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Soliton solutions to the curve shortening flow on the 2-dimensional hyperbolic space

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Abstract. We show that a curve is a soliton solution to the curve shortening flow on the 2-dimensional hyperbolic space if and only if its geodesic curvature can be written as the inner product between its tangent vector field and a fixed vector of the 3-dimensional Minkowski space. We show that for each fixed vector there is a 2-parameter family of soliton solutions to the flow. We prove that there are three classes of such curves. Moreover, we prove that each soliton is defined on the whole real line, it is embedded and its geodesic curvature, at each end, converges to a constant.

1. Introduction

A family of curves $\hat{X}^t : I \to M$, $t \in [0, T)$, on a 2-dimensional Riemannian manifold M^2 is said to be a solution to the *curve shortening flow* (CSF) with initial condition $\hat{X}^0(\cdot) = X(\cdot)$, if it satisfies the following equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{X}^t(\cdot) = \hat{k}^t(\cdot)\,\hat{N}^t(\cdot), \\ \hat{X}^0(\cdot) = X(\cdot), \end{cases} \tag{1.1}$$

where $\hat{k}^t(\cdot)$ is the geodesic curvature and $\hat{N}^t(\cdot)$ is the unit vector field normal to $\hat{X}^t(\cdot)$ for each $t \in [0, T)$. The geodesics are trivial solutions.

Epstein and Gage [5] showed that when $M^2 = \mathbb{R}^2$, the CSF is geometrically the same if tangential components are added to the right-hand side of the differential equation (1.1). Therefore, one can define the CSF to be a 1-parameter family of curves that satisfy $\big\langle \frac{\partial}{\partial t}\hat{X}^t(\cdot), \hat{N}^t(\cdot)\big\rangle = \hat{k}^t(\cdot)$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^2$.

The name curve shortening flow is justified by the fact that when the curves of the family $\hat{X}^t$ are closed, then the length of the curves decreases along the flow, i.e., it is a gradient type of flow for the arc length functional. Grayson [11] observed that the CSF is also known as the *curvature flow* or the *heat flow for isometric immersions*.

According to Epstein and Gage [5], the original motivation for studying equation (1.1) was to find a new and maybe more natural proof of the existence of closed geodesics on

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Riemannian manifolds, and the first results in this direction were obtained by Grayson [12] in 1989. However, equation (1.1) on the Euclidean plane was investigated earlier by several authors [1, 6–9, 11].

An important class of solutions to the CSF are those that evolve by isometries or homotheties. Such solutions are called *self-similar solutions*, and *solitons* if they evolve just by isometries. On the Euclidean plane, the *Grim Reaper* curve given by the graph of the function $f(s) = \ln(\cos(s))$ evolves by a flow of translations. Giga [10] proved that this is the unique curve on the plane that evolves by translations. An example of a plane curve that evolves by isometries of the plane is the *yin-yang* spiral. Abresch–Langer [1] and Epstein–Weinstein [6] investigated the closed curves, not necessarily simple, that evolve by homotheties. Halldorsson [13] completed the description of all self-similar solutions on the plane.

Gage [7, 8] showed that if a smooth closed convex curve evolves on $\mathbb{R}^2$ by the CSF, then the isoperimetric ratio L^2/A of the curve (the square of the length L divided by the enclosed area A) approaches 4π, as A approaches to zero. If each curve of the evolving family of curves is normalized by homothetic expansion of the plane, so that it encloses area π, then the resulting family of curves will converge to the unit circle. Then Gage and Hamilton [9] showed that convex embedded closed curves in $\mathbb{R}^2$ remain convex under the CSF, become circular, and then shrink to a point. Grayson [11] proved that closed embedded curves evolve to circular curves and then they collapse into a point at a finite time. Moreover, Angenent [2], under more general conditions, proved that the CSF evolves in a sense into a self-similar flow, showing the importance of self-similar solutions.

One should point out that, when the ambient space is not the Euclidean plane, there are very few results on self-similar solutions to the CSF. In 2015, Halldorsson [14] classifie d all the self-similar solutions on the Minkowski plane. Dos Reis and Tenenblat [4] characterized and described all the soliton solutions on the sphere $\mathbb{S}^2$. Some results on the CSF for Riemannian manifolds different from the plane can be found in [9, 12, 15, 17]. Moreover, Angenent [3] studied the topology of the closed geodesics on compact surfaces by using the CSF.

In this paper, we study the soliton solutions of the curve shortening flow on the 2-dimensional hyperbolic space $\mathbb{H}^2 \subset \mathbb{R}^3_1$, where $\mathbb{R}^3_1$ is the 3-dimensional Minkowski space. We show that a curve is a soliton solution to this flow if and only if its geodesic curvature can be written as the inner product between its tangent vector field and a fixed vector $v \in \mathbb{R}^3_1$. By considering the vector to be space-like, time-like or light-like, we show that there are three classes of such curves. In Proposition 3.1, we prove that the soliton solutions correspond to solving a system of ODEs, whose initial conditions are given in one of three disjoint sets. We show that for each fixed vector v, there is a 2-parameter family of soliton solutions to the curve shortening flow on $\mathbb{H}^2$. Moreover, we prove that each soliton is defined on the whole real line, it is embedded and its geodesic curvature, at each end, converges to a constant.

We observe that Halldorsson, when studying the mean curvature flow for curves on the Minkowski plane, showed that, in contrast to the case of the Euclidean plane, the soliton solutions may have finite Minkowski-length, without having end points. In this paper, we show that the soliton solutions to the CSF on $\mathbb{H}^2$ are defined on the whole real line. This is proved in a series of lemmas in Section 3, where we also prove our main results. In Section 4, we visualize some soliton solutions.

2. Main results

We consider the 2-dimensional hyperbolic space $\mathbb{H}^2 \subset \mathbb{R}^3_1$, with the Minkowski metric on $\mathbb{R}^3_1$ defined by $\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3$. One can show that the CSF on $\mathbb{H}^2$ is geometrically the same if tangential components are added to the right-hand side of the differential equation (1.1). Therefore, we consider the following definition.

Let $X \colon I \subset \mathbb{R} \to \mathbb{H}^2 \subset \mathbb{R}^3_1$ be a regular curve parametrized by arc length s. We denote by $T(s) = X'(s)$ the tangent vector field, and by $N(s) = X(s) \times T(s)$ the unit normal vector field (recall that in $\mathbb{R}^3_1$, $u \times v = u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$). The geodesic curvature of X is defined by $k(s) = \langle T'(s), N(s)\rangle$.

A one parameter family of curves $\hat{X} \colon I \times J \to \mathbb{H}^2$ is called a *curve shortening flow* (CSF) with initial condition X if

$$\left\{ \begin{array}{l} \left\langle \frac{\partial}{\partial t}\hat{X}(s,t), \hat{N}(s,t)\right\rangle = \hat{k}(s,t), \\ \hat{X}(s,0) = X(s), \end{array} \right. \tag{2.1}$$

where $\hat{k}^t(\cdot) = \hat{k}(\cdot, t)$ is the geodesic curvature and $\hat{N}^t(\cdot) = \hat{N}(\cdot, t)$ is the unit normal vector field of $\hat{X}^t(\cdot) = \hat{X}(\cdot, t)$.

One can show that it is a gradient type of flow for the arc length functional. Our goal is to study the case when $\hat{X}^t(s)$ evolves by a 1-parameter family of isometries of $\mathbb{H}^2$.

Definition 2.1. Let $\hat{X} \colon I \times J \to \mathbb{H}^2 \subset \mathbb{R}^3_1$ be a solution to the curve shortening flow (2.1), with initial condition $X \colon I \to \mathbb{H}^2$. We say that X is a *soliton solution* to the curve shortening flow if there is a 1-parameter family of isometries $M(t) \colon \mathbb{H}^2 \to \mathbb{H}^2$ such that $M(0) = \mathrm{Id}$ and $\hat{X}^t(s) = M(t)X(s)$ for all $t \in J$, where Id is the identity map.

We remark that an isometry of $\mathbb{H}^2$ is an element of the Lie group $O_1(3) = \{M \in \mathrm{GL}(3, \mathbb{R}) : M^T \varepsilon M = \varepsilon\}$ that preserves $\mathbb{H}^2$, where M^T is the transpose of M and

$$\varepsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2.2. *Let $X \colon I \to \mathbb{H}^2$ be a regular curve parametrized by arc length $s \in I$. Then X is a soliton solution to the curve shortening flow if, and only if, there is a vector $v \in \mathbb{R}^3_1 \setminus \{0\}$ such that*

$$\langle T(s), v\rangle = k(s), \tag{2.2}$$

where $T(s)$ is the unit tangent vector field and $k(s)$ is the geodesic curvature of X.

We observe that when X is a geodesic of $\mathbb{H}^2 \subset \mathbb{R}^3_1$, then it is a planar curve and hence there exists a vector $v \in \mathbb{R}^3_1 \setminus \{0\}$ such that $\langle T, v\rangle = 0$. The following theorem describes the soliton solutions to the CSF in $\mathbb{H}^2$.

Theorem 2.3. *For any vector $v \in \mathbb{R}^3_1 \setminus \{0\}$, there is a 2-parameter family of non-trivial soliton solutions to the curve shortening flow on $\mathbb{H}^2$. There are three classes of soliton curves on $\mathbb{H}^2$, determined by the type of the vector v. Each soliton solution is an embedded curve $X(s)$ on $\mathbb{H}^2$, defined for all $s \in \mathbb{R}$. Moreover, at each end, the curvature function $k(s)$ tends to one of the constants $\{-1, 0, 1\}$.*

3. Proofs of the main results

Proof of Theorem 2.2. Suppose that $X(s)$ is parametrized. It follows from the definition of the CSF that

$$\hat{k}(s,t) = \Big\langle \frac{\partial}{\partial t}\hat{X}(s,t), \hat{N}(s,t)\Big\rangle = \big\langle M'(t)X(s), M(t)N(s)\big\rangle.$$

In particular, for $t = 0$, we have $k(s) = \langle M'(0)X(s), N(s)\rangle$. $M'(0)$ is an element of the Lie algebra $\mathfrak{o}_1(3)$ of the Lie group $O_1(3)$. Let A_1, A_2, A_3 be a basis of $\mathfrak{o}_1(3)$, where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $M'(0) = c_1A_1 + c_2A_2 + c_3A_3$, for real numbers c_i, $i = 1,2,3$. We denote $X(s) = (x_1(s), x_2(s), x_3(s))$ and $N(s) = X(s) \times T(s)$. Then

$$M'(0)X(s) = (c_3x_2(s) + c_2x_3(s),\ c_3x_1(s) + (c_1+c_2)x_3(s),\ c_2x_1 - (c_1+c_2)x_2(s)).$$

Since $\langle X, X\rangle = -x_1^2 + x_2^2 + x_3^2 = -1$ and $-x_1x_1' + x_2x_2' + x_3x_3' = 0$, it follows that $\langle M'(0)X, N\rangle = \langle T, (c_1+c_2, c_2, -c_3)\rangle$. Therefore, taking $v = (c_1+c_2, c_2, -c_3)$, we have $k(s) = \langle T(s), v\rangle$.

Conversely, let $X(s)$ be a curve in $\mathbb{H}^2 \subset \mathbb{R}^3_1$ parametrized by arc length s, such that $\langle T(s), v\rangle = k(s)$ for a vector $v \in \mathbb{R}^3_1 \setminus \{0\}$. Without loss of generality, up to isometries of $\mathbb{H}^2$, we can consider v to be a multiple of $w_1 = (1,0,0)$ if v is a timelike vector, a multiple of $w_2 = (1,1,0)$ if v is a lightlike vector, and a multiple of $w_3 = (0,0,-1)$ if v is a spacelike vector. Thus, depending on the type of the vector v, we have the curvature as $k_i(s) = \langle T(s), v_i\rangle$, where $v_i = aw_i$, $a > 0$, and $i = 1,2,3$. Now, we define the evolution of X in $\mathbb{H}^2$ to be $\hat{X}_i(s,t) = M_i(t)X(s)$, where

$$M_1(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_1(t)) & \sin(\varphi_1(t)) \\ 0 & -\sin(\varphi_1(t)) & \cos(\varphi_1(t)) \end{pmatrix},$$

$$M_2(t) := \begin{pmatrix} 1 + (\varphi_2(t))^2/2 & -(\varphi_2(t))^2/2 & \varphi_2(t) \\ (\varphi_2(t))^2/2 & 1 - (\varphi_2(t))^2/2 & \varphi_2(t) \\ \varphi_2(t) & -\varphi_2(t) & 1 \end{pmatrix},$$

$$M_3(t) := \begin{pmatrix} \cosh(\varphi_3(t)) & \sinh(\varphi_3(t)) & 0 \\ \sinh(\varphi_3(t)) & \cosh(\varphi_3(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\varphi_i(t) = at$ for each $i = 1,2,3$. A straightforward computation shows that

$$\big\langle M_i'(t)X(s), M_i(t)N(s)\big\rangle = -\varphi_i'(t)\langle X(s) \times N(s), w_i\rangle = -\varphi_i'(t)\langle -T(s), w_i\rangle.$$

Thus,

$$\begin{aligned}\Big\langle \frac{\partial}{\partial t}\hat{X}_i(s,t), \hat{N}_i(s,t)\Big\rangle &= \big\langle M_i'(t)X(s), M_i(t)N(s)\big\rangle = -\varphi_i'(t)\langle -T(s), w_i\rangle \\ &= \langle T(s), v_i\rangle = k_i(s) = \hat{k}_i(s,t),\end{aligned}$$

where the last equality follows from the fact that isometries preserve geodesic curvature. Therefore, X is a soliton solution to the CSF. ■

It follows from Theorem 2.2 that the study of the solitons solutions to the CSF on $\mathbb{H}^2$ is reduced to describing the curves that satisfy equation (2.2) for some vector $v \in \mathbb{R}^3_1 \setminus \{0\}$. Up to isometries of $\mathbb{H}^2$, we consider v as being $v_i = ae_i$, where $a \in \mathbb{R}^+$, $e_1 = (-1, 0, 0)$ if v is a timelike vector, $e_2 = (-1, 1, 0)$ if v is a lightlike vector, and $e_3 = (0, 0, 1)$ if v is a spacelike vector. Our next result characterizes (2.2) in terms of a system of differential equations.

Proposition 3.1. *Let $X\colon I \to \mathbb{H}^2$ be a regular curve parametrized by arc length s. Consider the vectors*

$$e_1 = (-1, 0, 0), \quad e_2 = (-1, 1, 0) \quad \text{and} \quad e_3 = (0, 0, 1). \tag{3.1}$$

For each $i \in \{1, 2, 3\}$, define the functions

$$\alpha_i(s) = \langle X(s), e_i\rangle, \quad \tau_i(s) = \langle T(s), e_i\rangle \quad \text{and} \quad \eta_i(s) = \langle N(s), e_i\rangle,$$

where T and N are the unit vector fields tangent and normal to X, respectively. For a fixed $a > 0$,

$$k_i(s) = a\tau_i(s)$$

is satisfied for all $s \in I$ if, and only if, the functions $\alpha_i(s)$, $\tau_i(s)$ and $\eta_i(s)$ satisfy the system

$$\begin{cases} \alpha_i'(s) = \tau_i(s), \\ \tau_i'(s) = a\tau_i(s)\eta_i(s) + \alpha_i(s), \\ \eta_i'(s) = -a\tau_i^2(s), \end{cases} \tag{3.2}$$

with initial condition $(\alpha_i(0), \tau_i(0), \eta_i(0))$ satisfying

$$-\alpha_i^2(0) + \tau_i^2(0) + \eta_i^2(0) = \begin{cases} -1, & \text{if } i = 1, \\ 0, & \text{if } i = 2, \\ 1, & \text{if } i = 3. \end{cases} \tag{3.3}$$

For such functions, the expression $-\alpha_i^2(s) + \tau_i^2(s) + \eta_i^2(s)$ is equal to the right-hand side of (3.3), for all $s \in I$. Moreover, $\eta_i(s)$ is a decreasing function.

Proof. The vector fields X, T and N satisfy the following system of equations:

$$\begin{cases} X'(s) = T(s), \\ T'(s) = k(s)N(s) + X(s), \\ N'(s) = -k(s)T(s). \end{cases} \tag{3.4}$$

Taking the inner product with e_i, we get that $\alpha_i(s)$, $\tau_i(s)$ and $\eta_i(s)$ satisfy the system of equations

$$\begin{cases} \alpha_i'(s) = \tau_i(s), \\ \tau_i'(s) = k_i(s)\eta_i(s) + \alpha_i(s), \\ \eta_i'(s) = -k_i(s)\tau_i(s). \end{cases} \tag{3.5}$$

Suppose that $k_i(s) = a\tau_i(s)$ for all $s \in I$. Then from (3.5), we obtain (3.2). Note that

$$e_i = \alpha_i(s)X(s) + \tau_i(s)T(s) + \eta_i N(s).$$

Therefore, $\langle e_i, e_i\rangle = -\alpha_i^2(s) + \tau_i^2(s) + \eta_i^2(s)$ is constant for all $s \in I$. In particular, for $s = 0$, we obtain (3.5). Moreover, it follows from the third equation of the system (3.2) that the function $\eta_i(s)$ is decreasing.

Conversely, suppose that the functions $\alpha_i(s)$, $\tau_i(s)$ and $\eta_i(s)$ satisfy (3.2) and (3.3) for each $i \in \{1, 2, 3\}$. Since (3.5) holds, we have

$$\begin{cases} a\tau_i(s)\eta_i(s) + \alpha_i(s) = k_i(s)\eta_i(s) + \alpha_i(s), \\ -(a\tau_i(s))\tau_i(s) = -k_i(s)\tau_i(s), \end{cases}$$

i.e., $[a\tau_i(s) - k_i(s)]\eta_i(s) = 0$ and $[a\tau_i(s) - k_i(s)]\tau_i(s) = 0$, for all $s \in I$. For each i, in order to conclude that $k_i(s) = a\tau_i(s)$, for all s, we will assume that $k_i(s) \neq a\tau_i(s)$ at some point s_0. Then this will occur on some interval $J \subset I$ around s_0. Hence $\eta_i(s) = \tau_i(s) = 0$ for $s \in J$. Therefore, e_i will be orthogonal to $T(s)$ and $N(s)$ for all $s \in J$. Thus, e_i will be parallel to $X(s)$ for all $s \in J$. But e_i is a constant vector for each i, so this can only happen at some isolated points of a curve X on $\mathbb{H}^2$, which is a contradiction. Therefore, $k_i(s) = a\tau_i(s)$ for all $s \in I$ and for each $i \in \{1, 2, 3\}$. ∎

Our next proposition shows how a solution of the system (3.2), with initial conditions satisfying (3.3), is related to a soliton solution to the CSF.

Proposition 3.2. *Given a solution $(\alpha(s), \tau(s), \eta(s))$ to the system (3.2) on some interval J with fixed $a > 0$ and initial conditions $(\alpha(0), \tau(0), \eta(0))$ satisfying $-\alpha^2(0) + \tau^2(0) + \eta^2(0) = -1$ (respectively, 0 and 1), there exists a smooth curve $X\colon I \to \mathbb{H}^2$ parametrized by arc length s such that its tangent and normal unit vector fields T and N satisfy*

$$\alpha(s) = \langle X(s), e\rangle, \quad \tau(s) = \langle T(s), e\rangle \quad \text{and} \quad \eta(s) = \langle N(s), e\rangle, \tag{3.6}$$

where $e = (-1, 0, 0)$ (respectively, $e = (-1, 1, 0)$ and $e = (0, 0, 1)$).

Proof. Define $k(s) = a\tau(s)$. Then, up to isometries of $\mathbb{H}^2$, there exists a unique curve $X\colon I \to \mathbb{H}^2$ whose curvature is $k(s)$, i.e., $X(s)$ and its tangent and normal unit vector fields $T(s)$ and $N(s)$ satisfy the system (3.4). The curve $X(s)$ is uniquely determined by the initial conditions $X(0)$, $T(0)$ and $N(0)$, that can be chosen such that $-\alpha(0)X(0) + \tau(0)T(0) + \eta(0)N(0) = e$, where $e = (-1, 0, 0)$ (respectively, $e = (-1, 1, 0)$ and $e = (0, 0, 1)$). A straightforward computation shows that (3.3) and (3.4) imply $\frac{d}{ds}[-\alpha(s)X(s) + \tau(s)T(s) + \eta(s)N(s)] = 0$. Therefore, (3.6) is satisfied. ∎

Remark 3.3. Let $X\colon I \to \mathbb{H}^2$ be a regular curve parametrized by arc length s given by $X(s) = (x_1(s), x_2(s), x_3(s))$. The function $\alpha(s)$ defined by (3.6) has the following geometric interpretation.

- If $e = (-1, 0, 0)$ (timelike vector), then $\alpha(s) = x_1(s) > 0$ for all $s \in I$. Moreover, $\alpha(s)$ is the height function with respect to the vector $(1, 0, 0)$.
- If $e = (-1, 1, 0)$ (lightlike vector), then $\alpha(s) = x_1(s) + x_2(s) > 0$ for all $s \in I$. Moreover, $\alpha(s)$ is the height function with respect to the vector $(1, 1, 0)$.
- If $e = (0, 0, 1)$ (spacelike vector), then $\alpha(s) = x_3(s)$ for all $s \in I$. Moreover, $\alpha(s)$ is the height function (with sign) with respect to the vector $(0, 0, 1)$.

As we have seen in Propositions 3.1, 3.2 and Remark 3.3, the investigation of the soliton solutions to the CSF on the 2-dimensional hyperbolic space is equivalent to studying the solutions $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of the system

$$\begin{cases} \alpha'(s) = \tau(s), \\ \tau'(s) = a\tau(s)\eta(s) + \alpha(s), \\ \eta'(s) = -a\tau^2(s), \end{cases} \tag{3.7}$$

for each constant $a > 0$ and initial condition $\psi(0) \in H \cup C \cup S \subset \mathbb{R}^3$, where

$$\begin{aligned} H &:= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : -\alpha^2 + \tau^2 + \eta^2 = -1, \alpha > 0\}, \\ C &:= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 \setminus \{0\} : -\alpha^2 + \tau^2 + \eta^2 = 0, \alpha > 0\}, \\ S &:= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : -\alpha^2 + \tau^2 + \eta^2 = 1\}. \end{aligned} \tag{3.8}$$

These are disjoint sets and if the initial condition $\psi(0) \in H$ (respectively, C or S) then the solution $\psi(s)$ defined on the maximal interval I will be contained in H (respectively, C or S) for all $s \in I$.

From now on, using (3.7), we will prove a series of lemmas that will provide the proof of the main result (Theorem 2.3). Namely, we will prove that for any initial condition $\psi(0)$ given in one of the sets H, C or S defined by (3.8), the solutions $\psi(s)$ of (3.7) and hence the associated soliton solutions to the hyperbolic space are defined on the whole $\mathbb{R}$. Moreover, we will analyze the behaviour of the curvature function of the solitons at each end.

In the first lemma we will study the solution of (3.7) such that the function $\tau(s)$ is constant. As we will see, such solutions (that will be called trivial) only exist on S.

Lemma 3.4. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non null solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H \cup C \cup S$. Then the function $\tau(s) = b$, $s \in I$, where b is a real constant if, and only if, $b \in \{-1, 0, 1\}$, $I = \mathbb{R}$ and $\psi(s) \in S$ for all $s \in \mathbb{R}$. Moreover,*

i) *if $b = 0$, then $\psi(s) = (0, 0, \pm 1)$ are singular solutions of* (3.7) *in S,*

ii) *if $b^2 = 1$, then $a = 1$ and $\psi(s) = (\pm s + \alpha(0), \pm 1, -s \pm \alpha(0))$.*

Proof. If $b = 0$, it follows from (3.7) that $\alpha(s) = 0$ for all $s \in I$. Using the equation $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = \gamma$, where $\gamma \in \{-1, 0, 1\}$, we obtain $\eta^2(s) = 1$ for all $s \in I$. Hence, $\psi(s) = (0, 0, \pm 1)$ for all $s \in \mathbb{R}$ are singular solutions of (3.7) in S.

If $\psi(s)$ is not a singular solution, then $b \neq 0$ and it follows from (3.7) that $ab\eta(s) = -\alpha(s)$ and $\eta(s) = -ab^2 s + \eta(0)$, for all $s \in \mathbb{R}$. Using the relation $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = \gamma$, where $\gamma \in \{-1, 0, 1\}$, we conclude that $[-a^2 b^2 + 1]\eta^2(s) = \gamma - b^2$, and hence the function $[-a^2 b^2 + 1]\alpha^2(s)$ is also constant. Since $\psi(s)$ is not a singular solution, it follows that $a^2 b^2 = 1$ and $b^2 = \gamma$. Therefore, $\gamma = 1$, $b = \pm 1$, $a = 1$ and $\alpha(s) = \mp\eta(s)$, for all $s \in \mathbb{R}$. This concludes the proof. ∎

It follows from Lemma 3.4 that when $\tau(s)$ is a constant function, then the functions $\alpha(s)$ and $\eta(s)$ are linear in s and its corresponding soliton solutions to the CSF are curves of constant curvature, i.e., geodesics when $k(s) = \tau(s) = 0$, or planar curves with curvature $k(s) = \tau(s) = \pm 1$.

In this context, we define a *trivial solution* $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of (3.7), when $\tau(s)$ is a constant function. From now on, we will study only non trivial solutions of (3.7). It follows from Lemma 3.4 that there are no trivial solutions of (3.7) in $H \cup C$. In our next lemmas we will study the solutions $\psi(s)$ of (3.7) contained in $H \cup C$ and those contained in S separately.

Lemma 3.5. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H \cup C$.*

- i) *If $\alpha(s)$ has a critical point, then it is a global minimum point of α. Moreover, there exists always $\bar{s} \in I$ such that $\alpha(s)$ is strictly monotone on the intervals $(\omega_-, \bar{s}]$ and $[\bar{s}, \omega_+)$.*
- ii) *If s_0 is a critical point of $\tau(s)$, then $a^2\tau^2(s_0) > 1$ and s_0 is a local minimum (respectively, maximum) point of $\tau(s)$ if, and only if, $\tau(s_0) < 0$ (respectively, $\tau(s_0) > 0$).*

Proof. i) Let s_0 be a critical point of $\alpha(s)$. Note that $\alpha(s) > 0$ for all $s \in I$ whenever $\psi(0) \in H \cup C$. Taking the second derivative of $\alpha(s)$ and using (3.7) at $s = s_0$, we have

$$\alpha''(s_0) = \tau'(s_0) = a\tau(s_0)\eta(s_0) + \alpha(s_0) = \alpha(s_0) > 0.$$

Hence, s_0 is a global minimum point of $\alpha(s)$. Therefore, $\alpha(s)$ has at most one critical point. If there are no critical points then $\alpha(s)$ is trictly monotone on I.

ii) Let s_0 be a critical point of $\tau(s)$. Then $\tau'(s_0) = a\eta(s_0)\tau(s_0) + \alpha(s_0) = 0$ and $\eta(s_0)\tau(s_0) \neq 0$ because $\alpha(s) > 0$ for all $s \in I$. Since $\psi(0) \in H \cup C$, it follows that $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = \delta \leq 0$ for all $s \in I$, where $\delta \in \{-1, 0\}$. Thus,

$$\delta = -a^2\tau^2(s_0)\eta^2(s_0) + \tau^2(s_0) + \eta^2(s_0) = \eta^2(s_0)[-a^2\tau^2(s_0) + 1] + \tau^2(s_0) \leq 0.$$

Hence, $-a^2\tau^2(s_0) + 1 < 0$. Taking the second derivative of $\tau(s)$ and using (3.7) at $s = s_0$, we have

$$\tau''(s_0) = a\tau'(s_0)\eta(s_0) + a\tau(s_0)\eta'(s_0) + \alpha'(s_0) = \tau(s_0)[-a^2\tau^2(s_0) + 1]. \tag{3.9}$$

This concludes the proof of ii). ∎

Lemma 3.6. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in C$.*

- i) *If $\tau(s) > 0$ in I, then $\alpha(s)$ is strictly increasing in I, $\tau(s)$ is bounded and it has at most one critical point in I. Moreover, $\omega_- = -\infty$, $\lim_{s\to-\infty} \psi(s) = (0, 0, 0)$ and $\lim_{s\to\omega_+} -\eta(s) = \lim_{s\to\omega_+} \alpha(s) = +\infty$.*
- ii) *If $\tau(s) < 0$ in I, then $\alpha(s)$ is strictly decreasing in I, $\tau(s)$ is bounded and it has at most one critical point in I. Moreover, $\omega_+ = +\infty$, $\lim_{s\to+\infty} \psi(s) = (0, 0, 0)$ and $\lim_{s\to\omega_-} \eta(s) = \lim_{s\to\omega_-} \alpha(s) = +\infty$.*

Proof. i) If $\tau(s) > 0$, then it follows from Lemma 3.5 that $\tau(s)$ has only local maximum points, i.e., $\tau(s)$ has at most one critical point. The positive function $\alpha(s)$ is bounded and there exists $\bar{s} \in I$ such that α is strictly increasing on $(\omega_-, \bar{s})$. Thus, it follows from the equation $\alpha^2(s) = \tau^2(s) + \eta^2(s)$ that we can take $\bar{s}$ such that $\tau(s)$ and $\eta(s)$ are bounded and monotone on $(\omega_-, \bar{s})$. The interval I is maximal, hence $\omega_- = -\infty$. Since the limits $\lim_{s\to-\infty} \alpha(s)$, $\lim_{s\to-\infty} \alpha'(s) = \lim_{s\to-\infty} \tau(s)$ and $\lim_{s\to-\infty} \tau'(s)$ exist, we obtain that $\lim_{s\to-\infty} \tau(s) = 0$ and $\lim_{s\to-\infty} \alpha(s) = \lim_{s\to-\infty} \eta(s) = 0$. Using the fact that the function $\eta(s)$ is decreasing, we get that $\eta(s) < 0$ for all $s \in I$.

We claim that $\alpha(s)$ is unbounded on $(\bar{s}, \omega_+)$. In fact, assume by contradiction that the strictly increasing function $\alpha(s)$ is bounded on $(\bar{s}, \omega_+)$. Thus, it follows from the equation $\alpha^2(s) = \tau^2(s) + \eta^2(s)$ that we can take $\bar{s}$ such that $\tau(s)$ and $\eta(s)$ are bounded and monotone on $(\bar{s}, \omega_+)$. Hence, there exists $p \in C$ such that $\lim_{s\to\omega_+}(\alpha(s), \tau(s), \alpha(s)) = p$ and p is a singular (trivial) solution in C, which contradicts Lemma 3.4. Therefore, $\lim_{s\to\omega_+} \alpha(s) = +\infty$.

Now, assume by contradiction that the strictly decreasing and negative function $\eta(s)$ is bounded on $(\bar{s}, \omega_+)$. Since $\tau^2(s) + \eta^2(s) = \alpha^2(s)$, it follows that the function $\tau(s)$ is unbounded and positive on $(\bar{s}, \omega_+)$, because we showed that $\alpha(s)$ is unbounded on $(\bar{s}, \omega_+)$. Thus, we can choose $\bar{s}$ such that $2\tau(s) < a\tau^2(s)$ for all $s > \bar{s}$. Using the equations of (3.7), we obtain

$$2\alpha(s) - 2\alpha(\bar{s}) = 2\int_{\bar{s}}^{s} \tau(s)\,ds < \int_{\bar{s}}^{s} a\tau^2(s)\,ds = -\eta(s) + \eta(\bar{s}). \tag{3.10}$$

Hence $2\alpha(s) < -\eta(s) + \eta(\bar{s}) + 2\alpha(\bar{s})$, for each $s \in (\bar{s}, \omega_+)$. But this contradicts the fact that $\alpha(s)$ is unbounded. Therefore, $\eta(s)$ is unbounded and $\lim_{s\to\omega_+} \eta(s) = -\infty$.

Finally, if $\tau(s)$ is unbounded on $(\bar{s}, \omega_+)$, $\bar{s} \in I$, then we can choose again $\bar{s}$ such that $2\tau(s) < a\tau^2(s)$ for all $s > \bar{s}$. Using (3.10), we obtain $\alpha(s) + \eta(s) < 2\alpha(\bar{s}) - \eta(\bar{s}) - \alpha(s)$, and this is a contradiction, because $\alpha^2(s) = \tau^2(s) + \eta^2(s)$, that is, $\alpha(s) > -\eta(s)$ and $\lim_{s\to\omega_+} \alpha(s) = +\infty$.

ii) This proof is analogous to the proof of item i). ∎

Lemma 3.7. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of (3.7) defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H$. Then there exists a unique s_0 such that $\alpha'(s_0) = \tau(s_0) = 0$.*

Proof. Assume by contradiction that such an s_0 does not exist. Then, either $\tau(s) < 0$ or $\tau(s) > 0$ for all $s \in I$. If $\tau(s) < 0$, then $\alpha(s)$ is strictly decreasing. Taking $\bar{s} \in I$ we have $1 \le \alpha(s) \le \alpha(\bar{s})$ for all $s > \bar{s}$. Since $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = -1$, then $\tau^2(s) + \eta^2(s) < \alpha^2(s) < \alpha^2(\bar{s})$ for all $s > \bar{s}$. Hence, the functions $\alpha(s), \eta(s)$ are bounded and monotone in $[\bar{s}, \omega_+)$, and $\lim_{s\to\omega_+} \tau(s)$ exists, because $\tau^2(s) = -1 + \alpha^2(s) - \eta^2(s)$. Thus, there exists a point $p \in H$ such that $\lim_{s\to\omega_+}(\alpha(s), \tau(s), \alpha(s)) = p$. Therefore, $\omega_+ = +\infty$ and p is a singular (trivial) solution of (3.7), which contradicts Lemma 3.4. In a similar way, we can prove that $\tau(s) < 0$ for all $s \in I$ cannot occur.

Therefore, there is an $s_0 \in I$ such that $\alpha'(s_0) = \tau(s_0) = 0$. It follows from Lemma 3.5 that s_0 is a global minimum of the function $\alpha(s)$. Hence, s_0 is unique. ∎

In the next three lemmas, we will suppose that $\psi(0) \in H \cup C$ and that $\alpha(s)$ has only one critical point. Note that this hypothesis only excludes the case presented in Lemma 3.6 because it follows from Lemma 3.4 that $\alpha(s)$ has at most one critical point when $\psi(0) \in H \cup C$ and Lemma 3.7 shows that $\alpha(s)$ has a unique critical point, when $\psi(0) \in H$.

Lemma 3.8. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H \cup C$. If $\alpha(s)$ has one critical point, then $\lim_{s\to\omega_-} \alpha(s) = \lim_{s\to\omega_+} \alpha(s) = \infty$.*

Proof. Let s_0 be the global minimum point of $\alpha(s)$. Thus, $\alpha(s)$ is monotone on the intervals $(\omega_-, s_0]$ and $[s_0, \omega_+)$. Assume by contradiction that $\alpha(s)$ is bounded on the intervals $(\omega_-, s_0]$ and $[s_0, \omega_+)$. Since the functions $\alpha(s)$, $\tau(s)$ and $\eta(s)$ satisfy $\tau^2(s) + \eta^2(s) = \delta + \alpha^2(s) \le \alpha^2(s)$, where $\delta \in \{-1, 0\}$, then the functions $\alpha(s)$ and $\eta(s)$ are bounded and monotone on $(\omega_-, s_0]$ and $[s_0, \omega_+)$. The limits $\lim_{s\to\omega_-} \tau(s)$ and $\lim_{s\to\omega_+} \tau(s)$ exist, because $\tau^2(s) = \delta + \alpha^2(s) - \eta^2(s)$. Thus, there are points p_1 and $p_2 \in H \cup C$ such that $\lim_{s\to\omega_-}(\alpha(s), \tau(s), \alpha(s)) = p_1$ and $\lim_{s\to\omega_+}(\alpha(s), \tau(s), \alpha(s)) = p_2$. Hence, $\omega_- = -\infty$, $\omega_+ = +\infty$ and $\{p_1, p_2\}$ is a set of singular (trivial) solutions of (3.7), which contradicts Lemma 3.4.

Therefore, we conclude that the function $\alpha(s)$ is unbounded on the intervals $(\omega_-, s_0]$ and $[s_0, \omega_+)$, and moreover $\lim_{s\to\omega_-} \alpha(s) = \lim_{s\to\omega_+} \alpha(s) = \infty$. ∎

Lemma 3.9. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H \cup C$. If $\alpha(s)$ has one critical point, then $\lim_{s\to\omega_-} \eta(s) = \infty$ and $\lim_{s\to\omega_+} \eta(s) = -\infty$.*

Proof. Let s_0 be the global minimum point of $\alpha(s)$. Then $\tau(s) < 0$ for all $s < s_0$ and $\tau(s) > 0$ for all $s > s_0$. Moreover, $\alpha(s)$ is unbounded and monotone on the intervals $(\omega_-, s_0]$ and $[s_0, \omega_+)$.

Assume by contradiction that the function $\eta(s)$ is bounded on $(\omega_-, s_0]$. Since $\tau^2(s) + \eta^2(s) = \delta + \alpha^2(s)$, where $\delta \in \{-1, 0\}$, then it follows from Lemma 3.7 that the function $\tau(s)$ is unbounded and negative on (ω_-, s_0), i.e., there exists $s_1 \in (\omega_-, s_0]$ such that $a\tau(s) < -1$ and $-a\tau^2(s) < \tau(s)$ for all $s \in (\omega_-, s_1]$. Thus, using (3.7) for each $s \in (\omega_-, s_1]$, we obtain

$$\alpha(s) - \alpha(s_1) = -\int_s^{s_1} \tau(s) ds < \int_s^{s_1} a\tau^2(s)\, ds = \eta(s) - \eta(s_1),$$

that is, $\alpha(s) < \eta(s) - \eta(s_1) + \alpha(s_1)$ for each $s \in (\omega_-, s_1]$, which contradicts Lemma 3.8. Therefore, $\eta(s)$ is unbounded on $(\omega_-, s_1]$.

In a similar way, we can prove that the function $\eta(s)$ is unbounded on $[s_0, \omega_+)$. Since $\eta(s)$ is decreasing on (ω_-, ω_+), we have $\lim_{s\to\omega_-} \eta(s) = \infty$ and $\lim_{s\to\omega_+} \eta(s) = -\infty$. ∎

Lemma 3.10. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of* (3.7)*, with $a > 0$, defined on the maximal interval $I = (\omega_-, \omega_+)$ and initial condition $\psi(0) \in H \cup C$. If $\alpha(s)$ has one critical point, then the function $\tau(s)$ is bounded on I and it has only two critical points.*

Proof. Let $s_0 \in I$ be the global minimum point of $\alpha(s)$. The arguments consist in studying the existence and the properties of the critical points of $\tau(s)$.

Claim. If $\tau(s)$ does not have any critical point on I, then $-1 < a\tau(s) < 0$ on (ω_-, s_0) and $0 < a\tau(s) < 1$ on (s_0, ω_+).

In fact, suppose that $\tau'(s) \ne 0$ for all $s \in I$. At s_0, $\tau(s_0) = 0$ and $\tau'(s_0) = \alpha(s_0) > 0$. Moreover, we have that $\tau'(s) = a\tau(s)\eta(s) + \alpha(s) > 0$, i.e., $\tau(s)$ is an increasing function on I, $\lim_{s\to\omega_-} \tau(s) \ne 0$ and $\lim_{s\to\omega_+} \tau(s) \ne 0$. It follows from Lemma 3.9 that

$\lim_{s\to\omega_-}\eta(s) = +\infty$ and $\lim_{s\to\omega_+}\eta(s) = -\infty$. We also know that $\tau(s)$ is negative on (ω_-, s_0) and positive on (s_0, ω_+). Thus, there are $s_1 \in (\omega_-, s_0)$ and $s_2 \in (s_0, \omega_+)$ such that $-\alpha(s) < a\tau(s)\eta(s) < 0$ for all $s \in I \setminus [s_1, s_2]$. Hence, $-\alpha^2(s) < -a^2\tau^2(s)\eta^2(s)$ for all $s \in I \setminus [s_1, s_2]$ and

$$\delta = -\alpha^2(s) + \tau^2(s) + \eta^2(s) < -a^2\tau^2(s)\eta^2(s) + \tau^2(s) + \eta^2(s),$$

i.e., $a^2\tau^2(s)\eta^2(s) < -\delta + \tau^2(s) + \eta^2(s)$. Therefore,

$$1 < \frac{-\delta}{a^2\eta^2(s)\tau^2(s)} + \frac{1}{a^2\eta^2(s)} + \frac{1}{a^2\tau^2(s)}$$

for all $s \in I \setminus [s_1, s_2]$. Taking the limit when $s \to \omega_+$ and $s \to \omega_-$, using the fact that $\tau(s)$ is increasing and Lemma 3.9, we obtain that

$$\lim_{s\to\omega_+} a^2\tau^2(s) < 1 \quad \text{and} \quad \lim_{s\to\omega_-} a^2\tau^2(s) < 1.$$

Thus, using that $\tau(s)$ is increasing, we conclude that $-1 < a\tau(s) < 0$ on (ω_-, s_0) and $0 < a\tau(s) < 1$ on (s_0, ω_+). This proves our *Claim*.

Still assuming that $\tau(s)$ does not have any critical point, we define the positive functions $f(s) = \alpha(s) + \eta(s)$ and $g(s) = \alpha(s) - \eta(s)$ (observe that $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = \delta \le 0$). Taking the derivatives of f and g and using (3.7), we obtain $f'(s) = \tau(s)[1 - a\tau(s)]$ and $g'(s) = \tau(s)[1 + a\tau(s)]$. It follows from our *Claim* that the functions f and g are decreasing when $\tau(s) < 0$ and increasing when $\tau(s) > 0$, and that

$$0 < f(s) \cdot g(s) = -\delta + \tau^2(s) < \frac{-\delta a^2 + 1}{a^2}$$

for all $s \in I$, where $\delta \in \{-1, 0\}$. Hence, the functions f and g are positive, monotone and bounded on the interval (ω_-, s_0). Similarly, one shows that the functions f and g are positive, monotone and bounded on (s_0, ω_+). Thus, we conclude that there exist $M_1, M_2 \in \mathbb{R}$ such that

$$\begin{cases} \alpha(s) + \eta(s) \le M_1, \\ \alpha(s) - \eta(s) \le M_2, \end{cases} \qquad \forall s \in (\omega_-, s_0) \cup (s_0, \omega_+).$$

Hence, $2\alpha(s) \le M_1 + M_2,\ \forall s \in (\omega_-, s_0)$ and $2\alpha(s) \le M_1 + M_2,\ \forall s \in (s_0, \omega_+)$. These inequalities contradict Lemma 3.8. Hence, the function $\tau(s)$ has at least one critical point on each interval (ω_-, s_0) and (s_0, ω_+). From item ii) of Lemma 3.5 we have that $\tau(s)$ has only one local minimum $s_1 \in (\omega_-, s_0)$ and it has only one local maximum $s_2 \in (s_0, \omega_+)$, i.e., $\tau(s_1) \le \tau(s) \le \tau(s_2)$ for all $s \in I$ and $\tau(s)$ is bounded on the interval I. ∎

We will now study the solutions of the system (3.7), with initial conditions on S. We will first classify the singular points of the system that are in the set S.

Lemma 3.11. *Let $\Phi: S \to TS \subset \mathbb{R}^3$ be the differential vector field given by $\Phi(\alpha, \tau, \eta) = (\tau, a\tau\eta + \alpha, -a\tau^2)$, where $a > 0$. Then $p = (0, 0, 1)$ and $-p = (0, 0, -1)$ are the singular points of Φ, and the eigenvalues of $d\Phi_p$ and $d\Phi_{-p}$ are given respectively by*

$$\lambda_p = \frac{a \pm \sqrt{a^2+4}}{2} \quad \textit{and} \quad \lambda_{-p} = \frac{-a \pm \sqrt{a^2+4}}{2}. \tag{3.11}$$

Proof. Note that if $\Phi(\alpha,\tau,\eta)=0$, then $\alpha=\tau=0$ and $\eta=\pm1$. Hence, $p=(0,0,1)$ and $-p=(0,0,-1)$ are the singular points of Φ. The tangent plane in each singular point is defined by $T_{\pm p}S\approx\{(\alpha,\tau,\eta)\in\mathbb{R}^3:\eta=0\}$. Thus,

$$d\Phi=\begin{pmatrix}0 & 1 & 0\\ 1 & a\eta & a\tau\\ 0 & -2a\tau & 0\end{pmatrix}.$$

Hence λ is an eigenvalue of $d\Phi_{\pm p}$ if there is a non null vector $w=(w_1,w_2,0)\in T_{\pm p}S$ such that $d\Phi_{\pm p}(w)=\lambda w$. Hence, λ satisfies $\lambda^2\mp a\lambda-1=0$, which gives (3.11). ∎

In Lema 3.11, we saw that $(0,0,\pm1)$ are saddle points for the vector field Φ on S, i.e., $\psi(s)=(0,0,\pm1)$, $s\in I$, are singular solutions of (3.7). If the functions α and τ are identically zero, then the corresponding curve $X(s)$ is the intersection of the upper half hyperboloid with the plane going through the origin, orthogonal to $(0,0,1)$. Hence both singular solutions of the system correspond to the same curve.

In order to study the non trivial solutions of the system (3.7), we consider the singular point $p=(0,0,1)$ and a solution $\psi(s,q)$ of the system with initial condition $q\in S$. Since the eigenvalues of the linearized system at the singular point are not zero, it follows that the local behavior of the system (3.7) is equivalent to the linearized one. Hence, there exist initial conditions $q,\overline{q}\in S\setminus\{p\}$ such that $\lim_{s\to-\infty}\psi(s,q)=p$ and $\lim_{s\to+\infty}\psi(s,\overline{q})=p$. We define the unstable and stable sets as

$$\begin{aligned} W^u(p)&=\{q\in S:\lim_{s\to-\infty}\psi(s,q)=p\},\\ W^s(p)&=\{q\in S:\lim_{s\to+\infty}\psi(s,q)=p\}. \end{aligned}\tag{3.12}$$

From Lemma 3.4 we know that, if the function τ is a non zero constant, i.e., if $\tau(s)=b$, $b\in\mathbb{R}\setminus\{0\}$ for all $s\in I$, then $b^2=a=1$. Our next result provides two non trivial solutions of the system (3.7), $a=b^2=1$, defined on $\mathbb{R}$, with initial conditions on the set S. They are particular cases of the solutions obtained in Lemma 3.4 with the constant of integration being zero. Moreover, we also obtain the soliton solutions corresponding to these soltions.

Proposition 3.12. *Let $\psi(s)=(\alpha(s),\tau(s),\eta(s))$ be a solution of (3.7) defined on the maximal interval I, $a=1$ and $\psi(0)\in S$. Then, $\psi(s)=(-s,-1,-s)$ (respectively, $\overline{\psi}(s)=(s,1,-s)$), $s\in I=\mathbb{R}$, satisfy (3.7) with initial condition $(0,-1,0)$ (respectively, $(0,1,0)$). Moreover,*

i) *The curve*

$$X(s)=\Big(\frac{1+l^2+s^2}{2l},\frac{l^2-1-s^2}{2l},-s\Big),\tag{3.13}$$

where $l>0$, is the soliton solution to the CSF in $\mathbb{H}^2$ which corresponds to the solution $\psi(s)=(-s,-1,-s)$ of (3.7).

ii) *The curve*

$$\overline{X}(s)=\Big(\frac{1+l^2+s^2}{2l},\frac{l^2-1-s^2}{2l},s\Big),$$

where $l>0$, is the soliton solution to the CSF in $\mathbb{H}^2$ which corresponds to the solution $\psi(s)=(s,1,-s)$ of (3.7)

Proof. Straightforward computations show that $\psi(s)$ and $\overline{\psi}(s)$ satisfy (3.7) with initial conditions $(0,-1,0)$ and $(0,1,0)$ respectively.

i) Note that, if a curve $X(s)$ is given by (3.13), then

$$T(s) = \Big(\frac{s}{l}, -\frac{s}{l}, 1\Big), \quad N(s) = X(s) \times T(s) = \Big(\frac{-1+l^2+s^2}{2l}, \frac{l^2+1-s^2}{2l}, -s\Big),$$

$\alpha(s) = \langle X(s), (0,0,1)\rangle = s$, $\tau(s) = \langle T(s), (0,0,1)\rangle = 1$ and $\eta(s) = \langle N(s), (0,0,1)\rangle = -s$.

ii) The proof is similar to the case i). ∎

The plane curve $X(s)$ in $\mathbb{H}^2$ given by (3.13) is a soliton solution to the CSF with fixed vector $v = (0,0,1)$ and constant curvature.

In the following lemmas, we study the behavior of the functions $\alpha(s)$, $\tau(s)$ and $\eta(s)$ when $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ is a non trivial solution of the system (3.7) and the initial condition $\psi(0) \in S$.

Lemma 3.13. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in S$. If $s_0 \in I$ is a critical point of $\alpha(s)$, then s_0 is the global minimum* (*respectively, maximum*) *of $\alpha(s)$ if, and only if, $\alpha(s_0) > 0$* (*respectively, $\alpha(s_0) < 0$*)*. Moreover, there exists always $\bar{s}$ such that the function $\alpha(s)$ is monotone on the intervals $(\omega_-, \bar{s})$ and $(\bar{s}, \omega_+)$.*

Proof. Let s_0 be a critical point of $\alpha(s)$. Then $\tau(s_0) = 0$. It follows from (3.7) that

$$\alpha''(s_0) = \alpha(s_0). \tag{3.14}$$

Note that if $\alpha(s_0) = \tau(s_0) = 0$ and $\eta(s_0) = 1$, then $\alpha(s)$ is constant, because $(0,0,1)$ is a singular (trivial) solution of (3.7). Therefore, it follows from (3.14) that s_0 is a local minimum point of $\alpha(s)$ if $\alpha(s_0) > 0$. If there is another critical point s_1 of $\alpha(s)$ such that s_0 and s_1 are consecutive, then $\alpha(s_1) > \alpha(s_0) > 0$, because s_0 is a local minimum point. Thus, $\alpha''(s_1) = \alpha(s_1) > 0$ and s_1 is a local minimum point $\alpha(s)$; this is a contradiction. Therefore, if $\alpha(s_0) > 0$, then s_0 is a global minimum of $\alpha(s)$. If $\alpha(s_0) < 0$, it follows from (3.14) that s_0 a local maximum point of $\alpha(s)$. The proof that s_0 is a global maximum point of $\alpha(s)$ is analogue to the previous case.

Since the function $\alpha(s)$ has at most one critical point, there exists always $\bar{s}$ such that the function $\alpha(s)$ is monotone on the intervals $(\omega_-, \bar{s})$ and $(\bar{s}, \omega_+)$. ∎

We observe that if for a solution $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of (3.7) with $\psi(0) \in S$, the function $\alpha(s)$ does not have a critical point, then $\alpha(s)$ is monotone on I. Moreover, it follows from Lemma 3.13 that for any solution of (3.7) with $\psi(0) \in W^u \cup W^s \setminus \{(0,0,1)\}$, the functions $\alpha(s)$ and $\eta(s)$ do not have critical points.

Lemma 3.14. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in S$. Consider $W^u(p)$ and $W^s(p)$ given by* (3.12)*. If $\psi(0) \in S \setminus W^u(p)$* (*respectively, $S \setminus W^s(p)$*)*, then* $\lim_{s\to\omega_-} |\alpha(s)| = +\infty$ (*respectively,* $\lim_{s\to\omega_+} |\alpha(s)| = +\infty$).

Proof. It follows from Lemma 3.13 that there exits $\bar{s} \in I$ such that $\alpha(s)$ is monotone on the intervals $(\omega_-, \bar{s})$ and $(\bar{s}, \omega_+)$. If $\psi(0) \in S \setminus W^u(p)$, assume by contradiction that $\alpha(s)$

is bounded on $(\omega_-, \bar{s})$. Since $\tau^2(s) + \eta^2(s) = \alpha^2(s) + 1$, it follows that the functions $\alpha(s)$ and $\eta(s)$ are bounded and monotone on $(\omega_-, \bar{s})$ and the limit $\lim_{s\to\omega_-} \tau(s)$ exists. Hence, there exists $q \in \mathbb{R}^3_1$ such that $\lim_{s\to\omega_-}(\alpha(s), \tau(s), \alpha(s)) = q$, $\omega_- = -\infty$ and q is a singular solution of (3.7). But the system (3.7) does not have any singular solution on the set $S \setminus W^u(p)$. Therefore, $\lim_{s\to\omega_-} |\alpha(s)| = +\infty$. Similarly, when $\psi(0) \in S \setminus W^s(p)$ one proves that $\lim_{s\to\omega_+} |\alpha(s)| = +\infty$. ■

Lemma 3.15. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of (3.7) defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in S$.*

i) *If s_0 is a critical point of $\tau(s)$, then $\tau^2(s_0) \neq 1$ and $a^2\tau^2(s_0) \neq 1$. If $\tau(s_0) > 0$, then s_0 is a local minimum (respectively, maximum) point of $\tau(s)$ if, and only if, $\tau(s_0) > 1$ (respectively, $0 < \tau(s_0) < 1$). If $\tau(s_0) < 0$, then s_0 is a local minimum (respectively, maximum) point of $\tau(s)$ if, and only if, $\tau(s_0) < -1$ (respectively, $-1 < \tau(s_0) < 0$).*

ii) *The function $\tau(s)$ has at the most a finite number of critical points.*

Proof. i) Let s_0 be a critical point of $\tau(s)$. If $\tau^2(s_0) = 1$, it follows from $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = 1$ that $\alpha^2(s_0) = \eta^2(s_0)$. Thus, $0 = \tau'(s_0) = \pm a\eta(s_0) + \alpha(s_0)$, i.e., $a = 1$. Hence, it follows from Lemma 3.4 that the solution $\psi(s)$ of (3.7) with initial condition $\psi(s_0)$ is a trivial solution, which contradicts the hypothesis. Therefore, $\tau(s_0) \neq 1$. If $a^2\tau^2(s_0) = 1$, then $0 = \tau'(s_0) = \pm\eta(s_0) + \alpha(s_0)$ and from $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = 1$ we have that $\tau(s_0) = 1$, which also contradicts the hypothesis. Hence, $\tau^2(s_0) \neq 1$ and $a^2\tau^2(s_0) \neq 1$. Moreover, taking the second derivative of $\tau(s)$ at $s = s_0$, we obtain (3.9). This concludes the proof of the item i).

ii) Note that it follows from Lemma 3.13 that there exists always $\bar{s}$ such that $\tau(s)$ does not change sign on each interval $(\omega_-, \bar{s})$ and $(\bar{s}, \omega_+)$. We will prove for the interval $(\bar{s}, \omega_+)$, since similar arguments can be used for the interval $(\omega_-, \bar{s})$. If $\psi(0) \in W^s(p)$, i.e., $\lim_{s\to+\infty} \tau(s) = 0$, then it follows from item i) that $\tau(s)$ has at most a finite number of critical points on $(\bar{s}, \omega_+)$. If $\psi(0) \in S \setminus W^s(p)$, then it follows from Lemma 3.14 that $\lim_{s\to\omega_+} |\alpha(s)| = +\infty$.

Assume that $\tau(s) > 0$ for all $s \in (\bar{s}, \omega_+)$. Then there exists s_1 such that $\alpha(s) > 0$ for all $s > s_1$. If the function $\eta(s)$, which is monotone, is always positive, then $\tau'(s) > 0$ for all $s > s_1$, i.e., τ has no critical on (s_1, ω_+). Now, consider s_1 such that $\eta(s) < 0$ for all $s > s_1$. Assume by contradiction that there are $s_2, s_3 \in (s_1, \omega_+)$, two local maximum points of $\tau(s)$. From item i), we obtain that there are $b, d \in (s_2, s_3)$ such that $b < d$, $\tau(b) = \tau(d) = 1$, $\tau'(b) < 0$, $\tau'(d) > 0$. It follows from $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = 1$ that $\eta^2(b) = \alpha^2(b)$ and $\eta^2(d) = \alpha^2(d)$. Thus, from $\tau'(b) < 0$ we obtain $a\eta(b) < -\alpha(b) < 0$, i.e., $a > 1$ and from $\tau'(d) > 0$ we conclude $\alpha(d) > -a\eta(d) > 0$, i.e., $a < 1$, this is a contradiction. Therefore, τ has at most one local maximum point on the interval (s_1, ω_+).

Assuming that $\tau(s) < 0$ for all $s \in (\bar{s}, \omega_+)$, similar arguments show that τ has at most one local maximum point on the interval (s_1, ω_+).

Moreover, similar arguments for the interval $(\omega_-, \bar{s})$ imply that $\tau(s)$ has at most a finite number of critical points on I. ■

The following lemma shows that the function $\tau(s)$ is bounded and hence the curvature of the soliton $X(s)$ on $\mathbb{H}^2$ is bounded.

Lemma 3.16. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$ with initial condition $\psi(0) \in S$. Then the function $\tau(s)$ is bounded on I.*

Proof. It follows from Lemma 3.13 that there exists $\bar{s} \in I$ such that $\alpha(s)$ is monotone on the intervals $(\omega_-, \bar{s})$ and $(\bar{s}, \omega_+)$. Moreover, from Proposition 3.2 we have that $\eta(s)$ is monotone.

If $\psi(0) \in W^s(p)$, where p is a singular point, then $\lim_{s\to+\infty} \tau(s) = 0$ and $\tau(s)$ is bounded on $(\bar{s}, +\infty)$ for any $\bar{s} \in I$ fixed. Similarly, if $\psi(0) \in W^u(p)$, then $\lim_{s\to-\infty} \tau(s) = 0$ and $\tau(s)$ is bounded on $(-\infty, \bar{s})$, $\bar{s} \in I$ fixed.

We will now consider the cases when the initial condition $\psi(0)$ belongs to $S \setminus W^u(p)$ or $S \setminus W^s(p)$. If $\psi(0) \in S \setminus W^u(p)$, assume by contradiction that $\tau(s)$ is unbounded on $(\omega_-, \bar{s})$; then it follows from Lemma 3.15 that there exists $s_1 \in (\omega_-, \bar{s})$ such that $|\tau(s)| > 1$ and $a|\tau(s)| > 2$ for all $s \in (\omega_-, s_1)$. Thus, $|\alpha(s)| > |\eta(s)|$, because $\alpha^2(s) - \eta^2(s) = \tau^2(s) - 1 > 0$ and $a\tau^2(s) > 2|\tau(s)|$ for all $s \in (\omega_-, s_1)$. From Lemma 3.13 we have that $\tau(s)$ does not change sign on (ω_-, s_1).

If $\alpha'(s) = \tau(s) < 0$ on (ω_-, s_1), then $\alpha(s)$ is strictly decreasing on this interval. Thus, it follows from Lemma 3.14 that $\lim_{s\to\omega_-} \alpha(s) = +\infty$. Hence, s_1 can be chosen so that $\alpha(s)$ is decreasing and positive for all $s < s_1$. Therefore, using (3.7) and the fact that $a\tau^2(s) > 2|\tau(s)|$, we obtain

$$2\alpha(s) - 2\alpha(s_1) = -2\int_s^{s_1} \tau(s)\, ds < \int_s^{s_1} a\tau^2(s)\, ds = \eta(s) - \eta(s_1),$$

i.e., $0 < \alpha(s) - \eta(s) < 2\alpha(s_1) - \eta(s_1) - \alpha(s)$, which contradicts Lemma 3.14, because $\lim_{s\to\omega_-} \alpha(s) = +\infty$. Hence, $\tau(s)$ is bounded on (ω_-, s_1).

If $\alpha'(s) = \tau(s) > 0$ on (ω_-, s_1), then $\alpha(s)$ is strictly increasing on this interval. Thus, it follows from Lemma 3.14 that $\lim_{s\to\omega_-} \alpha(s) = -\infty$. Hence, s_1 can be chosen so that $\alpha(s)$ is increasing and negative for all $s < s_1$. Therefore, using (3.7) and the fact that $a\tau^2(s) > 2|\tau(s)|$, we obtain

$$2\alpha(s_1) - 2\alpha(s) = 2\int_s^{s_1} \tau(s)\, ds < \int_s^{s_1} a\tau^2(s)\, ds = -\eta(s_1) + \eta(s),$$

i.e., $0 < -\alpha(s) - \eta(s) < -2\alpha(s_1) - \eta(s_1) + \alpha(s)$, which contradicts Lemma 3.14, because $\lim_{s\to\omega_-} \alpha(s) = -\infty$. Hence, $\tau(s)$ is bounded on $(\omega_-, \bar{s})$.

When $\psi(0) \in S \setminus W^s(p)$, similar arguments show that $\tau(s)$ is bounded on $(\bar{s}, \omega_+)$. Therefore, the function $\tau(s)$ is bounded on I. ∎

Our next lemma provides the behavior of the function $\eta(s)$.

Lemma 3.17. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in S$. Consider $W^u(p)$ and $W^s(p)$ given by* (3.12)*. If $\psi(0) \in S \setminus W^u(p)$ (respectively, $S \setminus W^s(p)$), then $\lim_{s\to\omega_-} \eta(s) = +\infty$ (respectively, $\lim_{s\to\omega_+} \eta(s) = -\infty$).*

Proof. The proof follows from Lemmas 3.14 and 3.16 and the fact that $\tau^2(s) + \eta^2(s) = \alpha^2(s) + 1$. ∎

Lemma 3.18. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7) *defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in H \cup C \cup S$. Then $I = \mathbb{R}$.*

Proof. It follows from Lemmas 3.6, 3.10 and 3.16 that $\tau(s)$ is bounded on the interval I. Let $M > 0$ be such that $|\tau(s)| \leq M$ for all $s \in I$. Using (3.7), we obtain

$$|\alpha(s) - \alpha(s_0)| = \left| \int_{s_0}^{s} \tau(s)\, ds \right| \leq M|s - s_0| \tag{3.15}$$

for all $s \in I$. It follows from Lemmas 3.5 and 3.13 that there exists $\bar{s}$ such that $\alpha(s)$ is monotone on the intervals $(\bar{s}, \omega_+)$ and $(\omega_-, \bar{s})$. If $\alpha(s)$ is bounded on $(\bar{s}, \omega_+)$ (respectively, $(\omega_-, \bar{s})$), then it follows from $\tau^2(s) + \eta^2(s) = \alpha^2(s) + \gamma$, where $\gamma \in \{-1, 0, 1\}$, that $\tau(s)$ and $\eta(s)$ are bounded on $(\bar{s}, \omega_+)$ (respectively, $(\omega_-, \bar{s})$) and hence $\omega_+ = +\infty$ (respectively, $\omega_- = -\infty$). If $\alpha(s)$ is unbounded on $(\bar{s}, \omega_+)$ (respectively, $(\omega_-, \bar{s})$), then it follows from (3.15) that $\omega_+ = +\infty$ (resp. $\omega_- = -\infty$). Therefore, $I = \mathbb{R}$. ∎

Lemma 3.19. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a non trivial solution of* (3.7)*, with $a > 0$ and initial condition $\psi(0) \in H \cup C \cup S$, where H, C and S are defined by* (3.8)*. Then $\psi(s)$ and the corresponding soliton solution $X(s)$ to the CSF on $\mathbb{H}^2$ are defined for all $s \in \mathbb{R}$. Moreover, at each end the curvature $k(s)$ of X converges to one of the constants $\{-1, 0, 1\}$.*

Proof. Since $X(s)$ is a soliton solution to the CSF corresponding to $\psi(s)$, then $k(s) = a\tau(s)$. Thus, Lemmas 3.6, 3.10, 3.15 and 3.16 imply that $k(s)$ is bounded on $\mathbb{R}$ and it has at most a finite number of critical points. Thus, the limits $\lim_{s\to\pm\infty} k(s) = \lim_{s\to\pm\infty} a\tau(s)$ exist. In particular, when $\psi(0) \in W^u(p)$ then $\lim_{s\to-\infty} \tau(s) = 0$. Similarly, when $\psi(0) \in W^s(p)$ then $\lim_{s\to+\infty} \tau(s) = 0$. In these cases, the curvature function converges to zero at $-\infty$ and $+\infty$, respectively.

If $\lim_{s\to\pm\infty} \tau(s) \neq 0$, then $\lim_{s\to\pm\infty} |\alpha(s)| = +\infty$ and it follows from $-\alpha^2(s) + \tau^2(s) + \eta^2(s) = \delta$, where $\delta \in \{-1, 0, 1\}$, that

$$\lim_{s\to\pm\infty} \frac{\eta^2(s)}{\alpha^2(s)} = \lim_{s\to\pm\infty} \left(\frac{-\tau^2(s) + \delta}{\alpha^2(s)} + 1 \right) = 1.$$

Using (3.7), Lemmas 3.8, 3.9, 3.14, 3.17 and the L'Hospital rule, we obtain

$$\lim_{s\to\pm\infty} -\frac{\eta(s)}{\alpha(s)} = \lim_{s\to\pm\infty} a\tau(s) = \lim_{s\to\pm\infty} k(s).$$

Therefore, $\lim_{s\to-\infty} k(s) = \pm 1$ and $\lim_{s\to+\infty} k(s) = \pm 1$. ∎

Finally, we will prove our main theorem.

Proof of Theorem 2.3. For any vector $v \in \mathbb{R}^3_1 \setminus \{0\}$, without loss generality we can consider $v = ae$, where $a > 0$, and

$$e = \begin{cases} (-1, 0, 0) & \text{if } v \text{ is a timelike vector,} \\ (-1, 1, 0) & \text{if } v \text{ is a lightlike vector,} \\ (0, 0, 1) & \text{if } v \text{ is a spacelike vector.} \end{cases}$$

Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of (3.7) defined on the maximal interval $I = (\omega_-, \omega_+)$, $a > 0$, and initial condition $\psi(0) \in \mathbb{R}^3$ satisfying

$$-\alpha^2(0) + \tau^2(0) + \eta^2(0) = \begin{cases} -1 & \text{if } v \text{ is a timelike vector,} \\ 0 & \text{if } v \text{ is a lightlike vector,} \\ 1 & \text{if } v \text{ is a spacelike vector,} \end{cases}$$

i.e., $\psi(0) \in H \cup C \cup S$, where H, C and S are the disjoint sets given by (3.8). Moreover, it follows from Proposition 3.2 that there is a soliton solution $X(s)$ to the CSF, with curvature $k(s) = a\tau(s)$, such that the relations $\alpha(s) = \langle X(s), e\rangle$, $\tau(s) = \langle T(s), e\rangle$ and $\eta(s) = \langle N(s), e\rangle$ are satisfied, where T and N are the unit vector fields tangent and normal to X.

Thus, the initial conditions of (3.7), which are given by two constants, determine the soliton solution in each case. Therefore, for each fixed vector $v \in \mathbb{R}^3_1 \setminus \{0\}$, there is a 2-parameter family of non trivial soliton solutions to the CSF in $\mathbb{H}^2$. Moreover, it follows from Lemmas 3.4 and 3.18 that each soliton solution is defined for all $s \in \mathbb{R}$, i.e., $I = \mathbb{R}$, and Lemma 3.19 shows that the curvature at each end converges to one of the constants $\{-1, 0, 1\}$.

Note that from Lemmas 3.4, 3.5 and 3.13 we know that there exists $\bar{s} \in \mathbb{R}$ such that $\alpha(s)$ is strictly monotone on the intervals $(-\infty, \bar{s})$ and $(\bar{s}, +\infty)$. Since $\alpha(s)$ describes the Euclidean height of $X(s)$ with respect to a fixed plane, then $X(s)$ does not have self-intersections in each one of the intervals $(-\infty, \bar{s})$ and $(\bar{s}, +\infty)$. Therefore, $X(s)$ is embedded if $\alpha(s)$ is monotone in $\mathbb{R}$.

If $\alpha(s)$ is not monotone in $\mathbb{R}$, then $\alpha(s)$ has only one critical point. Suppose that $X(s)$ has some self-intersection, consider the simple region Σ bounded by $X([s_1, s_2])$ with $X(s_1) = X(s_2)$, $s_1 < \bar{s} < s_2$, and let θ be the external angle between the tangent vectors $T(s_1)$ and $T(s_2)$, which is at the most π. By the Gauss–Bonnet theorem, we obtain

$$\begin{aligned} 0 < 2\pi\,\chi(\Sigma) - \theta &= \int_\Sigma \kappa\, d\sigma + \int_{X([s_1,s_2])} k(s)\, ds = -\int_\Sigma d\sigma + \int_{X([s_1,s_2])} a\tau(s)\, ds \\ &= -\int_\Sigma d\sigma + a[\alpha(s_2) - \alpha(s_1)] = -\int_\Sigma d\sigma < 0. \end{aligned}$$

This is a contradiction. Hence, the soliton solution $X(s)$ to the CSF in $\mathbb{H}^2$ does not admit self-intersections. Note that $X(s)$ is already embedded on the intervals $(-\infty, \bar{s})$ and $(\bar{s}, +\infty)$ and from Lemmas 3.8 and 3.14 we have that the two ends of the curve are unbounded. Therefore, $X(s)$ is an embedded curve. ∎

4. Visualizing some soliton solutions to the CSF on $\mathbb{H}^2$

In this section, we visualize some examples of soliton solutions to the CSF on the hyperbolic space, which is parametrized by

$$\chi(u, w) = (\sqrt{1 + u^2 + w^2}, u, w).$$

If a curve $X(s) = \chi(u(s), w(s))$ on $\mathbb{H}^2$ is parametrized by arc length, then

$$T(s) = \Big(\frac{u(s)u'(s) + w(s)w'(s)}{\sqrt{1 + u^2(s) + w^2(s)}}, u'(s), w'(s)\Big)$$

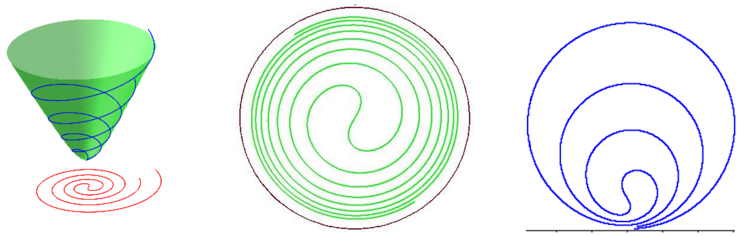
and the functions $u(s)$ and $w(s)$ satisfy the following system of ODEs:

$$\begin{cases} (u')^2 + (w')^2 + (u'w - uw')^2 = 1 + u^2 + w^2, \\ w''u' - u''w' + uw' - u'w = k(s)\sqrt{1 + u^2 + w^2}, \end{cases} \tag{4.1}$$

where $k(s)$ is the curvature of $X(s)$. The first equation follows from the fact that the curve is parametrized by arc length, and the second one from the expression of the curvature of X.

In Theorem 2.2, we saw that the curvature of a soliton solution to the CSF on $\mathbb{H}^2$ is determined by its tangent vector field and a non zero fixed vector v. We use (4.1) and the software `Maple` to plot examples of such solitons. In each example, we visualize the curve on the three models of the 2-dimensional hyperbolic space, namely the hyperboloid, the Poincaré disk and the upper half space.

In Figure 1 (a), the blue curve on the hyperboloid provides the visualization of a soliton solution $X(s)$ to the CSF on $\mathbb{H}^2$ whose curvature is given by $k(s) = \langle T(s), (-1, 0, 0)\rangle$ and $a = 1$. The red curve is the Euclidean orthogonal projection of $X(s)$ on the plane that contains the origin and it is orthogonal to the vector $(-1, 0, 0)$. In Figures 1 (b) and (c) we visualize the same soliton on the Poincaré disk and on the half space model, respectively.



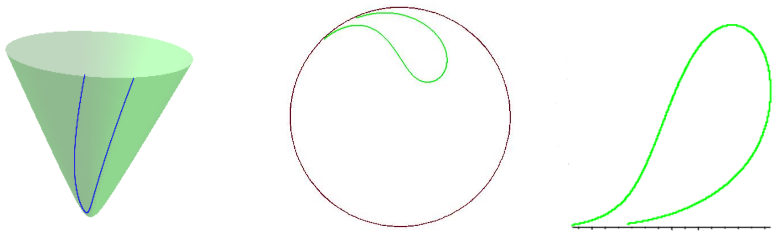
(a) **(b)** **(c)**

Figure 1. Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $v = (-1, 0, 0)$ and $a = 1$.

In Figure 2 (a), the blue curve provides the visualization of a soliton solution $X(s)$ to the CSF on $\mathbb{H}^2$ whose curvature is given by $k(s) = \langle T(s), (-1, 1, 0)\rangle$ and $a = 1$. In Figures 2 (b) and (c) we visualize the same soliton on the Poincaré disk and on the half space model, respectively.

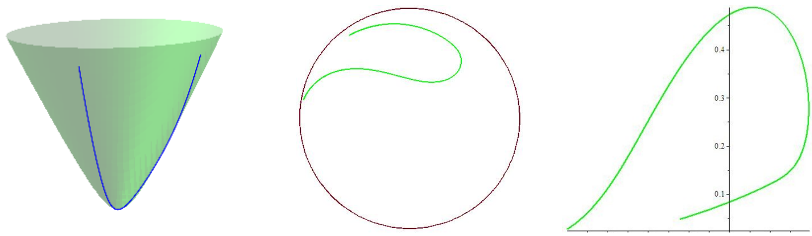
Finally, in Figure 3 (a), the blue curve provides the visualization of a soliton solution $X(s)$ to the CSF on $\mathbb{H}^2$ whose curvature is given by $k(s) = \langle T(s), (0, 0, 1)\rangle$ and $a = 1$. In Figures 3 (b) and (c) we visualize the same soliton on the Poincaré disk and on the half space model respectively. We point out that this soliton has non constant curvature and hence it is different from the one given in Proposition 3.12. In fact, in order to obtain Figure 3, we used initial condition $u(0) = w(0) = 0$ and $u'(0) = -w'(0) = -1/\sqrt{2}$ for the system (4.1), i.e., $T(0, 0) = (0, -1/\sqrt{2}, 1/\sqrt{2})$ and $\tau(0) = 1/\sqrt{2} \neq \pm 1$. Hence the curvature is not constant.

After obtaining our main results, we were informed that partial results on the solitons of the hyperbolic plane were obtained independently by E. Woolgar and R. Xie [16]. Namely, without considering the three types of soliton curves and assuming, without any



(a) (b) (c)

Figure 2. Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $(-1, 1, 0)$ and $a = 1$.



(a) (b) (c)

Figure 3. Soliton solution to the CSF on $\mathbb{H}^2$ with fixed vector $(0, 0, 1)$ and $a = 1$.

proof, that the solution curves are defined on the whole real line, they showed that the curves are embedded and the curvature tends to a constant at each end.

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