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Trace estimates of Toeplitz operators on Bergman spaces and applications to composition operators

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Abstract. Let Ω be a subdomain of $\mathbb{C}$ and let μ be a positive Borel measure on Ω. In this paper, we study the asymptotic behavior of the eigenvalues of compact Toeplitz operators T_μ acting on Bergman spaces on Ω. Let $(\lambda_n(T_\mu))$ be the decreasing sequence of the eigenvalues of T_μ, and let ρ be an increasing function such that $\rho(n)/n^A$ is decreasing for some $A > 0$. We give an explicit necessary and sufficient geometric condition on μ in order to have $\lambda_n(T_\mu) \asymp 1/\rho(n)$. As applications, we consider composition operators C_φ, acting on some standard analytic spaces on the unit disc $\mathbb{D}$. First, we give a general criterion ensuring that the singular values of C_φ satisfy $s_n(C_\varphi) \asymp 1/\rho(n)$. Next, we focus our attention on composition operators with univalent symbols, where we express our general criterion in terms of the harmonic measure of $\varphi(\mathbb{D})$. We finally study the case where $\partial\varphi(\mathbb{D})$ meets the unit circle in one point and give several concrete examples. Our method is based on upper and lower estimates of the trace of $h(T_\mu)$, where h is a suitable concave or convex function.

1. Introduction

Spectral properties of Toeplitz operators associated with positive measures play an important role in spectral theory of several operators: Hankel operators, composition operators and integration operators. In this paper, we are interested in the behavior of the eigenvalues of compact Toeplitz operators acting on analytic spaces on a subdomain Ω of $\mathbb{C}$ with applications to composition operators.

Let Ω be a domain of $\mathbb{C}$. We denote by $H(\Omega)$ the class of all holomorphic functions on Ω. Let $\omega\colon \Omega \to (0,\infty)$ be a continuous weight on Ω. The weighted Bergman space associated with ω is given by

$$\mathcal{A}^2_\omega = \Big\{ f \in H(\Omega) : \ \|f\|_\omega = \Big(\int_\Omega |f(z)|^2 \, dA_\omega(z) \Big)^{1/2} < \infty \Big\},$$

where $dA_\omega(z) = \omega^2(z)\, dA(z)$ and dA is the Lebesgue measure on $\mathbb{C}$.

Clearly, $\mathcal{A}^2_\omega$ is a reproducing kernel space. The reproducing kernel of $\mathcal{A}^2_\omega$ will be denoted by K (or K^ω if necessary).

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In this paper, we call the standard Bergman spaces, denoted by $\mathcal{A}^2_\alpha$, the Bergman spaces on $\mathbb{D}$ associated with $\omega^2(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha$, where $\alpha > -1$. The standard Fock spaces $\mathcal{F}^2_\alpha$ correspond to $\Omega = \mathbb{C}$ and $\omega^2(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}$, where $\alpha > 0$.

The Toeplitz operator T_μ, acting on $\mathcal{A}^2_\omega$, induced by a positive Borel measure μ on Ω is given by

$$T_\mu(f)(z) = \int_\Omega f(\zeta)\, K(z,\zeta)\, \omega^2(\zeta)\, d\mu(\zeta).$$

The boundedness, compactness and membership to Schatten classes of Toeplitz operators have been studied in several papers (see for instance [3, 11, 14, 23–25, 31, 37]). It has been proved that, under some regularity conditions on ω, T_μ is bounded (respectively, compact) if and only if $\mu(R_n)/A(R_n) = O(1)$ (respectively, $o(1)$), where (R_n) is a suitable lattice of Ω with respect to ω.

Our goal in this paper is to study the asymptotic behavior of the eigenvalues of compact Toeplitz operators on $\mathcal{A}^2_\omega$. First, we fix some notations. The class of weights on Ω considered in this paper, denoted by $\mathcal{W}(\Omega)$, contains all standard weights. Some examples are listed in Section 2. To each $\omega \in \mathcal{W}(\Omega)$, we associate a class of suitable lattices denoted by $\mathcal{L}_\omega$. The definitions of $\mathcal{W}$ and $\mathcal{L}_\omega$ are given in Section 2.

Throughout this paper we suppose that T_μ is compact. The decreasing sequence of the eigenvalues of T_μ will be denoted by $(\lambda_n(T_\mu))$. It is proved in [11] that $\lambda_n(T_\mu) = O(1/\log^\gamma(n))$ for some $\gamma > 0$ if and only if there exists $c > 0$ such that

$$\sum_n \exp\Big(-c\Big(\frac{A(R_n)}{\mu(R_n)}\Big)^{1/\gamma}\Big) < \infty,$$

for some $(R_n)_n \in \mathcal{L}_\omega$.

In this paper we are interested in compact Toeplitz operators T_μ such that $1/\lambda_n(T_\mu) = O(n^A)$ for some $A > 0$.

Recall that since T_μ is compact, we have $\lim_{n\to+\infty} \mu(R_n)/A(R_n) = 0$. Let $(R_n(\mu))$ be an enumeration of (R_n) such that the sequence

$$a_n(\mu) := \frac{\mu(R_n(\mu))}{A(R_n(\mu))}$$

is decreasing. First, we will prove the following result.

Theorem A. *Let $(R_n) \in \mathcal{L}_\omega$, where $\omega \in \mathcal{W}$. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing function such that $\rho(x)/x^A$ is decreasing for some $A > 0$. Let μ be a positive Borel measure on Ω such that T_μ defines a compact operator on $\mathcal{A}^2_\omega$. Then,*

(1) $\lambda_n(T_\mu) = O(1/\rho(n)) \iff a_n(\mu) = O(1/\rho(n))$.
(2) $\lambda_n(T_\mu) \asymp 1/\rho(n) \iff a_n(\mu) \asymp 1/\rho(n)$.

A preliminary version of this theorem, in the case of standard Bergman spaces of the unit disc, was announced in [6]. Before going on, two remarks on Theorem A are in order.

(i) The growth condition on ρ is, in some sense, necessary. Indeed, let ρ be an increasing function such that $\rho(x) = o(\rho(2x))$ when $x \to +\infty$. One can construct (see Section

4.5) a Toeplitz operator T_μ such that for any lattice $(R_n)_n$ we have

$$\limsup_{n\to\infty} \frac{\lambda_n(T_\mu)}{a_n(\mu)} = +\infty,$$

where $a_n(\mu)$ is the decreasing rearrangement of $(\mu(R_n)/A(R_n))$.

(ii) In general, the sequence $(a_n(\mu))_n$ is not sufficient to give asymptotic estimates of $(\lambda_n(T_\mu))_n$. Indeed, one can construct two positive Borel measures μ and ν on the unit disc $\mathbb{D}$ such that

$$a_n(\mu) = a_n(\nu) \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n(T_\mu)/\lambda_n(T_\nu) = \infty.$$

Next, we analyze the connection between the behavior of the eigenvalues of T_μ and the behavior of the Berezin transform of T_μ. Recall that the Berezin transform of a Toeplitz operator T_μ acting on $\mathcal{A}^2_\omega$ is given by

$$\tilde{\mu}(z) = \frac{\langle T_\mu K_z, K_z\rangle}{\|K_z\|^2}, \quad z \in \Omega. \tag{1.1}$$

Let $(R_n)_{n\geq 1} \in \mathcal{L}_\omega$ and let z_n be the center of R_n. It is known that T_μ is compact if and only if

$$\lim_{n\to\infty} \tilde{\mu}(z_n) = 0.$$

As before, let $(z_n(\mu))$ be an enumeration of (z_n) such that the sequence $(b_n(\mu))_n$, defined by

$$b_n(\mu) := \tilde{\mu}(z_n(\mu)), \tag{1.2}$$

is decreasing.

First, we consider Toeplitz operators T_μ such that $1/\lambda_n(T_\mu) = O(n^\gamma)$ for some $\gamma \in (0,1)$. We have the following.

Theorem B. *Let $\omega \in \mathcal{W}$ and let μ be a positive Borel measure on Ω such that T_μ is compact. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing positive function such that $\rho(x)/x^\gamma$ is decreasing for some $\gamma \in (0,1)$. We have*

$$\lambda_n(T_\mu) \asymp 1/\rho(n) \iff b_n(\mu) \asymp 1/\rho(n).$$

The case $\lambda_n(T_\mu) \lesssim 1/n^A$, for some $A > 1$, is rather different. Indeed, to have a description of the behavior of the eigenvalues of such Toeplitz operators in terms of $(b_n(\mu))$ it is necessary that

$$C_p(\mathcal{A}^2_\omega, (R_n)) := \sup_{n\geq 1} \sum_{j\geq 1} \tilde{\nu}_n^p(z_j) < \infty, \quad (p \in (0,1)), \tag{1.3}$$

where $d\nu_n = dA_{|R_n}$ (see Theorem 5.3).

We will prove the following converse.

Theorem C. *Let $\omega \in \mathcal{W}$, $(R_n)_n \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact. Suppose that $C_p(\mathcal{A}^2_\omega, (R_n)_n) < \infty$ for all $p \in (0,1)$. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing positive function satisfying $\rho(t)/t^\gamma$ is increasing for some $\gamma > 1$ and $\rho(t)/t^\beta$ is decreasing for some large β. Then we have*

$$\lambda_n(T_\mu) \asymp 1/\rho(n) \iff b_n(\mu) \asymp 1/\rho(n).$$

The proofs of these theorems are based on upper and lower estimates of the trace of $h(T_\mu)$ for convex and concave functions h.

As application, we consider composition operators on $\mathcal{H}_\alpha = \{f \in H(\mathbb{D}) : f' \in \mathcal{A}^2_\alpha\}$, which was the original motivation of this work. Let φ be an analytic self map of $\mathbb{D}$. The composition operator on $\mathcal{H}_\alpha$ induced by a symbol φ is defined by

$$C_\varphi(f) = f \circ \varphi, \quad (f \in \mathcal{H}_\alpha).$$

Using Theorem A and a standard connection between composition operators and Toeplitz operators, we give estimates of the singular values $s_n(C_\varphi, \mathcal{H}_\alpha)$ of general composition operators C_φ, when $1/s_n(C_\varphi, \mathcal{H}_\alpha)$ does not increase faster than all polynomials. These estimates are given in terms of the mean values of a generalized counting function associated with φ.

We also express these estimates in terms of the harmonic measure of $\varphi(\mathbb{D})$, when φ is univalent and $\varphi(\mathbb{D})$ is a Jordan domain.

Next, we consider composition operators C_φ induced by univalent symbols φ such that $\partial\varphi(\mathbb{D}) \cap \partial\mathbb{D}$ is reduced to one point. Namely, we suppose that $\partial\varphi(\mathbb{D})$ has, in a neighborhood of $+1$, a polar equation $1 - r = \gamma(|\theta|)$, where $\gamma\colon]0,\pi] \to]0,1]$ is a differentiable increasing function with $\gamma(0) = 0$, and satisfying the following conditions:

$$\frac{\gamma(t)}{t} \text{ is increasing}, \quad \gamma'(t) = O(\gamma(t)/t) \quad \text{as } t \to 0^+, \tag{1.4}$$

and

$$\gamma(t) = O(t/\log^\beta(1/t)) \quad \text{for some } \beta > 1/2. \tag{1.5}$$

Recall that by Tsuji–Warschwski's theorem, (see [33]), C_φ is compact if and only if

$$\int_0 \frac{\gamma(s)}{s^2}\, ds = \infty.$$

It is proved in [7] that the composition operator C_φ on $\mathcal{H}_\alpha$ is in the p-Schatten class ($p > 0$) if and only if

$$\int_0 \frac{e^{-\frac{p\alpha}{2}\Gamma(t)}}{\gamma(t)}\, dt < \infty, \tag{1.6}$$

where

$$\Gamma(t) = \frac{2}{\pi}\int_t^1 \frac{\gamma(s)}{s^2}\, ds. \tag{1.7}$$

We have the following result.

Theorem D. *Let $\alpha > 0$ and let Ω, γ and φ be as before. Suppose that $\int_0 \frac{\gamma(t)}{t^2} dt = \infty$. We have*:

(1) *If* $\lim_{t\to 0^+} \gamma(t)\log(1/t)/t = \infty$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) = O(1/n^A) \quad \text{for all } A > 0.$$

(2) *If* $\gamma(t)\log(1/t)/t = O(1)$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) \asymp \exp\Big(-\frac{\alpha}{2}\,\Gamma(x_n)\Big),$$

where x_n is given by $\int_{x_n}^2 \frac{dt}{\gamma(t)} = n$.

As examples, we obtain:

Corollary 1.1. *With the same notations as above, we have*:

(1) *If* $\gamma(t) = \kappa t/\log(e/t)$, *with* $\kappa > 0$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) \asymp \frac{1}{n^{\alpha\kappa/2\pi}}.$$

(2) *If* $\gamma(t) = \kappa t/\log(e/t)\log\log(e^2/t)$, *with* $\kappa > 0$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) \asymp \frac{1}{(\log n)^{\alpha\kappa/\pi}}.$$

The article is organized as follows. In Section 2, we recall some classical results on compact operators and introduce the weighted Bergman spaces considered throughout this paper. In Section 3, we show how to obtain estimates of the eigenvalues of a compact operator from trace estimates. Section 4 is devoted to proving the estimates of $\mathrm{Tr}(h(T_\mu))$, where h satisfies some concave/convex conditions. It is important to note that the proof presented in this paper, in particular in the concave case, is different from Luecking's proof [24] and does not require off-diagonal kernel estimates. This section contains the proof of Theorem A. In Section 5, we study the behavior of the eigenvalues of T_μ in terms of its Berezin transform. In Section 6, we consider composition operators C_φ with general symbol φ and give estimates of the singular values of C_φ in terms of the generalized Nevanlinna function associated with φ. Section 7 is devoted to composition operators with univalent symbols. We express the asymptotic behavior of the singular values of C_φ in terms of the harmonic measure of $\varphi(\mathbb{D})$ and we give explicit examples. In the last section we consider examples of composition operators acting on the Hardy space and on the classical Dirichlet space.

Notations. Throughout this paper, we will use the following notations:

- $x \lesssim y$ if there exists a constant $C > 0$ such that $x \le Cy$,
- $x \asymp y$ if $x \lesssim y$ and $y \lesssim x$,
- $C(x_1, \dots, x_n)$ is a constant which depends on $x_1, \dots, x_n$.

2. Preliminaries

2.1. Compact operators

Let H be a complex Hilbert space and let T be a bounded operator on H. The class of compact operators on H will be denoted $\mathcal{S}_\infty$ or ($\mathcal{S}_\infty(H)$ if necessary). Let $T \in \mathcal{S}_\infty$. The sequence $(s_n(T))_{n\geq 1}$ (or $(s_n(T,H))_n$) denotes the non increasing sequence of eigenvalues of $(T^*T)^{1/2}$. If T is positive, $(s_n(T))_{n\geq 1}$ is the sequence of eigenvalues of T and we write in this case $s_n(T) = \lambda_n(T)$.

By the spectral decomposition of compact operators, every compact operator T on H can be written as follows:

$$Tf = \sum_n s_n(T)\langle f, f_n\rangle g_n, \quad (f \in H),$$

where (f_n) and $(g_n)_{n\geq 1}$ are orthonormal systems of H.

So, it is easy to see that

$$s_n(T) = \inf\{\|T - R\|,\ \dim R(H) < n\}.$$

In particular, if T and S are two compact operators such that $T = XS$, where X is a contraction, then

$$s_n(T) \leq s_n(S), \quad \text{for all } n \geq 0.$$

Recall that a compact operator T on H belongs to the p-Schatten class $\mathcal{S}_p$ (for $p > 0$) if

$$\|T\|_p := \Big(\sum_{n\geq 1} s_n(T)^p\Big)^{1/p} < \infty.$$

The following result is known as the monotonicity Weyl lemma.

Lemma 2.1. *Let T, S be two positive bounded operators on a complex Hilbert space H such that $T \leq S$. If S is compact, then T is compact and $\lambda_n(T) \leq \lambda_n(S)$ for all $n \geq 1$.*

Let $h\colon [0,\infty) \to [0,\infty)$ be a continuous function such that $h(0) = 0$. For each positive compact operator $T = \sum_{n\geq 0} \lambda_n\langle ., e_n\rangle e_n$, the operator $h(T) =: \sum_{n\geq 0} h(\lambda_n)\langle ., e_n\rangle e_n$ is a positive compact operator and $\operatorname{Tr}(h(T)) = \sum_{n\geq 0} h(\lambda_n)$. We will also need the following general result, see [29].

Lemma 2.2. *Let $(T_n)_{n\geq 1}$ be a sequence of positive compact operators on a Hilbert space H and let $T = \sum_{n\geq 1} T_n$ (with norm-operator convergence). Let $h\colon [0,+\infty) \to [0,+\infty)$ be an increasing function such that $h(0) = 0$. Then:*

(1) *If h is convex, then $\operatorname{Tr}(h(T)) \geq \sum_n \operatorname{Tr}(h(T_n))$.*

(2) *If h is concave, then $\operatorname{Tr}(h(T)) \leq \sum_n \operatorname{Tr}(h(T_n))$.*

The following classical result will be used in Section 4.

Lemma 2.3. *Let $p \geq 1$ and let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be two positive decreasing sequences. Suppose that*

$$\sum_{k=1}^n a_k^{1/p} \leq \sum_{k=1}^n b_k^{1/p}, \quad \textit{for all } n \geq 1.$$

Then, for every increasing positive function h such that $h(t^p)$ is convex, we have

$$\sum_{k=1}^{n} h\,(a_n) \leq \sum_{k=1}^{n} h(b_n). \tag{2.1}$$

Proof. This is a direct consequence of Corollary 3.3 of Chapter IV in [13]. ∎

2.2. Weighted Bergman spaces

In this subsection we recall briefly the definition of the class of weights $\mathcal{W}$ introduced in [11]. Let Ω be a domain (bounded or not) of $\mathbb{C}$ and let $\partial\Omega$ denote the boundary of Ω. Let $\partial_\infty\Omega = \partial\Omega$ if Ω is bounded and $\partial_\infty\Omega = \partial\Omega \cup \{\infty\}$ if Ω is not bounded. Let ω be a positive continuous weight on Ω. In what follows, we suppose that the reproducing kernel K of $\mathcal{A}^2_\omega$ satisfies the following two conditions:

$$\lim_{z\to\partial_\infty\Omega} \|K_z\| = \infty, \tag{2.2}$$

and for every $\zeta \in \Omega$,

$$|K(\zeta, z)| = o(\|K_z\|) \quad \text{as } z \to \partial_\infty\Omega. \tag{2.3}$$

Let

$$\tau(z)(=\tau_\omega(z)) := \frac{1}{\omega(z)\|K_z\|}, \quad z \in \Omega. \tag{2.4}$$

Definition 2.4. Let ω be a weight such that (2.2) and (2.3) are satisfied. We say that $\omega \in \mathcal{W}$ (or $\mathcal{W}(\Omega)$) if there exists constants $a, C > 0$ such that for $z, \zeta \in \Omega$ satisfying $|z-\zeta| \leq a\tau(z)$, we have

$$\|K_z\|\|K_\zeta\| \leq C|K(\zeta, z)|, \quad \frac{1}{C}\tau(\zeta) \leq \tau(z) \leq C\tau(\zeta), \tag{2.5}$$

and

$$\tau(z) = O(\min(1, \operatorname{dist}(z, \partial_\infty\Omega))). \tag{2.6}$$

Now, we give some examples.

- *Standard Bergman spaces on the unit disc $\mathbb{D}$*. For $\alpha > -1$, define

$$\mathcal{A}^2_\alpha := \Big\{ f \in H(\mathbb{D}) : \|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2\, dA_\alpha(z) < \infty \Big\},$$

where $dA_\alpha = \frac{\alpha+1}{\pi} dA$. The reproducing kernel is given by

$$K^\alpha_z(w) = \frac{1}{(1-\bar{z}w)^{2+\alpha}}$$

and

$$\tau^2_\alpha(z) := \tau^2(z) = (1+\alpha)(1-|z|^2)^2.$$

- *Weighted Bergman spaces on* $\mathbb{D}$. Let $\mathcal{D}$ be the class of Oleinik–Perel'man weights on $\mathbb{D}$ (see [3, 11, 25]). It is easy to see from [3, 23], that if $\omega \in \mathcal{D}$, then $\omega \in \mathcal{W}$,
$$\|K_z^\omega\|^2 \asymp \omega^{-2}(z)\Delta(\log(1/\omega(z)) \quad \text{and} \quad \tau_\omega(z)^2 \asymp \frac{1}{\Delta(\log(1/\omega(z)))}.$$
For a more general situation, see [16].
- *Standard Fock spaces*. For $\alpha > 0$, define
$$\mathcal{F}_\alpha^2 := \mathcal{F}_\alpha^2(\mathbb{C}) = \Big\{ f \in H(\mathbb{C}) : \|f\|^2 = \frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2}\, dA(z) < \infty \Big\}. \tag{2.7}$$
Then the reproducing kernel is given by $K(z, w) = e^{\alpha z\bar{w}}$ and $\tau(z) \asymp 1$.
- *Weighted Fock spaces*. In this case, $\mathcal{A}_\omega^2$ will be denoted by $\mathcal{F}_\omega^2$. Let $\omega^2(z) = e^{-\Psi(|z|^2)}$ be a positive weight on $\mathbb{C}$. We say that $\omega \in \mathcal{R}$ if $\Psi\colon [0, +\infty) \to (0, +\infty) \in \mathcal{C}^3$ and satisfies the following conditions:
$$\Psi' > 0, \quad \Psi'' \geq 0, \quad \Psi''' \geq 0, \tag{2.8}$$
and
$$\Phi''(x) = O(x^{-1/2}(\Phi'(x))^{1+\eta}) \quad \text{for some } \eta < 1/2, \tag{2.9}$$
where $\Phi(x) = x\Psi'(x)$. This class of spaces was considered in [31] by K. Seip and E. H. Youssfi. One can see that since polynomials are dense in $\mathcal{F}_\omega^2$, conditions (2.2) and (2.3) are satisfied. It is proved in [31] that
$$\tau_\omega(z)^{-2} =: K(z, z)\omega^2(z) \asymp \Phi'(|z|^2).$$
Using Lemma 3.2 of [31], it is not hard to prove that $\omega \in \mathcal{W}$.

It is proved in [11] that, if $\omega \in \mathcal{W}$, then there exist $B_\omega > 1$, $\delta_\omega \in (0, a/4B_\omega)$ such that for all $\delta \in (0, \delta_\omega)$ there exists $(z_n) \in \Omega$ such that

- $\Omega = \cup_{n\geq 1} D(z_n, \delta\tau_\omega(z_n)) = \cup_{n\geq 1} D(z_n, B_\omega\delta\tau_\omega(z_n))$.
- $D(z_n, \frac{\delta}{B_\omega}\tau_\omega(z_n)) \cap D(z_m, \frac{\delta}{B_\omega}\tau_\omega(z_m)) = \emptyset$, for $n \neq m$.
- $z \in D(z_n, \delta\tau_\omega(z_n))$ implies that $D(z, \delta\tau_\omega(z)) \subset D(z_n, B_\omega\delta\tau_\omega(z_n))$.
- There exists an integer N such that every $D(z_n, B_\omega\delta\tau_\omega(z_n))$ cuts at most N sets of the family $(D(z_m, B_\omega\delta\tau_\omega(z_m)))_m$. We say that $(D(z_n, B_\omega\delta\tau_\omega(z_n)))_n$ is of finite multiplicity.

In the sequel, for $\omega \in \mathcal{W}$, the constants B_ω and δ_ω will be fixed.

Definition 2.5. We say that $(R_n)_n \in \mathcal{L}_\omega$ if $(R_n)_n = (D(z_n, \delta\tau_\omega(z_n)))_n$ satisfies the above four conditions.

In the following, we will consider $\omega \in \mathcal{W}$ and suppose that T_μ is compact. That is, $\mu(R_n)/A(R_n) = o(1)$ (see [11]). As mentioned before, $(R_n(\mu))$ will denote an enumeration of $(R_n)_n$ such that
$$a_n(\mu) := \mu(R_n(\mu))/A(R_n(\mu)), \tag{2.10}$$
is decreasing.

For $b > 0$, $bD(z,r)$ will denote the disc $D(z, br)$.

Lemma 2.6. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact on $\mathcal{A}^2_\omega$. Denote by $b = (1 + B_\omega)/2$ and by μ_n the restriction of μ to $\cup_{k\le n} R_k(\mu)$. Let ν_n and ν be the following measures:*

$$d\nu_n = \sum_{j=1}^{n} a_j(\mu)\, dA_{|bR_j(\mu)} \quad \text{and} \quad d\nu = \sum_{j\ge 1} a_j(\mu)\, dA_{|bR_j(\mu)}.$$

Then, there exists a constant $C = C(\omega, (R_n)) > 0$ such that

(1) $T_{\mu_n} \le C T_{\nu_n}$ *and* $T_\mu \le C T_\nu$.
(2) $\|T_\mu\| \le C a_1(\mu)$.
(3) $\lambda_k(T_{\mu_n}) \le C\lambda_k(T_{\nu_n})$, *for* $k \ge 1$.
(4) $\lambda_k(T_{\mu_n}) \le \lambda_k(T_\mu) \le \lambda_k(T_{\mu_n}) + C a_{n+1}(\mu)$, *for* $k \ge 1$.

Proof. By the subharmonicity inequality applied to the function $\zeta \to |f(\zeta)/K(\zeta, w)|^2$, we have

$$|f(\zeta)/K(\zeta,w)|^2 \lesssim \frac{1}{\tau^2(\zeta)} \int_{bR_n} \frac{|f(z)|^2}{|K(z,w)|^2}\, dA(z), \quad w, \zeta \in R_n.$$

In particular,

$$|f(\zeta)/K(\zeta,\zeta)|^2 \lesssim \frac{1}{\tau^2(\zeta)} \int_{bR_n} \frac{|f(z)|^2}{|K(z,\zeta)|^2}\, dA(z), \quad \zeta \in R_n.$$

Note that for $z, \zeta \in bR_n$, $\tau(\zeta) \asymp \tau(z) = \frac{1}{\omega(z)\|K_z\|}$ and $|K(\zeta, z) \asymp \|K_\zeta\|\|K_z\|$. Then we obtain

$$\frac{|f(\zeta)|^2}{\|K_\zeta\|^2} \lesssim \frac{1}{\tau^2(\zeta)} \int_{bR_n} \frac{|f(z)|^2}{\|K_z\|^2}\, dA(z) \asymp \int_{bR_n} |f(z)|^2\, \omega(z)^2\, dA(z).$$

Consequently, there exists a constant $C_1 > 0$, which depends on ω and $(R_n)_n$, such that

$$|f(\zeta)|^2 \omega^2(\zeta) \le \frac{C_1}{A(R_n)} \int_{bR_n} |f(z)|^2\, \omega^2(z)\, dA(z), \quad (\zeta \in R_n). \tag{2.11}$$

This gives that

$$\int_{R_n} |f(\zeta)|^2\, \omega^2(\zeta)\, d\mu(\zeta) \le C_1 a_n(\mu) \int_{bR_n} |f(z)|^2\, \omega^2(z)\, dA(z). \tag{2.12}$$

This implies

$$\begin{aligned}\langle T_{\mu_n} f, f\rangle &= \int_\Omega |f(\zeta)|^2\, \omega^2(\zeta)\, d\mu_n(\zeta) \le \sum_{j=1}^{n} \int_{R_j(\mu)} |f(\zeta)|^2\, \omega^2(\zeta)\, d\mu(\zeta) \\ &\le C_1 \sum_{j=1}^{n} a_j(\mu) \int_{bR_j(\mu)} |f(z)|^2\, \omega^2(z)\, dA(z) = C_1 \int_\Omega |f(z)|^2\, \omega^2(z) d\nu_n(z) \\ &= \langle C_1 T_{\nu_n} f, f\rangle.\end{aligned}$$

This means that $T_{\mu_n} \le C_1 T_{\nu_n}$, which proves the part (1) of the lemma.

Let N be the multiplicity of $(D(z_n, B_\omega \delta \tau_\omega(z_n)))_n$. From part (1) we have

$$0 \le T_\mu \le C_1 T_\nu \le N C_1 a_1(\mu) \, \mathrm{Id}_{\mathcal{A}^2_\omega}.$$

Then $\|T_\mu\| \le N C_1 a_1(\mu)$.

Clearly, part (3) is a consequence of part (1) and Lemma 2.1.

Since $\mu_n \le \mu$, we have $T_{\mu_n} \le T_\mu$. Then by Lemma 2.1, we get $\lambda_k(T_{\mu_n}) \le \lambda_k(T_\mu)$. For the second inequality note that $\lambda_k(T_\mu) \le \lambda_k(T_{\mu_n}) + \|T_\mu - T_{\mu_n}\|$. Using part (2), applied to $\mu - \mu_n$, we obtain

$$\|T_\mu - T_{\mu_n}\| = \|T_{\mu-\mu_n}\| \le C a_1(\mu - \mu_n) \le C a_{n+1}(\mu).$$

Combining the two last inequalities, we obtain $\lambda_k(T_\mu) \le \lambda_k(T_{\mu_n}) + C a_{n+1}(\mu)$. ∎

3. A general argument

Let $\beta > 0$ and let $\delta > 0$. The function $h_{\beta,\delta}$ defined on $[0, \infty)$ is given by

$$h_{\beta,\delta}(t) = (t^\beta - \delta)^+ := \max(t^\beta - \delta, 0).$$

The functions $h_{\beta,\delta}$, will play an important role in our study. First, note that $h_{\beta,\delta}$ is convex for $\beta \ge 1$ and if $\beta \in (0,1)$, we have $h_{\beta,\delta}(t^p)$ and $h^p_{\beta,\delta}$ are convex if and only if $p \ge 1/\beta$.

The following two lemmas will be used in the sequel to obtain estimates of eigenvalues of positive compact operator T from upper and lower estimates of the trace of $h(T)$ for some suitable functions h.

Lemma 3.1. *Let $\beta \in (0,1]$ and let $(a_n)_{n\ge1}$ be a decreasing sequence. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing positive function such that $\rho(x)/x^\gamma$ is decreasing for some $\gamma \in (0, 1/\beta)$. Suppose that there exists $B > 0$ such that for every $\delta \in (0,1)$, we have*

$$\sum_{n\ge1} h_{\beta,\delta}\Big(\frac{1}{B\rho(n)}\Big) \le \sum_{n\ge1} h_{\beta,\delta}(a_n) \le \sum_{n\ge1} h_{\beta,\delta}\Big(\frac{B}{\rho(n)}\Big). \tag{3.1}$$

Then $a_n \asymp 1/\rho(n)$.

Proof. Without loss of generality, we suppose that ρ is strictly increasing and $\beta = 1$. Let $\delta > 0$ and let $h_\delta(t) = (t-\delta)^+$. By (3.1) we have

$$\sum_{a_n \ge 2\delta} a_n \le 2 \sum_{a_n \ge 2\delta} (a_n - \delta) \le 2 \sum_n h_\delta(a_n) \le 2 \sum_n h_\delta(B/\rho(n)) \le \sum_{\rho(n) \le B/\delta} \frac{2B}{\rho(n)}$$

and

$$\sum_{\rho(n) \le \frac{1}{2B\delta}} \frac{1}{B\rho(n)} \le 2 \sum_n \Big(\frac{1}{B\rho(n)} - \delta\Big)^+ \le 2 \sum_n (a_n - \delta)^+ \le 2 \sum_{a_n \ge \delta} a_n.$$

These inequalities can be written as follows:

$$\sum_{\rho(n)\le 1/(2B\delta)} \frac{1}{\rho(n)} \lesssim \sum_{a_n\ge\delta} a_n \lesssim \sum_{\rho(n)\le 2B/\delta} \frac{1}{\rho(n)}. \tag{3.2}$$

We have

$$\sum_{\rho(n)\le x} \frac{1}{\rho(n)} \asymp \frac{\rho^{-1}(x)}{x}.$$

Indeed, obviously we have

$$\frac{\rho^{-1}(x)}{x} \lesssim \sum_{\rho(n)\le x} \frac{1}{\rho(n)}.$$

Conversely, using the fact that $\rho(x)/x^\gamma$ is decreasing, we have

$$\sum_{\rho(n)\le x} \frac{1}{\rho(n)} = \sum_{\rho(n)\le x} \frac{n^\gamma}{\rho(n)} \frac{1}{n^\gamma} \lesssim \frac{(\rho^{-1}(x))^\gamma}{x} \sum_{\rho(n)\le x} \frac{1}{n^\gamma} \asymp \frac{\rho^{-1}(x)}{x}.$$

Then

$$\sum_{a_n\ge\delta} a_n \lesssim \frac{\delta}{B}\rho^{-1}\Big(\frac{2B}{\delta}\Big).$$

Let $N(\delta) := \mathrm{Card}\{n : a_n \ge \delta\}$. Since $\delta N(\delta) \le \sum_{a_n\ge\delta} a_n$, we obtain

$$N(\delta) \lesssim \frac{1}{B}\rho^{-1}\Big(\frac{2B}{\delta}\Big).$$

In particular, for $\delta = a_n$, we get

$$n \lesssim \frac{1}{B}\,\rho^{-1}\Big(\frac{2B}{a_n}\Big).$$

This implies that

$$a_n \lesssim \frac{1}{\rho(n)}. \tag{3.3}$$

Let $A > 1$. Since $x^{1/\gamma}/\rho^{-1}(x)$ is decreasing, we have $\rho^{-1}(x/A) \le A^{-1/\gamma}\rho^{-1}(x)$. Then

$$\sum_{a_n\ge A\delta} a_n \lesssim \frac{A\delta}{B}\,\rho^{-1}\Big(\frac{2B}{A\delta}\Big) \lesssim \Big(\frac{1}{A}\Big)^{(1-\gamma)/\gamma} \frac{\delta}{B}\,\rho^{-1}\Big(\frac{2B}{\delta}\Big)$$

Then, for sufficiently large A, we have

$$\sum_{a_n\ge\delta} a_n \asymp \sum_{\delta\le a_n\le A\delta} a_n.$$

Using the left inequality of (3.2), we get

$$\delta\rho^{-1}(1/2B\delta) \lesssim \sum_{\rho(n)\le 1/(2B\delta)} \frac{1}{\rho(n)} \lesssim \sum_{\delta\le a_n\le A\delta} a_n \lesssim A\delta N(\delta).$$

In particular, for $\delta = a_n$, we obtain $\rho^{-1}(1/2Ba_n) \lesssim n$. Then

$$\frac{1}{\rho(n)} \lesssim a_n. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain $a_n \asymp 1/\rho(n)$. ■

The following lemma will be used in Section 5.

Lemma 3.2. *Let $(a_n)_{n\geq 1}$ be a positive decreasing sequence. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing positive function. Suppose that there exist $\beta > 1$ and $\gamma > 1$ such that $\rho(t)/t^\gamma$ is increasing and $\rho(t)/t^\beta$ is decreasing.*

Let $p \in (0, 1/\beta)$ and suppose that there exists $B > 0$ such that for every increasing concave function h satisfying that $h(t)/t^p$ is increasing, we have

$$\frac{1}{B}\sum_{n\geq 1} h\Big(\frac{1}{\rho(n)}\Big) \leq \sum_{n\geq 1} h(a_n) \leq B \sum_{n\geq 1} h\Big(\frac{1}{\rho(n)}\Big). \tag{3.5}$$

Then $a_n \asymp 1/\rho(n)$.

Proof. Let $\delta > 0$ and let h be the concave function given by

$$h(t) = \begin{cases} t, & t \in (0,\delta), \\ \delta^{1-p} t^p, & t \geq \delta. \end{cases}$$

Clearly $h(t)/t^p$ is increasing. Then (3.5) implies

$$\begin{aligned} \frac{1}{B}\Big(\sum_{\rho(n)>1/\delta} \frac{1}{\rho(n)} + \delta^{1-p} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p}\Big) &\leq \sum_{a_n<\delta} a_n + \delta^{1-p} \sum_{a_n\geq\delta} a_n^p \\ &\leq B\Big(\sum_{\rho(n)>1/\delta} \frac{1}{\rho(n)} + \delta^{1-p} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p}\Big). \end{aligned} \tag{3.6}$$

Now we proceed as in the proof of Lemma 3.1. Let $N(\delta) = \mathrm{Card}\{n : a_n \geq \delta\}$. It is clear that $\delta N(\delta) \leq \delta^{1-p} \sum_{a_n\geq\delta} a_n^p$. Then,

$$\delta N(\delta) \lesssim \sum_{\rho(n)>1/\delta} \frac{1}{\rho(n)} + \delta^{1-p} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p}.$$

Using the fact that $\rho(n)/n^\gamma$ is increasing and $\rho(n)^p/n^{p\beta}$ is decreasing with $\gamma > 1$ and $p\beta \in (0,1)$, we get

$$\sum_{\rho(n)\geq 1/\delta} \frac{1}{\rho(n)} \asymp \delta\rho^{-1}(1/\delta) \quad \text{and} \quad \delta^{1-p} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p} \asymp \delta\rho^{-1}(1/\delta).$$

Then $\delta N(\delta) \lesssim \delta\rho^{-1}(1/\delta)$, which implies that $a_n \lesssim 1/\rho(n)$.

For the reverse inequality, we repeat the argument used in the proof of Lemma 3.1. Indeed, one can verify that

$$\sum_{a_n<\delta/K} a_n + \delta^{1-p} \sum_{a_n\geq K\delta} a_n^p \leq C(K)\,\delta\,\rho^{-1}(1/\delta), \quad \text{with } \lim_{K\to\infty} C(K)=0.$$

So, from (3.6) we obtain

$$\begin{aligned}\frac{\delta^{1-p}}{B} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p} &\leq \sum_{a_n<\delta} a_n + \delta^{1-p} \sum_{a_n\geq\delta} a_n^p \\ &\leq \sum_{a_n<\delta/K} a_n + \delta^{1-p} \sum_{a_n\geq K\delta} a_n^p + \sum_{\delta/K\leq a_n<\delta} a_n + \delta^{1-p} \sum_{\delta\leq a_n<K\delta} a_n^p \\ &\leq C(K)\,\delta\,\rho^{-1}(1/\delta) + C(K,p)\,\delta\,N(\delta/K),\end{aligned}$$

Taking into account that

$$\delta^{1-p} \sum_{\rho(n)\leq 1/\delta} \frac{1}{\rho(n)^p} \asymp \delta\,\rho^{-1}(1/\delta),$$

we get, for large K, that $\rho^{-1}(1/\delta) \lesssim N(\delta/K)$, which implies that $1/\rho(n) \lesssim a_n$. This completes the proof. ∎

4. Estimates of the trace of $h(T_\mu)$

4.1. The convex case

The following result is implicitly proved in [11]. Here we give a direct and short proof. Recall that the sequence $(a_n(\mu))_n$ is given by equation (2.10).

Theorem 4.1. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact on $\mathcal{A}^2_\omega$. Let h be a convex increasing function such that $h(0)=0$. We have*

$$\sum_n h\Big(\frac{1}{B} a_n(\mu)\Big) \leq \sum_n h(\lambda_n(T_\mu)) \leq \sum_n h(B a_n(\mu)),$$

where B is a positive constant which depends on ω and (R_n).

Proof. We will use here the same notations as in Lemma 2.6. Since $\lim_{n\to\infty} a_n(\nu)=0$, T_ν is a compact operator, see [11]. Let $(f_n)_{n\geq 1}$ be an orthonormal basis of eigenfunctions of T_ν. We have

$$\begin{aligned}\sum_{n\geq 1} h(\lambda_n(T_\mu)) \leq \sum_{n\geq 1} h(C\lambda_n(T_\nu)) &= \sum_{n\geq 1} h(C\langle T_\nu f_n, f_n\rangle) \\ &= \sum_{n\geq 1} h\Big(C\int_\Omega |f_n(z)|^2\,\omega^2(z)\,d\nu(z)\Big) \\ &\leq \sum_{n\geq 1} h\Big(\sum_k NCa_k(\mu)\int_{R_k(\mu)} |f_n(z)|^2\,dA_\omega(z)\Big),\end{aligned}$$

where N is the multiplicity of $(bR_n)_n$.

Since h is convex, by Jensen's inequality, we get

$$\begin{aligned}\sum_{n\geq 1} h(\lambda_n(T_\mu)) &\leq \sum_{n,k} h(N^2 C a_k(\mu)) \frac{1}{N} \int_{R_k(\mu)} |f_n(z)|^2 \, dA_\omega(z) \\ &= \sum_{k} h(N^2 C a_k(\mu)) \frac{1}{N} \int_{R_k(\mu)} \sum_n |f_n(z)|^2 \, dA_\omega(z) \\ &\lesssim \sum_{k} h(N^2 C a_k(\mu)) \int_{R_k(\mu)} \|K_z\|^2 \, dA_\omega(z) \lesssim \sum_k h(N^2 C a_k(\mu)).\end{aligned}$$

Conversely, let $\bar{\mu}_j = \mu_{|R_j(\mu)}$ and put $\bar{\mu} = \sum_{j\geq 1} \bar{\mu}_j$. We have $T_{\bar{\mu}} \leq N T_\mu$. Note also that since $T_{\bar{\mu}}$ is a positive compact operator, $\lambda_1(T_{\bar{\mu}}) = \|T_{\bar{\mu}}\|$. So, by Lemma 2.6 and Lemma 2.2, we have

$$\begin{aligned}\mathrm{Tr}(h(T_\mu)) &\geq \mathrm{Tr}\Big(h\Big(\frac{1}{N} T_{\bar{\mu}}\Big)\Big) \geq \sum_{j\geq 1} \mathrm{Tr}\Big(h\Big(\frac{1}{N} T_{\bar{\mu}_j}\Big)\Big) \geq \sum_{j\geq 1} h\Big(\frac{1}{N}\lambda_1(T_{\bar{\mu}_j})\Big) \\ &= \sum_{j\geq 1} h\Big(\frac{1}{N}\|T_{\bar{\mu}_j}\|\Big) \geq \sum_{j\geq 1} h\Big(\frac{1}{N}\Big\langle T_{\bar{\mu}_j} \frac{K_{z_j}}{\|K_{z_j}\|}, \frac{K_{z_j}}{\|K_{z_j}\|}\Big\rangle\Big),\end{aligned}$$

where z_j is the center of $R_j(\mu)$.

Now, since

$$\Big\langle T_{\bar{\mu}_j} \frac{K_{z_j}}{\|K_{z_j}\|}, \frac{K_{z_j}}{\|K_{z_j}\|}\Big\rangle = \int_{R_j(\mu)} \Big|\frac{K_{z_j}(\zeta)}{\|K_{z_j}\|}\Big|^2 \omega^2(z)\, d\mu(z) \asymp a_j(\mu),$$

there exists $C > 0$ such that

$$\mathrm{Tr}(h(T_\mu)) \geq \sum_{j\geq 1} h\Big(\frac{1}{N}\Big\langle T_{\bar{\mu}_j} \frac{K_{z_j}}{\|K_{z_j}\|}, \frac{K_{z_j}}{\|K_{z_j}\|}\Big\rangle\Big) \geq \sum_{j\geq 1} h\Big(\frac{1}{NC} a_j\Big).$$

This ends the proof. ∎

4.2. The concave case

Theorem A will be obtained from the following result.

Theorem 4.2. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact on $\mathcal{A}^2_\omega$. Let h be a concave increasing function such that $h(0) = 0$. We have*

$$\frac{1}{B}\sum_{n\geq 1} h(a_n(\mu)) \leq \sum_{n\geq 1} h(\lambda_n(T_\mu)) \leq B \sum_{n\geq 1}\sum_{k\geq 0} h(a_n(\mu) e^{-\gamma k}).$$

In addition, if $h(t)/t^p$ is increasing for some $p \in (0, 1)$, then

$$\sum_{n\geq 1} h(\lambda_n(T_\mu)) \leq \frac{B}{p} \sum_{n\geq 1} h(a_n(\mu)),$$

where $B, \gamma > 0$ are constants which depend only on ω and (R_n).

We will need the following lemma in the proof of Theorem 4.2.

Lemma 4.3. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let $\nu_n = dA_{|R_n}$. Then T_{ν_n} is compact on $\mathcal{A}^2_\omega$ and*

$$\lambda_k(T_{\nu_n}) \le Be^{-\gamma k},$$

where $B, \gamma > 0$ depend on ω and (R_n).

Proof. Following the same proof of Theorem 3.8 of [23], there exist $B > 0$ and $\delta \in (0,1)$ such that

$$\|T_{\nu_n}\|_p^p \le B(1-\delta^p)^{-1} \le \frac{C(\delta, B)}{p}, \quad \text{for } p \in (0, 1/2).$$

This implies that $k\lambda_k^p(T_{\nu_n}) \le C/p$, where $C = C(\delta, B)$. Then, for $1/p = k/(eC)$, we obtain the desired result, with $\gamma = 1/(eC)$. ■

Proof of Theorem 4.2. We use the same notations as in Lemma 2.6. Let $\tilde{\mu}$ be the positive Borel measure given by

$$d\tilde{\mu} = \sum_k \frac{1}{NKa_k(\mu)} \mu_{|R_k(\mu)}, \quad \text{where } K > 0,$$

and with the convention

$$\frac{1}{NKa_k(\mu)} \mu_{|R_k(\mu)} = 0 \quad \text{if } a_k(\mu) = 0.$$

By Lemma 2.6, $T_{\tilde{\mu}}$ is bounded and

$$\|T_{\tilde{\mu}}\| \le C \sup_n \frac{\tilde{\mu}(R_n)}{A(R_n)}.$$

Note that if $R_n \cap R_k \neq \emptyset$ then $A(R_n) \asymp A(R_k)$. We have

$$\begin{aligned}
\frac{\tilde{\mu}(R_n)}{A(R_n)} &= \frac{1}{NK} \sum_{k:\ R_k \cap R_n \neq \emptyset} \frac{\mu(R_n \cap R_k)}{a_k(\mu) A(R_n)} \\
&\le \frac{1}{NK} \sum_{k:\ R_k \cap R_n \neq \emptyset} \frac{\mu(R_k)}{a_k(\mu) A(R_n)} \\
&\asymp \frac{1}{NK} \sum_{k:\ R_k \cap R_n \neq \emptyset} \frac{\mu(R_k)}{a_k(\mu) A(R_k)} \\
&\lesssim \frac{1}{K}.
\end{aligned}$$

Then, for large K, we have

$$\|T_{\tilde{\mu}}\| \le 1.$$

Let $(f_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{A}^2_\omega$ of eigenfunctions of T_μ. We have

$$\begin{aligned}
\sum_{n\geq 1} h(\lambda_n(T_\mu)) &= \sum_{n\geq 1} h(\langle T_\mu f_n, f_n\rangle) = \sum_{n\geq 1} h\Big(\int_\Omega |f_n(z)|^2\omega^2(z)\,d\mu(z)\Big) \\
&\geq \sum_{n\geq 1} h\Big(\frac{1}{N}\sum_k \int_{R_k(\mu)} |f_n(z)|^2\omega^2(z)\,d\mu(z)\Big) \\
&= \sum_{n\geq 1} h\Big(\sum_k CKa_k(\mu)\int_{R_k(\mu)} |f_n(z)|^2\omega^2(z)\,d\tilde{\mu}(z)\Big) \\
&\geq \sum_{n\geq 1}\sum_k h(CKa_k(\mu))\int_{R_k(\mu)} |f_n(z)|^2\omega^2(z)\,d\tilde{\mu}(z) \\
&= \sum_k h(CKa_k(\mu))\sum_{n\geq 1}\int_{R_k(\mu)} |f_n(z)|^2\omega^2(z)\,d\tilde{\mu}(z) \\
&= \sum_k h(CKa_k(\mu))\int_{R_k(\mu)} \|K_z\|^2\omega^2(z)\,d\tilde{\mu}(z) \asymp \sum_k h(a_k(\mu)),
\end{aligned}$$

which gives the first inequality.

Let $\bar{\mu}_j = \mu_{|R_j(\mu)}$. Since $\mu \leq \sum_{j=1}^\infty \bar{\mu}_j$, by Lemma 2.6 and Lemma 2.2 we have

$$\mathrm{Tr}(h(T_\mu)) \leq \mathrm{Tr}\Big(h\Big(\sum_{j=1}^\infty T_{\bar{\mu}_j}\Big)\Big) \leq \sum_{j=1}^\infty \mathrm{Tr}(h(T_{\bar{\mu}_j})).$$

By Lemma 2.6, $T_{\bar{\mu}_j} \leq CT_{\nu_j}$, where $d\nu_j = a_j(\mu)dA_{|bR_j(\mu)}$. Then by Lemma 4.3, we have

$$\mathrm{Tr}(h(T_{\bar{\mu}_j})) \leq \mathrm{Tr}(h(CT_{\nu_j})) \lesssim \sum_{k=1}^\infty h\big(Ca_j(\mu)e^{-\gamma k}\big) \lesssim \sum_{k=1}^\infty h\big(a_j(\mu)e^{-\gamma k}\big).$$

Then we obtain the second inequality of Theorem 4.2.

Now we prove the last inequality of Theorem 4.2 . Since $h(t)/t^p$ is increasing, we have

$$h(a_j(\mu)e^{-\gamma k}) \leq h(a_j(\mu))e^{-p\gamma k}.$$

It implies that

$$\sum_{j=1}^\infty\sum_{k=1}^\infty h\big(a_j(\mu)e^{-\gamma k}\big) \lesssim \sum_{j=1}^\infty\sum_{k=1}^\infty h(a_j(\mu))e^{-p\gamma k} \asymp \frac{1}{p}\sum_{j=1}^\infty h(a_j(\mu)).$$

Then we get

$$\sum_{n\geq 1} h(\lambda_n(T_\mu)) = \mathrm{Tr}(h(T_\mu)) \lesssim \frac{1}{p}\sum_{j=1}^\infty h(a_j(\mu)),$$

and the proof is complete. ∎

4.3. Remarks

(1) It is proved by P. Lin and R. Rochberg in [23] that if $\omega \in \mathcal{D}$ then for all $p \geq 1$, $T_\mu \in S_p$ if and only if $(a_n(\mu))_n \in \ell^p$. They also proved, for $p \in (0,1)$, that if $(a_n(\mu))_n \in \ell^p$ then $T_\mu \in S_p$. Since $\mathcal{D} \subset \mathcal{W}$, it is clear from Theorem 4.2 that the converse is also true. (See [2] for radial weights).

(2) For $\omega \in \mathcal{R}$, the class of weights introduced by K. Seip and E. H. Youssfi in [31], Theorem 4.2 completes the characterization of membership to Schatten classes given in [31].

(3) The factor $1/p$ in Theorem 4.2 can not be replaced by $1/p^{1-\varepsilon}$. Indeed, let $\Omega = \mathbb{D}$ and let $\omega = 1$. Suppose that

$$\sum_{n\geq 1} \lambda_n^p(T_\mu) \leq \frac{B}{p^{1-\varepsilon}} \sum_{n\geq 1} a_n^p(\mu), \quad \text{(for all } p > 0),$$

for every positive Borel measure μ on $\mathbb{D}$. Then if μ is of compact support, we have

$$n\lambda_n^p(T_\mu) \leq \sum_{j\geq 1} \lambda_j^p(T_\mu) \leq \frac{C}{p^{1-\varepsilon}}, \quad \text{(for all } p \in (0,1)),$$

for some constant $C > 0$. This implies that

$$\lambda_n(T_\mu) \leq e^{-Kn^{1/(1-\varepsilon)}}.$$

Now, for $d\mu = dA_{|D(0,\delta)}$, where $\delta \in (0,1)$, we have

$$\lambda_n(T_\mu) = 2\int_0^\delta r^{2n+1} dr \asymp \frac{1}{n+1}\delta^{2n+2},$$

which gives a contradiction.

4.4. Proof of Theorem A

The following corollary is somewhat more general than Theorem A.

Corollary 4.4. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact on $\mathcal{A}^2_\omega$. For $p \in (0,1)$ and for all $\varepsilon > 0$, we have*

$$p^{1+\varepsilon} C_1 \sum_{j=1}^n a_j^p(\mu) \leq \sum_{j=1}^n \lambda_j^p(T_\mu) \leq \frac{C_2}{p} \sum_{j=1}^n a_j^p(\mu),$$

where C_1 is a positive constant which depends on ε, ω and (R_n), and C_2 is a positive constant which depends on ω and (R_n).

Proof. Applying Theorem 4.2 to T_{μ_n} and taking into account the multiplicity of (R_n), there exists $B_1 > 0$, which depends only on ω and (R_n), such that

$$\sum_{j=1}^n \lambda_j^p(T_{\mu_n}) \leq \frac{B_1}{p} \sum_{j=1}^n a_j^p(\mu).$$

By Lemma 2.6,

$$\lambda_j(T_\mu) \le \lambda_j(T_{\mu_n}) + Ca_{n+1}(\mu).$$

We obtain

$$\sum_{j=1}^{n} \lambda_j^p(T_\mu) \le \frac{B_1}{p} \sum_{j=1}^{n} (a_j^p(\mu) + C^p a_{n+1}^p(\mu)) \le \frac{B_1 C'}{p} \sum_{j=1}^{n} a_j^p(\mu).$$

Conversely, let $q \in (0, 1)$. By Theorem 4.2, applied to T_{μ_n}, we have

$$\sum_{j=1}^{\infty} \lambda_j^{pq}(T_{\mu_n}) \le \frac{B_1}{pq} \sum_{j=1}^{n} a_j^{pq}(\mu).$$

Then

$$\lambda_j^p(T_{\mu_n}) \le \Big(\frac{B_1}{jpq} \sum_{k=1}^{n} a_k^{pq}(\mu)\Big)^{1/q} \le \Big(\frac{B_1}{q}\Big)^{1/q} \frac{n^{1/q-1}}{p^{1/q} j^{1/q}} \sum_{k=1}^{n} a_k^p(\mu).$$

Then, for $A > 0$ we have

$$\sum_{j \ge An+1} \lambda_j^p(T_{\mu_n})) \le \frac{C(B_1, q)}{p^{1/q} A^{1/q-1}} \sum_{k=1}^{n} a_k^p(\mu).$$

Once again, by Theorem 4.2, we have

$$\frac{1}{B} \sum_{j=1}^{n} a_j^p(\mu) \le \sum_{j=1}^{\infty} \lambda_j^p(T_{\mu_n}) \le \sum_{j=1}^{An} \lambda_j^p(T_{\mu_n}) + \sum_{j=An+1}^{\infty} \lambda_j^p(T_{\mu_n})$$
$$\le A \sum_{j=1}^{n} \lambda_j^p(T_{\mu_n}) + \frac{C(B_1, q)}{p^{1/q} A^{1/q-1}} \sum_{j=1}^{n} a_j^p(\mu) \le A \sum_{j=1}^{n} \lambda_j^p(T_\mu) + \frac{C(B_1, q)}{p^{1/q} A^{1/q-1}} \sum_{j=1}^{n} a_j^p(\mu).$$

For $q = \frac{1}{1+\varepsilon}$ and for A big enough we obtain the result. ■

Now we can state the following important consequence of Corollary 4.4.

Theorem 4.5. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on Ω such that T_μ is compact on $\mathcal{A}^2_\omega$. Let h be an increasing function on $[0, +\infty)$ such that $h(0) = 0$ and $h(t^p)$ is convex for some $p > 0$. We have*

$$\sum_{j=1}^{n} h\Big(\frac{1}{B} a_j(\mu)\Big) \le \sum_{j=1}^{n} h(\lambda_j(T_\mu)) \le \sum_{j=1}^{n} h(Ba_j(\mu)), \quad \text{for } n \ge 1,$$

where $B > 0$ is a positive constant which depends on ω, (R_n) and p.

Proof. This is a consequence of Lemma 2.3 and Corollary 4.4. ■

Proof of Theorem A. We prove (1). Suppose that $a_n(\mu) = O(1/\rho(n))$. Let $p \in (0,1)$ such that $pA < 1$. We have

$$\sum_{k=1}^{n} \frac{1}{\rho^p(k)} \asymp n/\rho^p(n).$$

By Corollary 4.4, since $a_n(\mu) = O(1/\rho(n))$, we obtain

$$n\lambda_n^p(T_\mu) \le \sum_{k=1}^{n} \lambda_k^p(T_\mu) \lesssim \sum_{k=1}^{n} a_k^p(\mu) \lesssim n/\rho^p(n).$$

This implies that $\lambda_n(T_\mu) = O(1/\rho(n))$. The reverse implication is obtained in the same way.

The second assertion comes from Theorem 4.5 and Lemma 3.1. ∎

4.5. Remarks on Theorem A

In this section we provide two examples. The first one shows that the condition $\rho(x)/x^A$ is decreasing, for some $A > 0$, is necessary and sharp. And in the second example we show that the sequence $(a_n(\mu))$ is not sufficient, in general, to describe the asymptotic behavior of the eigenvalues of T_μ.

(1) The conclusion of Theorem A is not valid if ρ increases faster than all polynomials. Namely, suppose that

$$\lim_{x\to+\infty} \frac{\rho(2x)}{\rho(x)} = +\infty. \tag{4.1}$$

Let

$$d\mu(re^{it}) = \frac{1}{\rho(1/(1-r))}\, r\,dr\,dt. \tag{4.2}$$

The Toeplitz operator T_μ defined on the unweighted Bergman space $\mathcal{A}^2(\mathbb{D})$ is compact. Since μ is radial, it is easy to see that $f_n = (n+1)^{1/2} z^n$ is an eigenfunction of T_μ and for all $M > 1$ we have

$$\begin{aligned}\lambda_n(T_\mu) &= 2\pi \int_0^1 r^{2n+1} \frac{1}{\rho(1/(1-r))}\, dr \\ &\ge 2\pi \int_0^{1-M/n} r^{2n+1} \frac{1}{\rho(1/(1-r))}\, dr \ge \frac{C(M)}{\rho(n/M)}.\end{aligned}$$

For p integer, let $(R_{n,j}(p))$ denote the p-adic decomposition of $\mathbb{D}$: for $0 \le j < p^{n+1}$,

$$R_{n,j}(p) = \Big\{ z \in \mathbb{D}\,;\ 1 - \frac{1}{p^n} \le |z| < 1 - \frac{1}{p^{n+1}} \text{ and } \frac{2j\pi}{p^{n+1}} \le \arg z < \frac{2(j+1)\pi}{p^{n+1}} \Big\}.$$

Let $(R_n)_n$ be a lattice of $\mathcal{A}^2(\mathbb{D})$. It is clear that for p big enough, then for all n there exists (k,j) such that $R_{k,j}(p) \subset R_n$. Note also that we have $A(R_{k,j}(p)) \asymp A(R_n)$ if $R_{k,j}(p)$ is maximal in R_n. Then we obtain

$$a_n(\mu) \lesssim a'_n(\mu), \quad n \ge 1,$$

where $(a_n(\mu))_n$ (respectively, $a'_n(\mu)$) is the decreasing rearrangement of $(\mu(R_n)/A(R_n))_n$ (respectively, $(\mu(R_{n,j}(p))/A(R_{n,j}(p)))_n$). We have

$$\frac{\mu(R_{n,j}(p))}{A(R_{n,j}(p))} \lesssim \frac{1}{\rho(p^n)}.$$

Then we have

$$a_n(\mu) \lesssim a'_n(\mu) \lesssim \frac{1}{\rho(n/1+p)}, \quad n \geq 1.$$

Using (4.1), we obtain

$$\lim_{n\to\infty} \frac{\lambda(T_\mu)}{a_n(\mu)} = \infty.$$

This proves our assertion.

(2) Now, we construct two positive Borel measures μ and ν on $\mathbb{D}$ such that

$$a_n(\mu) = a_n(\nu) \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n(T_\mu)/\lambda_n(T_\nu) = \infty.$$

To this end, let μ be the measure given in (4.2) and let $(R_j)_j$ be the dyadic $(p = 2)$ decomposition of $\mathbb{D}$. Let $\delta > 0$ be small enough and let $w_j \in R_j$ be such that $D(w_j, \delta\tau(w_j)) \subset R_j$. There exists a subsequence $(w_{j_n})_n$ which is an interpolating separated sequence of $\mathcal{A}^2 = \mathcal{A}^2_0$, see [30]. The sequence $(w_{j_n})_n$ satisfies

$$\left\| \sum c_n \frac{K_{w_n}}{\|K_{w_n}\|} \right\|^2 \asymp \sum |c_n|^2.$$

Let ν be the measure given by

$$\nu = \sum_{n\geq 1} c_n \delta_{w_{j_n}},$$

where $c_n = a_n(\mu)A(R_{j_n})$. Let $\nu_n = \sum_{k\geq n} c_k \delta_{w_{j_k}}$, we have

$$\lambda_n(T_\nu) \leq \|T_{\nu_n}\| \lesssim a_n(\mu).$$

This implies that $\liminf_{n\to\infty} \lambda_n(T_\nu)/\lambda_n(T_\mu) = 0$, while $a_n(\mu) = a_n(\nu)$.

5. The Berezin transform

5.1. Preliminaries

The Berezin transform of a bounded operator T acting on $\mathcal{A}^2_\omega$ is defined by

$$\tilde{T}(z) = \frac{\langle TK_z, K_z\rangle}{\|K_z\|^2}, \quad (z \in \Omega).$$

If T is positive and compact, then

$$\mathrm{Tr}(T) = \int_\Omega \tilde{T}(z) \frac{dA(z)}{\tau^2(z)}.$$

In particular, $T \in \mathcal{S}_1$ if and only if $\tilde{T} \in L^1(\Omega, dA/\tau^2)$.

The following general result is standard and well known (at least for $h(t) = t^p$, [36]).

Proposition 5.1. *Let T be a positive compact operator on $\mathcal{A}^2_\omega$. Let h be an increasing function such that $h(0) = 0$. We have*:

(1) *If h is convex, then*

$$\sum_n h(\lambda_n(T)) \geq \int_\Omega h(\tilde{T}(z)) \frac{dA(z)}{\tau^2(z)}.$$

(2) *If h is concave, then*

$$\sum_n h(\lambda_n(T)) \leq \int_\Omega h(\tilde{T}(z)) \frac{dA(z)}{\tau^2(z)}.$$

Proof. Let $(f_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{A}^2_\omega$ containing a maximal orthonormal system of eigenfunctions of T. Set $\lambda_n = \lambda_n(T)$ and write

$$\langle TK_z, K_z\rangle = \sum_n \lambda_n |\langle f_n, K_z\rangle|^2 = \sum_n \lambda_n |f_n(z)|^2.$$

If h is convex, then

$$\begin{aligned}\int_\Omega h(\tilde{T}(z)) \frac{dA(z)}{\tau^2_\omega(z)} &= \int_\Omega h\Big(\sum_n \lambda_n \frac{|f_n(z)|^2}{\|K_z\|^2}\Big) \frac{dA(z)}{\tau^2_\omega(z)} \leq \int_\Omega \sum_n h(\lambda_n) \frac{|f_n(z)|^2}{\|K_z\|^2} \frac{dA(z)}{\tau^2_\omega(z)} \\ &= \sum_n h(\lambda_n) \int_\Omega |f_n(z)|^2 \omega^2(z)\, dA(z) = \sum_n h(\lambda_n).\end{aligned}$$

The concave case is obtained in the same way. ∎

5.2. Trace estimates and consequences

Let $(R_n) \in \mathcal{L}_\omega$. In the sequel, z_n will denote the center of R_n. For the Toeplitz operator T_μ acting on $\mathcal{A}^2_\omega$, the Berezin transform of T_μ is denoted by $\tilde{\mu}$ and is given by (1.1). In this section we use the following notation:

$$\hat{\mu}(z_n) = \mu(R_n)/A(R_n).$$

Recall that $(a_n(\mu))_n$ is the decreasing rearrangement of $(\hat{\mu}(z_n))_n$. Our goal in this section is to estimate the eigenvalues of T_μ in terms of $\tilde{\mu}(z_n)$. To this end, by Lemma 3.1 and Lemma 3.2, it suffices to estimate $\mathrm{Tr}(h(T_\mu))$ in terms of $(h(\tilde{\mu}(z_n)))_{n\geq 1}$. For the convex case, we have the following result.

Theorem 5.2. *Let $\omega \in \mathcal{W}$. Let μ be a positive Borel measure on Ω and let h be a convex increasing function such that $h(0) = 0$. Then*

$$\sum_{n\geq 1} h\Big(\frac{1}{B}\tilde{\mu}(z_n)\Big) \leq \mathrm{Tr}(h(T_\mu)) \leq \sum_{n\geq 1} h(B\tilde{\mu}(z_n)),$$

where $B > 0$ does not depend on either μ or h.

Proof. By Theorem 4.1, we have

$$\sum_{n\geq 1} h\Big(\frac{1}{B_1}\hat{\mu}(z_n)\Big) \leq \mathrm{Tr}(h(T_\mu)) = \sum_{n\geq 1} h(\lambda_n(T_\mu)) \leq \sum_{n\geq 1} h(B_1\hat{\mu}(z_n)).$$

Since $\hat{\mu}(z_n) \lesssim \tilde{\mu}(z_n)$ (see Lemma 4.2 in [11]), we deduce that

$$\mathrm{Tr}(h(T_\mu)) \leq \sum_{n\geq 1} h(B\tilde{\mu}(z_n)).$$

On the other hand, by (2.11) we have

$$|K(z_n,\zeta)|^2\omega^2(z_n) \lesssim \frac{1}{A(R_n)}\int_{bR_n} |K(z,\zeta)|^2\,\omega^2(z)\,dA(z), \quad \zeta\in\Omega.$$

Since $\tau^2(z) = \frac{1}{\omega^2(z)\|K_z\|^2}$ and $A(R_n) \asymp \tau^2(z_n)$, we get

$$\frac{|K(z_n,\zeta)|^2}{\|K_{z_n}\|^2} \lesssim \int_{bR_n} \frac{|K(z,\zeta)|^2}{\|K_z\|^2}\,\frac{dA(z)}{\tau_\omega^2(z)}, \quad \zeta\in\Omega,$$

and

$$\begin{aligned}\tilde{\mu}(z_n) &= \int_\Omega \frac{|K(z_n,\zeta)|^2}{\|K_{z_n}\|^2}\,\omega^2(\zeta)\,dA(\zeta)\\ &\lesssim \int_\Omega\int_{bR_n} \frac{|K(z,\zeta)|^2}{\|K_z\|^2}\,\frac{dA(z)}{\tau_\omega^2(z)}\,\omega^2(\zeta)\,dA(\zeta) = \int_{bR_n} \tilde{\mu}(z)\,\frac{dA(z)}{\tau_\omega^2(z)}.\end{aligned}$$

Since h is convex, we obtain, for some $c>0$,

$$h(c\tilde{\mu}(z_n)) \leq \int_{bR_n} h(\tilde{\mu}(z))\,\frac{dA(z)}{\tau_\omega^2(z)}.$$

Taking into account that $(bR_n)_n$ is of finite multiplicity and using Theorem 5.1, we deduce

$$\sum_n h(c\tilde{\mu}(z_n)) \lesssim \int_\Omega h(\tilde{\mu}(z))\,\frac{dA(z)}{\tau_\omega^2(z)} \leq \sum_{n\geq 1} h(\lambda_n(T_\mu)) = \mathrm{Tr}(h(T_\mu)).$$

The proof is complete. ∎

Proof of Theorem B. By Theorem 5.2, we have

$$\sum_{n\geq 1} h\Big(\frac{1}{B}b_n(\mu)\Big) \leq \sum_{n\geq 1} h(\lambda_n(T_\mu)) \leq \sum_{n\geq 1} h(Bb_n(\mu)).$$

So, by Lemma 3.1 we have $b_n(\mu) \asymp 1/\rho(n)$ if and only if $\lambda_n(T_\mu) \asymp 1/\rho(n)$. ∎

Now, we turn to the concave case. Let $p>0$ and let $d\nu_n = dA_{|R_n}$. Recall that

$$C_p(\mathcal{A}^2_\omega,(R_n)) = \sup_{n\geq 1}\sum_{j\geq 1} \tilde{\nu}_n^p(z_j) \in [0,+\infty).$$

For the concave case we have the following result.

Theorem 5.3. *Let $\omega \in \mathcal{W}$, $(R_n)_n \in \mathcal{L}_\omega$ and $p \in (0,1)$. The following assertions are equivalent*:

(1) $C_p(\mathcal{A}^2_\omega, (R_n)_n) < \infty$.

(2) *There exists $B > 0$ such that for every Borel positive measure μ on Ω and every increasing concave function h such that $h(t)/t^p$ is increasing, we have*

$$\frac{1}{B} \sum_{n \geq 1} h(\tilde{\mu}(z_n)) \leq \operatorname{Tr}(h(T_\mu)) \leq B \sum_{n \geq 1} h(\tilde{\mu}(z_n)).$$

Proof. The same argument as before proves that $\operatorname{Tr}(h(T_\mu)) \leq C_2 \sum_{n \geq 1} h(\tilde{\mu}(z_n))$.

By Lemma 4.3, there exists $B > 0$ such that $\operatorname{Tr}(T^p_{\nu_j}) \leq B/p$. So, it is obvious that the condition $C_1 \sum_{n \geq 1} h(\tilde{\mu}(z_n)) \leq \operatorname{Tr}(h(T_\mu))$, applied with $\mu = \nu_n$ and $h(t) = t^p$, gives that $C_p(\mathcal{A}^2_\omega, (R_n)_n) < \infty$.

Conversely, suppose that $C_p(\mathcal{A}^2_\omega, (R_n)_n) < \infty$. A standard computation gives

$$\tilde{\mu}(z) \lesssim \sum_{j \geq 1} \hat{\mu}(z_j) \Big(\int_{R_j} \frac{|K(z,\zeta)|^2}{\|K_z\|^2} \, \omega^2(\zeta) \, dA(\zeta) \Big) \lesssim \sum_{j \geq 1} \hat{\mu}(z_j) \, \tilde{\nu}_j(z).$$

Since h is concave and $h(t)/t^p$ is increasing, we have

$$h(\tilde{\mu}(z)) \lesssim \sum_{j \geq 1} h(\hat{\mu}(z_j)) \, \tilde{\nu}_j(z)^p.$$

Consequently,

$$\begin{aligned} \sum_{n \geq 1} h(\tilde{\mu}(z_n)) &\lesssim \sum_{j \geq 1} h(\hat{\mu}(z_j)) \sum_{n \geq 1} \tilde{\nu}_j(z_n)^p. \\ &\lesssim C_p(\mathcal{A}^2_\omega, (R_n)_n) \sum_{j \geq 1} h(\hat{\mu}(z_j)) \lesssim C_p(\mathcal{A}^2_\omega, (R_n)_n) \operatorname{Tr}(h(T_\mu)). \end{aligned}$$

The proof is complete. ∎

Theorem C is a direct consequence of the following result.

Theorem 5.4. *Let $\omega \in \mathcal{W}$ and $(R_n)_n$. Let μ be a positive Borel measure on Ω such that T_μ is compact. Let $p \in (0,1)$ be such that $C_p(\mathcal{A}^2_\omega, (R_n)_n)) < \infty$. Let $\rho \colon [1, +\infty) \to (0, +\infty[$ be an increasing positive function. Suppose that there exist $\beta \in (0, 1/p)$ and $\gamma > 1$ such that $\rho(t)/t^\gamma$ is increasing and $\rho(t)/t^\beta$ is decreasing. Then*

$$\lambda_n(T_\mu) \asymp 1/\rho(n) \iff b_n(\mu) \asymp 1/\rho(n).$$

Proof. By Theorem 5.3, we have

$$\frac{1}{B} \sum_{n \geq 1} h(b_n(\mu)) \leq \sum_{n \geq 1} h(\lambda_n(T_\mu) \leq B \sum_{n \geq 1} h(b_n(\mu)).$$

So, by Lemma 3.2 we have $b_n(\mu) \asymp 1/\rho(n)$ if and only if $\lambda_n(T_\mu) \asymp 1/\rho(n)$. ∎

5.3. Examples

Now, we give some examples.

(1) Standard Fock spaces. Let $\alpha > 0$. Let $\mathcal{F}_\alpha^2$ be the standard Fock space given by (2.7). First, recall that the Berezin transform of T_μ is given by

$$\tilde{\mu}(z) = \int_{\mathbb{C}} e^{-\alpha|z-\zeta|^2} d\mu(\zeta), \quad z \in \mathbb{C}.$$

For more information on Fock spaces, see [38].

We have $C_p(\mathcal{F}_\alpha^2, (R_n)_n) < \infty$. for all $p \in (0, 1)$. Indeed,

$$\begin{aligned}\sum_{j\geq 1} \tilde{\nu}_n^p(z_j) &\asymp \sum_n \Big(\int_{R_n} e^{-\alpha|z_j-\zeta|^2} dA(\zeta)\Big)^p \\ &\asymp \sum_n \Big(\int_{R_n} e^{-\alpha|\zeta|^2} dA(\zeta)\Big)^p = O(1/p).\end{aligned}$$

(2) Weighted analytic spaces. Let Ω be a subdomain of $\mathbb{C}$ and let $\omega \in \mathcal{W}$. Let $M > 0$. We say that $\omega \in \mathcal{W}_M$ if the reproducing kernel of $\mathcal{A}_\omega^2$ satisfies

$$|K(z,\zeta)| \leq C(M)\|K_z\|\|K_\zeta\|\Big(\frac{\min(\tau_\omega(z), \tau_\omega(\zeta))}{|z-\zeta|}\Big)^M. \tag{5.1}$$

We will denote $\mathcal{W}_\infty = \cap_{M>0}\mathcal{W}_M$. Examples of such weights can be found in [3, 16, 31].

Proposition 5.5. *Let $M > 1$ and let $\omega \in \mathcal{W}_M$. For every $(R_n) \in \mathcal{L}_\omega$, we have*

$$C_p(\mathcal{A}_\omega^2, (R_n)_n) < \infty, \quad \textit{for all } p > 1/M.$$

In particular, if $\omega \in \mathcal{W}_\infty$, then $C_p(\mathcal{A}_\omega^2, (R_n)_n) < \infty$, for all $p > 0$.

Proof. Let $p > 1/M$ and let z_n be the center of R_n. We have

$$\sum_{j\geq 1} \tilde{\nu}_n^p(z_j) = \sum_{j\geq 1} \Big(\int_{R_n} \frac{|K_{z_j}(\zeta)|^2}{\|K_{z_j}\|^2} dA_\omega(\zeta)\Big)^p.$$

Since $(B_\omega R_j)$ is of finite multiplicity,

$$\Lambda_n := \{j : B_\omega R_j \cap B_\omega R_n \neq \emptyset\}$$

is finite. Then

$$\begin{aligned}\sum_{j\in\Lambda_n} \Big(\int_{R_n} \frac{|K_{z_j}(\zeta)|^2}{\|K_{z_j}\|^2} dA_\omega(\zeta)\Big)^p &\lesssim \sum_{j\in\Lambda_n} \Big(\int_{R_n} \|K_\zeta\|^2 dA_\omega(\zeta)\Big)^p \\ &\lesssim \sum_{j\in\Lambda_n} \Big(\int_{R_n} \frac{1}{\tau_\omega^2(\zeta)} dA(\zeta)\Big)^p = O(1).\end{aligned}$$

On the other hand, since $Mp > 1$, let $a > 0$ such that $(M - a)p = 1$. We have

$$\begin{aligned}\sum_{j\notin\Lambda_n}\Big(\int_{R_n}\frac{|K_{z_j}(\zeta)|^2}{\|K_{z_j}\|^2}\,dA_\omega(\zeta)\Big)^p &\lesssim \sum_{j\notin\Lambda_n}\Big(\int_{R_n}\|K_\zeta\|^2\Big(\frac{\min(\tau_\omega(z_j),\tau_\omega(\zeta))}{|z_j-\zeta|}\Big)^{2M}dA_\omega(\zeta)\Big)^p\\ &\asymp \sum_{j\notin\Lambda_n}\frac{\tau_\omega(z_j)^{(2M-2a)p}\,\tau_\omega^{2ap}(z_n)}{|z_j-z_n|^{2Mp}}\\ &\lesssim \int_{\Omega\setminus R_n}\frac{\tau_\omega(\zeta)^{(2M-2a)p-2}\,\tau_\omega^{2ap}(z_n)}{|\zeta-z_n|^{2Mp}}\,dA(\zeta)\\ &\lesssim \int_{\Omega\setminus R_n}\frac{\tau_\omega^{2ap}(z_n)}{|\zeta-z_n|^{2ap+2}}\,dA(\zeta) = O(1),\end{aligned}$$

which implies that $C_p(\mathcal{A}^2_\omega,(R_n)_n) < \infty$, whenever $p > 1/M$. ■

(3) Standard Bergman spaces on $\mathbb{D}$. Let $\alpha > -1$, let $\omega_\alpha^2(z) = (1+\alpha)(1-|z|^2)^\alpha$ and let $\mathcal{A}^2_\alpha$ be the associated standard Bergman spaces. Recall that the kernel of $\mathcal{A}^2_\alpha$ is given by

$$K_z^\alpha(w) = \frac{1}{(1-z\overline{w})^{2+\alpha}}.$$

We have the following.

Proposition 5.6. *Let $(R_n)_n \in \mathcal{L}_{\omega_\alpha}$ and let $p \in (0,1)$. We have $C_p(\mathcal{A}^2_\alpha,(R_n)_n)) < \infty$ if and only if $p > \frac{1}{2+\alpha}$.*

Proof. Let $(R_n) \in \mathcal{L}_\omega$. We have

$$\tilde{\nu}_n(z) = \int_{R_n}\frac{(1-|z|^2)^{2+\alpha}}{|1-\bar{z}\zeta|^{4+2\alpha}}\,dA_\alpha(\zeta).$$

Then

$$\begin{aligned}\sum_{j\ge1}\tilde{\nu}_n^p(z_j) &= \sum_{j\ge1}\Big(\int_{R_n}\frac{(1-|z_j|^2)^{2+\alpha}}{|1-\bar{z}_j\zeta|^{4+2\alpha}}\,dA_\alpha(\zeta)\Big)^p\\ &\asymp \sum_{j\ge1}\Big(\frac{(1-|z_j|^2)^{2+\alpha}(1-|z_n|^2)^{2+\alpha}}{|1-\bar{z}_j z_n|^{4+2\alpha}}\Big)^p\\ &\asymp \int_{\mathbb{D}}\Big(\frac{(1-|w|^2)^{2+\alpha}(1-|z_n|^2)^{2+\alpha}}{|1-\bar{w}z_n|^{4+2\alpha}}\Big)^p\frac{dA(w)}{(1-|w|^2)^2}\\ &\asymp \int_{\mathbb{D}}\frac{(1-|z_n|^2)^{(2+\alpha)p}}{|1-\bar{w}z_n|^{(4+2\alpha)p}}\,\frac{dA(w)}{(1-|w|^2)^{2-(2+\alpha)p}}.\end{aligned}$$

Then, the last integral is uniformly finite if and only if $p > \frac{1}{2+\alpha}$ (see Lemma 2 in [5]). ■

Proposition 5.6 implies that the Berezin transform is not sufficient to describe the behavior of the eigenvalues of Toeplitz operators. In what follows, we consider a modified Berezin transform which is more appropriate to our problem in this case (see for instance [34] and [26]).

Let T be a bounded operator on $\mathcal{A}^2_\alpha$ and let $s > -1$. The modified Berezin transform, $B_{\alpha,s}(T)$, of T is given by

$$B_{\alpha,s}(T)(z) = \frac{\langle TK^s_z, K^s_z\rangle}{\|K^s_z\|^2_\alpha}.$$

Let $\tau(z) = (1 - |z|^2)$. We have the following general result.

Proposition 5.7. *Let α and let s such that $s > (\alpha - 1)/2$. Let T be a positive compact operator on $\mathcal{A}^2_\alpha$. We have:*

(1) $\mathrm{Tr}(T) \asymp \displaystyle\int_{\mathbb{D}} B_{\alpha,s}(T)(z)\,\frac{dA(z)}{\tau^2(z)}.$

(2) *Let h be a concave function such that $h(0) = 0$. Then*

$$\mathrm{Tr}(h(T)) \lesssim \int_{\mathbb{D}} h\left(B_{\alpha,s}(T)(z)\right)\frac{dA(z)}{\tau^2(z)}.$$

(3) *Let h be a convex function such that $h(0) = 0$. Then*

$$\int_{\mathbb{D}} h\left(B_{\alpha,s}(T)(z)\right)\frac{dA(z)}{\tau^2(z)} \lesssim \mathrm{Tr}(h(T)).$$

All the implied constants depend on α and s.

Proof. Let $f = \sum_{n\geq 0} a_n z^n \in \mathcal{A}^2_\alpha$. Write $K^s_z(\zeta) = \sum_{n\geq 0} c_n(s)\,\overline{z}^n\,\zeta^n$. It is known that

$$c_n(s) \asymp (1+n)^{1+s}.$$

This implies that

$$\|K^s_{re^{it}}\|^2_\alpha = \|K^s_r\|^2_\alpha \asymp \frac{1}{(1-r)^{2+2s-\alpha}}.$$

Then we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\langle f, K^s_z\rangle|^2}{\|K^s_z\|^2_\alpha}\frac{dA(z)}{\tau^2(z)} &= \int_0^1 \Big(\int_0^{2\pi} |\langle f, K^s_{re^{it}}\rangle|^2\,\frac{dt}{2\pi}\Big)\frac{2r\,dr}{\|K^s_r\|^2_\alpha\,\tau^2(r)}\\ &\asymp \int_0^1 \Big(\sum_{n\geq 0}\frac{|a_n|^2\, r^{2n}\, c^2_n(s)}{(1+n)^{2+2\alpha}}\Big)(1-r)^{2s-\alpha}\,r\,dr\\ &\asymp \sum_{n\geq 0}\frac{|a_n|^2}{(1+n)^{2\alpha-2s}}\int_0^1 r^{2n+1}(1-r)^{2s-\alpha}dr \asymp \|f\|^2_\alpha.\end{aligned}$$

Let $(f_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{A}^2_\alpha$ containing a maximal orthonormal system of eigenfunctions of T. Write

$$\langle TK^s_z, K^s_z\rangle = \sum_n \lambda_n\,|\langle f_n K^s_z\rangle|^2, \quad (\lambda_n(T) = \lambda_n).$$

Then,

$$\int_{\mathbb{D}} B_{\alpha,s}(T)(z)\,\frac{dA(z)}{\tau^2(z)} = \sum_n \lambda_n \int_{\mathbb{D}} \frac{|\langle f_n, K^s_z\rangle|^2}{\|K^s_z\|^2_\alpha}\frac{dA(z)}{\tau^2(z)} \asymp \sum_n \lambda_n = \mathrm{Tr}(T).$$

To prove 2), suppose that h is concave. Then,

$$\begin{aligned}\int_{\mathbb{D}} h(B_{\alpha,s}(T)(z))\,\frac{dA(z)}{\tau^2(z)} &= \int_{\mathbb{D}} h\Big(\sum_n \lambda_n \frac{|\langle f_n, K_z^s\rangle|^2}{\|K_z^s\|_\alpha^2}\Big)\,\frac{dA(z)}{\tau^2(z)}\\ &\gtrsim \int_{\mathbb{D}} \sum_n h(\lambda_n)\,\frac{|\langle f_n, K_z^s\rangle|^2}{\|K_z^s\|_\alpha^2}\,\frac{dA(z)}{\tau^2(z)}\\ &= \sum_n h(\lambda_n) \int_{\mathbb{D}} \frac{|\langle f_n, K_z^s\rangle|^2}{\|K_z^s\|_\alpha^2}\,\frac{dA(z)}{\tau^2(z)} \asymp \sum_n h(\lambda_n).\end{aligned}$$

The convex case is obtained in the same way. ∎

Lemma 5.8. *Let $\alpha > -1$, and let $(R_n)_n \in \mathcal{L}_{\omega_\alpha}$. Let h be a concave function such that $h(t)/t^p$ is increasing for some $p \in (0,1)$. Let $s > \frac{1+p\alpha-2p}{2p}$ and let μ be a positive Borel measure on $\mathbb{D}$. Then,*

$$\operatorname{Tr} h(T_\mu) \asymp \sum_{n\geq 1} h(B_{\alpha,s}(T_\mu)(z_n)),$$

where the implied constants depend on α, s, p and $(R_n)_n$.

Proof. By Proposition 5.7 we have

$$\operatorname{Tr}(h(T_\mu)) \lesssim \int_{\mathbb{D}} h\left(B_{\alpha,s}(T_\mu)(z)\right)\frac{dA(z)}{\tau^2(z)} \asymp \sum_{n\geq 1} h(B_{\alpha,s}(T_\mu)(z_n)).$$

Conversely, by Theorem 4.2 we have

$$\operatorname{Tr} h(T_\mu) \asymp \sum_n h(a_n(\mu)).$$

So, it suffices to verify that

$$\int_{\mathbb{D}} h(B_{\alpha,s}(z))\,\frac{dA(z)}{\tau^2(z)} \lesssim \sum_n h(a_n(\mu)).$$

Let $\nu = \sum_n a_n(\mu)\, dA_{\alpha|bR_n}$. By Lemma 2.6, $T_\mu \lesssim T_\nu$. Then

$$\langle T_\mu K_z^s, K_z^s\rangle \lesssim \langle T_\nu K_z^s, K_z^s\rangle = \sum_n a_n(\mu) \int_{bR_n} |K_z^s(w)|^2\, dA_\alpha(w).$$

Using the concavity of h, we get

$$\begin{aligned}\int_{\mathbb{D}} h\left(B_{\alpha,s}(T_\mu)(z)\right)\frac{dA(z)}{\tau^2(z)} &= \int_{\mathbb{D}} h\Big(\frac{\langle T_\mu K_z^s, K_z^s\rangle}{\|K_z^s\|_\alpha^2}\Big)\,\frac{dA(z)}{\tau^2(z)}\\ &\lesssim \int_{\mathbb{D}} h\Big(\sum_n a_n(\mu)\int_{bR_n} \frac{|K_z^s(w)|^2}{\|K_z^s\|^2}\, dA_\alpha(w)\Big)\,\frac{dA(z)}{\tau^2(z)}\\ &\lesssim \int_{\mathbb{D}} \sum_n h\Big(a_n(\mu)\int_{bR_n} \frac{|K_z^s(w)|^2}{\|K_z^s\|^2}\, dA_\alpha(w)\Big)\,\frac{dA(z)}{\tau^2(z)}.\end{aligned}$$

On the other hand, we have

$$\int_{bR_n} \frac{|K_z^s(\zeta)|^2}{\|K_z^s\|_\alpha^2}\, dA_\alpha(\zeta) \asymp \int_{bR_n} \frac{(1-|z|^2)^{2+2s-\alpha}}{|1-\overline{z}\zeta|^{4+2s}}\, dA_\alpha(\zeta)$$
$$\asymp \frac{(1-|z|^2)^{2+2s-\alpha}}{|1-\overline{z}z_n|^{4+2s}}\,(1-|z_n|^2)^{2+\alpha}.$$

Using the assumption $h(t)/t^p$ is increasing, we get

$$h\Big(a_n(\mu)\int_{bR_n} \frac{|K_z^s(\zeta)|^2}{\|K_z^s\|_\alpha^2}\, dA_\alpha(w)\Big) \lesssim h(a_n(\mu))\Big(\int_{bR_n} \frac{|K_z^s(\zeta)|^2}{\|K_z^s\|_\alpha^2}\, dA_\alpha(\zeta)\Big)^p$$
$$\lesssim h(a_n(\mu))\Big(\frac{(1-|z|^2)^{2+2s-\alpha}(1-|z_n|^2)^{2+\alpha}}{|1-\overline{z}z_n|^{4+2s}}\Big)^p.$$

Since $s > (1-2p+\alpha p)/(2p)$, the integral

$$\int_{\mathbb{D}} \Big(\frac{(1-|z|^2)^{2+2s-\alpha}(1-|z_n|^2)^{2+\alpha}}{|1-\overline{z}z_n|^{4+2s}}\Big)^p \frac{dA(z)}{(1-|z|^2)^2}$$

is uniformly finite (see Lemma 2 in [5]). Combining all these inequalities, we obtain

$$\int_{\mathbb{D}} h\left(B_{\alpha,s}(T_\mu)(z)\right) \frac{dAz)}{\tau^2(z)}$$
$$\lesssim \sum_n h(a_n(\mu)) \int_{\mathbb{D}} \Big(\frac{(1-|z|^2)^{2+2s-\alpha}(1-|z_n|^2)^{2+\alpha}}{|1-\overline{z}z_n|^{4+2s}}\Big)^p \frac{dA(z)}{(1-|z|^2)^2} \lesssim \sum_n h(a_n(\mu)).$$

The proof is complete. ∎

Let $(b_n^{\alpha,s}(\mu))_n$ be the decreasing enumeration of $(B_{\alpha,s}(T_\mu)(z_n))_{n\geq 1}$. The following result is an improvement of Theorem C.

Theorem 5.9. *Let $\omega \in \mathcal{W}$ and $(R_n)_n \in \mathcal{L}_\omega$. Let μ be a positive Borel measure on $\mathbb{D}$ such that T_μ is compact on $\mathcal{A}^2_\alpha$. Let $\rho\colon [1,+\infty) \to (0,+\infty[$ be an increasing positive function. Suppose that there exist $\beta > 1$ and $\gamma > 1$ such that $\rho(t)/t^\gamma$ is increasing and $\rho(t)/t^\beta$ is decreasing. Then, for $s > \beta + \alpha - 2$, we have*

$$\lambda_n(T_\mu) \asymp 1/\rho(n) \iff b_n^{\alpha,s}(\mu) \asymp 1/\rho(n).$$

Proof. It is a consequence of Theorem 4.2 and Lemma 5.8. ∎

6. Composition operators

We consider composition operators on weighted analytic spaces on the unit disc $\mathbb{D}$. For $\omega \in \mathcal{W}$, we will denote $\mathcal{H}_\omega$ the space of analytic functions $f \in H(\mathbb{D})$ such that $f' \in \mathcal{A}^2_\omega$.

The space $\mathcal{H}_\omega$ becomes a Hilbert space if endowed with the norm $\|.\|_{\mathcal{H}_\omega}$, given by

$$\|f\|^2_{\mathcal{H}_\omega} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA_\omega(z).$$

For $\omega = \omega_\alpha$, the space $\mathcal{H}_{\omega_\alpha}$ will be denoted by $\mathcal{H}_\alpha$.

By the classical Littlewood–Paley identity, we have $\mathcal{H}_1 = H^2$, the Hardy space. Note also that for $\alpha \in [0, 1)$, $\mathcal{H}_\alpha := \mathcal{D}_\alpha$ are the weighted Dirichlet spaces, and for $\alpha > 1$, $\mathcal{H}_\alpha = \mathcal{A}^2_{\alpha-2}$ are the weighted standard Bergman spaces. For more information on these spaces, see [9, 12, 15].

Let φ be a holomorphic self map of $\mathbb{D}$. The composition operator C_φ with symbol φ acting on $\mathcal{H}_\omega$ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in \mathcal{H}_\omega.$$

Several papers have given some general criteria for boundedness, compactness and membership to Schatten classes of composition operators (see for instance [10, 17, 19, 24, 32, 34, 35]).

The Nevanlinna counting function, $N_{\varphi,\omega}$, of φ associated with $\mathcal{H}_\omega$ is defined by

$$N_{\varphi,\omega}(w) = \begin{cases} \sum_{z\in\varphi^{-1}(w)} \omega^2(z) \in (0,\infty], & \text{if } w \in \varphi(\mathbb{D}), \\ 0, & \text{if } w \notin \varphi(\mathbb{D}). \end{cases}$$

In what follows, $\mu_{\varphi,\omega}$ will denote the measure given by

$$d\mu_{\varphi,\omega}(w) = \frac{N_{\varphi,\omega}(w)}{\omega^2(w)}\, dA(w), \quad w \in \mathbb{D}.$$

The change of variable formula, see [1], can be written as follows:

$$\int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2\, dA_\omega(z) = \int_{\mathbb{D}} |f'(z)|^2 \omega^2(z)\, d\mu_{\varphi,\omega}(z).$$

Using this identity, it is clear that the composition operator C_φ on $\mathcal{H}_\omega$ is closely related to the Toeplitz operator $T_{\mu_{\varphi,\omega}}$ on $\mathcal{A}^2_\omega$. Indeed, if we suppose that $\varphi(0) = 0$, then the subspace $\mathcal{H}^0_\omega := \{f \in \mathcal{H}_\omega : f(0) = 0\}$ is reduced by C_φ. If $T\colon \mathcal{H}^0_\omega \to \mathcal{H}^0_\omega$ denotes the restriction of C_φ to $\mathcal{H}^0_\omega$, then T^*T is unitarily equivalent to $T_{\mu_{\varphi,\omega}}$ on $\mathcal{A}^2_\omega$. Namely,

$$T^*T = V^* T_{\mu_{\varphi,\omega}} V,$$

where $Vf = f'$ is the derivation operator which defines a unitary operator from $\mathcal{H}^0_\omega$ onto $\mathcal{A}^2_\omega$. As consequence, we have the following.

Proposition 6.1. *Let φ be an analytic self map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then C_φ is compact on $\mathcal{H}_\omega$ if and only if $T_{\mu_{\varphi,\omega}}$ is compact on $\mathcal{A}^2_\omega$. In this case, we have*

$$s_n^2(C_\varphi, \mathcal{H}_\omega) = \lambda_n(T_{\mu_{\varphi,\omega}}, \mathcal{A}^2_\omega).$$

As a direct consequence of Proposition 6.1 and trace estimates for Toeplitz operators, we obtain the following results.

Theorem 6.2. *Let $(R_n) \in \mathcal{L}_\omega$. Let $p \geq 1$ and let $h\colon [0,+\infty) \to [0,+\infty)$ be an increasing function such that $h(t^p)$ is convex and $h(0) = 0$. Let φ be an analytic self map of $\mathbb{D}$ satisfying $\varphi(0) = 0$. We have*

$$\sum_n h\Big(\frac{1}{B}\Big(\frac{\mu_{\varphi,\omega}(R_n)}{A(R_n)}\Big)\Big) \leq \sum_n h(s_n^2(C_\varphi, \mathcal{H}_\omega)) \leq \sum_n h\Big(B\Big(\frac{\mu_{\varphi,\omega}(R_n)}{A(R_n)}\Big)\Big),$$

where $B > 0$ depends on ω and p.

Corollary 6.3. *Let $\omega \in \mathcal{W}$ and let $(R_n) \in \mathcal{L}_\omega$. Let $\rho\colon [1, +\infty) \to (0, +\infty)$ be an increasing function such that $\rho(x)/x^A$ is decreasing for some $A > 0$. Let φ be an analytic self map of $\mathbb{D}$ such that $\varphi(0) = 0$ and C_φ is compact on $\mathcal{H}_\omega$. Then*

(1) $s_n(C_\varphi) = O(1/\rho(n)) \iff a_n(\mu_{\varphi,\omega}) \asymp O(1/\rho^2(n))$,

(2) $s_n(C_\varphi) \asymp 1/\rho(n) \iff a_n(\mu_{\varphi,\omega}) \asymp 1/\rho^2(n)$.

7. Composition operators with univalent symbol on $\mathcal{H}_\alpha$

The goal of this section is to provide some concrete examples. We will focus our attention on composition operators C_φ acting on $\mathcal{H}_\alpha$ such that φ is univalent. We will give estimates of the singular values of C_φ in terms of the pull-back measure induced by φ.

7.1. Composition operators with univalent symbol

Let φ be an analytic self map of $\mathbb{D}$. The pull-back measure associated with φ is the positive Borel measure on $\mathbb{D}$ defined by

$$m_\varphi(B) = m(\{\zeta \in \mathbb{T} : \varphi(\zeta) \in B\}),$$

where m is the normalized Lebesgue measure of $\mathbb{T}$ and where we still denote φ the boundary values of φ.

Let Ω be a simply connected subdomain of $\mathbb{D}$ which contains 0. Let φ be a conformal map of $\mathbb{D}$ onto Ω. Let σ be an automorphism of $\mathbb{D}$. Since C_σ is an invertible operator on $\mathcal{H}_\alpha$, we have $s_n(C_\varphi, \mathcal{H}_\alpha) \asymp s_n(C_{\varphi\circ\sigma}, \mathcal{H}_\alpha)$ as $n \to \infty$. So, without loss of generality we suppose, in the sequel, that $\varphi(0) = 0$.

Let n and j be integers such that $n \ge 1$ and $j \in \{0, 2, \dots, 2^n - 1\}$. The dyadic square $R_{n,j}$ is given by

$$R_{n,j} = \Big\{ z \in \mathbb{D} ;\ 1 - 2^{-n} \le |z| < 1 - \frac{1}{2^{n+1}} \ \text{and}\ \frac{2j\pi}{2^n} \le \arg z < \frac{2(j+1)\pi}{2^n} \Big\}.$$

By following the same proofs, in all the previous results, one can see that we can replace $(R_n)_n \in \mathcal{L}_{\omega_\alpha}$ by $(R_{n,j})_{n,j}$. For our purposes, it is more convenient to consider the Carleson boxes $W_{n,j}$, which are given by

$$W_{n,j} = \Big\{ z \in \mathbb{D} ;\ 1 - 2^{-n} \le |z| \ \text{and}\ \frac{2j\pi}{2^n} \le \arg z < \frac{2(j+1)\pi}{2^n} \Big\}.$$

The main result of this section is the following theorem.

Theorem 7.1. *Let φ be a univalent analytic self map of $\mathbb{D}$. Let $h\colon [0, +\infty) \to [0, +\infty)$ be an increasing function such that $h(0) = 0$. Suppose that there exists $p \ge 1$ such that $h(t^p)$ is convex, and let $\alpha > 0$. We have*

$$\sum_{n,j} h\Big(\frac{1}{B}(2^n m_\varphi(W_{n,j}))^\alpha\Big) \le \sum_n h(s_n^2(C_\varphi, \mathcal{H}_\alpha)) \le \sum_{n,j} h(B(2^n m_\varphi(W_{n,j}))^\alpha),$$

where $B > 0$ depends on α and p.

Let $(m_n(\varphi))_{n\geq 1}$ be the decreasing enumeration of $(2^n m_\varphi(W_{n,j}))_{n,j}$. As a consequence of Theorem 7.1, Lemma 3.1 and Lemma 3.2, we obtain the following result.

Corollary 7.2. *Let $\alpha > 0$. Let φ be a univalent analytic self map of $\mathbb{D}$. Let $\rho\colon [1,+\infty) \to (0,+\infty)$ be an increasing function such that $\rho(x)/x^A$ is decreasing for some $A > 0$. Then the following are equivalent*:

(1) $s_n(C_\varphi, \mathcal{H}_\alpha) \asymp 1/\rho(n)$.

(2) $m_n(\varphi) \asymp 1/\rho^{2/\alpha}(n)$.

To prove Theorem 7.1, we need some intermediate results. We begin by the following elementary lemma.

Lemma 7.3. *Let $p \geq 1$ and let $h\colon [0,+\infty) \to [0,+\infty)$ be an increasing function such that $h(0) = 0$ and $h(t^p)$ is convex. We have*

$$\sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(C\frac{\mu(R_{n,j})}{A(R_{n,j})}\Big) \leq \sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(2C\frac{\mu(W_{n,j})}{A(W_{n,j})}\Big) \leq B\sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(4C\frac{\mu(R_{n,j})}{A(R_{n,j})}\Big),$$

where $B > 0$ depends only on p.

Proof. The first inequality comes from the facts that h is increasing, $R_{n,j} \subset W_{n,j}$ and $A(W_{n,j}) = 2A(R_{n,j})$. For the reverse inequality, we follow the argument given in [18]. We have

$$W_{n,j} = \bigcup_{l\geq n}\bigcup_{k\in H_{l,n,j}} R_{l,k},$$

where

$$H_{l,n,j} = \Big\{k \in \{0,1,\dots,2^l-1\};\ \frac{j}{2^n} \leq \frac{k}{2^l} < \frac{j+1}{2^n}\Big\}.$$

From the above decomposition and the convexity of $h(t^p)$, we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} h\Big(2C\frac{\mu(W_{n,j})}{A(W_{n,j})}\Big) &= \sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} h\Big(\sum_{l\geq n}\sum_{k\in H_{l,n,j}} 2^{2n-2l-1}4C\frac{\mu(R_{l,k})}{A(R_{l,k})}\Big)\\
&\lesssim \sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} h\Big(\Big(\sum_{l\geq n}\sum_{k\in H_{l,n,j}} 2^{\frac{2n-2l}{p}}\Big(4C\frac{\mu(R_{l,k})}{A(R_{l,k})}\Big)^{1/p}\Big)^p\Big)\\
&\lesssim \sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} \Big(\sum_{l\geq n}\sum_{k\in H_{l,n,j}} 2^{\frac{2n-2l}{p}} h\Big(4C\frac{\mu(R_{l,k})}{A(R_{l,k})}\Big)\Big)\\
&\leq \sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1} \Big(\sum_{l\geq n}\sum_{k\in H_{l,n,j}} 2^{\frac{2n-2l+1}{p}}\Big) h\Big(4C\frac{\mu(R_{l,k})}{A(R_{l,k})}\Big)\\
&\leq B\sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1} h\Big(4C\frac{\mu(R_{l,k})}{A(R_{l,k})}\Big).
\end{aligned}$$

This ends the proof. ∎

In [20], P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza give an explicit relation between the Nevanlinna counting function of an analytic self map φ of $\mathbb{D}$ and its pull-back measure. Namely:

Theorem 7.4. *There exist absolute positive constants c_1, c_2, C_1 and C_2 such that for every analytic self map φ of $\mathbb{D}$, $\zeta \in \mathbb{T}$ and every $\delta \in (0, (1-|\varphi(0)|)/16)$, one has*

(1) $N_\varphi(w) \le C_1 m_\varphi(W(\zeta, c_1\delta))$, *for every $w \in W(\zeta,\delta)$.*

(2) $m_\varphi(W(\zeta,\delta)) \le \frac{C_2}{\delta^2}\int_{W(\zeta,c_2\delta)} N_\varphi(w)\,dA(w)$.

In particular, we have the following inequalities:

$$\frac{1}{C_2} m_\varphi(W(\zeta,\delta/c_2)) \le \sup_{z\in W(\zeta,\delta)} N_\varphi(z) \le C_1\, m_\varphi(W(\zeta,c_1\delta)), \tag{7.1}$$

For a simple proof of these results, see [8].

We also need a consequence of the well-known Hardy–Littlewood inequality.

Lemma 7.5. *Let φ be an analytic self map of $\mathbb{D}$, let $\alpha > 0$ and let $\zeta \in \mathbb{T}$. There exists an absolute constant $c > 0$ such that*

$$m_\varphi(W(\zeta,\delta))^\alpha \le \frac{C(\alpha)}{\delta^2}\int_{W(\zeta,\kappa\delta)\cap\mathbb{D}} N_\varphi^\alpha(z)\,dA(z), \quad \textit{for } 0<\delta<c(1-|\varphi(0)|),$$

where κ is an absolute constant and $C(\alpha)$ depends only on α.

Proof. Let $R \in (1,2)$ and let $\psi = \varphi/R$. By the Hardy–Littlewood inequality [27], for every $z \in \mathbb{D}$ such that $1-|z| < \frac{1}{2}(1-|\psi(0)|)$ and every $\delta \in (0, 1-|z|)$, we have

$$N_\psi(z)^\alpha \le \frac{C}{\delta^2}\int_{D(z,\delta)} N_\psi^\alpha(w)\,dA(w). \tag{7.2}$$

Let $z \in \mathbb{D}$ and let $\delta > 0$ be such that $\max(1-|z|,\delta) < \frac{1}{4}(1-|\varphi(0)|)$. Then, for $R = 1+\frac{1}{2}(1-|\varphi(0)|)$, we have $\delta < 1-|z|/R < \frac{1}{2}(1-|\psi(0)|)$. By (7.2), we get

$$N_\varphi^\alpha(z) = N_\psi^\alpha(z/R) \le \frac{C}{\delta^2}\int_{D(z/R,\delta)} N_\psi^\alpha(w)\,dA(w) \le \frac{4C}{\delta^2}\int_{D(z,2\delta)} N_\varphi^\alpha(w)\,dA(w).$$

Now let $\zeta \in \mathbb{T}$ and let $\delta < c(1-|\varphi(0)|)$, where $c = \frac{c_1}{4(2+c_1)}$ and c_1 is the constant appearing in (7.1).

For $z \in W(\zeta,\delta/c_1)$, we have $D(z,2\delta) \subset W(\zeta,(2+1/c_1)\delta)$. Then

$$N_\varphi^\alpha(z) \lesssim \frac{1}{\delta^2}\int_{W(\zeta,\kappa\delta)} N_\varphi^\alpha(w)\,dA(w), \quad \text{where } \kappa = 2+1/c_1,$$

and the result comes from (7.1). ∎

In the sequel, we will write $\mu_{\varphi,\alpha}$ and $N_{\varphi,\alpha}$ instead of $\mu_{\varphi,\omega_\alpha}$ and $N_{\varphi,\omega_\alpha}$.

Let $c > 0$ and let $W = W(\zeta,\delta)$ be a Carleson box. We will denote $W^c = W(\zeta, c\delta)$. Theorem 7.1 is a direct consequence of Theorem 6.2, Lemma 7.3 and the following inequalities.

Lemma 7.6. *Let $\alpha > 0$. Let $h: [0, +\infty) \to [0, +\infty)$ be an increasing positive function such that $h(t^p)$ is convex for some $p \geq 1$. Let φ be a univalent analytic self map of $\mathbb{D}$ and let $C > 0$. We have*

$$\sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(C_1 \frac{\mu_{\varphi,\alpha}(W_{n,j})}{A(W_{n,j})}\Big) \lesssim \sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\big(C(2^n\, m_\varphi(W_{n,j}))^\alpha\big)$$
$$\lesssim \sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(C_2 \frac{\mu_{\varphi,\alpha}(W_{n,j})}{A(W_{n,j})}\Big),$$

where the implied constants do not depend on h.

Proof. Since φ is univalent, $N_{\varphi,\alpha} = N_\varphi^\alpha$. Then, by equation (7.1) we have

$$\frac{\mu_{\varphi,\alpha}(R_{n,j})}{A(R_{n,j})} = \frac{1}{A(R_{n,j})}\int_{R_{n,j}} \frac{N_\varphi^\alpha(z)}{(1-|z|^2)^\alpha}\, dA(z) \lesssim 2^{\alpha n} \sup_{z\in W_{n,j}} N_\varphi^\alpha(z) \lesssim (2^n m_\varphi(W_{n,j}^{c_2}))^\alpha.$$

Then

$$\sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\Big(\frac{\mu_{\varphi,\alpha}(R_{n,j})}{A(R_{n,j})}\Big) \lesssim \sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\big(C_1'(2^n\, m_\varphi(W_{n,j}^{c_2}))^\alpha\big)$$
$$\lesssim \sum_{n\geq 1}\sum_{j=0}^{2^n-1} h\big(C_2(2^n\, m_\varphi(W_{n,j}))^\alpha\big).$$

Then the left inequality of Lemma 7.6 is obtained from Lemma 7.3.

Conversely, by Lemma 7.5, we have

$$\big(2^n m_\varphi(W_{n,j})\big)^\alpha \lesssim \frac{\mu_{\varphi,\alpha}(W_{n,j}^\kappa)}{A(W_{n,j}^\kappa)},$$

which gives the remaining inequality in order to finish the proof. ∎

7.2. Examples

Let Ω be a subdomain of $\mathbb{D}$ such that $0 \in \Omega$, $\partial\Omega \cap \partial\mathbb{D} = \{1\}$ and $\partial\Omega$ has, in a neighborhood of $+1$, a polar equation $1 - r = \gamma(|\theta|)$, where $\gamma: [0, \pi] \to [0, 1]$ is a differentiable continuous increasing function such that $\gamma(0) = 0$ and $\gamma'(t) = O(\gamma(t)/t)$ as $t \to 0^+$.

Let φ be a univalent map from $\mathbb{D}$ onto Ω with $\varphi(0) = 0$ and $\varphi(1) = 1$. By definition, the harmonic measure $\varpi(., E, \Omega)$ is the harmonic extension of χ_E on Ω, where E is closed subset of $\partial\Omega$. By conformal invariance of the harmonic measure, we have

$$\varpi(0, E, \Omega) = \varpi(0, \varphi^{-1}(E), \mathbb{D}) = m(\varphi^{-1}(E)) = m_\varphi(E).$$

So to use Theorem 7.1, we have to estimate the harmonic measure of our domains. To this end, we use Ahlfors–Warschawski type estimates. The following lemma is proved in

[7]. In the sequel of this subsection, we suppose that γ satisfies conditions (1.4) and (1.5). Recall that Γ is given by

$$\Gamma(t) = \frac{2}{\pi}\int_t^1 \frac{\gamma(s)}{s^2}\,ds.$$

Lemma 7.7. *Let γ, Ω and φ be as above. Then*

- $\varpi(0, W_{n,j}\cap\partial\Omega,\Omega) \lesssim \frac{C}{2^n}\exp\Big[-\Gamma\Big(\frac{2\pi(j+1)}{2^n}\Big)\Big]$, $0 \le j < j_n := \frac{2^n}{2\pi}\gamma^{-1}(2\pi/2^n)$.
- *There exists $\eta > 0$ such that for k satisfying $2^{k+1} \le j_n$, we have*

$$\operatorname{Card}\Big\{j\in\{2^k,\dots,2^{k+1}-1\} : \frac{\eta}{2^n}\exp\Big[-\Gamma\Big(\frac{2\pi(j+1)}{2^n}\Big)\Big] \lesssim \varpi(0, W_{n,j}\cap\partial\Omega,\Omega)\Big\} \asymp 2^k.$$

Now, we are able to prove the following estimates.

Theorem 7.8. *Let γ, Ω and φ be as above. Let $h\colon [0,+\infty) \to [0,+\infty)$ be an increasing function such that $h(0) = 0$. Suppose that there exists $p \ge 1$ such that $h(t^p)$ is convex and let $\alpha > 0$. We have*

$$B\int_0^1 \frac{h(be^{-\alpha\Gamma(s)})}{\gamma(s)}\,ds \le \sum_n h\left(s_n^2(C_\varphi,\mathcal{H}_\alpha)\right) \le A\int_0^1 \frac{h(ae^{-\alpha\Gamma(s)})}{\gamma(s)}\,ds,$$

where $A, B, a, b > 0$ depend on α and p.

Proof. By Theorem 7.1 and Lemma 7.7, it suffices to prove that

$$\int_0^1 \frac{h(\frac{C}{A}e^{-\alpha\Gamma(s)})}{\gamma(s)}\,ds \lesssim \sum_{n=1}^{\infty}\sum_{j=0}^{j_n} h\Big(C\exp\Big[-\alpha\Gamma\Big(\frac{2\pi(j+1)}{2^n}\Big)\Big]\Big)$$
$$\lesssim \int_0^1 \frac{h(ACe^{-\alpha\Gamma(s)})}{\gamma(s)}\,ds.$$

First, observe that

$$\int_{2\pi j/2^n}^{2\pi(j+1)/2^n} \frac{\gamma(s)}{s^2}\,ds \lesssim \int_{2\pi j/2^n}^{2\pi(j+1)/2^n} \frac{1}{s}\,ds = O(1). \tag{7.3}$$

So there exists $A > 0$ such that

$$\frac{1}{A}e^{-\alpha\Gamma(2\pi j/2^n)} \le e^{-\alpha\Gamma(s)} \le Ae^{-\alpha\Gamma(2\pi j/2^n)}, \quad \text{for } s \in \Big(\frac{2\pi j}{2^n}, \frac{2\pi(j+1)}{2^n}\Big).$$

Then

$$h\big(Ce^{-\alpha\Gamma(\frac{2\pi j}{2^n})}\big) \lesssim 2^n\int_{2j\pi/2^n}^{2(j+1)\pi/2^n} h\big(Ce^{-\alpha\Gamma(s)}\big)\,ds \lesssim h\big(ACe^{-\alpha\Gamma(\frac{2\pi(j+1)}{2^n})}\big),$$

and

$$\sum_{0}^{\infty}\sum_{j=1}^{j_n} h\big(C_1 e^{-\alpha\Gamma((2\pi j/2^n))}\big) \geq \sum_{0}^{\infty}\sum_{j=1}^{j_n} 2^n \int_{2\pi(j-1)/2^n}^{2\pi j/2^n} h\Big(\frac{C_1}{A}e^{-\alpha\Gamma(s)}\Big)\, ds$$
$$\asymp \sum_{0}^{\infty} 2^n \int_{0}^{\gamma^{-1}(2^{-n})} h\Big(\frac{C_1}{A}e^{-\alpha\Gamma(s)}\Big)\, ds \asymp C\sum_{n=0}^{\infty} 2^n \sum_{k=n}^{\infty} \int_{\gamma^{-1}(2^{-k-1})}^{\gamma^{-1}(2^{-k})} h\Big(\frac{C_1}{A}e^{-\alpha\Gamma(s)}\Big)\, ds$$
$$\asymp \sum_{k=0}^{\infty} 2^k \int_{\gamma^{-1}(2^{-k-1})}^{\gamma^{-1}(2^{-k})} h\Big(\frac{C_1}{A}e^{-\alpha\Gamma(s)}\Big)\, ds \asymp \int_0^1 \frac{h\big(\frac{C_1}{A}e^{-\alpha\Gamma(s)}\big)}{\gamma(s)}\, ds.$$

This proves the first inequality.

The second inequality can be obtained using similar computations. ■

Proof of Theorem D. The first assertion is a direct consequence of the characterization of membership to p-Schatten classes given in [7]. Indeed, suppose that

$$\lim_{t\to 0^+} \frac{\gamma(t)\log(1/t)}{t} = \infty.$$

Let $B > 0$ such that $Bp\alpha/2 = 3$. For small $t > 0$ we have

$$\gamma(t) \geq \frac{Bt}{\log(1/t)}.$$

This implies that $\Gamma(t) \geq B\log(\log(1/t))$. Then

$$\int_0 \frac{e^{-\frac{p\alpha}{2}\Gamma(t)}}{\gamma(t)}\, dt \lesssim \int_0 \frac{(\log(1/t))^{-Bp\alpha/2}}{\gamma(t)}\, dt \lesssim \int_0 \frac{1}{t\log^2(1/t)}\, dt < \infty, \quad \text{for all } p > 0.$$

Then $C_\varphi \in \cap_{p>0}\mathcal{S}_p(\mathcal{H}_\alpha)$. This is equivalent to $s_n(C_\varphi, \mathcal{H}_\alpha) = O(1/n^A)$ for all $A > 0$.

To prove the second assertion, let

$$\rho(x) = \exp\{\alpha\Gamma(\Lambda^{-1}(x))\}, \quad \text{where } \Lambda(t) = \int_t^2 \frac{ds}{\gamma(s)}.$$

First, we prove that $\rho(x)/x^A$ is decreasing, where A is such that $\gamma(t) \leq \frac{\pi A}{4\alpha}\frac{t}{\log(1/t)}$. Since $\gamma(t)/t$ is increasing, we have

$$\Lambda(t) = \int_t^2 \frac{dt}{\gamma(t)} \leq \frac{t}{\gamma(t)}\log(2/t) \leq \frac{2t}{\gamma(t)}\log(1/t) \leq \frac{\pi A}{2\alpha}\frac{t^2}{\gamma(t)^2}, \quad t < 1/2.$$

This implies that $t \to \Lambda(t)\exp(-\frac{\alpha}{A}\Gamma(t))$ is decreasing, since its derivative is negative, and then $\rho(x)/x^A$ is decreasing.

Note also that if h is an increasing positive function, then

$$\int_0^1 \frac{h(Ce^{-\alpha\Gamma(s)})}{\gamma(t)}\, dt \asymp \sum_{n\geq 1} h(Ce^{-\alpha\Gamma(x_n)}) \int_{x_{n+1}}^{x_n} \frac{dt}{\gamma(t)} = \sum_{n\geq 1} h(Ce^{-\alpha\Gamma(x_n)}).$$

Then, by Theorem 7.8 and Lemma 3.1, we obtain the result. ■

8. Concluding remarks

8.1. Composition operators on the Hardy space

The Hardy space H^2 is equal to $\mathcal{H}_1$. The problem of estimating the singular values of composition operators on H^2 was considered in several papers ([18, 21, 22, 28]). Using the same arguments as those given in Section 7, one can remove the condition that φ is univalent in Corollary 7.2. We have the following result.

Theorem 8.1. *Let φ be an analytic self map of $\mathbb{D}$. Let $\rho\colon [1,+\infty) \to (0,+\infty[$ be an increasing function such that $\rho(x)/x^A$ is decreasing for some $A > 0$. Then*

$$s_n(C_\varphi, H^2) \asymp 1/\rho(n) \iff m_n(\varphi) \asymp 1/\rho^2(n).$$

Note that our method can also be applied to composition operators with outer symbol. Such composition operators were considered in [4, 18, 28]. Namely, let φ be the outer function given by

$$\varphi(z) = \exp\Big(-\int_{\mathbb{T}} \frac{e^{it}-z}{e^{it}+z}\, U(|t|)\, \frac{dt}{2\pi}\Big), \tag{8.1}$$

where $U\colon [0,\pi] \to [0,\infty)$ is an increasing integrable function such that $U(0) = 0$. It is proved in [4, 28] that, under some regularity conditions on U, C_φ is compact if and only if

$$\int_0^1 \frac{U(s)}{s^2}\, ds = +\infty.$$

It is also proved in [4] that $C_\varphi \in S_p(H^2)$ if and only if

$$\int_0^1 \frac{dt}{U(t)\, q_U^{p/2-1}(t)} < \infty,$$

where $q_U(t) = \int_t^1 \frac{U(s)}{s^2} ds$.

One can extend this result. In accordance with [4], we say that U is admissible if U is concave or convex and if $U(t) \asymp U(2t) \asymp tU'(t)$. We have the following.

Theorem 8.2. *Let U be an admissible function such that $t^2 = o(U(t))$ and*

$$U(t) = o\Big(t\int_t^\pi \frac{U(s)}{s^2}\, ds\Big) \quad \text{as } t \to 0^+.$$

Let h be an increasing function such that $h(0) = 0$. Suppose that there exists $p \geq 1$ such that $h(t^p)$ and h^p are convex. We have

$$B\int_0^1 h\Big(\frac{b}{q_U(t)}\Big)\frac{q_U(t)}{U(t)}\, dt \leq \sum_n h(s_n^2(C_\varphi, H^2)) \leq A\int_0^1 h\Big(\frac{a}{q_U(t)}\Big)\frac{q_U(t)}{U(t)}\, dt,$$

where φ is given by (8.1) and $A, B, a, b > 0$ depend on p.

In [28], H. Queffélec and K. Seip gave some estimates of the singular values of such composition operators. They proved that if U is sufficiently regular, and if $q_U(t) = O(\log^\gamma \log(1/t))$ for some $\gamma > 0$, then

$$s_n(C_\varphi, H^2) \asymp \frac{1}{\sqrt{q_U(e^{-\sqrt{n}})}}.$$

The extremal decreasing case corresponds to $q_U(t) = \log^\gamma \log(1/t)$, for which they obtained that

$$s_n(C_\varphi, H^2) \asymp \frac{1}{\log^{\gamma/2} n}.$$

Using Theorem 8.2 and Lemma 3.1, we extend this result as follows.

Theorem 8.3. *Under the same hypothesis of Theorem 8.2, and supposing* $\int_0^1 \frac{U(s)}{s^2} ds = +\infty$*, we have*:

(1) *If* $\lim_{t \to 0^+} \dfrac{\log q_U(t)}{\log \log 1/t} = \infty$*, then*

$$s_n(C_\varphi, H^2) = O(1/n^A) \quad \textit{for all } A > 0.$$

(2) *If* $\dfrac{\log q_U(t)}{\log \log 1/t} = O(1)$*, then*

$$s_n(C_\varphi, H^2) \asymp \frac{1}{\sqrt{q_U(x_n)}},$$

where x_n *is given by*

$$\int_{x_n}^{\pi} \frac{q_U(t)}{U(t)}\, dt = n.$$

8.2. Composition operators on the Dirichlet space

The Dirichlet space, denoted by $\mathcal{D}$, is given by

$$\mathcal{D}(:= \mathcal{H}_0) = \{f \in H(\mathbb{D}) : \ f' \in L^2(\mathbb{D}, dA)\}.$$

The Nevanlinna counting function $N_{\varphi,0}$ induced by φ and associated with $\mathcal{D}$ is the counting function n_φ. That is,

$$N_{\varphi,0}(z) = n_\varphi(z) = \mathrm{Card}\{\varphi^{-1}(z)\}, \quad z \in \mathbb{D}.$$

In particular, if φ is univalent then

$$n_\varphi = \chi_\Omega \quad \text{and} \quad d\mu_{\varphi,0} = \chi_{\varphi(\Omega)}\, dA, \quad (\Omega = \varphi(\mathbb{D})).$$

Let Ω, γ and φ be as before. The compactness and membership to Schatten classes of C_φ is studied in [7]. Recall that C_φ is compact on $\mathcal{D}$ if and only if

$$\lim_{t \to 0^+} \frac{\gamma(t)}{t} = \infty,$$

and $C_\varphi \in S_p$ if and only if

$$\int_0 \Big(\frac{t}{\gamma(t)}\Big)^{p/2} \frac{\gamma'(t)}{\gamma(t)}\, dt < \infty.$$

Using Corollary 6.3 and the discussion above, one can prove easily that if $\gamma(t)/t = O(\log^\beta(1/t))$ for some $\beta > 0$, then

$$s_n(C_\varphi) \asymp \sqrt{e^n \gamma^{-1}(e^{-n})}.$$

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