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On the core of a low dimensional set-valued mapping

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Abstract. Let $\mathfrak{M} = (\mathcal{M}, \rho)$ be a metric space and let X be a Banach space. Let F be a set-valued mapping from $\mathcal{M}$ into the family $\mathcal{K}_m(X)$ of all compact convex subsets of X of dimension at most m. The main result in our recent joint paper with Charles Fefferman (which is referred to as a "finiteness principle for Lipschitz selections") provides efficient conditions for the existence of a Lipschitz selection of F, i.e., a Lipschitz mapping $f : \mathcal{M} \to X$ such that $f(x) \in F(x)$ for every $x \in \mathcal{M}$.

We give new alternative proofs of this result in two special cases. When $m = 2$, we prove it for $X = \mathbb{R}^2$, and when $m = 1$ we prove it for all choices of X. Both of these proofs make use of a simple reiteration formula for the "core" of a set-valued mapping F, i.e., for a mapping $G : \mathcal{M} \to \mathcal{K}_m(X)$ which is Lipschitz with respect to the Hausdorff distance, and such that $G(x) \subset F(x)$ for all $x \in \mathcal{M}$.

1. Introduction

Let $\mathfrak{M} = (\mathcal{M}, \rho)$ be a *pseudometric space*, i.e., suppose that the "distance function" $\rho : \mathcal{M} \times \mathcal{M} \to [0, +\infty]$ satisfies $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$, and $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ for all $x, y, z \in \mathcal{M}$.

Note that $\rho(x, y) = 0$ may hold with $x \ne y$, and $\rho(x, y)$ may be $+\infty$.

Let $(X, \|\cdot\|)$ be a real Banach space. Given a non-negative integer m, we let $\mathcal{K}_m(X)$ denote the family of all *non-empty compact convex subsets* $K \subset X$ of dimension at most m. (We say that a convex subset of X has dimension at most m if it is contained in an affine subspace of X of dimension at most m.) We let $\mathcal{K}(X) = \bigcup\{\mathcal{K}_m(X) : m = 0, 1, \dots\}$ denote the family of all non-empty compact convex finite-dimensional subsets of X.

By $\mathrm{Lip}(\mathcal{M}, X)$ we denote the space of all Lipschitz mappings from $\mathcal{M}$ to X equipped with the Lipschitz seminorm

$$\|f\|_{\mathrm{Lip}(\mathcal{M},X)} = \inf\big\{\lambda > 0 : \|f(x) - f(y)\| \le \lambda\, \rho(x, y) \text{ for all } x, y \in \mathcal{M}\big\}.$$

In this paper we study the following problem.

Problem 1.1. Suppose that we are given a set-valued mapping F which to each point $x \in \mathcal{M}$ assigns a set $F(x) \in \mathcal{K}_m(X)$. A *selection* of F is a map $f : \mathcal{M} \to X$ such that $f(x) \in F(x)$ for all $x \in \mathcal{M}$.

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We want to know *whether there exists a selection* f *of* F *in the space* $\mathrm{Lip}(\mathcal{M}, X)$. Such an f is called a *Lipschitz selection* of the set-valued mapping $F\colon \mathcal{M} \to \mathcal{K}_m(X)$.

If a Lipschitz selection f exists, then we ask *how small we can take its Lipschitz seminorm*.

The following result provides efficient conditions for the existence of a Lipschitz selection of an arbitrary set-valued mapping from a pseudometric space into the family $\mathcal{K}_m(X)$. We refer to it as a "finiteness principle for Lipschitz selections", or simply as a "finiteness principle".

Theorem 1.2 (Fefferman–Shvartsman, [16]). *Fix* $m \ge 1$. *Let* $(\mathcal{M}, \rho)$ *be a pseudometric space, and let* $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ *for a Banach space* X. *Let*

$$N(m, X) = 2^{\ell(m,X)}, \quad \text{where} \quad \ell(m, X) = \min\{m + 1, \dim X\}. \tag{1.1}$$

Suppose that for every subset $\mathcal{M}' \subset \mathcal{M}$ *consisting of at most* $N = N(m, X)$ *points, the restriction* $F|_{\mathcal{M}'}$ *of* F *to* $\mathcal{M}'$ *has a Lipschitz selection* $f_{\mathcal{M}'}$ *with Lipschitz seminorm* $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',X)} \le 1$.

Then F *has a Lipschitz selection* f *with Lipschitz seminorm* $\|f\|_{\mathrm{Lip}(\mathcal{M},X)} \le \gamma$, *where* $\gamma = \gamma(m)$ *is a positive constant depending only* m.

There is an extensive literature devoted to various versions of finiteness principles for Lipschitz selections and related topics. We refer the reader to the papers [1, 2, 4, 14–16, 19–21, 23–26] and references therein for numerous results in this direction.

We note that the "finiteness number" $N(m, X)$ in Theorem 1.2 is optimal; see [24].

For the case of the trivial distance function $\rho \equiv 0$, Theorem 1.2 agrees with the classical Helly's theorem [9], except that the optimal finiteness constant for $\rho \equiv 0$ is

$$n(m, X) = \ell(m, X) + 1 = \min\{m + 2, \dim X + 1\} \quad \text{in place of} \quad N(m, X) = 2^{\ell(m,X)}.$$

Thus, Theorem 1.2 may be regarded as a generalization of Helly's theorem.

Our interest in Helly-type criteria for the existence of Lipschitz selections was initially motivated by some intriguing close connections of this problem with the classical Whitney extension problem [28], namely, the problem of characterizing those functions defined on a closed subset, say $E \subset \mathbb{R}^n$, which are the restrictions to E of C^m-smooth functions on $\mathbb{R}^n$. We refer the reader to the papers [5–7, 10–13, 26] and references therein for numerous results and techniques concerning this topic.

One of the main ingredients of the proof of Theorem 1.2 is the construction of a special set-valued mapping $G\colon \mathcal{M} \to \mathcal{K}_m(X)$ introduced in [16] which we call a *"core"* of the set-valued mapping F. In fact, each core is associated with a positive constant. Here are the relevant definitions.

Definition 1.3. Let γ be a positive constant, and let $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ be a set-valued mapping. A set-valued mapping $G\colon \mathcal{M} \to \mathcal{K}_m(X)$ is said to be a γ-*core* of F if

(i) $G(x) \subset F(x)$ for all $x \in \mathcal{M}$;

(ii) G is γ-Lipschitz with respect to Hausdorff distance, i.e.,

$$\mathrm{d_H}(G(x), G(y)) \le \gamma\rho(x, y) \quad \text{for all} \quad x, y \in \mathcal{M}.$$

We refer to a map G as a *core of* F if G is a γ-core of F for some $\gamma > 0$.

Recall that the Hausdorff distance $\mathrm{d_H}(A, B)$ between two non-empty bounded subsets A and B of X is defined by

$$\mathrm{d_H}(A, B) = \inf\{r > 0 : A + B_X(0,r) \supset B \ \text{ and } \ B + B_X(0,r) \supset A\}. \tag{1.2}$$

Here and throughout this paper, for each $x \in X$ and $r > 0$, we use the standard notation $B_X(x,r)$ for the closed ball in X with center x and radius r. We also let $B_X = B_X(0,1)$ denote the unit ball in X, and we write rB_X to denote the ball $B_X(0,r)$.

In Definition 1.3, m can be any non-negative integer not exceeding the dimension of the Banach space X. It can happen that a core $G\colon \mathcal{M} \to \mathcal{K}_m(X)$ of a given set-valued mapping $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ in fact maps $\mathcal{M}$ into the smaller collection $\mathcal{K}_{m'}(X)$ for some integer $m' \in [0,m)$. The next claim shows that the existence of some core $G\colon \mathcal{M} \to \mathcal{K}_m(X)$ for F implies the existence of a (possibly different) core which maps $\mathcal{M}$ into $\mathcal{K}_0(X)$. Since $\mathcal{K}_0(X)$ is identified with X, that core is simply a Lipschitz selection of F.

Claim 1.4 ([16], Section 5). *Let γ be a positive constant, let m be a non-negative integer, and let $G\colon \mathcal{M} \to \mathcal{K}_m(X)$ be a γ-core of a set-valued mapping $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ for some Banach space X. Then F has a Lipschitz selection $f\colon \mathcal{M} \to X$ with $\|f\|_{\mathrm{Lip}(\mathcal{M},X)} \le C\gamma$, where $C = C(m)$ is a constant depending only on m.*

In [16] we showed that this claim follows from Definition 1.3 and the existence of the so-called *"Steiner-type point"* map $\mathrm{St}\colon \mathcal{K}_m(X) \to X$, [25]. See formula (1.12).

In [16], given a set-valued mapping $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ satisfying the hypothesis of Theorem 1.2, we constructed a γ-core G of F with a positive constant γ depending only on m. We produced the core G using a rather delicate and complicated procedure whose main ingredients are families of *basic convex sets* associated with F, metric spaces with bounded *Nagata dimension*, ideas and methods of the work [14] related to the case $\mathcal{M} = \mathbb{R}^n$, and Lipschitz selections on finite metric trees. See [16] for more details.

In the present paper we suggest and discuss a different new geometrical method for producing a core of a set-valued mapping. Its main ingredient is the so-called *balanced refinement* of a set-valued mapping, which we define as follows.

Definition 1.5. Let $\lambda \ge 0$, let $(\mathcal{M}, \rho)$ be a pseudometric space, let X be a Banach space, and let $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ be a set-valued mapping for some non-negative integer m. For each $x \in \mathcal{M}$, we consider the subset of $F(x)$ defined by

$$\mathcal{BR}[F:\lambda;\rho](x) = \bigcap_{z\in\mathcal{M}} [F(z) + \lambda\rho(x,z)B_X].$$

We refer to the mapping $\mathcal{BR}[F:\lambda;\rho] : \mathcal{M} \to \mathcal{K}_m(X) \cup \{\emptyset\}$ as the λ-*balanced refinement* of the set-valued mapping F.

We note that any Lipschitz selection f of a set-valued mapping $F\colon \mathcal{M} \to \mathcal{K}_m(X)$ with $\|f\|_{\mathrm{Lip}(\mathcal{M},X)} \le \lambda$ is also a Lipschitz selection of the λ-balanced refinement of F, i.e.,

$$f(x) \in \mathcal{BR}[F:\lambda;\rho](x) \quad \text{for all } x \in \mathcal{M}.$$

Various geometrical parameters of the set $\mathcal{BR}[F:\lambda;\rho](x)$ (such as diameter and width, etc.) may turn out to be smaller than the same parameters for the set $F(x)$ which contains it. When attempting to find Lipschitz selections of F, it may turn out to be convenient for our purposes to search for them in the more "concentrated" setting provided by the sets $\mathcal{BR}[F:\lambda;\rho](x)$. One can take this approach still further by searching in even smaller sets which can be obtained from *consecutive iterations of balanced refinements* of F, i.e., the sets which we describe in the following definition.

Definition 1.6. Let ℓ be a positive integer, and let $\vec{\lambda}=\{\lambda_k : 1\le k\le \ell\}$ be a finite sequence of ℓ non-negative numbers λ_k. We set $F^{[0]}=F$, and, for every $x\in\mathcal{M}$ and integer $k\in[0,\ell-1]$, we define

$$F^{[k+1]}(x)=\mathcal{BR}[F^{[k]}:\lambda_{k+1};\rho](x)=\bigcap_{z\in\mathcal{M}}\left[F^{[k]}(z)+\lambda_{k+1}\rho(x,z)B_X\right]. \tag{1.3}$$

We refer to the set-valued mapping $F^{[k]}\colon\mathcal{M}\to\mathcal{K}_m(X)\cup\{\emptyset\}$, $k\in[1,\ell]$, as *the k-th order $(\vec{\lambda},\rho)$-balanced refinement of F.*

Clearly,

$$F^{[k+1]}(x)\subset F^{[k]}(x)\quad \text{on } \mathcal{M} \text{ for every } k\in[0,\ell-1]. \tag{1.4}$$

(Put $z=x$ in the right-hand side of (1.3).)

Remark 1.7. Of course, for each integer $k\in[1,\ell]$, the set $F^{[k]}(x)$ also depends on the sequence $\vec{\lambda}=\{\lambda_k : 1\le k\le\ell\}$, on the pseudometric space $\mathfrak{M}=(\mathcal{M},\rho)$ and on the Banach space X. However, in all places where we use $F^{[k]}$'s, these objects, i.e., $\vec{\lambda}$, $\mathfrak{M}$ and X, are clear from the context. Therefore, in these cases, we omit any mention of $\vec{\lambda}$, $\mathfrak{M}$ and X in the notation of $F^{[k]}$'s.

We formulate the following conjecture.

Conjecture 1.8. *Let $(\mathcal{M},\rho)$ be a pseudometric space, and let X be a Banach space. Let m be a fixed positive integer and (as in the formula (1.1) of Theorem 1.2) let $N(m,X)$ denote the "finiteness number" $N(m,X)=2^\ell$, where $\ell=\ell(m,X)=\min\{m+1,\dim X\}$.*

There exist a constant $\gamma\ge 1$ and a sequence $\vec{\lambda}=\{\lambda_k : 1\le k\le\ell\}$ of ℓ numbers λ_k, all satisfying $\lambda_k\ge 1$, such that the following holds.

Let $F\colon\mathcal{M}\to\mathcal{K}_m(X)$ be a set-valued mapping such that, for every $\mathcal{M}'\subset\mathcal{M}$ with $\#\mathcal{M}'\le N(m,X)$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}\colon\mathcal{M}'\to X$ with Lipschitz seminorm $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',X)}\le 1$.

Then the ℓ-th order balanced refinement of the mapping F, namely the set-valued mapping $F^{[\ell]}\colon\mathcal{M}\to\mathcal{K}_m(X)$, is a γ-core of F.

Here $F^{[\ell]}$ is defined as in Definition 1.6 using the particular sequence $\vec{\lambda}$.

Our main results, Theorem 1.9 and Theorem 1.10 below, state that Conjecture 1.8 holds in two special cases, when either (i) $m=2$ and $\dim X=2$, or (ii) $m=1$ and X is an arbitrary Banach space. Note that in both of these cases the above mentioned finiteness number $N(m,X)$ equals 4.

Theorem 1.9. *Let $\mathfrak{M}=(\mathcal{M},\rho)$ be a pseudometric space, and let X be a two dimensional Banach space. Let $m=2$ so that the number $\ell(m,X)=2$. In this case, Conjecture 1.8* *holds for every* λ_1, λ_2 *and* γ *such that*

$$\lambda_1 \geq e(\mathfrak{M}, X), \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 \Big(\frac{3\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\Big)^2. \tag{1.5}$$

Here $e(\mathfrak{M}, X)$ *denotes the Lipschitz extension constant of* X *with respect to* $\mathfrak{M}$. (*See Definition* 3.1.)

Thus, the following statement is true. Let $F\colon \mathcal{M} \to \mathcal{K}(X)$ *be a set-valued mapping from a pseudometric space* $(\mathcal{M}, \rho)$ *into the family* $\mathcal{K}(X)$ *of all non-empty convex compact subsets of* X. *Given* $x \in \mathcal{M}$, *let*

$$F^{[1]}(x) = \bigcap_{z\in\mathcal{M}} \big[F(z) + \lambda_1 \rho(x,z) B_X\big], \quad F^{[2]}(x) = \bigcap_{z\in\mathcal{M}} \big[F^{[1]}(z) + \lambda_2 \rho(x,z) B_X\big]. \tag{1.6}$$

Suppose that for every subset $\mathcal{M}' \subset \mathcal{M}$ *with* $\#\mathcal{M}' \leq 4$, *the restriction* $F|_{\mathcal{M}'}$ *of* F *to* $\mathcal{M}'$ *has a Lipschitz selection with Lipschitz seminorm at most* 1.

Then for every λ_1, λ_2 *and* γ *satisfying* (1.5), *we have*

$$F^{[2]}(x) \neq \emptyset \quad \text{for every } x \in \mathcal{M}. \tag{1.7}$$

Furthermore,

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \leq \gamma \rho(x,y) \quad \text{for every } x, y \in \mathcal{M}. \tag{1.8}$$

If X *is a Euclidean two dimensional space,* (1.7) *and* (1.8) *hold when* (1.5) *is replaced by the weaker requirements that*

$$\lambda_1 \geq e(\mathfrak{M}, X), \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \lambda_2 \Big\{1 + \frac{2\lambda_2}{(\lambda_2^2 - \lambda_1^2)^{1/2}}\Big\}^2. \tag{1.9}$$

In particular, in Section 3 we show that the mapping $F^{[2]}$ satisfies (1.7) and (1.8) whenever X is an *arbitrary* two dimensional Banach space and $\lambda_1 = 4/3$, $\lambda_2 = 4$, and $\gamma = 100$. If X is also Euclidean, then one can set $\lambda_1 = 4/\pi$, $\lambda_2 = 12/\pi$ and $\gamma = 38$. Furthermore, we prove that if $\mathcal{M}$ is a subset of a Euclidean space E, ρ is the Euclidean metric in E, and X is a two dimensional Euclidean space, then properties (1.7) and (1.8) hold for $\lambda_1 = 1$, $\lambda_2 = 3$, and $\gamma = 25$.

In Section 5 we prove Theorem 5.5, which refines the result of Theorem 1.9 for the space $X = \ell^2_\infty$, i.e., for $\mathbb{R}^2$ equipped with the norm $\|x\| = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2)$. More specifically, we show that in this case properties (1.7) and (1.8) hold whenever $\lambda_1 \geq 1$, $\lambda_2 \geq 3\lambda_1$, and $\gamma \geq \lambda_2(3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1)$. In particular, these properties hold for $\lambda_1 = 1$, $\lambda_2 = 3$ and $\gamma = 15$.

Let us now explicitly formulate the above mentioned second main result of the paper. We prove it in Section 4. It deals with set-valued mappings from a pseudometric space into the family $\mathcal{K}_1(X)$ of all *bounded closed line segments* of an arbitrary Banach space X.

Theorem 1.10. *Let* $(\mathcal{M}, \rho)$ *be a pseudometric space. Let* $m = 1$ *and let* X *be a Banach space with* $\dim X > 1$; *thus,* $\ell(m, X) = 2$, *see* (1.1). *In this case, Conjecture* 1.8 *holds for every* λ_1, λ_2 *and* γ *such that*

$$\lambda_1 \geq 1, \quad \lambda_2 \geq 3\lambda_1, \quad \gamma \geq \frac{\lambda_2(3\lambda_2 + \lambda_1)}{\lambda_2 - \lambda_1}. \tag{1.10}$$

Thus, the following statement is true. Let $F\colon \mathcal{M} \to \mathcal{K}_1(X)$ *be a set-valued mapping such that for every subset* $\mathcal{M}' \subset \mathcal{M}$ *with* $\#\mathcal{M}' \le 4$*, the restriction* $F|_{\mathcal{M}'}$ *of* F *to* $\mathcal{M}'$ *has a Lipschitz selection with Lipschitz seminorm at most* 1.

Let $F^{[2]}$ *be the mapping defined by* (1.6)*. Then properties* (1.7) *and* (1.8) *hold whenever* λ_1*,* λ_2 *and* γ *satisfy* (1.10)*. In particular, one can set* $\lambda_1 = 1$*,* $\lambda_2 = 3$ *and* $\gamma = 15$*.*

If X *is a Euclidean space, the same statement is also true whenever, instead of* (1.10)*,* λ_1, λ_2 *and* γ *satisfy the weaker condition*

$$\lambda_1 \ge 1, \quad \lambda_2 \ge 3\lambda_1, \quad \gamma \ge \lambda_2 + \frac{2\lambda_2^2}{(\lambda_2^2 - \lambda_1^2)^{1/2}}. \tag{1.11}$$

In particular, in this case, (1.7) *and* (1.8) *hold whenever* $\lambda_1 = 1$*,* $\lambda_2 = 3$ *and* $\gamma = 10$*.*

In Section 5.1 we note that Conjecture 1.8 also holds for a one dimensional space X and $m = 1$. In this case, the statement of the conjecture is true for every $\lambda_1 \ge 1$ and $\gamma \ge 1$. See Proposition 5.1.

Note that Theorem 1.9 tells us that for every set-valued mapping F satisfying the hypothesis of this theorem, the mapping $F^{[2]}$ determined by (1.6) with $\lambda_1 = 4/3$ and $\lambda_2 = 4$ provides a γ-core of F with $\gamma = 100$. (See Definition 1.3.) In turn, Theorem 1.10 states that the mapping $F^{[2]}$ corresponding to the parameters $\lambda_1 = 1$ and $\lambda_2 = 3$ is a 15-core of any F satisfying the conditions of this theorem.

We note that the proofs of Theorem 1.9 and Theorem 1.10 rely on Helly's intersection theorem and a series of auxiliary results about neighborhoods of intersections of convex sets. See Section 2.

Remark 1.11. Let us compare Conjecture 1.8 (and Theorems 1.9 and 1.10) with the finiteness principle (FP) formulated in Theorem 1.2. First we note that FP is invariant with respect to the transition to an equivalent norm on X, while the statement of Conjecture 1.8 is not.

To express this more precisely, let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on X, i.e., suppose that for some $\alpha \ge 1$ the inequality $(1/\alpha)\|\cdot\|_1 \le \|\cdot\|_2 \le \alpha\|\cdot\|_1$ holds. Clearly, if FP holds for $(X, \|\cdot\|_1)$, then it immediately holds also for $(X, \|\cdot\|_2)$ (with the constant $\alpha^2\gamma$ instead of γ). However, the validity of Conjecture 1.8 for the norm $\|\cdot\|_1$ does not imply its validity for an equivalent norm $\|\cdot\|_2$ on X (at least we do not see any obvious way for obtaining such an implication). For example, the validity of Conjecture 1.8 in ℓ^n_∞ (i.e., $\mathbb{R}^n$ equipped with the uniform norm) does not automatically imply its validity in the space ℓ^n_2 (i.e., $\mathbb{R}^n$ with the Euclidean norm).

We also note the following: in a certain sense, the result of Theorem 1.9 is "stronger" than Theorem 1.2 (i.e., FP for the case of a two dimensional Banach space X). Indeed, in this case, the hypotheses of FP and Theorem 1.9 coincide. Moreover, Theorem 1.9 ensures that the set-valued mapping $F^{[2]}$ is a core of F. This property of $F^{[2]}$ implies, via arguments in [16], that the function

$$f(x) = \mathrm{St}(F^{[2]})(x), \quad x \in \mathcal{M}, \tag{1.12}$$

is a Lipschitz selection of F. Here $\mathrm{St}\colon \mathcal{K}_m(X) \to X$ is the Steiner-type point map [25].

Thus, FP (in the two dimensional case) follows immediately from Theorem 1.9. However, it is absolutely unclear how the statement of Theorem 1.9 can be deduced from FP. I want to thank Charles Fefferman who kindly drew my attention to this interesting fact.

Let us reformulate Conjecture 1.8 in a way which *does not require the use of the notion of a core of a set-valued mapping*. We recall that the mapping $F^{[\ell]}: \mathcal{M} \to \mathcal{K}_m(X)$ which appears in Conjecture 1.8 is a γ-core of F if

$$\mathrm{d_H}(F^{[\ell]}(x), F^{[\ell]}(y)) \le \gamma\rho(x,y) \quad \text{for all } x, y \in \mathcal{M}.$$

See part (ii) of Definition 1.3. Hence, given $x \in \mathcal{M}$,

$$F^{[\ell]}(x) \subset F^{[\ell]}(y) + \gamma\rho(x,y)B_X \quad \text{for every } y \in \mathcal{M}. \tag{1.13}$$

Let

$$F^{[\ell+1]}(x) = \mathcal{BR}[F^{[\ell]}:\gamma;\rho](x) = \bigcap_{y\in\mathcal{M}} \left[F^{[\ell]}(y) + \gamma\rho(x,y)B_X\right].$$

Cf. (1.3). This and (1.13) imply the inclusion $F^{[\ell+1]}(x) \supset F^{[\ell]}(x)$, $x \in \mathcal{M}$. On the other hand, (1.4) tells us that $F^{[\ell+1]}(x) \subset F^{[\ell]}(x)$, proving that $F^{[\ell+1]} = F^{[\ell]}$ on $\mathcal{M}$.

These observations enable us to reformulate Conjecture 1.8 as follows.

Conjecture 1.12. *Let $(\mathcal{M},\rho)$ be a pseudometric space, and let X be a Banach space. Let m be a fixed positive integer and let $\ell = \ell(m, X)$, see* (1.1).

There exists a sequence $\vec{\lambda} = \{\lambda_k : 1 \le k \le \ell+1\}$ of $\ell+1$ numbers λ_k, all satisfying $\lambda_k \ge 1$, such that, for every set-valued mapping $F: \mathcal{M} \to \mathcal{K}_m(X)$ satisfying the hypothesis of the finiteness principle (*Theorem* 1.2)*, the family $\{F^{[k]} : k = 1,\dots,\ell+1\}$ of set-valued mappings constructed by formula* (1.3) *has the following property*:

$$F^{[\ell]}(x) \ne \emptyset \quad \textit{and} \quad F^{[\ell+1]}(x) = F^{[\ell]}(x) \quad \textit{for all } x \in \mathcal{M}. \tag{1.14}$$

We refer to (1.14) as a *stabilization property* of balanced refinements.

Thus, Theorem 1.9 and Theorem 1.10 tell us that a stabilization property of balanced refinements holds whenever $\dim X = 2$ or $m = 1$ (and X is an arbitrary). More specifically, Theorem 1.9 shows that if $m = 2$ and $\dim X = 2$, Conjecture 1.12 holds with $\ell = 2$ and $\vec{\lambda} = \{4/3, 4, 10^2\}$.

In other words, in this case, $F^{[2]}(x) \ne \emptyset$ for each $x \in \mathcal{M}$ and $F^{[3]} = F^{[2]}$ on $\mathcal{M}$. In turn, Theorem 1.10 states that the same property holds whenever X is an arbitrary Banach space, $m = 1$, and $\vec{\lambda} = \{1, 3, 15\}$.

Readers might find it helpful to also consult a much more detailed version of this paper posted on the arXiv, [27].

2. Neighborhoods of intersections of convex sets in a Banach space

We first need to fix some notation.

Let $(X, \|\cdot\|)$ be a Banach space, and let B_X be the unit ball in X. Let A and B be non-empty subsets of X. We let $A + B = \{a + b : a \in A, b \in B\}$ denote the Minkowski sum of these sets.

Sometimes, for a given set $\mathcal{M}$, we will be looking simultaneously at two distinct pseudometrics on $\mathcal{M}$, say ρ and δ. In this case we will speak of a ρ-Lipschitz selection and ρ-Lipschitz seminorm, or a δ-Lipschitz selection and δ-Lipschitz seminorm, to make clear which pseudometric we are using. Furthermore, given a mapping $f: \mathcal{M} \to X$, we will write $\|f\|_{\mathrm{Lip}((\mathcal{M};\rho),X)}$ or $\|f\|_{\mathrm{Lip}((\mathcal{M};\delta),X)}$ to denote the Lipschitz seminorm of f with respect to the pseudometric ρ or δ, respectively.

For each finite set S, we let $\#S$ denote the number of elements of S.

Note two important geometric results that we use in the proof of the main theorems. Here is the first of them.

Theorem 2.1. *Let X be a Banach space, and let $C \subset X$ be a convex set. Let $a \in X$ and let $r \geq 0$. Suppose that*

$$C \cap B_X(a, r) \neq \emptyset. \tag{2.1}$$

Then for every $s > 0$ and $L > 1$,

$$[C \cap B_X(a, Lr)] + \theta(L) s B_X \supset (C + sB_X) \cap B_X(a, Lr + s), \tag{2.2}$$

where

$$\theta(L) = (3L + 1)/(L - 1). \tag{2.3}$$

If X is a Euclidean space, then (2.2) holds with $\theta(L) = 1 + 2L/\sqrt{L^2 - 1}$.

For the case of an arbitrary Banach space X, Theorem 2.1 was proved by Przesławski and Rybinski in [19], p. 279. For the case of a Euclidean space X, see Przesławski–Yost [21], Theorem 4. See also Section 2 in [27]. For similar results, we refer the reader to [1], [2], p. 369, and [4], p. 26.

The second of these geometrical results is the classical Helly intersection theorem for *two dimensional* Banach spaces. It can be formulated as follows.

Theorem 2.2. *Let $\mathcal{K}$ be a collection of convex closed subsets of a two dimensional Banach space X. Suppose that $\mathcal{K}$ is finite or at least one member of the family $\mathcal{K}$ is bounded.*

If every subfamily of $\mathcal{K}$ consisting of at most three elements has a common point, then there exists a point common to all of the family $\mathcal{K}$.

We conclude this section by stating and proving one more result which will be a third main tool for proving our main theorems. As we shall see, its proof makes use of the preceding two theorems.

Proposition 2.3. *Let X be a two dimensional Banach space. Let $C, C_1, C_2 \subset X$ be convex subsets, and let $r > 0$. Suppose that*

$$C_1 \cap C_2 \cap (C + rB_X) \neq \emptyset. \tag{2.4}$$

Then for every $L > 1$ and every $\varepsilon > 0$, the inclusion

$$\begin{aligned} &[\{(C_1 \cap C_2) + LrB_X\} \cap C] + \theta(L)\varepsilon B_X \\ &\supset [(C_1 \cap C_2) + (Lr + \varepsilon)B_X] \cap [\{(C_1 + rB_X) \cap C\} + \varepsilon B_X] \cap [\{(C_2 + rB_X) \cap C\} + \varepsilon B_X] \end{aligned}$$

holds. The function θ in the above inclusion is as defined in Theorem 2.1. I.e., for an arbitrary Banach space X, that inclusion holds for $\theta(L) = (3L + 1)/(L - 1)$, and if X is Euclidean, it also holds for $\theta(L) = 1 + 2L/\sqrt{L^2 - 1}$.

Proof. Let

$$A=[(C_1\cap C_2)+(Lr+\varepsilon)B_X]\cap[\{(C_1+rB_X)\cap C\}+\varepsilon B_X]\cap[\{(C_2+rB_X)\cap C\}+\varepsilon B_X]$$

and let $a\in A$. Let us prove that

$$a\in[\{(C_1\cap C_2)+LrB_X\}\cap C]+\theta(L)\varepsilon B_X. \tag{2.5}$$

Let us show that if

$$C_1\cap C_2\cap(C+rB_X)\cap B_X(a,Lr+\varepsilon)\neq\emptyset, \tag{2.6}$$

then (2.5) holds. Indeed, (2.6) provides the existence of a point $x\in X$ such that

$$x\in C_1\cap C_2\cap(C+rB_X)\cap B_X(a,Lr+\varepsilon). \tag{2.7}$$

In particular, $x\in C+rB_X$, so that $B_X(x,r)\cap C\neq\emptyset$, proving that condition (2.1) of Theorem 2.1 holds. This theorem tells us that

$$[B_X(x,Lr)\cap C]+\theta(L)\varepsilon B_X\supset(C+\varepsilon B_X)\cap B_X(x,Lr+\varepsilon). \tag{2.8}$$

Since $a\in A$, we have $a\in C+\varepsilon B_X$. From (2.7) we learn that and $a\in B_X(x,Lr+\varepsilon)$. Thus, the point a belongs to the set $(C+\varepsilon B_X)\cap B_X(x,Lr+\varepsilon)$. Therefore, by (2.8),

$$a\in[B_X(x,Lr)\cap C]+\theta(L)\varepsilon B_X=[(x+LrB_X)\cap C]+\theta(L)\varepsilon B_X.$$

But $x\in C_1\cap C_2$, see (2.7), and the required inclusion (2.5) follows.

Thus, it remains to prove (2.6). Helly's Theorem 2.2 tells us that this property holds provided any three sets in the left-hand side of (2.6) have a common point. Note that, thanks to (2.4), this is true for C_1, C_2 and $C+rB_X$. We also note that the point a belongs to A, so that $a\in[C_1\cap C_2]+(Lr+\varepsilon)B_X$, proving that $C_1\cap C_2\cap B_X(a,Lr+\varepsilon)\neq\emptyset$.

Let us prove that

$$C_1\cap(C+rB_X)\cap B_X(a,Lr+\varepsilon)\neq\emptyset. \tag{2.9}$$

The point $a\in A$ is so that $a\in(C_1+rB_X)\cap C+\varepsilon B_X$. Let b be a point nearest to a on $(C_1+rB_X)\cap C$, and let $b_1\in C_1$ be a point nearest to b on C_1. Let us prove that

$$b_1\in C_1\cap(C+rB_X)\cap B_X(a,Lr+\varepsilon). \tag{2.10}$$

Indeed, we have $\|a-b\|\le\varepsilon$ and $\|b_1-b\|\le r$. Thus,

$$b_1\in C_1\ \ \text{(by definition)}\ \ \text{and}\ \ b_1\in C+rB_X\ \ \text{(because } b\in C). \tag{2.11}$$

Furthermore,

$$\|a-b_1\|\le\|a-b\|+\|b-b_1\|\le\varepsilon+r\le\varepsilon+Lr,$$

proving that $b_1\in B_X(a,Lr+\varepsilon)$. Combining this property with (2.11), we obtain (2.10) and (2.9). In a similar way, we show that $C_2\cap(C+rB_X)\cap B_X(a,Lr+\varepsilon)\neq\emptyset$.

The proof of the proposition is complete. ∎

3. The main theorem for two dimensional Banach spaces

In this section we prove Theorem 1.9.

First, let us recall the notion of the Lipschitz extension constant $e(\mathfrak{M}, X)$ which we use in the formulation of this theorem.

Definition 3.1. Let $\mathfrak{M} = (\mathcal{M}, \rho)$ be a pseudometric space, and let X be a Banach space. We define the Lipschitz extension constant $e(\mathfrak{M}, X)$ of X with respect to $\mathfrak{M}$ as the infimum of the constants $\lambda > 0$ such that for every subset $\mathcal{M}' \subset \mathcal{M}$, and every Lipschitz mapping $f : \mathcal{M}' \to X$, there exists a Lipschitz extension $\tilde{f} : \mathcal{M} \to X$ of f to all of $\mathcal{M}$ such that $\|\tilde{f}\|_{\mathrm{Lip}(\mathcal{M},X)} \le \lambda \|f\|_{\mathrm{Lip}(\mathcal{M}',X)}$.

Remark 3.2. Recall several results about Lipschitz extension constants which we use in this paper. In particular, thanks to the McShane–Whitney extension theorem, $e(\mathfrak{M}, \mathbb{R}) = 1$ for every pseudometric space $\mathfrak{M} = (\mathcal{M}, \rho)$. Hence, $e(\mathfrak{M}, \ell^2_\infty) = 1$ as well.

It follows from [22] and [8] that $e(\mathfrak{M}, X) \le 4/3$ provided X is an *arbitrary two dimensional Banach space*. See also [3]. Furthermore, $e(\mathfrak{M}, X) \le 4/\pi$ whenever X is an arbitrary *two dimensional Euclidean space*. See [22] and [17]. We also note that, thanks to Kirszbraun's theorem [18], $e(\mathfrak{M}, X) = 1$ provided *X is a Euclidean space, $\mathcal{M}$ is a subset of a Euclidean space E*, and ρ is the Euclidean metric in E.

Proof of Theorem 1.9. Let $\mathfrak{M} = (\mathcal{M}, \rho)$ be a pseudometric space, and let X be a two dimensional Banach space. Let $F : \mathcal{M} \to \mathcal{K}(X)$ be a set-valued mapping satisfying the hypothesis of Theorem 1.9. We recall that, by this hypothesis, *for every subset $S \subset \mathcal{M}$ with $\#S \le 4$, the restriction $F|_S$ of F to S has a ρ-Lipschitz selection $f_S : S \to X$ with* $\|f_S\|_{\mathrm{Lip}((S,\rho),X)} \le 1$.

Let λ_1 and λ_2 be positive constants satisfying inequalities (1.5), i.e., $\lambda_1 \ge e(\mathfrak{M}, X)$ and $\lambda_2 \ge 3\lambda_1$. We set $L = \lambda_2/\lambda_1$. Thus, the inequalities

$$\lambda_1 \ge e(\mathfrak{M}, X), \quad L \ge 3, \tag{3.1}$$

hold. Then we introduce a new pseudometric on $\mathcal{M}$ defined by

$$\mathrm{d}(x, y) = \lambda_1 \rho(x, y), \quad x, y \in \mathcal{M}. \tag{3.2}$$

This definition, Definition 3.1, the above hypothesis of Theorem 1.9 and the inequality $\lambda_1 \ge e(\mathfrak{M}, X)$ imply the following claim.

Claim 3.3. *Let $\widetilde{S} \subset \mathcal{M}$ be a finite set, and let $S \subset \widetilde{S}$ be a set with $\#S \le 4$. Then there exists a* d*-Lipschitz mapping $\tilde{f}_S : \widetilde{S} \to X$ with $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \le 1$ such that $\tilde{f}_S(x) \in F(x)$ for every $x \in S$.*

We introduce set-valued mappings

$$F^{[1]}(x) = \bigcap_{z \in \mathcal{M}} [F(z) + \mathrm{d}(x, z) B_X], \quad x \in \mathcal{M}, \tag{3.3}$$

$$F^{[2]}(x) = \bigcap_{z \in \mathcal{M}} [F^{[1]}(z) + L\, \mathrm{d}(x, z) B_X], \quad x \in \mathcal{M}. \tag{3.4}$$

Thus, $F^{[1]}$ and $F^{[2]}$ are the first and the second order $(\{1, L\}, \mathrm{d})$-balanced refinements of F respectively. See Definition 1.6.

Our aim is to show that, if L and λ_1 satisfy inequality (3.1), then (i) $F^{[2]}(x) \neq \emptyset$ on $\mathcal{M}$, and (ii) the mapping $F^{[2]}$ is d-Lipschitz with respect to the Hausdorff distance. We prove the statements (i) and (ii) in Proposition 3.7 and Proposition 3.9 respectively.

We begin with the property (i). Its proof relies on a series of auxiliary lemmas.

Lemma 3.4. *Let X be a two dimensional Banach space, and let $\mathcal{K} \subset \mathcal{K}(X)$ be a collection of convex compact subsets of X with non-empty intersection.*

Given $\tau > 0$, let $B = \tau B_X$. Then

$$\Big(\bigcap_{K\in\mathcal{K}} K\Big) + B = \bigcap_{K,K'\in\mathcal{K}} \{(K \cap K') + B\}. \tag{3.5}$$

Proof. Obviously, the right-hand side of (3.5) contains its left-hand side. Let us prove the converse statement. Let us fix a point

$$x \in \bigcap_{K,K'\in\mathcal{K}} \{(K \cap K') + B\} \tag{3.6}$$

and prove that $x \in \cap\{K : K \in \mathcal{K}\} + B$. Clearly, it is true provided

$$B_X(x, \tau) \bigcap \Big(\bigcap_{K\in\mathcal{K}} K\Big) \neq \emptyset. \tag{3.7}$$

Let $S = \mathcal{K} \cup \{B_X(x, \tau)\}$. Helly's intersection Theorem 2.2 tells us that property (3.7) holds provided $\cap\{K : K \in S'\} \neq \emptyset$ for every subfamily $S' \subset S$ consisting of at most three elements. Clearly, this is true if $B_X(x, \tau) \notin S'$ because there exists a point common to all of the sets from $\mathcal{K}$.

Suppose that $B_X(x, \tau) \in S'$. Then $S' = \{B_X(x, \tau), K, K'\}$ for some $K, K' \in \mathcal{K}$. Thanks to (3.6), $x \in (K \cap K') + B$ so that $B_X(x, \tau) \cap K \cap K' \neq \emptyset$.

Thus, (3.7) holds, and the proof of the lemma is complete. ■

Lemma 3.5. *For each $x \in \mathcal{M}$, the set $F^{[1]}(x) \in \mathcal{K}(X)$, i.e., $F^{[1]}(x)$ is a non-empty convex compact subset of X.*

Furthermore, for every $x, z \in \mathcal{M}$, the set $\widetilde{F} = F^{[1]}(z) + L\,\mathrm{d}(x, z)$ has the following representation:

$$\widetilde{F} = \bigcap_{y',y''\in\mathcal{M}} \big\{[(F(y') + \mathrm{d}(z, y')B_X) \cap (F(y'') + \mathrm{d}(z, y'')B_X)] + L\,\mathrm{d}(x, z)B_X\big\}.$$

Proof. Let us prove that $F^{[1]}(x) \neq \emptyset$ for every $x \in \mathcal{M}$. Indeed, formula (3.3) and Helly's Theorem 2.2 tell us that $F^{[1]}(x) \neq \emptyset$ provided

$$[F(z_1) + \mathrm{d}(x, z_1)B_X] \cap [F(z_2) + \mathrm{d}(x, z_2)B_X] \cap [F(z_3) + \mathrm{d}(x, z_3)B_X] \neq \emptyset \tag{3.8}$$

for every $z_1, z_2, z_3 \in \mathcal{M}$.

Let $S = \{x, z_1, z_2, z_3\}$. To prove (3.8), let us apply the hypothesis of Theorem 1.9 to S. Thanks to this hypothesis, the restriction $F|_S$ has a ρ-Lipschitz selection $f_S : S \to X$ with ρ-Lipschitz seminorm at most 1. Hence,

$$\|f_S(x) - f_S(z_i)\| \le \rho(x, z_i) \le \lambda_1 \rho(x, z_i) = \mathrm{d}(x, z_i) \quad \text{for every } i = 1, 2, 3,$$

proving that $f_S(x)$ belongs to the left-hand side of (3.8). Thus, (3.8) holds for arbitrary $z_i \in \mathcal{M}$, $i = 1, 2, 3$, so that $F^{[1]}(x) \ne \emptyset$.

Clearly, $F^{[1]}(x)$ is a convex bounded subset of X. See (3.3).

Finally, the second statement of the lemma immediately follows from this property, Lemma 3.4 and formula (3.3). The proof of the lemma is complete. ■

Lemma 3.6. *For every $x \in \mathcal{M}$, the set $F^{[2]}(x)$ admits the following representation*:

$$F^{[2]}(x) = \bigcap_{u,u',u'' \in \mathcal{M}} \{[(F(u') + \mathrm{d}(u', u) B_X) \cap (F(u'') + \mathrm{d}(u'', u) B_X)] + L\, \mathrm{d}(u, x) B_X\}.$$

Proof. The lemma is immediate from (3.4) and Lemma 3.5. ■

Given $x, u, u', u'' \in \mathcal{M}$, we set

$$T_x(u, u', u'') = [(F(u') + \mathrm{d}(u', u) B_X) \cap (F(u'') + \mathrm{d}(u'', u) B_X)] + L\, \mathrm{d}(u, x) B_X. \tag{3.9}$$

In these settings, Lemma 3.6 reformulates as follows:

$$F^{[2]}(x) = \bigcap_{u,u',u'' \in \mathcal{M}} T_x(u, u', u''). \tag{3.10}$$

Proposition 3.7. *For every $x \in \mathcal{M}$, the set $F^{[2]}(x)$ is non-empty.*

Proof. Formula (3.10) and Helly's Theorem 2.2 tell us that $F^{[2]}(x) \ne \emptyset$ provided for every choice of elements $u_i, u'_i, u''_i \in \mathcal{M}$, $i = 1, 2, 3$, we have

$$T_x(u_1, u'_1, u''_1) \cap T_x(u_2, u'_2, u''_2) \cap T_x(u_3, u'_3, u''_3) \ne \emptyset. \tag{3.11}$$

We set $r_i = \mathrm{d}(x, u_i)$, $i = 1, 2, 3$; we may assume that $r_1 \le r_2 \le r_3$. For each $i \in \{1, 2, 3\}$, we also set

$$G(u'_i) = F(u'_i) + \mathrm{d}(u'_i, u_i) B_X \quad \text{and} \quad G(u''_i) = F(u''_i) + \mathrm{d}(u''_i, u_i) B_X. \tag{3.12}$$

We will prove that there exist points $y_i \in X$, $i = 1, 2, 3$, such that

$$y_i \in G(u'_i) \cap G(u''_i) \quad \text{for every } i = 1, 2, 3, \text{ and} \tag{3.13}$$

$$\|y_1 - y_2\| \le r_1 + r_2 \quad \text{and} \quad \|y_1 - y_3\| \le r_1 + 2r_2 + r_3. \tag{3.14}$$

Let us see that the existence of the points y_i with these properties implies (3.11). Indeed, since $\|y_1 - y_3\| \le r_1 + 2r_2 + r_3$, there exists a point $z \in [y_1, y_3]$ such that $\|z - y_1\| \le r_1$ and $\|z - y_3\| \le 2r_2 + r_3$. Hence,

$$\|y_2 - z\| \le \|y_2 - y_1\| + \|y_1 - z\| \le r_1 + r_2 + r_1 = 2r_1 + r_2.$$

Recall that $r_i = \mathrm{d}(x, u_i)$ and $r_1 \le r_2 \le r_3$. From this and the above inequalities, we have

$$\|z - y_i\| \le 3r_i = 3\,\mathrm{d}(x, u_i), \quad i = 1, 2, 3. \tag{3.15}$$

Let us prove that $z \in T_x(u_i, u_i', u_i'')$ for each $i \in \{1, 2, 3\}$. In fact, we know that $L \ge 3$, see (3.1). Furthermore, by (3.13), $y_i \in G(u_i') \cap G(u_i'')$, so that, thanks to (3.12), (3.15) and the definition in (3.9),

$$z \in [G(u_i') \cap G(u_i'')] + 3\,\mathrm{d}(x, u_i) B_X \subset [G(u_i') \cap G(u_i'')] + L\,\mathrm{d}(x, u_i) B_X = T_x(u_i, u_i', u_i'')$$

proving (3.11).

Thus, our aim is to prove the existence of points y_i satisfying (3.13) and (3.14). We will do this in three steps.

Step 1. We introduce sets $W_i \subset X$, $i = 1, \dots, 4$, defined by

$$\begin{aligned} &W_1 = G(u_1'), \quad W_2 = G(u_1''), \quad W_3 = [G(u_2') \cap G(u_2'')] + (r_1 + r_2) B_X, \\ &W_4 = [G(u_3') \cap G(u_3'')] + (r_1 + 2r_2 + r_3) B_X. \end{aligned} \tag{3.16}$$

Obviously, there exist the points y_i satisfying (3.13) and (3.14) whenever

$$W_1 \cap W_2 \cap W_3 \cap W_4 \ne \emptyset.$$

By Helly's Theorem 2.2, this property holds provided any three members of the family of sets $\{W_1, W_2, W_3, W_4\}$ have a common point.

Step 2. We prove that $W_1 \cap W_3 \cap W_4 \ne \emptyset$. To see this, we set

$$\begin{aligned} &V_1 = G(u_1') + (r_1 + r_2) B_X, \quad V_2 = G(u_2'), \quad V_3 = G(u_2''), \\ &V_4 = [G(u_3') \cap G(u_3'')] + (r_2 + r_3) B_X. \end{aligned} \tag{3.17}$$

Let us show that if

$$V_1 \cap V_2 \cap V_3 \cap V_4 \ne \emptyset, \tag{3.18}$$

then $W_1 \cap W_3 \cap W_4$ is non-empty as well.

Indeed, this property and the definitions in (3.17) imply the existence of points

$$z_1 \in G(u_1'), \quad z_2 \in G(u_2') \cap G(u_2''), \quad z_3 \in G(u_3') \cap G(u_3'')$$

such that $\|z_1 - z_2\| \le r_1 + r_2$ and $\|z_2 - z_3\| \le r_2 + r_3$. Hence,

$$\|z_1 - z_3\| \le \|z_1 - z_2\| + \|z_2 - z_3\| \le (r_1 + r_2) + (r_2 + r_3) = r_1 + 2r_2 + r_3.$$

Thus, thanks to (3.16), the point z_1 belongs to $W_1 \cap W_3 \cap W_4$, proving that this set is non-empty.

Let us prove (3.18). Helly's Theorem 2.2 tells us that (3.18) holds whenever every three members of the family $\mathcal{V} = \{V_1, V_2, V_3, V_4\}$ have a common point.

Let us prove this property. First, let us show that

$$V_1 \cap V_2 \cap V_4 \neq \emptyset. \tag{3.19}$$

Let $S = \{u_1', u_2', u_3', u_3''\}$ and let $\widetilde{S} = \{u_1', u_2, u_2', u_3, u_3', u_3''\}$. Thanks to Claim 3.3, there exists a d-Lipschitz mapping $\tilde{f}_S \colon \widetilde{S} \to X$ with $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \le 1$ such that

$$\tilde{f}_S(u_1') \in F(u_1'), \quad \tilde{f}_S(u_2') \in F(u_2'), \quad \tilde{f}_S(u_3') \in F(u_3') \quad \text{and} \quad \tilde{f}_S(u_3'') \in F(u_3'').$$

Let us prove that

$$\tilde{f}_S(u_2) \in V_1 \cap V_2 \cap V_4. \tag{3.20}$$

Indeed, $\tilde{f}_S(u_2') \in F(u_2')$ and $\|\tilde{f}_S(u_2') - \tilde{f}_S(u_2)\| \le \mathrm{d}(u_2', u_2)$, so that

$$\tilde{f}_S(u_2) \in F(u_2') + \mathrm{d}(u_2', u_2)B_X = G(u_2') = V_2.$$

In the same way, we prove that $\tilde{f}_S(u_1) \in G(u_1')$.

Note that $\|\tilde{f}_S(u_1) - \tilde{f}_S(u_2)\| \le \mathrm{d}(u_1, u_2)$ so that $\tilde{f}_S(u_2) \in G(u_1') + \mathrm{d}(u_1, u_2)B_X$. By the triangle inequality,

$$\mathrm{d}(u_1, u_2) \le \mathrm{d}(u_1, x) + \mathrm{d}(x, u_2) = r_1 + r_2,$$

proving that $\tilde{f}_S(u_2) \in G(u_1') + (r_1 + r_2)B_X = V_1$.

It remains to show that $\tilde{f}_S(u_2)$ belongs to V_4. Indeed, we know that $\tilde{f}_S(u_3') \in F(u_3')$, $\tilde{f}_S(u_3'') \in F(u_3'')$,

$$\|\tilde{f}_S(u_3) - \tilde{f}_S(u_3')\| \le \mathrm{d}(u_3, u_3') \quad \text{and} \quad \|\tilde{f}_S(u_3) - \tilde{f}_S(u_3'')\| \le \mathrm{d}(u_3, u_3'').$$

Hence,

$$\tilde{f}_S(u_3) \in [F(u_3') + \mathrm{d}(u_3', u_3)B_X] \cap [F(u_3'') + \mathrm{d}(u_3'', u_3)B_X] = G(u_3') \cap G(u_3'').$$

Furthermore, $\|\tilde{f}_S(u_2) - \tilde{f}_S(u_3)\| \le \mathrm{d}(u_2, u_3)$. These properties of $\tilde{f}_S(u_3)$ and the triangle inequality $\mathrm{d}(u_2, u_3) \le \mathrm{d}(u_2, x) + \mathrm{d}(x, u_3) = r_2 + r_3$ imply the following:

$$\tilde{f}_S(u_2) \in [G(u_3') \cap G(u_3'')] + \mathrm{d}(u_2, u_3)B_X \subset [G(u_3') \cap G(u_3'')] + (r_2 + r_3)B_X = V_4.$$

Thus, $\tilde{f}_S(u_2) \in V_1 \cap V_2 \cap V_4$, proving (3.19).

In the same fashion, we show that $V_1 \cap V_3 \cap V_4 \neq \emptyset$.

Next, we prove that

$$V_2 \cap V_3 \cap V_4 = G(u_2') \cap G(u_2'') \cap \{[G(u_3') \cap G(u_3'')] + (r_2 + r_3)B_X\} \neq \emptyset. \tag{3.21}$$

Following the scheme of the proof of (3.19), we set

$$S = \{u_2', u_2'', u_3', u_3''\} \quad \text{and} \quad \widetilde{S} = \{u_2, u_2', u_2'', u_3, u_3', u_3''\}.$$

Claim 3.3 tells us that there exists a d-Lipschitz mapping $\tilde{f}_S \colon \widetilde{S} \to X$ with Lipschitz seminorm $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \le 1$ such that $\tilde{f}_S(u_i') \in F(u_i')$ and $\tilde{f}_S(u_i'') \in F(u_i'')$, $i = 2, 3$.

Then, following the scheme of the proof of (3.20), we show that $\tilde{f}_S(u_2)$ belongs to $V_2 \cap V_3 \cap V_4$, proving the required property (3.21).

Finally, following the same approach, we prove that

$$V_1 \cap V_2 \cap V_3 = [G(u_1') + (r_1 + r_2)B_X] \cap G(u_2') \cap G(u_2'') \neq \emptyset. \tag{3.22}$$

More specifically, we set $S = \{u_1', u_2, u_2', u_2''\}$. Then, thanks to the hypothesis of Theorem 1.9, there exists a d-Lipschitz mapping $\tilde{f}_S : S \to X$ with $\|\tilde{f}_S\|_{\mathrm{Lip}((S,\mathrm{d}),X)} \leq 1$ such that $\tilde{f}_S(u_1') \in F(u_1')$, $\tilde{f}_S(u_2') \in F(u_2')$ and $\tilde{f}_S(u_2'') \in F(u_2'')$. Following the proof of (3.20), we show that $\tilde{f}_S(u_2) \in V_1 \cap V_2 \cap V_3$ completing the proof of (3.22).

Thus, (3.18) is proven, so that $W_1 \cap W_3 \cap W_4 \neq \emptyset$.

Step 3. First, using a similar approach, we show that $W_2 \cap W_3 \cap W_4 \neq \emptyset$.

Next, we prove that

$$W_1 \cap W_2 \cap W_4 = G(u_1') \cap G(u_1'') \cap \big\{[G(u_3') \cap G(u_3'')] + (r_1 + 2r_2 + r_3)B_X\big\} \neq \emptyset.$$

To see this, we set $S = \{u_1', u_1'', u_3', u_3''\}$ and $\widetilde{S} = \{u_1, u_1', u_1'', u_3, u_3', u_3''\}$. Claim 3.3 tells us that there exists a d-Lipschitz mapping $\tilde{f}_S : \widetilde{S} \to X$ with $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \leq 1$ such that $\tilde{f}_S(u_i') \in F(u_i')$ and $\tilde{f}_S(u_i'') \in F(u_i'')$, $i = 1, 3$. The reader can easily see that the point $\tilde{f}_S(u_1)$ belongs to $W_1 \cap W_2 \cap W_4$, proving the required property $W_1 \cap W_2 \cap W_4 \neq \emptyset$.

In a similar way, we prove that

$$W_1 \cap W_2 \cap W_3 = G(u_1') \cap G(u_1'') \cap \big\{[G(u_2') \cap G(u_2'')] + (r_1 + r_2)B_X\big\} \neq \emptyset. \tag{3.23}$$

More specifically, we set $S = \{u_1', u_1'', u_2', u_2''\}$, $\widetilde{S} = \{u_1, u_1', u_1'', u_2, u_2', u_2''\}$, and apply Claim 3.3 to S and $\widetilde{S}$. Thanks to this claim, there exists a d-Lipschitz mapping $\tilde{f}_S : \widetilde{S} \to X$ with $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \leq 1$ such that $\tilde{f}_S(u_i') \in F(u_i')$ and $\tilde{f}_S(u_i'') \in F(u_i'')$, $i = 1, 2$. One can readily see that the point $\tilde{f}_S(u_1) \in W_1 \cap W_2 \cap W_3$ so that (3.23) holds.

The proof of the proposition is complete. ∎

We turn to the proof of inequality (1.8).

Note that, thanks to formula (3.10), for every $x, y \in \mathcal{M}$ we have

$$F^{[2]}(x) = \bigcap_{u,u',u'' \in \mathcal{M}} T_x(u, u', u'') \quad \text{and} \quad F^{[2]}(y) = \bigcap_{u,u',u'' \in \mathcal{M}} T_y(u, u', u''). \tag{3.24}$$

Lemma 3.8. *For every $\tau > 0$ and every $x \in \mathcal{M}$, the representation*

$$F^{[2]}(x) + \tau B_X = \bigcap \big\{[T_x(u, u', u'') \cap T_x(v, v', v'')] + \tau B_X\big\} \tag{3.25}$$

holds. Here the first intersection in the right-hand side of (3.25) is taken over all elements $u, u', u'', v, v', v'' \in \mathcal{M}$.

Proof. The lemma is immediate from (3.24), Lemma 3.4 and Proposition 3.7. ∎

Proposition 3.9. *For every $x, y \in \mathcal{M}$, the inequality*

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \leq \gamma_0(L)\, \mathrm{d}(x, y) \tag{3.26}$$

holds. Here $\gamma_0(L) = L\theta(L)^2$ where $\theta(L)$ is the constant from Theorem 2.1.

Proof. Let $x, y \in \mathcal{M}$ and let $\tau = \gamma_0(L)\,\mathrm{d}(x, y)$. Let us prove that

$$F^{[2]}(x) + \gamma_0(L)\,\mathrm{d}(x, y) B_X = F^{[2]}(x) + \tau B_X \supset F^{[2]}(y). \tag{3.27}$$

Lemma 3.8 tells us that this inclusion holds provided

$$A = [T_x(u, u', u'') \cap T_x(v, v', v'')] + \tau B_X \supset F^{[2]}(y) \tag{3.28}$$

for arbitrary $u, u', u'', v, v', v'' \in \mathcal{M}$.

To prove this inclusion, we introduce the following sets:

$$C_1 = F(u') + \mathrm{d}(u', u) B_X, \quad C_2 = F(u'') + \mathrm{d}(u'', u) B_X, \quad C = T_x(v, v', v''). \tag{3.29}$$

Let

$$\varepsilon = L\theta(L)\,\mathrm{d}(x, y) \quad \text{and} \quad r = \mathrm{d}(x, u). \tag{3.30}$$

Then $\tau = \gamma_0(L)\,\mathrm{d}(x, y) = \theta(L)\,\varepsilon$, and

$$A = [T_x(u, u', u'') \cap T_x(v, v', v'')] + \tau B_X = [(C_1 \cap C_2) + LrB_X] \cap C + \theta(L)\varepsilon B_X.$$

We want to apply Proposition 2.3 to the set A. To do this, we have to verify condition (2.4) of this proposition, i.e., to show that

$$C_1 \cap C_2 \cap (C + rB_X) \neq \emptyset. \tag{3.31}$$

Let $S = \{u', u'', v', v''\}$ and let $\widetilde{S} = \{x, u, u', u'', v, v', v''\}$. Claim 3.3 tells us that there exists a d-Lipschitz mapping $\tilde{f}_S : \widetilde{S} \to X$ with d-Lipschitz seminorm $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \le 1$ such that $\tilde{f}_S(u') \in F(u')$, $\tilde{f}_S(u'') \in F(u'')$, $\tilde{f}_S(v') \in F(v')$ and $\tilde{f}_S(v'') \in F(v'')$.

Let us prove that $\tilde{f}_S(u)$ belongs to the left-hand side of (3.31). Indeed, thanks to the inequality $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)} \le 1$, we have $\|\tilde{f}_S(u') - \tilde{f}_S(u)\| \le \mathrm{d}(u', u)$,

$$\|\tilde{f}_S(u'') - \tilde{f}_S(u)\| \le \mathrm{d}(u'', u) \quad \text{and} \quad \|\tilde{f}_S(x) - \tilde{f}_S(u)\| \le \mathrm{d}(x, u) = r.$$

and $\|\tilde{f}_S(x) - \tilde{f}_S(u)\| \le \mathrm{d}(x, u) = r$. Thanks to these properties and (3.29), $\tilde{f}_S(u) \in C_1 \cap C_2$.

In a similar way, we show that $\tilde{f}_S(x) \in T_x(v, v', v'') = C$, see (3.29) and (3.9). Hence, we have $\tilde{f}_S(u) \in C + rB_X$. Thus, $C_1 \cap C_2 \cap (C + rB_X) \ni \tilde{f}_S(u)$ proving (3.31).

We see now that property (2.4) of Proposition 2.3 holds. This proposition tells us that

$$\begin{aligned} A &= [(C_1 \cap C_2) + LrB_X] \cap C + \theta(L)\varepsilon B_X \\ &\supset [(C_1 \cap C_2) + (Lr+\varepsilon)B_X] \cap \{[(C_1 + rB_X) \cap C] + \varepsilon B_X\} \cap \{[(C_2 + rB_X) \cap C] + \varepsilon B_X\} \\ &= A_1 \cap A_2 \cap A_3. \end{aligned}$$

Let us prove that

$$A_i \supset F^{[2]}(y) \quad \text{for every } i = 1, 2, 3. \tag{3.32}$$

We begin with the set $A_1 = [C_1 \cap C_2] + (Lr + \varepsilon) B_X$. Thus,

$$A_1 = [(F(u') + \mathrm{d}(u', u) B_X) \cap (F(u'') + \mathrm{d}(u'', u) B_X)] + (L\,\mathrm{d}(u, x) + L\theta(L)\,\mathrm{d}(x, y)) B_X.$$

See (3.29). By the triangle inequality,

$$\mathrm{d}(u,x)+\theta(L)\,\mathrm{d}(x,y)\ge \mathrm{d}(u,x)+\mathrm{d}(x,y)\ge \mathrm{d}(u,y).$$

Hence,

$$A_1\supset[(F(u')+\mathrm{d}(u',u)B_X)\cap(F(u'')+\mathrm{d}(u'',u)B_X)]+L\,\mathrm{d}(u,y)B_X=T_y(u,u',u'').$$

But $T_y(u,u',u'')\supset F^{[2]}(y)$, see (3.24), so that $A_1\supset F^{[2]}(y)$.

We turn to the proof of the inclusion $A_2\supset F^{[2]}(y)$. Note that A_2 is defined by

$$A_2=[(C_1+rB_X)\cap C]+\varepsilon B_X. \tag{3.33}$$

By the triangle inequality,

$$C_1+rB_X=F(u')+\mathrm{d}(u',u)B_X+\mathrm{d}(u,x)B_X\supset F(u')+\mathrm{d}(u',x)B_X. \tag{3.34}$$

Let

$$\widetilde{C}=F(u')+\mathrm{d}(u',x)B_X,\quad \widetilde{C}_1=F(v')+\mathrm{d}(v',v)B_X,\quad \widetilde{C}_2=F(v'')+\mathrm{d}(v'',v)B_X, \tag{3.35}$$

and let

$$\tilde{r}=\mathrm{d}(v,x). \tag{3.36}$$

In these settings, $C=T_x(v,v',v'')=[\widetilde{C}_1\cap\widetilde{C}_2]+L\tilde{r}B_X$.

Let

$$\widetilde{A}=\{[(\widetilde{C}_1\cap\widetilde{C}_2)+L\tilde{r}B_X]\cap\widetilde{C}\}+\varepsilon B_X. \tag{3.37}$$

Then, thanks to (3.33) and (3.34),

$$A_2\supset[(F(u')+\mathrm{d}(u',x)B_X)\cap C]+\varepsilon B_X=\{[(\widetilde{C}_1\cap\widetilde{C}_2)+L\tilde{r}B_X]\cap\widetilde{C}\}+\varepsilon B_X. \tag{3.38}$$

Let us prove that the set

$$\widetilde{A}=\{[(\widetilde{C}_1\cap\widetilde{C}_2)+L\tilde{r}B_X]\cap\widetilde{C}\}+\varepsilon B_X\supset F^{[2]}(y).$$

We will do this by applying Proposition 2.3 to the set $\widetilde{A}$. But first we must check the hypothesis of this proposition, i.e., we must show that

$$\widetilde{C}_1\cap\widetilde{C}_2\cap(\widetilde{C}+\tilde{r}B_X)\ne\emptyset. \tag{3.39}$$

To establish this property, we set $S=\{u',v',v''\}$ and $\widetilde{S}=\{x,u',v,v',v''\}$. Claim 3.3 tells us that there exists a d-Lipschitz mapping $\tilde{f}_S\colon\widetilde{S}\to X$, with d-Lipschitz seminorm $\|\tilde{f}_S\|_{\mathrm{Lip}((\widetilde{S},\mathrm{d}),X)}\le 1$, such that

$$\tilde{f}_S(u')\in F(u'),\quad \tilde{f}_S(v')\in F(v')\quad\text{and}\quad \tilde{f}_S(v'')\in F(v'').$$

Combining these properties of $\tilde{f}_S$ with the definitions in (3.35) and (3.36), we conclude that $\widetilde{C}_1\cap\widetilde{C}_2\cap(\widetilde{C}+\tilde{r}B_X)\ni\tilde{f}_S(v)$, proving (3.39).

We recall that $\varepsilon=L\theta(L)\,\mathrm{d}(x,y)$, see (3.30), so that

$$\widetilde{A}=[\{(\widetilde{C}_1\cap\widetilde{C}_2)+L\tilde{r}B_X\}\cap\widetilde{C}]+L\theta(L)\,\mathrm{d}(x,y)B_X\quad\text{(see (3.37))}.$$

We apply Proposition 2.3 to $\widetilde{A}$ and obtain the following:

$$\begin{aligned}\widetilde{A} &\supset \{(\widetilde{C}_1 \cap \widetilde{C}_2) + (L\tilde{r} + L\,\mathrm{d}(x,y))B_X\}\\ &\quad\cap \{[(\widetilde{C}_1 + \tilde{r}B_X) \cap \widetilde{C}] + L\,\mathrm{d}(x,y)B_X\} \cap \{[(\widetilde{C}_2 + \tilde{r}B_X) \cap \widetilde{C}] + L\,\mathrm{d}(x,y)B_X\}\\ &= \widetilde{A}_1 \cap \widetilde{A}_2 \cap \widetilde{A}_3.\end{aligned}$$

We prove that $\widetilde{A}_i \supset F^{[2]}(y)$ for every $i = 1, 2, 3$. First, let us show that

$$\widetilde{A}_1 = (\widetilde{C}_1 \cap \widetilde{C}_2) + (L\tilde{r} + L\,\mathrm{d}(x,y))B_X \supset F^{[2]}(y). \tag{3.40}$$

By (3.36) and the triangle inequality, $\tilde{r} + \mathrm{d}(x,y) = \mathrm{d}(v,x) + \mathrm{d}(x,y) \ge \mathrm{d}(v,y)$, so that

$$\begin{aligned}\widetilde{A}_1 &\supset (\widetilde{C}_1 \cap \widetilde{C}_2) + L\,\mathrm{d}(v,y)B_X\\ &= [(F(v') + \mathrm{d}(v',v)B_X) \cap (F(v'') + \mathrm{d}(v'',v)B_X)] + L\,\mathrm{d}(v,y)B_X\\ &= T_y(v,v',v'').\end{aligned}$$

See (3.35) and (3.9). This inclusion and (3.24) imply (3.40).

Next, let us show that

$$\widetilde{A}_2 = [(\widetilde{C}_1 + \tilde{r}B_X) \cap \widetilde{C}] + L\,\mathrm{d}(x,y)B_X \supset F^{[2]}(y)). \tag{3.41}$$

Thanks to (3.35), (3.36) and the triangle inequality,

$$\widetilde{C}_1 + \tilde{r}B_X = F(v') + \mathrm{d}(v',v)B_X + \mathrm{d}(v,x)B_X \supset F(v') + \mathrm{d}(v',x)B_X$$

so that

$$\widetilde{A}_2 \supset [(F(v') + \mathrm{d}(v',x)B_X) \cap (F(u') + \mathrm{d}(u',x)B_X)] + L\,\mathrm{d}(x,y)B_X = T_y(x,u',v').$$

See (3.9). From this and (3.24), we have $\widetilde{A}_2 \supset T_y(x,u',v') \supset F^{[2]}(y)$, proving (3.41).

In the same way, we can show that

$$\widetilde{A}_3 = [(\widetilde{C}_2 + \tilde{r}B_X) \cap \widetilde{C}] + L\,\mathrm{d}(x,y)B_X \supset T_y(x,u',v'') \supset F^{[2]}(y).$$

Combining this with (3.40) and (3.41), we obtain the required inclusion $\widetilde{A}_i \supset F^{[2]}(y)$ for every $i = 1, 2, 3$. In turn, this proves that

$$\widetilde{A} \supset F^{[2]}(y) \quad \text{because} \quad \widetilde{A} \supset \widetilde{A}_1 \cap \widetilde{A}_2 \cap \widetilde{A}_3 \supset F^{[2]}(y).$$

We know that $A_2 \supset \widetilde{A}$, see (3.38), so that $A_2 \supset F^{[2]}(y)$. In the same fashion, we show that

$$A_3 = [(C_2 + rB_X) \cap C] + L\varepsilon B_X \supset F^{[2]}(y)$$

proving (3.32). Hence, $A \supset A_1 \cap A_2 \cap A_3 \supset F^{[2]}(y)$, so that (3.28) holds.

Thus, (3.27) is proved. By interchanging the roles of elements x and y in this inclusion, we obtain the inclusion $F^{[2]}(y) + \gamma_0(L)\,\mathrm{d}(x,y)B_X \supset F^{[2]}(x)$. These two inclusions imply inequality (3.26), proving the proposition. ∎

We are in a position to complete the proof of Theorem 1.9.

Recall that λ_1 and λ_2 are parameters satisfying (1.5), and $L = \lambda_2/\lambda_1$. Thus, L and λ_1 satisfy (3.1). We also recall that $\mathrm{d} = \lambda_1 \rho$, see (3.2). Let γ be a parameter satisfying (1.5).

In these settings, the mappings $F^{[1]}$ and $F^{[2]}$ defined by formulae (3.3) and (3.4) are the first and the second order $(\{\lambda_1, \lambda_2\}, \rho)$-balanced refinements of F respectively. See Definition 1.6.

Proposition 3.7 tells us that, under these conditions, $F^{[2]}(x) \neq \emptyset$ on $\mathcal{M}$. In turn, Proposition 3.9 states that

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma_0(L)\,\mathrm{d}(x, y) \quad \text{for all } x, y \in \mathcal{M}.$$

Recall that $\gamma_0(L) = L \cdot \theta(L)^2$, where $\theta = \theta(L) = (3L + 1)/(L - 1)$, see (2.3). Hence, $\theta(L) = (3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1)$, so that

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma_0(L)\,\mathrm{d}(x, y) \le L \cdot \theta(L)^2\,\mathrm{d}(x, y)) = \lambda_2\,\frac{(3\lambda_2 + \lambda_1)^2}{(\lambda_2 - \lambda_1)^2}\,\rho(x, y).$$

Combining this inequality with the third inequality in (1.5), we obtain (1.8), proving Theorem 1.9 for the parameters λ_1, λ_2 and γ satisfying (1.5).

In particular, one can set $\lambda_1 = 4/3$, $\lambda_2 = 4$, and $\gamma = 100$. Indeed, in this case, we have $e(\mathfrak{M}, X) \le \lambda_1 = 4/3$, see Remark 3.2, so that λ_1, λ_2 and γ satisfy (1.5).

Next, let X be a two dimensional Euclidean space, and let λ_1, λ_2 and γ satisfy (1.9). In this case, we prove (1.7) and (1.8) by replacing in the proof of Theorem 1.9 the function $\theta = \theta(L)$ defined by (2.3) with the function $\theta(L) = 1 + 2L/\sqrt{L^2 - 1}$. We leave the details to the interested reader. See also Section 3 in [27].

In particular, we can set $\lambda_1 = 4/\pi$, $\lambda_2 = 12/\pi$, and $\gamma = 38$. Indeed, in this case, we have $e(\mathfrak{M}, X) \le 4/\pi$, see Remark 3.2, which implies (1.9) for these values of parameters λ_1, λ_2 and γ.

Finally, suppose that X is a Euclidean space, $\mathcal{M}$ is a subset of a Euclidean space E, and ρ is the Euclidean metric in E. In this case $e(\mathfrak{M}, X) = 1$, see Remark 3.2, so that one can set $\lambda_1 = 1$, $\lambda_2 = 3$ and $\gamma = 25$. Clearly, in this case inequalities (1.9) hold.

The proof of Theorem 1.9 is complete.

4. Balanced refinements of line segments in a Banach space

In this section, we prove Theorem 1.10. Let $(\mathcal{M}, \rho)$ be a pseudometric space, and let $(X, \|\cdot\|)$ be a Banach space. We assume that $\dim X > 1$. Let us recall that $\mathcal{K}_1(X)$ is the family of all non-empty convex compacts in X of dimension at most 1 (i.e., the family of all points and all bounded closed line segments in X).

We need the following version of one dimensional Helly's theorem.

Theorem 4.1. *Let $\mathcal{K}$ be a collection of closed convex subsets of X containing a set $K_0 \in \mathcal{K}_1(X)$. If K_0 has a common point with any two members of $\mathcal{K}$, then there exists a point common to all of the collection $\mathcal{K}$.*

Proof. We introduce a family $\widetilde{\mathcal{K}} = \{K \cap K_0 : K \in \mathcal{K}\}$, and apply to $\widetilde{\mathcal{K}}$ the one dimensional Helly theorem. (See part (a) of Lemma 5.2.) ∎

We also need the following variant of Proposition 2.3 for the family $\mathcal{K}_1(X)$.

Proposition 4.2. *Let X be a Banach space, and let $r \ge 0$. Let $C, C_1, C_2 \subset X$ be convex closed subsets, and let $C_1 \in \mathcal{K}_1(X)$. Suppose that*

$$C_1 \cap C_2 \cap (C + rB_X) \ne \emptyset. \tag{4.1}$$

Then for every $L > 1$ and every $\varepsilon > 0$, the inclusion

$$\begin{aligned}&[\{(C_1 \cap C_2) + LrB_X\} \cap C] + \theta(L)\varepsilon B_X \\ &\qquad \supset [(C_1 \cap C_2) + (Lr + \varepsilon)B_X] \cap [\{(C_1 + rB_X) \cap C\} + \varepsilon B_X]\end{aligned}$$

holds. Here $\theta(L)$ is the same as in Theorem 2.1, i.e., $\theta(L) = (3L+1)/(L-1)$ for an arbitrary X, and $\theta(L) = 1 + 2L/\sqrt{L^2-1}$ whenever X is a Euclidean space.

Proof. For the detailed proof of the proposition we refer the reader to Proposition 4.2 in [27]. This proof is a slight modification of the proof of Proposition 2.3, in which we use Theorem 4.1 (i.e., the one dimensional version of the Helly theorem) rather than Theorem 2.2 (i.e., the two dimensional Helly theorem). ■

We recall that $\dim X > 1$, so that the finiteness number $N(1, X) = \min\{2^2, 2^{\dim X}\} = 4$. Let $F\colon \mathcal{M} \to \mathcal{K}_1(X)$ be a set-valued mapping. We suppose that F satisfies the hypothesis of Theorem 1.10, i.e., that the following statement is true.

Claim 4.3. *For every subset $\mathcal{M}' \subset \mathcal{M}$ with $\#\mathcal{M}' \le 4$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}\colon \mathcal{M}' \to X$ with $\|f\|_{\mathrm{Lip}(\mathcal{M}',X)} \le 1$.*

Let $\vec{\lambda} = \{\lambda_1, \lambda_2\}$, and let $F^{[1]}$ and $F^{[2]}$ be the first and the second order $(\vec{\lambda}, \rho)$-balanced refinements of F. See Definition 1.6. Our aim is to show that if

$$\lambda_1 \ge 1, \quad \lambda_2 \ge 3\lambda_1, \quad \gamma \ge \lambda_2(3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1), \tag{4.2}$$

then the set-valued mapping $F^{[2]}$ is a γ-core of F (with respect to ρ), i.e., $F^{[2]}(x) \ne \emptyset$ for every $x \in \mathcal{M}$ and

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma\rho(x,y) \quad \text{for all } x, y \in \mathcal{M}.$$

To prove this, we set $L = \lambda_2/\lambda_1$ and introduce a new pseudometric on $\mathcal{M}$ defined by

$$\mathrm{d}(x,y) = \lambda_1\rho(x,y), \quad x, y \in \mathcal{M}.$$

Note that, thanks to (4.2),

$$L \ge 3 \quad \text{and} \quad \rho \le \mathrm{d} \quad \text{on } \mathcal{M}. \tag{4.3}$$

We also note that in these settings, for every $x \in \mathcal{M}$,

$$F^{[1]}(x) = \bigcap_{z\in\mathcal{M}} [F(z) + \mathrm{d}(x,z)B_X], \quad F^{[2]}(x) = \bigcap_{z\in\mathcal{M}} [F^{[1]}(z) + L\,\mathrm{d}(x,z)B_X]. \tag{4.4}$$

Next, we need the following analog of Lemma 3.4.

Lemma 4.4. *Let $\mathcal{K}$ be a family of convex closed subsets of X containing a set $K_0 \in \mathcal{K}_1(X)$. Suppose that $\cap\{K : K \in \mathcal{K}\} \neq \emptyset$. Then for every $r \geq 0$, we have*

$$\Big(\bigcap_{K\in\mathcal{K}} K\Big) + rB_X = \bigcap_{K\in\mathcal{K}} \{[K \cap K_0] + rB_X\}.$$

Proof. Let $\widetilde{\mathcal{K}} = \{K \cap K_0 : K \in \mathcal{K}\}$. Clearly, $\widetilde{\mathcal{K}} \subset \mathcal{K}_1(X)$. It is also clear that the statement of the lemma is equivalent to the equality

$$\Big(\bigcap_{\widetilde{K}\in\widetilde{\mathcal{K}}} \widetilde{K}\Big) + rB_X = \bigcap_{\widetilde{K}\in\widetilde{\mathcal{K}}} \{\widetilde{K} + rB_X\}$$

provided $\cap\{\widetilde{K} : \widetilde{K} \in \mathcal{K}\} \neq \emptyset$. We prove this equality by a slight modification of the proof of Lemma 3.4. More specifically, we obtain the result by using in that proof Helly's Theorem 4.1 instead of Theorem 2.2. We leave the details to the interested reader. ∎

Lemma 4.5. *For every $x \in \mathcal{M}$, the set $F^{[1]}(x)$ belongs to the family $\mathcal{K}_1(X)$. Moreover, for every $x, z \in \mathcal{M}$, we have*

$$F^{[1]}(z) + L\,\mathrm{d}(x,z)B_X = \bigcap_{v\in\mathcal{M}} \big\{[(F(v) + \mathrm{d}(z,v)B_X) \cap F(z)] + L\,\mathrm{d}(x,z)B_X\big\}. \tag{4.5}$$

Proof. Let $\mathcal{K} = \{F(z) + \mathrm{d}(z,x)B_X : z \in \mathcal{M}\}$. Clearly, $\mathcal{K}$ is a family of bounded closed convex subsets of X containing the set $F(x) \in \mathcal{K}_1(X)$. Theorem 4.1 tells us that the set $F^{[1]}(x) = \cap\{K : K \in \mathcal{K}\}$ is non-empty whenever, for every $z', z'' \in \mathcal{M}$, the set

$$E(x,z',z'') = F(x) \cap [F(z') + \mathrm{d}(z',x)B_X] \cap [F(z'') + \mathrm{d}(z'',x)B_X] \neq \emptyset. \tag{4.6}$$

Fix $z', z'' \in \mathcal{M}$ and set $\mathcal{M}' = \{x, z', z''\}$. Thanks to Claim 4.3, there exists a function $f_{\mathcal{M}'}\colon \mathcal{M}' \to X$ satisfying the following conditions: $f_{\mathcal{M}'}(x) \in F(x)$, $f_{\mathcal{M}'}(z') \in F(z')$, $f_{\mathcal{M}'}(z'') \in F(z'')$, $\|f_{\mathcal{M}'}(z') - f_{\mathcal{M}'}(x)\| \leq \rho(z',x) \leq \mathrm{d}(z',x)$, and

$$\|f_{\mathcal{M}'}(z'') - f_{\mathcal{M}'}(x)\| \leq \rho(z'',x) \leq \mathrm{d}(z'',x).$$

See (4.3). Then $f_{\mathcal{M}'}(x) \in E(x,z',z'')$, so that (4.6) holds. Hence, $F^{[1]}(x) \neq \emptyset$, proving that $F^{[1]}(x) \in \mathcal{K}_1(X)$.

It remains to note that equality (4.5) is immediate from (4.4) and Lemma 4.4.

The proof of the lemma is complete. ∎

Lemma 4.6. *For every $x \in \mathcal{M}$, the equality*

$$F^{[2]}(x) = \bigcap_{u,u'\in\mathcal{M}} \big\{[(F(u') + \mathrm{d}(u',u)B_X) \cap F(u)] + L\,\mathrm{d}(u,x)B_X\big\}$$

holds.

Proof. The statement of the lemma is immediate from (4.4) and Lemma 4.5. ∎

Given $x, u, u' \in \mathcal{M}$ we put

$$\widetilde{T}_x(u,u') = [(F(u') + \mathrm{d}(u',u)B_X) \cap F(u)] + L\,\mathrm{d}(u,x)B_X. \tag{4.7}$$

Now, Lemma 4.6 reformulates as follows:

$$F^{[2]}(x) = \bigcap_{u,u' \in \mathcal{M}} \widetilde{T}_x(u,u'). \tag{4.8}$$

Proposition 4.7. *For every $x \in \mathcal{M}$, the set $F^{[2]}(x)$ is non-empty.*

Proof. Recall that $F(x) \in \mathcal{K}_1(X)$. Furthermore, by (4.7), $F(x) = \widetilde{T}_x(x,x)$. Therefore, by (4.8) and Helly's Theorem 4.1, the set $F^{[2]}(x) \neq \emptyset$ whenever for every $u_i, u_i' \in \mathcal{M}$, $i = 1, 2$, we have

$$F(x) \cap \widetilde{T}_x(u_1,u_1') \cap \widetilde{T}_x(u_2,u_2') \neq \emptyset. \tag{4.9}$$

Thanks to (4.7),

$$\widetilde{T}_x(u_i,u_i') = [(F(u_i') + \mathrm{d}(u_i',u_i)B_X) \cap F(u_i)] + L\,\mathrm{d}(u_i,x)B_X, \quad i = 1, 2. \tag{4.10}$$

Let us fix elements $u_1, u_1', u_2, u_2' \in \mathcal{M}$ and prove that property (4.9) holds.

First we note that, without loss of generality, one may assume that $\rho(u_1,x) \ge \rho(u_2,x)$. Next, we introduce the following sets:

$$\begin{aligned} &G_1 = F(u_2), \quad G_2 = F(u_2') + \rho(u_2,u_2')B_X, \quad G_3 = F(x) + \rho(u_2,x)B_X,\\ &G_4 = [(F(u_1') + \rho(u_1',u_1)B_X) \cap F(u_1)] + \rho(u_1,u_2,)B_X. \end{aligned} \tag{4.11}$$

We prove now that if $G_1 \cap G_2 \cap G_3 \cap G_4 \neq \emptyset$, then (4.9) holds.

Indeed, let $\widetilde{\mathcal{M}} = \{u_1', u_1, x, u_2, u_2'\}$. This property and the definitions in (4.11) imply the existence of a mapping $g\colon \widetilde{\mathcal{M}} \to X$ with the following properties: $g(v) \in F(v)$ on $\widetilde{\mathcal{M}}$,

$$\|g(u_1) - g(u_1')\| \le \rho(u_1,u_1'), \quad \|g(u_1) - g(u_2)\| \le \rho(u_1,u_2), \tag{4.12}$$

and

$$\|g(u_2) - g(u_2')\| \le \rho(u_2,u_2'), \quad \|g(u_2) - g(x)\| \le \rho(u_2,x). \tag{4.13}$$

We establish (4.9) by showing that

$$g(x) \in F(x) \cap \widetilde{T}_x(u_1,u_1') \cap \widetilde{T}_x(u_2,u_2'). \tag{4.14}$$

In fact, from the above properties of g, it follows that $g(x) \in F(x)$. We also know that $g(u_2) \in F(u_2)$, $g(u_2') \in F(u_2')$. Thanks to (4.12), (4.13) and (4.3),

$$\|g(u_2) - g(u_2')\| \le \rho(u_2,u_2') \le \mathrm{d}(u_2,u_2'), \quad \text{and}$$
$$\|g(u_2) - g(x)\| \le \rho(u_2,x) \le L\,\mathrm{d}(u_2,x).$$

From these properties of g and (4.10), we have

$$g(x) \in [(F(u_2') + \mathrm{d}(u_2',u_2)B_X) \cap F(u_2)] + L\,\mathrm{d}(u_2,x)B_X = \widetilde{T}_x(u_2,u_2').$$

It remains to show that $g(x) \in \widetilde{T}_x(u_1, u'_1)$. As we know,

$$g(x) \in F(x), \quad g(u_1) \in F(u_1), \quad \text{and} \quad g(u'_1) \in F(u'_1). \tag{4.15}$$

Furthermore, thanks to (4.12) and (4.3),

$$\|g(u_1) - g(u'_1)\| \le \rho(u_1, u'_1) \le \mathrm{d}(u_1, u'_1). \tag{4.16}$$

Let us estimate $\|g(u_1) - g(x)\|$. From (4.12), (4.13) and the triangle inequality, we have

$$\begin{aligned}\|g(u_1) - g(x)\| &\le \|g(u_1) - g(u_2)\| + \|g(u_2) - g(x)\| \le \rho(u_1, u_2) + \rho(u_2, x)\\ &\le \rho(u_1, x) + \rho(x, u_2) + \rho(u_2, x) = \rho(u_1, x) + 2\rho(x, u_2).\end{aligned}$$

Recall that $\rho(u_1, x) \ge \rho(u_2, x)$. This and (4.3) yield

$$\|g(u_1) - g(x)\| \le 3\rho(u_1, x) \le L\,\mathrm{d}(u_1, x).$$

This inequality, (4.15), (4.16) and (4.10) imply the required property $g(x) \in \widetilde{T}_x(u_1, u'_1)$ proving (4.14).

Thus, to complete the proof of the proposition, we have to prove that the sets G_1, G_2, G_3 and G_4 have a common point. Note that $G_1 = F(u_2) \in \mathcal{K}_1(X)$ so that, by Theorem 4.1, this property holds provided

$$G_1 \cap G_i \cap G_j \ne \emptyset \quad \text{for every } 2 \le i, j \le 4,\ i \ne j. \tag{4.17}$$

Let us first prove that $G_1 \cap G_2 \cap G_3 \ne \emptyset$. Let $\mathcal{M}_1 = \{u'_2, u_2, x\}$. Thanks to Claim 4.3, there exists a mapping $f_1 \colon \mathcal{M}_1 \to X$ with the following properties: $f_1(x) \in F(x)$, $f_1(u_2) \in F(u_2)$, $f_1(u'_2) \in F(u'_2)$,

$$\|f_1(u_2) - f_1(x)\| \le \rho(u_2, x) \quad \text{and} \quad \|f_1(u_2) - f_1(u'_2)\| \le \rho(u_2, u'_2).$$

These properties of f_1 and the definitions in (4.11) tell us that $f_1(u_2) \in G_1 \cap G_2 \cap G_3$, proving that the sets G_1, G_2 and G_3 have a common point.

Let us prove that $G_1 \cap G_2 \cap G_4 \ne \emptyset$.

Let $\mathcal{M}_2 = \{u'_1, u_1, u'_2, u_2\}$. Using Claim 4.3, we produce a mapping $f_2 : \mathcal{M}_2 \to X$ such that $f_2(u_i) \in F(u_i)$, $f_2(u'_i) \in F(u'_i)$ for every $i = 1, 2$, $\|f_2(u_1) - f_2(u'_1)\| \le \rho(u_1, u'_1)$,

$$\|f_2(u_1) - f_2(u_2)\| \le \rho(u_1, u_2) \quad \text{and} \quad \|f_2(u_2) - f_2(u'_2)\| \le \rho(u_2, u'_2).$$

These properties of f_2 and (4.11) yield $f_2(u_2) \in G_1 \cap G_2 \cap G_4$, proving that the sets G_1, G_2 and G_4 have a common point.

In the same way, we show that $G_1 \cap G_3 \cap G_4 \ne \emptyset$. (We set $\mathcal{M}_3 = \{u'_1, u_1, x, u_2\}$, produce a corresponding function $f_3 \colon \mathcal{M}_3 \to X$ and show that $f_3(u_2) \in G_1 \cap G_3 \cap G_4$.)

Thus, (4.17) holds, proving that the sets G_i have a common point.

The proof of the proposition is complete. ■

In this section, we set $\gamma_0 = \gamma_0(L) = L\theta(L)$.

Proposition 4.8. *For every $x, y \in \mathcal{M}$, the inequality*

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma_0(L)\,\mathrm{d}(x,y)$$

holds.

Proof. Let $x, y \in \mathcal{M}$. Thanks to (4.8),

$$F^{[2]}(x) = \bigcap_{u,u' \in \mathcal{M}} \widetilde{T}_x(u,u') \quad \text{and} \quad F^{[2]}(y) = \bigcap_{u,u' \in \mathcal{M}} \widetilde{T}_y(u,u'). \tag{4.18}$$

Recall that

$$\widetilde{T}_x(u,u') = [(F(u') + \mathrm{d}(u',u)B_X) \cap F(u)] + L\,\mathrm{d}(u,x)B_X. \tag{4.19}$$

We know that $F^{[2]}(x) \ne \emptyset$, see Proposition 4.7, and $\widetilde{T}_x(x,x) = F(x) \in \mathcal{K}_1(X)$. These properties, (4.18) and Lemma 4.4 tell us that

$$F^{[2]}(x) + \gamma_0(L)\,\mathrm{d}(x,y)B_X = \bigcap_{u,u' \in \mathcal{M}} \{[\widetilde{T}_x(u,u') \cap F(x)] + \gamma_0(L)\,\mathrm{d}(x,y)B_X\}. \tag{4.20}$$

We fix $u, u' \in \mathcal{M}$ and introduce a set

$$\widetilde{A} = [\widetilde{T}_x(u,u') \cap F(x)] + \gamma_0(L)\,\mathrm{d}(x,y)B_X.$$

We also introduce sets

$$C_1 = F(u), \quad C_2 = F(u') + \mathrm{d}(u',u)B_X, \quad \text{and} \quad C = F(x). \tag{4.21}$$

Let

$$\varepsilon = L\,\mathrm{d}(x,y) \quad \text{and} \quad r = \mathrm{d}(x,u). \tag{4.22}$$

In these settings, $\gamma_0(L)\,\mathrm{d}(x,y) = \theta(L)\varepsilon$ and

$$\widetilde{A} = [\widetilde{T}_x(u,u') \cap F(x)] + \gamma_0(L)\,\mathrm{d}(x,y)B_X = [\{(C_1 \cap C_2) + LrB_X\} \cap C] + \theta(L)\varepsilon B_X.$$

Let us apply Proposition 4.2 to the set $\widetilde{A}$. To do this, we have to verify condition (4.1), i.e., to show that

$$C_1 \cap C_2 \cap (C + rB_X) \ne \emptyset. \tag{4.23}$$

Let $\widetilde{\mathcal{M}} = \{x, u, u'\}$. Thanks to Claim 4.3, there exists a ρ-Lipschitz selection $f_{\widetilde{\mathcal{M}}}$ of the restriction $F|_{\widetilde{\mathcal{M}}}$ with $\|f_{\widetilde{\mathcal{M}}}\|_{\mathrm{Lip}((\widetilde{\mathcal{M}};\rho),X)} \le 1$. Thus, $f_{\widetilde{\mathcal{M}}}(u') \in F(u')$, $f_{\widetilde{\mathcal{M}}}(u) \in F(u)$, $f_{\widetilde{\mathcal{M}}}(x) \in F(x)$,

$$\|f_{\widetilde{\mathcal{M}}}(u') - f_{\widetilde{\mathcal{M}}}(u)\| \le \rho(u',u) \quad \text{and} \quad \|f_{\widetilde{\mathcal{M}}}(x) - f_{\widetilde{\mathcal{M}}}(u)\| \le \rho(x,u).$$

Let us prove that

$$f_{\widetilde{\mathcal{M}}}(u) \in C_1 \cap C_2 \cap (C + rB_X). \tag{4.24}$$

Indeed, $f_{\widetilde{\mathcal{M}}}(u) \in F(u) = C_1$, see (4.21). Furthermore, $f_{\widetilde{\mathcal{M}}}(u') \in F(u')$ and, thanks to (4.3), $\rho \le \mathrm{d}$ on $\mathcal{M}$. Hence,

$$\|f_{\widetilde{\mathcal{M}}}(u') - f_{\widetilde{\mathcal{M}}}(u)\| \le \rho(u', u) \le \mathrm{d}(u', u)$$

proving that $f_{\widetilde{\mathcal{M}}}(u) \in C_2$, see (4.21). Finally, by (4.21) and (4.22), $f_{\widetilde{\mathcal{M}}}(x) \in F(x) = C$ and

$$\|f_{\widetilde{\mathcal{M}}}(x) - f_{\widetilde{\mathcal{M}}}(u)\| \le \rho(x, u) \le \mathrm{d}(x, u) = r, \quad \text{so that} \quad f_{\widetilde{\mathcal{M}}}(u) \in C + rB_X.$$

Thus, (4.24) is true so that property (4.23) holds. Furthermore, $C_1 = F(u) \in \mathcal{K}_1(X)$, so that all conditions of the hypothesis of Proposition 4.2 are satisfied. By this proposition,

$$\begin{aligned}\widetilde{A} &= [\{(C_1 \cap C_2) + LrB_X\} \cap C] + \theta(L)\varepsilon B_X \\ &\supset [(C_1 \cap C_2) + (Lr + \varepsilon)B_X] \cap [\{(C_1 + rB_X) \cap C\} + \varepsilon B_X] \\ &= \widetilde{A}_1 \cap \widetilde{A}_2.\end{aligned}$$

Let us prove that $\widetilde{A}_i \supset F^{[2]}(y)$ for every $i = 1, 2$.

We begin with the set

$$\widetilde{A}_1 = C_1 \cap C_2 + (Lr + \varepsilon)B_X.$$

Thanks to (4.21) and (4.22),

$$\widetilde{A}_1 = [\{F(u') + \mathrm{d}(u', u)B_X\} \cap F(u)] + (L\,\mathrm{d}(u, x) + L\,\mathrm{d}(x, y))B_X.$$

By the triangle inequality, $\mathrm{d}(u, x) + \mathrm{d}(x, y) \ge \mathrm{d}(u, y)$, so that

$$\widetilde{A}_1 \supset [\{F(u') + \mathrm{d}(u', u)B_X\} \cap F(u)] + L\,\mathrm{d}(u, y)B_X = \widetilde{T}_y(u, u'), \quad \text{see (4.19)}.$$

But, by (4.18), $\widetilde{T}_y(u, u') \supset F^{[2]}(y)$, which implies the required inclusion $\widetilde{A}_1 \supset F^{[2]}(y)$.

We turn to the set $\widetilde{A}_2 = [(C_1 + rB_X) \cap C] + \varepsilon B_X$. By (4.7), (4.21) and (4.22),

$$\widetilde{A}_2 = [(F(u) + \mathrm{d}(u, x)B_X) \cap F(x)] + L\,\mathrm{d}(x, y)B_X = T_y(x, u).$$

Thanks to (4.18), $\widetilde{T}_y(u, x) \supset F^{[2]}(y)$, proving that $\widetilde{A}_2 \supset F^{[2]}(y)$.

Thus,

$$\widetilde{A} = [\widetilde{T}_x(u, u') \cap F(x)] + \gamma_0(L)\,\mathrm{d}(x, y)B_X \supset \widetilde{A}_1 \cap \widetilde{A}_2 \supset F^{[2]}(y) \quad \text{for every } u, u' \in \mathcal{M}.$$

From this and the representation (4.20), we have $F^{[2]}(x) + \gamma_0(L)\,\mathrm{d}(x, y)B_X \supset F^{[2]}(y)$.

By interchanging the roles of x and y, we obtain also

$$F^{[2]}(y) + \gamma_0(L)\,\mathrm{d}(x, y)B_X \supset F^{[2]}(x).$$

These two inclusions and (1.2) imply the statement of the proposition. ∎

We complete the proof of Theorem 1.10 as follows: we fix λ_1, λ_2 and γ satisfying inequalities (1.10). Then, by Proposition 4.7, the set $F^{[2]}(x) \ne \emptyset$ for every $x \in \mathcal{M}$.

In turn, Proposition 4.8 tells us that $\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma_0(L)\,\mathrm{d}(x,y)$ on $\mathcal{M}$. We recall that $L = \lambda_2/\lambda_1$, $\mathrm{d} = \lambda_1\rho$, $\gamma_0(L) = L\theta(L)$ and $\theta(L) = (3L+1)/(L-1)$. Hence,

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \gamma_0(L)\,\mathrm{d}(x,y) = L\Big(\frac{3L+1}{L-1}\Big)\,\mathrm{d}(x,y) = \frac{\lambda_2(3\lambda_2+\lambda_1)}{\lambda_2-\lambda_1}\,\rho(x,y).$$

This inequality and (1.10) imply (1.8).

Thus, (1.7) and (1.8) hold provided λ_1, λ_2 and γ satisfy inequalities (1.10). In particular, we can set $\lambda_1 = 1$, $\lambda_2 = 3$ and $\gamma = \lambda_2(3\lambda_2+\lambda_1)/(\lambda_2-\lambda_1) = 15$.

Let now X be a Euclidean space, and let λ_1, λ_2 and γ be the parameters satisfying inequalities (1.11). In this case, we replace in the above calculations the constant $\theta(L) = (3L+1)/(L-1)$ with $\theta(L) = 1 + 2L/\sqrt{L^2-1}$. This leads us to the required estimate

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \Big(\lambda_2 + \frac{2\lambda_2^2}{(\lambda_2^2-\lambda_1^2)^{1/2}}\Big)\,\rho(x,y) \le \gamma\rho(x,y)$$

proving that (1.7) and (1.8) hold for λ_1, λ_2 and γ satisfying (1.11).

The proof of Theorem 1.10 is complete.

5. The main theorem in ℓ^2_∞

5.1. The case $X = \mathbb{R}$

Proposition 5.1. *Let $(\mathcal{M},\rho)$ be a pseudometric space. Let $m = 1$ and let $X = \mathbb{R}$; thus, $\ell = \ell(m,X) = 1$, see (1.1). In this case, Conjecture 1.8 holds for every $\lambda_1 \ge 1$ and $\gamma \ge 1$.*

Thus, the following statement is true. Let F be a set-valued mapping from $\mathcal{M}$ into the family $\mathcal{K}(\mathbb{R})$ of all closed bounded intervals in $\mathbb{R}$. Suppose that for every $x, y \in \mathcal{M}$ there exist points $g(x) \in F(x)$ and $g(y) \in F(y)$ such that $|g(x) - g(y)| \le \rho(x,y)$.

Let $F^{[1]}(x)$, $x \in \mathcal{M}$, be the λ_1-balanced refinement of the mapping F, i.e., the set

$$F^{[1]}(x) = \bigcap_{z\in\mathcal{M}} [F(z) + \lambda_1\rho(x,z)I_0], \quad \textit{where} \quad I_0 = [-1,1].$$

Then $F^{[1]}(x) \ne \emptyset$ for every $x \in \mathcal{M}$, and

$$\mathrm{d_H}(F^{[1]}(x), F^{[1]}(y)) \le \gamma\rho(x,y) \quad \textit{for all } x, y \in \mathcal{M}.$$

For a detailed proof of the proposition, we refer the reader to Section 5 in [27].

Here we only note that Proposition 5.1 easily follows from the one dimensional Helly theorem and a formula for a neighborhood of the intersection of intervals in $\mathbb{R}$. We formulate these statements in the following result.

Lemma 5.2. *Let $\mathcal{K} \subset \mathcal{K}(\mathbb{R})$ be a collection of closed bounded intervals in $\mathbb{R}$.*

(a) *(Helly's theorem in $\mathbb{R}$). If the intersection of every two intervals from $\mathcal{K}$ is nonempty, then there exists a point in $\mathbb{R}$ common to all of the family $\mathcal{K}$.*

(b) *Suppose that $\cap\{K : K \in \mathcal{K}\} \ne \emptyset$. Then for every $r \ge 0$ we have*

$$\Big(\bigcap_{K\in\mathcal{K}} K\Big) + rI_0 = \bigcap_{K\in\mathcal{K}} \{K + rI_0\}.$$

Proof. In Lemma 3.4 we proved an analog of property (b) of Lemma 5.2 for $\mathbb{R}^2$. We prove part (b) by replacing in that proof the Helly Theorem 2.2 with the one dimensional Helly theorem formulated in part (a) of the present lemma. We leave the details to the interested reader. ∎

5.2. Rectangular hulls of plane convex sets

Let us fix some additional notation. We let $\mathfrak{R}(\mathbb{R}^2)$ denote the family of all bounded closed rectangles in $\mathbb{R}^2$ *with sides parallel to the coordinate axes* Ox_1 and Ox_2.

Let $Q_0 = B_X$ be the unit ball of the Banach space $X = \ell^2_\infty$, i.e., the square $Q_0 = [-1,1]^2$. Given $a \in \mathbb{R}^2$ and $r \ge 0$, we set $rQ_0 = [-r,r]^2$ and $Q(a,r) = rQ_0 + a$.

Definition 5.3. Let S be a non-empty bounded convex closed subset in $\mathbb{R}^2$. We set

$$\mathcal{H}[S] = \cap\{\Pi : \Pi \in \mathfrak{R}(\mathbb{R}^2), \Pi \supset S\},$$

and refer to $\mathcal{H}[S]$ as the *"rectangular hull"* of the set S.

Note the following useful representation of the rectangular hull, which easily follows from Definition 5.3:

$$\mathcal{H}[S] = (S + Ox_1) \cap (S + Ox_2). \tag{5.1}$$

In the next section we will need the following auxiliary result.

Lemma 5.4. *Let $K_1, K_2 \in \mathcal{K}(\mathbb{R}^2)$ be two convex compacts in $\mathbb{R}^2$ with non-empty intersection. Let $\tau \ge 0$ and let $Q = [-\tau,\tau]^2$. Then*

$$(K_1 \cap K_2) + Q = (K_1 + Q) \cap (K_2 + Q) \cap \mathcal{H}[(K_1 \cap K_2) + Q]. \tag{5.2}$$

Proof. Obviously, the right-hand side of (5.2) contains its left-hand side.

Let us prove the converse statement. Fix a point

$$a \in (K_1 + Q) \cap (K_2 + Q) \cap \mathcal{H}[K_1 \cap K_2 + Q]. \tag{5.3}$$

Our aim is to prove that $a \in (K_1 \cap K_2) + Q$. Clearly, this property holds if and only if $Q(a,\tau) \cap K_1 \cap K_2 \ne \emptyset$. It is also clear that

$$Q(a,\tau) = \Pi_1(a) \cap \Pi_2(a) \quad \text{where} \quad \Pi_i(a) = Q(a,\tau) + Ox_i, \quad i = 1,2.$$

Thus,

$$a \in (K_1 \cap K_2) + Q \quad \text{provided} \quad K_1 \cap K_2 \cap \Pi_1(a) \cap \Pi_2(a) \ne \emptyset.$$

Thanks to Theorem 2.2, the family of sets $\{K_1, K_2, \Pi_1(a), \Pi_2(a)\}$ has a common point provided any three members of this family have a non-empty intersection.

Let us prove that it is true for a satisfying (5.3). Indeed, $a \in K_i + Q$, so that $K_i \cap Q(a,\tau) \ne \emptyset$, $i = 1,2$. Hence,

$$K_i \cap \Pi_1(a) \cap \Pi_2(a) = K_i \cap Q(a,\tau) \ne \emptyset, \quad i = 1,2.$$

Next, thanks to (5.1) and (5.3), for every $i = 1,2$,

$$a \in \mathcal{H}[(K_1 \cap K_2) + Q] \subset (K_1 \cap K_2) + Q + Ox_i.$$

Hence, $K_1 \cap K_2 \cap \Pi_i(x) \ne \emptyset$, $i = 1,2$, and the proof of the lemma is complete. ∎

5.3. Balanced refinements of set-valued mappings in ℓ^2_∞

In this section we refine the result of Theorem 1.9 for the space $X = \ell^2_\infty$.

Theorem 5.5. *In the settings of Theorem 1.9, properties (1.7) and (1.8) hold provided $X = \ell^2_\infty$,*

$$\lambda_1 \ge 1, \quad \lambda_2 \ge 3\lambda_1 \quad \text{and} \quad \gamma \ge \lambda_2(3\lambda_2+\lambda_1)/(\lambda_2-\lambda_1). \tag{5.4}$$

In particular, (1.7) and (1.8) hold whenever $\lambda_1 = 1$, $\lambda_2 = 3$ and $\gamma = 15$.

Proof. We mainly follow the scheme of the proof of Theorem 1.9 given in Section 3.

Let $F:\mathcal{M}\to\mathcal{K}(\mathbb{R}^2)$ be a set-valued mapping satisfying the hypothesis of Theorem 5.5. Thus, the next statement is true.

Claim 5.6. *For every $\mathcal{M}'\subset\mathcal{M}$, $\#\mathcal{M}'\le 4$, the restriction $F|_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a ρ-Lipschitz selection $f_{\mathcal{M}'}:\mathcal{M}'\to\ell^2_\infty$ with ρ-Lipschitz seminorm $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}((\mathcal{M}',\rho),\ell^2_\infty)}\le 1$.*

Let λ_1 and λ_2 be positive constants satisfying inequalities (5.4). We set $L = \lambda_2/\lambda_1$; thus, $L \ge 3$. Then we introduce a pseudometric on $\mathcal{M}$ defined by $\mathrm{d}(x,y) = \lambda_1\rho(x,y)$, $x,y\in\mathcal{M}$.

We let $F^{[1]}$ and $F^{[2]}$ denote the first and the second order $(\{1,L\},\mathrm{d})$-balanced refinements of F respectively. See Definition 1.6. Thus, for every $x\in\mathcal{M}$,

$$F^{[1]}(x) = \bigcap_{z\in\mathcal{M}}[F(z)+\mathrm{d}(x,z)Q_0] \quad \text{and} \quad F^{[2]}(x) = \bigcap_{z\in\mathcal{M}}[F^{[1]}(z)+L\,\mathrm{d}(x,z)Q_0].$$

We also recall that $e(\mathfrak{M},\ell^2_\infty) = 1$. In this case, Lemma 3.5 and Proposition 3.7 tell us that $F^{[1]}(x)\ne\emptyset$ and $F^{[2]}(x)\ne\emptyset$ for every $x\in\mathcal{M}$.

Let

$$\tilde\gamma(L) = L\,\theta(L), \quad \text{where} \quad \theta(L) = (3L+1)/(L-1).$$

Let us prove that

$$\mathrm{d_H}(F^{[2]}(x),F^{[2]}(y)) \le \tilde\gamma(L)\,\mathrm{d}(x,y) \quad \text{for every } x,y\in\mathcal{M}. \tag{5.5}$$

We recall that, thanks to formula (3.10), $F^{[2]}(x) = \cap\{T_x(u,u',u'') : u,u',u''\in\mathcal{M}\}$ where, given $u,u',u''\in\mathcal{M}$,

$$T_x(u,u',u'') = \{[F(u')+\mathrm{d}(u',u)Q_0]\cap[F(u'')+\mathrm{d}(u'',u)Q_0]\} + L\,\mathrm{d}(u,x)Q_0. \tag{5.6}$$

In particular, $T_x(u,u',u'')\ne\emptyset$ for all $u,u',u''\in\mathcal{M}$ (because $F^{[2]}(x)\ne\emptyset$).

The next lemma is a refinement of the formula (3.25) for the special case of $X = \ell^2_\infty$.

Lemma 5.7. *Let $\tau > 0$ and let $Q = \tau Q_0 = [-\tau,\tau]^2$. Then, for every $x\in\mathcal{M}$, we have*

$$F^{[2]}(x) + Q = \bigcap_{v,u,u',u''\in\mathcal{M}} \big\{[T_x(u,u',u'')\cap(F(v)+\mathrm{d}(x,v)Q_0)] + Q\big\}. \tag{5.7}$$

Proof. Lemma 3.8 tells us that

$$F^{[2]}(x) + Q = \bigcap \{[T_x(u_1, u_1', u_1'') \cap T_x(u_2, u_2', u_2'')] + Q\}, \tag{5.8}$$

where the intersection is taken over all $u_i, u_i', u_i'' \in \mathcal{M}$, $i = 1, 2$. Note also that, by (5.6),

$$F(v) + \mathrm{d}(x, v)Q_0 = T_x(x, v, v).$$

From this and (5.8) it follows that the right-hand side of (5.7) contains its left-hand side.

We prove the converse statement. Let us fix a point

$$a \in \bigcap_{v,u,u',u'' \in \mathcal{M}} \{[T_x(u, u', u'') \cap (F(v) + \mathrm{d}(x, v)Q_0)] + Q\} \tag{5.9}$$

and show that $a \in F^{[2]}(x) + Q$. In view of formula (5.8), it suffice to prove that for every $u_1, u_1', u_1'', u_2, u_2', u_2'' \in \mathcal{M}$, the point a belongs to the set A defined by

$$A = [T_x(u_1, u_1', u_1'') \cap T_x(u_2, u_2', u_2'')] + Q. \tag{5.10}$$

To see this, given $i \in \{1, 2\}$, we introduce the following sets: $Q_i = L\,\mathrm{d}(u_i, x)Q_0$,

$$K_i' = F(u_i') + \mathrm{d}(u_i, u_i')Q_0 \quad \text{and} \quad K_i'' = F(u_i'') + \mathrm{d}(u_i, u_i'')Q_0. \tag{5.11}$$

In these settings, $T_x(u_i, u_i', u_i'') = K_i' \cap K_i'' + Q_i$, $i = 1, 2$. See (5.6).

Note that $K_i' \cap K_i'' \neq \emptyset$ because $T_x(u_i, u_i', u_i'') \neq \emptyset$. Therefore, thanks to Lemma 5.4,

$$T_x(u_i, u_i', u_i'') = (K_i' + Q_i) \cap (K_i'' + Q_i) \cap \mathcal{H}[T_x(u_i, u_i', u_i'')], \quad i = 1, 2. \tag{5.12}$$

Now, let us introduce the following families of sets:

$$\mathcal{K}^+ = \{K_i' + Q_i, K_i'' + Q_i : i = 1, 2\}, \quad \mathcal{K}^{++} = \{\mathcal{H}[T_x(u_i, u_i', u_i'')] : i = 1, 2\},$$
$$\mathcal{K} = \mathcal{K}^+ \cup \mathcal{K}^{++}.$$

Then, thanks to (5.10) and (5.12), $A = [\cap\{K : K \in \mathcal{K}\}] + Q$.

We recall that, thanks to Proposition 3.7, the set $F^{[2]}(x) \neq \emptyset$, so that the left-hand side of (5.8) is non-empty as well. From this and (5.10) it follows that $A \neq \emptyset$, proving that $\cap\{K : K \in \mathcal{K}\} \neq \emptyset$. Therefore, thanks to Lemma 3.4,

$$A = \cap\{[K \cap K'] + Q : K, K' \in \mathcal{K}\}.$$

Thus, to prove that $a \in A$, it suffices to show that $a \in K \cap K' + Q$ for every $K, K' \in \mathcal{K}$.

To do this, first let us note that, thanks to (5.11),

$$K_i' + Q_i = F(u_i') + \mathrm{d}(u_i, u_i')Q_0 + L\,\mathrm{d}(u_i, x)Q_0 \supset F(u_i') + (\mathrm{d}(u_i, u_i') + \mathrm{d}(u_i, x))Q_0$$

for every $i = 1, 2$. Therefore, thanks to the triangle inequality,

$$K_i' + Q_i \supset F(u_i') + \mathrm{d}(u_i', x)Q_0. \tag{5.13}$$

In the same way, we prove that

$$K_i'' + Q_i \supset F(u_i'') + \mathrm{d}(u_i'', x)\, Q_0, \quad i = 1, 2. \tag{5.14}$$

Furthermore, we know that

$$\mathcal{H}[T_x(u_i, u_i', u_i'')] \supset T_x(u_i, u_i', u_i''), \quad i = 1, 2. \tag{5.15}$$

On the other hand, property (5.9) tells us that

$$a \in T_x(u, u', u'') \cap [F(v) + \mathrm{d}(x, v)\, Q_0] + Q \quad \text{for every } u, u', u'', v \in \mathcal{M}. \tag{5.16}$$

Combining this property with (5.13), (5.14) and (5.15), we conclude that

$$a \in K \cap K' + Q \quad \text{whenever either} \quad K \in \mathcal{K}^+, K' \in \mathcal{K}^{++} \quad \text{or} \quad K, K' \in \mathcal{K}^+.$$

It remains to prove that

$$a \in H_1 \cap H_2 + Q, \quad \text{where} \quad H_i = \mathcal{H}[T_x(u_i, u_i', u_i'')], \quad i = 1, 2. \tag{5.17}$$

It is immediate from Lemma 5.2, part (b), that

$$H_1 \cap H_2 + Q = (H_1 + Q) \cap (H_2 + Q),$$

so that

$$H_1 \cap H_2 + Q = \{\mathcal{H}[T_x(u_1, u_1', u_1'')] + Q\} \cap \{\mathcal{H}[T_x(u_2, u_2', u_2'')] + Q\}.$$

From this and (5.15), we have

$$H_1 \cap H_2 + Q \supset \{T_x(u_1, u_1', u_1'') + Q\} \cap \{T_x(u_2, u_2', u_2'') + Q\}. \tag{5.18}$$

But, thanks to (5.16), $a \in T_x(u_i, u_i', u_i'') + Q$, $i = 1, 2$. Combining this property with (5.18), we obtain the required property (5.17), completing the proof of the lemma. ∎

We are in a position to prove inequality (5.5). Our proof will follow the scheme of the proof of Proposition 3.9.

Let $x, y \in \mathcal{M}$, and let $\tau = \tilde{\gamma}(L)\, \mathrm{d}(x, y)$. (Recall that $\tilde{\gamma}(L) = L\theta(L)$ and $\mathrm{d} = \lambda_1 \rho$.) Lemma 5.7 tells us that

$$F^{[2]}(x) + \tau Q_0 = \bigcap_{v, u, u', u'' \in \mathcal{M}} \big\{[T_x(u, u', u'') \cap (F(v) + \mathrm{d}(x, v) Q_0)] + \tau Q_0\big\}. \tag{5.19}$$

Let us fix elements $u, u', u'', v \in \mathcal{M}$ and a set

$$\widetilde{A} = [T_x(u, u', u'') \cap (F(v) + \mathrm{d}(x, v) Q_0)] + \tau Q_0. \tag{5.20}$$

Let us prove that $\widetilde{A} \supset F^{[2]}(y)$. Let

$$C_1 = F(u') + \mathrm{d}(u', u) Q_0, \quad C_2 = F(u'') + \mathrm{d}(u'', u) Q_0, \quad C = F(v) + \mathrm{d}(x, v) Q_0, \tag{5.21}$$

and let $\varepsilon = L\,\mathrm{d}(x, y)$ and $r = \mathrm{d}(u, x)$. Then

$$\tau = \tilde{\gamma}(L)\,\mathrm{d}(x, y) = L\theta(L)\,\mathrm{d}(x, y) = \theta(L)\,\varepsilon.$$

In these settings, $T_x(u, u', u'') = (C_1 \cap C_2) + LrQ_0$, see (5.6), and

$$\widetilde{A} = [\{(C_1 \cap C_2) + LrQ_0\} \cap C] + \theta(L)\,\varepsilon\,Q_0, \quad \text{see (5.20)}.$$

Let us apply Proposition 2.3 to the sets C_1, C_2 and C defined by (5.21). To do this, we have to verify condition (2.4) of that proposition, i.e., to prove that

$$C_1 \cap C_2 \cap (C + rQ_0) \neq \emptyset. \tag{5.22}$$

Let $\mathcal{M}' = \{u, u', v\}$. Then, thanks to Claim 5.6, there exists a ρ-Lipschitz selection $f_{\mathcal{M}'}\colon \mathcal{M}' \to \ell^2_\infty$ of the restriction $F|_{\mathcal{M}'}$ with $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}((\mathcal{M}',\rho),\ell^2_\infty)} \le 1$.

Because $e(\mathfrak{M}, \ell^2_\infty) = 1$ and $\mathrm{d} = \lambda_1\rho \ge \rho$, the mapping $f_{\mathcal{M}'}\colon \mathcal{M}' \to \ell^2_\infty$ can be extended to a d-Lipschitz mapping $\tilde{f}\colon \mathcal{M} \to \ell^2_\infty$ defined on all of $\mathcal{M}$ with d-Lipschitz seminorm

$$\|\tilde{f}\|_{\mathrm{Lip}((\mathcal{M},\mathrm{d}),\ell^2_\infty)} \le \|f_{\mathcal{M}'}\|_{\mathrm{Lip}((\mathcal{M}',\rho),\ell^2_\infty)} \le 1.$$

Thus, $\tilde{f}(u') = f_{\mathcal{M}'}(u') \in F(u')$, $\tilde{f}(u'') = f_{\mathcal{M}'}(u'') \in F(u'')$, $\tilde{f}(v) = f_{\mathcal{M}'}(v) \in F(v)$,

$$\|\tilde{f}(u') - \tilde{f}(u)\| \le \mathrm{d}(u', u), \quad \|\tilde{f}(u'') - \tilde{f}(u)\| \le \mathrm{d}(u'', u)$$

and

$$\|\tilde{f}(x) - \tilde{f}(u)\| \le \mathrm{d}(u, x) = r, \quad \|\tilde{f}(x) - \tilde{f}(v)\| \le \mathrm{d}(v, x).$$

Hence, $\tilde{f}(u) \in C_1 \cap C_2$ and $\tilde{f}(x) \in C$, so that $C_1 \cap C_2 \cap (C + rQ_0) \ni \tilde{f}(u)$, proving (5.22).

This enables us to apply Proposition 2.3 to the sets C_1, C_2 and C. By this proposition,

$$\begin{aligned}\widetilde{A} &= [\{(C_1 \cap C_2) + LrQ_0\} \cap C] + \theta(L)\varepsilon Q_0\\ &\supset [(C_1 \cap C_2) + (Lr + \varepsilon)Q_0] \cap [\{(C_1 + rQ_0) \cap C\} + \varepsilon Q_0] \cap [\{(C_2 + rQ_0) \cap C\} + \varepsilon Q_0]\\ &= S_1 \cap S_2 \cap S_3.\end{aligned}$$

Let us prove that $S_i \supset F^{[2]}(y)$ for every $i = 1, 2, 3$. We begin with the set

$$\begin{aligned}S_1 &= (C_1 \cap C_2) + (Lr + \varepsilon)Q_0\\ &= [\{F(u') + \mathrm{d}(u', u)Q_0\} \cap \{F(u'') + \mathrm{d}(u'', u)Q_0\}] + (L\,\mathrm{d}(u, x) + L\,\mathrm{d}(x, y))Q_0.\end{aligned}$$

See (5.21). By the triangle inequality, $\mathrm{d}(u, x) + \mathrm{d}(x, y) \ge \mathrm{d}(u, y)$, so that

$$S_1 \supset \big[\{F(u') + \mathrm{d}(u', u)Q_0\} \cap \{F(u'') + \mathrm{d}(u'', u)Q_0\}\big] + L\,\mathrm{d}(u, y)Q_0 = T_y(u, u', u'').$$

See (5.6). But, thanks to (3.10), $T_y(u, u', u'') \supset F^{[2]}(y)$, proving the required inclusion $S_1 \supset F^{[2]}(y)$.

Let is prove that $S_2 \supset F^{[2]}(y)$. We have

$$\begin{aligned}S_2 &= [(C_1 + rQ_0) \cap C] + \varepsilon Q_0\\ &= \big[\{(F(u') + \mathrm{d}(u', u)Q_0) + \mathrm{d}(x, u)Q_0\} \cap \{F(v) + \mathrm{d}(x, v)Q_0\}\big] + L\,\mathrm{d}(x, y)Q_0.\end{aligned}$$

Therefore, thanks to the triangle inequality, (5.6) and (3.10),

$$S_2 \supset [(F(u') + \mathrm{d}(u', x)Q_0) \cap (F(v) + \mathrm{d}(x, v)Q_0)] + L\,\mathrm{d}(x, y)Q_0$$
$$= T_y(x, u', v) \supset F^{[2]}(y).$$

In the same way, we show that $S_3 \supset F^{[2]}(y)$. Hence, $\widetilde{A} \supset S_1 \cap S_2 \cap S_3 \supset F^{[2]}(y)$.

Combining this inclusion with the definition (5.20) and the representation (5.19), we conclude that

$$F^{[2]}(x) + \tau Q_0 \supset F^{[2]}(y).$$

By interchanging the roles of x and y, we obtain also the inclusion

$$F^{[2]}(y) + \tau Q_0 \supset F^{[2]}(x).$$

These two inclusions imply the inequality

$$\mathrm{d_H}(F^{[2]}(x), F^{[2]}(y)) \le \tau = \tilde{\gamma}(L)\,\mathrm{d}(x, y) = \lambda_1\,\tilde{\gamma}(L)\,\rho(x, y),$$

proving (1.8) with $\gamma = \lambda_1 L(3L + 1)/(L - 1)$. We recall that $L = \lambda_2/\lambda_1$, so that inequality (1.8) holds for any $\gamma \ge \lambda_2(3\lambda_2 + \lambda_1)/(\lambda_2 - \lambda_1)$.

The proof of Theorem 5.5 is complete. ■

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