound for each contour integral along Γ_k^{in}, $k = 1,\dots,K$. We first handle the case where $\mathcal{I}_k = \{k\}$, and assume without loss of generality that $\alpha_k > 0$. Then case I of Lemma 3.8 reads

$$\text{(3.12)} \qquad |\mathbf{G}_j(\tau)| \leq \begin{cases} C\exp(-c|j|) & \text{if } j \leq 0, \\ C\exp(j\,\mathrm{Re}(\varpi_k(\tau))) & \text{if } j \geq 1, \end{cases} \quad \forall \tau \in B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z},$$

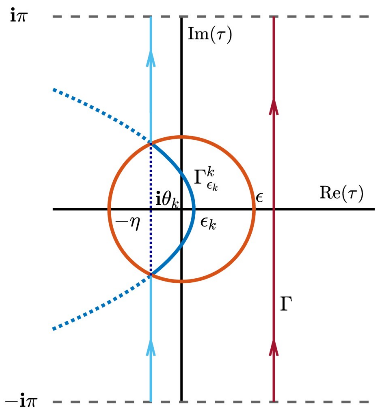

Figure 6. Illustration of the contour $\Gamma^k_{\epsilon_k}$ used in the proof of Lemma 3.9 and Lemma 3.10.

with

$$\mathrm{Re}(\varpi_k(\tau)) \le -\frac{\mathrm{Re}(\tau)}{\alpha_k} + \frac{\beta^*}{\alpha_k^{2\mu_k+1}}\mathrm{Re}(\tau)^{2\mu_k} - \frac{\beta_*}{\alpha_k^{2\mu_k+1}}(\mathrm{Im}(\tau)-\theta_k)^{2\mu_k} \quad \forall \tau \in B_\epsilon(\tau_k).$$

If $(j,n) \in \mathcal{D}_*$ is such that $j \le 0$, then we directly get

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_k^{\mathrm{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-n\eta-c|j|}.$$

From now on, we therefore consider $(j,n) \in \mathcal{D}_*$ with $j \ge 1$. As in the proof of Lemma 3.4, we use the above estimate (3.11) to get that

$$\mathrm{Re}(\varpi_k(\tau)) \le \frac{\eta}{\alpha_k} + \frac{\beta^*}{\alpha_k^{2\mu_k+1}}\,\eta^{2\mu_k}$$

for each $\tau \in \Gamma_k^{\mathrm{in}} \subset B_\epsilon(\tau_k)$. As a consequence, for all $(j,n) \in \mathcal{D}_*$ with $1 \le j \le n\underline{\delta}_k$, we have

$$\begin{aligned}-n\eta + j\,\mathrm{Re}(\varpi_k(\tau)) &\le n\eta\Big(-1 + \frac{j}{n\alpha_k} + \frac{j}{n}\,\frac{\beta^*}{\alpha_k^{2\mu_k+1}}\,\eta^{2\mu_k-1}\Big)\\ &\le -n\eta\Big(\underbrace{1-\frac{\underline{\delta}_k}{\alpha_k}}_{>0} - \frac{\beta^*}{\alpha_k^{2\mu_k}}\,\eta^{2\mu_k-1}\Big) \le -\frac{n\eta}{2}\Big(1-\frac{\underline{\delta}_k}{\alpha_k}\Big),\end{aligned}$$

since η is chosen such that $0<\eta<\eta_\epsilon<\epsilon<\Lambda_1^*$. And we have obtained the estimate

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_k^{\mathrm{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\,\exp\Big(-\frac{n\eta}{2}\Big(1-\frac{\underline{\delta}_k}{\alpha_k}\Big)\Big).$$

For the remaining cases $(j,n) \in \mathcal{D}_*$ and $n\overline{\delta}_k \le j \le nr$, we use a different contour near the ball $B_\epsilon(\tau_k)$. We refer to Figure 6 for an illustration. We introduce the contour

$$\Gamma^k_{\epsilon_k} := \Big\{\mathrm{Re}(\tau) - \frac{\beta^*}{\alpha_k^{2\mu_k}}\mathrm{Re}(\tau)^{2\mu_k} + \frac{\beta_*}{\alpha_k^{2\mu_k}}(\mathrm{Im}(\tau)-\theta_k)^{2\mu_k} = \Psi_k(\epsilon_k)\,\Big|\, -\eta \le \mathrm{Re}(\tau) \le \epsilon_k\Big\},$$

with $\Psi_k(\epsilon_k) := \epsilon_k - \frac{\beta^*}{\alpha_k^{2\mu}}\epsilon_k^{2\mu_k}$ and where $0 < \epsilon_k < \epsilon$ is chosen such that $\Gamma_{\epsilon_k}^k$ intersects Γ_k^{in} precisely on the boundary of $B_\epsilon(\tau_k)$. We note that there exists some constant $c_k > 0$ such that for any $\tau \in \Gamma_{\epsilon_k}^k \subset B_\epsilon(\tau_k)$, one has

$$\operatorname{Re}(\tau) \le \epsilon_k - c_k(\operatorname{Im}(\tau) - \theta_k)^{2\mu_k},$$

which yields

$$\begin{aligned}
&n \operatorname{Re}(\tau) + j \operatorname{Re}(\varpi_k(\tau)) \\
&\quad \le n \operatorname{Re}(\tau) + j\Big(-\frac{\operatorname{Re}(\tau)}{\alpha_k} + \frac{\beta^*}{\alpha_k^{2\mu_k+1}}\operatorname{Re}(\tau)^{2\mu_k} - \frac{\beta_*}{\alpha_k^{2\mu_k+1}}(\operatorname{Im}(\tau) - \theta_k)^{2\mu_k}\Big) \\
&\quad = n \operatorname{Re}(\tau) - \frac{j}{\alpha_k}\Psi(\epsilon_k) \\
&\quad \le -n c_k (\operatorname{Im}(\tau) - \theta_k)^{2\mu_k} + \frac{n\epsilon_k}{\alpha_k}\Big(\alpha_k - \frac{j}{n} + \frac{j}{n}\frac{\beta^*}{\alpha_k^{2\mu_k}}\epsilon_k^{2\mu_k-1}\Big) \\
&\quad \le -\frac{n\epsilon_k}{\alpha_k}\Big(\underbrace{\overline{\delta}_k - \alpha_k}_{>0} - r\frac{\beta^*}{\alpha_k^{2\mu_k}}\epsilon_k^{2\mu_k-1}\Big)
\end{aligned}$$

for each $\tau \in \Gamma_{\epsilon_k}^k$. Now, since $\overline{\delta}_k - \alpha_k > 0$ and $0 < \epsilon_k < \epsilon < \Lambda_2^*$, we have

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_k^{\text{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \le C\exp\Big(-n\,\frac{\epsilon_k(\overline{\delta}_k - \alpha_k)}{2\alpha_k}\Big),$$

which gives the desired estimate in the region $(j, n) \in \mathcal{D}_*$ and $n\overline{\delta}_k \le j \le nr$.

Let us finally comment on the case where $\mathcal{I}_k = \{\nu_{k,1}, \nu_{k,2}\}$ with $\alpha_{\nu_{k,1}} < 0 < \alpha_{\nu_{k,2}}$. This time, Lemma 3.8 gives

$$|\mathbf{G}_j(\tau)| \le \begin{cases} C\exp(j\operatorname{Re}(\varpi_{\nu_{k,1}}(\tau))) & \text{if } j \le 0, \\ C\exp(j\operatorname{Re}(\varpi_{\nu_{k,2}}(\tau))) & \text{if } j \ge 1, \end{cases} \qquad \forall \tau \in B_\epsilon(\tau_k),\ \forall j \in \mathbb{Z}.$$

The analysis for $(j, n) \in \mathcal{D}_*$ and $j \ge 1$ is unchanged, and we apply the same strategy for $(j, n) \in \mathcal{D}_*$ and $j \le 0$ without any difficulty. As a conclusion, we have obtained that there exist $C > 0$ and $\delta > 0$ such that for each $k = 1, \ldots, K$, we have

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_k^{\text{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \le C\mathrm{e}^{-n\delta} \quad \forall (j, n) \in \mathcal{D}_*,$$

which ends the proof. ∎

We prove in the next lemma that we obtain generalized Gaussian estimates in each sector $\mathcal{D}_k$, $k = 1, \ldots, K$.

Lemma 3.10. *There exists $\widehat{\epsilon}_* \in (0, \epsilon_*)$ such that for each $\epsilon \in (0, \widehat{\epsilon}_*)$, there are constants $C > 0$ and $c > 0$ such that for any $k = 1, \ldots, K$ and $(j, n) \in \mathcal{D}_k$, the following estimate holds*:

$$|\mathcal{G}_j^n| \le \frac{C}{n^{1/(2\mu_k)}}\exp\Big(-c\Big(\frac{|j - \alpha_k n|}{n^{1/(2\mu_k)}}\Big)^{\frac{2\mu_k}{2\mu_k-1}}\Big).$$

Proof. Let $(j,n) \in \mathcal{D}_k$, that is, $n \geq 1$ and $n\underline{\delta}_k \leq j \leq n\overline{\delta}_k$. Assume without loss of generality that $\alpha_k > 0$. We first consider the case where $\mathcal{I}_k = \{k\}$. Once again, we introduce a family of parametrized curves Γ_p^k given by

$$\Gamma_p^k := \left\{ \mathrm{Re}(\tau) - \frac{\beta^*}{\alpha_k^{2\mu_k}} \mathrm{Re}(\tau)^{2\mu_k} + \frac{\beta_*}{\alpha_k^{2\mu_k}} (\mathrm{Im}(\tau) - \theta_k)^{2\mu_k} = \Psi_k(\tau_{p,k}) \,\Big|\, -\eta \leq \mathrm{Re}(\tau) \leq \tau_{p,k} \right\},$$

where $\Psi_k(\tau_{p,k}) = \tau_{p,k} - \frac{\beta^*}{\alpha_k^{2\mu_k}} \tau_{p,k}^{2\mu_k}$ and $\eta > 0$, $\tau_{p,k} > -\eta$ are chosen as follows. For each $\epsilon \in (0, \epsilon_*)$, we fix $\eta \in (0, \eta_\epsilon)$ such that the curve Γ_p^k with $\tau_{p,k} = 0$ intersects the vertical segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ inside the ball $B_\epsilon(\tau_k)$. Furthermore, we let $0 < \epsilon_{0,k} < \epsilon$ be defined as the value of $\tau_{p,k}$ for which Γ_p^k, with $\tau_{p,k} = \epsilon_{0,k}$, intersects the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ on $\partial B_\epsilon(\tau_k)$, with η fixed previously. Finally, $\tau_{p,k}$ is now defined as

$$\tau_{p,k} := \begin{cases} \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) & \text{if } -\frac{\eta}{2} \leq \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \leq \epsilon_{0,k}, \\ \epsilon_{0,k} & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \epsilon_{0,k}, \\ -\frac{\eta}{2} & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}, \end{cases}$$

where ζ_k, γ_k and the function ρ_k are set to

$$\zeta_k := \frac{j - n\alpha_k}{2\mu_k n} \quad \text{and} \quad \gamma_k := \frac{j}{n} \frac{\beta^*}{\alpha_k^{2\mu_k}} > 0,$$

with $\rho_k(\zeta_k/\gamma_k)$ given by

$$\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) := \mathrm{sgn}\left(\frac{\zeta_k}{\gamma_k}\right)\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{1}{2\mu_k - 1}}.$$

In our estimate, we use a contour Γ_k, which consists of Γ_p^k in $B_\epsilon(\tau_k)$ and the vertical segment $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ otherwise (see Figure 5, left panel for an illustration in the case $K = 3$). Depending on the ratio ζ_k/γ_k, there exists (or not) a portion of the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$ within the ball $B_\epsilon(\tau_k)$ that we denote Γ_k^{in}. Note that when $\rho_k(\zeta_k/\gamma_k) > \epsilon_{0,k}$, we have $\Gamma_k^{\mathrm{in}} = \emptyset$. The analysis along $\Gamma_p^k \cup \Gamma_k^{\mathrm{in}}$ is exactly the same as the one conducted in the proof of Lemma 3.6, and we get that there exists $\epsilon_{**} \in (0, \epsilon_*)$ such that for all $\epsilon \in (0, \epsilon_{**})$, one has

$$\left| \frac{1}{2\mathbf{i}\pi} \int_{\Gamma_p^k \cup \Gamma_k^{\mathrm{in}}} \mathrm{e}^{n\tau} \mathbf{G}_j(\tau)\, \mathrm{d}\tau \right| \leq \frac{C}{n^{1/(2\mu_k)}} \exp\left(-c \left(\frac{|j - \alpha_k n|}{n^{1/(2\mu_k)}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right), \quad (j,n) \in \mathcal{D}_k.$$

The fact that one needs to eventually decrease the size of ϵ comes from Lemma 3.5, which is needed to obtain the generalized Gaussian bound and prove that

$$\int_{\Gamma_p^k} \mathrm{e}^{-n c_k^* (\mathrm{Im}(\tau) - \theta_k)^{2\mu_k}} |\mathrm{d}\tau| \leq \frac{C}{n^{1/(2\mu_k)}}.$$

Along the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi, \pi]\}$, we denote by Γ^{out} all portions that lie outside the balls $B_\epsilon(\tau_\upsilon)$ with $\upsilon \neq k$, and we get

$$\left| \frac{1}{2\mathbf{i}\pi} \int_{\Gamma^{\mathrm{out}}} \mathrm{e}^{n\tau} \mathbf{G}_j(\tau)\, \mathrm{d}\tau \right| \leq \mathrm{e}^{-n\eta - c|j|}, \quad (j,n) \in \mathcal{D}_k.$$

Thus, it only remains to estimate the integral along the ray $\{-\eta + \mathbf{i}\ell \mid \ell \in [-\pi,\pi]\}$ within a ball $B_\epsilon(\tau_\upsilon)$ with $\upsilon \neq k$, that we denote $\Gamma_\upsilon^{\text{in}}$. Let assume first that $\mathcal{I}_\upsilon = \{\upsilon\}$. We split the analysis in two cases.

(i) If $\alpha_\upsilon < 0$, then we have the estimate

$$|\mathbf{G}_j(\tau)| \le C\mathrm{e}^{-cj}, \quad \tau \in \Gamma_\upsilon^{\text{in}},$$

as $0 < n\underline{\delta}_k \le j \le n\overline{\delta}_k$, and we obtain

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\upsilon^{\text{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\mathrm{e}^{-n\eta-cj}, \quad (j,n) \in \mathcal{D}_k.$$

(ii) If $\alpha_\upsilon > 0$, then we have the following estimates for each $\tau \in \Gamma_\upsilon^{\text{in}} \subset B_\epsilon(\tau_\upsilon)$:

$$|\mathbf{G}_j(\tau)| \le C\mathrm{e}^{j\,\mathrm{Re}(\varpi_\upsilon(\tau))}, \quad \tau \in \Gamma_\upsilon^{\text{in}},$$

and

$$\mathrm{Re}(\varpi_\upsilon(\tau)) \le -\frac{\mathrm{Re}(\tau)}{\alpha_\upsilon} + \frac{\beta^*}{\alpha_\upsilon^{2\mu_\upsilon+1}}\mathrm{Re}(\tau)^{2\mu_\upsilon} - \frac{\beta_*}{\alpha_\upsilon^{2\mu_\upsilon+1}}(\mathrm{Im}(\tau)-\theta_\upsilon)^{2\mu_\upsilon}.$$

As a consequence, we readily obtain that

$$-n\eta + j\,\mathrm{Re}(\varpi_\upsilon(\tau)) \le n\eta\Big(-1 + \frac{j}{n\alpha_\upsilon} + \frac{j}{n}\frac{\beta^*}{\alpha_\upsilon^{2\mu_\upsilon+1}}\eta^{2\mu_\upsilon-1}\Big).$$

Thus, if $\alpha_\upsilon > \alpha_k$, we get that

$$-n\eta + j\,\mathrm{Re}(\varpi_\upsilon(\tau)) \le -n\eta\Big(\underbrace{1-\frac{\overline{\delta}_k}{\alpha_\upsilon}}_{>0} - \frac{\overline{\delta}_k\beta^*}{\alpha_\upsilon^{2\mu_\upsilon+1}}\eta^{2\mu_\upsilon-1}\Big) \le -\frac{n\eta}{2}\Big(1-\frac{\overline{\delta}_k}{\alpha_\upsilon}\Big),$$

provided that η is chosen small enough, which is always possible by eventually reducing the size of ϵ_{**}. This gives

$$\left|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma_\upsilon^{\text{in}}} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\right| \le C\mathrm{e}^{-n\frac{\eta}{2}(1-\frac{\overline{\delta}_k}{\alpha_\upsilon})}.$$

Finally, if $\alpha_\upsilon < \alpha_k$ we use a different contour inside the ball $B_\epsilon(\tau_\upsilon)$. Namely, we use the contour

$$\Gamma_{\epsilon_\upsilon}^\upsilon := \Big\{\mathrm{Re}(\tau) - \frac{\beta^*}{\alpha_\upsilon^{2\mu_\upsilon}}\mathrm{Re}(\tau)^{2\mu_\upsilon} + \frac{\beta_*}{\alpha_\upsilon^{2\mu_\upsilon}}(\mathrm{Im}(\tau)-\theta_\upsilon)^{2\mu_\upsilon} = \Psi_\upsilon(\epsilon_\upsilon) \mid -\eta \le \mathrm{Re}(\tau) \le \epsilon_\upsilon\Big\},$$

with $\Psi_\upsilon(\epsilon_\upsilon) := \epsilon_\upsilon - \frac{\beta^*}{\alpha_\upsilon^{2\mu_\upsilon}}\epsilon_\upsilon^{2\mu_\upsilon}$ and where $0 < \epsilon_\upsilon < \epsilon$ is chosen such that $\Gamma_{\epsilon_\upsilon}^\upsilon$ intersects the segment $\Gamma_\upsilon^{\text{in}}$ precisely on the boundary of $B_\epsilon(\tau_\upsilon)$. We note that there exists some constant $c_\upsilon > 0$ such that for any $\tau \in \Gamma_{\epsilon_\upsilon}^\upsilon \subset B_\epsilon(\tau_\upsilon)$, one has

$$\mathrm{Re}(\tau) \le \epsilon_\upsilon - c_\upsilon(\mathrm{Im}(\tau)-\theta_\upsilon)^{2\mu_\upsilon},$$

which yields

$$n\,\mathrm{Re}(\tau) + j\,\mathrm{Re}(\varpi_\upsilon(\tau)) \le -nc_\upsilon(\mathrm{Im}(\tau) - \theta_\upsilon)^{2\mu_\upsilon} - \frac{n\epsilon_\upsilon}{\alpha_\upsilon}\Big(\underline{\delta}_k - \alpha_\upsilon - \overline{\delta}_k\,\frac{\beta^*}{\alpha_\upsilon^{2\mu_\upsilon}}\,\epsilon_\upsilon^{2\mu_\upsilon-1}\Big),$$

for each $\tau \in \Gamma^\upsilon_{\epsilon_\upsilon}$. Now, since $\underline{\delta}_k - \alpha_\upsilon > \alpha_k - \alpha_\upsilon > 0$ and $0 < \epsilon_\upsilon < \epsilon$, we can always further reduce the size of ϵ_{**} so that

$$\Big|\frac{1}{2\mathbf{i}\pi}\int_{\Gamma^{\mathrm{in}}_\upsilon} \mathrm{e}^{n\tau}\mathbf{G}_j(\tau)\,\mathrm{d}\tau\Big| \le \frac{C}{n^{1/(2\mu_\upsilon)}}\exp\Big(-n\,\frac{\epsilon_\upsilon(\underline{\delta}_k - \alpha_\upsilon)}{2\alpha_\upsilon}\Big),$$

which gives the desired estimate.

If now $\mathcal{I}_\upsilon = \{\nu_{\upsilon,1}, \nu_{\upsilon,2}\}$, then we have $\alpha_{\nu_{\upsilon,1}} < 0 < \alpha_{\nu_{\upsilon,2}}$, and for $0 < n\underline{\delta}_k \le j \le n\overline{\delta}_k$, we get

$$|\mathbf{G}_j(\tau)| \le C\exp(j\,\mathrm{Re}(\varpi_{\nu_{\upsilon,2}}(\tau))), \quad \tau \in \Gamma^{\mathrm{in}}_\nu,$$

and the analysis is similar to the above case (ii). Finally, when $\mathcal{I}_k = \{\nu_{k,1}, \nu_{k,2}\}$, we necessarily have that $\alpha_{\nu_{k,1}} < 0 < \alpha_{\nu_{k,2}} = \alpha_k$ and the analysis remains unchanged. As there exists some $\widehat{\epsilon}_* \in (0, \epsilon_{**})$ such that for all $\epsilon \in (0, \widehat{\epsilon}_*)$ we have proved the desired generalized Gaussian bound. ∎

Case B. In this case, two or more α_k are equal. Note that for $(j,n) \in \mathcal{D}_*$ the analysis remains unchanged and Lemma 3.9 still holds true in this case. Let us assume for simplicity that $\alpha_{\nu_1} = \alpha_{\nu_2}$ for some couple of integers $\nu_1 \neq \nu_2$, and all other α_k's are distinct. The estimate from Lemma 3.10 is still valid for $(j,n) \in \mathcal{D}_k$ for each $k \notin \{\nu_1, \nu_2\}$. For $(j,n) \in \mathcal{D}_{\nu_1} = \mathcal{D}_{\nu_2}$, in the ball $B_\epsilon(\tau_{\nu_1})$, we use the contour $\Gamma^{\nu_1}_p$, and in the ball $B_\epsilon(\tau_{\nu_2})$, we use the contour $\Gamma^{\nu_2}_p$. We refer to Figure 5 for an illustration of such contours. Reproducing the analysis of Lemma 3.10, we obtain the existence of $C > 0$ and $c > 0$ such that for $(j,n) \in \mathcal{D}_{\nu_1} = \mathcal{D}_{\nu_2}$, the following estimate holds:

$$|\mathcal{G}^n_j| \le C\sum_{\nu\in\{\nu_1,\nu_2\}}\frac{1}{n^{1/(2\mu_\nu)}}\exp\Big(-c\Big(\frac{|j-\alpha_\nu n|}{n^{1/(2\mu_\nu)}}\Big)^{\frac{2\mu_\nu}{2\mu_\nu-1}}\Big).$$

Finally, we remark that Lemma 3.7 naturally extends to the case $K > 1$ in the implicit setting. This concludes the proof of Theorem 1.6.

4. Examples and extensions

We first give several examples of operators (1.2) that fit into the framework of Theorem 1.6, and that arise when discretizing the transport equation:

$$\partial_t u + \partial_x u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4.1}$$

with Cauchy data at $t = 0$. We refer to [13, 14] for a detailed analysis and more examples of finite difference schemes in that context.

4.1. Example 1: the Lax–Friedrichs scheme

The Lax–Friedrichs scheme is an explicit finite difference approximation of (4.1), which corresponds to the operators:

$$Q_1 := I, \quad Q_0 := \frac{1+\lambda}{2}\mathbf{S}^{-1} + \frac{1-\lambda}{2}\mathbf{S}, \tag{4.2}$$

where here and below, λ is a real parameter[11] and $\mathbf{S}$ still denotes the shift operator defined by

$$\mathbf{S}\colon\ (u_j)_{j\in\mathbb{Z}} \mapsto (u_{j+1})_{j\in\mathbb{Z}}.$$

We now restrict to $\lambda \in (0,1)$ so that both coefficients in the definition (4.2) are positive and they sum to 1. In probability theory, this corresponds to a random walk with probability $(1+\lambda)/2$ to jump of $+1$ and probability $(1-\lambda)/2$ to jump of -1 at each time iteration (recall our convention on the coefficients a_ℓ, which differs from the standard convolution product).

In the notation of (1.2), we have $r = p = 1$. Since we are dealing here with an explicit scheme, Assumptions 1.1 and 1.4 are trivially satisfied. The definition (1.6) reduces here to

$$F(\mathrm{e}^{\mathbf{i}\xi}) = \cos(\xi) - \mathbf{i}\lambda\sin(\xi).$$

Computing $|F(\mathrm{e}^{\mathbf{i}\xi})|^2 = \cos^2(\xi) + \lambda^2\sin^2(\xi)$, we find that $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$, and $F(\kappa)$ belongs to $\mathbb{S}^1$ for such κ if and only if $\kappa = \pm 1$. We thus have (1.7) with $\underline{\kappa}_1 := 1$ and $\underline{\kappa}_2 := -1$, and the reader can check that (1.8) is satisfied with

$$\alpha_1 = \alpha_2 = \lambda, \quad \beta_1 = \beta_2 = \frac{1-\lambda^2}{2}, \quad \mu_1 = \mu_2 = 1,$$

which means that Assumption 1.2 is satisfied. Since the modulus of $F(\kappa)$ attains its maximum at two points of $\mathbb{S}^1$, we cannot apply the uniform Gaussian bound from [9]. The improvements of Theorem 1.6 and Theorem 1.8 of [28] are relevant in that case. Note that Theorem 1.8 of [28] can be used for the Lax–Friedrichs scheme (4.2), since we are in the case where $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$. The spectral curve $F(\mathbb{S}^1)$ (an ellipse) is illustrated in Figure 7 in the case $\lambda = 1/2$.

We now turn to Assumption 1.3 and compute (see the general definition (1.9)):

$$\mathbb{A}_{-1}(z) = -\frac{1+\lambda}{2}, \quad \mathbb{A}_1(z) = -\frac{1-\lambda}{2}.$$

Hence Assumption 1.3 is satisfied too.

Finally, Assumption 1.5 is satisfied since we have $K = 2$, $\underline{z}_1 = 1$ and $\underline{z}_2 = -1 \neq \underline{z}_1$, which means that both sets $\mathcal{I}_1$ and $\mathcal{I}_2$ in (1.11) are singletons. Overall, the conclusion of Theorem 1.6 for the Lax–Friedrichs scheme in (4.2) is the uniform bound[12]

$$\mathcal{G}_j^n \le \frac{C}{\sqrt{n}}\exp\Big(-c\,\frac{(j-\lambda n)^2}{n}\Big).$$

This behavior is illustrated in Figure 7 in the case $\lambda = 1/2$.

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[11] In the theory of finite difference schemes, it is referred to as the Courant–Friedrichs–Lewy parameter [5].

[12] We do not use the absolute value here since all coefficients $\mathcal{G}_j^n$ are nonnegative.

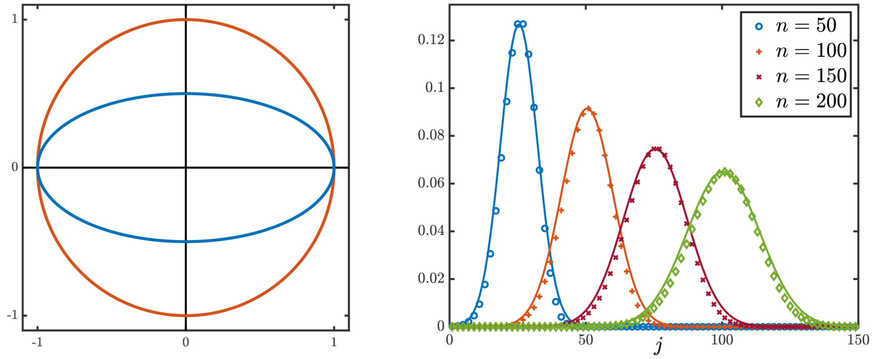


Figure 7. Left: The spectrum (blue curve) $\sigma(\mathscr{L}) = F(\mathbb{S}^1)$ for the Lax–Friedrichs scheme (4.2) with $\lambda = 1/2$. Right: The Green function (marked points) at different time iterations for the Lax–Friedrichs scheme (4.2) compared with a fixed Gaussian profile centered at $j = \lambda n$ (solid lines). We started with an initial condition given by the Dirac mass $\boldsymbol{\delta}$.

For readers who are familiar with the theory of the transport equation (see [14] otherwise), the parameter λ stands for the ratio $\Delta t/\Delta x$ of the time and space steps. Hence the bound of Theorem 1.6 equivalently reads (with new constants that are still denoted C and c)

$$\mathscr{G}_j^n \leq \frac{C}{\sqrt{n}} \exp\Big(-c\, \frac{(j\Delta x - n\Delta t)^2}{\Delta x (n \Delta t)}\Big),$$

which corresponds to the heat kernel at point $j\Delta x$, time $n\Delta t$ with a diffusion coefficient proportional to Δx.

4.2. Example 2: an implicit scheme

Our second example is based on the so-called method of lines for discretizing (4.1) (see [13, 14] for a detailed exposition of the method and its outcome). Here we first apply the centered finite difference for the spatial derivative and we then apply the implicit Euler scheme for the time integration. As in the case of the Lax–Friedrichs scheme (4.2), we introduce a positive parameter $\lambda > 0$ (which plays the role of the ratio $\Delta t/\Delta x$ but its origin is meaningless here), and we use the operators

$$Q_1 := I + \frac{\lambda}{2}(\mathbf{S} - \mathbf{S}^{-1}), \quad Q_0 := I. \tag{4.3}$$

In the notation of (1.2), this corresponds again to $r = p = 1$, but the scheme is now implicit because Q_1 is not the identity. We compute

$$\widehat{Q}_1(\mathrm{e}^{\mathbf{i}\xi}) = 1 + \mathbf{i}\lambda \sin(\xi) \neq 0,$$

which means that Q_1 is an isomorphism on $\ell^2(\mathbb{Z};\mathbb{C})$. The index condition (1.5) is also satisfied, since the complex number $\widehat{Q}_1(\kappa)$ has positive real part for all $\kappa \in \mathbb{S}^1$, so we can write

$$\widehat{Q}_1(\kappa) = \exp q(\kappa),$$

thanks to the standard determination of the logarithm (which implies the validity of (1.5)). The operator $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{\sqrt{1+\lambda^2}} \Big\{ \sum_{\ell \geq 0} x^\ell \mathbf{S}^{-\ell} + \sum_{\ell \geq 1} (-1)^\ell x^\ell \mathbf{S}^\ell \Big\},$$

where $x \in (0,1)$ is given by

$$x := \frac{\sqrt{1+\lambda^2} - 1}{\lambda}.$$

We are thus dealing with a convolution operator with infinite support.

We compute

$$F(\mathrm{e}^{\mathbf{i}\xi}) = \frac{1}{1 + \mathbf{i}\lambda \sin(\xi)},$$

which means that $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$ and, again, $F(\kappa)$ belongs to $\mathbb{S}^1$ if and only if $\kappa = \pm 1$ ($K = 2$ in the notation of Assumption 1.2). Setting $\underline{\kappa}_1 = 1$ and $\underline{\kappa}_2 = -1$, we find that the relation (1.8) is satisfied with

$$\alpha_1 = \lambda, \quad \beta_1 = \beta_2 = \frac{\lambda^2}{2}, \quad \alpha_2 = -\lambda.$$

Assumption 1.2 is thus satisfied but we now have $\underline{z}_1 = \underline{z}_2 = 1$, and we immediately see that Assumption 1.5 is also satisfied: both sets $\mathscr{I}_1$ and $\mathscr{I}_2$ equal $\{1,2\}$ and $\alpha_1\alpha_2 = -\lambda^2 < 0$. The spectral curve $F(\mathbb{S}^1)$ is illustrated in Figure 8 in the case $\lambda = 1/2$.

As far as Assumption 1.3 is concerned, we compute

$$\mathbb{A}_{-1}(z) = -\frac{\lambda}{2}\, z, \quad \mathbb{A}_1(z) = \frac{\lambda}{2}\, z,$$

so Assumption 1.3 is satisfied again. We also note that Assumption 1.4 is satisfied since we have $a_{-1,1} = -\lambda/2$ and $a_{1,1} = \lambda/2$. We can therefore apply Theorem 1.6 which, in the case of (4.3), yields the uniform Gaussian bound

$$|\mathscr{G}_j^n| \leq \frac{C}{\sqrt{n}} \Big(\exp\Big(-c\, \frac{(j+\lambda n)^2}{n}\Big) + \exp\Big(-c\, \frac{(j-\lambda n)^2}{n}\Big)\Big), \quad |j| \leq Ln.$$

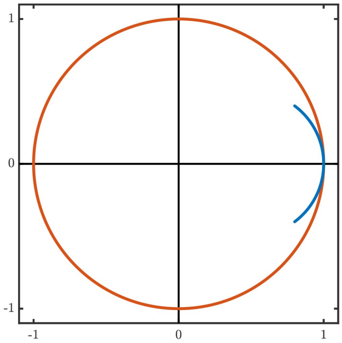
This behavior is illustrated in Figure 8 in the case $\lambda = 1/2$. Since we have an explicit formula for $\mathscr{G}_j^1$, it is clear that the bound

$$|\mathscr{G}_j^1| \leq C \exp(-c|j|),$$

for large j's, cannot be improved to some generalized Gaussian bound. This justifies why we need to distinguish the cases $|j|/n \gg 1$ and $|j|/n = O(1)$ in (1.13).

4.3. Example 3: the $O3$ scheme

Next, as a third example, we consider the $O3$ scheme [7, 8], which is an explicit scheme of order 3 obtained as the convex combination of the Lax–Wendroff scheme [21] and the



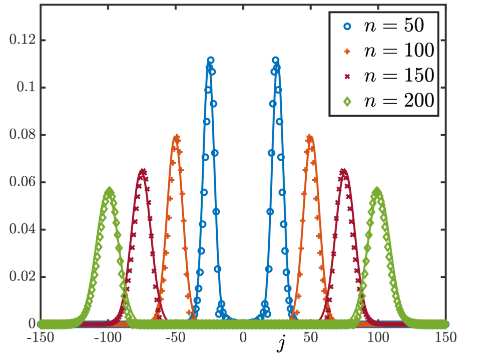


Figure 8. Left: The spectrum (blue curve) $\sigma(\mathcal{L}) = F(\mathbb{S}^1)$ for the implicit scheme (4.3) with $\lambda = 1/2$. Right: The absolute value of the Green function (marked points) at different time iterations for the implicit scheme (4.3) compared with two fixed Gaussian profiles centered at $j = \lambda n$ and $j = -\lambda n$ (solid lines). We started with an initial condition given by the Dirac mass $\boldsymbol{\delta}$.

Beam–Warming scheme [39]. The Lax–Wendroff scheme is an explicit finite approximation of (4.1) corresponding to the operators:

$$Q_1^{\mathrm{LW}} := I, \quad Q_0^{\mathrm{LW}} := (1-\lambda^2)I + \frac{\lambda+\lambda^2}{2}\mathbf{S}^{-1} + \frac{-\lambda+\lambda^2}{2}\mathbf{S},$$

for some $\lambda \in (0,1)$. On the other hand, the Beam–Warming scheme is an explicit scheme given by

$$Q_1^{\mathrm{BW}} := I, \quad Q_0^{\mathrm{BW}} := \Big(1 - \frac{3}{2}\lambda + \frac{1}{2}\lambda^2\Big)I + (2\lambda-\lambda^2)\mathbf{S}^{-1} + \frac{-\lambda+\lambda^2}{2}\mathbf{S}^{-2},$$

for some $\lambda \in (0,1)$. In both cases, λ stands for the ratio $\Delta t/\Delta x$. The $O3$ scheme is then defined as the following convex combination of the above two schemes:

$$Q_1^{O3} := I, \quad Q_0^{O3} := (1-\delta)Q_0^{\mathrm{LW}} + \delta Q_0^{\mathrm{BW}}, \quad \text{with } \delta = \frac{1+\lambda}{3}. \tag{4.4}$$

The expression for Q_0^{O3} can be simplified and we have

$$Q_0^{O3} = \frac{(2-\lambda)(1-\lambda^2)}{2}I + \frac{\lambda(2-\lambda)(1+\lambda)}{2}\mathbf{S}^{-1} - \frac{\lambda(1-\lambda^2)}{6}\mathbf{S}^{-2} + \frac{\lambda(2-\lambda)(\lambda-1)}{6}\mathbf{S}.$$

In the notation of (1.2), this corresponds to $r = 2$ and $p = 1$. Once again, since we are dealing here with an explicit scheme, Assumptions 1.1 and 1.4 are trivially satisfied. The definition (1.6) gives in that case

$$F^{O3}(\mathrm{e}^{\mathbf{i}\xi}) = (1-\delta)F^{\mathrm{LW}}(\mathrm{e}^{\mathbf{i}\xi}) + \delta F^{\mathrm{BW}}(\mathrm{e}^{\mathbf{i}\xi}),$$

with

$$F^{\mathrm{LW}}(\mathrm{e}^{\mathbf{i}\xi}) = 1 - \lambda^2 + \lambda^2\cos(\xi) - \mathbf{i}\lambda\sin(\xi)$$

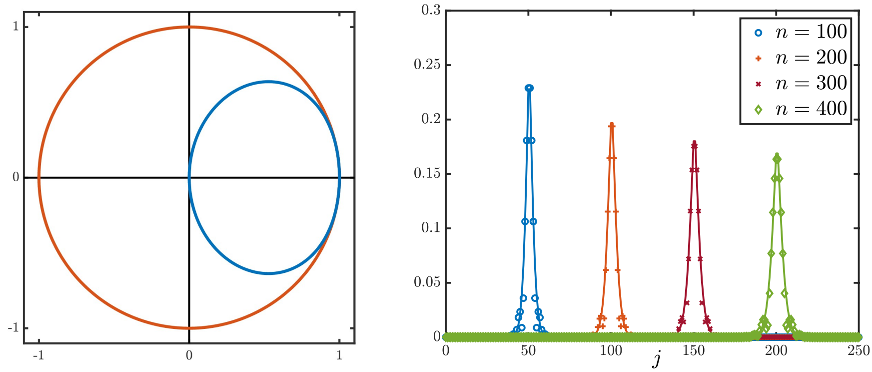


Figure 9. Left: The spectrum (blue curve) $\sigma(\mathcal{L}) = F(\mathbb{S}^1)$ for the $O3$ scheme (4.4) with $\lambda = 1/2$. Right: The absolute value of the Green function (marked points) at different time iterations for the $O3$ scheme (4.4) compared with a fixed generalized Gaussian profile $\mathcal{H}_j^n$ (4.5) centered at $j = \lambda n$ (solide lines). We started with an initial condition given by the Dirac mass $\boldsymbol{\delta}$.

and

$$F^{\mathrm{BW}}(\mathrm{e}^{\mathbf{i}\xi}) = 1 - \frac{3}{2}\lambda + \frac{1}{2}\lambda^2 + (2\lambda - \lambda^2)\mathrm{e}^{-\mathbf{i}\xi} + \frac{-\lambda + \lambda^2}{2}\mathrm{e}^{-\mathbf{i}2\xi}.$$

After some computations, we get

$$|F^{\mathrm{LW}}(\mathrm{e}^{i\xi})|^2 = 1 - 4\lambda^2(1-\lambda^2)\sin^4\Big(\frac{\xi}{2}\Big),$$
$$|F^{\mathrm{BW}}(\mathrm{e}^{i\xi})|^2 = 1 - 4\lambda(2-\lambda)(1-\lambda)^2\sin^4\Big(\frac{\xi}{2}\Big),$$

from which we deduce, by convexity, that $F^{O3}(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$. In fact, further computations lead to

$$|F^{O3}(\mathrm{e}^{\mathbf{i}\xi})|^2 = 1 - \frac{4}{9}\lambda(2-\lambda)(1-\lambda^2)\sin^4\Big(\frac{\xi}{2}\Big)\Big(3 + 4\lambda(1-\lambda)\sin^2\Big(\frac{\xi}{2}\Big)\Big),$$

so that $F^{O3}(\kappa)$ belongs to $\mathbb{S}^1$ only when $\kappa = 1$, that is, $K = 1$ in the notation of Assumption 1.3 with $\underline{\kappa}_1 = 1$. We find that the relation (1.8) is satisfied with

$$\alpha_1 = \lambda, \quad \beta_1 = \frac{\lambda(2-\lambda)(1-\lambda^2)}{24}, \quad \mu_1 = 2,$$

and that $\beta_1 > 0$ for $\lambda \in (0,1)$. Furthermore, note that Assumption 1.5 is trivially satisfied since $\mathcal{I}_1 = \{\underline{z}_1\}$, with $\underline{z}_1 = 1$. The spectral curve $F^{O3}(\mathbb{S}^1)$ is illustrated in Figure 9 in the case $\lambda = 1/2$.

Next, we have that

$$\mathbb{A}_{-2}(z) = \frac{\lambda(1-\lambda^2)}{6} \quad \text{and} \quad \mathbb{A}_1(z) = \frac{\lambda(2-\lambda)(1-\lambda)}{6},$$

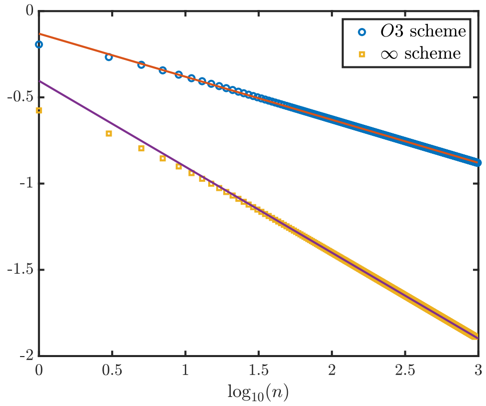


Figure 10. Illustration of the scaling factor in the generalized Gaussian bounds provided by Theorem 1.6 in the case of the $O3$ scheme (blue circles) and the ∞ scheme (orange squares). We plot $\log_{10}(\sup_{j\in\mathbb{Z}}|\mathscr{G}_j^n|)$ as a function of $\log_{10}(n)$ together with a best linear fit for each scheme for n ranging from 1 to 10^3. For the $O3$ scheme, we find a slope of -0.2496 while for the ∞ scheme, we find a slope of -0.4983, which compare well with the theoretical $-1/4$ and $-1/2$ scaling factors.

hence Assumption 1.3 also holds true for each $\lambda \in (0,1)$. We can therefore apply Theorem 1.6 which, in the case of (4.4), yields the uniform generalized Gaussian bound:

$$|\mathscr{G}_j^n| \le \frac{C}{n^{1/4}} \exp\Big(-c\,\frac{|j-\lambda n|^{4/3}}{n^{1/3}}\Big), \quad n \ge 1,\ j \in \mathbb{Z}.$$

This behavior is illustrated in Figure 9 in the case $\lambda = 1/2$, where we compare the Green function $\mathscr{G}_j^n$ to the generalized Gaussian profile $\mathscr{H}_j^n$ defined as

$$\mathscr{H}_j^n := \frac{C}{n^{1/4}} \exp\Big(-c\,\frac{|j-\lambda n|^{4/3}}{n^{1/3}}\Big), \quad n \ge 1,\ j \in \mathbb{Z}, \tag{4.5}$$

with two fixed constants $C > 0$ and $c > 0$ independent of j and n, which we have set to $C = 0.8$ and $c = 1.1765$, respectively. Note that the scaling factor $n^{-1/4}$ in the generalized Gaussian bound is further demonstrated in Figure 10, where we represent $\sup_{j\in\mathbb{Z}}|\mathscr{G}_j^n|$ in logarithmic scale. Using a best linear fit, we numerically obtain a slope of -0.2496, which is in good agreement with the theory.

4.4. Example 4: the ∞ scheme

We complete our series of examples with a last explicit scheme, which we shall call the ∞ scheme in reference to the spectral curve associated with it (see Figure 11 for an illustration). The ∞ scheme corresponds to the operators

$$Q_1^\infty := I, \quad Q_0^\infty := \frac{1}{16}\mathbf{S}^{-3} + \frac{1}{4}\mathbf{S}^{-2} + \frac{7}{16}\mathbf{S}^{-1} + \frac{7}{16}\mathbf{S} - \frac{1}{4}\mathbf{S}^2 + \frac{1}{16}\mathbf{S}^3, \tag{4.6}$$

so that we have $r = p = 3$ in that case. Here the definition (1.6) reduces to

$$F^\infty(\mathrm{e}^{\mathbf{i}\xi}) = \frac{7}{8}\cos(\xi) + \frac{1}{8}\cos(3\xi) - \frac{\mathbf{i}}{2}\sin(2\xi).$$

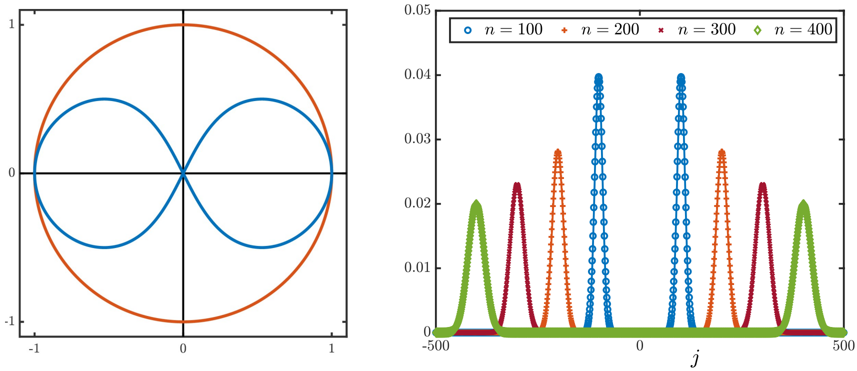


Figure 11. Left: The spectrum (blue curve) $\sigma(\mathcal{L}) = F(\mathbb{S}^1)$ for the ∞ scheme (4.6). Right: The absolute value of the Green function (marked points) at different time iterations for the ∞ scheme (4.6) compared with two fixed Gaussian profiles centered at $j = n$ and $j = -n$ (solid lines). We started with an initial condition given by the Dirac mass $\boldsymbol{\delta}$.

Computing

$$|F(\mathrm{e}^{\mathbf{i}\xi})|^2 = 1 - \sin^2(\xi)\Big(1 - \frac{\sin^2(2\xi)}{16}\Big),$$

we find that $F(\kappa)$ belongs to $\overline{\mathbb{D}}$ for all $\kappa \in \mathbb{S}^1$, and $F(\kappa)$ belongs to $\mathbb{S}^1$ for such κ if and only if $\kappa = \pm 1$. We thus have (1.7) with $\underline{\kappa}_1 = 1$ and $\underline{\kappa}_2 = -1$, and the reader can check that (1.8) is satisfied with

$$\alpha_1 = 1,\, \alpha_2 = -1, \quad \beta_1 = \beta_2 := \frac{1}{2}, \quad \mu_1 = \mu_2 = 1,$$

which means that Assumption 1.2 is satisfied. As a consequence, we also have $\underline{z}_1 = \underline{z}_2 = 1$, and we immediately see that Assumption 1.5 is also satisfied: both sets $\mathcal{I}_1$ and $\mathcal{I}_2$ equal $\{1, 2\}$ and $\alpha_1\alpha_2 = -1 < 0$. This is entirely similar to the behavior of the implicit scheme (4.3) except that (4.6) corresponds to a finitely supported convolution operator. The spectral curve $F(\mathbb{S}^1)$ is illustrated in Figure 11. Since the modulus of $F(\kappa)$ attains its maximum at two points of $\mathbb{S}^1$ and $\alpha_1 \neq \alpha_2$, we cannot apply the uniform Gaussian bound from [9] or [28]. The improvement of Theorem 1.6 is thus meaningful here.

As far as Assumption 1.3 is concerned, we compute

$$\mathbb{A}_{-3}(z) = -\frac{1}{16}, \quad \mathbb{A}_3(z) = -\frac{1}{16},$$

so Assumption 1.3 is satisfied again. We also note that Assumptions 1.1 and 1.4 are satisfied since the ∞ scheme is explicit. We can therefore apply Theorem 1.6, which, in the case of (4.6), yields the uniform Gaussian bound:

$$|\mathcal{G}_j^n| \leq \frac{C}{\sqrt{n}}\bigg(\exp\Big(-c\,\frac{(j+n)^2}{n}\Big) + \exp\Big(-c\,\frac{(j-n)^2}{n}\Big)\bigg), \quad n \geq 1,\ j \in \mathbb{Z}.$$

This behavior is illustrated in Figure 11. Note that the scaling factor $n^{-1/2}$ in the Gaussian bound is further demonstrated in Figure 10, where we represent $\sup_{j\in\mathbb{Z}}|\mathscr{G}_j^n|$ in logarithmic scale. Using a best linear fit, we numerically obtain a slope of -0.4983 which is in good agreement with the theory.

4.5. Extending Theorem 1.6 to the case where some α_k can be zero

In the explicit case, it is possible to extend Theorem 1.6 to the case where some α_k can be zero. The idea is to shift the operator $\mathscr{L} = Q_0$ so that for some integer $J \in \mathbb{Z}$, the shifted operator $\widetilde{\mathscr{L}} := \mathbf{S}^J Q_0$ verifies the assumptions of Theorem 1.6, for which we obtain a generalized Gaussian estimate which is necessarily of the form

$$|(\widetilde{\mathscr{L}}^n \boldsymbol{\delta})_j| \leq C \sum_{k=1}^{K} \frac{1}{n^{1/(2\mu_k)}} \exp\Big(-c\Big(\frac{|j-(\alpha_k+J)n|}{n^{1/(2\mu_k)}}\Big)^{\frac{2\mu_k}{2\mu_k-1}}\Big) \quad \forall n \in \mathbb{N}^*, \ \forall j \in \mathbb{Z}.$$

But we observe that

$$(\widetilde{\mathscr{L}}^n \boldsymbol{\delta})_j = (\mathscr{L}^n \boldsymbol{\delta})_{j-nJ} \quad \forall n \in \mathbb{N}^*, \ \forall j \in \mathbb{Z},$$

which gives a generalized Gaussian estimate for $\mathscr{L}^n \boldsymbol{\delta}$. More precisely, we introduce a relaxed version of Assumption 1.2 where some α_k could vanish.

Assumption 4.1. The function F defined in (1.6) satisfies Assumption 1.2 but possibly with some α_k in (1.8) that can be zero.

We then have the following corollary.

Corollary 4.2. *Assume that Q_1 is the identity, and that Q_0 satisfies Assumption 4.1. If there exists some $J \in \mathbb{Z}$ such that $\mathbf{S}^J Q_0$ satisfies Assumptions 1.2, 1.3 and 1.5, then there exist two constants $C > 0$ and $c > 0$ such that*

$$|(\mathscr{L}^n \boldsymbol{\delta})_j| \leq C \sum_{k=1}^{K} \frac{1}{n^{1/(2\mu_k)}} \exp\Big(-c\Big(\frac{|j-\alpha_k n|}{n^{1/(2\mu_k)}}\Big)^{\frac{2\mu_k}{2\mu_k-1}}\Big) \quad \forall n \in \mathbb{N}^*, \ \forall j \in \mathbb{Z}.$$

Typically, when Q_1 is the identity and $K = 1$ with $\alpha_1 = 0$ in Assumption 1.2, we can always apply Corollary 4.2 by choosing J sufficiently large (in that case, we have $r = 0$ and $p > 0$ for the shifted operator $\mathbf{S}^J Q_0$, $\mathbb{A}_0(z) = z$ and $\mathbb{A}_p(z)$ is a nonzero constant).

4.6. Further extensions

For the sake of clarity, we have focused here on the case of scalar iterations, but the techniques developed in this article should apply to some multistep iterations of the form

$$\begin{cases} Q_{s+1} u^{n+s+1} = Q_s u^{n+s} + \cdots + Q_0 u^n, & n \in \mathbb{N}, \\ u^0, \ldots, u^s \in \ell^2(\mathbb{Z}), \end{cases}$$

where $s \in \mathbb{N}$ is a given fixed integer and there are now $s+2$ convolution operators with finite support involved. Of course, the statement of the assumptions should be suitably modified (for instance, Assumption 1.1 now bears on Q_{s+1} and not on Q_1).

We have focused here on numerical schemes for which the modulus of the amplification factor F is not constant on $\mathbb{S}^1$ and such that the local behavior of F near a point where its modulus attains its maximum is dictated as in [36]. We recall that for explicit operators ($Q_1 = I$) of the form (1.2), the main result in [36] shows that Assumption 1.2 is necessary and sufficient for Q_0 to be power bounded from $\ell^1(\mathbb{Z};\mathbb{C})$ to $\ell^1(\mathbb{Z};\mathbb{C})$ (or equivalently from $\ell^\infty(\mathbb{Z};\mathbb{C})$ to $\ell^\infty(\mathbb{Z};\mathbb{C})$). A major advantage of the approach developed in this article is that it gets rid (more or less) of Fourier analysis. In particular, we have used the above strategy in [4] for proving a sharp stability result on a discretized transport equation with numerical boundary conditions under a degenerate version of the so-called Kreiss–Lopatinskii condition (see [14, 15, 20] for some background on numerical boundary conditions for hyperbolic equations). The problem considered in [4] is set in $\ell^2(\mathbb{N};\mathbb{C})$ rather than $\ell^2(\mathbb{Z};\mathbb{C})$, which makes many Fourier based techniques useless. Eventually, the above strategy is used in [1] to sharpen the local limit theorem of [27] and prove generalized Gaussian estimates for the remainder in the local limit theorem of [27].

A. Proof of intermediate and related results

A.1. Behavior of the amplification factor on the unit circle

The aim of this subsection is to prove the following result, which generalizes the classification obtained by Thomée [36], p. 280, for trigonometric polynomials. Lemma A.1 below shows that this classification only depends on the holomorphy of the considered function on a neighborhood of the unit circle.

Lemma A.1. *Let $\delta > 0$ and let f be a holomorphic function on the annulus*

$$\{\zeta \in \mathbb{C} \mid \mathrm{e}^{-\delta} < |\zeta| < \mathrm{e}^{\delta}\},$$

that satisfies

$$\sup_{\kappa \in \mathbb{S}^1} |f(\kappa)| = 1. \tag{A.1}$$

Then one of the following is satisfied:

- *$f(\kappa)$ has modulus 1 for any $\kappa \in \mathbb{S}^1$,*
- *there exists a finite set of points $\{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}$, $K \geq 1$, in $\mathbb{S}^1$ such that $f(\underline{\kappa}_k)$ has modulus 1 for any $k \in \{1, \dots, K\}$ and:*

$$|f(\kappa)| < 1 \quad \forall \kappa \in \mathbb{S}^1 \setminus \{\underline{\kappa}_1, \dots, \underline{\kappa}_K\}.$$

Assumption 1.2 in our work thus excludes the case where the rational function F in (1.6) has modulus 1 on the whole unit circle $\mathbb{S}^1$ (an example of such functions are the so-called Blaschke products [31]).

Proof of Lemma A.1. We first consider a point $\underline{\kappa} \in \mathbb{S}^1$ such that $f(\underline{\kappa})$ has modulus 1. Writing

$$f(\underline{\kappa}\mathrm{e}^{\zeta}) = f(\underline{\kappa}) \exp(g(\zeta)),$$

for some holomorphic function g on a neighborhood of 0, we can conclude that there exists a power series $\sum a_n x^n$ with real coefficients and a positive radius of convergence, such that for any sufficiently small real number ξ, we have

$$|f(\underline{\kappa}e^{\mathbf{i}\xi})| = \exp\left(\sum_{n=0}^{+\infty} a_n \xi^n\right).$$

Since $f(\underline{\kappa})$ has modulus 1, we have $a_0 = 0$. Using now the condition (A.1), we can conclude that either all the coefficients a_n, $n \in \mathbb{N}$, are zero or there exists a smallest nonzero *even* integer $2p$ such that

$$a_{2p} < 0 \quad \text{and} \quad a_0 = a_1 = \cdots = a_{2p-1} = 0.$$

In particular, for any ξ in some interval $(-\alpha, \alpha)$ with $\alpha > 0$, we have

$$|f(\underline{\kappa}e^{\mathbf{i}\xi})| \leq 1 - \frac{|a_{2p}|}{2}\,\xi^{2p}.$$

If all the coefficients a_n are zero, we can conclude that there exists an interval $(-\alpha, \alpha)$, with $\alpha > 0$, such that for any $\xi \in (-\alpha, \alpha)$, the modulus of $f(\underline{\kappa}\exp(\mathbf{i}\xi))$ equals 1. We have thus classified the two possible behaviors of the modulus of f near any point $\underline{\kappa} \in \mathbb{S}^1$ at which the modulus of f attains its maximum.

With this preliminary fact at our disposal, let us now consider the set

$$\mathcal{O} := \left\{\kappa \in \mathbb{S}^1 \mid \exists\alpha > 0,\ \forall\xi \in (-\alpha, \alpha),\ |f(\kappa e^{\mathbf{i}\xi})| = 1\right\}.$$

The set $\mathcal{O}$ is clearly open and the previous argument on the local behavior of $|f|$ near any point of $\mathbb{S}^1$ where it attains its maximum shows that the set $\mathcal{O}$ is closed. Since $\mathbb{S}^1$ is connected, $\mathcal{O}$ is either empty or equal to $\mathbb{S}^1$.

Let us now prove the claim of Lemma A.1. The case $\mathcal{O} = \mathbb{S}^1$ corresponds to the first possibility where $f(\kappa)$ has modulus 1 for any $\kappa \in \mathbb{S}^1$. We thus assume that the modulus of f is non-constant on $\mathbb{S}^1$ and therefore $\mathcal{O}$ is empty. Then any point $\underline{\kappa}$ admits an open neighborhood $\mathcal{V}$ in $\mathbb{S}^1$ such that

$$|f(\kappa)| < 1 \quad \forall\kappa \in \mathcal{V} \setminus \{\underline{\kappa}\}.$$

The conclusion follows from the compactness of $\mathbb{S}^1$. ∎

A.2. The Bernstein type inequality

This subsection is devoted to the proof of Lemma 3.1. A proof of Lemma 3.1 is provided in [4] in the particular case where the coefficients $a_{\ell,0}$ are real and $K = 1$, $\underline{\kappa}_1 = 1$. We explain below why the result holds in the broader context of complex-valued sequences and an arbitrary number K of tangency points. The result of Lemma 3.1 is a variation on the so-called Courant–Friedrichs–Lewy condition for numerical approximations of hyperbolic equations [5]. This condition, which bears on continuous and numerical domains of dependence, is known to be related to the von Neumann stability condition and the Bernstein inequality for trigonometric polynomials, see, for instance, [26] for a proof and historical comments on the Bernstein inequality. We refer to [33], p. 152, for the link between the CFL condition and the Bernstein inequality.

Proof of Lemma 3.1. Since Q_1 is the identity, we have $F = \widehat{Q}_0$. Given an integer $k \in \{1, \dots, K\}$, we introduce a polynomial function P_k defined by

$$P_k(z) := z^r \frac{F(z\underline{\kappa}_k)}{F(\underline{\kappa}_k)} = \frac{1}{\sum_{\ell=-r}^{p} a_{\ell,0}\,\underline{\kappa}_k^{\ell}} \sum_{\ell=-r}^{p} a_{\ell,0}\,\underline{\kappa}_k^{\ell}\, z^{\ell+r} \quad \forall z \in \mathbb{C}.$$

Assumption 1.2 implies that P_k is a non-constant holomorphic function on $\mathbb{C}$ and, furthermore, the modulus of P_k is not larger than 1 on $\mathbb{S}^1$. By the maximum principle for holomorphic functions, see Chapter 12 of [31], P_k maps $\mathbb{D}$ onto $\mathbb{D}$. In particular, we have

$$|P_k(1-\epsilon)| < 1 \quad \forall \epsilon \in (0,1),$$

and we easily obtain $P_k(1) = 1$. We now use (1.8) and compute

$$P_k(\mathrm{e}^{\mathbf{i}\xi}) = \exp\big(\mathbf{i}(r-\alpha_k)\xi - \beta_k \xi^{2\mu_k} + O(\xi^{2\mu_k+1})\big), \tag{A.2}$$

from which we get $P_k'(1) = r - \alpha_k$ and the asymptotic expansion

$$|P_k(1-\epsilon)| = 1 - (r-\alpha_k)\epsilon + O(\epsilon^2),$$

as (the real number) ϵ tends to zero. Arguing by contradiction, this gives $r - \alpha_k \geq 0$ since P_k maps $\mathbb{D}$ onto $\mathbb{D}$. We now show that α_k cannot equal r and argue again by contradiction. Assuming $\alpha_k = r$, (A.2) reduces to

$$P_k(\mathrm{e}^{\mathbf{i}\xi}) = \exp\big(-\beta_k \xi^{2\mu_k} + O(\xi^{2\mu_k+1})\big),$$

and we thus obtain,[13] for real positive values of ϵ tending to zero,

$$P_k(\mathrm{e}^{\mathbf{i}\epsilon\exp(\mathbf{i}\pi/(2\mu_k))}) = \exp\big(\beta_k \epsilon^{2\mu_k} + O(\epsilon^{2\mu_k+1})\big).$$

This leads to a contradiction because $\exp(\mathbf{i}\epsilon\exp(\mathbf{i}\pi/(2\mu_k)))$ belongs to $\mathbb{D}$ for any $\epsilon > 0$ and β_k has positive real part.

In order to prove the inequality $\alpha_k > -p$, we introduce the complex reciprocal polynomial Q_k of P_k (see, again, [26]):

$$Q_k(z) := z^{p+r}\,\overline{P}_k(1/\overline{z}) = \frac{1}{\sum_{\ell=-r}^{p} \overline{a}_{\ell,0}\,\overline{\underline{\kappa}}_k^{\ell}} \sum_{\ell=-r}^{p} \overline{a}_{\ell,0}\,\overline{\underline{\kappa}}_k^{\ell}\, z^{p-\ell}.$$

By the same argument as the one used for P_k, we have $Q_k(1) = 1$ and Q_k maps $\mathbb{D}$ onto $\mathbb{D}$. We also compute

$$Q_k'(1) = p + r - \overline{P_k'(1)} = p + \alpha_k,$$

and we therefore have $p + \alpha_k \geq 0$ because Q_k maps $\mathbb{D}$ onto $\mathbb{D}$. We now show that α_k can not equal $-p$ and argue again by contradiction. Assuming $\alpha_k = -p$, we use (A.2) and compute

$$Q_k(\mathrm{e}^{\mathbf{i}\epsilon\exp(\mathbf{i}\pi/(2\mu_k))}) = \mathrm{e}^{\mathbf{i}(p+r)\epsilon\exp(\mathbf{i}\pi/(2\mu_k))}\,\overline{P_k(\mathrm{e}^{\mathbf{i}\epsilon\exp(-\mathbf{i}\pi/(2\mu_k))})}$$

[13] The crucial fact here is that (A.2) holds for either real or complex values of ξ.

$$= \mathrm{e}^{\mathbf{i}(p+r)\epsilon\exp(\mathbf{i}\pi/(2\mu_k))}\,\overline{\exp\big(\mathbf{i}(r-\alpha_k)\epsilon\exp(-\mathbf{i}\pi/(2\mu_k)) + \beta_k\epsilon^{2\mu_k} + O(\epsilon^{2\mu_k+1})\big)}$$
$$= \exp(\overline{\beta_k}\epsilon^{2\mu_k} + O(\epsilon^{2\mu_k+1})).$$

We are again led to a contradiction. The proof of Lemma 3.1 is thus complete. ■

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