Poincaré series of multiplier and test ideals

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Abstract. We prove the rationality of the Poincaré series of multiplier ideals in any dimension thus extending the main results for surfaces of Galindo and Monserrat and Alberich-Carramiñana et al. Our results also hold for Poincaré series of test ideals. In order to do so, we introduce a theory of Hilbert functions indexed over $\mathbb{R}$ which gives a unified treatment of both cases.

1. Introduction

Let A be a commutative Noetherian ring containing a field $\mathbb{K}$. Assume that A is either local with maximal ideal $\mathfrak{m}$ such that the residue field is isomorphic to $\mathbb{K}$ or a graded ring with $A_0 = \mathbb{K}$ and maximal homogeneous ideal $\mathfrak{m}$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Depending on the characteristic of the base field we may find two parallel sets of invariants associated to the pair $(A, \mathfrak{a}^c)$ where c is a real parameter. In characteristic zero we have the theory of *multiplier ideals* which play a prominent role in birational geometry and are defined using resolution of singularities (see [22] for more insight). In positive characteristic we may find the so-called *test ideals* which originated from the theory of tight closure [17, 18] and are defined using the Frobenius endomorphism [5, 6, 27]. Despite their different origins, it is known that under some conditions on A, the reduction mod p of a multiplier ideal is the corresponding test ideal [9, 10, 15, 17, 23, 30, 31] (see also [4, 28]). Moreover, both theories share a lot of common properties which we summarize as saying that, under some assumptions on A, they form a filtration of $\mathfrak{m}$-primary ideals

$$\mathcal{J} : \quad A \supsetneq \mathcal{J}_{\alpha_1} \supsetneq \mathcal{J}_{\alpha_2} \supsetneq \cdots \supsetneq \mathcal{J}_{\alpha_i} \supsetneq \cdots,$$

and the indices where there is a strict inequality form a discrete set of rational numbers, see [6, 8, 9, 20, 22, 26, 29, 32]. The *multiplicity* of $c \in \mathbb{R}_{>0}$ is defined as $m(c) = \dim_{\mathbb{K}} (\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c)$, for $\varepsilon > 0$ small enough [13]. In order to gather the information given by these ideals and its multiplicities, we consider the *Poincaré series* of $\mathcal{J}$,

$$P_{\mathcal{J}}(T) = \sum_{c \in \mathbb{R}_{>0}} \dim_{\mathbb{K}} (\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c) \, T^c.$$

The natural question is whether this is a rational function, in the sense that it belongs to the field of fractional functions $\mathbb{Q}(z)$ where the indeterminate z corresponds to a fractional power $T^{1/e}$ for a suitable $e \in \mathbb{N}_{>0}$.

Galindo and Monserrat [14] proved that this rationality property holds for multiplier ideals associated to simple $\mathfrak{m}$-primary ideals in a complex smooth surface and provided an explicit formula. These results were extended later on by Alberich-Carramiñana et al. [1] (see also [2]) to the case of multiplier ideals associated to any $\mathfrak{m}$-primary ideal in a complex surface with rational singularities. The techniques used in both cases rely on the theory of singularities in dimension two and, in particular, the fact that the data coming from the log-resolution of any ideal can be encoded in a combinatorial object such as the *dual graph*. In the case of simple ideals, the divisors corresponding to the star vertices of the graph measure the difference between a multiplier ideal and its preceding. In general one needs the notion of *maximal jumping divisor* [1] to account for this difference. The formula obtained for the Poincaré series is then described in terms of the *excesses* of these maximal jumping divisors. During the preparation of this manuscript, we learned that Pande [24] has extended these results to the case of smooth varieties in arbitrary dimension.

In this work, we show the rationality of the Poincaré series of multiplier ideals of $\mathfrak{m}$-primary ideals in any normal variety in arbitrary dimension (see Theorem 3.2 and Corollary 4.8 for the Cohen–Macaulay case). Furthermore, we also prove the rationality of the Poincaré series for test ideals of $\mathfrak{m}$-primary ideals in F-finite rings (see Theorem 3.7 and Corollary 4.10 for the Cohen–Macaulay case). As a particular case, we obtain the rationality of $P_{\tau}(T)$ for ideals in normal surfaces in prime characteristic.

Our approach is completely algebraic, and it provides a unified proof of the rationality of the Poincaré series for both the multiplier and the test ideals in any dimension as long as we have discreteness of the jumping numbers and Skoda's theorem. We point out that our main results do not require the rationality of the jumping numbers. Examples of non-rational jumping numbers of multiplier ideals exist by work of Urbinati [33]. The rationality of the Poincaré series in this case means that it belongs to the field of fractional functions $\mathbb{Q}(T^{\alpha_1}, \ldots, T^{\alpha_s})$, where $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ is a finite set of jumping numbers.

To such purpose we develop a theory of Hilbert functions indexed over $\mathbb{R}$ that should be of independent interest. More precisely, in Section 2 we develop the notion of $\mathbb{R}$-good α-filtrations associated to a finitely generated A-module which is an extension of the well-known theory of good α-filtrations. In this general framework we can define the multiplicity of any module in the filtration and the corresponding Poincaré series. The main result is Theorem 2.6 where we prove the rationality of such a series. In Section 3 we specialize our main result to the case of multiplier ideals and test ideals. We also extend to arbitrary dimension the notion of maximal jumping divisor (see Definition 3.3) and give a formula for the multiplicity (see Proposition 3.5). In Section 4 we provide a different approach to the theory of $\mathbb{R}$-good α-filtrations in the case of Cohen–Macaulay rings that gives a simpler formula for the Poincaré series (see Theorem 4.5). By comparing our results with the ones previously obtained by geometric methods, we yield an algebraic formula for the excess associated to the maximal jumping divisor (see Proposition 4.13).

2. $\mathbb{R}$-good filtrations

Let A be a commutative Noetherian ring. Assume that A is either local or graded with maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. The theory of *good $\mathfrak{a}$-filtrations* gives an approach to the study of Hilbert functions that covers most of the classical results in a unified way. We start recalling briefly this notion but we refer to Rossi and Valla's monograph [25] and the references therein for more insight.

Let M be a finitely generated A-module. A *good $\mathfrak{a}$-filtration* on M is a decreasing filtration

$$\mathcal{M}: \quad M = M_0 \supseteq M_1 \supseteq \cdots$$

by A-submodules of M such that $\mathfrak{a} M_j \subseteq M_{j+1}$ for $j \geq 0$ and $\mathfrak{a} M_j = M_{j+1}$ for $j \gg 0$ large enough. Under these premises, we may consider the *Hilbert* and the *Hilbert–Samuel function* of M with respect to the filtration $\mathcal{M}$, defined as

$$H_{\mathcal{M}}(j) := \lambda(M_j/M_{j+1}) \quad \text{and} \quad H_{\mathcal{M}}^1(j) := \lambda(M/M_j)$$

respectively, where $\lambda(\cdot)$ denotes the length as A-module. Moreover, we consider the *Hilbert* and the *Hilbert–Samuel series*

$$\mathrm{HS}_{\mathcal{M}}(T) := \sum_{j \geq 0} \lambda(M_j/M_{j+1}) T^j \quad \text{and} \quad \mathrm{HS}_{\mathcal{M}}^1(T) := \sum_{j \geq 0} \lambda(M/M_j) T^j.$$

Notice that we have $\mathrm{HS}_{\mathcal{M}}(T) = (1 - T) \mathrm{HS}_{\mathcal{M}}^1(T)$. As a consequence of the Hilbert–Serre theorem, we can express them as rational functions:

$$\mathrm{HS}_{\mathcal{M}}(T) = (1 - T) \mathrm{HS}_{\mathcal{M}}^1(T) = (1 - T) \frac{h_{\mathcal{M}}(T)}{(1 - T)^{d+1}},$$

where $h_{\mathcal{M}}(T) \in \mathbb{Z}[T]$ satisfies $h_{\mathcal{M}}(1) \neq 0$ and d is the Krull dimension of M. The polynomial $h_{\mathcal{M}}(T)$ is the *h-polynomial* of $\mathcal{M}$.

The aim of this section is to extend the notion of good $\mathfrak{a}$-filtrations by allowing filtrations indexed over $\mathbb{R}$ and thus mimicking properties satisfied by filtrations given by multiplier and test ideals.

Definition 2.1. Let M be a finitely generated A-module and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. An $\mathbb{R}$-good $\mathfrak{a}$-filtration is a decreasing filtration $\mathcal{M} := \{M_\alpha\}_{\alpha \geq 0}$ of submodules of $M_0 = M$, indexed by positive real numbers such that

- $\mathfrak{a} M_\alpha \subseteq M_{\alpha+1}$ for all $j \geq 0$;
- $\mathfrak{a} M_\alpha = M_{\alpha+1}$ for all $\alpha > j$ with $j \gg 0$ large enough;
- $\forall \alpha, \exists \varepsilon > 0$ such that $M_\alpha = M_c$ for $c \in [\alpha, \alpha + \varepsilon)$.

We call it a $\mathbb{Q}$-good $\mathfrak{a}$-filtration when the set of indices is contained in $\mathbb{Q}$.

Remark 2.2. For a fixed $n \in \mathbb{N}$, we note that $\mathfrak{a}^n M \subseteq M_\alpha$ for every $\alpha \leq n$ and thus we have $\lambda(M/M_\alpha) < \infty$. In particular, $\#\{M_\alpha \mid \alpha \leq n\} \leq \lambda(M/\mathfrak{a}^n M) < \infty$.

From Remark 2.2, we may think of $\mathcal{M}$ as a filtration of submodules M_c indexed over $\mathbb{R}$ for which there exists an increasing sequence of real numbers $0 < \alpha_1 < \alpha_2 < \cdots$ such that

$M_{\alpha_i} = M_c \supsetneq M_{\alpha_{i+1}}$ for any $c \in [\alpha_i, \alpha_{i+1})$. In particular, we have a discrete filtration of submodules

$$\mathcal{M} : \quad M \supsetneq M_{\alpha_1} \supsetneq M_{\alpha_2} \supsetneq \cdots \supsetneq M_{\alpha_i} \supsetneq \cdots$$

and we say that the α_i are the *jumping numbers* of $\mathcal{M}$. A crucial observation is that, once we fix an index $c \in \mathbb{R}$, the filtration

$$\mathcal{M}_{c+\bullet} : \quad M_c \supseteq M_{c+1} \supseteq M_{c+2} \supseteq \cdots$$

is a good α-filtration.

Definition 2.3. Let $\mathcal{M} := \{M_c\}_{c \geq 0}$ be an $\mathbb{R}$-good α-filtration. We define the multiplicity of $c \in \mathbb{R}_{>0}$ as

$$m(c) := \lambda\left(M_{c-\varepsilon}/M_c\right)$$

for $\varepsilon > 0$ small enough. With this definition, it is clear that c is a jumping number if and only if $m(c) > 0$.

Definition 2.4. Let $\mathcal{M} := \{M_c\}_{c \geq 0}$ be an $\mathbb{R}$-good α-filtration. We define the Poincaré series of $\mathcal{M}$ as

$$P_{\mathcal{M}}(T) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, T^c.$$

The question that we want to address is whether the Poincaré series is rational in the sense that it belongs to the field of fractional functions $\mathbb{Q}(T^{\alpha_1}, \ldots, T^{\alpha_s})$, where $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ is a finite set of jumping numbers. In the case of $\mathbb{Q}$-good α-filtrations, the rationality of the Poincaré series means that it belongs to the field of fractional functions $\mathbb{Q}(T^{1/e})$, where $e \in \mathbb{N}_{>0}$ is the least common multiple of the denominators of all the jumping numbers.

Proposition 2.5. *Let $\mathcal{M} := \{M_c\}_{c \geq 0}$ be an $\mathbb{R}$-good α-filtration. Given $c \in \mathbb{R}_{>0}$, we have that*

$$\sum_{j \geq 0} m(c+j)\, T^j$$

is a rational function in $\mathbb{Q}(T)$.

Proof. Recall that the Hilbert series $\mathrm{HS}^1_{\mathcal{M}_{c-\varepsilon}}(T)$ and $\mathrm{HS}^1_{\mathcal{M}_c}(T)$ associated to the good α-filtrations $\mathcal{M}_{c-\varepsilon}$ and $\mathcal{M}_c$ are rational functions. From the short exact sequence

$$0 \longrightarrow M_c/M_{c+j} \longrightarrow M_{c-\varepsilon}/M_{c+j} \longrightarrow M_{c-\varepsilon}/M_c \longrightarrow 0$$

we get

$$\sum_{j \geq 0} \lambda\left(M_{c-\varepsilon}/M_{c+j}\right) T^j = \mathrm{HS}^1_{\mathcal{M}_c}(T) + m(c)\, \frac{1}{1-T} \cdot$$

Analogously, from the short exact sequence

$$0 \longrightarrow M_{c-\varepsilon+j}/M_{c+j} \longrightarrow M_{c-\varepsilon}/M_{c+j} \longrightarrow M_{c-\varepsilon}/M_{c-\varepsilon+j} \longrightarrow 0$$

we get

$$\sum_{j\geq 0} m(c+j)\,T^j = \sum_{j\geq 0} \lambda(M_{c-\varepsilon}/M_{c+j})\,T^j - \mathrm{HS}^1_{\mathcal{M}_{c-\varepsilon}}(T)$$

$$= m(c)\,\frac{1}{1-T} + \mathrm{HS}^1_{\mathcal{M}_c}(T) - \mathrm{HS}^1_{\mathcal{M}_{c-\varepsilon}}(T)$$

$$= \frac{m(c)}{1-T} + \frac{h_{\mathcal{M}_{c+\bullet}}(T) - h_{\mathcal{M}_{(c-\varepsilon)+\bullet}}(T)}{(1-T)^{d+1}},$$

and thus it is a rational function. Here, $h_{\mathcal{M}_{c+\bullet}}(T)$ and $h_{\mathcal{M}_{(c-\varepsilon)+\bullet}}(T)$ are the h-polynomials of the good α-filtrations $\mathcal{M}_{c+\bullet}$ and $\mathcal{M}_{(c-\varepsilon)+\bullet}$, respectively. ∎

Theorem 2.6. *Let $\mathcal{M} := \{M_c\}_{c\geq 0}$ be an $\mathbb{R}$-good α-filtration. Then, the Poincaré series $P_{\mathcal{M}}(T)$ is rational. Moreover, we have*

$$P_{\mathcal{M}}(T) = \sum_{c\in(0,1]} \left(\frac{m(c)}{1-T} + \frac{h_{\mathcal{M}_{c+\bullet}}(T) - h_{\mathcal{M}_{(c-\varepsilon)+\bullet}}(T)}{(1-T)^{d+1}} \right) T^c,$$

where $h_{\mathcal{M}_{c+\bullet}}(T)$ and $h_{\mathcal{M}_{(c-\varepsilon)+\bullet}}(T)$ are the h-polynomials of the good α-filtrations $\mathcal{M}_{c+\bullet}$ and $\mathcal{M}_{(c-\varepsilon)+\bullet}$ respectively.

Proof. From Remark 2.2, we note that the set of real numbers such that $m(c) \neq 0$ is a discrete set. In particular, there are only finitely many $c \in (0,1]$ such that $m(c) \neq 0$. Since $m(c+1) \neq 0$ implies $m(c) \neq 0$, we have that the nonzero contributors in the series have the form $m(c+j)$ for some $c \in (0,1]$.

Then we have

$$P_{\mathcal{M}}(T) = \sum_{c\in\mathbb{R}_{>0}} m(c)\,T^c = \sum_{c\in(0,1]} \left(\sum_{j\in\mathbb{Z}_{\geq 0}} m(c+j)\,T^j \right) T^c,$$

and thus the result follows from Proposition 2.5. ∎

3. Poincaré series of multiplier and test ideals

Let A be a commutative Noetherian ring containing a field $\mathbb{K}$. Throughout this section, we assume that A is either local or graded with maximal ideal $\mathfrak{m}$ such that the residue field is isomorphic to $\mathbb{K}$ and α is an $\mathfrak{m}$-primary ideal. Now we turn our attention to the case where the $\mathbb{R}$-good α-filtration that we consider is given by a filtration of ideals

$$\mathcal{J}: \quad A \supsetneq \mathcal{J}_{\alpha_1} \supsetneq \mathcal{J}_{\alpha_2} \supsetneq \cdots \supsetneq \mathcal{J}_{\alpha_i} \supsetneq \cdots$$

In this setting, the multiplicity of $c \in \mathbb{R}_{>0}$ is $m(c) = \dim_{\mathbb{K}}(\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c)$, for $\varepsilon > 0$ small enough, and the Poincaré series of $\mathcal{J}$ is

$$P_{\mathcal{J}}(T) = \sum_{c\in\mathbb{R}_{>0}} \dim_{\mathbb{K}}(\mathcal{J}_{c-\varepsilon}/\mathcal{J}_c)\,T^c.$$

Our aim is to specialize the results we obtained in the previous section to the case of multiplier ideals and test ideals.

3.1. Multiplier ideals

Let $(A, \mathfrak{m})$ be a normal local ring essentially of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero and let $\mathfrak{a} \subseteq A$ be an ideal. Under these general assumptions, we ensure the existence of canonical divisors K_X on $X = \operatorname{Spec} A$ which are not necessarily $\mathbb{Q}$-Cartier. Then we may find some effective boundary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index m large enough. Now, given a *log-resolution* $\pi \colon X' \to X$ of the triple $(X, \Delta, \mathfrak{a})$, we pick a canonical divisor $K_{X'}$ in X' such that $\pi_* K_{X'} = K_X$ and let F be an effective divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$.

The *multiplier ideal* associated to the triple $(X, \Delta, \mathfrak{a}^c)$ for some real number $c \in \mathbb{R}_{>0}$ is defined as

$$\mathcal{J}(X, \Delta, \mathfrak{a}^c) = \pi_* \mathcal{O}_{X'}\left(\left\lceil K_{X'} - \frac{1}{m}\,\pi^*(m(K_X + \Delta)) - cF \right\rceil\right).$$

This construction allowed de Fernex and Hacon, see [12], to define the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ associated to $\mathfrak{a}$ and c as the unique maximal element of the set of multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^c)$, where Δ varies among all the effective divisors such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The key point in their proof is the existence of such a divisor Δ that realizes the multiplier ideal as $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(X, \Delta, \mathfrak{a}^c)$. In this general framework, we have that the *local vanishing theorem* still holds, see Theorem 4.1.19 in [11]. Namely, for any $c \in \mathbb{R}_{>0}$ we have

$$R^1 \pi_* \mathcal{O}_{X'}\left(\left\lceil K_{X'} - \frac{1}{m}\,\pi^*(m(K_X + \Delta)) - cF \right\rceil\right) = 0.$$

Remark 3.1. If A is $\mathbb{Q}$-Gorenstein, the canonical module K_X is $\mathbb{Q}$-Cartier, so no boundary Δ is required in the definition of multiplier ideal. Namely, we have

$$\mathcal{J}(\mathfrak{a}^c) = \pi_* \mathcal{O}_{X'}\left(\left\lceil K_{X'} - \frac{1}{m}\,\pi^*(m K_X) - cF \right\rceil\right).$$

From its construction, we have that the multiplier ideals form a filtration

$$A \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_2}) \supsetneq \cdots \supsetneq \mathcal{J}(\mathfrak{a}^{\alpha_i}) \supsetneq \cdots,$$

and the α_i where we have a strict inclusion of ideals are the *jumping numbers* of the ideal $\mathfrak{a}$.

Assume in addition that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal and thus F is a divisor with exceptional support. Then any multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is $\mathfrak{m}$-primary as well. To ensure that $\mathcal{J} = \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ is an $\mathbb{R}$-good $\mathfrak{a}$-filtration, we notice the following:

- *Skoda's theorem* ([12], Corollary 5.7): We have $\mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{c-1}) \subseteq \mathcal{J}(\mathfrak{a}^c)$ for every $c > 0$, and equality holds for $c > \dim A$.

- *Discreteness*: If $\mathfrak{a}$ is $\mathfrak{m}$-primary, the number of multiplier ideals in any interval $[c_1, c_2]$ is smaller or equal than $\dim_{\mathbb{K}}(\mathcal{J}(\mathfrak{a}^{c_1})/\mathcal{J}(\mathfrak{a}^{c_2}))$.

There are cases where the jumping numbers are not rational, as shown by Urbinati in [33]. Known cases where the jumping numbers form a discrete set of rational numbers and thus the filtration $\mathcal{J} = \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ is a $\mathbb{Q}$-good $\mathfrak{a}$-filtration are:

- X is $\mathbb{Q}$-Gorenstein.
- The *symbolic Rees algebra* $\mathcal{R}(-(K_X + \Delta)) := \bigoplus_{n \geq 0} \mathcal{O}_X(-n(K_X + \Delta))$ is finitely generated ([9], Remark 2.26).

Theorem 3.2. *Let* $(A, \mathfrak{m})$ *be a normal local ring of dimension* d *essentially of finite type over an algebraically closed field* $\mathbb{K}$ *of characteristic zero, let* $\mathfrak{a} \subseteq A$ *be an* $\mathfrak{m}$-*primary ideal and let* $\mathcal{J} := \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ *be the filtration given by multiplier ideals. For any given* $c \geq 0$, *we have a good* $\mathfrak{a}$- *filtration*

$$\mathcal{J}_{c+\bullet}: \quad \mathcal{J}(\mathfrak{a}^c) \supseteq \mathcal{J}(\mathfrak{a}^{c+1}) \supseteq \mathcal{J}(\mathfrak{a}^{c+2}) \supseteq \cdots$$

Then, the Poincaré series $P_{\mathcal{J}}(T)$ *is rational. Indeed, we have*

$$P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \left(\frac{m(c)}{1-T} + \frac{h_{\mathcal{J}_{c+\bullet}}(T) - h_{\mathcal{J}_{(c-\varepsilon)+\bullet}}(T)}{(1-T)^{d+1}} \right) T^c,$$

where $h_{\mathcal{J}_{c+\bullet}}(T)$ *and* $h_{\mathcal{J}_{(c-\varepsilon)+\bullet}}(T)$ *are the h-polynomials of* $\mathcal{J}_{c+\bullet}$ *and* $\mathcal{J}_{(c-\varepsilon)+\bullet}$, *respectively.*

Proof. The result follows from Theorem 2.6. ∎

When A is the local ring at a rational singularity of a surface, Alberich-Carramiñana et al. ([1], Theorem 4.1) gave a precise formula for the multiplicity $m(c)$ of any given $c \in \mathbb{R}_{>0}$, and consequently an explicit description of the Poincaré series. We may follow the same approach to get a partial extension of their formula.

Definition 3.3. Let $(X, \Delta, \mathfrak{a}^c)$ be a triple. The *maximal jumping divisor* associated to $c \in \mathbb{R}_{>0}$ is

$$H_c = \left\lceil K_{X'} - \frac{1}{m} \pi^*(m(K_X + \Delta)) - (c - \varepsilon)F \right\rceil - \left\lceil K_{X'} - \frac{1}{m} \pi^*(m(K_X + \Delta)) - cF \right\rceil,$$

where ε is small enough.

Remark 3.4. Denote

$$K_{X'} - \frac{1}{m} \pi^*(m(K_X + \Delta)) = \sum_i k_i E_i \quad \text{and} \quad F = \sum_i e_i E_i,$$

where the E_i's are the exceptional components of π. Then H_c can be defined as the reduced divisor whose components are the E_i such that $k_i - ce_i \in \mathbb{Z}$. In particular, we have $H_c = H_{c+1}$ for all $c \in \mathbb{R}_{>0}$.

Proposition 3.5. *Let* $(X, \Delta, \mathfrak{a}^c)$ *be a triple. Then, the multiplicity of* $c \in \mathbb{R}_{>0}$ *is*

$$m(c) = h^0 \left(H_c, \mathcal{O}_{H_c} \left(\left\lceil K_{X'} - \frac{1}{m} \pi^*(m(K_X + \Delta)) - cF \right\rceil + H_c \right) \right).$$

Proof. To avoid heavy notation, let

$$K_\pi := K_{X'} - \frac{1}{m} \pi^*(m(K_X + \Delta)).$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF\rceil) \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF\rceil + H_c) \longrightarrow \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c) \longrightarrow 0.$$

Pushing it forward to X and applying local vanishing for multiplier ideals, we get the short exact sequence

$$0 \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF\rceil) \longrightarrow \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF\rceil + H_c)$$
$$\longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c)) \otimes \mathbb{K}_O \longrightarrow 0,$$

or equivalently,

$$0 \longrightarrow \mathcal{J}(\mathfrak{a}^c) \longrightarrow \mathcal{J}(\mathfrak{a}^{(c-\varepsilon)}) \longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c)) \otimes \mathbb{K}_O \longrightarrow 0.$$

Therefore the multiplicity of c is just $m(c) = h^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c))$. ∎

Question 3.6. The key ingredient for the explicit formula of the Poincaré series of multiplier ideals in dimension 2 given by Alberich-Carramiñana et al. [1] is that the multiplicities satisfy $m(c + k) - m(c) = k\rho_c$, where $\rho_c := -F \cdot H_c$ are the *excesses* associated to the maximal jumping divisor H_c. Pande [24] proved that $m(c + j)$ is a polynomial function in j of degree less than d in the case of smooth varieties in arbitrary dimension d. These results motivate the following question regarding multiplicities for $\mathfrak{m}$-primary ideals in normal rings. Is there a polynomial expression in terms of j for

$$m(c + j) - m(c)$$
$$= h^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c + jF)) - h^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF\rceil + H_c))?$$

3.2. Test ideals

Let A be a commutative Noetherian ring containing a field $\mathbb{K}$ of characteristic $p > 0$. The theory of test ideals has its origins in the work of Hochster and Huneke on tight closure [18]. In the case of A being a regular ring, Hara and Yoshida [17] extended the notion of test ideals to pairs $(A, \mathfrak{a}^c)$, where $\mathfrak{a} \subseteq A$ is an ideal. Their construction has been generalized in subsequent works [5–8, 27, 32] using the theory of *Cartier operators*.

Assume that A is F-finite. Then, the *test ideal* $\tau(\mathfrak{a}^c)$ associated to $\mathfrak{a}$ and some real number $c \in \mathbb{R}_{\geq 0}$ is the smallest nonzero ideal which is compatible with any Cartier operator $\phi \in \bigoplus_{e \geq 0} \mathrm{Hom}_A(F_*^e A, A) \cdot F_*^e \mathfrak{a}^{\lceil cp^e\rceil}$, where F_*^e is the Frobenius functor. In this situation we also have a filtration

$$A \supsetneq \tau(\mathfrak{a}^{\alpha_1}) \supsetneq \tau(\mathfrak{a}^{\alpha_2}) \supsetneq \cdots \supsetneq \tau(\mathfrak{a}^{\alpha_i}) \supsetneq \cdots$$

and the α_i where we have a strict inclusion of ideals are called the *F-jumping numbers* of the ideal $\mathfrak{a}$.

Even though the test ideals of an $\mathfrak{m}$-primary ideal may not be $\mathfrak{m}$-primary, we still have that $\tau = \{\tau(\mathfrak{a}^c)\}_{c \geq 0}$ is an $\mathbb{R}$-good $\mathfrak{a}$-filtration because of Skoda's theorem and Remark 2.2.

- *Skoda's theorem* [5, 16, 29]: We have $\mathfrak{a} \cdot \tau(\mathfrak{a}^{c-1}) \subseteq \tau(\mathfrak{a}^c)$ for every $c > 0$, and equality holds for $c > \dim A$.

- *Discreteness*: If α is $\mathfrak{m}$-primary, the number of test ideals in any interval $[c_1, c_2]$ is smaller or equal than $\dim_{\mathbb{K}}(\tau(\alpha^{c_1})/\tau(\alpha^{c_2}))$.

Known cases where the F-jumping numbers form a discrete set of rational numbers and thus the filtration $\tau = \{\tau(\alpha^c)\}_{c \geq 0}$ is a $\mathbb{Q}$-good α-filtration are:

- $(A, \mathfrak{m})$ is an F-finite, normal $\mathbb{Q}$-Gorenstein local domain [6, 8, 20, 26, 29, 32].
- A is an F-finite ring which is a direct summand of a regular ring [3].

Theorem 3.7. *Let $(A, \mathfrak{m})$ be an F-finite local ring of dimension d containing a field $\mathbb{K}$ of characteristic $p > 0$ and let α be an $\mathfrak{m}$-primary ideal. Let $\tau = \{\tau(\alpha^c)\}_{c \geq 0}$ be the filtration given by test ideals and, for any given $c \geq 0$, consider the good α-filtration*

$$\tau_{c+\bullet}: \quad \tau(\alpha^c) \supseteq \tau(\alpha^{c+1}) \supseteq \tau(\alpha^{c+2}) \supseteq \cdots$$

Then, the Poincaré series $P_{\mathcal{J}}(T)$ is rational. Indeed, we have

$$P_\tau(T) = \sum_{c \in (0,1]} \left(\frac{m(c)}{1-T} + \frac{h_{\tau_{c+\bullet}}(T) - h_{\tau_{(c-\varepsilon)+\bullet}}(T)}{(1-T)^{d+1}} \right) T^c,$$

where $h_{\tau_{c+\bullet}}(T)$ and $h_{\tau_{(c-\varepsilon)+\bullet}}(T)$ are the h-polynomials of $\tau_{c+\bullet}$ and $\tau_{(c-\varepsilon)+\bullet}$, respectively.

Proof. Under these assumptions on A, we have that $\tau = \{\tau(\alpha^c)\}_{c \geq 0}$ is a $\mathbb{R}$-good α-filtration. The result follows from Theorem 2.6. ∎

Motivated by the case of multiplier ideals for smooth varieties [14, 24] and varieties with rational singularities [1], we would like to have a precise description of the multiplicities of F-jumping numbers since it would yield a more explicit formula for the Poincaré series. More precisely, we ask the following.

Question 3.8. Is the multiplicity of test ideals of $\mathfrak{m}$-primary ideals in a strongly F-regular ring, $m(c + j)$, a polynomial function in j of degree less than d?

4. Poincaré series in Cohen–Macaulay rings

Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Let α be an $\mathfrak{m}$-primary ideal generated by a regular sequence $f_1, \ldots, f_d$. Let $\mathcal{J} = \{\mathcal{J}_c\}_{c \geq 0}$ be an $\mathbb{R}$-good α-filtration of $\mathfrak{m}$-primary ideals satisfying Skoda's theorem, so $\mathcal{J}_c = \alpha \mathcal{J}_{c-1}$ for all $c > d$. The Poincaré series of $\mathcal{J}$ is

$$P_{\mathcal{J}}(T) = \sum_{c \in \mathbb{R}_{>0}} m(c) \, T^c = \sum_{c \in (0,1]} \left(\sum_{j \geq 0} m(c+j) T^j \right) T^c,$$

and zooming in the summands we have

$$\sum_{j \geq 0} m(c+j) \, T^j = m(c) + m(c+1) \, T + \cdots + m(c+d-2) \, T^{d-2}$$

$$+ T^{d-1} \sum_{j \geq 0} \lambda(\alpha^j \mathcal{J}_{c+d-1-\varepsilon}/\alpha^j \mathcal{J}_{c+d-1}) \, T^j.$$

The aim of this section is to work towards finding a more explicit formula for the Poincaré series in Cohen–Macaulay rings, especially in the case that $\mathcal{J}$ is a filtration of multiplier or test ideals where we require that the residue field is infinite. Namely, let $(A, \mathfrak{m})$ be a local Noetherian ring with infinite residue field and let $\mathfrak{a}$ be any $\mathfrak{m}$-primary ideal. Every minimal reduction of $\mathfrak{a}$ can be generated by a superficial sequence of length equal to the analytical spread of $\mathfrak{a}$ [19], Theorem 8.6.3. Since $\mathfrak{a}$ is $\mathfrak{m}$-primary, the analytical spread is $\ell(\mathfrak{a}) = \dim(A)$. If A is Cohen–Macaulay, this superficial sequence is indeed a regular sequence. Therefore we have $\bar{\mathfrak{a}} = \overline{(f_1, \ldots, f_d)}$, where $\overline{(\cdot)}$ denotes the integral closure. Multiplier ideals and test ideal are invariant up to integral closure, so we may assume that $\mathfrak{a}$ is generated by a regular sequence.

Setup 4.1. Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Let $J \subseteq A$ be an $\mathfrak{m}$-primary ideal and let $\mathfrak{a} = (f_1, \ldots, f_d)$ be a parameter ideal. Consider a free resolution

$$(4.1) \qquad \cdots \longrightarrow A^{\beta_2} \longrightarrow A^{\beta_1} \longrightarrow A \longrightarrow A/\mathfrak{a}^j \longrightarrow 0 \,,$$

where $\beta_1 = \binom{j+(d-1)}{d-1}$ is the number of generators of $\mathfrak{a}^j$. After tensoring with A/J, we get

$$(4.2) \quad \cdots \longrightarrow (A/J)^{\beta_2} \xrightarrow{\phi_j^J} (A/J)^{\beta_1} \xrightarrow{\varphi_j^J} (A/J) \longrightarrow A/(\mathfrak{a}^j + J) \longrightarrow 0 \,,$$

The morphisms φ_j^J and ϕ_j^J play a role in what follows. If the ideal J is clear from the context, we simply denote φ_j and ϕ_j. Notice also that $\phi_j = 0$ for $j = 0$.

Lemma 4.1. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Let $J \subseteq A$ be an $\mathfrak{m}$-primary ideal and let $\mathfrak{a} = (f_1, \ldots, f_d)$ be a parameter ideal. Then, for every $j \in \mathbb{Z}_{>0}$ we have*

$$\lambda(J/\mathfrak{a}^j J) = \lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j) + (\beta_1 - 1)\,\lambda(A/J),$$

where $\beta_1 = \binom{j+(d-1)}{d-1}$.

Proof. From the short exact sequence, $0 \to J \to A \to A/J \to 0$, we have the induced long exact sequence

$$0 \to \operatorname{Tor}_1^A(A/\mathfrak{a}^j, A/J) \to J/\mathfrak{a}^j J \to A/\mathfrak{a}^j \to A/(\mathfrak{a}^j + J) \to 0.$$

Following Notation 4.1, we have $\operatorname{Tor}_1^A(A/\mathfrak{a}^j, A/J) = \ker \varphi_j / \operatorname{Im} \phi_j$ and $A/(\mathfrak{a}^j + J) = (A/J)/\operatorname{Im} \varphi_j$. Then,

$$\begin{aligned}
\lambda(J/\mathfrak{a}^j J) &= \lambda(A/\mathfrak{a}^j) + \lambda(\operatorname{Tor}_1^A(A/\mathfrak{a}^j, A/J)) - \lambda(A/(\mathfrak{a}^j + J)) \\
&= \lambda(A/\mathfrak{a}^j) + [\lambda(\ker \varphi_j) - \lambda(\operatorname{Im} \phi_j)] - [\lambda(A/J) - \lambda(\operatorname{Im} \varphi_j)] \\
&= \lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j) - \lambda(A/J) + [\lambda(\ker \varphi_j) + \lambda(\operatorname{Im} \varphi_j)] \\
&= \lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j) - \lambda(A/J) + \lambda((A/J)^{\beta_1}) \\
&= \lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j) - \lambda(A/J) + \beta_1 \lambda(A/J) \\
&= \lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j) + (\beta_1 - 1)\lambda(A/J). \quad \blacksquare
\end{aligned}$$

Lemma 4.2. *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d. *Let* $J \subseteq K \subseteq A$ *be* $\mathfrak{m}$-*primary ideals and let* $\mathfrak{a} = (f_1, \ldots, f_d)$ *be a parameter ideal. Then,*

$$\sum_{j \geq 0} \lambda(\mathfrak{a}^j K / \mathfrak{a}^j J) \, T^j = \frac{\lambda(K/J)}{(1-T)^d} + \sum_{j \geq 1} \left[\lambda(\operatorname{Im} \phi_j^K) - \lambda(\operatorname{Im} \phi_j^J) \right] T^j.$$

Proof. From the short exact sequences

$$0 \to \mathfrak{a}^j K / \mathfrak{a}^j J \to K / \mathfrak{a}^j J \to K / \mathfrak{a}^j K \to 0, \quad 0 \to J / \mathfrak{a}^j J \to K / \mathfrak{a}^j J \to K / J \to 0,$$

we get

$$\lambda(\mathfrak{a}^j K / \mathfrak{a}^j J) = \lambda(K/J) + \lambda(J / \mathfrak{a}^j J) - \lambda(K / \mathfrak{a}^j K).$$

Thus, applying Lemma 4.1 to the ideals J and K, we get

$$
\begin{aligned}
\lambda(\mathfrak{a}^j K / \mathfrak{a}^j J) &= \lambda(K/J) + [\lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j^J) + (\beta_1 - 1)\lambda(A/J)] \\
&\quad - [\lambda(A/\mathfrak{a}^j) - \lambda(\operatorname{Im} \phi_j^K) + (\beta_1 - 1)\lambda(A/K)] \\
&= \lambda(K/J) + (\beta_1 - 1)(\lambda(A/J) - \lambda(A/K)) + [\lambda(\operatorname{Im} \phi_j^K) - \lambda(\operatorname{Im} \phi_j^J)] \\
&= \beta_1 \lambda(K/J) + [\lambda(\operatorname{Im} \phi_j^K) - \lambda(\operatorname{Im} \phi_j^J)],
\end{aligned}
$$

where $\beta_1 = \binom{j + (d-1)}{d-1}$. Then the result follows since $\sum_{j \geq 0} \binom{j+(d-1)}{d-1} T^j = \frac{1}{(1-T)^d}$. ∎

In order to get some control on $\lambda(\operatorname{Im} \phi_j)$, we use the following result of Kodiyalam, see Theorem 2 in [21], in the form that we need in the present work.

Proposition 4.3. *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d, *and let* $\mathfrak{a}$ *and* J *be* $\mathfrak{m}$-*primary ideals. Then, for all* $i \geq 0$, *the function* $\lambda(\operatorname{Tor}_i^A(A/\mathfrak{a}^j, A/J))$ *is a polynomial of degree* $d-1$ *for* $j \gg 0$ *large enough.*

Using the additivity of the function λ and the fact that Tor modules are the homology modules of the complex (4.2), we get the following.

Corollary 4.4. *Under Setup 4.1, the function* $\lambda(\operatorname{Im} \phi_j^J)$ *is a polynomial of degree* $d - 1$ *for* $j \gg 0$ *large enough.*

The main result of this section is the following.

Theorem 4.5. *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* d *containing an infinite field* $\mathbb{K}$ *isomorphic to its residue field. Let* $\mathfrak{a} = (f_1, \ldots, f_d)$ *be a parameter ideal and let* $\mathcal{J} = \{\mathcal{J}_c\}_{c \geq 0}$ *be an* $\mathbb{R}$-*good* $\mathfrak{a}$-*filtration of* $\mathfrak{m}$-*primary ideals satisfying* $\mathcal{J}_c = \mathfrak{a}\mathcal{J}_{c-1}$ *for all* $c > d$. *Then, there exist* $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ *and* $p(T) \in \mathbb{Z}[T]$ *such that*

$$
\begin{aligned}
P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \Big(& m(c) + \cdots + m(c+d-2)\, T^{d-2} + \frac{m(c+d-1)T^{d-1}}{(1-T)^d} \\
& + T^d \Big(\frac{\alpha_d}{(1-T)^d} + \cdots + \frac{\alpha_1}{(1-T)} + p(T) \Big) \Big) T^c.
\end{aligned}
$$

Proof. We have

$$\sum_{j\geq 0} m(c+j)\,T^j = m(c) + m(c+1)\,T + \cdots + m(c+d-2)\,T^{d-2}$$

$$+\, T^{d-1} \sum_{j\geq 0} \dim_{\mathbb{K}}(\mathfrak{a}^j\,\mathcal{J}_{c+d-1-\varepsilon}/\mathfrak{a}^j\,\mathcal{J}_{c+d-1})\,T^j,$$

so applying Lemma 4.2 with $K = \mathcal{J}_{c+d-1-\varepsilon}$ and $J = \mathcal{J}_{c+d-1}$ we get

$$P_{\mathcal{J}}(T) = \sum_{c\in(0,1]} \left(m(c) + \cdots + m(c+d-2)\,T^{d-2} + \frac{m(c+d-1)\,T^{d-1}}{(1-T)^d} \right.$$

$$\left. +\, T^{d-1} \sum_{j\geq 1} \left[\dim_{\mathbb{K}}(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1-\varepsilon}}) - \dim_{\mathbb{K}}(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1}}) \right] T^j \right) T^c.$$

Using Corollary 4.4, we have that for $j \gg 0$ large enough,

$$\dim_{\mathbb{K}}\left(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1-\varepsilon}} \right) - \dim_{\mathbb{K}}\left(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1}} \right)$$

is a polynomial of degree $d-1$ that can be written as

$$\alpha_d \binom{(j-1)+d-1}{d-1} + \cdots + \alpha_3 \binom{(j-1)+2}{2} + \alpha_2\, j + \alpha_1.$$

Therefore, there exists $k \in \mathbb{Z}_{>0}$ such that

$$T^{d-1} \sum_{j\geq 1} \left[\lambda(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1-\varepsilon}}) - \lambda(\operatorname{Im}\phi_j^{\mathcal{J}_{c+d-1}}) \right] T^j$$

$$= T^d \left(q(T) + \sum_{j\geq k} \left[\alpha_d \binom{(j-1)+d-1}{d-1} + \cdots \right. \right.$$

$$\left. \left. +\, \alpha_3 \binom{(j-1)+2}{2} + \alpha_2\, j + \alpha_1 \right] T^{j-1} \right)$$

$$= T^d \left(q(T) + \left(\frac{\alpha_d}{(1-T)^d} - q_d(T) \right) + \cdots + \left(\frac{\alpha_1}{(1-T)} - q_1(T) \right) \right),$$

where $q(T), q_d(T), \ldots, q_1(T) \in \mathbb{Z}(T)$ have degree $\leq k-2$ and the result follows after taking $p(T) = q(T) - q_d(T) - \cdots - q_1(T)$. ∎

The following result is a direct consequence of Theorem 4.5.

Corollary 4.6. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d containing an infinite field $\mathbb{K}$ isomorphic to its residue field. Let $\mathfrak{a} = (f_1, \ldots, f_d)$ be a parameter ideal and let $\mathcal{J} = \{\mathcal{J}_c\}_{c\geq 0}$ be an $\mathbb{R}$-good $\mathfrak{a}$-filtration of $\mathfrak{m}$-primary ideals satisfying $\mathcal{J}_c = \mathfrak{a}\mathcal{J}_{c-1}$ for all $c > d$. Then, the function $m(c+j)$ is a polynomial function on j of degree less than d for $j \gg 0$ large enough.*

Remark 4.7. In the case of multiplier ideals in a smooth variety, Pande proved that this result holds for all j, see Theorem 3.2 in [24].

Now we also specialize Theorem 4.5 to the case of multiplier and test ideals.

Corollary 4.8. *Suppose* $(A, \mathfrak{m})$ *is a normal Cohen–Macaulay local ring of dimension* d *essentially of finite type over an algebraically closed field of characteristic zero,* $\mathfrak{a} \subseteq A$ *is any* $\mathfrak{m}$*-primary ideal, and* $\mathcal{J} := \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ *is the filtration given by multiplier ideals. Then,*

$$
P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \Big(m(c) + \cdots + m(c + d - 2) \, T^{d-2} + \frac{m(c + d - 1) T^{d-1}}{(1 - T)^d}
$$
$$
+ \, T^d \Big(\frac{\alpha_d}{(1 - T)^d} + \cdots + \frac{\alpha_1}{(1 - T)} + p(T) \Big) \Big) T^c.
$$

Proof. For every $\mathfrak{m}$-primary ideal $\mathfrak{a}$, there exist a parameter ideal with the same integral closure. Since the multiplier ideals are the same for an ideal and its integral closure (see Variation 9.6.39 in [22]; see also Corollary 5.7 in [12]), the result follows from Theorem 4.5. ∎

For test ideals, we have to be careful because they might not be $\mathfrak{m}$-primary. A sufficient condition for this to happen is the following.

Lemma 4.9. *Let* $(A, \mathfrak{m})$ *be a local* F*-finite Noetherian ring containing a field* $\mathbb{K}$ *of characteristic* $p > 0$ *and let* $\mathfrak{a} \subseteq A$ *be an* $\mathfrak{m}$*-primary ideal. Assume that* $A_{\mathfrak{p}}$ *is a strongly* F*-regular ring for all prime ideals* $\mathfrak{p} \neq \mathfrak{m}$*. Then, the test ideals* $\tau(\mathfrak{a}^c)$ *are* $\mathfrak{m}$*-primary or* A*.*

Proof. Since test ideals localize, see Proposition 3.2 in [5], we have that $\tau(\mathfrak{a}^c)_{\mathfrak{p}} = \tau(\mathfrak{a}_{\mathfrak{p}}^c) = \tau(A_{\mathfrak{p}}^c) = \tau(A_{\mathfrak{p}}) = A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$, because $A_{\mathfrak{p}}$ is strongly F-regular. Therefore we have $\mathrm{rad}(\tau(\mathfrak{a}^c)) \supseteq \mathfrak{m}$. ∎

Corollary 4.10. *Suppose that* $(A, \mathfrak{m})$ *is an* F*-finite Cohen–Macaulay local domain of dimension* d *containing an infinite field* $\mathbb{K}$ *of characteristic* $p > 0$ *isomorphic to its residue field,* $A_{\mathfrak{p}}$ *is a strongly* F*-regular ring for all prime ideals* $\mathfrak{p} \neq \mathfrak{m}$*,* $\mathfrak{a}$ *is any* $\mathfrak{m}$*-primary ideal and* $\tau = \{\tau(\mathfrak{a}^c)\}_{c \geq 0}$ *is the filtration given by test ideals. Then,*

$$
P_{\tau}(T) = \sum_{c \in (0,1]} \Big(m(c) + \cdots + m(c + d - 2) \, T^{d-2} + \frac{m(c + d - 1) T^{d-1}}{(1 - T)^d}
$$
$$
+ \, T^d \Big(\frac{\alpha_d}{(1 - T)^d} + \cdots + \frac{\alpha_1}{(1 - T)} + p(T) \Big) \Big) T^c.
$$

Proof. For every $\mathfrak{m}$-primary ideal $\mathfrak{a}$, there exist a parameter ideal with the same integral closure, because $\mathbb{K}$ is infinite. Since the test ideals are the same for an ideal and its integral closure (see the proof of Theorem 4.1 in [16]; see also Lemma 2.27 in [6]), the result follows from Theorem 4.5. ∎

Remark 4.11. Let $(A, \mathfrak{m})$ be an F-finite normal local ring of dimension 2. Then the condition of being strongly F-regular in the punctured spectrum and being Cohen–Macaulay is automatically satisfied.

4.1. The case of multiplier ideals in dimension two revisited

Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension 2 essentially of finite type over an algebraically closed field $\mathbb{K}$. Let $\mathfrak{a} = (f_1, f_2)$ be a parameter ideal and $\mathcal{J} = \{\mathcal{J}_c\}_{c \geq 0}$ an $\mathbb{R}$-good $\mathfrak{a}$-filtration of $\mathfrak{m}$-primary ideals satisfying $\mathcal{J}_c = \mathfrak{a}\mathcal{J}_{c-1}$ for all $c > 2$. Using Theorem 4.5, we get the Poincaré series

$$(4.3) \quad P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \left(m(c) + \frac{m(c+1)T}{(1-T)^2} + T^2 \left(\frac{\alpha_2}{(1-T)^2} + \frac{\alpha_1}{(1-T)} + p(T) \right) \right) T^c.$$

We see that, at least for the case of multiplier ideals in a complex surface with a rational singularity, this formula is much simpler. To do so we compare our formula with the one obtained in that case.

Theorem 4.12 (Theorem 6.1 in [1]). *Let $(A, \mathfrak{m})$ be the local ring of a complex surface with a rational singularity, let $\mathfrak{a} \subseteq A$ be an $\mathfrak{m}$-primary ideal, and let $\mathcal{J} := \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ be the filtration given by multiplier ideals. Then*

$$P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \left(\frac{m(c)}{1-T} + \frac{\rho_c T}{(1-T)^2} \right) T^c,$$

where $\rho_c := -F \cdot H_c$ is the excess associated to the maximal jumping divisor H_c.

If we compare both formulas, we observe that

$$P_{\mathcal{J}}(T) = \sum_{c \in (0,1]} \left(\frac{m(c) + (m(c+1) - 2m(c))T + m(c)T^2}{(1-T)^2} \right.$$
$$\left. + T^2 \left(\frac{\alpha_2}{(1-T)^2} + \frac{\alpha_1}{(1-T)} + p(T) \right) \right) T^c.$$
$$= \sum_{c \in (0,1]} \left(\frac{m(c)}{1-T} + \frac{\rho_c T}{(1-T)^2} + \frac{T^2}{(1-T)^2} \right.$$
$$\left. \times \left(m(c) + \alpha_2 + \alpha_1(1-T) + p(T)(1-T)^2 \right) \right) T^c,$$

and we conclude that $m(c) = -\alpha_2$, $\alpha_1 = 0$ and $p(T) = 0$. If we take a closer look to these conditions, we obtain a reformulation of Proposition 4.5 in [1] which, in particular, gives an algebraic formula for the excesses.

Proposition 4.13. *Let $(A, \mathfrak{m})$ be the local ring of a complex surface with a rational singularity, let $\mathfrak{a} \subseteq A$ be an $\mathfrak{m}$-primary ideal, and let $\mathcal{J} := \{\mathcal{J}(\mathfrak{a}^c)\}_{c \geq 0}$ be the filtration given by multiplier ideals. Then,*

$$\rho_c = \frac{1}{j} \left(\lambda(\text{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1}))) - \lambda(\text{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon}))) \right)$$

for every $j \geq 1$, where ρ_c is the excess associated to the maximal jumping divisor H_c. In particular,

$$m(c+j) - m(c) = \lambda(\text{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1}))) - \lambda(\text{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon})))$$

for every $j \geq 1$.

Proof. First recall that the morphisms ϕ_j^J in Setup 4.1 for an $\mathfrak{m}$-primary ideal $J \subseteq A$ are

$$0 \longrightarrow (A/J)^j \xrightarrow{\phi_j^J} (A/J)^{j+1} \xrightarrow{\varphi_j^J} (A/J) \longrightarrow A/(\mathfrak{a}^j + J) \longrightarrow 0 ,$$

and thus

$$\lambda(\operatorname{Im}\phi_j^J) = \lambda((A/J)^j) - \lambda(\ker\phi_j^J) = j\lambda(A/J) - \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/J)).$$

For simplicity, we denote λ_j^{c+1} and $\lambda_j^{c+1-\varepsilon}$ when we refer to $\lambda(\operatorname{Im}\phi_j^J)$ with J being the multiplier ideals $\mathcal{J}(\mathfrak{a}^{c+1})$ and $\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon})$ respectively. Then, as in the proof of Theorem 4.5, we have

$$\sum_{j \geq 1} [\lambda_j^{c+1-\varepsilon} - \lambda_j^{c+1}] T^{j-1} = q(T) + \left(\frac{\alpha_2}{(1-T)^2} - q_2(T)\right) + \left(\frac{\alpha_1}{(1-T)} - q_1(T)\right),$$

where, for some $k \gg 0$,

$$q(T) = (\lambda_1^{c+1-\varepsilon} - \lambda_1^{c+1}) + (\lambda_2^{c+1-\varepsilon} - \lambda_2^{c+1})T + \cdots + (\lambda_{k-1}^{c+1-\varepsilon} - \lambda_{k-1}^{c+1})T^{k-2},$$

$$q_2(T) = \alpha_2(1 + 2T + \cdots + (k-1)T^{k-2}),$$

$$q_1(T) = \alpha_1(1 + T + \cdots + T^{k-2}).$$

Since $\alpha_1 = 0, \alpha_2 = -m(c)$ and

$$0 = p(T) = (\lambda_1^{c+1-\varepsilon} - \lambda_1^{c+1} + m(c)) + (\lambda_2^{c+1-\varepsilon} - \lambda_2^{c+1} + 2m(c))T$$
$$+ \cdots + (\lambda_{k-1}^{c+1-\varepsilon} - \lambda_{k-1}^{c+1} + (k-1)m(c))T^{k-2},$$

we get for $j = 1, \ldots, k-1$,

$$jm(c) = \lambda_j^{c+1} - \lambda_1^{c+1-\varepsilon}$$
$$= j\lambda(A/\mathcal{J}(\mathfrak{a}^{c+1})) - \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1})))$$
$$- j\lambda(A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon})) + \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon})))$$
$$= jm(c+1) + \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon}))) - \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1}))).$$

Therefore,

$$j\rho_c = \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1}))) - \lambda(\operatorname{Tor}_2^A(A/\mathfrak{a}^j, A/\mathcal{J}(\mathfrak{a}^{c+1-\varepsilon}))).$$

The same formula also holds for $j \geq k$ since we have

$$\lambda_j^{c+1-\varepsilon} - \lambda_j^{c+1} = \alpha_2 j = -m(c)j. \qquad \blacksquare$$

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