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Closed G_2-eigenforms and exact G_2-structures

Marco Freibert and Simon Salamon

Abstract. A study is made of left-invariant G_2-structures with an exact 3-form on a Lie group G whose Lie algebra $\mathfrak{g}$ admits a codimension-one nilpotent ideal $\mathfrak{h}$. It is shown that such a Lie group G cannot admit a left-invariant closed G_2-eigenform for the Laplacian and that any compact solvmanifold $\Gamma\backslash G$ arising from G does not admit an (invariant) exact G_2-structure. We also classify the seven-dimensional Lie algebras $\mathfrak{g}$ with codimension-one ideal equal to the complex Heisenberg Lie algebra which admit exact G_2-structures with or without special torsion. To achieve these goals, we first determine the six-dimensional nilpotent Lie algebras $\mathfrak{h}$ admitting an exact $\mathrm{SL}(3,\mathbb{C})$-structure ρ or a half-flat $\mathrm{SU}(3)$-structure (ω,ρ) with exact ρ, respectively.

1. Introduction

The group G_2 is one of the exceptional cases in Berger's celebrated list [4] of restricted holonomy groups of non-locally symmetric irreducible Riemannian manifolds and only occurs in dimension seven. For over 30 years, it was unknown whether such manifolds exist at all until Bryant found local examples [6], Bryant and the second author found complete ones [8], and Joyce [25] constructed compact manifolds with G_2 holonomy.

The construction of these examples relies on the fact that the metric is encoded in a certain type of three-form, which we shall refer to as a G_2-structure. More exactly, a G_2-*structure* on a seven-dimensional manifold M is a three-form $\varphi \in \Omega^3 M$ on M with pointwise stabilizer conjugate to $G_2 \subseteq \mathrm{SO}(7) \subseteq \mathrm{GL}(7,\mathbb{R})$. The form φ induces a Riemannian metric g_φ, an orientation and a Hodge star operator $\star_\varphi$ on M. The holonomy group of g_φ is contained in G_2 if the structure is torsion-free, meaning that φ is parallel for the Levi-Civita connection, which is the case if and only if φ is closed and coclosed [15].

The G_2-structures that are closed but *not* coclosed constitute a basic intrinsic torsion class in the Fernández–Gray classification, and play a natural role in the construction of compact manifolds with holonomy equal to G_2. Joyce's examples were found by first constructing closed G_2-structures on smooth manifolds with sufficiently small intrinsic torsion and then proving analytically that such closed G_2-structures may be deformed to torsion-free ones.

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Closed G_2-structures are the initial values for the *Laplacian flow* $\dot{\varphi}_t = \Delta_{\varphi_t}\varphi_t$ for one-parameter families of closed G_2-structures $(\varphi_t)_{t\in I}$ introduced by Bryant in [7]. The critical points of this flow are precisely the torsion-free G_2-structures [29], and the hope is to use the Laplacian flow to deform a closed G_2-structure (without any smallness assumption on the intrinsic torsion) to a torsion-free one for $t \to \infty$.

Short-time existence and uniqueness of the Laplacian flow were established in [9], and other foundational properties were proven in a series of papers by Lotay and Wei [29–31]. However, a lot is still unknown about the long-time behaviour of the flow, and it is important to characterise finite-time singularities. One expects that, like for the Ricci flow, these singularities are modeled by self-similar solutions of the Laplacian flow. The initial values φ_0 of these self-similar solutions are called *Laplacian solitons*, and a special class of them is given by *closed* G_2*-eigenforms* characterised by

$$\Delta_{\varphi_0}\varphi_0 = \mu\varphi_0$$

for some $\mu \in \mathbb{R} \setminus \{0\}$. Although this equation looks quite easy, no examples of these structures are known. Moreover, compact manifolds cannot admit a closed G_2-eigenform [29].

Closed G_2-eigenforms are also of interest from another point of view: they constitute a special class of the so-called λ*-quadratic closed* G_2*-structures*, $\lambda \in \mathbb{R}$, namely those with $\lambda = 0$. In general, quadratic closed G_2-structures are exactly the closed G_2-structures for which the exterior derivative $d\tau$ of the associated torsion two-form τ depends quadratically on τ. These structures have been studied by Ball [2, 3] and include many other interesting closed G_2-structures. For example, the case $\lambda = 1/6$ corresponds to the so-called *extremally Ricci-pinched (ERP)* closed G_2-structures, and the case $\lambda = 1/2$ is equivalent to the induced metric being Einstein.

By Lauret's work [26], homogeneous λ-quadratically closed G_2-structures on homogeneous manifolds can only exist for $\lambda \in \{0, 1/6, 1/2\}$. The homogeneous ERP closed G_2-structures were classified in [3] using the classification of left-invariant such structures on Lie groups in [27]. Moreover, [13] shows that no solvable Lie group can admit a left-invariant closed Einstein G_2-structure. Since the *strong Alekseevsky conjecture* is true in dimension seven (i.e., any simply-connected homogeneous 7-Einstein manifold of negative scalar curvature is isometric to a left-invariant metric on a simply-connected solvable Lie group) [1], there are no closed homogeneous Einstein G_2-structures. So the homogeneous case is settled for $\lambda \in \{1/6, 1/2\}$, leaving open the case $\lambda = 0$. Concerning the latter case, almost nothing was before this article, although [33] had shown that there are no closed G_2-eigenforms in a very specific family of closed G_2-structures on very special types of solvable Lie algebras, including a few almost nilpotent ones.

We shall fill this gap as follows. Let G be a 7-dimensional Lie group with Lie algebra $\mathfrak{g}$. We prove that G cannot admit a left-invariant closed G_2-eigenform if $\mathfrak{g}$ is *almost nilpotent*, i.e., if it admits a codimension-one nilpotent ideal. We are led to focus on ideals of two types, $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$, with the former of step 4, and the latter of step 2 and isomorphic to the real Lie algebra underlying the complex Heisenberg group. It is striking that our non-existence proof is at the limit of, but just within, the realm of computations that can be checked by hand. This fact has enabled us to complement our conclusions with more positive ones relating to $\mathfrak{n}_{28}$, mentioned below.

We are naturally led to the class of almost nilpotent Lie algebras by the following facts.

Lauret and Nicolini [28] showed that any Lie algebra $\mathfrak{g}$ that possesses a closed G_2-structure has a codimension-one unimodular ideal $\mathfrak{h}$. Hence, it is quite natural to start with those for which $\mathfrak{h}$ is nilpotent.

Motivation is also provided by the results of Podestà and Raffero [36] on closed G_2-structures on seven-manifolds with a transtive reductive group of symmetries.

Moreover, a closed G_2-eigenform φ is always (cohomologically) exact, and the existence problem of exact G_2-structures on a restricted class of almost nilpotent Lie algebras has been studied in [14]: there are no exact G_2-structures on strongly unimodular Lie algebras $\mathfrak{g}$ with $b_2(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$. The latter implies that $\mathfrak{g}$ is almost nilpotent [32], whereas 'strongly unimodular' is a technical condition, necessary for the existence of a cocompact lattice in the associated simply-connected Lie group G.

We also answer negatively the existence problem for exact G_2-structures on strongly unimodular almost nilpotent Lie algebras, and so on compact almost nilpotent (completely solvable) solvmanifolds, thereby generalising the result of [14][1]. It is not known if there exists *any* compact manifold with an exact G_2-structure, though it *is* known that (in contrast to other situations) nilmanifolds cannot serve as examples.

We do succeed in classifying all almost nilpotent Lie algebras admitting an exact G_2-structure for which the codimension-one nilpotent ideal is isomorphic to $\mathfrak{n}_{28}$. For such almost nilpotent Lie algebras, we also classify those that admit exact G_2-structures with special torsion of positive or negative type, a notion introduced by Ball in [3].

To prove our results, we split our almost nilpotent Lie algebra $\mathfrak{g}$ as a vector space into $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}e_7$, with $\mathfrak{h}$ being the codimension-one nilpotent ideal and $e_7 \in \mathfrak{h}^\perp$ of norm one. Then the equations determining a closed G_2-eigenform or an exact G_2-structure can be encoded into conditions on the induced SU(3)-structure (ω, ρ) on $\mathfrak{h}$. In particular, for an exact G_2-structure, the $SL(3, \mathbb{C})$-structure ρ has to be exact and for a closed G_2-eigenform, (ω, ρ) has to be half-flat with ρ being the exterior derivative of a primitive $(1, 1)$-form ν. The extra equation $\nu \wedge \omega^2 = 0$ turns out to be of crucial importance in enabling us to rule out solutions to the eigenform equations.

We show that exactly five out of 34 six-dimensional nilpotent Lie algebras admit an exact $SL(3, \mathbb{C})$-structure and that exactly two of them admit a half-flat SU(3)-structure (ω, ρ) with ρ exact, namely $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$. Both results have independent interest because special kinds of closed and exact $SL(3, \mathbb{C})$-structures on six-dimensional nilpotent Lie algebras have been studied in [19], and the six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure (ω, ρ) with $d\omega = \rho$ were determined in [18]. Moreover, the result on exact $SL(3, \mathbb{C})$-structures implies that if an almost nilpotent Lie algebra $\mathfrak{g}$ admits an exact G_2-structure, then the codimension-one nilpotent ideal $\mathfrak{h}$ has to be one of the five Lie algebras. We provide examples of exact G_2-structures on almost nilpotent Lie algebras with codimension-one nilpotent ideal $\mathfrak{h}$ for all possible nilpotent Lie algebras $\mathfrak{h}$ except when $\mathfrak{h}$ equals the nilpotent Lie algebra called $\mathfrak{n}_4$.

This leaves open the question to be studied in future work: is there an almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_4$ that admits an exact G_2-structure?

[1] After the submission of this paper, A. Fino, L. Martín-Merchán and F. Salvatore generalised our result further to any compact quotient $\Gamma\backslash G$ of a Lie group G by a cocompact lattice Γ, [17].

The paper is organised as follows.

In Section 2, we summarise basic facts about $\mathrm{SL}(3,\mathbb{C})$-, $\mathrm{SU}(3)$- and G_2-structures that are relevant to our investigation. In Section 3, we show how one can reduce the existence problem of a closed G_2-eigenform or an exact G_2-structure on a seven-dimensional Lie algebra $\mathfrak{g}$ to the existence problem of $\mathrm{SL}(3,\mathbb{C})$- or $\mathrm{SU}(3)$-structures satisfying certain equations on a six-dimensional ideal $\mathfrak{h}$ in $\mathfrak{g}$. Next, in Section 4, we prove our results on exact $\mathrm{SL}(3,\mathbb{C})$-structures and on half-flat $\mathrm{SU}(3)$-structures (ω,ρ) with exact ρ. We use these results to prove in Section 5 that no strongly unimodular almost nilpotent Lie algebra, and so also no compact almost nilpotent (completely solvable) solvmanifold, can admit an exact G_2-structure. Finally, in Section 6, we carry out a detailed analysis of the respective cases $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ in order to show that no almost nilpotent Lie algebra can admit a closed G_2-eigenform. Moreover, we prove the mentioned classification results of almost nilpotent Lie algebras with a codimension ideal isomorphic to $\mathfrak{n}_{28}$ admitting exact G_2-structures.

2. Preliminaries

2.1. G-structures in six and seven-dimensions

In this subsection, we define three different types of G-structures in six and seven dimensions, and recall some of their basic properties. Proofs of the relevant facts and more information may be found, for example, in [6, 7, 23].

In all cases, the G-structure is defined by one or two differential forms which are pointwise isomorphic to one or two 'model' forms on $\mathbb{R}^n$, $n=6$ or $n=7$, whose $\mathrm{GL}(n,\mathbb{R})$-stabiliser is G. Here, *pointwise isomorphic* means that for each $p\in M$ there is a vector space isomorphism $u\colon T_pM\to\mathbb{R}^n$ that identifies the differential forms at the point $p\in M$ with the model forms on $\mathbb{R}^n$.

Definition 2.1. (1) An $\mathrm{SL}(3,\mathbb{C})$-*structure* on an oriented six-dimensional manifold is a three-form $\rho\in\Omega^3M$ which is pointwise isomorphic to

$$\rho_0 := e^{135}-e^{146}-e^{236}-e^{245}\in\Lambda^3(\mathbb{R}^6)^*.$$

(2) An $\mathrm{SU}(3)$-*structure* on a six-dimensional manifold is a pair (ω,ρ) of a two-form $\omega\in\Omega^2M$ and a three-form $\rho\in\Omega^3M$ which is pointwise isomorphic to (ω_0,ρ_0) with

$$\omega_0 := e^{12}+e^{34}+e^{56}\in\Lambda^2(\mathbb{R}^6)^*.$$

(3) A G_2-*structure* on a seven-dimensional manifold M is a three-form $\varphi\in\Omega^3M$ which is pointwise isomorphic to

$$\varphi_0 := \omega_0\wedge e^7+\rho_0\in\Lambda^3(\mathbb{R}^7)^*.$$

In all cases, if $u\colon T_pM\to\mathbb{R}^n$ is one of the pointwise isomorphisms, then the basis $(u^{-1}(e_1),\dots,u^{-1}(e_n))$ of T_pM is called an *adapted* basis for the G-structure in question. Sometimes, we will also call the dual basis of $(u^{-1}(e_1),\dots,u^{-1}(e_n))$ an *adapted* basis for the G-structure in question.

Since $\mathrm{SL}(3,\mathbb{C}) \subseteq \mathrm{GL}(3,\mathbb{C})$ and $\mathrm{GL}(3,\mathbb{C})$-structures are almost complex structures, an $\mathrm{SL}(3,\mathbb{C})$-structure ρ has to induce an almost complex structure J_ρ. Explicitly, J_ρ is obtained as follows.

Definition 2.2. Let ρ be an $\mathrm{SL}(3,\mathbb{C})$-structure on an oriented six-dimensional manifold M. Then ρ induces an almost complex structure $J = J_\rho$ on M defined in $p \in M$ to be the unique endomorphism J_p of T_pM satisfying

$$J_p f_{2i-1} = -f_{2i} \quad \text{and} \quad J_p f_{2i} = f_{2i-1}$$

for one, and so for any, adapted oriented basis $(f_1, \ldots, f_6)$ of T_pM.

Moreover, set $\hat{\rho} := J^*\rho \in \Omega^3 M$. Then

$$\hat{\rho}_p = f^{246} - f^{235} - f^{145} - f^{136},$$

where $(f^1, \ldots, f^6)$ is the dual basis of the adapted basis $(f_1, \ldots, f_6)$ at $p \in M$. Furthermore, $\Psi := \rho + i\hat{\rho} \in \Omega^3(M,\mathbb{C})$ is a non-zero $(3,0)$-form.

We give now an equivalent characterisation of an $\mathrm{SL}(3,\mathbb{C})$-structure, and for this we have to introduce a quartic invariant λ of a three-form on a vector space.

Definition 2.3. Let V be a six-dimensional vector space. Let $\kappa\colon \Lambda^5 V^* \to V \otimes \Lambda^6 V^*$ be the natural $\mathrm{GL}(V)$-equivariant isomorphism, i.e., $\kappa^{-1}(v \otimes \nu) = v \lrcorner \nu$. Next, let $\rho \in \Lambda^3 V^*$ and define $K_\rho \in \mathrm{End}(V) \otimes \Lambda^6 V^*$ by

$$K_\rho(v) = \kappa((v \lrcorner \rho) \wedge \rho),$$

and finally set

$$\lambda(\rho) := \tfrac{1}{6}\,\mathrm{tr}(K_\rho^2) \in (\Lambda^6 V^*)^{\otimes 2}.$$

It makes sense to say that $\lambda(\rho) > 0$, meaning that $\lambda(\rho)$ is the square of some element in $\Lambda^6 V^*$. Thus, one may also speak of $\lambda(\rho) < 0$. Using this notation, Hitchin [23] gives the following characterisation of an $\mathrm{SL}(3,\mathbb{C})$-structure.

Lemma 2.4. *Let $\rho \in \Omega^3 M$ be a three-form on an oriented six-dimensional manifold M. Then ρ is an $\mathrm{SL}(3,\mathbb{C})$-structure if and only if $\lambda(\rho_p) < 0$ for all $p \in M$.*

We will also need the following technical statements in the sequel.

Lemma 2.5. *Let $\rho \in \Omega^3 M$ be an $\mathrm{SL}(3,\mathbb{C})$-structure on a seven-dimensional manifold. Let $p \in M$ and $v, w \in T_pM$, and set $J := J_\rho$. Then:*

(a) *$\rho_p(v, w, \cdot) = 0$ if and only if v and w are $\mathbb{C}$-linearly dependent.*

(b) *If $v \neq 0$, then the two-forms $\omega_1 := \rho_p(v, \cdot, \cdot) \in \Lambda^2 T_p^* M$ and $\omega_2 := \rho_p(J_p v, \cdot, \cdot) \in \Lambda^2 T_p^* M$ satisfy*

$$\ker(\omega_i) = \mathrm{span}(v, J_p v), \quad \omega_i \wedge \omega_j = \delta_{ij}\omega_1^2 \quad \textit{for all } i, j = 1, 2.$$

Proof. (a) First, let v and w be $\mathbb{C}$-linearly dependent. Without loss of generality, we may assume that $w = cJv$ for some $c \in \mathbb{R}$. Since Ψ is a $(3,0)$-form, we do have $\Psi(v, Jv, u) = -\Psi(Jv, J^2 v, u) = \Psi(Jv, v, u) = -\Psi(v, Jv, u)$, i.e., $\Psi(v, Jv, u) = 0$ for any $u \in T_pM$. As $\rho = \mathrm{Re}(\Psi)$, this implies $\rho(v, w, \cdot) = c\rho(v, Jv, \cdot) = 0$.

Next, assume that v and w are $\mathbb{C}$-linearly independent. Then we may extend v and w to a $\mathbb{C}$-basis of (T_pM, J) by an element $u \in T_pM$. Then $v_{\mathbb{C}} := v - iJv$, $w_{\mathbb{C}} := w - iJw$ and $u_{\mathbb{C}} := u - iJu$ form a basis of $(T_pM)^{1,0}$ and so $0 \neq \Psi(v_{\mathbb{C}}, w_{\mathbb{C}}, u_{\mathbb{C}})$, since Ψ is a $(3,0)$-form. This property of Ψ also shows

$$\Psi(z - iJz, \cdot, \cdot) = \Psi(z, \cdot, \cdot) - i\Psi(Jz, \cdot, \cdot) = \Psi(z, \cdot, \cdot) - i^2\Psi(z, \cdot, \cdot) = 2\Psi(z, \cdot, \cdot)$$

for any $z \in T_pM$ and so $\Psi(v_{\mathbb{C}}, w_{\mathbb{C}}, u_{\mathbb{C}}) = 8\Psi(v, w, u)$. Consequently, $\Psi(v, w, u) \neq 0$. Thus, $\rho(v, w, u) \neq 0$ or $\hat{\rho}(v, w, u) \neq 0$. In the latter case, we do have

$$\begin{aligned}\rho(v, w, Ju) + i\hat{\rho}(v, w, Ju) &= \Psi(v, w, Ju) = i\Psi(v, w, u) = i^2\hat{\rho}(v, w, u)\\ &= -\hat{\rho}(v, w, u) \neq 0,\end{aligned}$$

i.e., $\rho(v, w, Ju) \neq 0$.

(b) Since all equations are invariant under non-zero rescalings, we may assume that v has norm one. Now SU(3) acts transitively on the six-sphere S^6. Consequently, there is an adapted basis $(f_1, \dots, f_6)$ of ρ at $p \in M$ with $f_1 = v$, and so $f_2 = J_pf_1 = J_pv$. But so

$$\begin{aligned}\omega_1 &= v \lrcorner \rho_p = f_1 \lrcorner (f^{135} - f^{146} - f^{236} - f^{245}) = f^{35} - f^{46},\\ \omega_2 &= (J_pv) \lrcorner \rho_p = f_2 \lrcorner (f^{135} - f^{146} - f^{236} - f^{245}) = -f^{36} - f^{45}.\end{aligned}$$

Then a straightforward computation shows that ω_1 and ω_2 have the desired properties. ■

Similarly, one knows that $\mathrm{SU}(3) \subseteq \mathrm{SO}(6)$, and therefore an SU(3)-structure induces a Riemannian metric g as follows:

Definition 2.6. Let (ω, ρ) be an SU(3)-structure on a six-dimensional manifold M. Define $g = g_{(\omega,\rho)}$ to be the Riemannian metric on M for which any adapted basis $(f_1, \dots, f_6)$ at any point $p \in M$ is orthonormal. Now ω^3 is a volume form on M and so ω induces an orientation on M. We get an induced almost complex structure J_ρ, which relates g to ω by the equation

$$g = \omega(J_\rho \cdot, \cdot).$$

Hence, (g, J, ω) is an almost Hermitian structure on M. Moreover, $\Psi = \rho + i\hat{\rho}$ is of constant length.

Next, we turn to the special class of SU(3)-structures defined in [10].

Definition 2.7. An SU(3)-structure (ω, ρ) on a six-dimensional manifold M is called *half-flat* if $d\omega^2 = 0$ and $d\rho = 0$.

Finally, we turn to G_2-structures and use that $\mathrm{G}_2 \subseteq \mathrm{SO}(7) \subseteq \mathrm{GL}^+(7, \mathbb{R})$.

Definition 2.8. Let $\varphi \in \Omega^3 M$ be a G_2-structure on a seven-dimensional manifold. Define $g = g_\varphi$ to be the Riemannian metric on M for which any adapted basis $(f_1, \dots, f_7)$ at any point $p \in M$ is orthonormal. Similarly, define an orientation on M by requiring that any adapted basis $(f_1, \dots, f_7)$ at any point $p \in M$ is oriented. We get an induced Hodge star operator $\star_\varphi$ and

$$(\star_\varphi \varphi)_p = f^{1234} + f^{1256} + f^{3456} + f^{1367} + f^{1457} + f^{2357} - f^{2467}$$

for any $p \in M$ and any adapted basis $(f_1, \ldots, f_7)$ at p with dual basis $(f^1, \ldots, f^7)$. Moreover, $\star_\varphi \varphi$ is pointwise isomorphic to

$$\star_{\varphi_0}\varphi_0 = \tfrac{1}{2}\omega_0^2 + e^7 \wedge \hat{\rho}_0 \in \Lambda^4(\mathbb{R}^7)^*.$$

A G_2*-structure on a seven-dimensional vector space* V is simply a constant G_2-structure on the manifold V, or said more directly, a three-form $\varphi \in \Lambda^3 V^*$ for which there exists an vector space isomorphism $u\colon V \to \mathbb{R}^7$ with $u^*\varphi_0 = \varphi$. So G_2-structures $\varphi \in \Omega^3 M$ on seven-dimensional manifolds M are those for which φ_p is a G_2-structure on the seven-dimensional vector space T_pM for all $p \in M$.

Similarly, we define an SU(3)*-structure on a six-dimensional vector space* V and get the following result.

Lemma 2.9. *Let $\varphi \in \Lambda^3 V^*$ be a G_2-structure on a seven-dimensional vector space V. Moreover, let $v \in V$ be of norm one with respect to g_φ and let $W := v^{\perp_{g_\varphi}}$. Then there is a unique* SU(3)*-structure $(\omega, \rho) \in \Lambda^2 W^* \times \Lambda^3 W^*$ on W such that*

$$\varphi = \omega \wedge \alpha + \rho \quad \textit{and} \quad \star_\varphi \varphi = \tfrac{1}{2}\omega^2 + \alpha \wedge \hat{\rho},$$

with $\alpha \in V^*$ uniquely defined by $\alpha(W) = 0$ and $\alpha(v) = 1$, and W^* identified with the annihilator of v.

Proof. The group G_2 acts transitively on the unit sphere in $\mathbb{R}^7$. Hence, we may assume that φ has an adapted basis $(f_1, \ldots, f_7)$ with $v = f_7$ and so $\alpha = f^7$. Since $(f_1, \ldots, f_7)$ is an orthonormal basis of V, we have $W = \operatorname{span}(f_1, \ldots, f_6)$ and the statements follow from the relations $\varphi_0 = \omega_0 \wedge e^7 + \rho_0$ and $\star_{\varphi_0}\varphi_0 = \frac{1}{2}\omega_0^2 + e^7 \wedge \hat{\rho}_0$ between the model forms of a G_2- and an SU(3)-structure. ∎

Now note that G_2 also acts transitively on the Grassmanian of oriented 2-planes in $\mathbb{R}^7$, which appears again on the next page as the quadric Q_5. Since $\varphi_0(e_1, e_2, \cdot) = e^7 \neq 0$, we have the following.

Lemma 2.10. *Let $\varphi \in \Lambda^3 V^*$ be a G_2-structure on a seven-dimensional vector space V, and let $v, w \in V$ be linearly independent. Then $\varphi(v, w, \cdot) \neq 0$.*

Next, we consider some representation theory of G_2. Consider the G_2-representation $\Lambda^k(\mathbb{R}^7)^*$ for $k \in \{0, \ldots, 7\}$. We decompose this representation into its irreducible components, which are by now well known. Since the Hodge star operator $\star_{\varphi_0}$ is an isomorphism of G_2-representations between $\Lambda^k(\mathbb{R}^7)^*$ and $\Lambda^{7-k}(\mathbb{R}^7)^*$, it suffices to do this for $k \in \{0, \ldots, 3\}$. Obviously, $\Lambda^0(\mathbb{R}^7)^*$ is trivial, and $\Lambda^1(\mathbb{R}^7)^* \cong \mathbb{R}^7$ is also irreducible. For $k = 2, 3$, we have

$$\Lambda^2(\mathbb{R}^7)^* = \Lambda^2_7 \oplus \Lambda^2_{14} \quad \text{and} \quad \Lambda^3(\mathbb{R}^7)^* = \mathbb{R}\varphi_0 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

with

$$\begin{aligned}
\Lambda^2_7 &= \{v \lrcorner \varphi_0 \mid v \in \mathbb{R}^7\}, & \Lambda^2_{14} &= \{\nu \in \Lambda^2(\mathbb{R}^7)^* \mid \nu \wedge \varphi_0 = -\star_{\varphi_0} \nu\},\\
\Lambda^3_7 &= \{v \lrcorner \star_{\varphi_0} \varphi_0 \mid v \in \mathbb{R}^7\}, & \Lambda^3_{27} &= \{\gamma \in \Lambda^3(\mathbb{R}^7)^* \mid \gamma \wedge \varphi_0 = 0,\ \gamma \wedge \star_{\varphi_0}\varphi_0 = 0\}.
\end{aligned}$$

The subscript denotes the dimension of the irreducible representation. For example, Λ^2_{14} is isomorphic to the adjoint representation $\mathfrak{g}_2$, as can be seen by using the metric g_{φ_0} to identify a two-form $\nu \in \Lambda^2_{14}$ with an endomorphism of $\mathbb{R}^7$. The decompositions of $\Lambda^k(\mathbb{R}^7)^*$ into irreducible G_2-representations give rise to corresponding decompositions of $\Omega^k(M)$. In particular, $\Omega^2_{14}(M) = \{\nu \in \Omega^2(M) | \nu \wedge \varphi = \star_\varphi \nu\}$ is a $C^\infty(M)$-submodule of $\Omega^2(M)$.

2.2. Closed G_2-structures

In this subsection, we consider the following situation.

Definition 2.11. A G_2-structure $\varphi \in \Omega^3 M$ on a seven-dimensional manifold M is called *closed* if $d\varphi = 0$. If φ is a closed G_2-structure, then

$$d \star_\varphi \varphi = \tau \wedge \varphi$$

for a unique two-form $\tau \in \Omega^2_{14}(M)$. The two-form τ is called the *torsion two-form* of φ and encodes the intrinsic torsion of φ.

The notion of a closed G_2-structure with *special torsion of positive or negative type* was introduced by Ball in [3] in an attempt to study so-called *quadratic closed* G_2*-structures*, as explained in the next subsection.

To motivate the definition of closed G_2-structure with special torsion, let $\varphi \in \Omega^3 M$ be a closed G_2-structure with associated torsion two-form τ. Then τ pointwise lies in the adjoint representation $\mathfrak{g}_2$. The adjoint action of G_2 on $\mathfrak{g}_2$ has three different types of orbits, distinguished by the conjugacy classes of the G_2-stabiliser at some point in the orbit:

There is the ‘generic’ case, where the stabiliser is a maximal torus T^2 inside of G_2.

Then there are two exceptional orbits, where the stabiliser is some copy of U(2), but the two copies $\mathrm{U}(2)^+$ and $\mathrm{U}(2)^-$ are not conjugate to each other. The first orbit $\mathrm{G}_2/\mathrm{U}(2)^+$ is the twistor space of the Wolf space $\mathrm{G}_2/\mathrm{SO}(4)$, whereas $\mathrm{G}_2/\mathrm{U}(2)^-$ can be regarded as a twistor space of $S^6 \cong \mathrm{G}_2/\mathrm{SU}(3)$ and as such is biholomorphic to the complex quadric Q_5, see [5].

We say that φ has *special torsion of positive type* or *negative type*, respectively, if the pointwise stabiliser of τ is in each point conjugate to $\mathrm{U}(2)^+$ or $\mathrm{U}(2)^-$, respectively. As shown in [3], these conditions can be characterised by properties of τ^3:

Definition 2.12. Let $\varphi \in \Omega^3 M$ be a closed G_2-structure on a seven-dimensional manifold M with associated torsion two-form $\tau \in \Omega^2 M$. Then φ has *special torsion of positive type* if $\tau^3 = 0$, and φ has *special torsion of negative type* if $|\tau^3|^2_\varphi = \frac{2}{3}|\tau|^6_\varphi$.

2.3. Closed G_2-eigenforms for the Laplacian

In this subsection, we discuss properties of closed G_2-eigenforms (from now on, we shall omit the words ‘for the Laplacian’), and their relation to other structures. We repeat the definition.

Definition 2.13. A G_2-structure φ on a seven-dimensional manifold M is called a *closed* G_2*-eigenform* if $d\varphi = 0$ and there exists some $\mu \in \mathbb{R} \setminus \{0\}$ such that

$$\Delta_\varphi \varphi = \mu \varphi.$$

Remark 2.14. (1) Any closed G_2-structure φ (on a connected manifold) with $\Delta_\varphi \varphi = f\varphi$ for some $f \in C^\infty(M)$, $f \neq 0$, is a closed G_2-eigenform. For, differentiation gives

$$df \wedge \varphi = d(f\varphi) = d\Delta_\varphi \varphi = d^2\delta_\varphi \varphi = 0,$$

and so $df = 0$ since wedging with φ injects $\Omega^1 M$ into $\Omega^4 M$. Hence, f is constant.

(2) A closed G_2-eigenform φ is an exact G_2-structure since

$$\varphi = \tfrac{1}{\mu}\Delta_\varphi \varphi = d\left(\tfrac{\delta_\varphi \varphi}{\mu}\right).$$

(3) A closed G_2-eigenform φ is an example of a *Laplacian soliton*, i.e., a soliton for the *Laplacian flow* (*of closed* G_2*-structures*) given by

$$\dot{\varphi}_t = \Delta_{\varphi_t}\varphi_t.$$

More generally, a *Laplacian soliton* φ is a closed G_2-structure satisfying

$$\Delta_\varphi \varphi = \mu\varphi + \mathcal{L}_X \varphi$$

for some $X \in \mathfrak{X}(M)$, $\mu \in \mathbb{R}$.

(4) Lotay and Wei showed in [29] that a compact seven-dimensional manifold cannot support any Laplacian soliton φ with $X = 0$ unless φ is torsion-free. In particular, there do not exist any closed G_2-eigenforms on compact manifolds.

(5) In [29] it is also shown that for a closed G_2-eigenform $\varphi \in \Omega^3 M$ with $\Delta_\varphi \varphi = \mu\varphi$ we must have $\mu > 0$.

Let φ be a closed G_2-eigenform on a seven-dimensional manifold M. By the last remark, we then have $\Delta_\varphi \varphi = \mu\varphi$ for some $\mu > 0$. Hence, by scaling φ by an appropriate non-zero factor, we may and will assume for the rest of the article (unless stated otherwise) that $\mu = 1$, i.e., that

$$\Delta_\varphi \varphi = \varphi \tag{2.1}$$

In the last subsection, we introduced the torsion 2-form $\tau \in \Omega^2_{14}(M)$ uniquely defined by

$$d \star_\varphi \varphi = \tau \wedge \varphi.$$

Since $\tau \in \Omega^2_{14}(M)$, we do have $\tau \wedge \varphi = - \star_\varphi \tau$ and so

$$\tau = \star_\varphi \star_\varphi \tau = - \star_\varphi (\tau \wedge \varphi) = - \star_\varphi d \star_\varphi \varphi = \delta_\varphi \varphi,$$

which yields

$$\Delta_\varphi \varphi = d\delta_\varphi \varphi = d\tau \tag{2.2}$$

Remark 2.15. Let φ be a closed G_2-eigenform. Then $d\tau = \Delta_\varphi \varphi = \mu\varphi$ for some $\mu > 0$, which we do not assume to be equal to 1 in this remark. Since wedging with $\star_\varphi \varphi$ is pointwise G_2-equivariant from $\Omega^2(M)$ to $\Omega^6 M$, and $\Omega^6 M$ is pointwise isomorphic to the G_2-representation $\mathbb{R}^7$, we must have $\tau \wedge \star_\varphi \varphi = 0$. Thus,

$$7\,\mathrm{vol}_\varphi = \varphi \wedge \star_\varphi \varphi = \tfrac{1}{\mu}\, d\tau \wedge \star_\varphi \varphi = -\tfrac{1}{\mu}\, \tau \wedge d \star_\varphi \varphi = -\tfrac{1}{\mu}\, \tau^2 \wedge \varphi$$
$$= \tfrac{1}{\mu}\, \tau \wedge \star_\varphi \tau = \tfrac{|\tau|_\varphi^2}{\mu}\, \mathrm{vol}_\varphi,$$

i.e., $\mu = \frac{1}{7}|\tau|_\varphi^2$, and so

$$d\tau = \tfrac{1}{7}\,|\tau|_\varphi^2\,\varphi.$$

In particular, $|\tau|_\varphi$ is constant. Moreover, closed G_2-eigenforms are special kinds of so-called *λ-quadratic closed* G_2*-structures*, which are closed G_2-structures $\varphi \in \Omega^3 M$ fulfilling

$$d\tau = \tfrac{1}{7}\,|\tau|_\varphi^2 \varphi + \lambda \left(\tfrac{1}{7}\,|\tau|_\varphi^2 \varphi + \star_\varphi(\tau \wedge \tau)\right)$$

for $\lambda \in \mathbb{R}$, namely those for $\lambda = 0$.

Note that $\lambda(\frac{1}{7}\,|\tau|_\varphi \varphi + \star_\varphi(\tau \wedge \tau))$ lies in $\Omega^3_{27}(M)$, so the above decomposition can be seen as one of $d\tau \in \Omega^3 M$ into the three components $\Omega^3_1(M) := C^\infty(M) \cdot \varphi$, $\Omega^3_7(M)$ and $\Omega^3_{27}(M)$ of $\Omega^3 M$, with the $\Omega^3_7(M)$-component being zero.

More generally, for any closed G_2-structure φ, the $\Omega^3_1(M)$-part of $d\tau$ equals $\frac{1}{7}|\tau|_\varphi^2\varphi$ and the $\Omega^3_7(M)$-part of $d\tau$ vanishes, i.e., we always have $d\tau = \frac{1}{7}\,|\tau|_\varphi^2\varphi + \gamma$ for some $\gamma \in \Omega^3_{27}(M)$.

One can show that λ-quadratic closed G_2-structure are exactly those closed G_2-structures for which $\gamma \in \Omega^3_{27}(M)$, and so the entire three-form $d\tau$ depends quadratically on τ, explaining the naming of these structures.

We restrict now to left-invariant G_2-structures on seven-dimensional Lie groups G. These will from now on be identified with the corresponding structures on the associated seven-dimensional Lie algebra $\mathfrak{g}$.

As stated already, it is well known that a seven-dimensional nilpotent Lie algebra cannot admit an exact G_2-structure, see e.g. [12], and so we have to look for exact G_2-structures on the more general class of solvable Lie algebras. We will give now a new proof of this fact, and in the process prove a slightly stronger result. Fist, we recall the following.

Definition 2.16. Let $\mathfrak{k}$ be a Lie algebra.

The *descending central series* $\mathfrak{k}^0, \mathfrak{k}^1, \ldots$ of $\mathfrak{k}$ is defined by $\mathfrak{k}^0 := \mathfrak{k}$, $\mathfrak{k}^1 := [\mathfrak{k}, \mathfrak{k}]$ and inductively by $\mathfrak{k}^k := [\mathfrak{k}, \mathfrak{k}^{k-1}]$ for all $k \in \mathbb{N}$.

The *ascending central series* $\mathfrak{k}_0, \mathfrak{k}_1, \ldots$ of $\mathfrak{k}$ is defined by $\mathfrak{k}_0 := \{0\}$, $\mathfrak{k}_1 := \mathfrak{z}(\mathfrak{k})$ and inductively by $\mathfrak{k}_k := \left\{X \in \mathfrak{k} | [X, \mathfrak{k}] \subseteq \mathfrak{k}^{k-1}\right\}$ for all $k \in \mathbb{N}$.

Note that $\mathfrak{k}$ is nilpotent if and only if $\mathfrak{k}^r = \{0\}$ for some $r \in \mathbb{N}$ or, equivalently, if and only if $\mathfrak{k}_s = \mathfrak{k}$ for some $s \in \mathbb{N}$ (and then $r = s$).

This allows us to prove the following.

Proposition 2.17. *Let $\mathfrak{k}$ be a seven-dimensional Lie algebra. If $\mathfrak{k}$ admits an exact* G_2*-structure, then* $\dim(\mathfrak{z}(\mathfrak{k})) \le 1$ *and* $\mathfrak{k}_2 = \mathfrak{z}(\mathfrak{k})$. *In particular, if $\mathfrak{k}$ is nilpotent, or, more generally, if $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, with $\mathfrak{b}$ nilpotent and* $\dim(\mathfrak{b}) \ge 2$*, then it cannot admit any exact* G_2*-structure.*

Proof. Assume that $\varphi \in \Lambda^2\mathfrak{k}^*$ is an exact G_2-structure on $\mathfrak{k}$, i.e., that $d\chi = \varphi$ for some $\chi \in \Lambda^2\mathfrak{k}^*$. Let $X, Y \in \mathfrak{z}(\mathfrak{k})$. Then

$$0 = d\chi(X, Y, Z) = \varphi(X, Y, Z)$$

for any $Z \in \mathfrak{k}$. Hence, by Lemma 2.10, the vectors X and Y have to be linearly dependent. Thus, $\dim(\mathfrak{z}(\mathfrak{k})) \leq 1$.

If $\dim(\mathfrak{z}(\mathfrak{k})) = 0$, then trivially $\mathfrak{k}_2 = \mathfrak{z}(\mathfrak{k})$. So let us assume that $\dim(\mathfrak{z}(\mathfrak{k})) = 1$ and take $X \in \mathfrak{z}(\mathfrak{k}) \setminus \{0\}$. If $\mathfrak{k}_2 \neq \mathfrak{z}(\mathfrak{k})$, then there exists $Y \in \mathfrak{k}_2$ linearly independent of X and we get $[Y, Z] \in \mathrm{span}(X)$ and so

$$0 = -\chi([Y, Z], X) = d\chi(X, Y, Z) = \varphi(X, Y, Z)$$

for any $Z \in \mathfrak{k}$, which contradicts Lemma 2.10. Thus, $\mathfrak{k}_2 = \mathfrak{z}(\mathfrak{k})$.

So $\mathfrak{k}$ certainly cannot be nilpotent nor can it be of the form $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{b}$ being nilpotent and $\dim(\mathfrak{b}) \geq 2$. ∎

3. Reduction to six dimensions

In this section, we reduce the existence problem of closed G_2-eigenforms or exact G_2-structures on a seven-dimensional Lie algebra $\mathfrak{g}$ to the existence of SU(3)-structures of a certain type on a codimension-one ideal $\mathfrak{h}$ satisfying specific equations. By Proposition 3.2 in [28], such an ideal $\mathfrak{h}$ always exists:

Proposition 3.1. *Let $\mathfrak{g}$ be a seven-dimensional Lie algebra admitting a closed G_2-structure $\varphi \in \Lambda^3\mathfrak{g}^*$. Then $\mathfrak{g}$ admits a unimodular codimension-one ideal $\mathfrak{h}$.*

To obtain the reduction from seven to six dimensions, we need to recall what a derivation of a Lie algebra $\mathfrak{h}$ is and how an endomorphism of $\mathfrak{h}$ acts on $\Lambda^*\mathfrak{h}^*$:

Definition 3.2. Let $\mathfrak{h}$ be a Lie algebra, let $f \in \mathrm{End}(\mathfrak{h})$ be an (vector space) endomorphism of $\mathfrak{h}$ and let $\alpha \in \Lambda^k\mathfrak{h}^*$ be a k-form on $\mathfrak{h}$. Then the k-form $f.\alpha \in \Lambda^k\mathfrak{h}^*$ is defined by

$$(f.\alpha)(X_1, \ldots, X_k) := -\left(\alpha(f(X_1), X_2, \ldots, X_k) + \cdots + \alpha(X_1, \ldots, X_{k-1}, f(X_k))\right).$$

A *derivation of* $\mathfrak{h}$ is a (vector space) endomorphism $f \in \mathrm{End}(\mathfrak{h})$ of $\mathfrak{h}$ such that $f([X, Y]) = [f(X), Y] + [X, f(Y)]$ for all $X, Y \in \mathfrak{h}$.

Remark 3.3. Let $\mathfrak{h}$ be a Lie algebra, $\alpha \in \Lambda^k\mathfrak{h}^*$, $\beta \in \Lambda^l\mathfrak{h}^*$ and $f, g \in \mathrm{End}(\mathfrak{h})$. Then:

- Due to the global minus sign in the definition of $f.\alpha$, we have $[f, g].\alpha = f.(g.\alpha) - g.(f.\alpha)$, i.e., $\mathrm{End}(\mathfrak{h}) \ni f \mapsto f. \in \mathrm{End}(\Lambda^k\mathfrak{h}^*)$ is a representation of the Lie algebra $\mathrm{End}(\mathfrak{h})$ on $\Lambda^k\mathfrak{h}^*$.
- Moreover, we have
$$f.(\alpha \wedge \beta) = f.\alpha \wedge \beta + \alpha \wedge f.\beta$$
and so $f.(\alpha \wedge \alpha) = 2\alpha \wedge f.\alpha$ if k is even.
- f is a derivation if and only if $f.d\gamma = d(f.\gamma)$ for all one-forms $\gamma \in \mathfrak{h}^*$ on $\mathfrak{h}$ and then the same formula holds for forms of arbitrary degree on $\mathfrak{h}$. Moreover, the vector

space $\mathrm{Der}(\mathfrak{h})$ of all derivations of $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathrm{End}(\mathfrak{h})$ of all (vector space) endomorphisms of $\mathfrak{h}$ and it is the Lie algebra of the Lie group $\mathrm{Aut}(\mathfrak{h})$ of all Lie algebra automorphisms of $\mathfrak{h}$, i.e., of

$$\mathrm{Aut}(\mathfrak{h}) := \{F \in \mathrm{GL}(\mathfrak{h}) | F([X,Y]) = [F(x), F(Y)] \text{ for all } X, Y \in \mathfrak{h}\} \subseteq \mathrm{GL}(\mathfrak{h}).$$

Let us begin with the reduction to six dimensions.

For this, let φ be a closed G_2-eigenform on a seven-dimensional Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a codimension-one ideal. Choose $e_7 \in \mathfrak{g}$ of norm one in the orthogonal complement $\mathfrak{h}^{\perp_{g_\varphi}}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Split now $\mathfrak{g} = \mathfrak{h} \oplus \mathrm{span}(e_7)$ and, similarly, $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathrm{span}(e^7)$, where e^7 is the unique element in the annihilator of $\mathfrak{h}$ with $e^7(e_7) = 1$ and $\mathfrak{h}^*$ is identified with the annihilator of e_7. Set $f := \mathrm{ad}(e_7)|_{\mathfrak{h}}$ and note that f is a derivation of $\mathfrak{h}$. Then

$$d\alpha = d_{\mathfrak{h}}\alpha + e^7 \wedge f.\alpha, \quad d(\alpha \wedge e^7) = d_{\mathfrak{h}}\alpha \wedge e^7, \quad de^7 = 0$$

for any $\alpha \in \Lambda^k \mathfrak{h}^*$, where $d_{\mathfrak{h}}$ is the differential of $\mathfrak{h}$. Next, decompose φ according to the splitting, i.e., write

$$\varphi = \omega \wedge e^7 + \rho \tag{3.1}$$

with $\omega \in \Lambda^2\mathfrak{h}^*$, $\rho \in \Lambda^3\mathfrak{h}^*$. Then (ω, ρ) is an SU(3)-structure on $\mathfrak{h}$ by Lemma 2.9 and one has

$$\star_\varphi \varphi = \tfrac{1}{2}\omega^2 + e^7 \wedge \hat{\rho}.$$

We do the same for the torsion-two form $\tau \in \Lambda^2_{14}\mathfrak{g}^*$ of φ, i.e., we write

$$\tau = \nu + \alpha \wedge e^7 \tag{3.2}$$

with $\nu \in \Lambda^2\mathfrak{h}^*$ and $\alpha \in \mathfrak{h}^*$. Now the G_2-representation $\Lambda^2_{14}\mathfrak{g}^*$ splits as SU(3)-representations into $\Lambda^2_{14}\mathfrak{g}^* = \mathfrak{h}^* \wedge e^7 \oplus [\Lambda^{1,1}_0\mathfrak{h}^*]$ as $\Lambda^2_{14}\mathfrak{g}^*$ is the adjoint representation of G_2 and $[\Lambda^{1,1}_0\mathfrak{h}^*]$ is the adjoint representation of SU(3). Thus, $\nu \in [\Lambda^{1,1}_0\mathfrak{h}^*]$ and so $\nu \wedge \rho = 0$. Consequently,

$$\begin{aligned}\tfrac{1}{2} d_{\mathfrak{h}}(\omega^2) + e^7 \wedge (\omega \wedge f.\omega - d_{\mathfrak{h}}\hat{\rho}) = d \star_\varphi \varphi = \tau \wedge \varphi &= (\nu + \alpha \wedge e^7) \wedge (\omega \wedge e^7 + \rho) \\ &= e^7 \wedge (\omega \wedge \nu - \alpha \wedge \rho),\end{aligned}$$

i.e.,

$$d_{\mathfrak{h}}(\omega^2) = 0, \quad \omega \wedge f.\omega - d_{\mathfrak{h}}\hat{\rho} = \omega \wedge \nu - \alpha \wedge \rho.$$

Moreover,

$$d_{\mathfrak{h}}\nu + e^7 \wedge (f.\nu - d_{\mathfrak{h}}\alpha) = d\tau = \varphi = \omega \wedge e^7 + \rho,$$

i.e.,

$$d_{\mathfrak{h}}\nu = \rho, \quad f.\nu - d_{\mathfrak{h}}\alpha = \omega,$$

Hence, $d_{\mathfrak{h}}\rho = 0$ and $d_{\mathfrak{h}}(\omega^2) = 0$ and so the SU(3)-structure (ω, ρ) on $\mathfrak{h}$ is half-flat with exact ρ. Moreover, we necessarily have $\alpha = 0$. For this, note that $\tau \in \Omega^2_{14}M = \{\beta \in \Omega^2 M | \star_\varphi \beta = -\beta \wedge \varphi\}$ and so

$$\tau = \star_\varphi^2 \tau = -\star_\varphi (\tau \wedge \varphi) = -\star_\varphi (e^7 \wedge (\omega \wedge \nu - \alpha \wedge \rho)) \in \Lambda^2\mathfrak{h}^*$$

since e^7 is perpendicular to $\mathfrak{h}^*$ by assumption. Consequently, $\tau = \nu \in \Lambda^3\mathfrak{h}^*$ and $\alpha = 0$.

Summarizing, we have arrived at:

Theorem 3.4. *Let $\mathfrak{g}$ be a seven-dimensional Lie algebra, let $\varphi \in \Lambda^3\mathfrak{g}^*$ be a G_2-structure on $\mathfrak{g}$, let $\mathfrak{h}$ be a codimension-one unimodular ideal in $\mathfrak{g}$, let $e_7 \in \mathfrak{h}^{\perp_{g_\varphi}}$ be of norm one, $e^7 \in \mathrm{Ann}(\mathfrak{h})$ with $e^7(e_7) = 1$, and set $f := \mathrm{ad}(e_7)|_{\mathfrak{h}}$. Write $\varphi = \omega \wedge e^7 + \rho$ with $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$. Then φ is a closed G_2-eigenform with $\Delta_\varphi\varphi = \varphi$ if and only if (ω, ρ) is half-flat and there exists a primitive $(1,1)$-form $\nu \in [\Lambda_0^{1,1}\mathfrak{h}^*]$ on $\mathfrak{h}$ such that $\rho = d_{\mathfrak{h}}\nu$ and*

$$f.\nu = \omega, \tag{3.3}$$

$$\omega \wedge f.\omega - d_{\mathfrak{h}}\hat{\rho} = \omega \wedge \nu. \tag{3.4}$$

If we only look for exact G_2-structures $\varphi \in \Lambda^3\mathfrak{g}^*$, the same calculations as above show:

Theorem 3.5. *Let $\mathfrak{g}$ be a seven-dimensional Lie algebra, let $\varphi \in \Lambda^3\mathfrak{g}^*$ be a G_2-structure on $\mathfrak{g}$, let $\mathfrak{h}$ be a codimension-one unimodular ideal in $\mathfrak{g}$, let $e_7 \in \mathfrak{h}^{\perp_{g_\varphi}}$ of norm one, $e^7 \in \mathrm{Ann}(\mathfrak{h})$ with $e^7(e_7) = 1$, and set $f := \mathrm{ad}(e_7)|_{\mathfrak{h}}$. Write $\varphi = \omega \wedge e^7 + \rho$ with $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$. Then φ is an exact G_2-structure if and only if there exist a two-form $\nu \in \Lambda^2\mathfrak{h}^*$ on $\mathfrak{h}$ and a one-form $\alpha \in \mathfrak{h}^*$ with $\rho = d_{\mathfrak{h}}\nu$ and*

$$f.\nu - d_{\mathfrak{h}}\alpha = \omega. \tag{3.5}$$

4. Results in dimension six

From now on, we restrict to ourselves to a special class of Lie algebras:

Definition 4.1. A Lie algebra $\mathfrak{g}$ is called *almost nilpotent* if it admits a codimension-one nilpotent ideal $\mathfrak{h}$. Note that then $\mathfrak{g} \cong \mathfrak{h} \rtimes_f \mathbb{R}$ for a derivation $f \in \mathrm{Der}(\mathfrak{h})$, where $\mathfrak{h} \rtimes_f \mathbb{R}$ denotes the semi-direct product of $\mathbb{R}$ with $\mathfrak{h}$ and Lie algebra representation $\rho : \mathbb{R} \to \mathrm{Der}(\mathfrak{h})$ of $\mathbb{R}$ on $\mathfrak{h}$ given by $\rho(t) = tf$ for all $t \in \mathbb{R}$.

In order to investigate the existence of exact G_2-structures and closed G_2-eigenforms on seven-dimensional almost nilpotent Lie algebras $\mathfrak{g}$, we first have to determine which six-dimensional nilpotent Lie algebras admit exact $SL(3,\mathbb{C})$-structures or half-flat $SU(3)$-structures (ω, ρ) for which there exists a primitive $(1,1)$-form ν with $\rho = d\nu$.

4.1. Exact $SL(3,\mathbb{C})$-structures on nilpotent Lie algebras

We start by determining the six-dimensional nilpotent Lie algebras $\mathfrak{h}$ admitting an exact $SL(3,\mathbb{C})$-structure $\rho \in \Lambda^3\mathfrak{h}^*$. To this aim, we rephrase the condition of being exact in the following way:

Proposition 4.2. *Let $\mathfrak{h}$ be a six-dimensional nilpotent Lie algebra. Then $\mathfrak{h}$ admits an exact $SL(3,\mathbb{C})$-structure $\rho \in \Lambda^3\mathfrak{h}^*$ if and only if there exist linear independent one-forms $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ and two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{h}^*$ with $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$ for $i, j \in \{1,2\}$ such that $\ker(\omega_1) = \ker(\omega_2)$ is a complement of $\ker(\alpha_1) \cap \ker(\alpha_2)$ in $\mathfrak{h}$ and such that either*

(a) *$\dim(\mathfrak{z}(\mathfrak{h})) = 1$, $\dim(\mathfrak{h}_2) = 2$, $\ker(\omega_1) = \ker(\omega_2) = \mathfrak{h}_2$ and there exists a closed non-zero one-form $\gamma \in \mathfrak{h}^* \setminus \{0\}$ with $\gamma(\mathfrak{h}_2) = \{0\}$ such that*

$$d\alpha_1 = \omega_1, \quad d\alpha_2 = \omega_2 + \gamma \wedge \alpha_1,$$

(b) *or* $\dim(\mathfrak{z}(\mathfrak{h}))=2$, $\ker(\omega_1)=\ker(\omega_2)=\mathfrak{z}(\mathfrak{h})$ *and*

$$d\alpha_1=\omega_1, \quad d\alpha_2=\omega_2.$$

In the first case, $\mathfrak{h}_2$ is J-invariant, and in the second case $\mathfrak{z}(\mathfrak{h})$ is J-invariant for the almost complex structure J induced by ρ.

Proof. The forward implication. Assume that $\mathfrak{h}$ admits an exact $\mathrm{SL}(3,\mathbb{C})$-structure ρ, i.e., $\rho=d\nu$ for some $\nu\in\Lambda^2\mathfrak{h}^*$. Let $X,Y\in\mathfrak{z}(\mathfrak{h})$. Then

$$\rho(X,Y,Z)=d\nu(X,Y,Z)=0$$

for any $Z\in\mathfrak{h}$, which implies that X and Y are $\mathbb{C}$-linearly dependent by Lemma 2.5 (a). So $\dim(\mathfrak{z}(\mathfrak{h}))\in\{1,2\}$.

If $\dim(\mathfrak{z}(\mathfrak{h}))=2$, $\mathfrak{z}(\mathfrak{h})$ is J-invariant, so we may choose a basis X, JX of $\mathfrak{z}(\mathfrak{h})$. By Lemma 2.5 (b), $\omega_1:=\rho(X,\cdot,\cdot)$ and $\omega_2:=\rho(JX,\cdot,\cdot)$ have two-dimensional common kernel $\mathfrak{z}(\mathfrak{h})$ and fulfill $\omega_i\wedge\omega_j=\delta_{ij}\omega_1^2$ for $i,j\in\{1,2\}$.

Moreover, setting $\alpha_1:=-\nu(X,\cdot)\in\mathfrak{h}^*$ and $\alpha_2:=-\nu(JX,\cdot)\in\mathfrak{h}^*$, we have

$$d\alpha_1(Y,Z)=-\alpha_1([Y,Z])=-\nu([Y,Z],X)=d\nu(X,Y,Z)=\rho(X,Y,Z)=\omega_1(Y,Z)$$

for all $Y,Z\in\mathfrak{h}$, i.e., $d\alpha_1=\omega_1$. In the same way, one obtains $d\alpha_2=\omega_2$, which then also shows that α_1 and α_2 are linearly independent. Now choose $Y\in\mathfrak{h}_2$ linearly independent of X and JX. By Lemma 2.5 (a), there exists $Z\in\mathfrak{h}$ with $\rho(X,Y,Z)\neq 0$ and so

$$0\neq\rho(X,Y,Z)=\nu(X,[Y,Z]).$$

Since $[Y,Z]\in\mathfrak{z}(\mathfrak{h})=\mathrm{span}(X,JX)$, this shows $\alpha_1(JX)=-\alpha_2(X)=\nu(X,JX)\neq 0$. Thus $\ker(\alpha_1)\cap\ker(\alpha_2)$ is complementary to $\ker(\omega_1)=\ker(\omega_2)=\mathfrak{z}(\mathfrak{h})$.

Next, consider the case $\dim(\mathfrak{z}(\mathfrak{h}))=1$ and choose $X\in\mathfrak{z}(\mathfrak{h})$ and $Y\in\mathfrak{h}_2$ linearly independent. Then we have

$$\rho(X,Y,Z)=d\nu(X,Y,Z)=\nu(X,[Y,Z])=0$$

for any $Z\in\mathfrak{h}$, that is, X and Y are $\mathbb{C}$-linearly dependent by Lemma 2.5 (a). Hence $\dim(\mathfrak{h}_2)=2$ and $\mathfrak{h}_2$ is J-invariant.

Choose a basis X, JX of $\mathfrak{h}_2$ such that $X\in\mathfrak{z}(\mathfrak{h})$ and set again $\omega_1:=\rho(X,\cdot,\cdot)$, $\omega_2:=\rho(JX,\cdot,\cdot)$, $\alpha_1:=-\nu(X,\cdot)$ and $\alpha_2:=-\nu(JX,\cdot)$. As in the case $\dim(\mathfrak{z}(\mathfrak{h}))=2$, we get $\ker(\omega_1)=\ker(\omega_2)=\mathrm{span}(X,JX)=\mathfrak{h}_2$, $\omega_i\wedge\omega_j=\delta_{ij}\omega_1^2$ for $i,j=1,2$ and $d\alpha_1=\omega_1$.

Next, let $Y\in\mathfrak{h}_3$ be linearly independent of X and JX. By Lemma 2.5 (a), we again have some $Z\in\mathfrak{h}$ with $0\neq\rho(X,Y,Z)=\nu(X,[Y,Z])$ and from $[Y,Z]\in\mathfrak{h}_2=\mathrm{span}(X,JX)$ we get again that $\alpha_1(JX)=-\alpha_2(X)=\nu(X,JX)\neq 0$, i.e., that $\ker(\alpha_1)\cap\ker(\alpha_2)$ is complementary to $\ker(\omega_1)=\ker(\omega_2)=\mathfrak{z}(\mathfrak{h})$. So we finally have to prove the equation for $d\alpha_2$ in this case. Thereto, let $\gamma\in\mathfrak{h}^*\setminus\{0\}$ be the one-form uniquely defined by $[JX,Y]=-\gamma(Y)X$ for all $Z\in\mathfrak{h}$. Obviously, $\gamma(X)=\gamma(JX)=0$, i.e., $\gamma(\mathfrak{h}_2)=\{0\}$. Moreover, $d\gamma=0$ as

$$\begin{aligned}d\gamma(Z,W)X=-\gamma([Z,W])X=[JX,[Z,W]]&=[Z,[W,JX]]+[W,[JX,Z]]\\&=-\gamma(W)[Z,X]+\gamma(Z)[W,X]=0\end{aligned}$$

for all $Z, W \in \mathfrak{h}$. Furthermore,

$$\begin{aligned} d\alpha_2(Y,Z) = -\alpha_2([Y,Z]) &= -\nu([Y,Z],JX) \\ &= d\nu(JX,Y,Z) + \nu([JX,Y],Z) - \nu([JX,Z],Y) \\ &= \rho(JX,Y,Z) - \nu(\gamma(Y)X,Z) + \nu(\gamma(Z)X,Y) \\ &= \omega_2(Y,Z) + \gamma(Y)\alpha_1(Z) - \gamma(Z)\alpha_1(Y) \\ &= \omega_2(Y,Z) + (\gamma\wedge\alpha_1)(Y,Z), \end{aligned}$$

as claimed.

The backwards implication. Assume that there exist linear independent one-forms $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ and two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{h}^*$ as in the statement. Note that $\dim(\ker(\alpha_1) \cap \ker(\alpha_2)) = 4$ since α_1, α_2 are linearly independent. Consequently, we have $\dim(\ker(\omega_1)) = \dim(\ker(\omega_2)) = 2$ and so ω_1, ω_2 are non-degenerate two-forms on $V := \ker(\alpha_1) \cap \ker(\alpha_2)$ satisfying $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$ for $i, j = 1, 2$ and $\omega_1^2 \neq 0$. Then it is well known that there is a basis $(v_1, \ldots, v_4)$ such that, with respect to the dual basis $(v^1, \ldots, v^4)$, we have

$$\omega_1 = v^{12} + v^{34}, \quad \omega_2 = v^{13} - v^{24},$$

see, e.g., the proof of Lemma 2.2 in [16]. Consider $(v^1, \ldots, v^4)$ as one-forms on $\mathfrak{h}$ by identifying V^* with the annihilator of $\ker(\omega_1) = \ker(\omega_2)$ and set

$$\rho := \alpha_2 \wedge \omega_1 - \alpha_1 \wedge \omega_2 = -\alpha^1 \wedge v^{13} + \alpha^1 \wedge v^{24} + \alpha^2 \wedge v^{12} + \alpha^2 \wedge v^{34}.$$

The three-form ρ is an $\mathrm{SL}(3,\mathbb{C})$-structure on $\mathfrak{h}^*$, since an adapted basis is given by $(\alpha_1, \alpha_2, v^3, v^2, v^1, -v^4)$. Moreover, ρ is exact since $\nu := \alpha^{12} \in \Lambda^2\mathfrak{h}^*$ satisfies $d\nu = d\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge d\alpha^2 = \alpha_2 \wedge \omega_1 - \alpha_1 \wedge \omega_2 = \rho$ in both cases. ∎

There are 34 (isomorphism classes of) real six-dimensional nilpotent Lie algebras. Of these, exactly those five that admit an exact $\mathrm{SL}(3,\mathbb{C})$-structures are listed in Table 1. The notation for these Lie algebras is obtained by numbering the 34 six-dimensional nilpotent Lie algebras from $\mathfrak{n}_1$ to $\mathfrak{n}_{34}$ in the order in which they occur in Table A.1 in [37].

Corollary 4.3. *Let $\mathfrak{h}$ be a six-dimensional nilpotent Lie algebra. Then $\mathfrak{h}$ admits an exact $\mathrm{SL}(3,\mathbb{C})$-structure if and only if $\mathfrak{h}$ is one of the five Lie algebras listed in Table 1.*

$\mathfrak{g}$	differentials
$\mathfrak{n}_1$	$(0,0,12,13,14+23,34-25)$
$\mathfrak{n}_4$	$(0,0,12,13,14+23,24+15)$
$\mathfrak{n}_9$	$(0,0,0,12,14-23,15+34)$
$\mathfrak{n}_{18}$	$(0,0,0,12,13+42,14+23)$
$\mathfrak{n}_{28}$	$(0,0,0,0,13+42,14+23)$

Table 1. Six-dimensional nilpotent Lie algebras admitting an exact $\mathrm{SL}(3,\mathbb{C})$-structure.

Proof. By Proposition 4.2, we either have $\dim(\mathfrak{z}(\mathfrak{h}))=2$ or $\dim(\mathfrak{z}(\mathfrak{h}))=1$ and $\dim(\mathfrak{h}_2)=2$.

Let us first assume that $\dim(\mathfrak{z}(\mathfrak{h})) = 2$. By Proposition 4.2, there are closed two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{h}^*$ with common kernel $\ker(\omega_1) = \ker(\omega_2) = \mathfrak{z}(\mathfrak{h})$ such that $\omega_i \wedge \omega_j =$

$\delta_{ij}\omega_1^2$. Moreover, there are linearly independent one-forms $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ such that $d\alpha_i = \omega_i$ for $i = 1, 2$ and such that $\ker(\alpha_1) \cap \ker(\alpha_2)$ is complementary to $\mathfrak{z}(\mathfrak{h})$. Now $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is a four-dimensional nilpotent Lie algebra and ω_1, ω_2 descend to closed two-forms on $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$, again called ω_1, ω_2. It is well known, see e.g. [35], that there are exactly three four-dimensional nilpotent Lie algebras, namely $(0,0,0,0)$, $(0,0,12,0)$ and $(0,0,12,13)$. One easily checks that only $(0,0,0,0)$ and $(0,0,12,0)$ admit closed two-forms ω_1, ω_2 with $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$.

If $\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \cong (0,0,0,0)$, then one may choose (cf. the proof of Proposition 4.2) a basis $e^1, \ldots, e^4$ of the dual space of $(0,0,0,0)$ such that $\omega_1 = e^{13} - e^{24}$ and $\omega_2 = e^{14} + e^{23}$. We may extend this basis to a basis $e^1, \ldots, e^6$ of $\mathfrak{h}^*$ by $e^5 := \alpha^1$ and $e^6 := \alpha^2$ and so $\mathfrak{h} \cong (0,0,0,0,13+42,14+23) = \mathfrak{n}_{28}$.

If $\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \cong (0,0,12,0)$, then ω_1 is a symplectic form on $(0,0,12,0)$ and so symplectomorphic to $e^{14} + e^{23}$ by [35], i.e., we may assume that $\omega_1 = e^{14} + e^{23}$. Then, since ω_2 is closed, $\omega_1 \wedge \omega_2 = 0$ and $\omega_2^2 = \omega_1^2$, one checks that $\omega_2 = a(e^{14} - e^{23}) + b_1 e^{13} - b_2 e^{24} + c e^{12}$ for certain $a, b_1, b_2, c \in \mathbb{R}$ with $b_1 b_2 - a^2 = 1$. But so

$$f := \begin{pmatrix} 1 & \frac{a}{b_1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{a}{b_1} \\ -\frac{c}{b_2} & -\frac{ac}{b_1 b_2} & 0 & 1 \end{pmatrix}$$

is an automorphism of $(0,0,12,0)$ with $f^*\omega_1 = \omega_1$ and $f^*\omega_2 = b_1 e^{13} - \frac{1}{b_1} e^{24}$. Next,

$$g := \operatorname{diag}\left(b_1^{-2/3}, b_1^{1/3}, b_1^{-1/3}, b_1^{2/3}\right)$$

fulfills $g^* f^* \omega_1 = \omega_1$ and $g^* f^* \omega_2 = e^{13} - e^{24}$. Extending $e^1, \ldots, e^4$ to a basis $e^1, \ldots, e^6$ by setting $e^5 := \alpha_1$ and $e^6 := \alpha_2$, we do get $\mathfrak{h} \cong (0,0,12,0,14+23,13+42) \cong (0,0,0,12,13+42,14+23) = \mathfrak{n}_{18}$, where the latter isomorphism F is, e.g., given by the one with $F(e_1) = -e_2$, $F(e_2) = e_1$, $F(e_3) = e_4$, $F(e_4) = -e_3$, $F(e_5) = e_6$ and $F(e_6) = e_5$.

Next, let $\dim(\mathfrak{z}(\mathfrak{h})) = 1$ and $\dim(\mathfrak{h}_2) = 2$. By Proposition 4.2, there are two-forms $\omega_1, \omega_2 \in \Lambda^2 \mathfrak{h}^*$ with common kernel $\ker(\omega_1) = \ker(\omega_2) = \mathfrak{h}_2$ such that $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$. Moreover, there are linearly independent one-forms $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ and $\gamma \in \mathfrak{h}^* \setminus \{0\}$ closed with $\gamma(\mathfrak{h}_2) = \{0\}$ such that $d\alpha_1 = \omega_1$, $d\alpha_2 = \omega_2 + \gamma \wedge \alpha_1$ and such that $\ker(\alpha_1) \cap \ker(\alpha_2)$ is complementary to $\mathfrak{z}(\mathfrak{h})$. Note that then ω_1 is closed and

$$d\omega_2 = \gamma \wedge \omega_1.$$

Hence, $\mathfrak{h}_2 \lrcorner\, d\omega_2 = 0$ and so $\omega_1, \omega_2, \gamma$ descend to forms on $\mathfrak{a} := \mathfrak{h}/\mathfrak{h}_2$ with $d\omega_1 = 0$ and $d\omega_2 = \gamma \wedge \omega_1$. Since $d\omega_2 \neq 0$ on $\mathfrak{a}$, $\mathfrak{a}$ cannot be Abelian and we must either have $\mathfrak{a} \cong (0,0,12,0)$ or $\mathfrak{a} \cong (0,0,12,13)$.

Let us first assume that $\mathfrak{a} \cong (0,0,12,0)$. By the results in [35], all symplectic forms on $\mathfrak{a}$ are symplectomorphic to each other. Hence, we may assume that $\omega_1 = e^{13} - e^{24}$. Then $\omega_2 = a_1 e^{12} + a_2 e^{34} + b(e^{13} + e^{24}) + c_1 e^{14} + c_2 e^{23}$ for certain $a_1, a_2, b, c_1, c_2 \in \mathbb{R}$ with $a_1 a_2 + c_1 c_2 - b^2 = 1$. We must have $a_2 \neq 0$ as otherwise $d\omega_2 = 0$, a contradiction.

But so the automorphism

$$\begin{pmatrix} -\frac{1}{a_2} & 0 & 0 & 0 \\ 0 & a_2^2 & 0 & 0 \\ \frac{c_1}{a_2^2} & -ba_2 & -a_2 & 0 \\ -\frac{b}{a_2^2} & c_2a_2 & 0 & \frac{1}{a_2^2} \end{pmatrix}$$

of $(0,0,12,0)$ is well-defined. This automorphism fixes ω_1 and transforms ω_2 into $-e^{12}-e^{34}$. Hence, we may assume that $\omega_1 = e^{13}-e^{24}$ and that $\omega_2 = -e^{12}-e^{34}$. Then $d\omega_2 = -e^{124} = e^1 \wedge \omega_1$, i.e., $\gamma = e^1$. Thus, extending $e^1,\dots,e^4$ to a basis $e^1,\dots,e^6$ of $\mathfrak{h}^*$ by $e^5 := \alpha_1$ and $e^6 := \alpha_2 + e^3$, we have $\mathfrak{h} = (0,0,12,0,13-24,15-34) \cong (0,0,0,12,14-23,15+34) = \mathfrak{n}_9$, where the latter isomorphism fixes e^i for $i \notin \{3,4\}$ and interchanges e^3 and e^4.

Next, let us consider the case $\mathfrak{a} \cong (0,0,12,13)$. By [35], all symplectic two-forms on $\mathfrak{a}$ are symplectomorphic. Thus, we may assume that $\omega_1 = e^{14}+e^{23}$. Then $\omega_2 = a_1e^{12}+a_2e^{34}+b_1e^{13}-b_2e^{24}+c(e^{14}-e^{23})$ for certain $a_1,a_2,b_1,b_2,c \in \mathbb{R}$ with $a_1a_2+b_1b_2-c^2 = 1$.

Let us first assume that $a_2 \neq 0$. Then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{b_2}{a_2} & 1 & 0 & 0 \\ -\frac{a_2c+b_2^2}{a_2^2} & \frac{b_2}{a_2} & 1 & 0 \\ \frac{a_2b_1+b_2c}{a_2^2} & -\frac{c}{a_2} & \frac{b_2}{a_2} & 1 \end{pmatrix}$$

is an automorphism of $(0,0,12,13)$ which fixes ω_1 and maps ω_2 to $\frac{1}{a_2}e^{12}+a_2e^{34}$, i.e., we may assume that $\omega = \frac{1}{a_2}e^{12}+a_2e^{34}$. Then $d\omega_2 = a_2e^{124} = -a_2e^2 \wedge \omega_1$, i.e., $\gamma = -a_2e^2$. Hence, extending $e^1,\dots,e^4$ to a basis $e^1,\dots,e^6$ of $\mathfrak{h}^*$ by $e^5 := \alpha_1$ and $e^6 := \frac{1}{a_2}(\alpha_2 - \frac{1}{a_2}e^3)$, we have $\mathfrak{h} = (0,0,12,13,14+23,34-25) = \mathfrak{n}_1$.

Finally, we consider the case $a_2 = 0$. Then $b_1b_2 - c^2 = 1$ and so, in particular, $b_2 \neq 0$. Thus,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{c}{b_2} & 1 & 0 & 0 \\ 0 & -\frac{c}{b_2} & 1 & 0 \\ \frac{-a_1b_2^2+c}{b_2^3} & \frac{c^2}{b_2^2} & -\frac{c}{b_2} & 1 \end{pmatrix}$$

is a well-defined automorphism of $(0,0,12,13)$ which fixes ω_1 and maps ω_2 to $\frac{1}{b_2}e^{13} - b_2e^{24}$. Hence, we may assume that $\omega_2 = \frac{1}{b_2}e^{13}-b_2e^{24}$ and then $\gamma = -b_2e^1$ as $d\omega_2 = -b_2e^{123} = -b_2e^1 \wedge \omega_1$. Thus, extending $e^1,\dots,e^4$ to a basis $e^1,\dots,e^6$ of $\mathfrak{h}^*$ by $e^5 := \alpha_1$ and $e^6 := -\frac{1}{b_2}(\alpha_2 - \frac{1}{b_2}e^4)$, we have $\mathfrak{h} = (0,0,12,13,14+23,24+15) = \mathfrak{n}_4$.

Conversely, the existence of forms as in Proposition 4.2 follows from the discussion above on any of the Lie algebras $\mathfrak{n}_1, \mathfrak{n}_4, \mathfrak{n}_9, \mathfrak{n}_{18}$ and $\mathfrak{n}_{28}$. ■

4.2. Half-flat SU(3)-structures (ω, ρ) with exact ρ

Here, we determine the six-dimensional nilpotent Lie algebras which admit a half-flat SU(3)-structure (ω,ρ) for which $\rho = d\nu$ for a primitive $(1,1)$-form ν. In fact, we will determine all nilpotent Lie algebras which admit a half-flat SU(3)-structure (ω, ρ) with exact ρ and show that these are the same for which $\rho = d\nu$ with a primitive $(1,1)$-form ν.

For this, note that by Corollary 4.3, only $\mathfrak{n}_1$, $\mathfrak{n}_4$, $\mathfrak{n}_9$, $\mathfrak{n}_{18}$, $\mathfrak{n}_{28}$ may admit a half-flat SU(3)-structure (ω, ρ) with exact ρ. Now Conti determined the six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structures in [11] and his results reduce the possible cases to $\mathfrak{n}_4$, $\mathfrak{n}_9$, $\mathfrak{n}_{28}$. We will show that $\mathfrak{n}_4$ cannot admit a half-flat SU(3)-structure (ω, ρ) with exact ρ, and for the proof we use the following obstruction by Schulte-Hengesbach and the first author [20] adapted to our setting. Note that this obstruction is a refinement of one used by Conti in [11].

Lemma 4.4. *Let $\mathfrak{h}$ be a six-dimensional Lie algebra and let $\nu \in \Lambda^6\mathfrak{h}^* \setminus \{0\}$. If there is a non-zero one-form $\alpha \in \mathfrak{h}^*$ satisfying*

$$\alpha \wedge \tilde{J}_\tau^*\alpha \wedge \sigma = 0 \tag{4.1}$$

for all exact three-forms $\tau \in \Lambda^3\mathfrak{h}^*$ and all closed four-forms $\sigma \in \Lambda^4\mathfrak{h}^*$, where $\tilde{J}_\tau^*\alpha$ is defined for $X \in \mathfrak{h}^*$ by

$$\tilde{J}_\tau^*\alpha(X)\,\nu = \alpha \wedge (X \lrcorner \tau) \wedge \tau, \tag{4.2}$$

then $\mathfrak{g}$ does not admit a half-flat SU(3)*-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ with exact ρ.*

This allows us now to prove:

Theorem 4.5. *Let $\mathfrak{h}$ be a six-dimensional nilpotent Lie algebra. Then $\mathfrak{h}$ admits a half-flat* SU(3)*-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ with exact ρ if and only if $\mathfrak{h}$ is isomorphic to $\mathfrak{n}_9$ or $\mathfrak{n}_{28}$. In these cases, $\mathfrak{h}$ also admits a half-flat* SU(3)*-structure $(\tilde{\omega}, \tilde{\rho}) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$, with $\tilde{\rho} = d\nu$ for some primitive $(1,1)$-form $\nu \in [\Lambda_0^{1,1}\mathfrak{h}^*]$.*

Proof. As explained above, by the results of [11] and Corollary 4.3, only $\mathfrak{n}_4$, $\mathfrak{n}_9$ or $\mathfrak{n}_{28}$ may admit a half-flat $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ with exact ρ

Now a direct computation, efficiently carried out with a computer algebra system like MAPLE, shows that one may use the obstruction in Lemma 4.4 with, e.g., $\alpha = e^1$ or $\alpha = e^2$, to exclude the existence of a half-flat SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ with exact ρ on $\mathfrak{n}_4$.

For the other two cases, we provide a half-flat SU(3)-structure $(\tilde{\omega}, \tilde{\rho}) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ and some $\nu \in [\Lambda_0^{1,1}\mathfrak{h}^*]$ with $\tilde{\rho} = d\nu$.

Case $\mathfrak{n}_9$. Here, we may take the SU(3)-structure defined by the adapted basis $(e^1, e^3, e^2, e^4, e^5, -e^6)$, i.e.,

$$\omega = e^{13} + e^{24} - e^{56}, \quad \rho = e^{125} + e^{146} - e^{236} - e^{345}.$$

Then one checks that $d(\omega^2) = 0$. Moreover, set $\nu := e^{13} + \frac{1}{2}e^{26} + \frac{1}{2}e^{45} + e^{56}$. Then $d\nu = \rho$ and ρ is a $(1,1)$-form. Since $\nu \wedge \omega^2 = 0$, ν is primitive as well, i.e., $\nu \in [\Lambda_0^{1,1}\mathfrak{h}^*]$.

Case $\mathfrak{n}_{28}$. Take the SU(3)-structure defined by the adapted basis $(e^1, e^2, e^3, e^4, e^6, e^5)$, i.e.,

$$\omega = e^{12} + e^{34} - e^{56}, \quad \rho = e^{136} - e^{145} - e^{235} - e^{246}.$$

Then $d(\omega^2) = 0$. Setting now $\nu := e^{12} + e^{56}$, we get $d\nu = \rho$ and that ρ is a $(1,1)$-form. Again $\nu \wedge \omega^2 = 0$ and so ν is primitive, i.e., $\nu \in [\Lambda_0^{1,1}\mathfrak{h}^*]$. ∎

Remark 4.6. Fino and Raffero determined in [18] all six-dimensional nilpotent Lie algebras admitting a so-called *coupled* half-flat SU(3)-structure, i.e., a half-flat SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ with $d\omega = \rho$. Interestingly, the six-dimensional nilpotent Lie algebras admitting a coupled half-flat SU(3)-structure are also $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$.

Our proof of Theorem 4.5 is independent of the coupled approach, and in some sense more direct. Fino and Raffero compute with a computer algebra system, for all 24 six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure, the most general exact three-form ρ and check if the quartic invariant λ of ρ can be negative. This way they obtain that, of the six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure, those which admit a maybe non half-flat SU(3)-structure with exact three-form part are precisely $\mathfrak{n}_4$, $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$. Then they show by different methods that $\mathfrak{n}_4$ cannot admit a coupled SU(3)-structure.

5. Exact G_2-structures on compact almost nilpotent solvmanifolds

Here, we prove that a compact almost nilpotent solvmanifold cannot admit an invariant exact G_2-structure. For this, note first that Corollary 4.3 implies the following.

Corollary 5.1. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal $\mathfrak{h}$. If $\mathfrak{g}$ admits an exact* G_2*-structure, then $\mathfrak{h}$ is isomorphic to $\mathfrak{n}_1$, $\mathfrak{n}_4$, $\mathfrak{n}_9$, $\mathfrak{n}_{18}$ or $\mathfrak{n}_{28}$.*

We show now that four of the five cases of a codimension-one nilpotent ideal, namely $\mathfrak{n}_1$, $\mathfrak{n}_9$, $\mathfrak{n}_{18}$ and $\mathfrak{n}_{28}$, may occur in Corollary 5.1, leaving open if there is an almost nilpotent Lie algebra with codimension one ideal $\mathfrak{n}_4$ which admits an exact G_2-structure.

For this, note that Theorem 6.9 below even classifies all the almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$ which admit an exact G_2-structure. For $\mathfrak{h} \in \{\mathfrak{n}_1, \mathfrak{n}_9\mathfrak{n}_{18}\}$, we provide now one example of an exact G_2-structure on a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal $\mathfrak{h}$:

Example 5.2. For the six-dimensional nilpotent Lie algebras $\mathfrak{h} \in \{\mathfrak{n}_1, \mathfrak{n}_9, \mathfrak{n}_{18}\}$, we give an SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$, a two-form $\nu \in \Lambda^2\mathfrak{h}^*$, a one-form $\alpha \in \mathfrak{h}^*$ and a derivation $f \in \mathrm{Der}(\mathfrak{h})$ such that $\rho = d\nu$ and such that (3.5) is valid.

Case $\mathfrak{n}_1$. Take the SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{n}_1^* \times \Lambda^3\mathfrak{n}_1^*$ defined by the adapted basis $(-e^1 + ae^5, e^3 + ae^6, e^2, e^4, e^5, e^6)$ with $a := (3 + \sqrt{5})/2$, i.e.,

$$\omega = -e^{13} - a\,e^{16} + e^{24} - a\,e^{35} + (1 + a^2)e^{56}, \quad \rho = -e^{125} + e^{146} + e^{236} - e^{345}.$$

Setting

$$\nu := \Big(2 - \frac{2}{3}\,a\Big)e^{15} - \frac{1}{2}e^{16} + \Big(2 - \frac{2}{3}\,a\Big)e^{24} - \frac{1}{2}e^{35} + e^{56},$$

one gets $d\nu = \rho$. Moreover,

$$f := \begin{pmatrix} -\frac{a}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{a}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a^2+1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{4}\,a & 0 \\ -1 & 0 & 0 & 0 & 0 & -\frac{7}{4}\,a \end{pmatrix}$$

is a derivation of $\mathfrak{n}_1$ and one computes $f.\nu = \omega + e^{13}$. Hence, choosing $\alpha := e^4$, we have $d\alpha = e^{13}$ and so $f.\nu - d\alpha = \omega$, i.e., (3.5) is fulfilled.

Case $\mathfrak{n}_9$. In this case, we choose the SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{n}_9^* \times \Lambda^3\mathfrak{n}_9^*$ defined by the adapted basis $(e^1, e^3, e^2, e^4, e^5, -e^6)$, i.e.,

$$\omega = e^{13} + e^{24} - e^{56}, \quad \rho = e^{125} + e^{146} - e^{236} - e^{345}.$$

Setting

$$\nu := -\frac{2}{3}e^{13} + e^{24} + \frac{1}{2}e^{26} + \frac{1}{2}e^{45} + e^{56},$$

one obtains $d\nu = \rho$. Moreover,

$$f := \operatorname{diag}\Big(\frac{1}{2}, -\frac{3}{4}, 1, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\Big)$$

is a derivation of $\mathfrak{n}_9$ and $f.\nu = e^{13} + e^{24} - e^{56} = \omega$. Thus, for $\alpha := 0$, (3.5) is satisfied.

Case $\mathfrak{n}_{18}$. Here, we look at the SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{n}_{18}^* \times \Lambda^3\mathfrak{n}_{18}^*$ defined by the adapted basis $(e^1, e^2, e^3, e^4, e^6, e^5)$, i.e.,

$$\omega = e^{12} + e^{34} - e^{56}, \quad \rho = e^{136} - e^{145} - e^{235} - e^{246}.$$

Taking

$$\nu := \frac{3}{2}e^{16} - \frac{3}{2}e^{34} + e^{56},$$

we get $d\nu = \rho$. Now one checks that

$$f := \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

is a derivation of $\mathfrak{n}_{18}$ and that $f.\nu = e^{34} - e^{56}$. Thus, for $\alpha := -e^4$, we have $d\alpha = -e^{12}$ and so $f.\nu - d\alpha = e^{12} + e^{34} - e^{56} = \omega$, i.e., (3.5) is valid for our choices.

Next, we look at compact almost nilpotent solvmanifolds, i.e., manifolds of the form $\Gamma\backslash G$, where G is a simply-connected almost nilpotent Lie group and Γ a cocompact lattice in G. A necessary condition for the existence of such a lattice is that the associated Lie algebra $\mathfrak{g}$ is *strongly unimodular*, cf. [21].

Definition 5.3. Let $\mathfrak{g}$ be a solvable Lie algebra, let $\mathfrak{n}$ be its nilradical, and let $\mathfrak{n}^0, \mathfrak{n}^1, \ldots$ be the descending central series of $\mathfrak{n}$. One checks that ad_X preserves $\mathfrak{n}^i$ for all $X \in \mathfrak{g}$ and all $i \in \mathbb{N}$. The Lie algebra $\mathfrak{g}$ is called *strongly unimodular* if $\mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}^i/\mathfrak{n}^{i+1}}) = 0$ for all $i \in \mathbb{N}$ and all $X \in \mathfrak{g}$.

Remark 5.4. Since the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ of a solvable Lie algebra $\mathfrak{g}$ is nilpotent, the nilradical $\mathfrak{n}$ contains the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. Hence, if $\mathfrak{g}$ is strongly unimodular, one has $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$, i.e., $\mathfrak{g}$ is unimodular.

Theorem 5.5. *Let $\mathfrak{g}$ be a seven-dimensional strongly unimodular almost nilpotent Lie algebra. Then $\mathfrak{g}$ does not admit an exact* G_2*-structure.*

Proof. Assume the contrary. By Corollary 5.1, the Lie algebra $\mathfrak{g}$ then admits a codimension-one nilpotent ideal $\mathfrak{h}$ which is isomorphic to $\mathfrak{n}_1, \mathfrak{n}_4, \mathfrak{n}_9, \mathfrak{n}_{18}$ or $\mathfrak{n}_{28}$. Moreover, $\mathfrak{h}$ is the nilradical as the entire Lie algebra cannot be nilpotent according to Proposition 2.17. Furthermore, we have an induced SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ on $\mathfrak{h}$ with exact ρ, i.e., there is some $\nu \in \Lambda^2\mathfrak{h}^*$ with $d\nu = \rho$ which has to fulfill

$$f.\nu - d_{\mathfrak{h}}\alpha = \omega$$

for some one-form $\alpha \in \mathfrak{h}^*$ by Theorem 3.5. Now we know that in the cases $\mathfrak{n}_1$, $\mathfrak{n}_4$ and $\mathfrak{n}_9$, we have $\dim(\mathfrak{z}(\mathfrak{h})) = 1$ and $\dim(\mathfrak{h}_2) = 2$, with $\mathfrak{h}_2$ being J-invariant by Proposition 4.2 for the almost complex structure J induced by ρ. Moreover, in all theses cases, one checks that $\mathfrak{h}_2$ is the sum of quotient spaces of the form $\mathfrak{h}^i/\mathfrak{h}^{i+1}$, i.e., the trace of each ad_X, $X \in \mathfrak{g}$, has to be trace-free on $\mathfrak{h}_2$. In these cases, we set $\mathfrak{a} := \mathfrak{h}_2$.

In the cases $\mathfrak{n}_{18}$ and $\mathfrak{n}_{28}$, we have $\dim(\mathfrak{z}(\mathfrak{h})) = 2$ and $\mathfrak{z}(\mathfrak{h})$ is J-invariant by Proposition 4.2. Moreover, $\mathfrak{z}(\mathfrak{h})$ equals in both cases the last non-zero $\mathfrak{h}^i$, so is of the form $\mathfrak{h}^i/\mathfrak{h}^{i+1}$. Hence, each ad_X, $X \in \mathfrak{g}$, has to be trace-free when restricted to $\mathfrak{z}(\mathfrak{h})$. Here, we set $\mathfrak{a} := \mathfrak{z}(\mathfrak{h})$.

Now coming back to general case, we choose some $0 \neq X \in \mathfrak{z}(\mathfrak{h}) \subseteq \mathfrak{a}$. Then we get

$$0 = -\mathrm{tr}(f|_{\mathfrak{a}})\,\nu(X, JX) = (f.\nu - d_{\mathfrak{h}}\alpha)(X, JX) = \omega(X, JX) = -\|X\|^2 \neq 0,$$

since f has to preserve $\mathfrak{a} = \mathrm{span}(X, JX)$. This yields the desired contradiction and so $\mathfrak{g}$ cannot admit an exact G_2-structure. ■

In general, if G is a simply-connected solvable Lie group which admits a cocompact lattice Γ, then any left-invariant differential form β induces a differential form $\tilde{\beta}$ on the compact quotient $\Gamma\backslash G$. We then call $\tilde{\beta}$ *invariant*. By Theorem 3.2.10 in [34], the assignment $\beta \mapsto \tilde{\beta}$ induces an injection $H^*(\mathfrak{g}) \to H^*_{dR}(\Gamma\backslash G)$.

Hence, Theorem 5.5 implies that no compact almost nilpotent solvmanifold can admit an *invariant* exact G_2-structure. If G is *completely solvable*, i.e., if ad_X has only real eigenvalues for all $X \in \mathfrak{g}$, then $H^*(\mathfrak{g}) \to H^*_{dR}(\Gamma\backslash G)$ is an isomorphism by [22] and so one may skip the word 'invariant' in the following statement.

Corollary 5.6. *Let $M = \Gamma\backslash G$ be an almost nilpotent solvmanifold, i.e., G is a simply-connected almost nilpotent Lie group and Γ is a cocompact lattice in G. Then M does not admit an invariant exact G_2-structure. If G is completely solvable, then M does not admit any exact G_2-structure at all.*

6. Closed G_2-eigenforms on almost nilpotent Lie algebras

In this section, we establish:

Theorem 6.1. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra. Then $\mathfrak{g}$ does not admit a closed G_2-eigenform.*

To start the proof, note that by Theorem 3.4 and Theorem 4.5, the codimension-one nilpotent ideal $\mathfrak{h}$ of an almost nilpotent Lie algebra admitting a closed G_2-eigenform has to be isomorphic to $\mathfrak{n}_9$ or to $\mathfrak{n}_{28}$.

In Subsection 6.1, we will show in Theorem 6.6 that no almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_9$ can admit a closed G_2-eigenform and in Subsection 6.2, we will show in Theorem 6.12 that no almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$ can admit a closed G_2-eigenform. This work completes the proof of Theorem 6.1.

In Subsection 6.2, we also give a classification of all almost nilpotent Lie algebras with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$ that admit an exact G_2-structure, and we distinguish those with special torsion of positive type or of negative type, respectively.

6.1. The case $\mathfrak{n}_9$

Note first that the Lie algebra $\mathrm{Der}(\mathfrak{n}_9)$ of all derivations of $\mathfrak{n}_9$ is given by

$$\mathrm{Der}(\mathfrak{n}_9) = \left\{ \left. \begin{pmatrix} f_{5,5}-f_{4,4} & 0 & 0 & 0 & 0 & 0 \\ f_{4,3} & -f_{5,5}+2f_{4,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2f_{5,5}-2f_{4,4} & 0 & 0 & 0 \\ f_{5,3} & f_{5,4} & f_{4,3} & f_{4,4} & 0 & 0 \\ f_{5,1} & f_{6,4} & f_{5,3} & f_{5,4} & f_{5,5} & 0 \\ f_{6,1} & f_{6,2} & f_{6,3} & f_{6,4} & f_{5,4} & 2f_{5,5}-f_{4,4} \end{pmatrix} \right| f_{i,j} \in \mathbb{R} \right\} \tag{6.1}$$

with respect to the basis $(e_1, \ldots, e_6)$ of $\mathfrak{n}_9$. This can be checked by a lengthy but straightforward calculation done efficiently with a computer algebra system like MAPLE. Exponentials of these derivations are then (inner) automorphisms of the Lie algebra $\mathfrak{n}_9$. Using these automorphisms, one obtains:

Lemma 6.2. *Let $(\omega, \rho) \in \Lambda^2\mathfrak{n}_9^* \times \Lambda^3\mathfrak{n}_9^*$ be an $\mathrm{SU}(3)$-structure on $\mathfrak{n}_9$ with exact ρ, i.e., there exists some $\nu \in \Lambda^2\mathfrak{n}_9^*$ with $d\nu = \rho$. Then ω, ρ and ν are given, up to automorphism, by*

$$\begin{cases} \rho = \varepsilon(e^{125} + e^{146} - e^{236} - e^{345}) =: \varepsilon\rho_0, \\ \omega = a_1e^{13} + a_2e^{24} + a_3e^{56} + a_4(e^{12} + e^{34}) + a_5(e^{15} - e^{36}) + a_6(e^{25} - e^{46}) \\ \qquad + a_7(e^{26} + e^{45}), \\ \nu = b_1e^{12} + b_2e^{13} + b_3e^{14} + b_4(e^{15} + e^{34}) + b_5(e^{16} + e^{35}) + b_6e^{23} + b_7e^{24} \\ \qquad + \frac{\varepsilon}{2}(e^{26} + e^{45}) + \varepsilon\, e^{56}, \end{cases} \tag{6.2}$$

for certain $\varepsilon \in \{1, -1\}$, $a_1, \ldots, a_7 \in \mathbb{R}$ with $a_1a_2 > 0$, $a_1a_3 < 0$ and certain $b_1, \ldots, b_7 \in \mathbb{R}$. If (ω, ρ) is half-flat and ν is of type $(1,1)$, then we may assume that $a_4 = a_5 = 0$, $b_1 = b_4 = 0$ and $b_6 = b_3$ in (6.2).

Proof. First of all, observe that the most general exact three-form ρ is given by

$$\rho = c_1 e^{123} + c_2 e^{124} + c_3 e^{125} + c_4 e^{126} + c_5 e^{134} + c_6 e^{135} + c_4 e^{145} + c_7 e^{146} + c_8 e^{234} \\ - c_7 e^{236} - c_7 e^{345}$$

for certain $c_i \in \mathbb{R}$, $i = 1, \ldots, 8$. The quartic invariant $\lambda(\rho)$ of ρ computes to be equal to

$$\lambda(\rho) = -4c_7^2(c_3c_7 - c_4^2)\,(e^{1234567})^{\otimes 2}.$$

As $\lambda(\rho)$ has to be negative, we surely must have $c_7 \neq 0$. Now exponentials of matrices as in (6.1) give automorphisms of $\mathfrak{n}_9$. We consider first the automorphism $F_1 := \exp(A_1)$ for the matrix A_1 as in (6.1) with $f_{4,3} = 0$, $f_{4,4} = 0$, $f_{5,5} = 0$, $f_{6,1} = 0$, $f_{6,2} := 0$, $f_{6,3} = 0$ and $f_{5,1} = -f_{5,3}\, f_{5,4}/2$. For notational simplicity, we set $a := f_{5,3}$, $b := f_{5,4}$, $s := f_{6,4}$ and so have

$$A_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ -\frac{ab}{2} & s & a & b & 0 & 0 \\ 0 & 0 & 0 & s & b & 0 \end{pmatrix}.$$

The exponential of this matrix is easily computed to be

$$F_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a & b & 0 & 1 & 0 & 0 \\ 0 & S & a & b & 1 & 0 \\ A & B & C & S & b & 1 \end{pmatrix}$$

with $A = \frac{1}{2}as - \frac{1}{12}ab^2$, $B = bs + \frac{b^3}{6}$, $C = \frac{ab}{2}$ and $S = s + \frac{1}{2}b^2$. Then

$$(F_1^*\rho)(e_1, e_2, e_6) = c_4 + b\,c_7, \quad (F_1^*\rho)(e_1, e_3, e_5) = c_6 + a\,c_7,$$
$$(F_1^*\rho)(e_2, e_3, e_4) = c_8 - 2s\,c_7$$

So, setting $a := -\frac{c_6}{c_7}$ and $b := -\frac{c_4}{c_7}$ and $s := \frac{c_8}{2c_7}$, we get

$$(F_1^*\rho)(e_1, e_4, e_5) = (F_1^*\rho)(e_1, e_2, e_6) = 0, \quad (F_1^*\rho)(e_1, e_3, e_5) = 0,$$
$$(F_1^*\rho)(e_2, e_3, e_4) = 0,$$

i.e., we have

$$F_1^*\rho = \tilde{c}_1 e^{123} + \tilde{c}_2 e^{124} + \tilde{c}_3 e^{125} + \tilde{c}_4 e^{134} + \tilde{c}_5 (e^{146} - e^{236} - e^{345}),$$

for certain $\tilde{c}_i \in \mathbb{R}$, $i = 1, \ldots, 5$, now with $\tilde{c}_5 \neq 0$ (in fact, $\tilde{c}_5 = c_7$). Next, we consider the automorphism $F_2 := \exp(A_2)$ with the matrix A_2 as in (6.1) with $f_{4,3} := 0$, $f_{4,4} = 0$, $f_{5,5} = 0$, $f_{5,3} = 0$, $f_{5,4} = 0$, $f_{5,1} = 0$ and $f_{6,4} = 0$, i.e., we have

$$A_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p & q & r & 0 & 0 & 0 \end{pmatrix},$$

where we set $p := f_{6,1}$, $q := f_{6,2}$ and $r := f_{6,3}$. With $F_2 := \exp(A_2) = I_6 + A_2$, we obtain

$$(F_2^* F_1^*\rho)(e_1, e_2, e_3) = \tilde{c}_1 - \tilde{c}_5\, p, \quad (F_2^* F_1^*\rho)(e_1, e_2, e_4) = \tilde{c}_2 - \tilde{c}_5\, q,$$
$$(F_2^* F_1^*\rho)(e_1, e_3, e_4) = \tilde{c}_4 - \tilde{c}_5\, r.$$

Thus, setting $p := \tilde{c}_1/\tilde{c}_5$, $q := \tilde{c}_2/\tilde{c}_5$ and $r := \tilde{c}_4/\tilde{c}_5$, we get that

$$F_2^* F_1^*\rho = Ae^{125} + B(e^{146} - e^{236} - e^{345}).$$

Then $\lambda(F_2^* F_1^* \rho) = -4AB^3$, i.e., we must have $A \cdot B > 0$. It is now fairly easy to see that the exponential $F_3 := \exp(A_3)$ of a diagonal matrix A_3 as in (6.1) allows to normalise $A = B \in \{-1, 1\}$, i.e., we have

$$(F_3^* F_2^* F_1^* \rho) = \varepsilon (e^{125} + e^{146} - e^{236} - e^{345}) = \varepsilon \rho_0$$

for some $\varepsilon \in \{-1, 1\}$.

From now on, we will use again ρ for $F_3^* F_2^* F_1^* \rho$, so that $\rho = \varepsilon \rho_0$. Notice that $\varepsilon(e_1, -e_3, e_2, -e_4, e_5, e_6)$ or $\varepsilon(e_1, e_3, e_2, e_4, e_5, -e_6)$ is an oriented adapted basis for $\rho = \varepsilon\rho_0$, depending on the orientation induced by ω^3. Hence, the induced almost complex structure $J = J_\rho$ is either given by $J_0 e_1 = -e_3$, $J_0 e_2 = -e_4$ and $J_0 e_5 = e_6$ or by $-J_0$. A straightforward computation shows that a two-form $\nu \in \Lambda^2 \mathfrak{n}_9^*$ with $d\nu = \rho$ has to be as claimed and ν is of type $(1, 1)$ precisely when $b_1 = b_4 = 0$ and $b_6 = b_3$.

Next, we are interested in bringing ω into a canonical form. To this aim, note first that ω has to be a $(1, 1)$-form with respect to J_0 and so

$$\begin{aligned}\omega = a_1 e^{13} + a_2 e^{24} + a_3 e^{56} + a_4 (e^{12} + e^{34}) + a_5 (e^{14} + e^{23}) + a_6 (e^{15} - e^{36}) \\ + a_7 (e^{16} + e^{35}) + a_8 (e^{25} - e^{46}) + a_9 (e^{26} + e^{45})\end{aligned}$$

for certain $a_1, \ldots, a_9 \in \mathbb{R}$. Observe that $a_1 = \omega(e_1, e_3) = \omega(e_1, \mp J e_1) = \pm g(e_1, e_1)$, $a_2 = \omega(e_2, e_4) = \pm g(e_2, e_2)$ and $a_3 = \omega(e_5, e_6) = \mp g(e_5, e_5)$, and so $a_1 a_2 > 0$, $a_1 a_3 < 0$ as claimed.

In order to bring ω into a form with less parameters without changing ρ, we need to look at those matrices A in (6.1) that are in the Lie algebra $\mathfrak{a}$ of the stabiliser group of ρ_0. Such an A has to commute with J_0, which is the case if and only if $f_{5,5} = f_{4,4}$, $f_{5,3} = 0$, $f_{5,4} = 0$, $f_{6,1} = 0$, $f_{6,3} = -f_{5,1}$, $f_{6,2} = 0$, $f_{6,4} = 0$. Moreover, the complex $\pm J_0$-trace of A must be equal to zero, which additionally gives us $f_{4,4} = 0$. So A is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y & 0 & 0 & 0 \end{pmatrix}$$

for $x := f_{4,3}$ and $y := f_{5,1}$. Then $F := \exp(A) = I_6 + A$ and

$$(F^*\omega)(e_1, e_4) = a_5 + x a_2 - y a_9, \quad (F^*\omega)(e_1, e_6) = a_7 + x a_9 + y a_3.$$

Now $g(e_2, e_2) = \pm a_2$, $g(e_5, e_5) = \mp a_3$, as we observed above, and $g(e_2, e_5) = \omega(e_2, J e_5) = \pm\omega(e_2, e_6) = \pm a_9$. Since any minor of g has to be non-zero, we then get that $0 \neq -(g(e_2, e_2) g(e_5, e_5) - g(e_2, e_5)^2) = a_2 a_3 + a_9^2$. Thus, setting

$$x := -\frac{a_3 a_5 + a_7 a_9}{a_2 a_3 + a_9^2} \quad \text{and} \quad y := -\frac{a_2 a_7 - a_5 a_9}{a_2 a_3 + a_9^2}$$

yields $(F^*\omega)(e_2, e_3) = (F^*\omega)(e_1, e_4) = 0$ and $(F^*\omega)(e_3, e_5) = (F^*\omega)(e_1, e_6) = 0$. Hence, renaming $F^*\omega$ by ω and using again coefficients labeled $a_1, \ldots, a_7$, we have

$$\begin{aligned}\omega = a_1 e^{13} + a_2 e^{24} + a_3 e^{56} + a_4 (e^{12} + e^{34}) + a_5 (e^{15} - e^{36}) \\ + a_6 (e^{25} - e^{46}) + a_7 (e^{26} + e^{45}).\end{aligned}$$

Then

$$d\left(\frac{1}{2}\omega^2\right) = (a_2a_5 + a_4a_7 + (a_5a_7 - a_3a_4))e^{123435} + (a_5a_7 - a_3a_4)e^{12356},$$

i.e., (ω, ρ) is half-flat if and only if

$$a_2a_5 + a_4a_7 = 0, \quad a_5a_7 - a_3a_4 = 0.$$

The first equation gives us $a_5 = -\frac{a_4a_7}{a_2}$ and inserting this into the second equation yields

$$0 = a_5a_7 - a_3a_4 = -\frac{a_4a_7^2}{a_2} - \frac{a_2a_3a_4}{a_2} = -\frac{a_4(a_2a_3 + a_7^2)}{a_2}.$$

Since a_7 plays the role of the former a_9, we showed above that $a_2a_3 + a_7^2 \neq 0$. Thus, $a_4 = 0$ and so $a_5 = 0$. The equations $a_2a_5 + a_4a_7 = 0$, $a_5a_7 - a_3a_4 = 0$ are now fulfilled, so this finishes the proof. ∎

For the rest of this subsection, we assume that (ω, ρ, ν) is as in (6.2) with (ω, ρ) being half-flat and ν being a primitive form of type $(1, 1)$. Moreover, we assume that (3.3) and (3.4) are valid. We show that these assumptions give rise to a contradiction. First note the following:

(i) We may assume that $\omega^3 \in \mathbb{R}_+ \cdot e^{123456}$, i.e., ω induces the orientation in which the ordered basis $(e_1, \ldots, e_6)$ is oriented. This follows from the observation that with (ω, ρ, ν, f) also $(-\omega, \rho, \nu, -f)$ satisfies (3.3) and (3.4).

(ii) Moreover, we may assume that $\varepsilon = 1$ as with (ω, ρ, ν, f) also $(\omega, -\rho, -\nu, -f)$ fulfills (3.3) and (3.4)

Using these simplifications, we obtain:

Lemma 6.3. *We have*

$$f_{5,3} = f_{6,1} = f_{6,4} = 0, \quad f_{6,3} = -f_{5,1}, \quad f_{6,2} = 2f_{5,4},$$
$$a_3 = f_{4,4} - 3f_{5,5}, \quad a_6 = f_{5,4}, \quad a_7 = -\frac{f_{4,4} + f_{5,5}}{2}$$

and $b_3 = b_5 = 0$ *or* $f_{5,4} = 0$*,*

Proof. First of all,

$$0 = (\omega - f.\nu)(e_3, e_6) = f_{5,3}, \quad 0 = (\omega - f.\nu)(e_3, e_4) = b_5 f_{5,4} - \tfrac{f_{5,3}}{2},$$
$$0 = (\omega - f.\nu)(e_1, e_5) = b_5 f_{5,4} - f_{6,1} + \tfrac{f_{5,3}}{2},$$

i.e., $f_{5,3} = f_{6,1} = 0$ and $b_5 f_{5,4} = 0$. Moreover, we get

$$0 = (\omega - f.\nu)(e_3, e_5) = f.\nu = -f_{6,3} + \tfrac{f_{4,3}}{2} + b_5(3f_{5,5} - 2f_{4,4}),$$
$$0 = (\omega - f.\nu)(e_1, e_6) = f.\nu = f_{5,1} + \tfrac{f_{4,3}}{2} + b_5(3f_{5,5} - 2f_{4,4}),$$

which yields $f_{6,3} = -f_{5,1}$. Furthermore,

$$0 = (\omega - f.\nu)(e_2, e_6) = a_7 + \tfrac{f_{4,4}+f_{5,5}}{2} + f_{6,4},$$
$$0 = (\omega - f.\nu)(e_4, e_5) = a_7 + \tfrac{f_{4,4}+f_{5,5}}{2} - f_{6,4},$$

which gives $f_{6,4} = 0$ as well as $a_7 = -(f_{4,4} + f_{5,5})/2$. Next, we have

$$0 = (\omega - f.\nu)(e_2, e_5) = a_6 - f_{6,2} + f_{5,4}, \quad 0 = (\omega - f.\nu)(e_4, e_6) = -a_6 + f_{5,4},$$

i.e., $f_{6,2} = 2f_{5,4}$ and $a_6 = f_{5,4}$. Moreover, we get

$$0 = (\omega - f.\nu)(e_5, e_6) = a_3 - (f_{4,4} - 3f_{5,5}), \quad 0 = (\omega - f.\nu)(e_1, e_2) = f_{5,4}(b_3 + 2b_5),$$

i.e., $a_3 = f_{4,4} - 3f_{5,5}$, and, since also $b_5 f_{5,4} = 0$, $f_{5,4} = 0$ or $b_3 = b_5 = 0$. ■

Lemma 6.4. *In Lemma 6.3, we must have* $f_{5,4} = 0$.

Proof. Assume that $f_{5,4} \neq 0$. Then $b_3 = b_5 = 0$ by Lemma 6.3 and so

$$0 = (\omega - f.\nu)(e_1, e_5) = \tfrac{f_{4,3}}{2} + f_{5,1}, \quad 0 = (\omega - f.\nu)(e_1, e_3) = a_1 + 3b_2(f_{5,5} - f_{4,4}),$$
$$0 = (\omega - f.\nu)(e_2, e_4) = a_2 + b_7(3f_{4,4} - f_{5,5}),$$

i.e., $a_1 = 3b_2(f_{4,4} - f_{5,5})$, $a_2 = b_7(f_{5,5} - 3f_{4,4})$ and $f_{5,1} = -f_{4,3}/2$. Imposing these identities, we get

$$0 = (\omega - f.\nu)(e_1, e_4) = \tfrac{f_{4,3}(1+4b_7)}{4},$$

and so either $f_{4,3} = 0$ or $b_7 = -1/4$ holds.

We show that $f_{4,3} = 0$ and argue by contradiction, i.e., we assume that $f_{4,3} \neq 0$ and so $b_7 = -1/4$. Then (3.4) gives us

$$0 = (f.\omega \wedge \omega - \omega \wedge \nu - d\hat{\rho})(e_1, e_4, e_5, e_6) = -f_{4,3}((f_{4,4} - f_{5,5})^2 + f_{5,4}^2),$$

so that $f_{5,5} = f_{4,4}$ and $f_{5,4} = 0$ due to $f_{4,3} \neq 0$. But then one checks that $\omega^3 = 0$, a contradiction. Hence, we must have $f_{4,3} = 0$.

Assuming $f_{4,3} = 0$, one computes

$$\begin{aligned} 0 &= (f.\omega \wedge \omega - \omega \wedge \nu - d\hat{\rho})(e_1, e_2, e_3, e_5) \\ &= -\varepsilon b_2\, f_{5,4}\,(12\, f_{4,4}^2 - 36\, f_{4,4}\, f_{5,5} + 24\, f_{5,5}^2 - 1), \\ 0 &= (f.\omega \wedge \omega - \omega \wedge \nu - d\hat{\rho})(e_1, e_3, e_5, e_6) \\ &= 2b_2\,(3\, f_{4,4}^2 - 12\, f_{4,4} f_{5,5} + 9\, f_{5,5}^2 - 1)\,(2f_{4,4} - 3f_{5,5}). \end{aligned} \tag{6.3}$$

One checks that $b_2 = 0$ implies $\omega^3 = 0$, and so we must have $b_2 \neq 0$. Since $f_{5,4} \neq 0$ by assumption, (6.3) yields

$$0 = 12\, f_{4,4}^2 - 36\, f_{4,4}\, f_{5,5} + 24\, f_{5,5}^2 - 1,$$
$$0 = (3\, f_{4,4}^2 - 12\, f_{4,4}\, f_{5,5} + 9\, f_{5,5}^2 - 1)\,(2\, f_{4,4} - 3f_{5,5}).$$

So either $3f_{4,4}^2 - 12f_{4,4}f_{5,5} + 9f_{5,5}^2 - 1 = 0$ or $2f_{4,4} - 3f_{5,5} = 0$. However, both cases give us a contradiction.

Namely, if $3f_{4,4}^2 - 12f_{4,4}f_{5,5} + 9f_{5,5}^2 - 1 = 0$, then

$$3f_{4,4}^2 - 12f_{4,4}f_{5,5} + 9f_{5,5}^2 = 1 = 12f_{4,4}^2 - 36f_{4,4}f_{5,5} + 24f_{5,5}^2,$$

and so

$$0 = 9f_{4,4}^2 - 24f_{4,4}f_{5,5} + 15f_{5,5}^2 = (3f_{4,4} - 4f_{5,5})^2 - f_{5,5}^2,$$

i.e.,

$$3f_{4,4} - 4f_{5,5} = \pm f_{5,5}.$$

So either $f_{4,4} - f_{5,5} = 0$ or $f_{5,5} = \frac{3}{5}f_{4,4}$. However, in the first case, one checks that $\omega^3 = 0$, a contradiction. Thus, we must have $f_{5,5} = \frac{3}{5}f_{4,4}$. But then

$$1 = 3f_{4,4}^2 - 12f_{4,4}f_{5,5} + 9f_{5,5}^2 = 3f_{4,4}^2 - \tfrac{36}{5}f_{4,4}^2 + \tfrac{81}{25}f_{4,4}^2 = -\tfrac{24}{25}f_{4,4}^2 \leq 0,$$

again a contradiction.

Consider now the case $2f_{4,4} - 3f_{5,5} = 0$. Then $f_{5,5} = \frac{2}{3}f_{4,4}$ and so

$$1 = 12f_{4,4}^2 - 36f_{4,4}f_{5,5} + 24f_{5,5}^2 = -\tfrac{4}{3}f_{4,4}^2 \leq 0,$$

which is a also a contradiction.

Thus, we must have $f_{5,4} = 0$. ∎

To simplify the notation, we set from now on

$$x := f_{4,4}, \quad y := f_{5,5}, \quad z := f_{4,3}.$$

With this new notation, one gets:

Lemma 6.5. *We have* $a_2 = (y - 3x)b_7$, $f_{5,1} = (2x - 3y)b_5 - \frac{z}{2}$, $x + y \neq 0$, $4b_7 + 1 \neq 0$, $b_2 \neq 0$ *and* $a_1 = \frac{b_2}{2(x+y)}$.

Proof. Firstly

$$0 = (\omega - f.\nu)(e_2, e_4) = a_2 + b_7(3x - y),$$
$$0 = (\omega - f.\nu)(e_3, e_5) = (3y - 2x)b_5 + \tfrac{z}{2} + f_{5,1},$$

i.e., $a_2 = (y - 3x)b_7$, $f_{5,1} = (2x - 3y)b_5 - z/2$. Next, set

$$A := (12b_7 - 1)x^2 - (40b_7 + 2)xy + (12b_7 - 1)y^2.$$

Then

$$0 \neq \omega^3 = \tfrac{3}{2}a_1 A e^{123456},$$

which implies $a_1 \neq 0$ and $A \neq 0$. Since

$$0 = \nu \wedge \omega^2 = \left(\tfrac{Ab_2}{2} + a_1(4b_7 + 1)(x + y)\right)e^{123456},$$

we either have $b_2 = 0$, and then $(4b_7 + 1)(x + y) = 0$ as well, or $b_2 \neq 0$, and so $x + y \neq 0$, $4b_7 + 1 \neq 0$, and $A = -\frac{2a_1(4b_7+1)(x+y)}{b_2}$. Moreover, we have

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_2, e_4, e_5, e_6) = \tfrac{1}{2}(x + y)(A + 4b_7 + 1)\,. \tag{6.4}$$

Assume now first that $b_2 = 0$. Then we must have $x + y = 0$, as otherwise $4b_7 + 1 = 0$ and so $(x + y)A = 0$, a contradiction to $x + y \neq 0$ and $A \neq 0$. But then

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_1, e_2, e_3, e_6) = -1 + \tfrac{a_1}{2},$$
$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_1, e_3, e_5, e_6) = a_1(40x^2 - 1),$$

from which we obtain $a_1 = 2$ and $x = \delta\sqrt{1/40}$ for some $\delta \in \{-1, 1\}$. But then

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_1, e_2, e_3, e_4) = \tfrac{12}{5}b_7 - 3,$$

i.e., $b_7 = 5/4$. Hence,

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_1, e_4, e_5, e_6) = \tfrac{z}{2} - \delta\tfrac{\sqrt{10}}{5}b_3,$$
$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)\,(e_2, e_3, e_4, e_5) = -\tfrac{\delta}{8}(3\sqrt{10}b_5 - 2\delta z),$$

and so $b_3 = \frac{5\delta}{2\sqrt{10}}z$, $b_5 = \frac{2\delta}{3\sqrt{10}}z$. However,

$$0 = (\omega - f.\nu)(e_1, e_4) = \tfrac{31}{24}z^2,$$

i.e., $z = 0$, and so

$$0 = (\omega - f.\nu)(e_1, e_3) = 2,$$

a contradiction.

Hence, we must have $b_2 \neq 0$, $x + y \neq 0$, $4b_7 + 1 \neq 0$ and $A = -\frac{2a_1(4b_7+1)(x+y)}{b_2}$. But then (6.4) gives us $A = -(4b_7 + 1)$. Thus,

$$\tfrac{2a_1(4b_7+1)(x+y)}{b_2} = -A = 4b_7 + 1$$

and so, since $4b_7 + 1 \neq 0$,

$$a_1 = \tfrac{b_2}{2(x+y)}\,. \qquad \blacksquare$$

This allows us now to prove:

Theorem 6.6. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_9$. Then $\mathfrak{g}$ does not admit a closed G_2-eigenform.*

Proof. We assume that the parameters fulfill all the conditions that we derived in all the previous lemmas. Then we first get

$$0 = (\omega - f.\nu)(e_2, e_3) = -\tfrac{(4x-6y)b_5-4yb_3-(4b_7+1)z}{4},$$

i.e.,

$$z = \tfrac{(4x-6y)b_5-4yb_3}{4b_7+1}$$

since $4b_7 + 1 \neq 0$ by Lemma 6.5. Then one computes

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_2, e_3, e_6) = -\tfrac{b_2(6xy+6y^2-1)+4(x+y)}{4(x+y)},$$

i.e., $b_2(6xy + 6y^2 - 1) = -4(x + y)$. Since $x + y \neq 0$, also $6xy + 6y^2 - 1 \neq 0$, and so

$$b_2 = -\tfrac{4(x+y)}{6xy+6y^2-1}.$$

Moreover,

$$\begin{aligned}0 &= (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_2, e_4, e_5, e_6)\\ &= \tfrac{x+y}{2}\left(b_7(12x^2 - 40xy + 12y^2 + 4) + 1 - (x + y)^2\right),\\ 0 &= (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_3, e_5, e_6) = -\tfrac{b_2}{2(6xy+6y^2-1)}(2x^2-14xy+24y^2-1),\end{aligned}$$

that is,

$$2x^2 - 14xy + 24y^2 - 1 = 0, \quad b_7(12x^2 - 40xy + 12y^2 + 4) = (x + y)^2 - 1,$$

since $b_2 \neq 0$, $x + y \neq 0$ by Lemma 6.5.

We show now that $12x^2 - 40xy + 12y^2 + 4 \neq 0$.

If this is not the true, then $(x + y)^2 = 1$, i.e., $y = \delta - x$ for some $\delta \in \{-1, 1\}$. But then $0 = 12x^2 - 40xy + 12y^2 + 4 = 16\,(2x - \delta)^2$, i.e., $x = \delta/2 = y$ and so $2x^2 - 14xy + 24y^2 - 1 = 2 \neq 0$, a contradiction.

Thus, $12x^2 - 40xy + 12y^2 + 4 \neq 0$ and we have

$$b_7 = \tfrac{(x+y)^2-1}{12x^2-40xy+12y^2+4}.$$

One then computes

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_2, e_3, e_5, e_6) = \tfrac{(b_5+2b_3)(x-4y)}{2}.$$

Hence, $b_5 = -2b_3$ or $x = 4y$. However, $x = 4y$ is impossible since then

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_3, e_5, e_6) = \tfrac{2}{30y^2-1},$$

a contradiction. Thus $b_5 = -2b_3$ and one gets

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_2, e_4, e_6) = -\tfrac{2b_3(x-y)^2(x+2y)}{3x^2-10xy+3y^2+1}$$

and

$$0 \neq \omega^3 = \tfrac{12(x-y)^2}{(6xy+6y^2-1)\cdot(3x^2-10xy+3y^2+1)}\, e^{123456}.$$

Thus, $x - y \neq 0$ and so $b_3(x + 2y) = 0$. We show that $b_3 \neq 0$ and, consequently, $x = -2y$.

If $b_3 = 0$, then $b_5 = 0$ as well and we do get

$$\begin{aligned}0 &= (\omega - f.\nu)(e_1, e_3) = \tfrac{12x^2-12y^2-2}{6xy+6y^2-1},\\ 0 &= (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_3, e_5, e_6) = \tfrac{-4x^2+28xy-48y^2+2}{6xy+6y^2-1}.\end{aligned}$$

One easily checks that all solutions of

$$-12x^2 + 12y^2 + 2 = 0, \quad -4x^2 + 28xy - 48y^2 + 2 = 0$$

are given by

$$(x, y) = \delta_1 \tfrac{\sqrt{30}}{15} \left(\tfrac{3}{2}, 1\right), \quad (x, y) = \tfrac{\delta_2}{12}(5, -1)$$

for $\delta_1, \delta_2 \in \{-1, 1\}$. However, in the first case, one computes

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_2, e_3, e_4) = -\tfrac{98}{27}$$

and in the second case one obtains

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_2, e_3, e_4) = -\tfrac{365}{119},$$

and so a contradiction in both cases.

Hence, $b_3 \neq 0$ and so $x = -2y$. But then

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_3, e_5, e_6) = \tfrac{120y^2-2}{6y^2+1},$$

i.e., $y = \delta\sqrt{1/60}$ for some $\delta \in \{-1, 1\}$ and we finally obtain

$$0 = (f.\omega \wedge \omega - d\hat{\rho} - \nu \wedge \omega)(e_1, e_2, e_3, e_4) = -\tfrac{686}{209},$$

a contradiction.

Hence, $\mathfrak{g}$ does not admit a closed G_2-eigenform. ■

6.2. The case $\mathfrak{n}_{28}$

In this subsection, we are considering exact G_2-structures and closed G_2-eigenforms on seven-dimensional almost nilpotent Lie algebras with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$. We will determine all such Lie algebras which admit an exact G_2-structure and we will show that no such Lie algebra can admit a closed G_2-eigenform.

First of all, note that $\mathfrak{n}_{28}$ is a well-known real six-dimensional nilpotent Lie algebra, namely the one underlying the complex three-dimensional Heisenberg Lie algebra, and the Iwasawa manifold, see e.g. [24]. Moreover, for this subsection, denote by J_0 the almost complex structure on $\mathfrak{n}_{28}$ uniquely defined by $J_0 e_{2i-1} = e_{2i}$ for $i = 1, 2$ and $J_0 e_5 = -e_6$.

The Lie algebra of all derivations of $\mathfrak{n}_{28}$ is given by

$$\mathrm{Der}(\mathfrak{n}_{28}) = \left\{ \begin{pmatrix} A & 0 \\ B & \mathrm{tr}_{\mathbb{C}} A \end{pmatrix} \,\middle|\, A \in \mathbb{C}^{2\times 2},\ B \in \mathbb{R}^{2\times 4} \right\}, \tag{6.5}$$

with respect to the basis $(e_1, \ldots, e_6)$, where we consider $A \in \mathbb{C}^{2\times 2}$ as a real 4×4-matrix and $\mathrm{tr}_{\mathbb{C}} A \in \mathbb{C}$ as a real 2×2-matrix. The Lie group $\mathrm{Inn}(\mathfrak{n}_{28})$ of inner automorphism of $\mathfrak{n}_{28}$ is given by the Lie group generated by the exponentials of elements in $\mathrm{Der}(\mathfrak{n}_{28})$, and so equals

$$\mathrm{Inn}(\mathfrak{n}_{28}) = \left\{ \begin{pmatrix} C & 0 \\ D & \det_{\mathbb{C}}(C) \end{pmatrix} \,\middle|\, C \in \mathrm{GL}(2, \mathbb{C}),\ D \in \mathbb{R}^{2\times 4} \right\}. \tag{6.6}$$

Now split $\mathfrak{n}_{28} = V \oplus W$ with $V := \mathrm{span}(e_1, e_2, e_3, e_4)$ and $W := \mathrm{span}(e_5, e_6)$ and do the same for the dual space $\mathfrak{n}_{28}^* = V^* \oplus W^*$. Let an SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{n}_{28}^* \times \Lambda^3\mathfrak{n}_{28}^*$ with exact ρ be given, i.e., there exists $\nu \in \Lambda^2\mathfrak{n}_{28}^*$ with $d\nu = \rho$. Write

$$\omega = ae^{56} + e^5 \wedge \alpha_1 + e^6 \wedge \alpha_2 + \tilde{\omega}$$

for $a \in \mathbb{R}$, $\alpha_1, \alpha_2 \in V^*$ and $\tilde{\omega} \in \Lambda^2 V^*$ and, similarly,

$$\nu = be^{56} + e^5 \wedge \beta_1 + e^6 \wedge \beta_2 + \tilde{\nu}$$

for $b \in \mathbb{R}$, $\beta_1, \beta_2 \in V^*$ and $\tilde{\nu} \in \Lambda^2 V^*$. Moreover, let $\sigma_2 := de^5$ and $\sigma_3 := de^6$, and note that $\sigma_2, \sigma_3 \in [[\Lambda^{2,0} V^*]]$ with respect to the almost complex structure J_0 on V. Set

$$\rho_0 := e^6 \wedge \sigma_2 - e^5 \wedge \sigma_3.$$

Then ρ_0 induces the complex structure J_0 (if one chooses the right orientation on $\mathfrak{n}_{28}$) and we have

$$\rho = d\nu = b\rho_0 + \sigma_2 \wedge \beta_1 + \sigma_3 \wedge \beta_2.$$

This shows that $b \neq 0$ as otherwise $\rho(e_5, \cdot, \cdot) = 0$, contradicting Lemma 2.5 (a). Moreover, $\rho(e_5, e_6, \cdot) = 0$, i.e., e_5 and e_6 are $J := J_\rho$-linearly dependent by Lemma 2.5 (a). But so

$$0 \neq g(e_5, e_5) = \omega(Je_5, e_5) = ae^{56}(Je_5, e_5),$$

implies $a \neq 0$ and we may apply an inner automorphism F as in (6.6) with $C = I_2$ and suitable $D \in \mathbb{R}^{2\times 4}$ to get rid of α_1 and α_2 in ω, i.e., we may assume that $\omega = ae^{56} + \tilde{\omega}$ with $\tilde{\omega} \neq 0$ due to the non-degeneracy of ω. Now we must have

$$0 = \omega \wedge \rho = ae^{56} \wedge (\sigma_2 \wedge \beta_1 + \sigma_3 \wedge \beta_2) + be^6 \wedge \sigma_2 \wedge \tilde{\omega} - be^5 \wedge \sigma_3 \wedge \tilde{\omega}$$

Thus, $\sigma_2 \wedge \beta_1 + \sigma_3 \wedge \beta_2 = 0$, and so $\rho = b\rho_0$, and $\sigma_i \wedge \tilde{\omega} = 0$ for $i = 2, 3$. Now σ_1, σ_2 span $[[\Lambda^{2,0} V^*]]$ and so the latter identity shows $\tilde{\omega} \in [\Lambda^{1,1} V^*]$. A straightforward computation shows that $\sigma_3 \wedge \beta_2 = -\sigma_2 \wedge \beta_1$ implies $\beta_2 = J_0^* \beta_1$.

Next, $(\sigma_1, J_0|_V)$, $\sigma_1 := e^{12} + e^{34}$, defines an almost Hermitian structure on V and SU(2) preserves σ_1 and acts as SO(3) on $[\Lambda_0^{1,1} V^*]$. Since matrices of the block-diagonal form $\mathrm{diag}(A, I_2)$ with $A \in \mathrm{SU}(2)$ are in $\mathrm{Inn}(\mathfrak{n}_{28})$ and preserve ρ_0, we may thus assume that $\tilde{\omega} = a_1e^{12} + a_2e^{34}$. Moreover, an element of the form $\mathrm{diag}(b_1, b_1, b_2, b_2, b_1b_2, b_1b_2)$ is in $\mathrm{Inn}(\mathfrak{n}_{28})$ and only scales ρ_0, and so we may even assume that $|a_1| = |a_2| = |a|$, i.e., there are $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ such that $\tilde{\omega} = \varepsilon_1(a\varepsilon_2 e^{12} + e^{34})$. Since $J = \pm J_0$, we do get

$$\varepsilon_2 a^2 = \omega(e_1, e_2)\,\omega(e_3, e_4) = \omega(Je_2, e_2)\,\omega(Je_4, e_4) = g(e_2, e_2)\,g(e_4, e_4) > 0,$$

and so $\varepsilon_2 = 1$, i.e., $\omega = a\varepsilon_1(e^{12} + e^{34}) + ae^{56}$. Moreover,

$$\varepsilon_1 a^2 = \omega(e_1, e_2)\,\omega(e_5, e_6) = -\omega(Je_2, e_2)\,\omega(Je_6, e_6) = -g(e_2, e_2)\,g(e_6, e_6) < 0,$$

i.e., $\varepsilon_1 = -1$ and, consequently, $\omega = a(-e^{12} + e^{34}) + e^{56}) =: a\omega_0$. Noting that for $a > 0$, the ordered basis $(e_1, -e_2, e_3, -e_4, e_5, -e_6)$ is oriented and so $J = J_0$, and otherwise $J = -J_0$, the normalisation condition reads $|a|^3 = b^2$. Hence, we may write $a = \varepsilon\lambda^2$ and $b = \lambda^3$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ and some $\varepsilon \in \{-1, 1\}$.

Finally, we may use block-diagonal matrices $\mathrm{diag}(A, I_2)$ with $A \in \mathrm{SU}(2)$ to bring $\tilde{\nu}$ into a canonical form. For this, note that $\mathrm{diag}(A, I_2)$ is in $\mathrm{Inn}(\mathfrak{n}_{28})$ and preserves (ω, ρ) and so we may assume that $\tilde{\nu} = c_1 e^{12} + c_2 e^{34} + c_3\sigma_2 + c_4\sigma_3$ for certain $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Thus,

$$\nu = \lambda^3 e^{56} + e^5 \wedge \beta + e^6 \wedge J^*\beta + c_1 e^{12} + c_2 e^{34} + c_3\sigma_2 + c_4\sigma_3$$

for $\beta := \beta_1 \in V^*$. If $\nu \in [\Lambda_0^{1,1}\mathfrak{n}_{28}^*]$, then $c_3 = c_4 = 0$ and $c_2 = \lambda^3 - c_1$. Summarizing, we have arrived at:

Lemma 6.7. *Let $(\omega, \rho) \in \Lambda^2\mathfrak{n}_{28}^* \times \Lambda^3\mathfrak{n}_{28}^*$ be an* SU(3)*-structure for which there exists $\nu \in \Lambda^2\mathfrak{n}_{28}^*$ with $\rho = d\nu$. Then (ω, ρ, ν), are, up to automorphism, given by*

$$\begin{cases} \omega = \varepsilon\lambda^2\omega_0 = \varepsilon\lambda^2(-e^{12} - e^{34} + e^{56}), \\ \rho = \lambda^3\rho_0 = \lambda^3(e^{136} - e^{246} - e^{145} - e^{235}), \\ \nu = \lambda^3 e^{56} + e^5 \wedge \beta + e^6 \wedge J^*\beta + c_1 e^{12} + c_2 e^{34} + c_3\sigma_2 + c_4\sigma_3, \end{cases} \tag{6.7}$$

for certain $c_1, c_2, c_3, c_4 \in \mathbb{R}, \lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in \{-1, 1\}$ and $\beta \in V^*$.

If $\nu \in [\Lambda_0^{1,1}\mathfrak{n}_{28}^*]$, then, up to an automorphism, (ω, ρ) take the form as in (6.7) and

$$\nu = \lambda^3 e^{56} + e^5 \wedge \beta + e^6 \wedge J_0^*\beta + c e^{12} + (\lambda^3 - c) e^{34} \tag{6.8}$$

for some $c \in \mathbb{R}$ and $\beta \in \mathfrak{n}_{28}^*$.

Next, we determine those seven-dimensional almost nilpotent Lie algebras $\mathfrak{g}$ with codimension-one nilpotent ideal $\mathfrak{n}_{28}$ which admit an exact G_2-structure. First of all, we get some restriction on f if $\mathfrak{g}$ admits an exact G_2-structure, i.e., if (3.5) is valid:

Lemma 6.8. *Let $(\omega, \rho) \in \Lambda^2\mathfrak{n}_{28}^* \times \Lambda^3\mathfrak{n}_{28}^*$ be as in (6.7) and assume that $\nu \in \Lambda^2\mathfrak{n}_{28}^*$ satisfies $d\nu = \rho$. Then $\alpha \in \mathfrak{n}_{28}^*$ and $f \in \mathrm{Der}(\mathfrak{n}_{28})$ fulfill (3.5) if and only if*

$$f.\nu^{1,1} = \omega, \quad f.\nu^{2,0} = d\alpha, \quad [f, J_0] = 0,$$

where $\nu^{1,1}$ is the $(1,1)$-part and $\nu^{2,0}$ is the $(2,0) + (0,2)$-part of ν. If this is the case, then no eigenvalue of f is purely imaginary.

Proof. We decompose $f = f_1 + f_2$ into its J_0-invariant part f_1 and its J_0-anti-invariant part f_2. Then f_1 preserves the splitting $\Lambda^2\mathfrak{n}_{28}^* = [\Lambda^{1,1}\mathfrak{n}_{28}^*] \oplus [[\Lambda^{2,0}\mathfrak{n}_{28}^*]]$, while f_2 interchanges the two summands. As $d\alpha$ is of type $(2,0) + (0,2)$, (3.5) is equivalent to

$$f_1.\nu^{1,1} + f_2.\nu^{2,0} = \omega, \quad f_1.\nu^{2,0} + f_2.\nu^{1,1} = d\alpha.$$

if we decompose $\nu = \nu^{1,1} + \nu^{2,0}$ as in the statement. Note that f_2 is a strictly lower triangular block matrix with respect to the splitting $\mathfrak{n}_{28} = V \oplus W$, while f_1 is a lower triangular block matrix with respect to the same splitting. Moreover, $\nu^{2,0}$, $d\alpha \in \Lambda^2 V^*$ and $\nu^{1,1}$ has a non-trivial $\Lambda^2 W^*$-part. Thus, $f_1.\nu^{2,0} \in \Lambda^2 V^*$ and $f_2.\nu^{1,1} \in W^* \wedge V^* \oplus \Lambda^2 V^*$ with non-trivial $W^* \wedge V^*$-part if $f_2 \neq 0$. Hence, $f_2 = 0$, i.e., $f = f_1$, and so $[f, J_0] = 0$, and the above equations simplify to

$$f.\nu^{1,1} = \omega, \quad f.\nu^{2,0} = d\alpha$$

as stated.

Finally, assume that there is an eigenvector $X \in \mathfrak{n}_{28} \setminus \{0\}$ of f with purely imaginary eigenvalue ic, $c \in \mathbb{R}$. Then $f(X) = cJ_0X$, and so $f(J_0X) = J_0 f(X) = -cX$, and we get

$$0 \neq g(X,X) = \omega(J_0X,X) = f.\nu^{1,1}(J_0X,X) = -\nu^{1,1}(f(J_0X),X) - \nu^{1,1}(J_0X, f(X)) \\ = c\nu^{1,1}(X,X) - c\nu^{1,1}(J_0X,J_0X) = 0$$

a contradiction. Hence, no eigenvalue of f can be purely imaginary. ∎

We are now in the position to give a classification of those almost nilpotent Lie algebras with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$ which admit an exact G_2-structure:

Theorem 6.9. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal $\mathfrak{n}_{28}$, that is, $\mathfrak{g} \cong \mathfrak{n}_{28} \rtimes_f \mathbb{R}$ for some $f \in \mathrm{Der}(\mathfrak{n}_{28})$. Then $\mathfrak{g}$ admits an exact G_2-structure if and only if f has no purely imaginary eigenvalues. Equivalently, $\mathfrak{g}$ admits an exact G_2-structure if and only if $\mathfrak{g} \cong \mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ for certain $a \in [-1/4, \infty) \setminus \{0\}$, $b_1, b_2 \in \mathbb{R}$ or $\mathfrak{g} \cong \mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$ for some $b \in \mathbb{R}$, where*

$$f_{a,b_1,b_2} := \begin{pmatrix} a+ib_1 & & \\ & -\frac{1}{2}-a+ib_2 & \\ & & -\frac{1}{2}+i(b_1+b_2) \end{pmatrix}, \quad h_b := \begin{pmatrix} -\frac{1}{4}+ib & 1 & \\ & -\frac{1}{4}+ib & \\ & & -\frac{1}{2}+2ib \end{pmatrix}.$$

Proof. We note that the forward implication in the first statement is incorporated in Lemma 6.8.

So let $\mathfrak{g} \cong \mathfrak{n}_{28} \rtimes_f \mathbb{R}$ for some $f \in \mathrm{Der}(\mathfrak{n}_{28})$ which has no purely imaginary eigenvalues. By (6.5), we know that

$$f = \begin{pmatrix} A & 0 \\ B & \mathrm{tr}_{\mathbb{C}} A \end{pmatrix}$$

for some $A \in \mathbb{C}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 4}$. If we conjugate f with an automorphism F of $\mathfrak{n}_{28}$ as in (6.6) with $C = I_2$, we surely get a Lie algebra $\mathfrak{n}_{28} \rtimes_{FfF^{-1}} \mathbb{R}$ which is isomorphic to $\mathfrak{g}$, where

$$FfF^{-1} = \begin{pmatrix} A & 0 \\ B + D(A - (\mathrm{tr}_{\mathbb{C}} A)I_2) & \mathrm{tr}_{\mathbb{C}} A \end{pmatrix}.$$

Thus, FfF^{-1} is block-diagonal if $A - (\mathrm{tr}_{\mathbb{C}} A)I_2$ is invertible, i.e., if $\mathrm{tr}_{\mathbb{C}} A$ is not a complex eigenvalue of the complex matrix A. However, if $\mathrm{tr}_{\mathbb{C}} A$ would be an eigenvalue of the complex matrix A, then the other complex eigenvalue would have to be zero and so the real matrix f would have one eigenvalue equal to zero, which is excluded since f has no purely imaginary eigenvalues.

Thus, calling FfF^{-1} again f, we may assume that $f = \mathrm{diag}(A, \mathrm{tr}_{\mathbb{C}} A)$. But then we may use an automorphism of $\mathfrak{n}_{28}$ as in (6.6) to bring A into complex Jordan normal form. Hence, we may assume that either $A = \mathrm{diag}(w_1, w_2)$ for $w_1, w_2 \in \mathbb{C}$ with $\mathrm{Re}(w_1) \neq 0$, $\mathrm{Re}(w_2) \neq 0$, $\mathrm{Re}(w_1 + w_2) = \mathrm{tr}_{\mathbb{C}} A \neq 0$ or

$$A = \begin{pmatrix} w & 1 \\ 0 & w \end{pmatrix}$$

for some $w \in \mathbb{C}$ with $\mathrm{Re}(w) \neq 0$. We provide now in both cases an example of an SU(3)-structure $(\omega, \rho) \in \Lambda^2\mathfrak{n}_{28}^* \times \Lambda^3\mathfrak{n}_{28}^*$, a two-form $\nu \in \Lambda^2\mathfrak{n}_{28}^*$, and a one-form $\alpha \in \mathfrak{n}_{28}^*$ such that $\rho = d\nu$ and such that (3.5) is fullfilled, where the latter equation is valid by Lemma 6.8 if and only if $f.\nu^{1,1} = \omega$, $f.\nu^{2,0} = d\alpha$. We will always choose $\alpha = 0$ and a $(1,1)$-form ν, so that the second equation is automatically fulfilled and we only have to deal with the first one.

In the first case, one checks by a straightforward computation that

$$\omega = \lambda^2\omega_0, \quad \rho = \lambda^3\rho_0, \quad \nu = \frac{\lambda^2}{2\,\mathrm{Re}(w_1)}e^{12} + \frac{\lambda^2}{2\,\mathrm{Re}(w_2)}e^{34} + \lambda^3 e^{56}$$

with $\lambda := -\frac{1}{2\,\mathrm{Re}(w_1+w_2)}$ fulfills all necessary equations, whereas in the second case

$$\omega = \lambda^2\omega_0, \quad \rho = \lambda^3\rho_0, \quad \nu = -2\lambda^3e^{12} - (16\lambda^5 + 2\lambda^3)e^{34} - 4\lambda^4\cdot(e^{14} - e^{23}) + \lambda^3e^{56}$$

with $\lambda := -\frac{1}{4\mathrm{Re}(w)}$ does the job.

The second statement in the assertion follows immediately from the considerations above by noting that rescaling f by a non-zero scalar gives an isomorphic Lie algebra and by noting that we may order the real parts of the eigenvalues of A in such a way that the first one is greater or equal to the second one. ∎

Remark 6.10. Note that f_{a,b_1,b_2} and h_b in Theorem 6.9 both fix e^{56} and so one easily sees that (ω, ρ) as in Lemma 6.7 with $\lambda = 1$, i.e., $\omega = -e^{12} - e^{34} + e^{56}$, $\rho = e^{136} - e^{246} - e^{145} - e^{235}$, give rise to an exact G_2-structure on $\mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ and $\mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$, respectively. This explains the strange 'normalisation' of the endomorphisms in Theorem 6.9.

Looking for exact G_2-structures of special torsion, we obtain:

Theorem 6.11. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$ which admits an exact G_2-structure. Then*

(a) *$\mathfrak{g}$ admits an exact G_2-structure with special torsion of negative type,*

(b) *$\mathfrak{g}$ admits an exact G_2-structure with special torsion of positive type if and only if $\mathfrak{g} \not\cong \mathfrak{n}_{28} \rtimes_{f_{-1/4,b,b}} \mathbb{R}$ for all $b \in \mathbb{R}$.*

Proof. By Theorem 6.9, we may assume that $\mathfrak{g} = \mathfrak{n}_{28} \rtimes_f \mathbb{R}$ with either $f = f_{a,b_1,b_2}$ for certain $a \in [-1/4, \infty)$, $b_1, b_2 \in \mathbb{R}$ or $f = h_b$ for some $b \in \mathbb{R}$. We divide the proof into four different parts.

Part (I). We first show that $\mathfrak{g} = \mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ for $a, b_1, b_2 \in \big([-1/4, 1/2] \setminus \{0\}\big) \times \mathbb{R}^2$ with $a \neq -1/4$, or $b_1 \neq b_2$ and $\mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$ for $b \in \mathbb{R}$ admits an exact G_2-structure with special torsion of positive type.

For this, note that under the assumptions on a, b_1, b_2, the Lie algebra $\mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ is isomorphic to $\mathfrak{n}_{28} \rtimes_g \mathbb{R}$ with

$$g = g_{a,b_1,b_2,c} = \begin{pmatrix} a+ib_1 & c & \\ & -\frac{1}{2}-a+ib_2 & \\ & & -\frac{1}{2}+i(b_1+b_2) \end{pmatrix}$$

If $a = -1/4$ and $b_1 = b_2 =: b$, then, for any $c \in \mathbb{R} \setminus \{0\}$, we have $\mathfrak{n}_{28} \rtimes_{g_{-1/4,b,b,c}} \mathbb{R} \cong \mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$.

So we are looking for exact G_2-structures with special torsion of positive type on $\mathfrak{n}_{28} \rtimes_g \mathbb{R}$. For this, note that $a \neq 0$ and $2a+1 \neq 0$ by assumption. Thus,

$$\nu = \frac{1}{2a}e^{12} + \frac{2c^2-4a(b_1-b_2)^2+1)}{a(4(b_1-b_2)^2+1)(2a+1)}e^{34} + e^{56} - \frac{2(b_1-b_2)c}{a(4(b_1-b_2)^2+1)}(e^{13}+e^{24}) + \frac{c}{a(4(b_1-b_2)^2+1)}(e^{14}-e^{23}) \in \Lambda^2\mathfrak{n}_{28}^*$$

is well-defined and one checks that $d\nu = \rho_0 = e^{136} - e^{246} - e^{145} - e^{235}$ and $g.\nu = \omega_0 = -e^{12} - e^{34} + e^{56}$. Thus, the pair (ω, ρ) gives rise to an exact G_2-structure φ. As $\hat{\rho} = e^{135} - e^{146} - e^{236} - e^{245}$, we get

$$\begin{aligned} d\star_\varphi \varphi &= d\left(\tfrac{1}{2}\omega^2 + e^7 \wedge \hat{\rho}\right) = e^7 \wedge g.\left(\tfrac{1}{2}\omega^2\right) - e^7 \wedge d_{\mathfrak{n}_{28}}\hat{\rho} \\ &= e^7 \wedge \left(-3e^{1234} + (2a-1)e^{1256} - (2+2a)e^{3456} + c(e^{1456} - e^{2356})\right) \end{aligned}$$

due to $d(\omega^2) = 0$. Hence, the torsion two-form τ is given by

$$\tau = -\star_\varphi d \star_\varphi \varphi = (2+2a)e^{12} - (2a-1)e^{34} + c(e^{14} - e^{23}) + 3e^{56}.$$

Now the exact G_2-structure φ has special torsion of positive type if and only if $\tau^3 = 0$, which is equivalent to $((2+2a)e^{12} - (2a-1)e^{34} + c(e^{14} - e^{23}))^2 = 0$, and so to

$$0 = (2+2a)(2a-1) + c^2 = 4a^2 + 2a - 2 + c^2.$$

Here, a is fixed and we are searching for a solution of this equation for c, which is possible if $4a^2 + 2a - 2 \leq 0$, i.e., if $a \in [-1/4, 1/2]$. Note that for $a = -1/4$, we have $c = \pm 3/2 \neq 0$, and so $\mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$ admits an exact G_2-structure with special torsion of positive type for any $b \in \mathbb{R}$.

Part (II). We show that (ω_0, ρ_0) defines also an exact G_2-structure on $\mathfrak{n}_{28} \rtimes_{g_{-1/4,b,b,c}} \mathbb{R} \cong \mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$ with special torsion of negative type for a suitable chosen $c \in \mathbb{R} \setminus \{0\}$.

The computations in (I) show that $\tau = \frac{3}{2}(e^{12} + e^{34}) + c(e^{14} - e^{23}) + 3e^{56}$. Hence, φ has special torsion of negative type if and only if $\frac{2}{3}|\tau|_\varphi^6 = |\tau^3|_\varphi^2$, which here is equivalent to

$$\tfrac{2}{3}\left(\tfrac{27}{2} + 2c^2\right)^3 = \left(18\left(\tfrac{9}{4} - c^2\right)\right)^2 \quad \Longleftrightarrow \quad c^2\left(\tfrac{16}{3}c^4 - 216c^2 + 2187\right) = 0,$$

i.e., to $c = 0$ or $c = \pm 9/2$. Thus, for $c = 9/2$, we get an exact G_2-structure with special torsion of negative type on $\mathfrak{n}_{28} \rtimes_{g_{-1/4,b,b,9/2}} \mathbb{R} \cong \mathfrak{n}_{28} \rtimes_{h_b} \mathbb{R}$.

Part (III). Next, we show that $\mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ admits an exact G_2-structure with special torsion of positive type if $(a, b_1, b_2) \in (1/2, \infty) \times \mathbb{R}^2$ and that it admits an exact G_2-structure with special torsion of negative type for any possible values of (a, b_1, b_2), i.e., for any $(a, b_1, b_2) \in ([1/4, \infty) \setminus \{0\}) \times \mathbb{R}^2$.

For this, we note that $\mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ is isomorphic to $\mathfrak{n}_{28} \rtimes_h \mathbb{R}$ with

$$h := h_{a,b_1,b_2,r} := \begin{pmatrix} a & -b_1 & & & & \\ b_1 & a & & & & \\ & & -\frac{1}{2}-a & -b_2 & & \\ & & b_2 & -\frac{1}{2}-a & & \\ & & r & & -\frac{1}{2} & -(b_1+b_2) \\ & & & -r & b_1+b_2 & -\frac{1}{2} \end{pmatrix}$$

for any $r \in \mathbb{R}$. The minus sign occurring before one of the rs is due to $(e_1, -e_2, e_3, -e_4, e_6, -e_5)$ being a complex basis, i.e., due to the shift in the order of e_5 and e_6.

We have $d\nu = \rho_0$ and $h.\nu = \omega_0$ for

$$\nu = \frac{1}{2a}e^{12} - \frac{(b_1+2b_2)^2 + (2r^2+a+1)(a+1)}{(2a+1)((a+1)^2+(b_1+2b_2)^2)}e^{34} + e^{56} + \frac{r(b_1+2b_2)}{(a+1)^2+(b_1+2b_2)^2}(e^{35}-e^{46}) + \frac{r(1+a)}{(a+1)^2+(b_1+2b_2)^2}(e^{36}+e^{45}).$$

Hence, the pair (ω_0, ρ_0) defines an exact G_2-structure φ on $\mathfrak{n}_{28} \rtimes_h \mathbb{R}$ for any value of $r \in \mathbb{R}$. Moreover, we have

$$\begin{aligned} d \star_\varphi \varphi &= e^7 \wedge \left(h.\left(\tfrac{1}{2}\omega^2\right) - d_{\mathfrak{n}_{28}}\hat{\rho}\right) \\ &= e^7 \wedge \left(-3e^{1234} + (2a-1)e^{1256} - (2+2a)e^{3456} + r\,(e^{1236}+e^{1245})\right) \end{aligned}$$

and so the torsion two-form τ is given by

$$\tau = -\star_\varphi d \star_\varphi \varphi = (2+2a)e^{12} - (2a-1)e^{34} - r\,(e^{36}+e^{45}) + 3e^{56}.$$

Hence,

$$\tau^3 = -12(1+a)(6a-3-r^2)e^{123456}$$

and $\tau^3 = 0$, i.e., φ has special torsion of positive type, if and only if $r^2 = 6a - 3$. But we assumed $a > 1/2$ and so have $6a - 3 > 0$ and, consequently, φ has special torsion of positive type for $r = \sqrt{6a-3} \in \mathbb{R}$.

Moreover, φ has special torsion of negative type if and only if

$$\tfrac{2}{3}((2+2a)^2 + (2a-1)^2 + 2r^2 + 9)^3 = |\tau|_\varphi^6 = |\tau^3|_\varphi^2 = \left(12(1+a)(6a-3-r^2)\right)^2.$$

Bringing both terms on one side and factorising gives

$$\tfrac{16}{3}\,((a-2)^2 + r^2)\cdot(8a^2 + 22a + 5 - r^2)^2 = 0.$$

So one may find some $r \in \mathbb{R}$ such that φ has special torsion of negative type if $(2a+5)(4a+1) = 8a^2 + 22a + 5 \geq 0$. But this is the case if $a \geq -1/4$ and so $\mathfrak{n}_{28} \rtimes_{f_{a,b_1,b_2}} \mathbb{R}$ admits an exact G_2-structure with special torsion of negative type for any possible values of (a, b_1, b_2).

Part (IV). Finally, we need to show that for any $b \in \mathbb{R}$, the Lie algebra $\mathfrak{n}_{28} \rtimes_{f_{-1/4,b,b}} \mathbb{R}$ does not admit an exact G_2-structure with special torsion of positive type.

For this, let (ω, ρ) be a half-flat SU(3)-structure which determines an exact G_2-structure φ on $\mathfrak{n}_{28} \rtimes_{f_{-1/4,b,b}} \mathbb{R}$ and let $\nu \in \Lambda^2\mathfrak{n}_{28}^*$ and $\alpha \in \mathfrak{n}_{28}$ be such that (3.5) holds. By Lemma 6.7, we may assume that

$$\omega = \varepsilon\lambda^2\omega_0 = \varepsilon\lambda^2(-e^{12} - e^{34} + e^{56}), \quad \rho = \lambda^3\rho_0 = \lambda^3(e^{136} - e^{246} - e^{145} - e^{235})$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$, up to an automorphism F of $\mathfrak{n}_{28}$, i.e., (ω, ρ) are of this form on $\mathfrak{n}_{28} \rtimes_{Ff_{-1/4,b,b}F^{-1}} \mathbb{R}$. Now one computes that $f := Ff_{-1/4,b,b}F^{-1}$ is of the form

$$f = \begin{pmatrix} \left(-\frac{1}{4} + ib\right) I_2 & \\ B & -\frac{1}{2} + 2ib \end{pmatrix}$$

for some $B \in \mathbb{R}^{2\times 4}$. By Lemma 6.8, we have $[f, J_0] = 0$, which amounts to B being of the form

$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & a_4 & -a_3 \end{pmatrix}$$

for certain $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Moreover, by Lemma 6.8, we have $f.\nu^{1,1} = \omega$ and

$$\nu^{1,1} = \lambda^3 e^{56} + e^5 \wedge \beta + e^6 \wedge J^*\beta + c_1 e^{12} + c_2 e^{34}$$

for certain $\beta \in \mathrm{span}(e^1, e^2, e^3, e^4)$ and $c_1, c_2 \in \mathbb{R}$ by Lemma 6.7. Thus,

$$\varepsilon\lambda^2(-e^{12} - e^{34} + e^{56}) = \omega = f.\nu^{1,1} = \lambda^3 e^{56} + e^5 \wedge \gamma + e^6 \wedge J^*\gamma + \tfrac{c_1}{2} e^{12} + \tfrac{c_2}{2} e^{34}$$

for some $\gamma \in \mathrm{span}(e^1, e^2, e^3, e^4)$, which implies, in particular, $\lambda = \varepsilon$, i.e., $\omega = \varepsilon\omega_0$ and $\rho = \varepsilon\rho_0$. Since for $\varepsilon = -1$, the induced orientation is the opposite of that for $\varepsilon = 1$, we always have $\hat{\rho} = e^{135} - e^{146} - e^{236} - e^{245}$. Thus, one computes

$$\begin{aligned} d \star_\varphi \varphi &= e^7 \wedge \big(f.\big(\tfrac{1}{2}\omega^2\big) - d_{\mathfrak{n}_{28}}\hat{\rho}\big) \\ &= e^7 \wedge \big(-3e^{1234} - \tfrac{3}{2}e^{1256} - \tfrac{3}{2}e^{3456} + a_1(e^{2345} + e^{1346}) + a_2(e^{2346} - e^{1345}) \\ &\qquad + a_3(e^{1245} + e^{1236}) + a_4(e^{1246} - e^{1235})\big) \end{aligned}$$

independently of ε. Hence,

$$\begin{aligned} \tau &= -\star_\varphi d \star_\varphi \varphi \\ &= \varepsilon\big(\tfrac{3}{2}(e^{12} + e^{34}) + 3e^{56} - a_1(e^{16} + e^{25}) + a_2(e^{15} - e^{26}) - a_3(e^{36} + e^{45}) \\ &\quad + a_4(e^{35} - e^{46})\big), \end{aligned}$$

and so

$$\tau^3 = 9\varepsilon\left(\tfrac{9}{2} + a_1^2 + a_2^2 + a_3^2 + a_4^2\right) e^{123456} \neq 0,$$

i.e., φ does not have special torsion of positive type. ∎

Finally, we show that a Lie algebra of the form $\mathfrak{g} = \mathfrak{n}_{28} \rtimes_f \mathbb{R}$ cannot admit a closed G_2-eigenform:

Theorem 6.12. *Let $\mathfrak{g}$ be a seven-dimensional almost nilpotent Lie algebra with codimension-one nilpotent ideal isomorphic to $\mathfrak{n}_{28}$. Then $\mathfrak{g}$ does not admit a closed G_2-eigenform.*

Proof. We assume the contrary. Then, by Lemma 6.7, there exist $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in \{-1, 1\}$ such that $\omega = \varepsilon\lambda\omega_0$, $\rho = \lambda^3\rho_0$ and $\nu \in [\Lambda_0^{1,1}\mathfrak{n}_{28}^*]$ is as in (6.8), i.e.,

$$\nu = c e^{12} + (\lambda^3 - c) e^{34} + \lambda^3 e^{56} + e^5 \wedge \beta + e^6 \wedge J_0^*\beta$$

for certain $c \in \mathbb{R}$ and $\beta \in V^*$. Moreover, we may assume that

$$f.\nu = \omega, \quad \omega \wedge (f.\omega - \nu) = d\hat{\rho}$$

for some $f \in \mathrm{Der}(\mathfrak{n}_{28})$. Since then also $(-\omega, \rho, \nu, -f)$ fulfills all necessary equations, and so defines a closed G_2-eigenform, we may assume that $\varepsilon = 1$.

But then one computes

$$d\hat{\rho} = \lambda^3\, d\hat{\rho}_0 = 4\lambda^3\, e^{1234} = \lambda^2(-e^{12} - e^{34} + e^{56}) \wedge -2\lambda\,(e^{12} + e^{34} + e^{56}) \\ = \omega \wedge -2\lambda\,(e^{12} + e^{34} + e^{56}).$$

Since wedging with ω is an isomorphism from $\Lambda^2\mathfrak{h}^*$ to $\Lambda^4\mathfrak{h}^*$, the latter equation implies

$$f.\omega - \nu = -2\lambda\,(e^{12} + e^{34} + e^{56})$$

By (6.5), we know that

$$f = \begin{pmatrix} A & 0 \\ B & \mathrm{tr}_{\mathbb{C}} A \end{pmatrix}$$

for some $A = (a_{ij})_{i,j} \in \mathbb{C}^{2\times 2}$ and some $B \in \mathbb{R}^{2\times 4}$. Thus, inserting (e_1, e_2) into the equality $f.\omega - \nu = -2\lambda(e^{12} + e^{34} + e^{56})$ yields

$$2\,\mathrm{Re}(a_{11})\,\lambda^2 - c = -2\,\mathrm{Re}(a_{11})\,\omega(e_1, e_2) - \nu(e_1, e_2) = (f.\omega - \nu)(e_1, e_2) = -2\lambda.$$

Similarly, we obtain

$$2\,\mathrm{Re}(a_{22})\,\lambda^2 - (\lambda^3 - c) = (f.\omega - \nu)(e_3, e_4) = -2\lambda.$$

by inserting (e_3, e_4). Adding these two equations yields

$$2\,\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)\,\lambda^2 - \lambda^3 = -4\lambda. \tag{6.9}$$

Moreover, inserting (e_5, e_6), we do get

$$-2\,\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)\,\lambda^2 - \lambda^3 = (f.\omega - \nu)(e_5, e_6) = -2\lambda.$$

Adding (6.9) to this equation, we obtain $-2\lambda^3 = -6\lambda$, and so, since $\lambda \neq 0$, that $\lambda^2 = 3$.

However, we also get

$$-2\,\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)\,\lambda^3 = -2\,\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)\,\nu(e_5, e_6) = f.\nu(e_5, e_6) = \omega(e_5, e_6) = \lambda^2,$$

i.e., $2\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)\lambda^2 = \lambda$, which, together with (6.9), yields $\lambda - \lambda^3 = -4\lambda$, i.e., $\lambda^2 = 5$, a contradiction.

Thus, $\mathfrak{g}$ does not admit a closed G_2-eigenform. ■

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Marco Freibert
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel,
Heinrich-Hecht-Platz 6, 24118 Kiel, Germany;
freibert@math.uni-kiel.de

Simon Salamon
Mathematics Department, King’s College London,
Strand, London, WC2R 2LS, United Kingdom;
simon.salamon@kcl.ac.uk