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On secant defectiveness and identifiability of Segre–Veronese varieties

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Abstract. We give an almost asymptotically sharp bound for the non-secant defectiveness and identifiability of Segre–Veronese varieties. We also provide new examples of defective Segre–Veronese varieties, and implement our methods in Magma. Finally, we give two applications of our techniques: we classify possibly singular 2-secant defective toric surfaces and we study secant defectiveness of Losev–Manin spaces.

Contents

1. Introduction

The *h-secant variety* $\mathbb{S}\mathrm{ec}_h(X)$ of a non-degenerate n-dimensional variety $X \subseteq \mathbb{P}^N$ is the Zariski closure of the union of all linear spaces spanned by collections of h points of X. The *expected dimension* of $\mathbb{S}\mathrm{ec}_h(X)$ is $\mathrm{expdim}(\mathbb{S}\mathrm{ec}_h(X)) := \min\{nh + h - 1, N\}$. In general, the actual dimension of $\mathbb{S}\mathrm{ec}_h(X)$ may be smaller than the expected one. In this case, following Section 2 of [17], we say that X is *h-defective*. The problem of determining the actual dimension of secant varieties, and its relation with the dimension of certain linear systems of hypersurfaces with double points, has a very long history in algebraic geometry [44, 45, 49]. Since then the geometry of secant varieties has been studied and

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used by many authors in various contexts [17, 43], and the problem of secant defectiveness has been widely studied for Segre–Veronese varieties, Grassmannians, Lagrangian Grassmannians, spinor varieties and flag varieties [4, 5, 7–9, 11, 15, 16, 26, 27, 34, 41, 50].

An important concept related to the theory of secant varieties is that of *identifiability*. We say that a point $p \in \mathbb{P}^N$ is h-identifiable, with respect to a non-degenerated variety $X \subseteq \mathbb{P}^N$, if it lies on a unique $(h-1)$-plane in $\mathbb{P}^N$ that is h-secant to X. Especially when $\mathbb{P}^N$ can be interpreted as a tensor space, identifiability and tensor decomposition algorithms are central in applications, for instance, in biology, blind signal separation, data compression algorithms, analysis of mixture models psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis [18, 21–23, 32, 33, 37, 40, 46].

Let $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r}$ be the Segre–Veronese variety given as the image, in $\mathbb{P}^N$ with

$$N = \prod_{i=1}^{r} \binom{n_i + d_i}{d_i} - 1,$$

of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ via the embedding induced by $|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}}(d_1, \dots, d_r)|$.

For Segre–Veronese varieties, the problem of secant defectiveness has been solved in some very special cases, mostly for products of few factors [1–4, 9, 11, 28, 50]. Secant defectiveness for Segre–Veronese products $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, with arbitrary number of factors and degrees, was completely settled in [34]. Furthermore, h-defectiveness of Segre products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \subseteq \mathbb{P}^N$ is classified only for $h \leq 6$, see [5].

In this paper we go in a somehow orthogonal direction and give a general bound on h for the non-defectiveness of $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r}$ in term of the d_i and the n_i. In general, h-defectiveness is classified only for small values of h, see Proposition 3.2 of [50] and Theorem 4.8 of [8].

Our main results on non-secant defectiveness and identifiability of Segre–Veronese varieties in Theorem 3.1 and Corollary 3.6 can be summarized as follows.

Theorem 1.1. *The Segre–Veronese variety* $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r} \subseteq \mathbb{P}^N$ *is not h-defective for*

$$h \leq \frac{d_j}{n_j + d_j} \frac{1}{1 + \sum_{i=1}^{r} n_i} \prod_{i=1}^{r} \binom{n_i + d_i}{d_i},$$

where $n_j/d_j = \max_{1 \leq i \leq r}\{n_i/d_i\}$. *Furthermore, if in addition*

$$2\sum_{i=1}^{r} n_i < \frac{d_j}{n_j + d_j} \frac{1}{1 + \sum_{i=1}^{r} n_i} \prod_{i=1}^{r} \binom{n_i + d_i}{d_i},$$

under the bound above, $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r} \subseteq \mathbb{P}^N$ *is* $(h-1)$*-identifiable.*

Note that Theorem 1.1 gives a polynomial bound of degree $\sum_i n_i$ in the d_i, while in the n_i we have a polynomial bound of degree $\sum_i d_i - 2$. For Segre–Veronese varieties, the expected generic rank is given by a polynomial of degree $\sum_i n_i$ in the d_i and of degree $\sum_i d_i - 1$ in the n_i. At the best of our knowledge, the bound in Theorem 1.1 is the best general bound so far for non-secant defectiveness and identifiability of Segre–Veronese varieties. In order to help the reader grasp the difference among the general bounds of this kind available in the literature, we work out explicitly some cases in Table 1.

$n_1 = n_2 = n_3$	d_1	d_2	d_3	Theorem 1.1	Theorem 4.8 of [8]	Proposition 3.2 of [50]
2	3	3	3	$h \leq 85$	$h \leq 19$	$h \leq 3$
2	3	4	4	$h \leq 193$	$h \leq 21$	$h \leq 3$
2	3	5	5	$h \leq 378$	$h \leq 25$	$h \leq 3$
3	5	5	5	$h \leq 10976$	$h \leq 64$	$h \leq 4$
10	5	6	6	$h \leq 2070715873$	$h \leq 13311$	$h \leq 11$
30	5	5	7	$h \leq 1703293480928730$	$h \leq 893731$	$h \leq 31$

Table 1. General bounds for h in the literature.

The proof of Theorem 1.1 passes through the bound for non-secant defectiveness of a toric variety in Theorem 2.12. The toric approach we present in Section 2 has been implemented as a Magma algorithm to check non-defectiveness of a projective toric variety.

Organization of the paper

The paper is organized as follows. In Section 2 we introduce a technique to study secant defectiveness based on polytope triangulations. In Section 3 we prove Theorem 1.1, and in Proposition 3.3 we recover, with our techniques, a previously known classification of some special secant defective two factors Segre–Veronese varieties. In Section 4 we give new examples of defective Segre–Veronese varieties. In Section 5 we discuss a Magma (see [12]) implementation of our techniques. Finally, in Section 6 we give two applications of our techniques: we classify possibly singular 2-secant defective toric surfaces and we study secant defectiveness of Losev–Manin spaces.

2. A convex geometry translation of Terracini's lemma

Let N be a rank n free abelian group, $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding rational vector space. Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, that is the convex hull of finitely many points in M which do not lie on a hyperplane. The polytope P defines a polarized pair (X_P, H) consisting of the toric variety X_P together with a very ample Cartier divisor H of X_P. More precisely, X_P is the Zariski closure of the image of the monomial map

$$\phi_P : (\mathbb{C}^*)^n \to \mathbb{P}^{|P\cap M|-1}, \quad u \mapsto [\chi^m(u) : m \in P \cap M],$$

where $\chi^m(u)$ denotes the Laurent monomial in the variables $(u_1, \dots, u_n)$ defined by the point m, and H is a hyperplane section of X_P. The defining fan $\Sigma := \Sigma(X) \subseteq N_{\mathbb{Q}}$ of the normalization $\tilde{X}_P$ of X_P is the normal fan of P and $H = -\Sigma_{\rho\in\Sigma(1)} \min_{m\in P}\langle m, \rho\rangle D_\rho$, where each ρ denotes the primitive generator of a 1-dimensional cone of Σ and D_ρ is the corresponding torus invariant divisor. Each element $v \in N$ defines a 1-parameter subgroup of $(\mathbb{C}^*)^n$ via the homomorphism $\eta_v : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by $t \mapsto t^v$. We denote by $\Gamma_v \subseteq X$ the Zariski closure of the curve $(\phi_P \circ \eta_v)(\mathbb{C}^*)$.

Given $a \in \mathbb{C}^*$, denote by $\Gamma_v(a)$ the point $\phi_P(\eta_v(a))$, and by $m_1, \dots, m_r$ the integer points of $P \cap M$.

Lemma 2.1. *Given a point $a \in \mathbb{C}^*$, the tangent space of X at $\Gamma_v(a)$ is the projectivization of the vector subspace of $\mathbb{C}^{|P\cap M|}$ generated by the rows of the following matrix*:

$$M_v(a) := \begin{pmatrix} a^{\langle m_1, v\rangle} & \dots & a^{\langle m_r, v\rangle} \\ \langle m_1, e_1\rangle a^{\langle m_1 - e_1^*, v\rangle} & \dots & \langle m_r, e_1\rangle a^{\langle m_r - e_1^*, v\rangle} \\ \vdots & & \vdots \\ \langle m_1, e_n\rangle a^{\langle m_1 - e_n^*, v\rangle} & \dots & \langle m_r, e_n\rangle a^{\langle m_r - e_n^*, v\rangle} \end{pmatrix}.$$

Proof. The point $\Gamma_v(a)$ is in the image of ϕ_P, so that we can use this parametrization to compute the tangent space. Observe that since P is full-dimensional, the map ϕ_P is finite; moreover, it is étale being equivariant with respect to the torus action. It follows that ϕ_P is smooth and thus the tangent space of X at $\Gamma_v(a)$ is spanned by the partial derivatives of order less than or equal to one of the monomials $\chi^{m_1}(u), \dots, \chi^{m_r}(u)$ evaluated at a^v. ■

Remark 2.2. Given a subset $\Delta := \{m_{i_0}, \dots, m_{i_n}\}$ of cardinality $n+1$ of $P \cap M$, the corresponding $(n+1) \times (n+1)$ minor of the matrix $M_v(a)$, whenever $a \neq 0$, is

$$\delta_{v,\Delta}(a) := \frac{a^{\langle m_{i_0} + \dots + m_{i_n}, v\rangle}}{a^{\langle e_1^* + \dots + e_n^*, v\rangle}} \begin{vmatrix} 1 & \dots & 1 \\ \langle m_{i_0}, e_1\rangle & \dots & \langle m_{i_n}, e_1\rangle \\ \vdots & & \vdots \\ \langle m_{i_0}, e_n\rangle & \dots & \langle m_{i_n}, e_n\rangle \end{vmatrix}.$$

Observe that $\delta_{v,\Delta}(a)$ is non-zero exactly when the points of Δ do not lie on a hyperplane.

Our strategy now is to consider vectors $v_1, \dots, v_k \in N$, not necessarily primitive, and study when the linear span $\Lambda_{v_1,\dots,v_k}(a)$ of the tangent spaces of X at the points $\Gamma_{v_1}(a), \dots, \Gamma_{v_k}(a)$ has the expected dimension. By Lemma 2.1, the space $\Lambda_{v_1,\dots,v_k}(a)$ is the linear span of the vertical join $M_{v_1,\dots,v_k}(a)$ of the matrices $M_{v_1}(a), \dots, M_{v_k}(a)$. Given a set Δ of cardinality $n+1$, we denote by

$$b(\Delta) := \frac{1}{n+1} \sum_{m \in \Delta} m \tag{2.1}$$

its barycenter.

We will need the following result [31]. Given $K, L \subseteq [n] = \{1, 2, \dots, n\}$ and an $n \times n$ matrix A, we denote by $A_{K,L}$ the determinant of the submatrix obtained from A whose rows and columns are indexed by the set K and L, respectively.

Proposition 2.3 (Laplace's generalized expansion for the determinant). *Let A be an $n \times n$ matrix, $m < n$ a positive integer and fix a set of rows $J = \{j_1 < \dots < j_m\}$. Then*

$$\det(A) = \sum_{I = \{i_1 < \dots < i_m\} \subseteq [n]} (-1)^{i_1 + \dots + i_m + j_1 + \dots + j_m} A_{J,I} A_{J',I'},$$

where $I' = [n] \setminus I$ and $J' = [n] \setminus J$.

The following is the main technical tool in our strategy.

Proposition 2.4. *Let S be a subset of $P \cap M$ and assume the following.*

(1) *There are disjoint subsets $S_1, \dots, S_k$ of S of cardinality $n+1$ each of which is not contained in a hyperplane.*
(2) *There are $v_1, \dots, v_k \in N$ such that for each $1 \leq i \leq k$ and each subset Δ not contained in a hyperplane of cardinality $n+1$ of $S \setminus S_1 \cup \cdots \cup S_{i-1}$, the value $\langle b(\Delta), v_i \rangle$ attains its maximum exactly at $\Delta = S_i$.*

Then, up to a rescaling of the v_i if needed, the matrix $M_{v_1,\dots,v_k}(a)$ has maximal rank $(n+1)k$ for any a big enough.

Moreover, if in addition $S = P \cap M$ and $P \cap M \setminus S_1 \cup \cdots \cup S_k$ is affinely independent, then the matrix $M_{v_1,\dots,v_k,v_{k+1}}(a)$ has maximal rank $|P \cap M|$ for any a big enough and any vector $v_{k+1} \neq 0$.

Proof. First of all observe that the rank of $M_{v_1,\dots,v_k}(a)$ does not change if we multiply one of its rows by a non-zero constant. We apply this modification to the matrix by multiplying the $(i+1)$-th row of $M_v(a)$ by $a^{\langle e_i^*, v\rangle}$ for $i = 1, \dots, n$. In this way, for each subset $\Delta := \{m_{i_0}, \dots, m_{i_n}\} \subseteq S$ of cardinality $n+1$, the minor $\delta_{v,\Delta}(a)$ becomes

$$\widetilde{\delta}_{v,\Delta}(a) := a^{(n+1)\langle b(\Delta),v\rangle} \begin{vmatrix} 1 & \dots & 1 \\ \langle m_{i_0}, e_1\rangle & \dots & \langle m_{i_n}, e_1\rangle \\ \vdots & & \vdots \\ \langle m_{i_0}, e_n\rangle & \dots & \langle m_{i_n}, e_n\rangle \end{vmatrix}.$$

Let $\widetilde{M}_{v_1,\dots,v_k}(a)$ be the modified matrix and let $\widetilde{M}$ be the $(n+1) \times k$ square submatrix whose columns correspond to the points of the set S.

We denote by $\mathcal{P}(n+1, k)$ the set of partitions of S into k disjoint subsets of cardinality $n+1$. The determinant of $\widetilde{M}$ is a Laurent polynomial in $\mathbb{C}[a^{\pm 1}]$ with exponents given by sums of k terms of the form $(n+1)\langle b(\Delta), v_i\rangle$. Applying the Laplace expansion in Proposition 2.3 several times we can write this determinant as follows:

$$\det(\widetilde{M}) = \sum_{(I_1,\dots,I_k)\in\mathcal{P}(n+1,k)} \operatorname{sign}(I_1, \dots, I_k) M_{I_1} M_{I_2} \cdots M_{I_k},$$

where

$$\operatorname{sign}(I_1, \dots, I_k) = (-1)^{1+2+\cdots+(k-1)(n+1)+\sum_{j\in I_1\cup\cdots\cup I_{k-1}} j}$$

and M_{I_j} is the determinant of the $(n+1) \times (n+1)$ submatrix of $\widetilde{M}$ whose columns and rows are labeled, respectively, by I_j and $\{(j-1)(n+1)+1, \dots, j(n+1)\}$.

By the first assumption in the hypothesis, one of its terms is the non-zero product

$$\widetilde{\delta}_{v_1,S_1}(a) \cdots \widetilde{\delta}_{v_k,S_k}(a).$$

Moreover, observe that each term of the determinant has the above form for some partition of S into k disjoint subsets of cardinality $n+1$. We will show that, up to rescaling the vectors $v_1, \dots, v_k$, the above product is the leading term of the determinant and thus the matrix has maximal rank. By the second assumption in the hypothesis, the degree of $\widetilde{\delta}_{v_1,S_1}(a)$ is bigger than the degree of $\widetilde{\delta}_{v_1,\Delta}(a)$ for any $\Delta \neq S_1$. Multiplying v_1 by a positive integer, we can also assume that the degree of $\widetilde{\delta}_{v_1,S_1}(a)$ is bigger than the degree of $\widetilde{\delta}_{v_j,\Delta}(a)$ for any $j > 1$ and any $\Delta \subseteq S$ of cardinality $n+1$. In a similar way one proves inductively that, up to re-scaling v_i, the following inequalities hold:

$$\deg \widetilde{\delta}_{v_i,S_i}(a) > \begin{cases} \deg \widetilde{\delta}_{v_i,\Delta}(a) & \text{for any } \Delta \subseteq S \setminus S_1 \cup \cdots \cup S_{i-1} \text{ (by hypothesis (2))}, \\ \deg \widetilde{\delta}_{v_j,\Delta}(a) & \text{for any } j > i \text{ and any } \Delta \subseteq S \text{ (taking a multiple of } v_i). \end{cases}$$

Note that we can choose the v_i all distinct. The claim follows by comparing the degree of $\widetilde{\delta}_{v_1,S_1}(a)\cdots\widetilde{\delta}_{v_k,S_k}(a)$ with the degree of any other term of the determinant coming from a different partition of S.

Finally, if in addition $S = P \cap M$ and $P \cap M \setminus S_1 \cup \cdots \cup S_k$ is affinely independent, the matrix $M_{v_1,\ldots,v_k,v_{k+1}}(a)$ has rank at most r for any $a \neq 0$ and any vector $v_{k+1} \neq 0$, since this is the dimension of the subspace spanned by its rows. Now, consider $S_{k+1} := P \cap M \setminus S_1 \cup \cdots \cup S_k = \{m_{j_1}, \ldots, m_{j_s}\}$ and $s = r - (n+1)k$. Since S_{k+1} is affinely independent, there are $k_1, \ldots, k_{s-1}$ such that the $s \times s$ matrix

$$N = \begin{pmatrix} 1 & \ldots & 1 \\ \langle m_{j_1}, e_{k_1}\rangle & \ldots & \langle m_{j_s}, e_{k_1}\rangle \\ \vdots & & \vdots \\ \langle m_{j_1}, e_{k_{s-1}}\rangle & \ldots & \langle m_{j_s}, e_{k_{s-1}}\rangle \end{pmatrix}$$

has rank s. Consider the submatrix

$$N_{v_{k+1}}(a) := \begin{pmatrix} a^{\langle m_{j_1}, v\rangle} & \ldots & a^{\langle m_{j_s}, v\rangle} \\ \langle m_{j_1}, e_{k_1}\rangle a^{\langle m_{j_1} - e^*_{k_1}, v\rangle} & \ldots & \langle m_{j_s}, e_{k_1}\rangle a^{\langle m_{j_s} - e^*_{k_1}, v\rangle} \\ \vdots & & \vdots \\ \langle m_{j_1}, e_{k_{s-1}}\rangle a^{\langle m_{j_1} - e^*_{k_{s-1}}, v\rangle} & \ldots & \langle m_{j_s}, e_{k_{s-1}}\rangle a^{\langle m_{j_s} - e^*_{k_{s-1}}, v\rangle} \end{pmatrix}$$

of $M_{v_{k+1}}(a)$ obtained from $M_{v_{k+1}}(a)$ taking only the rows $1, k_1+1, \ldots, k_{s-1}+1$. Now, we repeat this construction using $N_{v_{k+1}}(a)$ instead of $M_{v_{k+1}}(a)$ and obtain a $r \times r$ matrix with non-zero determinant. Since it is a submatrix of $M_{v_1,\ldots,v_k,v_{k+1}}(a)$, we conclude that $M_{v_1,\ldots,v_k,v_{k+1}}(a)$ has rank r. ∎

The following is inspired by Proposition 2.4.

Definition 2.5. We say that $\Delta \subseteq M$ is a simplex if Δ contains $n+1$ integer points and it is not contained in an affine hyperplane. For any vector $v \in N$, consider the linear form $\varphi_v \colon M \to \mathbb{R}$ given by $\varphi_v(p) = \langle p, v\rangle$. We will write

$$\varphi_v(\Delta) = \frac{1}{n+1}\sum_{p\in\Delta}\varphi_v(p).$$

We say that v separates the simplex Δ in a subset $S \subseteq M$ if

$$\max\{\varphi_v(T); T \subseteq S \mid T \text{ is a simplex}\} = \varphi_v(\Delta)$$

and the maximum is attained only at Δ.

Remark 2.6. Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, $\Delta_1, \ldots, \Delta_k$ disjoint simplexes contained in $P \cap M$ and $v_1, \ldots, v_k$ vectors in N. Assume that v_i separates Δ_i in $\Delta_i \cup \cdots \cup \Delta_k$ for any $i = 1 \ldots k$. Since the maximum in Definition 2.5 is attained just once, if we take vectors $w_1, \ldots, w_k$ in N close enough to the v_i, then w_i separates Δ_i in $\Delta_i \cup \cdots \cup \Delta_k$ for any $i = 1 \ldots k$. Therefore, we may assume without loss of generality that the v_i are distinct. Observe that if v_i separates Δ_i in $\Delta_i \cup \cdots \cup \Delta_k$ for any $i = 1 \ldots k$, then any multiple of the v_i will do so as well.

As a consequence of Proposition 2.4, we get the following criterion for non-secant defectiveness of toric varieties.

Theorem 2.7. *Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, X_P the corresponding toric variety, $\Delta_1, \ldots, \Delta_k$ disjoint simplexes contained in $P \cap M$ and $v_1, \ldots, v_k$ vectors in N. If v_i separates Δ_i in $\Delta_i \cup \cdots \cup \Delta_k$ for any $i = 1 \ldots k$, then X_P is not k-defective. Moreover, if $(P \cap M) \setminus \Delta_1, \ldots, \Delta_k$ is affinely independent, then X_P is not defective. In particular, if $P \cap M = \Delta_1 \cup \cdots \cup \Delta_k$, then X_P is not defective.*

Proof. Without loss of generality we can assume that P is contained in the positive quadrant and contains the origin. Applying Proposition 2.4 with $S = \Delta_1 \cup \cdots \cup \Delta_k$, we get that $M_{v_1,\ldots,v_k}(a)$ has maximal rank for any a big enough and the v_i are distinct, taking multiples if necessary. Take any a big enough, then the tangent spaces of X_P at the points $\Gamma_{v_1}(a), \ldots, \Gamma_{v_k}(a)$ are in general position. By Terracini's lemma [49], we conclude that X_P is not k-defective. For the second statement just use the second part of Proposition 2.4. ∎

Since Theorem 2.7 in principle can be applied to any toric variety, in particular, to Segre–Veronese varieties, one just need to describe the vectors $v_1, \ldots, v_k$. Due to its recursive nature Theorem 2.7 can be algorithmically implemented.[1] The algorithm is quickly explained in Algorithm 1. In what follows $M \simeq \mathbb{Z}^n$ and $N := \mathrm{Hom}(M, \mathbb{Z})$ is its dual. Denote by $M_{\mathbb{Q}}$ the corresponding rational vector space. Giving a subset $S \subseteq M_{\mathbb{Q}}$ we say that S is independent if it is affinely independent and we say that it is full-dimensional if its affine span is the whole space.

Algorithm 1 can show that a toric variety is not defective but can not determine whether it is defective. Furthermore, some details must be considered. That is if the output is `false`, then all the secant varieties of the toric variety X_S are not defective. On the other hand there is no guarantee that if the output is `true`, then X_S admits a defective r-secant variety for some r. Due to this we sometimes apply Algorithm 1 several times to improve the possibility of getting a correct result in case the output is `true`.

We where able to use an implementation of this algorithm in MAGMA [12] in order to find several non-defective Segre–Veronese varieties, see Section 5.

Definition 2.8. Given a finite subset $S \subseteq M$, the *barycentric polytope* of S, denoted by $B(S) \subseteq M_{\mathbb{Q}}$, is the convex hull of all the points $b(\Delta)$, where Δ varies among all the subsets of S of cardinality $n + 1$ which are not contained in a hyperplane and $b(\Delta)$ is as in (2.1).

[1] A Magma library which implements an algorithm based on Theorem 2.7 can be downloaded from the following link: https://github.com/alaface/secant-algorithm

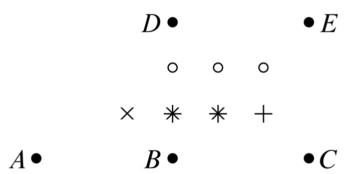
Input : a finite, full-dimensional subset $S \subseteq M$
while S *is full-dimensional* **do**
  choose $v \in N_{\mathbb{Q}}$ such that φ_v is injective on S;
  reorder S increasingly according to φ_v;
  define $\Delta := \{\max(S)\}$;
  repeat
    $x := \max\{u \in S \setminus \Delta : \Delta \cup \{u\}$ is independent$\}$;
    $\Delta := \Delta \cup \{x\}$;
  until Δ *is full-dimensional*;
  $S := S \setminus \Delta$;
end
if S *is independent* **then return** false ;
else return true ;

Algorithm 1: Algorithm to check non-defectiveness based on Theorem 2.7.

Example 2.9. Consider

$$S = \{A = (0,0), B = (1,0), C = (2,0), D = (1,1), E = (2,1)\}$$

as in the picture below. We have nine possible ways to form simplexes $\Delta \subseteq S$ and the barycentric polytope $B(S)$ is a trapezoid. In the picture we draw circles in the barycenters of simplexes Δ with $D, E \in \Delta$, we draw $+$ on barycenters of simplexes with $E \in \Delta$ but $D \notin \Delta$, and finally we draw $\times$ in barycenters of simplexes with $D \in \Delta$ but $E \notin \Delta$.



There are exactly two shared barycenters, corresponding to the pairs of simplexes

$$\Delta = \{A, D, C\} \quad \text{and} \quad \Delta' := \{A, B, E\},$$
$$\Delta = \{B, C, D\} \quad \text{and} \quad \Delta' := \{A, C, E\}.$$

Note that neither of these shared barycenters are vertexes of $B(S)$. The next lemma shows that this is always the case.

Now, we prove two technical lemmas in order to get a general bound for non-secant defectiveness of toric varieties from Theorem 2.7. In Section 3 we will specialize this bound to Segre–Veronese varieties.

Lemma 2.10. *Let Δ, Δ' be two simplexes in $S \subseteq M$. If $b(\Delta) = b(\Delta')$, then $b(\Delta)$ is not a vertex of $B(S)$.*

Proof. Let $\Delta = \{p_1, \dots, p_{n+1}\}$ and $\Delta' = \{p'_1, \dots, p'_{n+1}\}$. We say that the pair (p_i, p'_j) is good if

$$\Delta_{ij} := (\Delta \setminus \{p_i\}) \cup \{p'_j\} \quad \text{and} \quad \Delta'_{ij} := (\Delta' \setminus \{p'_j\}) \cup \{p_i\}$$

are simplexes. Observe that it is enough to show that there exists a good pair with $\Delta_{ij} \neq \Delta$, since in this case $b(\Delta) = b(\Delta')$ is the mid point of the segment with vertexes $b(\Delta_{ij})$ and $b(\Delta'_{ij})$. To show the existence of a good pair let us denote by Λ_i the hyperplane spanned by $\Delta \setminus \{p_i\}$, and by Λ'_i the hyperplane spanned by $\Delta' \setminus \{p'_i\}$.

Note that if either $p_i \in \Lambda'_j$ or $p'_j \in \Lambda_i$ then the pair (p_i, p'_j) is not good and vice versa. Assume $p_1 \notin \Delta'$. We now show that at least one pair (p_1, p'_i) is good. Indeed, assuming the contrary, we can partition the set $\{1, \dots, n+1\}$ into a disjoint union $I \cup J$ of two subsets such that $p_1 \in \Lambda'_j$ for any $j \in I$ and $p'_i \in \Lambda_1$ for any $i \in I$. Then we would get

$$p_1 \in \bigcap_{j \in J} \Lambda'_j = \langle p'_i : i \in I \rangle \subseteq \Lambda_1,$$

a contradiction. ■

Lemma 2.11. *Let $S \subseteq P \cap M$ be a subset not contained in a hyperplane. Then there exists a vector in N, with non-negative entries, separating a simplex in S.*

Proof. Without loss of generality we can assume that P is contained in the positive quadrant. First, assume that there is a vertex $b(\Delta)$ of $b(S)$ whose i-th coordinate is strictly bigger than those of the other vertexes of $b(S)$. In this case we may simply take $v = e_i^*$. Now, if there are several vertexes with the same i-th coordinate, say for $i = 1$, then among these we check if there is only one maximizing the 2-th coordinate. If so, we choose $v = ae_1^* + e_2^*$ with $a \gg 0$. If not, among the vertexes maximizing also the 2-coordinate, we consider those maximizing the 3-th coordinate. As before we have two cases. In the first, we take $v = ae_1^* + be_2^* + e_3^*$ with $a \gg b \gg 0$, while in the second case, among these vertexes, we consider those maximizing the 4-th coordinate. Proceeding recursively in this way and noting that a vertex of $b(S)$ corresponds to a, unique by Lemma 2.10, barycenter of a simplex in S, we get the claim. ■

We provide a bound for non-secant defectiveness of the projective toric variety X associated to a polytope P by counting the maximum number of integer points on a hyperplane section of P.

Theorem 2.12. *Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, $X_P \subseteq \mathbb{P}^{|P \cap M|-1}$ the corresponding n-dimensional toric variety, and m the maximum number of integer points in a hyperplane section of P. If*

$$h \leq \frac{|P \cap M| - m}{n+1},$$

then X_P is not h-defective.

Proof. Set $S := P \cap M$. By Lemma 2.11, there is a vector $v_1 \in N$ separating a simplex Δ_1 in P. Now, consider $S \setminus \Delta_1$. If $|S \setminus \Delta_1| > m$, then S is not contained in a hyperplane and we may apply again Lemma 2.11 to get a second vector $v_2 \in N$ separating a simplex Δ_2 in $S \setminus \Delta_1$. Proceeding recursively in this way, as long as $|S \setminus (\Delta_1 \cup \dots \cup \Delta_k)| > m$, we get the statement by Theorem 2.7. ■

In order to apply Theorem 2.12 in specific cases we will make use of the following result asserting that the maximum number of integer points of P lying on a hyperplane is attained on a facet.

Proposition 2.13. *Let $P \subseteq M_{\mathbb{Q}}$ be full-dimensional lattice polytope such that there exist linearly independent $v_1, \dots, v_n \in N$ and facets $F_1, \dots, F_n$ such that for any i, we have $v_j(F_i \cap M) = v_j(P \cap M)$ for any $j \neq i$. Then, given a hyperplane $H \subseteq M_{\mathbb{Q}}$, there exists a facet F_i, with $1 \leq i \leq n$, such that $|H \cap P \cap M| \leq |F_i \cap M|$.*

Proof. Consider the map

$$\pi_i \colon M_{\mathbb{Q}} \to \mathbb{Q}^{n-1}, \quad x \mapsto (v_1(x), \dots, v_{i-1}(x), v_{i+1}(x), \dots, v_n(x)).$$

Note that, by hypothesis, $\pi_i(F_i \cap M) = \pi_i(P \cap M)$. Observe that there exists an index i such that the restriction of π_i to H is injective. Then $|H \cap P \cap M| = |\pi_i(H \cap P \cap M)| \leq |\pi_i(F_i \cap M)| \leq |F_i \cap M|$. ∎

2.1. An alternative proof of Theorem 2.12

The bound in Theorem 2.12 is, to the best of our knowledge, the first general bound for non-secant defectiveness of toric varieties appearing in the literature.

A machinery based on tropical geometry was introduced to study secant defectiveness by J. Draisma in [24]. In order to use this tropical technique, one has to produce a regular partition of the polytope P that is a subdivision into polyhedral cones such that none of the integer points of P lies on the boundaries.

We thank J. Draisma for explaining this to us. In this section we give another proof of Theorem 2.12 based on Draisma's tropical approach.

Lemma 2.14. *Let $P \subset \mathbb{R}^n$ be a convex lattice polytope. There exist a lattice simplex $\Delta \subset P$ and an affine hyperplane $H \subset \mathbb{R}^n$ separating Δ from the convex hull of the integer points of $P \setminus \Delta$.*

This is equivalent to say that there exists a degree one polynomial $h \colon \mathbb{R}^n \to \mathbb{R}$ that is positive on all the integer points of Δ and negative on all the integer points of $P \setminus \Delta$.

Proof. We proceed by induction on n. Assume that the statement holds for all the polytopes of dimension at most $n-1$.

Let $v \in P$ be a vertex, and denote by $v_1, \dots, v_m$ the end-points of the edges of P starting at v. Let P' be the convex hull of $v, v_1, \dots, v_m$ and P'' the convex hull of the integer points of P' except v. Note that $v \notin P''$. Consider a facet of P'' that can be connected with v by a segment that does not intersect the interior of P''. If P'' has dimension $n-1$, then we can take the whole P'' as such a facet. Let H be the hyperplane containing this face. Then H intersects only the edges of P that are adjacent to v.

Now, cut P along H and denote by Q the part that contains v, and by F the face of Q that lies in H. By construction, the integer points of Q are the integer points of F and v. By the induction hypothesis, we can cut out a simplex Δ' in F by a hyperplane H' of dimension $n-2$ contained in H. Finally, consider a hyperplane obtained by performing an infinitesimal rotation of H around H'. Such a hyperplane separates the simplex Δ generated by Δ' and v from the convex hull of the integer points of $P \setminus \Delta$. ∎

Before stating the next result, we recall the definition of regular subdivision of a lattice polytope Definition 2.2.10 of [20]. Let $P \subseteq \mathbb{R}^n$ be a lattice polytope, J the set of indexes of the lattice points of P and $w\colon J \to \mathbb{R}$ a function. Let $P^w \subseteq \mathbb{R}^{n+1}$ be the convex hull of the points $p_i^w := (p_i, w(p_i))$ for each $i \in J$.

The *regular subdivision* of P produced by w is the set of projected lower faces of P^w. This regular subdivision is denoted by $\mathcal{S}(P, w)$.

Theorem 2.15. *Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope and X_P the corresponding n-dimensional toric variety. Consider a regular subdivision of P into k open simplexes such that no integer point of P lies on the boundaries. Assume that among these simplexes exactly k_i are i-dimensional. Then*

$$\dim(\mathbb{S}\mathrm{ec}_k(X_P)) \geq \sum_{i=0}^{n} k_i(i+1) - 1.$$

In particular, if in the regular subdivision of P there are k full-dimensional simplexes, then X_P is not k-defective.

Proof. In the terminology of Section 2 of [24], the integer points of P lying in a simplex are a set of winning directions. Therefore, the statement follows from Corollary 2.3 of [24]. ∎

Lemma 2.16. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope and let $\Delta \subseteq P$ be a lattice simplex which can be separated from the convex hull P_0 of the lattice points of $P \setminus \Delta$ by a hyperplane H. Then, given a regular subdivision $\mathcal{S}(P_0, w_0)$ of P_0, there exists a regular subdivision $\mathcal{S}(P, w)$ of P which contains all the polytopes in $\mathcal{S}(P_0, w_0)$ and such that $\Delta \in \mathcal{S}(P, w)$.*

Proof. We denote by J_0 and J the indexes for the set of lattice points of P_0 and P, respectively. By definition, we have $J_0 \subseteq J$. We define $w\colon J \to \mathbb{R}$ as $w_{|J_0} := w_0$ and extend it to $J \setminus J_0$ as follows.

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be the function which defines H. After possibly perturbing h, we can assume that it takes distinct values on the set of vertexes $\{p_i : i \in J \setminus J_0\}$ of Δ. As a consequence this set is totally ordered. After possibly relabeling J, we can assume $J \setminus J_0 = \{0, \ldots, n\}$ and $h(p_i) < h(p_j)$ if $i < j$, and both indexes are in $J \setminus J_0$.

Define $w(p_0)$ in such a way that it is bigger than $w(p_i)$ for any $i \in J_0$. In this way the convex hull of $\{p_0^w\} \cup P_0^w$ contains all the lower facets of P_0^w. Now, assume that w has been defined on p_i for $0 \leq i < r$ and define $w(p_r) > w(p_{r-1})$ so that for each point (p, α) in the convex hull of $\{p_0^w, \ldots, p_r^w\}$ and each point (p, β) in the convex hull of $\{p_0^w, \ldots, p_{r-1}^w\} \cup P_0^w$, the inequality $\alpha \geq \beta$ holds.

By construction, the convex hull of $\{p_0, \ldots, p_r\}$ is in the latter regular subdivision. Moreover, due to the fact that all the points $p_0^w, \ldots, p_r^w$ have last coordinate bigger than those of the remaining lifted lattice points of P_0, it follows that any lower face of P_0 is in the latter regular subdivision. The statement follows by induction on r. ∎

Remark 2.17. While applying the inductive procedure to produce the new regular subdivision in Lemma 2.16 several new regular subdivisions can be created and destroyed along the way as shown in Figure 1. Note that at each step the regular subdivision of P_0 is left unaltered.

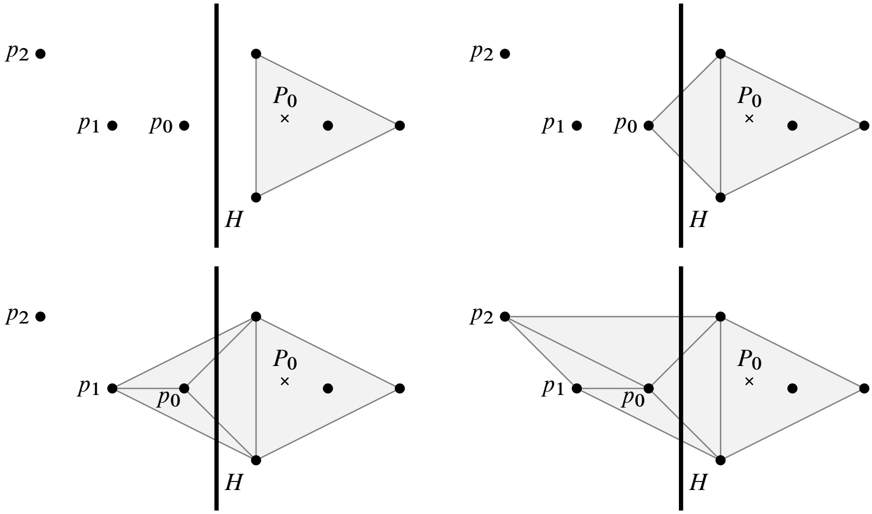


Figure 1. Regular subdivisions of a polytope in the proof of Lemma 2.16.

Alternative proof of Theorem 2.12. Set $S := P \cap M$. By Lemma 2.14, there is a hyperplane H_1 separating a simplex Δ_1 in P. Now, consider $S \setminus \Delta_1$. If $|S \setminus \Delta_1| > m$, then S is not contained in a hyperplane and we may apply again Lemma 2.14 to get a second hyperplane H_2 separating a simplex Δ_2 in $S \setminus \Delta_1$. Proceeding recursively in this way, as long as $|S \setminus (\Delta_1 \cup \cdots \cup \Delta_k)| > m$, and applying Lemma 2.16, we get the statement by Theorem 2.15. ∎

Remark 2.18. The main difference between our method for checking non-defectiveness and the tropical one described in Theorem 2.12 is the following. In both methods one has to separate a lattice simplex Δ from the convex hull of the set S of lattice points. In our case this means that one has to separate a vertex of the barycentric polytope, while in the tropical case one has to separate the lattice points in Δ from the remaining ones by means of a hyperplane. It is clear that the latter separation implies the former but the converse is not true in general as shown by the following example. Let

$$S_1 := \{P_1, P_2, P_3\}, \quad S_2 := \{Q_1, Q_2, Q_3\},$$

where

$$P_1 = (0,0),\ P_2 = (3,1),\ P_3 = (4,0),\ Q_1 = (-1,-2),\ Q_2 = (1,3),\ Q_3 = (2,2).$$

Then $v = (1,0)$ separates S_1 in $S_1 \cup S_2$. However, the convex hulls of S_1 and S_2 overlap as shown in Figure 2. In particular, the convex hulls of the simplexes in Proposition 2.4 may overlap. Our method thus starts from determining a general linear form ϕ on the linear span $\langle S \rangle$ and then separating the simplex whose barycenter has the biggest value with respect to ϕ. In the tropical approach one has to check whether the $n+1$ lattice points corresponding to the biggest $n+1$ values of ϕ span a simplex. Otherwise, another ϕ has to

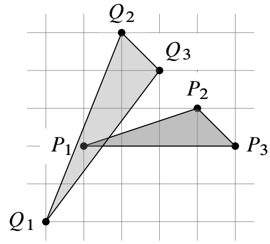


Figure 2. The convex hulls of S_1 and S_2.

be chosen. In the above example, the form corresponding to $(1, 0)$ does not work with the tropical method, while the form corresponding to $(-1, 1)$ gives a hyperplane separating $\{P_1, Q_2, Q_3\}$ from $\{P_2, P_3, Q_1\}$.

3. Bounds for Segre–Veronese varieties

Let $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r}$ be the Segre–Veronese variety given as the image in $\mathbb{P}^N$ with

$$N = \prod_{i=1}^{r} \binom{n_i + d_i}{d_i} - 1$$

of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ under the embedding induced by $|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}}(d_1, \dots, d_r)|$. In the following, we prove our main result.

Theorem 3.1. *The Segre–Veronese variety* $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r} \subseteq \mathbb{P}^N$ *is not h-defective for*

$$h \le \frac{d_j}{n_j + d_j} \frac{1}{1 + \sum_{i=1}^{r} n_i} \prod_{i=1}^{r} \binom{n_i + d_i}{d_i},$$

where $n_j/d_j = \max_{1\le i\le r}\{n_i/d_i\}$.

Proof. Let $\Delta^{n_i}_{d_i} \subseteq \mathbb{Q}^{n_i+1}$ be the standard simplex. The polytope $P = \Delta^{n_1}_{d_1} \times \cdots \times \Delta^{n_r}_{d_r}$ has

$$\prod_{i=1}^{r} \binom{d_i + n_i}{d_i}$$

integer points, and each facet is given by the Cartesian product of a facet of one of the $\Delta^{n_j}_{d_j}$ and the remaining $\Delta^{n_i}_{d_i}$ for $i \ne j$. Therefore, each facet contains

$$f_j = \binom{d_j + n_j - 1}{d_j} \prod_{i\ne j}^{r} \binom{d_i + n_i}{d_i}$$

points for some j. Now, we compare the number of integer points on each facet:

$$f_j \le f_k,$$

$$\binom{d_j+n_j-1}{d_j}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i} \leq \binom{d_k+n_k-1}{d_k}\prod_{i\neq k}^{r}\binom{d_i+n_i}{d_i},$$
$$\binom{d_j+n_j-1}{d_j}\binom{d_k+n_k}{d_k} \leq \binom{d_k+n_k-1}{d_k}\binom{d_j+n_j}{d_j},$$
$$\frac{d_k+n_k}{n_k} \leq \frac{d_j+n_j}{n_j},$$
$$\frac{d_k}{n_k} \leq \frac{d_j}{n_j}.$$

Therefore, the facet with maximum number of integer points is the one which minimizes d_i/n_i and so maximizes n_i/d_i. Assume that $n_j/d_j = \max_{1\leq i\leq r}\{n_i/d_i\}$.

Since P satisfies the conditions in Proposition 2.13 the maximum number of integer points in a hyperplane section of P is attained on a facet and in this case it is given by

$$\binom{d_j+n_j-1}{d_j}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i}.$$

Finally, to conclude it is enough to note that

$$\frac{1}{1+\sum_i n_i}\left(\prod_{i=1}^{r}\binom{d_i+n_i}{d_i} - \binom{d_j+n_j-1}{d_j}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i}\right)$$
$$= \frac{1}{1+\sum_i n_i}\binom{d_j+n_j-1}{d_j-1}\prod_{i\neq j}^{r}\binom{d_i+n_i}{d_i}$$
$$= \frac{1}{1+\sum_i n_i}\,\frac{d_j}{d_j+n_j}\prod_{i=1}^{r}\binom{d_i+n_i}{d_i} = \frac{1}{1+n_j/d_j}\,\frac{1}{1+\sum_i n_i}\prod_{i=1}^{r}\binom{d_i+n_i}{d_i}$$

and to apply Theorem 2.12. ■

Remark 3.2. According to Theorem 3.1, we have a polynomial bound of degree $\sum_i n_i$ in the d_i, while in the n_i we have a polynomial bound of degree $\sum_i d_i - 2$.

A bound for non-secant defectiveness of Segre varieties was given in Theorem 1.1 of [29] using the inductive machinery developed in [5]. When the numbers $n_i + 1$ are powers of two Corollary 5.1 of [29] gives a sharp asymptotic bound for non-secant defectiveness of Segre varieties. However, for general values of the n_i the bound in Theorem 1.1 of [29] tends to zero when r goes to infinity.

Proposition 3.3. *The Segre–Veronese variety* $\mathrm{SV}^{1,n}_{2k+1,2}$ *is not defective. Furthermore,* $\mathrm{SV}^{1,n}_{2k,2}$ *is not h-defective for* $h \leq k(n+1)$.

Proof. Let us begin with $\mathrm{SV}^{1,n}_{2k+1,2}$. The corresponding polytope is $P = \Delta^1_{2k+1}\times\Delta^n_2$, where

$$\Delta^1_{2k+1} = \{0,1,\dots,2k+1\} \quad\text{and}\quad \Delta^n_2 = \{(x_1,\dots,x_n)\in\mathbb{Z}_{\geq 0};\textstyle\sum x_j \leq 2\}.$$

We view P as a union of $2k+2$ floors labeled by Δ^1_{2k+1}. We will triangulate each pair of floors. Note that it is enough to do this in the case $k=0$, where we have just two floors.

Consider the following disjoint subsets of P:

$$\begin{aligned} S_1 &= \{e_1+e_2\} \cup \{e_1+e_2+e_j;\, j=2\dots n+1\} \cup \{e_2+e_2\},\\ S_2 &= \{e_1+e_3\} \cup \{e_1+e_3+e_j;\, j=3\dots n+1\} \cup \{e_3+e_j;\, j=2,3\},\\ &\vdots\\ S_n &= \{e_1+e_{n+1}\} \cup \{e_1+e_{n+1}+e_{n+1}\} \cup \{e_{n+1}+e_j;\, j=2,\dots,n+1\},\\ S_{n+1} &= \{(0,\dots,0)\} \cup \{e_j;\, j=1\dots n+1\}. \end{aligned}$$

Note that each set S_i has cardinality $n+2$, and since $|P| = 2\binom{n+2}{2} = (n+1)(n+2)$, we have $P = \bigcup_{i=1}^{n+1} S_i$. Moreover, each S_i is an $(n+1)$-simplex in $\mathbb{Q}^{n+1}$.

Now, consider integers

$$b_1 \gg b_2 \gg \cdots \gg b_{n+1} > 0$$

and vectors

$$\begin{aligned} v_1 &= (b_1, b_2, 0, \dots, 0),\\ v_2 &= (b_1, b_3, b_2, 0, \dots, 0),\\ v_3 &= (b_1, b_4, b_3, b_2, 0, \dots, 0),\\ &\vdots\\ v_n &= (b_1, b_{n+1}, \dots, b_2),\\ v_{n+1} &= (1, 1, \dots, 1). \end{aligned}$$

We will show that these vectors and simplexes make Theorem 2.7 work. In the first step, in order to maximize $\langle b(\Delta), v_1\rangle$, we need that Δ has the maximum possible number of points on the top floor, corresponding to e_1. Furthermore, since e_2 appears in all the vectors of S_1 and $b_2 \gg b_3 \gg \cdots \gg b_{n+1}$ among the simplexes having $n+1$ points on the top floor, the one maximizing $\langle b(\Delta), v_1\rangle$ is S_1. Therefore, v_1 separates S_1.

Now, note that the remaining points on the top floor are exactly the ones in the hyperplane $x_2 = 0$. Then, among the simplexes with points in $S \setminus S_1$, the ones maximizing $\langle b(\Delta), v_2\rangle$ must have n points on the top floor and two on the bottom floor. Since $b_2 \gg b_3$, the points on the top floor must have the third coordinate non-zero, and since there are exactly n of these, we have to take all of them. By the same argument on the bottom floor, we have to take $(0,1,1,0,\dots,0)$ and $(0,0,2,0,\dots,0)$. Hence, v_2 separates S_2.

Now, the remaining points on the top floor are in the linear space $x_2 = x_3 = 0$. Arguing similarly, we see that $v_1,\dots,v_n$ separate $S_1,\dots,S_n$. In the last step there are just $n+2$ points left and these form a simplex. Setting $S_{n+1} = \Delta \setminus \bigcup_{i=1}^n S_i$ any vector v_{n+1} will do.

Therefore, for each pair of floors, we construct $n+1$ simplexes and since we have $k+1$ pairs of floors, Theorem 2.7 yields that $SV^{1,n}_{2k+1,2} \subseteq \mathbb{P}^N$ is not h-defective for $h \le (k+1)(n+1)$. Then

$$\dim \mathbb{S}\mathrm{ec}_{(k+1)(n+1)}(SV^{1,n}_{2k+1,2}) = (k+1)(n+1)^2 + (k+1)(n+1) - 1 = N$$

and $SV^{1,n}_{2k+1,2} \subseteq \mathbb{P}^N$ is not defective.

Now, consider $\mathrm{SV}^{1,n}_{2k,2}$. In this case we have $2k+1$ floors. Considering just the first $2k$ of them and arguing as in the previous case, we get that $\mathrm{SV}^{1,n}_{2k,2}$ is not h-defective for $h \leq k(n+1)$. ∎

Remark 3.4. The non-secant defectiveness of $\mathrm{SV}^{1,n}_{2k+1,2}$ was proven, by different methods, in Proposition 3.1 of [6]. Furthermore, by Proposition 3.2 of [6], $\mathrm{SV}^{1,n}_{2k,2}$ is h-defective for $k(n+1)+1 \leq h \leq k(n+1)+n$.

3.1. Identifiability

Let $X \subseteq \mathbb{P}^N$ be an irreducible non-degenerated variety. A point $p \in \mathbb{P}^N$ is said to be h-identifiable, with respect to X, if it lies on a unique $(h-1)$-plane h-secant to X. We say that X is h-identifiable if the general point of $\mathbb{S}ec_h(X)$ is h-identifiable.

Corollary 3.5. *Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, X_P the corresponding n-dimensional toric variety, and m the maximum number of points on a hyperplane section of $P \cap M$. Assume that $2n < \frac{|P\cap M|-m}{n+1}$. Then X_P is $(h-1)$-identifiable for*

$$h \leq \frac{|P \cap M| - m}{n+1}.$$

Proof. It is enough to apply Theorem 2.12 and Theorem 3 of [14]. ∎

Corollary 3.6. *Consider the Segre–Veronese variety $\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r} \subseteq \mathbb{P}^N$, set*

$$\frac{n_j}{d_j} = \max_{1\leq i\leq r}\left\{\frac{n_i}{d_i}\right\}$$

and assume that $2\sum_{i=1}^r n_i < \frac{d_j}{n_j+d_j}\frac{1}{1+\sum_{i=1}^r n_i}\prod_{i=1}^r \binom{n_i+d_i}{d_i}$. Then, for

$$h \leq \frac{d_j}{n_j+d_j}\,\frac{1}{1+\sum_{i=1}^r n_i}\prod_{i=1}^r \binom{n_i+d_i}{d_i},$$

$\mathrm{SV}^{n_1,\dots,n_r}_{d_1,\dots,d_r} \subseteq \mathbb{P}^N$ is $(h-1)$-identifiable.

Proof. It is enough to apply Theorem 3.1 and Theorem 3 of [14]. ∎

Results on the identifiability of Segre–Veronese varieties have been recently given in [25], and in [10] under hypotheses on non-secant defectiveness.

4. New examples of defective Segre–Veronese varieties

In this section we give examples of defective Segre–Veronese varieties using three different methods. Namely, by the general theory of flattenigs in Section 4.1, by constructing low degree rational normal curves in Segre–Veronese varieties in Section 4.2, and by producing special Cremona transformations of product of projective lines in Section 4.3. As noticed in Remarks 4.8 and 4.9, the defective Segre–Veronese varieties in Sections 4.2 and 4.3 were already well known even though the methods we present are new.

4.1. Flattenings

Let $V_1, \dots, V_p$ be vector spaces of finite dimension, and consider the tensor product $V_1 \otimes \dots \otimes V_p = (V_{a_1} \otimes \dots \otimes V_{a_s}) \otimes (V_{b_1} \otimes \dots \otimes V_{b_{p-s}}) = V_A \otimes V_B$ with $A \cup B = \{1, \dots, p\}$, $B = A^c$. Then we may interpret a tensor

$$T \in V_1 \otimes \dots \otimes V_p = V_A \otimes V_B$$

as a linear map $\widetilde{T}: V_A^* \to V_{A^c}$. Clearly, if the rank of T is at most r, then the rank of $\widetilde{T}$ is at most r as well. Indeed, a decomposition of T as a linear combination of r rank one tensors yields a linear subspace of V_{A^c}, generated by the corresponding rank one tensors, containing $\widetilde{T}(V_A^*) \subseteq V_{A^c}$. The matrix associated to the linear map $\widetilde{T}$ is called an (A, B)-*flattening* of T.

In the case of mixed tensors, we can consider the embedding

$$\mathrm{Sym}^{d_1} V_1 \otimes \dots \otimes \mathrm{Sym}^{d_p} V_p \hookrightarrow V_A \otimes V_B,$$

where $V_A = \mathrm{Sym}^{a_1} V_1 \otimes \dots \otimes \mathrm{Sym}^{a_p} V_p$ and $V_B = \mathrm{Sym}^{b_1} V_1 \otimes \dots \otimes \mathrm{Sym}^{b_p} V_p$, with $d_i = a_i + b_i$ for any $i = 1, \dots, p$. In particular, if $n = 1$, we may interpret a tensor $F \in \mathrm{Sym}^{d_1} V_1$ as a degree d_1 homogeneous polynomial on $\mathbb{P}(V_1^*)$. In this case, the matrix associated to the linear map $\widetilde{F}: V_A^* \to V_B$ is nothing but the a_1-th *catalecticant matrix* of F, that is, the matrix whose rows are the coefficient of the partial derivatives of order a_1 of F.

Remark 4.1. Consider a tensor $T \in \mathrm{Sym}^{d_1} \mathbb{C}^{n_1+1} \otimes \mathrm{Sym}^{d_2} \mathbb{C}^{n_2+1} \otimes \mathrm{Sym}^{d_3} \mathbb{C}^{n_3+1}$ and the flattening

$$\mathrm{Sym}^{d_1} \mathbb{C}^{n_1+1} \otimes \mathrm{Sym}^{d_2-k} \mathbb{C}^{n_2+1} \to \mathrm{Sym}^{k} \mathbb{C}^{n_2+1} \otimes \mathrm{Sym}^{d_3} \mathbb{C}^{n_3+1}.$$

Fix coordinates $x_0, \dots, x_{n_2}$ on $\mathbb{C}^{n_2+1}$ and $v_0, \dots, v_{n_3}$ on $\mathbb{C}^{n_3+1}$. Then the matrix of the above flattening has the following form:

$$\begin{pmatrix} \frac{\partial^{d_3}}{\partial v_0^{d_3}} \frac{\partial^k}{\partial x_0^k} T \\ \vdots \\ \frac{\partial^{d_3}}{\partial v_0^{d_3}} \frac{\partial^k}{\partial x_{n_2}^k} T \\ \vdots \\ \frac{\partial^{d_3}}{\partial v_{n_3}^{d_3}} \frac{\partial^k}{\partial x_0^k} T \\ \vdots \\ \frac{\partial^{d_3}}{\partial v_{n_3}^{d_3}} \frac{\partial^k}{\partial x_{n_2}^k} T \end{pmatrix}.$$

Note that T has $\binom{n_2+k}{n_2}$ partial derivatives of order k with respect to $x_0, \dots, x_{n_2}$ and each of these derivatives has in turn $\binom{n_3+d_3}{n_3}$ partial derivatives of order d_3 with respect to $v_0, \dots, v_{n_3}$. Therefore, this is a matrix of size $\binom{n_3+d_3}{n_3}\binom{n_2+k}{n_2} \times \binom{n_2+d_2-k}{n_2}\binom{n_1+d_1}{n_1}$.

In general, the $(h+1) \times (h+1)$ minors of the above matrix yield equations for the secant variety $\mathbb{S}\mathrm{ec}_h(\mathrm{SV}^{(n_1,n_2,n_3)}_{(d_1,d_2,d_3)})$. However, in practice it is hard to compute the codimension of the variety cut out by these minors. In a Magma script, that can be found as an ancillary file in the arXiv version of the paper, we manage to simplify the computations. The script reduces the equations given by the minors to positive characteristic. The variety cut out by these reduced equations has dimension greater or equal than our original variety. So if this dimension is strictly less that the expected dimension of $\mathbb{S}ec_h(SV^{(n_1,n_2,n_3)}_{(d_1,d_2,d_3)})$, we get that $SV^{(n_1,n_2,n_3)}_{(d_1,d_2,d_3)}$ is h-defective.

Proposition 4.2. *The Segre–Veronese variety* $SV^{(1,1,2)}_{(1,5a+3,1)}$ *is* $(6a+5)$*-defective for all* $a \geq 0$*, and the Segre–Veronese variety* $SV^{(1,1,2)}_{(1,5a+5,1)}$ *is* $(6a+7)$*-defective for all* $a \geq 0$.

Proof. We begin with $SV^{(1,1,2)}_{(1,5a+3,1)} \subset \mathbb{P}^{30a+23}$. The $(6a+5)$-secant variety of $SV^{(1,1,2)}_{(1,5a+3,1)}$ is expected to fill the ambient projective space. On the other hand, we may consider the following flattening:

$$\mathbb{C}^2 \otimes \mathrm{Sym}^{3a+2}\,\mathbb{C}^2 \to \mathrm{Sym}^{2a+1}\,\mathbb{C}^2 \otimes \mathbb{C}^3.$$

By Remark 4.1, the matrix associated to this linear map is a $(6a+6) \times (6a+6)$ block matrix where the blocks are catalecticant matrices. So the determinant of this matrix yields a non-trivial equation for $\mathbb{S}ec_{6a+5}(SV^{(1,1,2)}_{(1,5a+3,1)}) \subset \mathbb{P}^{30a+23}$.

Now, consider $SV^{(1,1,2)}_{(1,5a+5,1)} \subset \mathbb{P}^{30a+35}$. In this case $\mathbb{S}ec_{6a+7}(SV^{(1,1,2)}_{(1,5a+3,1)})$ is expected to be a hypersurface in $\mathbb{P}^{30a+35}$. Consider the following flattening:

$$\mathbb{C}^2 \otimes \mathrm{Sym}^{3a+3}\,\mathbb{C}^2 \to \mathrm{Sym}^{2a+2}\,\mathbb{C}^2 \otimes \mathbb{C}^3.$$

Note that the source and the target vector spaces have dimension $6a+8$ and $6a+9$, respectively. By Remark 4.1, if we take the minors of size $6a+8$ of the corresponding $(6a+9) \times (6a+8)$ matrix, then we get at least two independent equations for $\mathbb{S}ec_{6a+7}(SV^{(1,1,2)}_{(1,5a+5,1)}) \subset \mathbb{P}^{30a+35}$. ∎

Proposition 4.3. *Let* $n, d \geq 2$ *and assume that there exist* $d_1, d_2 \geq 1$ *such that* $2(d_1+1) = (d_2+1)(n+1)$*. Then* $SV^{(1,1,n)}_{(1,d,1)}$ *is* $(2d_1+1)$*-defective.*

Proof. Proceeding as in the first part of the proof of Proposition 4.2, we consider the flattening

$$\mathbb{C}^2 \otimes \mathrm{Sym}^{d_1}\,\mathbb{C}^2 \to \mathrm{Sym}^{d_2}\,\mathbb{C}^2 \otimes \mathbb{C}^n.$$

Note that $\mathbb{S}ec_{(2(d_1+1)-1)}(SV^{(1,1,n)}_{(1,d,1)})$ is expected to fill the ambient projective space. However, the above flattening yields at least one non-trivial equation for this variety. ∎

Corollary 4.4. *The Segre–Veronese variety* $SV^{(1,1,n)}_{(1,a(n+3)-2,1)}$ *is* $(2a(n+1)-1)$*-defective for all* $a \geq 0$*. Moreover, if* n *is odd then the Segre–Veronese variety* $SV^{(1,1,n)}_{(1,a(n+3)/2-2,1)}$ *is* $(a(n+1)-1)$*-defective for all* $a \geq 0$*. In particular,* $SV^{(1,1,3)}_{(1,7,1)}$ *is* 11*-defective.*

Proof. Take $d_1 = a(n+1)-1$ and $d_1 = \frac{a(n+1)}{2} - 1$ in Proposition 4.3. ∎

Proposition 4.5. *For* $n = 3$ *and* $d \in \{3,6,9\}$*,* $n = 4$ *and* $d \in \{4,7\}$*,* $n = 5$ *and* $d \in \{4,5\}$*, the Segre–Veronese variety* $SV^{(1,1,n)}_{(1,d,1)}$ *is* h*-defective with* $h = 5$*,* $h = 9$*,* $h = 13$*,* $h = 7$*,* $h = 11$*,* $h = 7$ *and* $h = 9$*, respectively.*

Proof. The proof follows from an application of the Magma script described in the last part of Remark 4.1. ∎

4.2. Rational normal curves and defectiveness

In some particular cases defectiveness can be proved by producing low degree rational curves through a certain number of general points on a Segre–Veronese variety.

Lemma 4.6. *Consider the product $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $n_1 < n_2 \leq \cdots \leq n_r$. There exists a rational curve in X of multi-degree $(n_1, \ldots, n_r)$ through $n_1 + 3$ general points $p_1, \ldots, p_{n_1+3} \in X$.*

Proof. Let us begin with the case $n_2 = \cdots = n_r = n_1 + 1$. We view $\mathbb{P}^{n_1}$ as a linear subspace of $\mathbb{P}^{n_2} \subseteq \cdots \subseteq \mathbb{P}^{n_r}$, and write $p_i = (p_i^1, \ldots, p_i^r)$, where $p_i^j \in \mathbb{P}^{n_j}$. Without loss of generality, we may assume that $p_1^1, \ldots, p_{n_1+2}^1 \in \mathbb{P}^{n_1}$ are the projections from $p_{n_1+3}^j$ of $p_1^j, \ldots, p_{n_1+2}^j$ for all $j = 2, \ldots, r$.

Let $C_1 \subseteq \mathbb{P}^{n_1}$ be the unique rational normal curve of degree n_1 through $p_1^1, \ldots, p_{n_1+3}^1$. This is the image of a morphism $\gamma_1 : \mathbb{P}^1 \to C_1 \subseteq \mathbb{P}^{n_1}$ of degree n_1 such that $\gamma_1(x_k) = p_k^1$ for $k = 1, \ldots, n_1 + 3$, where $x_1, \ldots, x_{n_1+3} \in \mathbb{P}^1$.

Now, consider a projective space $\mathbb{P}^{n_i}$ with $i > 1$. The rational normal curves in $\mathbb{P}^{n_i}$ through $p_1^i, \ldots, p_{n_1+2}^i$ form a family of dimension greater than or equal to $n_1 - 1$, and the equality holds if and only if $n_i = n_1 + 1$. Among these curves there is one $\gamma_i : \mathbb{P}^1 \to C_i \subseteq \mathbb{P}^{n_i}$ whose tangent direction at $p_{n_i+3}^j$ is given by the line $\langle p_{n_1+3}^j, p_{n_1+3}^1 \rangle$ and such curve is unique if and only if $n_i = n_1 + 1$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\ \gamma_i\ } & C_i \\ & \underset{\gamma_1}{\searrow} & \downarrow{\scriptstyle \pi_i} \\ & & C_1, \end{array}$$

where $\pi_i : C_i \to C_1$ is the morphism induced by the projection from $p_{n_1+3}^i$. Consider the points $y_j = \gamma_i^{-1}(p_j^i)$ for $j = 1, \ldots, n_1 + 3$. The automorphism $\gamma_1^{-1} \circ \pi_i \circ \gamma_i \in \mathrm{PGL}(2)$ maps y_j to x_j, and we may use it to reparametrize γ_i to a curve $\pi_i^{-1} \circ \gamma_1 : \mathbb{P}^1 \to C_i \subseteq \mathbb{P}^{n_i}$ such that $(\pi_i^{-1} \circ \gamma_1)(x_j) = p_j^i$ for $j = 1, \ldots, n_1 + 3$.

Finally, the map

$$\gamma : \mathbb{P}^1 \to C \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}, \quad t \mapsto (\gamma_1(t), \ldots, \gamma_r(t)),$$

yields a curve of multi-degree $(n_1, \ldots, n_r)$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ such that $\gamma(x_i) = p_i = (p_i^1, \ldots, p_i^r)$ for $i = 1, \ldots, n_1 + 3$.

When $n_i > n_1 + 1$ first we project C_i from a certain number of general points in order to reach a projective space of dimension $n_1 + 1$ and then we apply the argument above. ■

Proposition 4.7. *The Segre–Veronese varieties $\mathrm{SV}^{(2,2,2)}_{(1,1,1)}$ and $\mathrm{SV}^{(2,3,3)}_{(1,1,1)}$ are, respectively, 4-defective and 5-defective.*

Proof. Let us begin with $\mathrm{SV}^{(2,3,3)}_{(1,1,1)}$. Let $p \in \mathbb{S}\mathrm{ec}_5(\mathrm{SV}^{(2,3,3)}_{(1,1,1)})$ be a general point lying on the span of general points $p_1, \ldots, p_5 \in \mathrm{SV}^{(2,3,3)}_{(1,1,1)}$. By Lemma 4.6 there is a rational curve in $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$ of multi-degree $(2,3,3)$ through 5 general points and via the Segre–Veronese embedding we get a rational normal curve $C \subseteq SV^{(2,3,3)}_{(1,1,1)}$ of degree eight through $p_1, \dots, p_5$.

Now, C spans a linear space $\Pi \cong \mathbb{P}^8$ passing through p. Any 4-dimensional linear subspace of Π passing through p that is 5-secant to C is 5-secant to $SV^{(2,3,3)}_{(1,1,1)}$ as well. Hence, if this family of 4-dimensional linear spaces has positive dimension, we get that $SV^{(2,3,3)}_{(1,1,1)}$ is 5-defective. To conclude it is enough to observe that by Theorem 3.1 of [39] such family has dimension one.

Now, consider $SV^{(2,2,2)}_{(1,1,1)}$. We may move four general points of $SV^{(2,2,2)}_{(1,1,1)}$ on the diagonal. This a Veronese variety V^2_3 spanning a linear subspace $\Pi \cong \mathbb{P}^9$. Arguing as in the first part of the proof, we have that if the family of 3-dimensional linear subspaces of Π through a general point of Π and 4-secant to V^2_3 form a family of positive dimension, then $SV^{(2,2,2)}_{(1,1,1)}$ is 4-defective. To conclude it is enough to observe that, by Proposition 1.2 of [39], such family is 2-dimensional. ■

Remark 4.8. The 4-defectiveness of $SV^{(2,2,2)}_{(1,1,1)}$ was already well known thanks to an explicit equation for $\mathbb{S}ec_4(SV^{(2,2,2)}_{(1,1,1)})$ originally worked out by V. Strassen [48] and then generalized by J. M. Landsberg, L. Manivel and G. Ottaviani [13, 35, 36, 42]. The 5-defectiveness of $SV^{(2,3,3)}_{(1,1,1)}$ was already known Proposition 4.10 of [5].

4.3. On secant defectiveness of $SV^{(1,1,1)}_{(d_1,d_2,d_3)}$

Let $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, let H_i be the pull-back of a hyperplane on the i-th factor of $\mathbb{P}$, and let $p_1, p_2 \in \mathbb{P}$ be general points. Denote by $\mathcal{L}(a,b,c;2^r)$ the non-complete linear system $|aH_1 + bH_2 + cH_3 - \sum_{i=1}^r 2p_i|$ on $\mathbb{P}$, and let $X \to \mathbb{P}$ be the blow-up of $\mathbb{P}$ at p_1, p_2 with exceptional divisors E_1, E_2. Without loss of generality, we may take $p_1 = ([0:1],[0:1],[0:1])$, $p_2 = ([1:0],[1:0],[1:0])$. Consider the rational map

$$\phi\colon \mathbb{P} \dashrightarrow \mathbb{P}, \quad ([x_0:x_1],[y_0:y_1],[z_0:z_1]) \mapsto ([x_1y_0 : x_0y_1],[y_0:y_1],[y_0z_1:y_1z_0]).$$

Note that $\phi^2([x_0:x_1],[y_0:y_1],[z_0:z_1]) = \phi([x_1y_0:x_0y_1],[y_0:y_1],[y_0z_1:y_1z_0]) = ([x_0y_1y_0 : x_1y_0y_1],[y_0 : y :_1],[y_0y_1z_0 : y_1y_0z_0]) = ([x_0:x_1],[y_0:y_1],[z_0:z_1])$. So ϕ is an involution.

Then the exceptional locus of ϕ is the inverse image via ϕ of the indeterminacy locus of $\phi^{-1} = \phi$. Such indeterminacy locus is given by

$$\{x_1y_0 = x_0y_1 = 0\} = \{[0:1]\times[0:1]\times\mathbb{P}^1\} \cup \{[1:0]\times[1:0]\times\mathbb{P}^1\},$$
$$\{y_0z_1 = y_1z_0 = 0\} = \{\mathbb{P}^1\times[0:1]\times[0:1]\} \cup \{\mathbb{P}^1\times[1:0]\times[1:0]\}.$$

Hence, the exceptional locus of ϕ is given by

$$\{\mathbb{P}^1\times[0:1]\times\mathbb{P}^1\} \cup \{\mathbb{P}^1\times[1:0]\times\mathbb{P}^1\}.$$

In particular, ϕ lifts to a birational, but not biregular, involution $\widetilde{\phi}\colon X \dashrightarrow X$, mapping $\{\mathbb{P}^1\times[0:1]\times\mathbb{P}^1\}$ to E_1 and $\{\mathbb{P}^1\times[1:0]\times\mathbb{P}^1\}$ to E_2, which is an isomorphism in codimension one. The action of $\widetilde{\phi}$ on $\operatorname{Pic}(X) \cong \mathbb{Z}[H_1, H_2, H_3, E_1, E_2]$ is given by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 \end{pmatrix},$$

where we keep denoting by H_1, H_2, H_3 their pull-backs on X. Therefore, $\widetilde{\phi}$ maps the linear system $\mathcal{L}(d_1, d_2, d_3; m_1, m_2)$ to the linear system $\mathcal{L}(d_1, d_1 + d_2 + d_3 - m_1 - m_2, d_3; d_1 + d_3 - m_1, d_1 + d_3 - m_2)$.

Now, consider a linear system of the form $\mathcal{L}(d_1, d_2, d_3; 2^{2r})$ that is with $2r$ double base points. Applying the map ϕ centered at two of the double points, we get $\mathcal{L}(d_1, d_1 + d_2 + d_3 - 4, d_3; d_1 + d_3 - 2, d_1 + d_3 - 2, 2^{2r-2})$. Now, applying again the map ϕ centered at two of the remaining double points to this new linear system, we get $\mathcal{L}(d_1, 2d_1 + d_2 + 2d_3 - 8, d_3; d_1 + d_3 - 2, d_1 + d_3 - 2, d_1 + d_3 - 2, d_1 + d_3 - 2, 2^{2r-4})$. Proceeding in this way, after r steps, we get the linear system $\mathcal{L}(d_1, rd_1 + d_2 + rd_3 - 4r, d_3; (d_1 + d_3 - 2)^{2r})$. Summing up applying r maps of type ϕ, we have

$$\mathcal{L}(d_1, d_2, d_3; 2^{2r}) \mapsto \mathcal{L}(d_1, rd_1 + d_2 + rd_3 - 4r, d_3; (d_1 + d_3 - 2)^{2r}). \tag{4.1}$$

Similarly, applying r maps of type ϕ to a linear system with an odd number of double base points, we get

$$\mathcal{L}(d_1, d_2, d_3; 2^{2r+1}) \mapsto \mathcal{L}(d_1, rd_1 + d_2 + rd_3 - 4r, d_3; (d_1 + d_3 - 2)^{2r}, 2). \tag{4.2}$$

For instance, (4.1) yields that $\mathcal{L}(1, d, 1; 2^{2r})$ goes to $\mathcal{L}(1, d - 2r, 1)$ and this last linear system has the expected dimension. So, by Terracini's lemma [49], $SV^{(1,1,1)}_{(1,d,1)}$ is not $2r$-defective for any r. Note that since $SV^{(1,1,1)}_{(1,d,1)} \subseteq \mathbb{P}^{4(d+1)-1}$ when d is odd we get that $SV^{(1,1,1)}_{(1,d,1)}$ is not h-defective for any h while when $d = 2a$ is even we miss the last secant variety, namely the $(2a + 1)$-secant variety, which is indeed defective. To see this note that the linear system $\mathcal{L}(1, 2a, 1; 2^{2a+1})$ is equivalent to $\mathcal{L}(1, 0, 1; 2)$, the $(2a + 1)$-secant variety of $SV^{(1,1,1)}_{(1,2a,1)}$ is expected to fill the ambient space $\mathbb{P}^{8a+3}$ but by considering the tangent plane to the quadric surface given by the first and the third copies of $\mathbb{P}^1$, we see that $\mathcal{L}(1, 0, 1; 2)$ has one non-trivial section.

Similarly, $\mathcal{L}(1, d, 2; 2^{2r})$ goes to $\mathcal{L}(1, d - r, 2; 1^{2r})$, which has the expected dimension. In this case, we get that $SV^{(1,1,1)}_{(1,d,2)} \subseteq \mathbb{P}^{6(d+1)-1}$ is not h-defective for any $h \leq \overline{h}$, where $\overline{h}$ is the biggest even number such that $\overline{h} \leq \frac{3}{2}(d + 1)$.

Furthermore, (4.2) yields that $\mathcal{L}(1, d, 1; 2^{2r+1})$ goes to $\mathcal{L}(1, d - 2r, 1; 2)$, which is empty for $2r > d$. So $\mathbb{S}\mathrm{ec}_{d+2}(SV^{(1,1,1)}_{(1,d,1)})$ fills the ambient space $\mathbb{P}^{4(d+1)-1}$. However, as we have seen $\mathbb{S}\mathrm{ec}_{d+1}(SV^{(1,1,1)}_{(1,d,1)})$ does not fill the ambient space when d is even.

Finally, $\mathcal{L}(1, d, 2; 2^{2r+1})$ goes to $\mathcal{L}(1, d - r, 2; 1^{2r}, 2)$, which is empty for $r > d$. Hence, $\mathbb{S}\mathrm{ec}_{2d+3}(SV^{(1,1,1)}_{(1,d,2)})$ fills the ambient space $\mathbb{P}^{6(d+1)-1}$.

Remark 4.9. We believe that it should be possible to produce rational maps, in the same spirit of what we did in Section 4.3 for the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, in order to explain most of the possible new defective cases in the tables in Section 5.

(n_1, n_2)	$(d_1, d_2) \neq (1,1)$	known defective cases	possible new defective cases
$(1,2)$	$d_1 + d_2 \leq 40$	$(1,3), (2k,2), 1 \leq k \leq 19$	none
$(1,3)$	$d_1 + d_2 \leq 20$	$(2k,2), 1 \leq k \leq 9$	none
$(1,4)$	$d_1 + d_2 \leq 10$	$(2k,2), 1 \leq k \leq 4$	none
$(1,5)$	$d_1 + d_2 \leq 9$	$(2k,2), 1 \leq k \leq 3$	none
$(1,6)$	$d_1 + d_2 \leq 5$	$(2,2)$	none
$(1,7)$	$d_1 + d_2 \leq 3$	none	none

Table 2. Script results for $SV^{(1,n_2)}_{(d_1,d_2)}$.

(n_1, n_2)	$(d_1, d_2) \neq (1,1)$	known defective cases	possible new defective cases
$(2,2)$	$d_1 + d_2 \leq 23$	$(2,2)$	none
$(2,3)$	$d_1 + d_2 \leq 10$	$(1,2), (2,2)$	none
$(2,4)$	$d_1 + d_2 \leq 6$	$(2,2)$	none
$(2,5)$	$d_1 + d_2 \leq 4$	$(1,2), (2,1), (2,2)$	none
$(2,6)$	$d_1 + d_2 \leq 3$	$(2,1)$	none

Table 3. Script results for $SV^{(2,n_2)}_{(d_1,d_2)}$.

Finally, we would like to stress that the defectiveness of the Segre–Veronese varieties considered in Section 4.3 was already well known Theorem 2.1 of [34].

5. Segre–Veronese varieties with two or three factors

We look at Segre–Veronese varieties with two factors $SV^{(n_1,n_2)}_{(d_1,d_2)}$. We assume that $n_1 \leq n_2$ and $n_2 > 1$ since, by Theorem 2.2 of [34], $SV^{(1,1)}_{(d_1,d_2)}$ is defective if and only if $d_1 = 2$ and d_2 is even. We also assume that $(d_1, d_2) \neq (1,1)$ since Segre varieties with two factors are almost always defective.

If either $n_1 = 1$ or $n_1 = 2$, we get the results listed in Tables 2 and 3. The only cases where the script was unable to prove the non-defectiveness are the already known ones, Conjecture 5.5 (b), (d) of [4] and Conjecture 5.5 (a), (c), (e) of [4], respectively.

For $3 \leq n_1 \leq 4, n_1 \leq n_2 \leq 5$, we found six cases, listed in Table 4, where the computer was unable to check whether the corresponding Segre–Veronese variety is defective or not. Again these cases already appeared in the literature, Conjecture 5.5 (c), (e) of [4].

Now, we proceed with Segre–Veronese varieties with three factors $SV^{(n_1,n_2,n_3)}_{(d_1,d_2,d_3)}$. We assume that $n_1 \leq n_2 \leq n_3$ and $n_3 > 1$, since Theorem 2.2 of [34] classifies defective products of $\mathbb{P}^1$. If $n_1 = n_2$, we assume that $d_1 \leq d_2$ and, similarly, for $n_2 = n_3$, we assume that $d_2 \leq d_3$. By [50], the following Segre–Veronese varieties are defective:

$$SV^{(1,1,2)}_{(1,1,2)},\ SV^{(1,1,3)}_{(1,1,2)},\ SV^{(1,1,4)}_{(1,1,2)},\ SV^{(1,1,5)}_{(1,1,2)},\ SV^{(1,1,6)}_{(1,1,2)},\ SV^{(1,1,2)}_{(2,2,2)},\ SV^{(1,1,3)}_{(2,2,2)},\ SV^{(1,1,2)}_{(1,3,1)},$$
$$SV^{(1,1,3)}_{(1,4,1)},\ SV^{(1,1,4)}_{(1,5,1)},\ SV^{(1,2,2)}_{(2k,1,1)},\ SV^{(1,2,3)}_{(5,1,1)},\ SV^{(1,2,4)}_{(6,1,1)},\ SV^{(1,3,3)}_{(2k,1,1)},\ SV^{(2,2,2)}_{(2,1,1)},\ SV^{(2,3,3)}_{(2,1,1)}.$$

(n_1, n_2)	$(d_1, d_2) \neq (1, 1)$	known defective cases	possible new defective cases
$(3, 3)$	$d_1 + d_2 \leq 8$	$(2, 2)$	none
$(3, 4)$	$d_1 + d_2 \leq 5$	$(2, 1), (2, 2)$	none
$(3, 5)$	$d_1 + d_2 \leq 4$	$(2, 2), (3, 1)$	none
$(4, 4)$	$d_1 + d_2 \leq 5$	$(2, 2)$	none
$(4, 5)$	$d_1 + d_2 \leq 3$	none	none

Table 4. Script results for $\mathrm{SV}^{(n_1,n_2)}_{(d_1,d_2)}$, $3 \leq n_1 \leq 4, n_1 \leq n_2 \leq 5$.

(n_1, n_2, n_3)	(d_1, d_2, d_3), $d_1 \leq d_2$	known defective cases	possible new defective cases	new defective cases
$(1, 1, 2)$	$d_1 + d_2 + d_3 \leq 13$	$(1, 1, 2)$, $(1, 3, 1), (2, 2, 2)$	none	$(1, 5, 1)$, $(1, 8, 1), (1, 10, 1)$
$(1, 1, 3)$	$d_1 + d_2 + d_3 \leq 11$	$(1, 1, 1), (1, 1, 2)$ $(1, 4, 1), (2, 2, 2)$	none	$(1, 3, 1), (1, 6, 1)$ $(1, 7, 1), (1, 9, 1)$
$(1, 1, 4)$	$d_1 + d_2 + d_3 \leq 9$	$(1, 1, 1)$, $(1, 1, 2), (1, 5, 1)$	none	$(1, 4, 1), (1, 7, 1)$
$(1, 1, 5)$	$d_1 + d_2 + d_3 \leq 7$	$(1, 1, 1)$, $(1, 1, 2), (1, 2, 1)$	none	$(1, 4, 1), (1, 5, 1)$
$(1, 1, 6)$	$d_1 + d_2 + d_3 \leq 4$	$(1, 1, 1)$, $(1, 1, 2), (1, 2, 1)$	none	none

Table 5. Script results for $\mathrm{SV}^{(1,1,n_3)}_{(d_1,d_2,d_3)}$.

The following ones are also defective by Theorem 2.4 of [15] since they are unbalanced:

$$\mathrm{SV}^{(1,1,3)}_{(1,1,1)},\ \mathrm{SV}^{(1,1,4)}_{(1,1,1)},\ \mathrm{SV}^{(1,1,5)}_{(1,1,1)},\ \mathrm{SV}^{(1,1,5)}_{(1,2,1)},\ \mathrm{SV}^{(1,2,4)}_{(1,1,1)},\ \mathrm{SV}^{(1,1,6)}_{(1,1,1)},\ \mathrm{SV}^{(1,1,6)}_{(1,2,1)},\ \mathrm{SV}^{(1,2,5)}_{(1,1,1)}.$$

The variety $\mathrm{SV}^{(2,2,2)}_{(1,1,1)}$ is defective by Theorem 3.1 of [35] and $\mathrm{SV}^{(2,3,3)}_{(1,1,1)}$ is defective by Proposition 4.10 of [5]. In Tables 5, 6 and 7 we present the results found for Segre–Veronese of three factors. We were unable to check, using our script, whether the following Segre–Veronese varieties are defective or not:

$$\begin{gathered}\mathrm{SV}^{(1,1,2)}_{(1,5,1)},\ \mathrm{SV}^{(1,1,2)}_{(1,8,1)},\ \mathrm{SV}^{(1,1,2)}_{(1,10,1)},\ \mathrm{SV}^{(1,1,3)}_{(1,3,1)},\ \mathrm{SV}^{(1,1,3)}_{(1,6,1)},\ \mathrm{SV}^{(1,1,3)}_{(1,7,1)},\\ \mathrm{SV}^{(1,1,3)}_{(1,9,1)},\ \mathrm{SV}^{(1,1,4)}_{(1,4,1)},\ \mathrm{SV}^{(1,1,4)}_{(1,7,1)},\ \mathrm{SV}^{(1,1,5)}_{(1,4,1)},\ \mathrm{SV}^{(1,1,5)}_{(1,5,1)},\ \mathrm{SV}^{(1,2,3)}_{(2,1,1)},\\ \mathrm{SV}^{(1,2,3)}_{(3,1,1)},\ \mathrm{SV}^{(1,2,3)}_{(7,1,1)},\ \mathrm{SV}^{(1,2,4)}_{(3,1,1)},\ \mathrm{SV}^{(1,2,4)}_{(5,1,1)},\ \mathrm{SV}^{(1,3,4)}_{(2,1,1)}.\end{gathered}$$

The defectiveness of the cases in the last column of Table 5 is proved in Propositions 4.2, 4.5 and Corollary 4.4. We did not manage to prove that the cases in the last column of Table 6 are indeed defective.

The following Magma script shows how to check the results listed in the above tables. In the specific case we are listing the defective Segre–Veronese varieties with $[n_1, n_2] = [1, 2]$ and $1 \leq d_1, d_2 \leq 10$.

(n_1, n_2, n_3)	(d_1, d_2, d_3)	known defective cases	possible new defective cases
$(1, 2, 2)$	$d_1 + d_2 + d_3 \leq 11$	(2, 1, 1), (4, 1, 1), (6, 1, 1), (8, 1, 1)	none
$(1, 2, 3)$	$d_1 + d_2 + d_3 \leq 9$	(5, 1, 1)	(2, 1, 1), (3, 1, 1), (7, 1, 1)
$(1, 2, 4)$	$d_1 + d_2 + d_3 \leq 7$	(1, 1, 1)	(3, 1, 1), (5, 1, 1)
$(1, 2, 5)$	$d_1 + d_2 + d_3 \leq 4$	(1, 1, 1)	none
$(1, 3, 3)$	$d_1 + d_2 + d_3 \leq 7$	(2, 1, 1), (4, 1, 1)	none
$(1, 3, 4)$	$d_1 + d_2 + d_3 \leq 4$	none	(2, 1, 1)

Table 6. Script results for $SV^{(1,2,n_3)}_{(d_1,d_2,d_3)}$ and $SV^{(1,3,n_3)}_{(d_1,d_2,d_3)}$.

(n_1, n_2, n_3)	(d_1, d_2, d_3)	known defective cases	possible new defective cases
$(2, 2, 2)$	$d_1 + d_2 + d_3 \leq 9$	(1, 1, 1), (1, 1, 2)	none
$(2, 2, 3)$	$d_1 + d_2 + d_3 \leq 6$	none	none
$(2, 2, 4)$	$d_1 + d_2 + d_3 \leq 4$	none	none
$(2, 3, 3)$	$d_1 + d_2 + d_3 \leq 4$	(1, 1, 1), (2, 1, 1)	none

Table 7. Script results for $SV^{(2,n_2,n_3)}_{(d_1,d_2,d_3)}$.

```
> load "library.m";
> dd := [[d1,d2] : d1,d2 in [1..10]];
> for d in dd do
  if IsSVDef([1,2],d,5) then d; end if;
  end for;

[ 1, 3 ]
[ 2, 2 ]
[ 4, 2 ]
[ 6, 2 ]
[ 8, 1 ]
[ 8, 2 ]
[ 10, 2 ]
```

Observe that the case $[d_1, d_2] = [8, 1]$ has been recognized by the program as a defective one. Anyway if one runs the function `IsSVDef([1,2],[8,1],5)` enough times, then at some point the output will be `false`.

Our second Magma example compares the running times for checking non-speciality of the Segre–Veronese varieties $[n_1, n_2] = [1, 2]$ embedded with multidegrees $[d_1, d_2] = [13, 13]$, and $[n_1, n_2, n_3] = [2, 2, 2]$ embedded with multidegrees $[d_1, d_2, d_3] = [2, 2, 6]$. The first function `IsSVDef` is based on our algorithm. The second function makes use of the classical Terracini's lemma, which reduces the defectiveness checking to the calculation of the dimension of a linear system of affine hypersurfaces through double points in general position.

```
> load "library.m";
> time IsSVDef([1,2],[13,13],5);
false
Time: 4.480
> time IsSpecial(ProjSpaces([1,2],[13,13]));
false
Time: 198.190

> time IsSVDef([2,2,2],[2,2,6],10);
false
Time: 3.510
> time IsSpecial(ProjSpaces([2,2,2],[2,2,6]));
false
Time: 30.110
```

According to our tests we found that the difference between the computational times of the above two functions increases according to the number of points of the Riemann–Roch polytope of the toric variety.

6. Applications

In this section we provide two applications of our methods to secant varieties of toric surfaces and Losev–Manin spaces. We would like to mention that 2-secant defective smooth toric varieties were classified in [19].

Proposition 6.1. *Let $P \subseteq M_{\mathbb{Q}}$ be a 2-dimensional lattice polytope and X_P the corresponding 2-dimensional toric variety. Then X_P is 2-defective if and only if either X_P is a cone or P is contained in V_2^2.*

Proof. Clearly, if X_P is a cone or P is contained in the polytope of V_2^2, then X_P is 2-defective. Assume that neither X_P is a cone nor P is contained in the polytope of V_2^2. We may assume that $M = \mathbb{Z}^2$, P has at least 6 points, $A = (0,0)$, $B = (0,1)$, $C = (1,0) \in P$, and P is contained in the first quadrant.

To simplify the notation, let us write $D = (2,0)$, $E = (1,1)$, $F = (0,2)$, $\Delta_0 = \{A, B, C\}$. We distinguish three cases depending on how many points there are in $P \cap \{D, E, F\}$.

First assume that there are two points p, q in $\{D, E, F\} \cap P$. Then there is at least one point $r \in (P \cap M) \setminus \Delta_2^2$. Hence, using $\Delta_1 = \Delta_0$, $v_1 = (-1,-1)$, $\Delta_2 = \{p, q, r\}$ and any v_2, we see that X_P is not 2-defective by Theorem 2.7.

Now, assume that $\{p\} = \{D, E, F\} \cap P$ has exactly one point. Then there are at least two points $q, r \in (P \cap M) \setminus \Delta_2^2$. If there are such two points making $\Delta_2 = \{p, q, r\}$ a simplex, we are done as in the previous case. We therefore can assume that all points in $(P \cap M) \setminus \Delta_0$ are collinear. We will prove that $p = E$. Indeed, the points of $P \setminus \Delta_0$ can not all lie in the segment $\{(x, 0), x \geq 2\}$ since X_P is not a cone, and similarly they can not all lie on the segment $\{(0, y), y \geq 2\}$. Therefore, there is a point $G = (x, y) \in P$ with $x \geq 1$ and $y \geq 1$. Since $E \in \overline{BCG}$ and P is convex, we conclude that $E \in (P \cap M)$.

Now, either the points in $(P \cap M) \setminus \Delta_0$ are contained in the vertical line $\{(1, y), y \geq 1\}$ or $q = (x_q, y_q), r = (x_r, y_r)$ for some $2 \leq x_q < x_r$ and $1 \leq y_q < y_r$. In the first case we may use

$$\Delta_1 = \{A, B, E\}, \quad v_1 = (1, -1), \quad \Delta_2 = \{(1,3), (1,2), C\},$$

with v_2 arbitrary, and in the second case we may use

$$\Delta_1 = \{B, q, r\}, \quad v_1 = (a, 1), \quad \Delta_2 = \{A, C, E\}, \quad \text{with } a \gg 0,$$

again with v_2 arbitrary.

Finally, assume that $\{D, E, F\} \cap (P \cap M) = \emptyset$. Then none of the points of $P \cap M$ lies on the segments $\{(x, 0), x \geq 2\}$ and $\{(0, y), y \geq 2\}$ and, as in the second case, we can prove that $E \in (P \cap M)$. ■

Remark 6.2. In higher dimensions, the analogue of Proposition 6.1 does not hold. Consider the polytope $P \subseteq \mathbb{Q}^3$ with vertexes $(0,0,1), (1,0,2), (0,2,1), (2,2,1), (1,1,0)$. The lattice points of P are

$$(0,0,1),\ (1,0,2),\ (0,2,1),\ (2,2,1),\ (1,1,0),\ (1,1,1),\ (1,2,1),\ (0,1,1),$$

and hence the corresponding map to a projective space is given by

$$(6.1) \qquad (\mathbb{C}^*)^3 \to \mathbb{P}^7, \quad (x, y, z) \mapsto (xyz, x^2y^2z, z, xz^2, y^2z, xy, xy^2z, yz).$$

Note that P contains $(1,1,1)$ as an interior point, and hence it is not equivalent, modulo $\mathrm{GL}(3, \mathbb{Z})$ and translations, to a polytope contained in the polytope of the degree two Veronese embedding of $\mathbb{P}^3$. Furthermore, X_P is 2-defective by Terracini's lemma. Now, the singular locus of X_P is the union of seven invariant curves, which correspond to the singular 2-dimensional cones of the normal fan, and it is stabilized by the action of the torus. Hence, it corresponds via (6.1) to the locus stabilized by the action of the torus on $\mathbb{C}^3$. Computing the differential of (6.1), we get that the line L corresponding to the plane $\{z = 0\} \subseteq \mathbb{C}^3$ is in the singular locus of X_P. Hence, if X_P is a cone, this line must be contained in its vertex. However, a line going through a general point of L and the point $(1, \dots, 1) \in X_P$ is not entirely contained in X_P, and hence X_P cannot be a cone. The variety X_P is a Gorenstein canonical toric Fano 3-fold of degree 10. Its entry in the Graded Ring Database is 523456.[2]

6.0.1. An application to Losev–Manin spaces. Let LM_n be the blown-up of $\mathbb{P}^n$ at all the linear spaces of codimension at least two spanned by subsets of the $n+1$ torus fixed points of $\mathbb{P}^n$. The variety LM_n is the Losev–Manin's moduli space introduced in [38]. This moduli space parametrizes $(n+1)$-pointed chains of projective lines $(C, x_0, x_\infty, x_1, \dots, x_{n+1})$, where:

- C is a chain of smooth rational curves with two fixed points x_0, x_∞ on the extremal components,
- $x_1, \dots, x_{n+1}$ are smooth marked points different from x_0, x_∞ but non-necessarily distinct,
- there is at least one marked point on each component.

The n-dimensional permutohedron P_n is the n-dimensional polytope in $\mathbb{R}^{n+1}$ given as the convex hull of all the points obtained by permuting the coordinates of $(1, 2, \dots, n+1)$.

[2] See http://www.grdb.co.uk/forms/toricf3c.

n	$F(n)$	Proposition 6.3
3	38	$h \le 5$
4	291	$h \le 33$
5	2932	$h \le 272$
6	36961	$h \le 2879$
7	561948	$h \le 37475$

Table 8. The bound in Proposition 6.3 for small values of n.

Note that P_n is contained in the hyperplane $\{z_1 + \cdots + z_{n+1} = (n+1)(n+2)/2\}$. The Losev–Manin moduli space LM_n is the toric variety associated to the permutohedron P_n.

By Theorem 1.3 of [30], the permutohedron P_n has the *Integer Decomposition Property*. This means that for all $r \ge 1$ and $m \in P_n \cap M$, there are $m_1, \dots, m_r \in P_n \cap M$ such that $m = m_1 + \cdots + m_r$. In particular, P_n is very ample, and then the sections associated to its integer points yield an embedding

$$LM_n \hookrightarrow \mathbb{P}^{|P_n \cap M|-1}.$$

Proposition 6.3. *Let $LM_n \subset \mathbb{P}^{|P_n \cap M|-1}$ the n-dimensional Losev–Manin moduli space in the embedding induced by the permutohedron P_n. Then LM_n is not h-defective for*

$$h \le \frac{F(n) - (n+1)^{n-1}}{n+1},$$

where $F(n) = |P_n \cap M|$ is the number of forests of trees on $n+1$ labeled nodes.

Proof. By Section 3 of [47], the number of integer points of P_n is the number of forests of trees on $n+1$ labeled nodes. Our aim is to estimate the maximum number of integer points of P_n lying on a hyperplane. Note that $P_n \subset H_n$, where $H_n \subset \mathbb{R}^n$ is the hypercube defined by $1 \le z_i \le n+1$ for $i = 1, \dots, n$. Hence, if m_{P_n} and m_{H_n} are the maximum number of integer points of P_n and H_n, respectively, lying on a hyperplane, then we have that $m_{P_n} \le m_{H_n}$.

Note that H_n is the polytope associated to the Segre–Veronese embedding of $(\mathbb{P}^1)^n$ with multi-degree $(n, \dots, n)$, and H_n satisfies the conditions of Proposition 2.13. Therefore, the maximum number of points of H_n lying on a hyperplane is attained on a facet of H_n. So $m_{H_n} = (n+1)^{n-1}$. Finally, to get the bound in the statement, it is enough to apply Theorem 2.12. ■

In Table 8 we work out the bound in Proposition 6.3 for small values of n. The values of $F(n)$ are given by the OEIS sequence A001858.

Remark 6.4. Furthermore, thanks to the Magma script, we got that $LM_n \subset \mathbb{P}^{|P_n \cap M|-1}$ is never defective for $n \le 5$. Here we are applying our algorithm, based on Theorem 2.7, to the case $n = 2$. The output consists of a boolean, which in this case implies that the variety $LM_2 \subseteq \mathbb{P}^6$ is not defective, and a subdivision of the set of points into two simplexes plus a residue point.

```
load "library.m";
P2 := Permutohedron(2);
TestDef(Points(P2),2);
false <<[
    (1, 2),
    (1, 3),
    (2, 1)
], [
    (2, 2),
    (2, 3),
    (3, 1)
]>, {
    (3, 2)
}>
```

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