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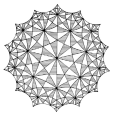

An example concerning Fourier analytic criteria for translational tiling

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Abstract. It is well known that the functions $f \in L^1(\mathbb{R}^d)$ whose translates along a lattice Λ form a tiling, can be completely characterized in terms of the zero set of their Fourier transform. We construct an example of a discrete set $\Lambda \subset \mathbb{R}$ (a small perturbation of the integers) for which no characterization of this kind is possible: there are two functions $f, g \in L^1(\mathbb{R})$ whose Fourier transforms have the same set of zeros, but such that $f + \Lambda$ is a tiling while $g + \Lambda$ is not.

1. Introduction

1.1. Let f be a function in $L^1(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}$ be a discrete set. We say that f *tiles* $\mathbb{R}$ *at level* w with the translation set Λ, or that $f + \Lambda$ *is a tiling of* $\mathbb{R}$ *at level* w (where w is a constant), if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w \quad \text{a.e.} \tag{1.1}$$

and the series in (1.1) converges absolutely a.e.

In the same way, one can define tiling of $\mathbb{R}^d$ by translates of a function $f \in L^1(\mathbb{R}^d)$.

For example, if $f = \mathbb{1}_\Omega$ is the indicator function of a set Ω, and $f + \Lambda$ is a tiling at level 1, then this means that the translated copies $\Omega + \lambda$, $\lambda \in \Lambda$, fill the whole space without overlaps up to measure zero. To the contrary, for tiling by a general real or complex-valued function f, the translated copies may have overlapping supports.

Tilings by translates of a function have been studied by several authors, see, in particular, [9–14].

1.2. It is well known that in the study of translational tilings, the set

$$Z(\widehat{f}) := \{t : \widehat{f}(t) = 0\}$$

of the zeros of the Fourier transform

$$\widehat{f}(t) = \int f(x) \exp(-2\pi i t x)\, dx$$

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plays an important role. For example, let Λ be a lattice in $\mathbb{R}^d$; then $f + \Lambda$ is a tiling if and only if the set $Z(\widehat{f})$ contains $\Lambda^* \setminus \{0\}$, where Λ^* is the dual lattice. This means that the functions f that tile by a lattice Λ can be completely characterized in terms of the zero set $Z(\widehat{f})$. (One can show that the tiling level is given by $w = \widehat{f}(0)\det(\Lambda)^{-1}$.)

The necessity of the condition for tiling in the last example can be generalized as follows. For a discrete set $\Lambda \subset \mathbb{R}$, we consider the measure

$$\delta_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda.$$

We will assume that Λ has *bounded density*, which means that

$$\sup_{x\in\mathbb{R}} \#(\Lambda \cap [x, x+1)) < +\infty. \tag{1.2}$$

In particular, (1.2) implies that the measure δ_Λ is a temperate distribution on $\mathbb{R}$, so it has a well-defined Fourier transform $\widehat{\delta}_\Lambda$ in the distributional sense.

Theorem 1.1 ([11]). *Let $f \in L^1(\mathbb{R})$, and let $\Lambda \subset \mathbb{R}$ be a discrete set of bounded density. If $f + \Lambda$ is a tiling at some level w, then*

$$\operatorname{supp}(\widehat{\delta}_\Lambda) \setminus \{0\} \subset Z(\widehat{f}). \tag{1.3}$$

A similar result is true also in $\mathbb{R}^d$. In earlier works, [7, 8, 10], this result was proved under various extra assumptions.

If Λ is a lattice, then $\widehat{\delta}_\Lambda = \det(\Lambda)^{-1} \cdot \delta_{\Lambda^*}$, by the Poisson summation formula. This implies that $\operatorname{supp}(\widehat{\delta}_\Lambda) = \Lambda^*$. Hence, in this case, the condition (1.3) is not only necessary, but also sufficient, for $f + \Lambda$ to be a tiling at some level w.

However, for a general discrete set Λ of bounded density, the sufficiency of the condition (1.3) for tiling has remained an open problem. In this paper, we settle this problem in the negative. Our main result is the following.

Theorem 1.2. *There is a discrete set $\Lambda \subset \mathbb{R}$ of bounded density (a small perturbation of the integers) with the following property: given any real scalar w, there are two real-valued functions $f, g \in L^1(\mathbb{R})$ whose Fourier transforms have the same set of zeros, but such that $f + \Lambda$ is a tiling at level w, while $g + \Lambda$ is not a tiling at any level.*

Moreover, we will show that if the given scalar w is *positive*, then the functions f and g can be chosen positive as well.

It follows that the necessary condition (1.3) is generally not sufficient for tiling:

Corollary 1.3. *There exist a set $\Lambda \subset \mathbb{R}$ of bounded density and a positive function $f \in L^1(\mathbb{R})$ such that (1.3) is satisfied, but $f + \Lambda$ is not a tiling at any level.*

But even stronger, Theorem 1.2 shows that also no other condition can be given in terms of the Fourier zero set $Z(\widehat{f})$ that would characterize the functions $f \in L^1(\mathbb{R})$ such that $f + \Lambda$ is a tiling, even under the extra assumption that f is positive.

Our approach is based on the relation of the problem to Malliavin's non-spectral synthesis example [17]. The proof involves the implicit function method due to Kargaev [5], who proved the existence of a set $\Omega \subset \mathbb{R}$ of finite measure such that the Fourier transform of its indicator function vanishes on some interval.

2. Preliminaries and notation

In this section we recall some preliminary background and fix notation that will be used later on. For further details, we refer the reader to [3].

The closed support of a Schwartz distribution S, or a function ϕ, on the real line $\mathbb{R}$ or on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, is denoted by $\operatorname{supp}(S)$ or $\operatorname{supp}(\phi)$, respectively.

If S is a Schwartz distribution on $\mathbb{T}$, its Fourier coefficients $\widehat{S}(n)$ are defined by

$$\widehat{S}(n) = \langle S, e^{-2\pi i n t} \rangle, \quad n \in \mathbb{Z}.$$

The action of S on a function $\phi \in C^\infty(\mathbb{T})$ is denoted by $\langle S, \phi \rangle$. We have

$$\langle S, \phi \rangle = \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{\phi}(-n). \tag{2.1}$$

Let $A(\mathbb{T})$ be the Wiener space of continuous functions ϕ on $\mathbb{T}$ whose Fourier series converges absolutely. It is a Banach space endowed with the norm

$$\|\phi\|_{A(\mathbb{T})} = \sum_{n \in \mathbb{Z}} |\widehat{\phi}(n)|.$$

A distribution S on $\mathbb{T}$ is called a *pseudomeasure* if S can be extended to a continuous linear functional on $A(\mathbb{T})$. This is the case if and only if the Fourier coefficients $\widehat{S}(n)$ are bounded. The space $PM(\mathbb{T})$ of all pseudomeasures is a Banach space with the norm

$$\|S\|_{PM(\mathbb{T})} = \sup_{n \in \mathbb{Z}} |\widehat{S}(n)|.$$

The duality between the spaces $A(\mathbb{T})$ and $PM(\mathbb{T})$ is given by

$$\langle S, \phi \rangle = \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{\phi}(-n), \quad S \in PM(\mathbb{T}),\ \phi \in A(\mathbb{T}), \tag{2.2}$$

which is consistent with (2.1).

In a similar way, we will denote by $A(\mathbb{R})$ the space of Fourier transforms of functions in $L^1(\mathbb{R})$, that is, $\phi \in A(\mathbb{R})$ if and only if

$$\phi(t) = \int_{\mathbb{R}} \widehat{\phi}(x) e^{2\pi i t x}\, dx, \quad \widehat{\phi} \in L^1(\mathbb{R}), \quad \|\phi\|_{A(\mathbb{R})} = \|\widehat{\phi}\|_{L^1(\mathbb{R})}.$$

The Banach space dual to $A(\mathbb{R})$ is then the space $PM(\mathbb{R})$ of temperate distributions S on $\mathbb{R}$ whose Fourier transform $\widehat{S}$ is in $L^\infty(\mathbb{R})$. The space $PM(\mathbb{R})$ is normed as

$$\|S\|_{PM(\mathbb{R})} = \|\widehat{S}\|_{L^\infty(\mathbb{R})},$$

and the duality between the spaces $A(\mathbb{R})$ and $PM(\mathbb{R})$ is given by

$$\langle S, \phi \rangle = \int_{\mathbb{R}} \widehat{S}(x) \widehat{\phi}(-x)\, dx, \quad S \in PM(\mathbb{R}),\ \phi \in A(\mathbb{R}). \tag{2.3}$$

The elements of the space $PM(\mathbb{R})$ are called *pseudomeasures* on $\mathbb{R}$.

The product $\phi\psi$ of two functions $\phi, \psi \in A$ (on either $\mathbb{T}$ or $\mathbb{R}$) is also in A, and

$$\|\phi\psi\|_A \leqslant \|\phi\|_A \, \|\psi\|_A.$$

If $S \in PM$ and $\phi \in A$, then the product $S\phi$ is a pseudomeasure defined by

$$\langle S\phi, \psi\rangle = \langle S, \phi\psi\rangle, \quad \psi \in A,$$

and we have

$$\|S\phi\|_{PM} \leqslant \|S\|_{PM} \, \|\phi\|_A.$$

If $S \in PM$, $\phi \in A$ and if ϕ vanishes in a *neighborhood* of $\operatorname{supp}(S)$, then $S\phi = 0$. This is obvious from the definition of $\operatorname{supp}(S)$ if ϕ is a smooth function of compact support, while for a general $\phi \in A$, this follows by approximation.

If S is a Schwartz distribution on $\mathbb{R}$ supported on a compact interval $I = [a, b]$, then its Fourier transform $\widehat{S}$ is an infinitely smooth function on $\mathbb{R}$ given by

$$\widehat{S}(x) = \langle S, e^{-2\pi i x t}\rangle, \quad x \in \mathbb{R}.$$

(In fact, $\widehat{S}$ is the restriction to $\mathbb{R}$ of an entire function of exponential type).

If S is a distribution on $\mathbb{R}$ supported on an interval I of length $|I| < 1$, then S may be considered also as a distribution on $\mathbb{T}$, and in this case we have $S \in PM(\mathbb{T})$ if and only if $S \in PM(\mathbb{R})$. If, in addition, ϕ is a function on $\mathbb{R}$ such that $\operatorname{supp}(\phi) \subset I$, then $\phi \in A(\mathbb{T})$ if and only if $\phi \in A(\mathbb{R})$, and the action $\langle S, \phi\rangle$ then has the same value with respect to either definition (2.2) or (2.3).

3. Malliavin's non-spectral synthesis phenomenon

3.1. The *spectral synthesis problem*, posed by Beurling, asks the following: let V be a closed, linear subspace of the space $\ell^\infty(\mathbb{Z})$ endowed with the weak* topology (as the dual of ℓ^1). We say that V is *translation-invariant* if whenever a sequence $\{c(n)\}$ belongs to V, then so do all of the translates of $\{c(n)\}$. Define the *spectrum* $\sigma(V)$ of a translation-invariant subspace V to be the (closed) set of points $t \in \mathbb{T}$ such that the sequence $e_t := \{\exp(2\pi i n t)\}$ is in V. Is it true that V is generated by the exponentials e_t, $t \in \sigma(V)$?; i.e., is V the weak* closure of the linear span of these exponentials?

There are also other equivalent formulations of the spectral synthesis problem, see [4], Chapter IX. One of them is the following: let $S \in PM(\mathbb{T})$, $\phi \in A(\mathbb{T})$, and assume that ϕ vanishes on $\operatorname{supp}(S)$. Does it follow that $\langle S, \phi\rangle = 0$?

The answer to the last question is affirmative if ϕ is smooth, or, more generally, if $\phi \in A(\mathbb{T}) \cap \operatorname{Lip}(1/2)$. This result is due to Beurling and Pollard, see e.g. Section 5 in Chapter V of [3]. However, it was proved by Malliavin that in the general case, the question admits a negative answer:

Theorem 3.1 (Malliavin [15,16]). *There exist a pseudomeasure $S \in PM(\mathbb{T})$ and a function $\phi \in A(\mathbb{T})$ such that ϕ vanishes on* $\operatorname{supp}(S)$*, but $\langle S, \phi\rangle \neq 0$.*

The spectral synthesis problem can be posed more generally in any locally compact abelian group G (where the case discussed above corresponds to the group $G = \mathbb{Z}$). For

compact groups, the problem admits a positive answer; while Malliavin showed [17] that the answer is negative for all non-compact groups G.

For more details on the subject, we refer the reader to Chapter IX of [4], Chapter V of [3], Chapter 7 of [18], [1], and Chapter 3 of [2].

3.2. Let $S \in PM(\mathbb{T})$ and let $\phi \in A(\mathbb{T})$ be given by Malliavin's theorem (Theorem 3.1), that is, ϕ vanishes on $\mathrm{supp}(S)$ while $\langle S, \phi\rangle \neq 0$. Since ϕ does not vanish everywhere on the circle $\mathbb{T}$, there is an open interval I of length $|I| < 1$ such that $\mathrm{supp}(S) \subset I$. Hence we may regard S also as a distribution on $\mathbb{R}$, and we have $S \in PM(\mathbb{R})$. By multiplying ϕ on a smooth function supported on I and which is equal to 1 in a neighborhood of $\mathrm{supp}(S)$, we may assume that $\mathrm{supp}(\phi) \subset I$ as well, and consequently $\phi \in A(\mathbb{R})$.

Furthermore, by applying a linear change of variable to S and ϕ, we may actually suppose that I is an arbitrary open interval on $\mathbb{R}$. We shall take $I = (a, b)$, where a and b are any two numbers satisfying $0 < a < b < 1/2$.

For each $r > 0$, we now define a distribution $T_r \in PM(\mathbb{R})$ by

$$T_r := \delta_0 + r(S + \widetilde{S}), \tag{3.1}$$

where $\widetilde{S}(t) := \overline{S(-t)}$.[1] We will prove the following result.

Theorem 3.2. *Given any $\varepsilon > 0$, there exists a real sequence $\Lambda = \{\lambda_n\}$, $n \in \mathbb{Z}$, satisfying $|\lambda_n - n| \leqslant \varepsilon$ for all n, such that for some $r > 0$ we have $\widehat{\delta_\Lambda} = T_r$ in the interval $(-b, b)$.*

The proof of this theorem will be given in the next section. Our goal in the present section is to complete the proof of Theorem 1.2 based on this result. We will show that Λ has the property from the statement of the theorem: given any real scalar w, there are two real-valued functions $f, g \in L^1(\mathbb{R})$ whose Fourier transforms have the same set of zeros, but such that $f + \Lambda$ is a tiling at level w while $g + \Lambda$ is not a tiling at any level. Moreover, if the given scalar w is *positive*, then the functions f and g can be chosen positive as well.

3.3. Since the set Λ has bounded density, for any $h \in L^1(\mathbb{R})$ the convolution $h * \delta_\Lambda$ is a locally integrable function satisfying

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |(h * \delta_\Lambda)(y)|\, dy < +\infty,$$

see Lemma 2.2 in [10]. This implies that $h * \delta_\Lambda$ is a temperate distribution on $\mathbb{R}$.

Lemma 3.3. *Let h be a function in $L^1(\mathbb{R})$ such that $\mathrm{supp}(\widehat{h}) \subset (-b, b)$. Then the Fourier transform of $h * \delta_\Lambda$ is the pseudomeasure $T_r \cdot \widehat{h}$.*

Proof. The assertion means that for any Schwartz function β we have

$$\int_{\mathbb{R}} (h * \delta_\Lambda)(x)\, \widehat{\beta}(x)\, dx = \langle T_r, \widehat{h} \cdot \beta\rangle. \tag{3.2}$$

Let χ be a Schwartz function whose Fourier transform $\widehat{\chi}$ is nonnegative, has compact support, and $\int \widehat{\chi}(t)dt = 1$, and for each $\varepsilon > 0$, let $\chi_\varepsilon(x) := \chi(\varepsilon x)$. Let $q_\varepsilon := (\widehat{h} \cdot \beta) * \widehat{\chi}_\varepsilon$.

[1] The distribution $\widetilde{S}$ can be more formally defined by $\langle \widetilde{S}, \psi\rangle := \overline{\langle S, \widetilde{\psi}\rangle}$, where $\widetilde{\psi}(t) := \overline{\psi(-t)}$.

Then q_ε is an infinitely smooth function with compact support. As $\varepsilon \to 0$, the function q_ε remains supported on a certain closed interval J contained in $(-b, b)$, and q_ε converges to $\widehat{h} \cdot \beta$ in the space $A(\mathbb{R})$. The assumption that $\widehat{\delta}_\Lambda = T_r$ in $(-b, b)$ thus implies that

$$\lim_{\varepsilon \to 0} \langle \widehat{\delta}_\Lambda, q_\varepsilon \rangle = \lim_{\varepsilon \to 0} \langle T_r, q_\varepsilon \rangle = \langle T_r, \widehat{h} \cdot \beta \rangle. \tag{3.3}$$

The function β is the Fourier transform of some function α in the Schwartz class. Let $p_\varepsilon := (h * \alpha) \cdot \chi_\varepsilon$. Then p_ε is a smooth function in $L^1(\mathbb{R})$ and we have $\widehat{p}_\varepsilon = q_\varepsilon$. Since q_ε belongs to the Schwartz space, the same is true for p_ε, and it follows that

$$\begin{aligned} \langle \widehat{\delta}_\Lambda, q_\varepsilon \rangle &= \langle \delta_\Lambda, \widehat{q}_\varepsilon \rangle = \sum_{\lambda \in \Lambda} p_\varepsilon(-\lambda) = \sum_{\lambda \in \Lambda} (h * \alpha)(-\lambda)\, \chi_\varepsilon(-\lambda) \\ &= \sum_{\lambda \in \Lambda} \chi_\varepsilon(-\lambda) \int_{\mathbb{R}} \alpha(-x) h(x - \lambda)\, dx. \end{aligned} \tag{3.4}$$

Now we need the following:

Claim. *We have*

$$\sum_{\lambda \in \Lambda} \int_{\mathbb{R}} |\alpha(-x)| \cdot |h(x - \lambda)|\, dx < +\infty.$$

We observe that $|\chi_\varepsilon(-\lambda)| \leqslant 1$ and $\chi_\varepsilon(-\lambda) \to 1$ as $\varepsilon \to 0$ for each λ. Hence the claim allows us to apply the dominated convergence theorem to the sum (3.4), which yields

$$\lim_{\varepsilon \to 0} \langle \widehat{\delta}_\Lambda, q_\varepsilon \rangle = \sum_{\lambda \in \Lambda} \int_{\mathbb{R}} \alpha(-x) h(x - \lambda)\, dx. \tag{3.5}$$

The claim also allows us to exchange the sum and integral in (3.5), and it follows that

$$\lim_{\varepsilon \to 0} \langle \widehat{\delta}_\Lambda, q_\varepsilon \rangle = \int_{\mathbb{R}} \alpha(-x) \sum_{\lambda \in \Lambda} h(x - \lambda)\, dx = \int_{\mathbb{R}} (h * \delta_\Lambda)(x)\, \widehat{\beta}(x)\, dx. \tag{3.6}$$

Comparing (3.3) and (3.6), we see that (3.2) holds.

It remains to prove the claim. Indeed, we have

$$\sum_{\lambda \in \Lambda} \int_{\mathbb{R}} |\alpha(-x)| \cdot |h(x - \lambda)|\, dx = \int_{\mathbb{R}} |h(-x)| \sum_{\lambda \in \Lambda} |\alpha(x - \lambda)|\, dx. \tag{3.7}$$

The inner sum on the right-hand side of (3.7) is a bounded function of x, since α is a Schwartz function and Λ has bounded density, while h is a function in $L^1(\mathbb{R})$. Hence the integral in (3.7) converges, and this completes the proof of the lemma. ∎

3.4. Recall that $S \in PM(\mathbb{R})$, $\operatorname{supp}(S) \subset (a, b)$ where $0 < a < b < 1/2$, $\phi \in A(\mathbb{R})$ is a function with $\operatorname{supp}(\phi) \subset (a, b)$, ϕ vanishes on $\operatorname{supp}(S)$, and $\langle S, \phi \rangle \neq 0$. Let ψ be a smooth function whose zero set $Z(\psi)$ is the same as $Z(\phi)$. In particular, we have $\operatorname{supp}(\psi) \subset (a, b)$ and ψ vanishes on $\operatorname{supp}(S)$ as well. Let also τ be a smooth function satisfying $\tau(-t) = \overline{\tau(t)}$, $\operatorname{supp}(\tau) \subset (-a, a)$, and $\tau(0) = 1$.

Given a real scalar w, we define two functions $f, g \in L^1(\mathbb{R})$ by the conditions

$$\widehat{f}(t) = w \cdot \tau(t) + \psi(t) + \overline{\psi(-t)}, \tag{3.8}$$

$$\widehat{g}(t) = w \cdot \tau(t) + \phi(t) + \overline{\phi(-t)}. \tag{3.9}$$

Then f and g are real-valued and their Fourier transforms have the same set of zeros.

By Lemma 3.3, the Fourier transform of $f * \delta_\Lambda$ is the pseudomeasure

$$\widehat{f} \cdot T_r = w\delta_0 + r(S\psi + \widetilde{(S\psi)}) = w\delta_0,$$

where the first equality is due to (3.1) and (3.8), while the second equality is true since ψ is smooth and vanishes on $\mathrm{supp}(S)$, hence $S\psi = 0$. We conclude that $f * \delta_\Lambda = w$ a.e., which means that $f + \Lambda$ is a tiling at level w.

In the same way, Lemma 3.3 implies that the Fourier transform of $g * \delta_\Lambda$ is

$$\widehat{g} \cdot T_r = w\delta_0 + r(S\phi + \widetilde{(S\phi)}).$$

However, in this case $S\phi$ is not the zero distribution, since $\langle S\phi, 1\rangle = \langle S, \phi\rangle \neq 0$. This shows that the Fourier transform of $g * \delta_\Lambda$ is not a scalar multiple of δ_0, and it follows that $g + \Lambda$ is not a tiling at any level.

3.5. The above construction yields real-valued functions f and g, but these two functions need not be positive. We will now show that if the given scalar w is positive, then the construction can be modified so as to yield *everywhere positive* functions f and g.

In what follows, ϕ and ψ continue to denote the same two functions as above.

Step 1. We show that there is a nonnegative sequence $\{c(k)\} \in \ell^1(\mathbb{Z})$, such that

$$|\widehat{\phi}(x)| \leqslant c(k), \quad k \in \mathbb{Z},\ |x-k| \leqslant 1/2. \tag{3.10}$$

Indeed, we have $\phi \in A(\mathbb{R})$ and $\mathrm{supp}(\phi) \subset (a,b)$. Considered as a function in $A(\mathbb{T})$, ϕ may be expressed on (a,b) as the sum of an absolutely convergent Fourier series. Hence there is a finite (complex) measure μ supported on $\mathbb{Z}$ such that $\phi(t) = \widehat{\mu}(-t)$, $t \in (a,b)$. Let Φ be an infinitely smooth function such that $\Phi(t) = 1$ if $t \in \mathrm{supp}(\phi)$, while $\Phi(t) = 0$ for $t \in \mathbb{R} \setminus (a,b)$. Then $\phi(t) = \widehat{\mu}(-t)\Phi(t)$ for every $t \in \mathbb{R}$, which implies that

$$\widehat{\phi}(x) = (\mu * \widehat{\Phi})(x) = \sum_{n\in\mathbb{Z}} \mu(n)\,\widehat{\Phi}(x-n), \quad x \in \mathbb{R}.$$

Since the Fourier transform $\widehat{\Phi}$ has fast decay, there is a sequence $\{\gamma(k)\} \in \ell^1(\mathbb{Z})$ such that $|\widehat{\Phi}(x)| \leqslant |\gamma(k)|$ whenever $|x-k| \leqslant 1/2$. It follows that

$$|\widehat{\phi}(x)| \leqslant c(k) := \sum_{n\in\mathbb{Z}} |\mu(n)| \cdot |\gamma(k-n)|, \quad |x-k| \leqslant 1/2,$$

which establishes (3.10).

Step 2. We may assume that the same sequence $\{c(k)\}$ also satisfies

$$|\widehat{\psi}(x)| \leqslant c(k), \quad k \in \mathbb{Z},\ |x-k| \leqslant 1/2. \tag{3.11}$$

Indeed, we may apply the same procedure from Step 1 also to the function ψ, and then define $\{c(k)\}$ to be the maximum of the two sequences obtained from both steps.

Step 3. Let $\{d(k)\}$ be any positive sequence in $\ell^1(\mathbb{Z})$ such that

$$d(k) > c(k), \quad k \in \mathbb{Z}. \tag{3.12}$$

We show that there is $\tau \in A(\mathbb{R})$ such that $\operatorname{supp}(\tau) \subset (-a, a)$ and

$$\widehat{\tau}(x) \geqslant d(k), \quad k \in \mathbb{Z},\ |x-k| \leqslant 1/2. \tag{3.13}$$

Let χ be an infinitely smooth function, $\operatorname{supp}(\chi) \subset (-a, a)$, whose Fourier transform $\widehat{\chi}$ is nonnegative and satisfies $\widehat{\chi}(x) \geqslant 1$ on the interval $[-1/2, 1/2]$. Let $\tau(t) := \widehat{\nu}(-t)\chi(t)$, where ν is a positive, finite measure supported on $\mathbb{Z}$ defined by $\nu := \sum_n d(n)\delta_n$. Then the function $\widehat{\tau} = \nu * \widehat{\chi}$ is in $L^1(\mathbb{R})$, so that we have $\tau \in A(\mathbb{R})$, and

$$\widehat{\tau}(x) = \sum_{n\in\mathbb{Z}} d(n)\,\widehat{\chi}(x-n) \geqslant d(k)\,\widehat{\chi}(x-k) \geqslant d(k), \quad |x-k| \leqslant 1/2,$$

which gives (3.13).

Step 4. Now suppose that we are given a *positive* scalar w. We then define the two functions f and g by the conditions

$$\widehat{f}(t) = w \cdot \tau(0)^{-1} \cdot \Big[\tau(t) + \frac{\psi(t) + \overline{\psi(-t)}}{2}\Big],$$

$$\widehat{g}(t) = w \cdot \tau(0)^{-1} \cdot \Big[\tau(t) + \frac{\phi(t) + \overline{\phi(-t)}}{2}\Big].$$

We observe that by the definition of the function τ we have

$$\tau(0) = \widehat{\nu}(0)\,\chi(0) = \Big(\int d\nu\Big)\Big(\int \widehat{\chi}(x)\,dx\Big) > 0,$$

and in particular $\tau(0)$ is nonzero. Then f and g are in $L^1(\mathbb{R})$, their Fourier transforms have the same set of zeros, and by the same argument as before, one can verify that $f + \Lambda$ is a tiling at level w, while $g + \Lambda$ is not a tiling at any level.

Finally, we check that f and g are everywhere positive functions. Indeed, we have

$$f(x) = w \cdot \tau(0)^{-1} \cdot \big[\widehat{\tau}(-x) + \operatorname{Re}(\widehat{\psi}(-x))\big],$$

$$g(x) = w \cdot \tau(0)^{-1} \cdot \big[\widehat{\tau}(-x) + \operatorname{Re}(\widehat{\phi}(-x))\big],$$

and by (3.10), (3.11), (3.12) and (3.13), it follows that $f(x), g(x) > 0$ for every $x \in \mathbb{R}$.

This completes the proof of Theorem 1.2 based on Theorem 3.2. ∎

It remains to prove Theorem 3.2. This will be done in the next section.

4. Kargaev's implicit function method

4.1. In [19], Sapogov posed the following question: does there exist a set $\Omega \subset \mathbb{R}$ of positive and finite measure, such that the Fourier transform of its indicator function $\mathbb{1}_\Omega$ vanishes on some open interval (a, b)?

The question was answered in the affirmative by Kargaev [5]. The solution was based on an innovative application of the infinite-dimensional implicit function theorem, which established the existence of a set of the form $\Omega = \bigcup_{n\in\mathbb{Z}}[n + \alpha_n, n + \beta_n]$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in ℓ^1, that has the above mentioned property.

The approach was later used in [11] in order to prove the existence of non-periodic tilings of $\mathbb{R}$ by translates of a function f. In that paper, a self-contained presentation of the method was given in a simplified form, that does not invoke the infinite-dimensional implicit function theorem.

In this section, we use an adapted version of Kargaev's method in order to prove Theorem 3.2 (and more, in fact). The presentation below generally follows the lines of [11], Sections 2 and 3, but the proof also requires some additional arguments.

4.2. Let $\{\alpha_n\}$, $n \in \mathbb{Z}$, be a bounded sequence of real numbers. To such a sequence we associate a function F on the real line, defined by

$$F(x) = \sum_{n\in\mathbb{Z}} F_n(x), \quad x \in \mathbb{R}, \tag{4.1}$$

where F_n is the function $\mathbb{1}_{[n,n+\alpha_n]}$ if $\alpha_n \geqslant 0$, or $-\mathbb{1}_{[n+\alpha_n,n]}$ if $\alpha_n < 0$.

Since the sequence $\{\alpha_n\}$ is bounded, the series (4.1) is easily seen to converge in the space of temperate distributions to a bounded function F on $\mathbb{R}$. In particular, F is a temperate distribution.

Theorem 4.1. *Given two numbers $b \in (0, 1/2)$ and $\varepsilon > 0$, there is $\delta > 0$ with the following property. Let S be a Schwartz distribution on $\mathbb{R}$ satisfying*

$$S(-t) = \overline{S(t)}, \quad \operatorname{supp}(S) \subset (-b, b), \quad \sup_{k\in\mathbb{Z}} |\widehat{S}(k)| \leqslant \delta. \tag{4.2}$$

Then there is a bounded, real sequence $\alpha = \{\alpha_n\}$, $n \in \mathbb{Z}$, such that $\|\alpha\|_\infty \leqslant \varepsilon$ and

$$\widehat{F} = S \quad \text{in } (-b, b),$$

where F is the function defined by (4.1).

The proof of Theorem 4.1 is given below. It is divided into a series of lemmas.

4.3. Given a number $b \in (0, 1/2)$, we choose $l = l(b)$ such that $b < l < 1/2$. We also choose an infinitely smooth function Φ satisfying the conditions $\Phi(-t) = \overline{\Phi(t)}$ for all $t \in \mathbb{R}$, $\Phi(t) = 1$ for $t \in [-b, b]$, and $\Phi(t) = 0$ for $t \in \mathbb{R} \setminus (-l, l)$.

Let $I := [-1/2, 1/2]$. Denote by $\widehat{\psi}(k)$ the kth Fourier coefficient of a function ψ on I:

$$\widehat{\psi}(k) = \int_I \psi(t)\, e^{-2\pi i k t}\, dt, \quad k \in \mathbb{Z}.$$

The following lemma is inspired by Lemma 2.2 in [6].

Lemma 4.2. *Let $\varphi \in C^\infty(\mathbb{R})$, with $\varphi(0) = 0$, and denote $\psi_s(t) := \varphi(st)\,\Phi(t)$. Then*

$$|\widehat{\psi}_s(k)| \leqslant \frac{C\,|s|}{1 + |k|^m} \tag{4.3}$$

for every $s \in [-1, 1]$ and $k \in \mathbb{Z}$, where $C = C(\Phi, \varphi, m) > 0$ is a constant which does not depend on s or k.

Proof. First suppose that $k = 0$. We have $|\varphi(t)| \leqslant C|t|$ for $t \in I$, hence

$$|\widehat{\psi}_s(0)| \leqslant C|s| \int_I |t\,\Phi(t)|\,dt = C|s|.$$

Next we assume that $k \neq 0$. We integrate by parts m times and use the fact that the function ψ_s vanishes in a neighborhood of the points $\pm 1/2$. This yields

$$\widehat{\psi}_s(k) = \frac{1}{(2\pi i k)^m} \int_I \psi_s^{(m)}(t)\, e^{-2\pi i k t}\,dt. \tag{4.4}$$

By the product rule for the mth derivative we have

$$\psi_s^{(m)}(t) = \sum_{j=0}^{m} \binom{m}{j} s^j \varphi^{(j)}(st)\,\Phi^{(m-j)}(t). \tag{4.5}$$

Combining (4.4) and (4.5) yields the estimate

$$|\widehat{\psi}_s(k)| \leqslant \frac{1}{(2\pi|k|)^m} \sum_{j=0}^{m} \binom{m}{j} |s|^j \int_I |\varphi^{(j)}(st)\,\Phi^{(m-j)}(t)|\,dt.$$

Since the derivatives $\varphi', \varphi'', \ldots, \varphi^{(m)}$ are bounded on I, each one of the terms in the sum corresponding to $j = 1, 2, \ldots, m$ is bounded by $C|s|$, while the term corresponding to $j = 0$ can be estimated using $|\varphi(t)| \leqslant C|t|$, $t \in I$, which again yields $C|s|$. ∎

Lemma 4.3. *Let $\varphi \in C^\infty(\mathbb{R})$, with $\varphi'(0) = 0$, and denote*

$$\psi_{u,v}(t) := \frac{\varphi(vt) - \varphi(ut)}{t} \cdot \Phi(t).$$

Then

$$|\widehat{\psi}_{u,v}(k)| \leqslant \max\{|u|, |v|\} \cdot \frac{C|v-u|}{1+|k|^m} \tag{4.6}$$

for every $u, v \in [-1, 1]$ and $k \in \mathbb{Z}$, where $C = C(\Phi, \varphi, m) > 0$ is a constant which does not depend on u, v or k.

Proof. We may suppose that $u < v$. We observe that

$$\psi_{u,v}(t) = \int_u^v \varphi'(st)\,\Phi(t)\,ds = \int_u^v \psi_s(t)\,ds,$$

where we define $\psi_s(t) := \varphi'(st)\,\Phi(t)$. Hence

$$\widehat{\psi}_{u,v}(k) = \int_u^v \widehat{\psi}_s(k)\,ds, \quad k \in \mathbb{Z}.$$

By Lemma 4.2, the estimate (4.3) is valid for every $s \in [u, v]$, where $C = C(\Phi, \varphi, m) > 0$ is a constant which does not depend on s or k. Hence

$$|\widehat{\psi}_{u,v}(k)| \leqslant \frac{C}{1+|k|^m} \int_u^v |s|\,ds,$$

from which (4.6) follows. ∎

4.4. If T is a Schwartz distribution supported on $[-l,l]$, then $\widehat{T}(k)$ denotes the kth Fourier coefficients of T:

$$\widehat{T}(k) = \langle T, e^{-2\pi ikt}\rangle, \quad k \in \mathbb{Z}.$$

Lemma 4.4. *Let T be a distribution supported on $[-l,l]$. Then the series*

$$\sum_{n\in\mathbb{Z}} \widehat{T}(n)\, e^{2\pi int} \tag{4.7}$$

converges unconditionally in the distributional sense to T in the open interval $(-1/2,1/2)$.

This follows from the unconditional convergence of the series (4.7) to T, considered as a distribution on the circle $\mathbb{T}$.

4.5. Let X be the space of all bounded sequences of real numbers $\alpha = \{\alpha_n\}$, $n \in \mathbb{Z}$, endowed with the norm

$$\|\alpha\|_X := \sup_{n\in\mathbb{Z}} |\alpha_n|$$

that makes X into a real Banach space.

Let Y be the space of distributions T supported on $[-l,l]$ whose Fourier coefficients $\widehat{T}(k)$, $k \in \mathbb{Z}$, are real and bounded. If we endow Y with the norm

$$\|T\|_Y := \sup_{k\in\mathbb{Z}} |\widehat{T}(k)|$$

then also Y is a real Banach space, which can be viewed as a closed subspace of $PM(\mathbb{T})$.

We observe that a distribution T supported on $[-l,l]$ has real Fourier coefficients (that is, $\widehat{T}(k) \in \mathbb{R}$ for every $k \in \mathbb{Z}$) if and only if $T(-t) = \overline{T(t)}$.

Lemma 4.5. *Let $\{T_n\}$, $n \in \mathbb{Z}$, be a sequence of elements of Y. Assume that there is a sequence $\gamma \in X$ such that*

$$|\widehat{T}_n(k)| \leq \frac{|\gamma_n|}{1+|k-n|^2} \tag{4.8}$$

for every n and k in $\mathbb{Z}$. Then the series

$$T = \sum_{n\in\mathbb{Z}} T_n \tag{4.9}$$

converges unconditionally in the distributional sense to an element $T \in Y$ satisfying

$$\|T\|_Y \leq K\|\gamma\|_X, \tag{4.10}$$

where K is an absolute constant.

Proof. Indeed, the condition (4.8) implies that for any $\phi \in A(\mathbb{T})$ we have

$$|\langle T_n, \phi\rangle| = \Big|\sum_{k\in\mathbb{Z}} \widehat{T}_n(k)\,\widehat{\phi}(-k)\Big| \leq |\gamma_n| \sum_{k\in\mathbb{Z}} \frac{|\widehat{\phi}(-k)|}{1+|k-n|^2},$$

and hence

$$\sum_{n\in\mathbb{Z}} |\langle T_n, \phi\rangle| \leqslant K\,\|\gamma\|_X\,\|\phi\|_{A(\mathbb{T})}, \quad K := \sum_{n\in\mathbb{Z}} \frac{1}{1+|n|^2}\cdot$$

This shows that the series (4.9) converges unconditionally in the weak* topology of the space $PM(\mathbb{T})$ (the dual of $A(\mathbb{T})$) to an element $T \in Y$ satisfying (4.10). ∎

4.6. Let $\alpha = \{\alpha_n\}$, $n \in \mathbb{Z}$, be a sequence in X such that $\|\alpha\|_X \leqslant 1$. Define

$$(R\alpha)(t) := \sum_{n\in\mathbb{Z}} e^{2\pi i n t} \cdot \frac{e^{2\pi i\alpha_n t} - 1 - 2\pi i\alpha_n t}{2\pi i t} \cdot \Phi(t). \tag{4.11}$$

Let T_n be the nth term of the series (4.11). We observe that $T_n \in Y$. If we apply Lemma 4.2 to the function

$$\varphi(t) := \frac{e^{2\pi i t} - 1 - 2\pi i t}{2\pi i t},$$

with $s = \alpha_n$ and $m = 2$, then it follows from the lemma that condition (4.8) is satisfied with $\gamma_n := C\alpha_n^2$, where $C > 0$ does not depend on α, k or n. Hence by Lemma 4.5, the series (4.11) converges in the distributional sense to an element of the space Y, and we have

$$\|R\alpha\|_Y \leqslant C\,\|\alpha\|_X^2, \quad \alpha \in X,\ \|\alpha\|_X \leqslant 1, \tag{4.12}$$

where the constant C does not depend on α.

We note that the mapping R defined by (4.11) is *nonlinear*.

4.7. For each $r > 0$, let U_r denote the closed ball of radius r around the origin in X:

$$U_r := \{\alpha \in X : \|\alpha\|_X \leqslant r\}.$$

Lemma 4.6. *Given any $\rho > 0$, there is $0 < r < 1$ such that*

$$\|R\beta - R\alpha\|_Y \leqslant \rho\,\|\beta - \alpha\|_X, \quad \alpha, \beta \in U_r. \tag{4.13}$$

In particular, if r is small enough, then R is a contractive (nonlinear) mapping on U_r.

Proof. Let $\alpha, \beta \in U_r$ $(0 < r < 1)$. Then using (4.11) we have

$$(R\beta - R\alpha)(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n t} \cdot \frac{(e^{2\pi i\beta_n t} - 2\pi i\beta_n t) - (e^{2\pi i\alpha_n t} - 2\pi i\alpha_n t)}{2\pi i t} \cdot \Phi(t). \tag{4.14}$$

Let T_n be the nth element of the series (4.14). We apply Lemma 4.3 to the function $\varphi(t) := (e^{2\pi i t} - 2\pi i t)/(2\pi i)$ with $u = \alpha_n$, $v = \beta_n$ and $m = 2$. The lemma implies that the condition (4.8) is satisfied with $\gamma_n := Cr \cdot (\beta_n - \alpha_n)$, where the constant C does not depend on r, α, β, k or n. It therefore follows from Lemma 4.5 that we have the estimate $\|R\beta - R\alpha\|_Y \leqslant Cr\,\|\beta - \alpha\|_X$, where C is a constant not depending on r, α or β. Hence it suffices to choose r small enough so that $Cr \leqslant \rho$. ∎

4.8. For each element $T \in Y$ we denote by $\mathcal{F}(T)$ the sequence of Fourier coefficients of T, namely, the sequence $\{\widehat{T}(k)\}$, $k \in \mathbb{Z}$. This defines a linear mapping $\mathcal{F}: Y \to X$ satisfying $\|\mathcal{F}(T)\|_X = \|T\|_Y$.

Lemma 4.7. *Given any $\varepsilon > 0$, there is $\delta > 0$ with the following property. Let $S \in Y$, $\|S\|_Y \leqslant \delta$. Then one can find an element $T \in Y$, $\|T - S\|_Y \leqslant \varepsilon\|S\|_Y$, which solves the equation $T + R(\mathcal{F}(T)) = S$.*

Proof. Fix $S \in Y$ such that $\|S\|_Y \leqslant \delta$, and let

$$B = B(S, \varepsilon) := \{T \in Y : \|T - S\|_Y \leqslant \varepsilon\|S\|_Y\}.$$

We observe that if $T \in B$ then $\|T\|_Y \leqslant (1 + \varepsilon)\|S\|_Y$. Define a map $H: B \to Y$ by

$$H(T) := S - R(\mathcal{F}(T)), \quad T \in B,$$

and notice that an element $T \in B$ is a solution to the equation $T + R(\mathcal{F}(T)) = S$ if and only if T is a fixed point of the map H.

Let us show that, if δ is small enough, then $H(B) \subset B$. Indeed, if $T \in B$, then using (4.12) we have

$$\|H(T) - S\|_Y = \|R(\mathcal{F}(T))\|_Y \leqslant C\|\mathcal{F}(T)\|_X^2 = C\|T\|_Y^2 \leqslant C(1 + \varepsilon)^2\|S\|_Y^2.$$

Hence if we choose δ such that $C(1 + \varepsilon)^2\delta \leqslant \varepsilon$, then we obtain

$$\|H(T) - S\|_Y \leqslant \varepsilon\|S\|_Y,$$

and it follows that $H(B) \subset B$.

It also follows from Lemma 4.6 that if δ is small enough, then H is a contractive mapping from the closed set B into itself. Indeed, let $T_1, T_2 \in B$. Then we have

$$\begin{aligned}\|H(T_2) - H(T_1)\|_Y &= \|R(\mathcal{F}(T_2)) - R(\mathcal{F}(T_1))\|_Y \leqslant \rho\|\mathcal{F}(T_2) - \mathcal{F}(T_1)\|_X \\ &= \rho\|T_2 - T_1\|_Y,\end{aligned}$$

where $0 < \rho < 1$. Then the Banach fixed point theorem implies that H has a (unique) fixed point $T \in B$, which yields the desired solution. ∎

4.9. *Proof of Theorem* 4.1. Let S be a Schwartz distribution satisfying (4.2). Then $S \in Y$ and $\|S\|_Y \leqslant \delta$. Define $S_1(t) := S(-t)$. Then also S_1 is a distribution in Y and we have $\|S_1\|_Y = \|S\|_Y$. By Lemma 4.7, if δ is small enough, then there is an element $T \in Y$, $\|T - S_1\|_Y \leqslant \varepsilon\|S_1\|_Y$, which solves the equation $T + R(\mathcal{F}(T)) = S_1$.

Let $\alpha \in X$ be the sequence defined by $\alpha_n = \widehat{T}(n)$, $n \in \mathbb{Z}$. Then $\|\alpha\|_X \leqslant \varepsilon$ provided that δ is small enough. Let F be the function given by (4.1) that is associated to this sequence $\alpha = \{\alpha_n\}$. We have

$$\widehat{F}(-t) = \lim_{N\to\infty} \sum_{|n|\leqslant N} \widehat{F}_n(-t)$$

in the sense of distributions, and

$$\widehat{F}_n(-t) = e^{2\pi i n t} \cdot \frac{e^{2\pi i \alpha_n t} - 1}{2\pi i t}.$$

Hence

$$\widehat{F}(-t) = \lim_{N\to\infty}\Big[\sum_{|n|\leqslant N}\alpha_n e^{2\pi i n t} + \sum_{|n|\leqslant N} e^{2\pi i n t}\cdot\frac{e^{2\pi i\alpha_n t}-1-2\pi i\alpha_n t}{2\pi i t}\Big].$$

The first sum converges to T in $(-b,b)$ according to Lemma 4.4; while the second sum converges to $R\alpha$ in $(-b,b)$, due to (4.11) and the fact that $\Phi(t)=1$ on $(-b,b)$. We conclude that

$$\widehat{F}(-t) = (T+R\alpha)(t) = S_1(t) \quad \text{in } (-b,b).$$

This means that $\widehat{F} = S$ in $(-b,b)$, and thus Theorem 4.1 is proved. ∎

4.10. The theorem just proved will now be used to deduce the following one:

Theorem 4.8. *Given two numbers a and b such that $0<a<b<1/2$, and given $\varepsilon>0$, there is $\delta>0$ with the following property. Let S be a distribution on $\mathbb{R}$ satisfying*

$$S(-t) = \overline{S(t)}, \quad \operatorname{supp}(S)\subset(-b,-a)\cup(a,b), \quad \sup_{k\in\mathbb{Z}}|\widehat{S}(k)|\leqslant\delta. \tag{4.15}$$

Then there is a real sequence $\Lambda=\{\lambda_n\}$, $n\in\mathbb{Z}$, such that $|\lambda_n-n|\leqslant\varepsilon$ for all n, and

$$\widehat{\delta}_\Lambda = \delta_0 + S \quad \text{in } (-b,b).$$

Proof. We choose an infinitely smooth function Ψ such that $\Psi(-t)=\overline{\Psi(t)}$ for all $t\in\mathbb{R}$, $\Psi(t)=-1/(2\pi i t)$ in $(-b,-a)\cup(a,b)$, and $\Psi(t)=0$ for $t\in\mathbb{R}\setminus(-l,l)$.

Let S be a distribution satisfying (4.15). Then $S\in Y$ and $\|S\|_Y\leqslant\delta$. Define a new distribution $S_1 := S\cdot\Psi$; then also $S_1\in Y$. We have

$$\|S_1\|_Y \leqslant M\|S\|_Y, \quad \text{where } M := \|\Psi\|_{A(\mathbb{T})}.$$

By Theorem 4.1, if δ is small enough, then there is a sequence $\alpha\in X$, $\|\alpha\|_X\leqslant\varepsilon$, such that the function F defined by (4.1) satisfies $\widehat{F}=S_1$ in $(-b,b)$. It follows that the distributional derivative F' of the function F satisfies

$$\widehat{F}'(t) = 2\pi i t\,\widehat{F}(t) = 2\pi i t\,S_1(t) = 2\pi i t\,\Psi(t)\,S(t) = -S(t) \quad \text{in } (-b,b), \tag{4.16}$$

which is true since $2\pi i t\,\Psi(t)=-1$ in a neighborhood of $\operatorname{supp}(S)$.

Let $\Lambda=\{\lambda_n\}$, $n\in\mathbb{Z}$, be defined by $\lambda_n := n+\alpha_n$. Then we have $|\lambda_n-n|\leqslant\varepsilon$ for all n. It follows from the definition (4.1) of F that

$$F' = \sum_{n\in\mathbb{Z}}(\delta_n-\delta_{\lambda_n}) = \delta_{\mathbb{Z}}-\delta_\Lambda,$$

that is, $\delta_\Lambda = \delta_{\mathbb{Z}} - F'$. By Poisson's summation formula, $\widehat{\delta}_{\mathbb{Z}}=\delta_{\mathbb{Z}}$, hence we have

$$\widehat{\delta}_\Lambda = \delta_{\mathbb{Z}} - \widehat{F}' = \delta_0 + S \quad \text{in } (-b,b)$$

due to (4.16). The proof of Theorem 4.8 is thus concluded. ∎

4.11. Finally, we observe that Theorem 3.2 follows from Theorem 4.8. Indeed, if S is a pseudomeasure on $\mathbb{R}$ such that $\mathrm{supp}(S) \subset (a, b)$, then the distribution $r(S + \widetilde{S})$ satisfies the conditions (4.15) if $r > 0$ is sufficiently small. Hence Theorem 4.8 yields a sequence $\Lambda = \{\lambda_n\}$ with the properties as in the statement of Theorem 3.2.

5. Addendum: A problem of Kolountzakis

The following question was posed to us by Kolountzakis: does there exist a real sequence $\Lambda = \{\lambda_n\}$, $n \in \mathbb{Z}$, satisfying

$$A \leqslant \lambda_{n+1} - \lambda_n \leqslant B, \quad n \in \mathbb{Z}, \tag{5.1}$$

where $A, B > 0$ are constants, such that $f + \Lambda$ is a tiling for some nonzero $f \in L^1(\mathbb{R})$, but there is no *nonnegative* f with this property?

The answer turns out to depend on the level of the tiling. Suppose first that there is a tiling $f + \Lambda$ at some *nonzero* level w. Then f must have nonzero integral, see Lemma 2.3(i) in [10]. In turn this implies (see Section 4 in [11]) that $\widehat{\delta}_\Lambda = c \cdot \delta_0$ in some neighborhood $(-\eta, \eta)$ of the origin, where c is a nonzero, positive scalar. It follows that $f + \Lambda$ is a tiling whenever f is a Schwartz function with $\mathrm{supp}(\widehat{f}) \subset (-\eta, \eta)$. In particular, there exist tilings $f + \Lambda$ at level one with f nonnegative.

To the contrary, we will construct an example showing that the same is not true if Λ is only assumed to admit a tiling at level zero. We will prove the following result.

Theorem 5.1. *There is a real sequence $\Lambda = \{\lambda_n\}$, $n \in \mathbb{Z}$, satisfying (5.1) for which there exist tilings $f + \Lambda$ with nonzero $f \in L^1(\mathbb{R})$, but any such a tiling is necessarily a tiling at level zero. In particular, Λ cannot tile with any nonnegative (nonzero) f.*

Proof. Let a and b be two numbers such that $0 < a < b < 1/2$. Let ψ be a smooth even function, $\psi(t) > 0$ on $(-a, a)$, and $\psi(t) = 0$ outside $(-a, a)$. By Theorem 4.1, given any $\varepsilon > 0$, there is a real sequence $\alpha = \{\alpha_n\}$, $n \in \mathbb{Z}$, satisfying $|\alpha_n| \leqslant \varepsilon$ for all n, and such that $\widehat{F}(t) = r\psi(t)$ in $(-b, b)$ for some $r > 0$, where F is the function defined by (4.1).

Let the sequence $\Lambda = \{\lambda_n\}$, $n \in \mathbb{Z}$, be defined by $\lambda_n := n + \alpha_n$. Then

$$\widehat{\delta}_\Lambda = \delta_0 - 2\pi i r t \psi(t) \quad \text{in } (-b, b). \tag{5.2}$$

This can be shown in the same way as done in the proof of Theorem 4.8 above.

In particular, we have $\widehat{\delta}_\Lambda = 0$ in the open set $G := (-b, -a) \cup (a, b)$, so there exist nonzero real-valued Schwartz functions f such that $f + \Lambda$ is a tiling (at level zero). It suffices to choose f such that $\mathrm{supp}(\widehat{f})$ is contained in G.

On the other hand, suppose that there is $f \in L^1(\mathbb{R})$ such that $f + \Lambda$ is a tiling at some *nonzero* level w. Then, as before, this implies that $\widehat{\delta}_\Lambda = c \cdot \delta_0$ in some interval $(-\eta, \eta)$, where c is a nonzero (positive) scalar. This contradicts (5.2), hence no such f exists. In particular, if f is nonnegative and $f + \Lambda$ is a tiling, then the tiling level must be zero and f vanishes a.e. ∎

6. Remarks

6.1. Let $\Lambda \subset \mathbb{R}$ be a discrete set of bounded density. If the temperate distribution $\widehat{\delta}_\Lambda$ is a *measure* on $\mathbb{R}$, then condition (1.3) is not only necessary, but also sufficient, for a function $f \in L^1(\mathbb{R})$ to tile at some level w with the translation set Λ. In this case, the tiling level is given by $w = c(\Lambda)\widehat{f}(0)$, where $c(\Lambda)$ is the mass that the measure $\widehat{\delta}_\Lambda$ assigns to the origin (see Theorem 2.2 in [12]).

For example, if Λ is a *periodic set*, then $\widehat{\delta}_\Lambda$ is a (pure point) measure, and $f + \Lambda$ is a tiling if and only if (1.3) holds. It follows that the set Λ in Theorem 1.2 is not periodic, nor can it be represented as a finite union of periodic sets.

6.2. If f has fast decay, e.g., $|f(x)| = o(|x|^{-N})$ as $|x| \to +\infty$ for every N, then $\widehat{f}$ is a smooth function and again condition (1.3) is both necessary and sufficient for $f + \Lambda$ to be a tiling at some level w. It follows that the function f in Corollary 1.3 cannot be chosen to have fast decay.

6.3. Theorem 1.2 also holds in $\mathbb{R}^d$ for every $d \geqslant 1$. This can be easily deduced from the one-dimensional result by taking cartesian products. For example, in $\mathbb{R}^2$ one may take $F(x, y) = f(x)h(y)$, $G(x, y) = g(x)h(y)$, where f, g are the functions from Theorem 1.2 and where $h \in L^1(\mathbb{R})$ is such that $h + \mathbb{Z}$ is a tiling at level one. Then $\widehat{F}$ and $\widehat{G}$ have the same set of zeros, but F tiles with the translation set $\Lambda \times \mathbb{Z}$ while g does not.

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References

[1] Benedetto, J. J.: *Spectral synthesis*. Mathematische Leitfäden, B. G. Teubner, Stuttgart, 1975.

[2] Graham, C. C. and McGehee, O. C.: *Essays in commutative harmonic analysis*. Grundlehren der Mathematischen Wissenschaften 238, Springer-Verlag, New York-Berlin, 1979.

[3] Kahane, J.-P.: *Séries de Fourier absolument convergentes* (French). Ergebnisse der Mathematik und ihrer Grenzgebiete 50, Springer-Verlag, Berlin-New York, 1970.

[4] Kahane, J.-P. and Salem, R.: *Ensembles parfaits et séries trigonométriques*. Second edition (French). Hermann, Paris, 1994.

[5] Kargaev, P. P.: The Fourier transform of the characteristic function of a set vanishing on an interval (Russian). *Mat. Sb. (N.S.)* **117(159)** (1982), no. 3, 397–411. English translation in *Math. USSR-Sb.* **45** (1983), no. 3, 397–410.

[6] Kargaev, P. P. and Vol'berg, A. L.: Three results concerning the support of functions and their Fourier transforms. *Indiana Univ. Math. J.* **41** (1992), no. 4, 1143–1164.

[7] Kolountzakis, M. N.: Non-symmetric convex domains have no basis of exponentials. *Illinois J. Math.* **44** (2000), no. 3, 542–550.

[8] Kolountzakis, M. N.: Packing, tiling, orthogonality and completeness. *Bull. London Math. Soc.* **32** (2000), no. 5, 589–599.

[9] Kolountzakis, M. N.: The study of translational tiling with Fourier analysis. In *Fourier analysis and convexity*, pp. 131–187. Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2004.

[10] Kolountzakis, M. N. and Lagarias, J. C.: Structure of tilings of the line by a function. *Duke Math. J.* **82** (1996), no. 3, 653–678.

[11] Kolountzakis, M. N. and Lev, N.: On non-periodic tilings of the real line by a function. *Int. Math. Res. Not. IMRN* 2016, no. 15, 4588–4601.

[12] Kolountzakis, M. N. and Lev, N.: Tiling by translates of a function: results and open problems. *Discrete Anal.* (2021), Paper no. 12, 24 pp.

[13] Leptin, H. and Müller, D.: Uniform partitions of unity on locally compact groups. *Adv. Math.* **90** (1991), no. 1, 1–14.

[14] Liu, B.: Periodic structure of translational multi-tilings in the plane. *Amer. J. Math.* **143** (2021), no. 6, 1841–1862.

[15] Malliavin, P.: Sur l'impossibilité de la synthèse spectrale dans une algèbre de fonctions presque périodiques (French). *C. R. Acad. Sci. Paris* **248** (1959), 1756–1759.

[16] Malliavin, P.: Sur l'impossibilité de la synthèse spectrale sur la droite (French). *C. R. Acad. Sci. Paris* **248** (1959), 2155–2157.

[17] Malliavin, P.: Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts (French). *Inst. Hautes Études Sci. Publ. Math.* 1959 (1959), 85–92.

[18] Rudin, W.: *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics 12, Interscience Publishers, New York-London, 1962.

[19] Sapogov, N. A.: Fourier transform of the characteristic function of a set of finite Lebesgue measure in $\mathbb{R}^n$. *Zap. Nauchn. Sem. LOMI* **81** (1978), p. 73 (Russian). English translation in *J. Soviet Math.* **26** (1984), no. 5, p. 2135.

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