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Convex bodies and graded families of monomial ideals

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Abstract. We show that the mixed volumes of arbitrary convex bodies are equal to mixed multiplicities of graded families of monomial ideals, and to normalized limits of mixed multiplicities of monomial ideals. This result evinces the close relation between the theories of mixed volumes from convex geometry and mixed multiplicities from commutative algebra.

1. Introduction

The connection between volumes of convex bodies and algebraic-geometric invariants has long been explored by researchers and it has led to numerous applications in various fields of mathematics. To highlight some of these, we have the Bernstein–Koushnirenko–Khovanskii theorem [1, 19, 20], Huh's proof of the log-concavity of characteristic polynomials of matroids [14], and the theory of Newton–Okounkov bodies [17, 21] and its applications to limits in commutative algebra [6].

The goal of this paper is to expand on this fruitful research line by showing that the mixed volumes of arbitrary convex bodies are equal to mixed multiplicities of graded families of monomial ideals, and also equal to normalized limits of mixed multiplicities of monomial ideals (Theorem C). This is an extension of the main result in [24], where the case of lattice polytopes is treated. Our proof is based on two intermediate results of interest in their own right (Theorems A and B).

Let R be a d-dimensional standard graded polynomial ring over a field $\Bbbk$ and let $\mathfrak{m} = [R]_+$ be its graded irrelevant ideal. For homogeneous ideals $J_1, \ldots, J_r$ and for an $\mathfrak{m}$-primary homogeneous ideal I, there exist integers $e_{(d_0,\mathbf{d})}(I \mid J_1, \ldots, J_r) \geq 0$ for every $d_0 \in \mathbb{N}$, $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ with $d_0 + |\mathbf{d}| = d - 1$, called the *mixed multiplicities* of $J_1, \ldots, J_r$ with respect to I, such that

$$\lim_{m\to\infty} \frac{\dim_{\Bbbk}\left(\frac{J_1^{mn_1}\cdots J_r^{mn_r}}{I^{n_0 m} J_1^{mn_1}\cdots J_r^{mn_r}}\right)}{m^d} = \sum_{d_0+|\mathbf{d}|=d-1} \frac{e_{(d_0,\mathbf{d})}(I \mid J_1, \ldots, J_r)}{(d_0+1)!\, d_1! \cdots d_r!}\, n_0^{d_0+1} n_1^{d_1} \cdots n_r^{d_r},$$

for every $n_0, n_1, \ldots, n_r \geq 0$ (see [25] for a survey).

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On the other hand, by Minkowski's theorem, for a sequence $K_1, \ldots, K_r$ of convex bodies in $\mathbb{R}^d$, the *mixed volumes* $\mathrm{MV}_d(K_{\rho_1}, \ldots, K_{\rho_d})$ of sequences $(K_{\rho_1}, \ldots, K_{\rho_d})$ of convex bodies with $1 \le \rho_1, \ldots, \rho_d \le r$ satisfy the following equation:

$$\mathrm{Vol}_d(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{\substack{\mathbf{d}=(d_1,\ldots,d_r)\in\mathbb{N}^r \\ d_1+\cdots+d_r=d}} \frac{1}{d_1!\cdots d_r!} \mathrm{MV}_d(K_1^{d_1}, \ldots, K_r^{d_r})\, \lambda_1^{d_1} \cdots \lambda_r^{d_r}$$

for every $\lambda_1, \ldots, \lambda_r \ge 0$ (see Theorem 3.3 on p. 116 of [10]); here $K_i^{d_i}$ denotes d_i copies of K_i. The relation between these two sets of invariants is established in [24], where the authors show that mixed volumes of lattice polytopes coincide with the mixed multiplicities of certain monomial ideals. However, this relation does not extend to arbitrary convex bodies as the associated sequences of ideals are no longer powers of ideals but rather (not necessarily Noetherian) graded families of ideals.

A sequence of ideals $\mathbb{I} = \{I_n\}_{n\in\mathbb{N}}$ is a *graded family* if $I_0 = R$ and $I_i I_j \subseteq I_{i+j}$ for every $i, j \in \mathbb{N}$. The family is *Noetherian* if the corresponding Rees algebra $\mathcal{R}(\mathbb{I}) = \oplus_{n\in\mathbb{N}} I_n t^n \subseteq R[t]$ is Noetherian. The study of mixed multiplicities of graded families was pioneered by Cutkosky–Sarkar–Srinivasan [8] for the case of $\mathfrak{m}$*-primary filtrations* (in more general rings), that is, when each I_n is $\mathfrak{m}$-primary and $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{N}$. Their strategy is to first show the existence of these multiplicities for Noetherian filtrations, and then pass to arbitrary filtrations using the theory of Newton–Okounkov bodies [17] (see also [6] and [21]). In our first result, we prove the existence of mixed multiplicities for arbitrary graded families of monomial ideals under a mild assumption. Our approach differs from the one of [8] in that we exploit Minkowski's theorem to show the existence of the polynomial leading to the definition of mixed multiplicities.

In order to present our first result, we need to introduce some prior notation. Let $\mathbb{I} = \{I_n\}_{n\in\mathbb{N}}$ be a (not necessarily Noetherian) graded family of $\mathfrak{m}$-primary monomial ideals and let $\mathbb{J}(1) = \{J(1)_n\}_{n\in\mathbb{N}}, \ldots, \mathbb{J}(r) = \{J(r)_n\}_{n\in\mathbb{N}}$ be (not necessarily Noetherian) graded families of monomial ideals in R. We further assume that the degrees of the generators of $J(i)_n$ are bounded by a linear function on n for each $1 \le i \le r$. We note that the latter condition is similar to others that have been considered in previous works regarding limits of graded families of ideals (see, e.g., Theorem 6.1 in [6]).

Theorem A (Theorem 3.13, Lemma 3.14). *Under the notations and assumptions above, the function*

$$F(n_0, n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}\left(J(1)_{mn_1}\cdots J(r)_{mn_r} \,/\, I_{mn_0} J(1)_{mn_1}\cdots J(r)_{mn_r}\right)}{m^d}$$

is equal to a homogeneous polynomial $G(n_0, \mathbf{n}) = G(n_0, n_1, \ldots, n_r)$ of total degree d with non-negative real coefficients for all $n_0 \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. Additionally, $G(n_0, \mathbf{n})$ has no term of the form $\alpha \mathbf{n}^{\mathbf{d}} = \alpha n_1^{d_1} \cdots n_r^{d_r}$ with $0 \ne \alpha \in \mathbb{R}$, $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ and $|\mathbf{d}| = d$.

Furthermore, the coefficients of the polynomial $G(n_0, \mathbf{n})$ can be explicitly described in terms of mixed volumes of certain Newton–Okounkov bodies.

We note that Theorem A is new even when the graded families are all $\mathfrak{m}$-primary. In this case, our theorem is an extension of that of [8] for monomial ideals (also, see [18]). For this reason, we isolate the $\mathfrak{m}$-primary case in Theorem 3.5.

The polynomial $G(n_0, \mathbf{n})$ from Theorem A can be written as

$$G(n_0, \mathbf{n}) = \sum_{\substack{(d_0,\mathbf{d})\in\mathbb{N}^{r+1} \\ d_0+|\mathbf{d}|=d-1}} \frac{1}{(d_0+1)!\mathbf{d}!} e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1),\ldots,\mathbb{J}(r))\, n_0^{d_0+1}\mathbf{n}^{\mathbf{d}};$$

here, if $\mathbf{d} = (d_1,\ldots,d_r)$, then $\mathbf{d}! = d_1!\cdots d_r!$. For each $(d_0,\mathbf{d}) \in \mathbb{N}^{r+1}$ with $d_0 + |\mathbf{d}| = d-1$, we define the real numbers $e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1),\ldots,\mathbb{J}(r)) \geq 0$ to be the *mixed multiplicities* of $\mathbb{J}(1),\ldots,\mathbb{J}(r)$ with respect to $\mathbb{I}$ (see Definition 3.15).

The *volume* and *multiplicity* of a graded family $\mathbb{B} = \{B_n\}_{n\in\mathbb{N}}$ of zero dimensional ideals in a Noetherian local ring S of dimension s are defined, respectively, as

$$\mathrm{vol}_S(\mathbb{B}) = \limsup_{n\to\infty} \frac{\lambda(S/B_n)}{n^s/s!} \quad \text{and} \quad e_S(\mathbb{B}) = \lim_{p\to\infty} \frac{e_S(B_p)}{p^s},$$

where $\lambda(N)$ denotes length of an S-module N and $e_S(J)$ denotes the Hilbert–Samuel multiplicity of an ideal J. Several works prove the equality of these two invariants under certain assumptions (see [5, 6, 9, 21, 22]). The general version of the so called *volume = multiplicity formula* is due to Cutkosky, and it is shown on any S for which the limit in the definition of volume exists [7]. In our next result we show the existence of a “volume = multiplicity formula” for mixed multiplicities of graded families of monomial ideals.

Theorem B (Theorem 4.7). *With the above assumptions and notations, for each $d_0 \in \mathbb{N}$ and $\mathbf{d} = (d_1,\ldots,d_r) \in \mathbb{N}^r$ with $d_0 + |\mathbf{d}| = d-1$, we have the equality*

$$e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1),\ldots,\mathbb{J}(r)) = \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(I_p|J(1)_p,\ldots,J(r)_p)}{p^d}.$$

With the previous results in hand, we are ready to present the main result of this paper. Here we express mixed volumes of arbitrary convex bodies as mixed multiplicities of graded families of monomial ideals and as normalized limits of mixed-multiplicities of ideals.

In a subsequent work [4], we extended the results of Theorems A and B to more general settings. We showed the existence of mixed multiplicities and a “volume = multiplicity formula” for arbitrary graded families of ideals on Noetherian local rings under mild assumptions.

We now fix the following slightly different notation. Let $(K_1,\ldots,K_r)$ be a sequence of convex bodies in $\mathbb{R}^d_{\geq 0}$ and let $K_0 \subset \mathbb{R}^d$ be the convex hull of the points $\mathbf{0}, \mathbf{e}_1,\ldots,\mathbf{e}_d \in \mathbb{R}^d$, where $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^d$ and $\mathbf{e}_i = (0,\ldots,1,\ldots,0)$ denotes the i-th elementary basis vector for $1 \leq i \leq d$. Denote by $\mathbf{K}$ the sequence of convex bodies $\mathbf{K} = (K_0, K_1,\ldots,K_r)$. For each $(d_0,\mathbf{d}) = (d_0,\ldots,d_r) \in \mathbb{N}^{r+1}$, let $\mathbf{K}_{(d_0,\mathbf{d})}$ be the multiset $\mathbf{K}_{(d_0,\mathbf{d})} = \bigcup_{i=0}^{r}\bigcup_{j=1}^{d_i}\{K_i\}$ of d_i copies of K_i for each $0 \leq i \leq r$. Here let R be a $(d+1)$-dimensional standard graded polynomial ring over a field $\mathbb{k}$, and let $\mathfrak{m} = [R]_+$ be its graded irrelevant ideal. We let $\mathbb{M}$ be the graded family $\mathbb{M} = \{\mathfrak{m}^n\}_{n\in\mathbb{N}}$.

Theorem C (Theorem 5.5). *Under the notations and the assumptions above, there exist graded families of monomial ideals* $\mathbb{J}(1)\ldots,\mathbb{J}(r)$ *in* R *such that, for each* $(d_0,\mathbf{d})\in\mathbb{N}^{r+1}$ *with* $d_0+|\mathbf{d}|=d$*, we have the equalities*

$$\begin{aligned}\mathrm{MV}_d(\mathbf{K}_{(d_0,\mathbf{d})}) = e_{(d_0,\mathbf{d})}(\mathbb{M}\mid \mathbb{J}(1),\ldots,\mathbb{J}(r)) &= \lim_{p\to\infty}\frac{e_{(d_0,\mathbf{d})}(\mathfrak{m}^p\mid J(1)_p,\ldots,J(r)_p)}{p^{d+1}}\\ &= \lim_{p\to\infty}\frac{e_{(d_0,\mathbf{d})}(\mathfrak{m}\mid J(1)_p,\ldots,J(r)_p)}{p^{|\mathbf{d}|}}.\end{aligned}$$

In particular, when $r=d$*, we obtain the equalities*

$$\begin{aligned}\mathrm{MV}_d(K_1,\ldots,K_d) &= e_{(0,1,\ldots,1)}(\mathbb{M}\mid \mathbb{J}(1),\ldots,\mathbb{J}(d))\\ &= \lim_{p\to\infty}\frac{e_{(0,1,\ldots,1)}(\mathfrak{m}^p\mid J(1)_p,\ldots,J(d)_p)}{p^{d+1}}\\ &= \lim_{p\to\infty}\frac{e_{(0,1,\ldots,1)}(\mathfrak{m}\mid J(1)_p,\ldots,J(d)_p)}{p^{d}}.\end{aligned}$$

Finally, we briefly describe the content of the paper. In Section 2 we set up the notation and include some preliminary results that are used in the rest of the paper. In Section 3 we include the proof of Theorem A, and in Section 4 the one of Theorem B. Lastly, Section 5 includes the proof of our main result Theorem C.

2. Notation and preliminaries

In this section, we set up the notation that is used throughout the article. We also include some preliminary information needed for our results.

For a vector $\mathbf{n}=(n_1,\ldots,n_r)\in\mathbb{N}^r$, we denote by $|\mathbf{n}|$ the sum of its entries. We also denote by $\mathbf{0}$ the vector $(0,\ldots,0)\in\mathbb{N}^r$. For $\mathbf{n}=(n_1\ldots,n_r)$ and $\mathbf{m}=(m_1,\ldots,m_r)$ in $\mathbb{N}^r$, we write $\mathbf{n}\geq\mathbf{m}$ if $n_i\geq m_i$ for every $1\leq i\leq r$. Moreover, we write $\mathbf{n}\gg\mathbf{0}$ if $n_i\gg 0$ for every $1\leq i\leq r$. We also use the abbreviations $\mathbf{n}^{\mathbf{m}}=n_1^{m_1}\cdots n_r^{m_r}$ and $\mathbf{n}!=n_1!\cdots n_r!$.

Below we recall the definitions of graded families of ideals and filtrations of ideals.

Definition 2.1. A *graded family* of ideals $\{I_i\}_{i\in\mathbb{N}}$ in a ring R is a family of ideals indexed by the natural numbers such that $I_0=R$ and $I_iI_j\subset I_{i+j}$ for all $i,j\in\mathbb{N}$.

- (i) If $(R,\mathfrak{m})$ is a local ring (or a positively graded ring with $\mathfrak{m}=[R]_+$) and I_i is $\mathfrak{m}$-primary for $i>0$, then we say that $\{I_i\}_{i\in\mathbb{N}}$ is a *graded family of* $\mathfrak{m}$*-primary ideals*.
- (ii) If we have the inclusion $I_i\supseteq I_{i+1}$ for all $i\in\mathbb{N}$, then we say that $\{I_i\}_{i\in\mathbb{N}}$ is a *filtration of ideals* in R.
- (iii) We say that $\{I_i\}_{i\in\mathbb{N}}$ is *Noetherian* when the corresponding Rees algebra $\bigoplus_{i\in\mathbb{N}}I_it^i\subset R[t]$ is Noetherian.
- (iv) When $R=\mathbb{k}[x_1,\ldots,x_d]$ is a standard graded polynomial ring over a field $\mathbb{k}$ and each I_i is a monomial ideal, we say that $\{I_i\}_{i\in\mathbb{N}}$ is a *graded family of monomial ideals*.

2.1. Mixed volumes of convex bodies

Let $\mathbf{K} = (K_1, \dots, K_r)$ be a sequence of convex bodies in $\mathbb{R}^d$. For any sequence $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$ of non-negative integers, we denote by $\lambda\mathbf{K}$ the Minkowski sum $\lambda\mathbf{K} := \lambda_1 K_1 + \cdots + \lambda_r K_r$, and by $\mathbf{K}_\lambda$ the multiset $\mathbf{K}_\lambda := \bigcup_{i=1}^r \bigcup_{j=1}^{\lambda_i} \{K_i\}$ of λ_i copies of K_i for each $1 \le i \le r$.

For any convex body $K \subset \mathbb{R}^d$, we denote by $\mathrm{Vol}_d(K)$ the d-dimensional volume. The following important and classical theorem says that the volume $\mathrm{Vol}_d(\lambda\mathbf{K})$ of the convex body $\lambda\mathbf{K}$ is a polynomial of degree d in λ (see Theorem 3.2 on p. 116 of [10]). For more details regarding the topic of mixed volumes the reader is referred to [10], Chapter IV.

Theorem 2.2 (Minkowski). *$\mathrm{Vol}_d(\lambda\mathbf{K})$ is a homogeneous polynomial of degree d that satisfies*

$$\mathrm{Vol}_d(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{\rho_1=1,\dots,\rho_d=1}^{r} V(K_{\rho_1}, \dots, K_{\rho_d})\, \lambda_{\rho_1}, \dots, \lambda_{\rho_d},$$

for certain coefficients $V(K_{\rho_1}, \dots, K_{\rho_d})$, where the summation is carried out independently over the ρ_i for $1 \le i \le d$.

Theorem 2.2 leads to the following definition (see Theorem 3.3 on p. 116 of [10]).

Definition 2.3. The *mixed volume* of d convex bodies $K_1, \dots, K_d \subset \mathbb{R}^d$ is defined by

$$\mathrm{MV}_d(K_1, \dots, K_d) := d!\, V(K_1, \dots, K_d).$$

Note that under the current notations, we have the following equation:

$$\mathrm{Vol}_d(\lambda\mathbf{K}) = \sum_{\substack{\mathbf{d}\in\mathbb{N}^r \\ |\mathbf{d}|=d}} \frac{1}{\mathbf{d}!} \mathrm{MV}_d(\mathbf{K}_\mathbf{d})\, \lambda^{\mathbf{d}}. \tag{2.1}$$

2.2. Semigroups, Newton–Okounkov bodies, and limits of lengths

In this subsection, we describe the notions and methods of Newton–Okounkov bodies and recall some important results from [17].

Here we use a slightly simpler setting. Suppose that $S \subset \mathbb{Z}^{d+1}$ is a semigroup in $\mathbb{Z}^{d+1}$. Fix a linear map $\pi\colon \mathbb{R}^{d+1} \to \mathbb{R}$ with integral coefficients, that is, $\pi(\mathbb{Z}^{d+1}) \subset \mathbb{Z}$.

Let $L = L(S)$ be the linear subspace of R^{d+1} which is generated by S. Let $M = M(S)$ be the rational half-space $M(S) := L(S) \cap \pi^{-1}(\mathbb{R}_{\ge 0})$, and let $\partial M_\mathbb{Z} = \partial M \cap \mathbb{Z}^{d+1}$. Let $\mathrm{Con}(S) \subset L(S)$ be the closed convex cone which is the closure of the set of all linear combinations $\sum_i \lambda_i s_i$ with $s_i \in S$ and $\lambda_i \ge 0$. Let $G(S) \subset L(S)$ be the group generated by S.

We say that the pair (S, M) is *admissible* if $S \subset M$; additionally, if $\mathrm{Con}(S)$ is strictly convex and intersects the space ∂M only at the origin, then (S, M) is called a *strongly admissible* pair (see Definition 1.9 in [17]).

Following [17], when (S, M) is an admissible pair we fix the following notation:

- $[S]_k := S \cap \pi^{-1}(k)$.
- $m = \mathrm{ind}(S, M) := [\mathbb{Z} : \pi(G(S))]$.

- $\operatorname{ind}(S, \partial M) := [\partial M_{\mathbb{Z}} : G(S) \cap \partial M]$.
- $\Delta(S, M) := \operatorname{Con}(S) \cap \pi^{-1}(m)$ (the Newton–Okounkov body of (S, M)).
- $q = \dim(\partial M)$.
- $\operatorname{Vol}_q(\Delta(S, M))$ is the *integral volume* of $\Delta(M, S)$ (see Definition 1.13 in [17]); this volume is computed using the translation of the *integral measure* on ∂M.

The following result is of fundamental importance in our approach.

Theorem 2.4 (Kaveh–Khovanskii, Corollary 1.16 in [17]). *Suppose that the pair (S, M) is strongly admissible. Then*

$$\lim_{k\to\infty} \frac{\#[S]_{km}}{k^q} = \frac{\operatorname{Vol}_q(\Delta(S, M))}{\operatorname{ind}(S, \partial M)}.$$

Remark 2.5. Whenever the rational half-space M is implicit from the context, we write $\Delta(S)$ instead of $\Delta(S, M)$.

3. Mixed multiplicities of graded families of monomial ideals

Throughout the present section we use the data below.

Setup 3.1. Let $\Bbbk$ be a field, let $R = \Bbbk[x_1, \ldots, x_d]$ be the standard graded polynomial ring, and let $\mathfrak{m} \subset R$ be the graded irrelevant ideal $\mathfrak{m} = (x_1, \ldots, x_d)$. Following the notation in Section 2.2, we fix the linear map $\pi\colon \mathbb{R}^{d+1} \to \mathbb{R}$ given by the projection $(\alpha_1, \ldots, \alpha_d, \alpha_{d+1}) \in \mathbb{R}^{d+1} \mapsto \alpha_{d+1} \in R$. Let $\pi_1\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection given by $(\alpha_1, \ldots, \alpha_d, \alpha_{d+1}) \in \mathbb{R}^{d+1} \mapsto (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Let M be the rational half-space $M = \pi^{-1}(\mathbb{R}_{\geq 0}) = \mathbb{R}^d \times \mathbb{R}_{\geq 0}$.

For a semigroup $S \subset \mathbb{N}^{d+1}$ and $m \in \mathbb{N}$, we denote by $[S]_m$ the level set

$$[S]_m = S \cap \pi^{-1}(m) = S \cap (\mathbb{N}^d \times \{m\}).$$

3.1. Mixed multiplicities of $\mathfrak{m}$-primary graded families of monomial ideals

In this subsection, we prove the existence of mixed multiplicities of graded families of monomial $\mathfrak{m}$-primary ideals in a polynomial ring. This extends the main result from [8] in the setting of monomial ideals. Here our proof depends directly on the Minkowski theorem (Theorem 2.2).

We begin by introducing the following setup.

Setup 3.2. Adopt Setup 3.1. Let $\mathbb{J}(1) = \{J(1)_n\}_{n\in\mathbb{N}}, \ldots, \mathbb{J}(r) = \{J(r)_n\}_{n\in\mathbb{N}}$ be (not necessarily Noetherian) graded families of $\mathfrak{m}$-primary monomial ideals in R. Let $c \in \mathbb{N}$ be a positive integer such that

$$J(i)_1 \supset \mathfrak{m}^{c-1} \quad \text{for all } 1 \leq i \leq r. \tag{3.1}$$

and $c \geq 2$. Thus, it follows that

$$J(i)_n \supset \mathfrak{m}^{cn} \quad \text{for all } 1 \leq i \leq r \text{ and } n \in \mathbb{N}. \tag{3.2}$$

For a vector $\mathbf{n} = (n_1, \ldots, n_r)$ in $\mathbb{N}^r$, we shall abbreviate $\mathbf{J_n} = J(1)_{n_1} \cdots J(r)_{n_r}$. We identify each monomial $\mathbf{x^m} = x_1^{m_1} \cdots x_d^{m_d} \in R$ with the corresponding vector $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. We now connect our setting with the information in Subsection 2.2. Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ be an r-tuple of non-negative integers. Thus, for each $m \geq 1$, equation (3.2) yields the following:

$$\dim_{\mathbb{k}}(R/\mathbf{J}_{m\mathbf{n}}) = \dim_{\mathbb{k}}(R/\mathfrak{m}^{cm|\mathbf{n}|+1}) - \dim_{\mathbb{k}}(\mathbf{J}_{m\mathbf{n}}/\mathfrak{m}^{cm|\mathbf{n}|+1}). \tag{3.3}$$

Motivated by the last term in (3.3), we define the following set:

$$\Gamma_{\mathbf{n}} := \left\{(\mathbf{m}, m) = (m_1, \ldots, m_d, m) \in \mathbb{N}^{d+1} \mid \mathbf{x^m} \in \mathbf{J}_{m\mathbf{n}} \text{ and } |\mathbf{m}| \leq cm|\mathbf{n}|\right\}.$$

The next lemma provides some basic properties of $\Gamma_{\mathbf{n}}$.

Lemma 3.3. *The following statements hold*:

(i) $\Gamma_{\mathbf{n}}$ *is a subsemigroup of the semigroup* $\mathbb{N}^{d+1}$.

(ii) $G(\Gamma_{\mathbf{n}}) = \mathbb{Z}^{d+1}$, *and so* $L(\Gamma_{\mathbf{n}}) = \mathbb{R}^{d+1}$.

(iii) *The pair* $(\Gamma_{\mathbf{n}}, M)$ *is strongly admissible, with* $\dim(\partial M) = d$ *and* $\operatorname{ind}(\Gamma_{\mathbf{n}}, M) = \operatorname{ind}(\Gamma_{\mathbf{n}}, \partial M) = 1$.

(iv) *For any* $n \in \mathbb{N}$ *and* $1 \leq i \leq r$, *we have*

$$\Delta\left(\Gamma_{n\mathbf{e}_i}\right) = \left(n\pi_1\left(\Delta\left(\Gamma_{\mathbf{e}_i}\right)\right), 1\right).$$

Proof. (i) Suppose that $(\mathbf{m}, m)$, $(\mathbf{m}', m') \in \Gamma_{\mathbf{n}}$, that is, $\mathbf{x^m} \in \mathbf{J}_{m\mathbf{n}}$, $\mathbf{x}^{\mathbf{m}'} \in \mathbf{J}_{m'\mathbf{n}}$, $\mathbf{m} \leq cm|\mathbf{n}|$ and $\mathbf{m}' \leq cm'|\mathbf{n}|$. As $\mathbb{J}(1), \ldots, \mathbb{J}(r)$ are graded families of ideals, it follows that $\mathbf{x}^{\mathbf{m}+\mathbf{m}'} \in \mathbf{J}_{(m+m')\mathbf{n}}$. Thus, the inequality $|\mathbf{m}+\mathbf{m}'| = |\mathbf{m}| + |\mathbf{m}'| \leq cm|\mathbf{n}| + cm'|\mathbf{n}| = c(m+m')|\mathbf{n}|$ yields the result.

(ii) By (3.1), we can choose $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ such that $\mathbf{x^m} \in \mathbf{J_n}$ and $|\mathbf{m}| = c|\mathbf{n}| - 1$. Since $\mathbf{x}^{\mathbf{m}+\mathbf{e}_i} \in \mathbf{J_n}$ and $|\mathbf{m}+\mathbf{e}_i| = c|\mathbf{n}|$ for all $1 \leq i \leq d$, it follows that $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \in G(\Gamma_{\mathbf{n}})$ (here $\mathbf{e}_i$ denotes the i-th elementary basis vector in $\mathbb{N}^{d+1}$). The equation

$$\mathbf{e}_{d+1} = (\mathbf{m}, 1) - m_1\mathbf{e}_1 - \cdots - m_d\mathbf{e}_d$$

implies that $\mathbf{e}_{d+1} \in G(\Gamma_{\mathbf{n}})$, and so the result follows.

(iii) The fact that $(\Gamma_{\mathbf{n}}, M)$ is strongly admissible follows from the way that $\Gamma_{\mathbf{n}}$ was defined. The other claims are obtained directly from part (ii).

(iv) By definition, for all $m \geq 0$ we have $\pi_1\left([\Gamma_{n\mathbf{e}_i}]_m\right) = \pi_1\left([\Gamma_{\mathbf{e}_i}]_{nm}\right)$. Hence, one obtains

$$\pi_1(\Delta(\Gamma_{n\mathbf{e}_i})) = \pi_1(\operatorname{Con}(\Gamma_{n\mathbf{e}_i}) \cap \pi^{-1}(1)) = \pi_1(\operatorname{Con}(\Gamma_{\mathbf{e}_i}) \cap \pi^{-1}(n)) = n\pi_1(\Delta(\Gamma_{\mathbf{e}_i})),$$

and so the result follows. ■

Let $A = \bigcup_{m \geq 0}(A_m, m)$ and $B = \bigcup_{m \geq 0}(B_m, m)$ be two subsets of $\mathbb{N}^{d+1} = \mathbb{N}^d \times \mathbb{N}$. Following §1.6 in [17], we define the *levelwise addition* of A and B as the set $A \oplus_t B \subseteq \mathbb{N}^{d+1}$ such that $(A \oplus_t B)_m = A_m + B_m$ for every $m \in \mathbb{N}$. The following proposition decomposes $\Gamma_{\mathbf{n}}$ as a levelwise sum of simpler semigroups. This basic result can be seen as the main step in our proof.

Proposition 3.4. *Assume Setup 3.2. We have the equality* $\Gamma_{\mathbf{n}} = \Gamma_{n_1\mathbf{e}_1} \oplus_t \cdots \oplus_t \Gamma_{n_r\mathbf{e}_r}$.

Proof. The result is obtained from Proposition 3.11 (ii) and the fact that $\beta(J(i)_n) \le cn$ for all $1 \le i \le r$ and $n \in \mathbb{N}$ (see the assumptions and notations in Setup 3.8). ■

From (3.3), and the fact that $\mathbb{J}(1), \ldots, \mathbb{J}(r)$ are graded families of monomial ideals, we obtain that

$$\begin{aligned} \dim_{\mathbb{k}}(R/\mathbf{J}_{m\mathbf{n}}) &= \dim_{\mathbb{k}}(R/\mathfrak{m}^{cm|\mathbf{n}|+1}) - \dim_{\mathbb{k}}(\mathbf{J}_{m\mathbf{n}}/\mathfrak{m}^{cm|\mathbf{n}|+1}) \\ &= \binom{cm|\mathbf{n}|+d}{d} - \#[\Gamma_{\mathbf{n}}]_m. \end{aligned} \tag{3.4}$$

After the previous preparatory results, we are ready for the main result of this subsection. The following theorem shows the existence of a homogeneous polynomial that can be used to define the mixed multiplicities of the graded families $\mathbb{J}(1), \ldots, \mathbb{J}(r)$. As a consequence of the proof, we describe the coefficients of the polynomial explicitly in terms of the mixed volumes of certain Newton–Okounkov bodies.

Theorem 3.5. *Assume Setup 3.2. The function*

$$F(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}\left(R/J(1)_{mn_1}\cdots J(r)_{mn_r}\right)}{m^d}$$

is equal to a homogeneous polynomial $G(\mathbf{n}) = G(n_1, \ldots, n_r)$ *of total degree* d *with real coefficients for all* $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$*. Explicitly, the polynomial* $G(\mathbf{n})$ *is given by*

$$G(\mathbf{n}) = \sum_{|\mathbf{d}|=d} \frac{1}{\mathbf{d}!}\left(c^d - \mathrm{MV}_d(\Delta(\Gamma)_{\mathbf{d}})\right)\mathbf{n}^{\mathbf{d}},$$

where $\Delta(\Gamma)$ *denotes the sequence* $\Delta(\Gamma) = (\Delta(\Gamma_{\mathbf{e}_1}), \ldots, \Delta(\Gamma_{\mathbf{e}_r}))$ *of Newton–Okounkov bodies.*

Proof. Let $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. By using Theorem 2.4, Lemma 3.3 (iii) and (3.4), we obtain the equation

$$F(\mathbf{n}) = \lim_{m\to\infty} \frac{\binom{cm|\mathbf{n}|+d}{d}}{m^d} - \lim_{m\to\infty}\frac{\#[\Gamma_{\mathbf{n}}]_m}{m^d} = \frac{c^d|\mathbf{n}|^d}{d!} - \mathrm{Vol}_d(\Delta(\Gamma_{\mathbf{n}})). \tag{3.5}$$

Due to Proposition 3.4, Lemma 3.3 (iv) and Proposition 1.32 in [17], we get the equality

$$\begin{aligned} \pi_1(\Delta(\Gamma_{\mathbf{n}})) &= \pi_1(\Delta(\Gamma_{n_1\mathbf{e}_1})) + \cdots + \pi_1(\Delta(\Gamma_{n_r\mathbf{e}_r})) \\ &= n_1\pi_1(\Delta(\Gamma_{\mathbf{e}_1})) + \cdots + n_r\pi_1(\Delta(\Gamma_{\mathbf{e}_r})). \end{aligned}$$

Thus, (2.1) implies that

$$\begin{aligned} \mathrm{Vol}_d(\Delta(\Gamma_{\mathbf{n}})) &= \mathrm{Vol}_d(\pi_1(\Delta(\Gamma_{\mathbf{n}}))) \\ &= \sum_{|\mathbf{d}|=d}\frac{1}{\mathbf{d}!}\mathrm{MV}_d(\pi_1(\Delta(\Gamma))_{\mathbf{d}})\,\mathbf{n}^{\mathbf{d}} = \sum_{|\mathbf{d}|=d}\frac{1}{\mathbf{d}!}\mathrm{MV}_d(\Delta(\Gamma)_{\mathbf{d}})\,\mathbf{n}^{\mathbf{d}}, \end{aligned} \tag{3.6}$$

where $\pi_1(\Delta(\Gamma))$ denotes the sequence $\pi_1(\Delta(\Gamma)) = (\pi_1(\Delta(\Gamma_{\mathbf{e}_1})), \ldots, \pi_1(\Delta(\Gamma_{\mathbf{e}_r})))$ of convex bodies. Finally, the result follows by combining (3.5) and (3.6). ■

Proposition 3.6. *Assume Setup 3.2 and use the same notation of Theorem 3.5. Let* $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ *with* $|\mathbf{d}| = d$*. Then, one has that* $c^d - \mathrm{MV}_d(\Delta(\Gamma)_{\mathbf{d}}) \geq 0$.

Proof. Let $\Sigma \subset \mathbb{R}^d$ be the polytope given as the convex hull of the points $\mathbf{0}, c\,\mathbf{e}_1, \ldots, c\,\mathbf{e}_d \subset \mathbb{R}^d$. Consider the polytope $\Delta = \Sigma \times \{1\} \subset \mathbb{R}^{d+1}$, and notice that by construction we have $\Delta \supset \Delta(\Gamma_{\mathbf{e}_i})$ for all $1 \leq i \leq r$. Since $\mathrm{Vol}_d(\Delta) = c^d/d!$ and $\mathrm{MV}_d(\Delta, \ldots, \Delta) = d!\,\mathrm{Vol}_d(\Delta)$, the inequality

$$c^d - \mathrm{MV}_d(\Delta(\Gamma)_{(d_1,\ldots,d_r)}) = \mathrm{MV}_d(\Delta, \ldots, \Delta) - \mathrm{MV}_d(\Delta(\Gamma)_{(d_1,\ldots,d_r)}) \geq 0$$

follows from the monotonicity of mixed volumes (see, e.g., equation 5.25 in [23]). ∎

With Theorem 3.5 in hand, we are ready to define the mixed multiplicities of graded families of $\mathfrak{m}$-primary monomial ideals. Due to Proposition 3.6, the mixed multiplicities defined below are always non-negative.

Definition 3.7. Assume Setup 3.2 and let $G(\mathbf{n})$ be as in Theorem 3.5. Write

$$G(\mathbf{n}) = \sum_{|\mathbf{d}|=d} \frac{1}{\mathbf{d}!}\, e_{\mathbf{d}}(\mathbb{J}(1), \ldots, \mathbb{J}(r))\, \mathbf{n}^{\mathbf{d}}.$$

For each $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ with $|\mathbf{d}| = d$, we define the non-negative real number

$$e_{\mathbf{d}}(\mathbb{J}(1), \ldots, \mathbb{J}(r)) \geq 0$$

to be the *mixed multiplicities of type* $\mathbf{d}$ *of* $\mathbb{J}(1) = \{J(1)_n\}_{n\in\mathbb{N}}, \ldots, \mathbb{J}(r) = \{J(r)_n\}_{n\in\mathbb{N}}$.

3.2. Mixed multiplicities of arbitrary graded families of monomial ideals

In this subsection, we introduce the notion of mixed multiplicities for arbitrary graded families of monomial ideals under mild conditions. We begin with the following setup that is used in our results.

Setup 3.8. Adopt Setup 3.1. Let $\mathbb{I} = \{I_n\}_{n\in\mathbb{N}}$ be a (not necessarily Noetherian) graded family of $\mathfrak{m}$-primary monomial ideals. Let $\mathbb{J}(1) = \{J(1)_n\}_{n\in\mathbb{N}}, \ldots, \mathbb{J}(r) = \{J(r)_n\}_{n\in\mathbb{N}}$ be (not necessarily Noetherian) graded families of monomial ideals in R.

For a homogeneous ideal J, we denote $\beta(J) = \max\{j \mid [J \otimes_R \Bbbk]_j \neq 0\}$, that is, the maximum degree of a minimal set of homogeneous generators of J. We assume that there exists $\beta \in \mathbb{N}$ satisfying

$$\beta(J(i)_n) \leq \beta n$$

for all $1 \leq i \leq r$ and $n \in \mathbb{N}$; similar assumptions have been considered in previous works regarding limits of graded families of ideals (see [6], Theorem 6.1). Let $c' \in \mathbb{N}$ be a positive integer such that $I_1 \supset \mathfrak{m}^{c'}$; in particular, $I_{n_0} \supset \mathfrak{m}^{n_0 c'}$. We set $c = \max\{\beta + 1, c'\}$.

We have the following simple observation that plays an important role in our approach.

Lemma 3.9. *We have that*

$$\mathfrak{m}^{c(n_0+|\mathbf{n}|)} \cap \mathbf{J_n} = \mathfrak{m}^{c(n_0+|\mathbf{n}|)} \cap I_{n_0}\mathbf{J_n}$$

for all $n_0 \in \mathbb{N}$ *and* $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$.

Proof. Since $\beta(\mathbf{J}_\mathbf{n}) \le \beta|\mathbf{n}|$, we obtain the following inclusion:

$$\mathfrak{m}^{c(n_0+|\mathbf{n}|)} \cap I_{n_0}\mathbf{J}_\mathbf{n} \supset \mathfrak{m}^{c(n_0+|\mathbf{n}|)} \cap \mathfrak{m}^{cn_0}\mathbf{J}_\mathbf{n} = \mathfrak{m}^{c(n_0+|\mathbf{n}|)} \cap \mathbf{J}_\mathbf{n}.$$

The result follows. ∎

Let $n_0 \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. We define the sets

$$\Gamma_{n_0,\mathbf{n}} := \left\{(\mathbf{m}, m) = (m_1, \ldots, m_d, m) \in \mathbb{N}^{d+1} \mid \mathbf{x}^\mathbf{m} \in \mathbf{J}_{m\mathbf{n}} \text{ and } |\mathbf{m}| \le cm(n_0 + |\mathbf{n}|)\right\} \tag{3.7}$$

and

$$\widehat{\Gamma}_{n_0,\mathbf{n}} := \left\{(\mathbf{m}, m) = (m_1, \ldots, m_d, m) \in \mathbb{N}^{d+1} \mid \mathbf{x}^\mathbf{m} \in I_{mn_0}\mathbf{J}_{m\mathbf{n}} \text{ and } |\mathbf{m}| \le cm(n_0 + |\mathbf{n}|)\right\}. \tag{3.8}$$

The lemma below is equivalent to Lemma 3.3, and its proof follows verbatim.

Lemma 3.10. *Let $S \subset \mathbb{N}^{d+1}$ be equal to either $\Gamma_{n_0,\mathbf{n}}$ or $\widehat{\Gamma}_{n_0,\mathbf{n}}$. The following statements hold*:

- (i) *S is a subsemigroup of the semigroup $\mathbb{N}^{d+1}$.*
- (ii) *$G(S) = \mathbb{Z}^{d+1}$, and so $L(S) = \mathbb{R}^{d+1}$.*
- (iii) *The pair (S, M) is strongly admissible, with $\dim(\partial M) = d$ and $\operatorname{ind}(S, M) = \operatorname{ind}(S, \partial M) = 1$.*
- (iv) *For any $n \in \mathbb{N}$ and $1 \le i \le r$, we have*

$$\Delta(\Gamma_{0,n\mathbf{e}_i}) = (n\pi_1(\Delta(\Gamma_{0,\mathbf{e}_i})), 1) \quad \text{and} \quad \Delta(\widehat{\Gamma}_{0,n\mathbf{e}_i}) = (n\pi_1(\Delta(\widehat{\Gamma}_{0,\mathbf{e}_i})), 1).$$

- (v) *For any $n \in \mathbb{N}$, $\Delta(\Gamma_{n,\mathbf{0}}) = (n\pi_1(\Delta(\Gamma_{1,\mathbf{0}})), 1)$ and $\Delta(\widehat{\Gamma}_{n,\mathbf{0}}) = (n\pi_1(\Delta(\widehat{\Gamma}_{1,\mathbf{0}})), 1)$.*

The next proposition decomposes $\Gamma_{n_0,\mathbf{n}}$ and $\widehat{\Gamma}_{n_0,\mathbf{n}}$ as the levelwise sum of simpler semigroups (this result plays the same role that Proposition 3.4 played in the previous subsection).

Proposition 3.11. *Assume Setup 3.8. We have the following equalities*:

- (i) $\Gamma_{n_0,\mathbf{n}} = \Gamma_{n_0,\mathbf{0}} \oplus_t \Gamma_{0,n_1\mathbf{e}_1} \oplus_t \cdots \oplus_t \Gamma_{0,n_r\mathbf{e}_r}$.
- (ii) $\widehat{\Gamma}_{n_0,\mathbf{n}} = \widehat{\Gamma}_{n_0,\mathbf{0}} \oplus_t \widehat{\Gamma}_{0,n_1\mathbf{e}_1} \oplus_t \cdots \oplus_t \widehat{\Gamma}_{0,n_r\mathbf{e}_r}$.

Proof. (ii) For each $m \ge 0$, we need to show that

$$\pi_1([\widehat{\Gamma}_{n_0,\mathbf{n}}]_m) = \pi_1([\widehat{\Gamma}_{n_0,\mathbf{0}}]_m) + \pi_1([\widehat{\Gamma}_{0,n_1\mathbf{e}_1}]_m) + \cdots + \pi_1([\widehat{\Gamma}_{0,n_r\mathbf{e}_r}]_m).$$

Fix $m \in \mathbb{Z}_{>0}$.

First, we concentrate on the inclusion "$\supset$". Let $w_0 \in [\widehat{\Gamma}_{n_0,\mathbf{0}}]_m$ and, for each $1 \le i \le r$, let $w_i \in [\widehat{\Gamma}_{0,n_i\mathbf{e}_i}]_m$. Note that, for each $1 \le i \le r$, there exists $\mathbf{x}^{\mathbf{m}_i} \in J(i)_{mn_i}$ such that $w_i = (\mathbf{m}_i, m) \in \mathbb{N}^{d+1}$. Similarly, there exists $\mathbf{x}^{\mathbf{m}_0} \in I_{mn_0}$ such that $w_0 = (\mathbf{m}_0, m) \in \mathbb{N}^{d+1}$. Since $|\mathbf{m}_i| \le cmn_i$ for $0 \le i \le r$, it is clear that $\mathbf{x}^{\mathbf{m}_0} \cdots \mathbf{x}^{\mathbf{m}_r} \in I_{mn_0}\mathbf{J}_{m\mathbf{n}}$ and $|\mathbf{m}_0 + \cdots + \mathbf{m}_r| = |\mathbf{m}_0| + \cdots + |\mathbf{m}_r| \le cm(n_0 + |\mathbf{n}|)$. Therefore, it follows that $\widehat{\Gamma}_{n_0,\mathbf{n}} \supset \widehat{\Gamma}_{n_0,\mathbf{0}} \oplus_t \widehat{\Gamma}_{0,n_1\mathbf{e}_1} \oplus_t \cdots \oplus_t \widehat{\Gamma}_{0,n_r\mathbf{e}_r}$.

Next, we focus on the inclusion "$\subset$". Let $w \in [\widehat{\Gamma}_{n_0,\mathbf{n}}]_m$. Since $\mathbb{I}, \mathbb{J}(1), \ldots, \mathbb{J}(r)$ are graded families of monomial ideals, there exist $\mathbf{x}^{\mathbf{m}_0} \in I_{mn_0}$ and $\mathbf{x}^{\mathbf{m}_i} \in J(i)_{mn_i}$ such that $w = (\mathbf{m}_0 + \cdots + \mathbf{m}_r, m) \in \mathbb{N}^{d+1}$.

By assumption we have $\sum_{i=0}^{r} |\mathbf{m}_i| \le cm(n_0 + |\mathbf{n}|)$. For ease of notation, set $J(0)_n = I_n$ for all $n \in \mathbb{N}$.

Let $l(\mathbf{m}_0, \ldots, \mathbf{m}_r) := \sum_{i=0}^{r} \max\{|\mathbf{m}_i| - cmn_i,\ 0\}$. If $l(\mathbf{m}_0, \ldots, \mathbf{m}_r) = 0$, it then follows that $|\mathbf{m}_i| \le cmn_i$ for $0 \le i \le r$, and so we obtain that $\pi_1(w) = \pi_1(w_0) + \cdots + \pi_1(w_r)$, where $w_0 = (\mathbf{m}_0, m) \in [\widehat{\Gamma}_{n_0,\mathbf{0}}]_m$ and $w_i = (\mathbf{m}_i, m) \in [\widehat{\Gamma}_{0,n_i\mathbf{e}_i}]_m$ for $1 \le i \le r$. On the other hand, suppose that $l(\mathbf{m}_0, \ldots, \mathbf{m}_r) > 0$. Thus, there exist $0 \le j_1, j_2 \le r$ such that $|\mathbf{m}_{j_1}| > cmn_{j_1}$ and $|\mathbf{m}_{j_2}| < cmn_{j_2}$. From the fact that $\beta(J(j_1)_{mn_{j_1}}) \le cmn_{j_1}$, we can choose $1 \le k \le d$ such that $\mathbf{x}^{\mathbf{m}_{j_1} - \mathbf{e}_k} \in J(j_1)_{mn_{j_1}}$. For $0 \le i \le r$, we now set

$$\mathbf{x}^{\mathbf{m}'_i} \in J(i)_{mn_i} \quad \text{by} \quad \mathbf{m}'_i = \begin{cases} \mathbf{m}_{j_1} - \mathbf{e}_k & \text{if } i = j_1, \\ \mathbf{m}_{j_2} + \mathbf{e}_k & \text{if } i = j_2, \\ \mathbf{m}_i & \text{otherwise.} \end{cases}$$

Notice that $\pi_1(w) = \mathbf{m}'_0 + \cdots + \mathbf{m}'_r$ and $l(\mathbf{m}'_0, \ldots, \mathbf{m}'_r) = l(\mathbf{m}_0, \ldots, \mathbf{m}_r) - 1$. Therefore, by inducting on $l(\mathbf{m}_1, \ldots, \mathbf{m}_r)$, we obtain the other inclusion $\widehat{\Gamma}_{n_0,\mathbf{n}} \subset \widehat{\Gamma}_{n_0,\mathbf{0}} \oplus_t \widehat{\Gamma}_{0,n_1\mathbf{e}_1} \oplus_t \cdots \oplus_t \widehat{\Gamma}_{0,n_r\mathbf{e}_r}$.

(i) This part follows similarly, for example by following the arguments of part (ii) with $I_n = R$ for all $n \in \mathbb{N}$. ■

From Lemma 3.9 and the fact that $\mathbb{I}, \mathbb{J}(1), \ldots, \mathbb{J}(r)$ are graded families of monomial ideals, we obtain the following equalities:

$$\begin{aligned} \dim_{\mathbb{k}} (\mathbf{J}_{m\mathbf{n}}/I_{mn_0}\mathbf{J}_{m\mathbf{n}}) &= \dim_{\mathbb{k}} \left(\mathbf{J}_{m\mathbf{n}}/(\mathfrak{m}^{cm(n_0+|\mathbf{n}|)+1} \cap \mathbf{J}_{m\mathbf{n}})\right) \\ &\quad - \dim_{\mathbb{k}} \left(I_{mn_0}\mathbf{J}_{m\mathbf{n}}/(\mathfrak{m}^{cm(n_0+|\mathbf{n}|)+1} \cap I_{mn_0}\mathbf{J}_{m\mathbf{n}})\right) \\ &= \#[\Gamma_{n_0,\mathbf{n}}]_m - \#[\widehat{\Gamma}_{n_0,\mathbf{n}}]_m. \end{aligned} \tag{3.9}$$

We are now ready for the main result of this section. We show the existence of a homogeneous polynomial that allows us to define the mixed multiplicities of the graded families $\mathbb{I}, \mathbb{J}(1), \ldots, \mathbb{J}(r)$. Additionally, we explicitly describe this polynomial in terms of the mixed volume of certain Newton–Okounkov bodies.

Remark 3.12. We briefly describe the basic idea behind the proof of the following theorem. By utilizing the semigroups $\Gamma_{n_0,\mathbf{n}}$ and $\widehat{\Gamma}_{n_0,\mathbf{n}}$ and Theorem 2.4, we can study the asymptotic growth of the graded families $\mathbb{J}(1), \ldots, \mathbb{J}(r)$ with respect to $\mathbb{I}$ (see (3.9)). Due to Proposition 3.11, we can decompose the Newton–Okounkov body of $\Gamma_{n_0,\mathbf{n}}$ in terms of the Newton–Okounkov bodies of semigroups that depend on each individual filtration (a similar statement holds for the $\widehat{\Gamma}_{n_0,\mathbf{n}}$). Then the existence of a polynomial that coincides with the asymptotic function is a direct consequence of Minkowski's theorem (2.1).

Theorem 3.13. *Assume Setup 3.8. The function*

$$F(n_0, n_1, \ldots, n_r) = \lim_{m \to \infty} \frac{\dim_{\mathbb{k}}(J(1)_{mn_1} \cdots J(r)_{mn_r} \,/\, I_{mn_0} J(1)_{mn_1} \cdots J(r)_{mn_r})}{m^d}$$

is equal to a homogeneous polynomial $G(n_0,\mathbf{n}) = G(n_0,n_1,\ldots,n_r)$ *of total degree* d *with real coefficients for all* $n_0 \in \mathbb{N}$ *and* $\mathbf{n} = (n_1,\ldots,n_r) \in \mathbb{N}^r$*. Explicitly, the polynomial* $G(n_0,\mathbf{n})$ *is given by*

$$G(n_0,\mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d} \frac{1}{d_0!\mathbf{d}!}\left(\mathrm{MV}_d(\Delta(\Gamma)_{(d_0,\mathbf{d})}) - \mathrm{MV}_d(\Delta(\widehat{\Gamma})_{(d_0,\mathbf{d})})\right) n_0^{d_0}\,\mathbf{n}^{\mathbf{d}},$$

where $\Delta(\Gamma)$ *and* $\Delta(\widehat{\Gamma})$ *denote the sequences of Newton–Okounkov bodies*

$$\begin{aligned}\Delta(\Gamma) &= (\Delta(\Gamma_{1,\mathbf{0}}), \Delta(\Gamma_{0,\mathbf{e}_1}),\ldots,\Delta(\Gamma_{0,\mathbf{e}_r})) \quad \textit{and}\\ \Delta(\widehat{\Gamma}) &= (\Delta(\widehat{\Gamma}_{1,\mathbf{0}}), \Delta(\widehat{\Gamma}_{0,\mathbf{e}_1}),\ldots,\Delta(\widehat{\Gamma}_{0,\mathbf{e}_r})),\end{aligned}$$

respectively.

Proof. The proof follows along the same lines of Theorem 3.5. Let $n_0 \in \mathbb{N}$ and $\mathbf{n} = (n_1,\ldots,n_r) \in \mathbb{N}^r$. By using Theorem 2.4, Lemma 3.10 (iii) and (3.9), we obtain the equation

$$\begin{aligned}F(n_0,\mathbf{n}) &= \lim_{m\to\infty}\frac{\#[\Gamma_{n_0,\mathbf{n}}]_m}{m^d} - \lim_{m\to\infty}\frac{\#[\widehat{\Gamma}_{n_0,\mathbf{n}}]_m}{m^d}\\ &= \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})). \qquad (3.10)\end{aligned}$$

From Proposition 3.11, Lemma 3.10 (iv), (v), Proposition 1.32 in [17] and (2.1), we obtain that

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) = \sum_{d_0+|\mathbf{d}|=d}\frac{1}{d_0!\mathbf{d}!}\mathrm{MV}_d(\Delta(\Gamma)_{(d_0,\mathbf{d})})\, n_0^{d_0}\,\mathbf{n}^{\mathbf{d}}$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) = \sum_{d_0+|\mathbf{d}|=d}\frac{1}{d_0!\mathbf{d}!}\mathrm{MV}_d(\Delta(\widehat{\Gamma})_{(d_0,\mathbf{d})})\, n_0^{d_0}\,\mathbf{n}^{\mathbf{d}}.$$

So, the result follows. ∎

Lemma 3.14. *Assume Setup 3.8 and use the same notation of Theorem 3.13. Let* $d_0 \in \mathbb{N}$ *and* $\mathbf{d} = (d_1,\ldots,d_r) \in \mathbb{N}^r$ *with* $d_0 + |\mathbf{d}| = d$*. Then:*

(i) $\mathrm{MV}_d(\Delta(\Gamma)_{(d_0,\mathbf{d})}) - \mathrm{MV}_d(\Delta(\widehat{\Gamma})_{(d_0,\mathbf{d})}) \geq 0.$

(ii) $\mathrm{MV}_d(\Delta(\Gamma)_{(d_0,\mathbf{d})}) - \mathrm{MV}_d(\Delta(\widehat{\Gamma})_{(d_0,\mathbf{d})}) = 0$ *when* $d_0 = 0$*.*

Proof. Notice that $\Delta(\Gamma_{1,\mathbf{0}}) \supset \Delta(\widehat{\Gamma}_{1,\mathbf{0}})$ and that $\Delta(\Gamma_{0,\mathbf{e}_i}) = \Delta(\widehat{\Gamma}_{0,\mathbf{e}_i})$ for all $1 \leq i \leq r$. The result follows from the monotonicity of mixed volumes (see, e.g., equation 5.25 in [23]). ∎

After proving Theorem 3.13, we can define the mixed multiplicities of graded families of monomial ideals. As a consequence of Lemma 3.14, these mixed multiplicities are always non-negative and we can restrict ourselves to the terms of the form $n_0^{d_0+1}\mathbf{n}^{\mathbf{d}}$ in the definition below.

Definition 3.15. Assume Setup 3.8 and let $G(n_0, \mathbf{n})$ be as in Theorem 3.13. Write

$$G(n_0, \mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d-1} \frac{1}{(d_0+1)!\mathbf{d}!} e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1), \dots, \mathbb{J}(r))\, n_0^{d_0+1}\mathbf{n}^{\mathbf{d}}.$$

For each $d_0 \in \mathbb{N}$ and $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{N}^r$ with $d_0 + |\mathbf{d}| = d - 1$, we define the non-negative real number

$$e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1), \dots, \mathbb{J}(r)) \geq 0$$

to be the *mixed multiplicity of type* $(d_0, \mathbf{d})$ *of* $\mathbb{J}(1) = \{J(1)_n\}_{n\in\mathbb{N}}, \dots, \mathbb{J}(r) = \{J(r)_n\}_{n\in\mathbb{N}}$ *with respect to* $\mathbb{I} = \{I_n\}_{n\in\mathbb{N}}$.

The following remark shows that Definitions 3.7 and 3.15 agree in the $\mathfrak{m}$-primary case.

Remark 3.16. Assume Setup 3.8 and suppose that $\mathbb{J}(1), \dots, \mathbb{J}(r)$ are also graded families of $\mathfrak{m}$-primary monomial ideals. For all $m, n_0 \in \mathbb{N}$ and $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, we have the short exact sequence

$$0 \to \mathbf{J}_{m\mathbf{n}}/I_{mn_0}\mathbf{J}_{m\mathbf{n}} \to R/I_{mn_0}\mathbf{J}_{m\mathbf{n}} \to R/\mathbf{J}_{m\mathbf{n}} \to 0.$$

So, for each $d_0 \in \mathbb{N}$ and $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{N}^r$ with $d_0 + |\mathbf{d}| = d$, we can deduce the following:

(i) If $d_0 = 0$, then $e_{(d_0,\mathbf{d})}(\mathbb{I}, \mathbb{J}(1), \dots, \mathbb{J}(r)) = e_{\mathbf{d}}(\mathbb{J}(1), \dots, \mathbb{J}(r))$.

(ii) If $d_0 > 0$, then $e_{(d_0,\mathbf{d})}(\mathbb{I}, \mathbb{J}(1), \dots, \mathbb{J}(r)) = e_{(d_0-1,\mathbf{d})}(\mathbb{I} \mid \mathbb{J}(1), \dots, \mathbb{J}(r))$.

4. A "volume = multiplicity formula" for mixed multiplicities

In this section, we focus on proving Theorem B (see Theorem 4.7) which gives a "volume = multiplicity formula" for mixed multiplicities. This can be seen as an extension of the usual "volume = multiplicity formula" for graded families of ideals (see, e.g., Theorem 6.5 in [6]). Before that, we need to briefly recall the notion of mixed multiplicities for the case of ideals (for more details, see, e.g., [24]).

Throughout this subsection, we adopt Setup 3.8 and the following extra piece of notation.

Notation 4.1. Assume Setup 3.8. For every $p \in \mathbb{N}$ and $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, let $\mathbf{J}(p)^{\mathbf{n}}$ denote the ideal $J(1)_p^{n_1} \cdots J(r)_p^{n_r}$.

Let $I \subset R$ be a homogeneous $\mathfrak{m}$-primary ideal and let $J_1, \dots, J_r \subset R$ be homogeneous ideals. Since I is $\mathfrak{m}$-primary, we have that

$$T = T(I \mid J_1, \dots, J_r) := \bigoplus_{n_0\geq 0, n_1\geq 0\dots, n_r\geq 0} I^{n_0}J_1^{n_1}\cdots J_r^{n_r}/I^{n_0+1}J_1^{n_1}\cdots J_r^{n_r} \tag{4.1}$$

is a finitely generated standard $\mathbb{N}^{r+1}$-graded algebra over the Artinian local ring R/I. From Theorem 1.2 (a) in [24], one has a polynomial $P_T(n_0, n_1, \dots, n_r) \in \mathbb{Q}[n_0, n_1, \dots, n_r]$

of degree $d-1=\dim(R)-1$ such that $P_T(\nu)=\dim_{\mathbb{k}}([T]_\nu)$ for all $\nu\in\mathbb{N}^{r+1}$ with $\nu\gg\mathbf{0}$. Furthermore, if we write

(4.2)
$$P_T(n_0,n_1,\ldots,n_r)=\sum_{d_0,d_1,\ldots,d_r\geq 0} e(d_0,d_1,\ldots,d_r)\binom{n_0+d_0}{d_0}\binom{n_1+d_1}{d_1}\cdots\binom{n_r+d_r}{d_r},$$

then $0\leq e(d_0,d_1,\ldots,d_r)\in\mathbb{Z}$ for all $d_0+d_1+\cdots+d_r=d-1$. For each $d_0\in\mathbb{N}$ and $\mathbf{d}=(d_1,\ldots,d_r)\in\mathbb{N}^r$ with $d_0+|\mathbf{d}|=d-1$, we say that

$$e_{(d_0,\mathbf{d})}(I\mid J_1,\ldots,J_r):=e(d_0,d_1,\ldots,d_r) \tag{4.3}$$

is the *mixed multiplicity of type* $(d_0,\mathbf{d})$ *of* $J_1,\ldots,J_r$ *with respect to* I. The following lemma shows that the definition given in (4.3) agrees with the one given in Definition 3.15.

Lemma 4.2. *Let $I\subset R$ be a monomial $\mathfrak{m}$-primary ideal and let $J_1,\ldots,J_r\subset R$ be monomial ideals. Consider the monomial filtrations $\{I^n\}_{n\in\mathbb{N}},\{J_1^n\}_{n\in\mathbb{N}},\ldots,\{J_r^n\}_{n\in\mathbb{N}}$ given by the powers of $I,J_1,\ldots,J_r$. Then, we have the equality*

$$e_{(d_0,\mathbf{d})}\big(\{I^n\}_{n\in\mathbb{N}}\mid\{J_1^n\}_{n\in\mathbb{N}},\ldots,\{J_r^n\}_{n\in\mathbb{N}}\big)=e_{(d_0,\mathbf{d})}(I\mid J_1,\ldots,J_r)$$

for each $d_0\in\mathbb{N}$ and $\mathbf{d}=(d_1,\ldots,d_r)\in\mathbb{N}^r$ with $d_0+|\mathbf{d}|=d-1$.

Proof. Let $F(n_0,\mathbf{n})$ be the function $F(n_0,\mathbf{n})=\lim_{m\to\infty}\dim_{\mathbb{k}}(\mathbf{J}^{m\mathbf{n}}/I^{mn_0}\mathbf{J}^{m\mathbf{n}})/m^d$, where $\mathbf{n}=(n_1,\ldots,n_r)$ and $\mathbf{J}^{m\mathbf{n}}$ denotes the ideal $\mathbf{J}^{m\mathbf{n}}=J_1^{mn_1}\cdots J_r^{mn_r}\subset R$.

For each $n_0\in\mathbb{N}$ and $\mathbf{n}\in\mathbb{N}^r$, we have the following equality:

$$\dim_{\mathbb{k}}(\mathbf{J}^{\mathbf{n}}/I^{n_0}\mathbf{J}^{\mathbf{n}})=\sum_{k=0}^{n_0-1}\dim_{\mathbb{k}}(I^k\mathbf{J}^{\mathbf{n}}/I^{k+1}\mathbf{J}^{\mathbf{n}})=\sum_{k=0}^{n_0-1}\dim_{\mathbb{k}}([T]_{(k,\mathbf{n})})$$

(where T is the algebra introduced in (4.1)). Let $\nu=(\nu_0,\ldots,\nu_r)\in\mathbb{N}^{r+1}$ be such that $P_T(n_0,\mathbf{n})=\dim_{\mathbb{k}}\big([T]_{(n_0,\mathbf{n})}\big)$ for all $(n_0,\mathbf{n})\geq\nu$. Thus, for all $(n_0,\mathbf{n})\geq\nu$, we can write

$$\dim_{\mathbb{k}}(\mathbf{J}^{\mathbf{n}}/I^{n_0}\mathbf{J}^{\mathbf{n}})=\sum_{k=0}^{\nu_0-1}\dim_{\mathbb{k}}([T]_{(k,\mathbf{n})})+\sum_{k=\nu_1}^{n_0-1}P_T(k,\mathbf{n}). \tag{4.4}$$

For any $k\in\mathbb{N}$, one has that $[T]_{(k,*,\ldots,*)}=\bigoplus_{\mathbf{n}\geq\mathbf{0}}I^k\mathbf{J}^{\mathbf{n}}/I^{k+1}\mathbf{J}^{\mathbf{n}}$ is a finitely generated $\mathbb{N}^r$-graded module over the finitely generated standard $\mathbb{N}^r$-graded algebra $[T]_{(0,*,\ldots,*)}=\bigoplus_{\mathbf{n}\geq\mathbf{0}}\mathbf{J}^{\mathbf{n}}/I\mathbf{J}^{\mathbf{n}}$. From Theorem 4.1 in [12] (also, see Theorem 3.4 in [3]), for all $\mathbf{n}\gg\mathbf{0}$, we obtain that

$$\dim_{\mathbb{k}}([T]_{(k,\mathbf{n})})=P_{[T]_{(k,*,\ldots,*)}}(\mathbf{n})\quad\text{for some polynomial }P_{[T]_{(k,*,\ldots,*)}}(\mathbf{n})$$

with degree bounded by $\dim(\mathrm{MultiProj}([T]_{(0,*,\ldots,*)}))$ (see §1 of [16], and Definition 2.2 in [2]). Since I is an $\mathfrak{m}$-primary ideal, we have the equality

$$\dim(\mathrm{MultiProj}([T]_{(0,*,\ldots,*)}))=\dim(\mathrm{MultiProj}(\mathcal{F}(J_1,\ldots,J_r))),$$

where $\mathcal{F}(J_1,\ldots,J_r)$ denotes the special fiber ring $\mathcal{F}(J_1,\ldots,J_r) = \mathcal{R}(J_1,\ldots,J_r) \otimes_R R/\mathfrak{m}$. By using the Segre embedding, we get the isomorphism

$$\operatorname{MultiProj}(\mathcal{R}(J_1,\ldots,J_r) \otimes_R R/\mathfrak{m}) \cong \operatorname{Proj}(\mathcal{R}(J_1\cdots J_r) \otimes_R R/\mathfrak{m}).$$

Therefore, for all $\mathbf{n} \gg \mathbf{0}$, we obtain that $\dim_{\mathbb{k}}([T]_{(k,\mathbf{n})}) = P_{[T]_{(k,*,\ldots,*)}}(\mathbf{n})$ and

$$\deg(P_{[T]_{(k,*,\ldots,*)}}(\mathbf{n})) \le \dim(\operatorname{Proj}(\mathcal{R}(J_1\cdots J_r)\otimes_R R/\mathfrak{m})) = \ell(J_1\cdots J_r)-1 \le d-1, \tag{4.5}$$

where $\ell(J_1\cdots J_r)$ denotes the analytic spread of $J_1\cdots J_r$ and the last inequality follows from Proposition 5.1.6 in [15].

By combining (4.4) and (4.5), we obtain the following equality:

$$\begin{aligned} F(n_0,\mathbf{n}) &= \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}(\mathbf{J}^{m\mathbf{n}} \,/\, I_{mn_0}\mathbf{J}^{m\mathbf{n}})}{m^d} \\ &= \lim_{m\to\infty} \frac{\sum_{k=0}^{\nu_0-1}\dim_{\mathbb{k}}([T]_{(k,m\mathbf{n})}) + \sum_{k=\nu_1}^{mn_0-1} P_T(k,m\mathbf{n})}{m^d} \\ &= \lim_{m\to\infty} \frac{\sum_{k=\nu_1}^{mn_0-1} P_T(k,m\mathbf{n})}{m^d}. \end{aligned}$$

Since $\deg(P_T(n_0,\mathbf{n})) = d-1$, we can write

$$F(n_0,\mathbf{n}) = \lim_{m\to\infty} \frac{\sum_{k=0}^{mn_0-1} P_T(k,m\mathbf{n})}{m^d}.$$

Notice that

$$\begin{aligned} &\sum_{k=0}^{mn_0-1} P_T(k,m\mathbf{n}) \\ &\quad= \sum_{d_0,\ldots,d_r\ge 0} e_{(d_0,\mathbf{d})}(I \mid J_1,\ldots,J_r)\binom{mn_0+d_0}{d_0+1}\binom{mn_1+d_1}{d_1}\cdots\binom{mn_r+d_r}{d_r}. \end{aligned}$$

Therefore, we obtain that $F(n_0,\mathbf{n})$ coincides with the following polynomial:

$$F(n_0,\mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d-1} \frac{1}{(d_0+1)!\,\mathbf{d}!}\, e_{(d_0,\mathbf{d})}(I \mid J_1,\ldots,J_r)\, n_0^{d_0+1}\mathbf{n}^{\mathbf{d}},$$

and so the result follows. ∎

Let c be as in Setup 3.8. For ease of notation, we define the following functions (recall Notation 4.1):

$$H_p(n_0,\mathbf{n}) := \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}(\mathbf{J}(p)^{m\mathbf{n}}/(\mathfrak{m}^{cmp(n_0+|\mathbf{n}|)+1}\cap \mathbf{J}(p)^{m\mathbf{n}}))}{m^d\,p^d} \tag{4.6}$$

and

$$\widehat{H}_p(n_0,\mathbf{n}) := \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}(I_p^{mn_0}\mathbf{J}(p)^{m\mathbf{n}}/(\mathfrak{m}^{cmp(n_0+|\mathbf{n}|)+1}\cap I_p^{mn_0}\mathbf{J}(p)^{m\mathbf{n}}))}{m^d\,p^d}. \tag{4.7}$$

We note that the existence of these limits follow as in (3.10).

The following technical proposition is needed to treat the Noetherian case of the formula.

Proposition 4.3. *Assume Setup 3.8. In addition, suppose that $\mathbb{I}, \mathbb{J}(1), \ldots, \mathbb{J}(r)$ are Noetherian graded families. Then, for fixed $n_0 \in \mathbb{N}$, $\mathbf{n} \in \mathbb{N}^r$ and $\varepsilon \in \mathbb{R}_{>0}$, there exists $p_0 \in \mathbb{N}$ such that if $p \geq p_0$ then*

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) \geq H_p(n_0,\mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \varepsilon$$

and

$$\mathrm{Vol}_d\big(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})\big) \geq \widehat{H}_p(n_0,\mathbf{n}) \geq \mathrm{Vol}_d\big(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})\big) - \varepsilon.$$

Proof. Similarly to (3.7) and (3.8), we now define

$$\Gamma_{n_0,\mathbf{n}}(p) := \big\{(\mathbf{m}, mp) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in \mathbf{J}(p)^{m\mathbf{n}} \text{ and } |\mathbf{m}| \leq cmp(n_0 + |\mathbf{n}|)\big\}$$

and

$$\widehat{\Gamma}_{n_0,\mathbf{n}}(p) := \big\{(\mathbf{m}, mp) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in I_p^{mn_0}\mathbf{J}(p)^{m\mathbf{n}} \text{ and } |\mathbf{m}| \leq cmp(n_0 + |\mathbf{n}|)\big\}.$$

For each $(n_0, \mathbf{n}) \in \mathbb{N}^{d+1}$, we consider the graded families of monomial ideals $\{\mathbf{J}(p)^{\mathbf{n}}\}_{p\in\mathbb{N}} = \{J(1)_p^{n_1} \cdots J(r)_p^{n_r}\}_{p\in\mathbb{N}}$ and $\{I_p^{n_0}\mathbf{J}(p)^{\mathbf{n}}\}_{p\in\mathbb{N}} = \{I_p^{n_0} J(1)_p^{n_1} \cdots J(r)_p^{n_r}\}_{p\in\mathbb{N}}$. From these graded families, we define the semigroups

$$\mathfrak{A}_{n_0,\mathbf{n}} := \big\{(\mathbf{m}, p) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in \mathbf{J}(p)^{\mathbf{n}} \text{ and } |\mathbf{m}| \leq cp(n_0 + |\mathbf{n}|)\big\}$$

and

$$\mathfrak{B}_{n_0,\mathbf{n}} := \big\{(\mathbf{m}, p) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in I_p^{n_0}\mathbf{J}(p)^{\mathbf{n}} \text{ and } |\mathbf{m}| \leq cp(n_0 + |\mathbf{n}|)\big\}.$$

By construction, for all $p, m \geq 1$ we have the inclusions

$$m \star [\mathfrak{A}_{n_0,\mathbf{n}}]_p \subset [\Gamma_{n_0,\mathbf{n}}(p)]_{mp} \subset [\mathfrak{A}_{n_0,\mathbf{n}}]_{mp}$$

and

$$m \star [\mathfrak{B}_{n_0,\mathbf{n}}]_p \subset [\widehat{\Gamma}_{n_0,\mathbf{n}}(p)]_{mp} \subset [\mathfrak{B}_{n_0,\mathbf{n}}]_{mp}.$$

As a consequence of Proposition 3.1 in [21] (see also Theorem 3.3 in [6]) and Theorem 2.4, for a fixed $\varepsilon \in \mathbb{R}_{>0}$, there exists $p_0 \in \mathbb{N}$ such that if $p \geq p_0$ then

$$\begin{aligned}\mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}})) &\geq \lim_{m\to\infty} \frac{\#\big([\Gamma_{n_0,\mathbf{n}}(p)]_{mp}\big)}{m^d\, p^d} \geq \lim_{m\to\infty} \frac{\#\big(m \star [\mathfrak{A}_{n_0,\mathbf{n}}]_p\big)}{m^d\, p^d} \\ &\geq \mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}})) - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\mathrm{Vol}_d(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})) &\geq \lim_{m\to\infty} \frac{\#\big([\widehat{\Gamma}_{n_0,\mathbf{n}}(p)]_{mp}\big)}{m^d\, p^d} \geq \lim_{m\to\infty} \frac{\#\big(m \star [\mathfrak{B}_{n_0,\mathbf{n}}]_p\big)}{m^d\, p^d} \\ &\geq \mathrm{Vol}_d\,(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})) - \varepsilon.\end{aligned}$$

Therefore, using the notation of (4.6) and (4.7), we obtain the inequalities $\mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}})) \geq H_p(n_0,\mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}})) - \varepsilon$ and $\mathrm{Vol}_d(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})) \geq \widehat{H}_p(n_0,\mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})) - \varepsilon$ for all $p \geq p_0$.

To conclude the proof, we only need to show that the equalities $\mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}})) = \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}}))$ and $\mathrm{Vol}_d(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})) = \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}}))$ hold.

By the Noetherian assumption, there exists $q > 0$ such that

$$J(i)_q^n = J(i)_{nq} \quad \text{and} \quad I_q^n = I_{nq} \quad \text{for every } n \geq 0, 1 \leq i \leq r \tag{4.8}$$

(see, e.g., Lemma 13.10 in [11] and Theorem 2.1 in [13]). Hence $\mathbf{J}(mq)^{\mathbf{n}} = \mathbf{J}_{mq\mathbf{n}}$ and $I_{mq}^{n_0}\mathbf{J}(mq)^{\mathbf{n}} = I_{mqn_0}\mathbf{J}_{mq\mathbf{n}}$ for all $m \geq 0$, and so Theorem 2.4 yields the required equalities

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) = \lim_{m\to\infty} \frac{\#[\Gamma_{n_0,\mathbf{n}}]_{mq}}{m^d q^d} = \lim_{m\to\infty} \frac{\#[\mathfrak{A}_{n_0,\mathbf{n}}]_{mq}}{m^d q^d} = \mathrm{Vol}_d(\Delta(\mathfrak{A}_{n_0,\mathbf{n}}))$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) = \lim_{m\to\infty} \frac{\#[\widehat{\Gamma}_{n_0,\mathbf{n}}]_{mq}}{m^d q^d} = \lim_{m\to\infty} \frac{\#[\mathfrak{B}_{n_0,\mathbf{n}}]_{mq}}{m^d q^d} = \mathrm{Vol}_d(\Delta(\mathfrak{B}_{n_0,\mathbf{n}})).$$

Therefore, the proof of the proposition is now complete. ∎

We now focus on approximating the graded families $\mathbb{I}, \mathbb{J}(1), \ldots, \mathbb{J}(r)$ by using successive truncations of them. For that, we need to introduce some additional notation.

Notation 4.4. Let $a > 0$ be a positive integer. Let $\mathbb{I}_a = \{I_{a,n}\}_{n\in\mathbb{N}}$ be the Noetherian graded family generated by $I_1, \ldots, I_a$, that is, for $n > a$ one has $I_{a,n} = \sum_{i=1}^{n-1} I_{a,i} I_{a,n-i}$. Likewise, define $\mathbb{J}(i)_a = \{J(i)_{a,n}\}_{n\in\mathbb{N}}$ for all $1 \leq i \leq r$.

For a vector $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we abbreviate $\mathbf{J}_{a,\mathbf{n}} = J(1)_{a,n_1} \cdots J(r)_{a,n_r}$. As in (3.7) and (3.8), we now define

$$\Gamma_{a,n_0,\mathbf{n}} := \left\{(\mathbf{m}, m) = (m_1, \ldots, m_d, m) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in \mathbf{J}_{a,m\mathbf{n}} \text{ and } |\mathbf{m}| \leq cm(n_0 + |\mathbf{n}|)\right\}$$

and

$$\widehat{\Gamma}_{a,n_0,\mathbf{n}} := \left\{(\mathbf{m}, m) = (m_1, \ldots, m_d, m) \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in I_{a,mn_0}\mathbf{J}_{a,m\mathbf{n}} \text{ and } |\mathbf{m}| \leq cm(n_0 + |\mathbf{n}|)\right\}.$$

For every $p \in \mathbb{N}$ and every $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, let $\mathbf{J}(a,p)^{\mathbf{n}}$ denote the ideal $J(1)_{a,p}^{n_1} \cdots J(r)_{a,p}^{n_r}$. Additionally, we have the following functions:

$$H_{a,p}(n_0,\mathbf{n}) := \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}\left(\mathbf{J}(a,p)^{m\mathbf{n}} / (\mathfrak{m}^{cmp(n_0+|\mathbf{n}|)+1} \cap \mathbf{J}(a,p)^{m\mathbf{n}})\right)}{m^d p^d}$$

and

$$\widehat{H}_{a,p}(n_0,\mathbf{n}) := \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}\left(I_{a,p}^{mn_0}\mathbf{J}(a,p)^{m\mathbf{n}} / (\mathfrak{m}^{cmp(n_0+|\mathbf{n}|)+1} \cap I_{a,p}^{mn_0}\mathbf{J}(a,p)^{m\mathbf{n}})\right)}{m^d p^d}.$$

The next technical proposition is used in the proof of Theorem 4.7. For its proof we use an argument quite similar to the one used in Proposition 4.3 of [8].

Proposition 4.5. *Assume Setup 3.8. Then, for fixed $n_0 \in \mathbb{N}$, $\mathbf{n} \in \mathbb{N}^r$ and $\varepsilon \in \mathbb{R}_{>0}$, there exists $a_0 \in \mathbb{N}$ such that if $a \geq a_0$, then*

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\Gamma_{a,n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \varepsilon$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) - \varepsilon.$$

Proof. For a positive integer $a > 0$, since $I_n = I_{a,n}$ and $J(i) = J(i)_{a,n}$ for all $n \leq a$, for all $m \geq 1$ we obtain the following inclusions:

$$m \star [\Gamma_{n_0,\mathbf{n}}]_a \subset [\Gamma_{a,n_0,\mathbf{n}}]_{ma} \subset [\Gamma_{n_0,\mathbf{n}}]_{ma} \quad \text{and} \quad m \star [\widehat{\Gamma}_{n_0,\mathbf{n}}]_a \subset [\widehat{\Gamma}_{a,n_0,\mathbf{n}}]_{ma} \subset [\widehat{\Gamma}_{n_0,\mathbf{n}}]_{ma}.$$

Then, by Proposition 3.1 in [21] (see also Theorem 3.3 in [6]) and Theorem 2.4, for a fixed $\varepsilon \in \mathbb{R}_{>0}$, there exists $a_0 \in \mathbb{N}$ such that if $a \geq a_0$, then

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\Gamma_{a,n_0,\mathbf{n}})) \geq \lim_{m\to\infty} \frac{\#(m \star [\Gamma_{n_0,\mathbf{n}}]_a)}{m^d a^d} \geq \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \varepsilon$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) \geq \lim_{m\to\infty} \frac{\#(m \star [\widehat{\Gamma}_{n_0,\mathbf{n}}]_a)}{m^d a^d} \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) - \varepsilon.$$

So, the result follows. ∎

Finally, we are ready for our analog of the "volume = multiplicity formula" in the case of mixed multiplicities. The following theorem expresses the mixed multiplicities of graded families as normalized limits of mixed multiplicities of ideals.

Remark 4.6. We first provide a couple of general words regarding the proof of Theorem 4.7. Despite a number of technical steps in the proof, the idea behind is quite simple: we perform a double approximation process. First, we approximate the graded families with successive truncations, which are Noetherian graded families. Then, under this Noetherian hypothesis, we can choose suitable Veronese subalgebras of the Rees algebras corresponding to these truncations that are standard graded (see, e.g., Lemma 13.10 in [11], Theorem 2.1 in [13]). This means that these Veronese subalgebras are simply Rees algebras of ideals. Finally, by certain technical steps, we can complete the proof.

We further extended this double approximation technique in [4], where we managed to show the existence of mixed multiplicities of arbitrary graded families of ideals on Noetherian local rings whose completion is reduced at minimal primes of maximal dimension (in particular, it holds for analytically unramified local rings).

Theorem 4.7. *Assume Setup 3.8. Then, for each $d_0 \in \mathbb{N}$ and $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ with $d_0 + |\mathbf{d}| = d - 1$, we have the equality*

$$e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1), \ldots, \mathbb{J}(r)) = \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(I_p|J(1)_p, \ldots, J(r)_p)}{p^d}.$$

Proof. Let $p \geq 1$, and consider the filtrations

$$\mathbb{I}(p) = \{I_p^n\}_{n\in\mathbb{N}},\ \mathbb{J}(1)(p) = \{J(1)_p^n\}_{n\in\mathbb{N}}, \ldots, \mathbb{J}(r)(p) = \{J(r)_p^n\}_{n\in\mathbb{N}}.$$

By applying Theorem 3.13 to the filtrations

$$\mathbb{I}(p) = \{I_p^n\}_{n\in\mathbb{N}},\ \mathbb{J}(1)(p) = \{J(1)_p^n\}_{n\in\mathbb{N}},\ \ldots,\ \mathbb{J}(r)(p) = \{J(r)_p^n\}_{n\in\mathbb{N}},$$

let $F_p(n_0, \mathbf{n})$ be the function

$$F_p(n_0, \mathbf{n}) = \lim_{m\to\infty} \frac{\dim_{\Bbbk}(\mathbf{J}(p)^{m\mathbf{n}}/I_p^{mn_0}\mathbf{J}(p)^{m\mathbf{n}})}{m^d} \tag{4.9}$$

and let $G_p(n_0, \mathbf{n})$ be the polynomial of total degree d that coincides with $F_p(n_0, \mathbf{n})$. From Lemma 4.2, we can write

$$G_p(n_0, \mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d-1} \frac{1}{(d_0+1)!\,\mathbf{d}!}\, e_{(d_0,\mathbf{d})}(I_p|J(1)_p, \ldots, J(r)_p)\, n_0^{d_0+1}\,\mathbf{n}^{\mathbf{d}}. \tag{4.10}$$

Notice that we have the equation

$$\frac{1}{p^d}\, F_p(n_0, \mathbf{n}) = H_p(n_0, \mathbf{n}) - \widehat{H}_p(n_0, \mathbf{n}); \tag{4.11}$$

see (4.6) and (4.7).

Fix a positive real number $\varepsilon > 0$. By using Proposition 4.5, choose $a \gg 0$ such that

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\Gamma_{a,n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \varepsilon/2 \tag{4.12}$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) - \varepsilon/2. \tag{4.13}$$

From Proposition 4.3, applied to the Noetherian graded families $\mathbb{I}_a, \mathbb{J}(1)_a, \ldots, \mathbb{J}(r)_a$, choose $p \gg 0$ such that

$$\mathrm{Vol}_d(\Delta(\Gamma_{a,n_0,\mathbf{n}})) \geq H_{a,p}(n_0, \mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\Gamma_{a,n_0,\mathbf{n}})) - \varepsilon/2 \tag{4.14}$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) \geq \widehat{H}_{a,p}(n_0, \mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{a,n_0,\mathbf{n}})) - \varepsilon/2. \tag{4.15}$$

Since $I_{a,n} \subset I_n$ and $J(i)_{a,n} \subset J(i)_n$ for all $n \in \mathbb{N}$, one has $H_p(n_0, \mathbf{n}) \geq H_{a,p}(n_0, \mathbf{n})$ and $\widehat{H}_p(n_0, \mathbf{n}) \geq \widehat{H}_{a,p}(n_0, \mathbf{n})$, and so from (4.12), (4.13), (4.14) and (4.15) one obtains

$$\mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) \geq H_p(n_0, \mathbf{n}) \geq H_{a,p}(n_0, \mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \varepsilon \tag{4.16}$$

and

$$\mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) \geq \widehat{H}_p(n_0, \mathbf{n}) \geq \widehat{H}_{a,p}(n_0, \mathbf{n}) \geq \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) - \varepsilon. \tag{4.17}$$

Therefore, by combining (3.10), (4.11), (4.16) and (4.17), we obtain the equalities

$$\begin{aligned} F(n_0, \mathbf{n}) &= \mathrm{Vol}_d(\Delta(\Gamma_{n_0,\mathbf{n}})) - \mathrm{Vol}_d(\Delta(\widehat{\Gamma}_{n_0,\mathbf{n}})) \\ &= \lim_{p\to\infty} (H_p(n_0,\mathbf{n}) - \widehat{H}_p(n_0,\mathbf{n})) = \lim_{p\to\infty} \frac{1}{p^d} F_p(n_0,\mathbf{n}) \end{aligned}$$

for all n_0 and $\mathbf{n} \in \mathbb{N}^r$. Accordingly, it necessarily follows that the coefficients of the polynomials $\frac{1}{p^d} G_p(n_0,\mathbf{n})$ converge to the ones of the polynomial $G(n_0,\mathbf{n})$ (see, e.g., Lemma 3.2 in [8]). Therefore, by Definition 3.15 and (4.10), we obtain

$$e_{(d_0,\mathbf{d})}(\mathbb{I}|\mathbb{J}(1),\ldots,\mathbb{J}(r)) = \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(I_p|J(1)_p,\ldots,J(r)_p)}{p^d},$$

and so the result follows. ∎

5. Mixed volumes of convex bodies as mixed multiplicities

This section includes the proof Theorem C, which is the main result of this paper (see Theorem 5.5). In this result, we describe the mixed volumes of (arbitrary) convex bodies as the mixed multiplicities of certain (not necessarily Noetherian) graded families of ideals, and as the normalized limits of the mixed multiplicities of certain monomial ideals.

Throughout this section we fix the following setup.

Setup 5.1. Let $\Bbbk$ be a field, let $R = \Bbbk[x_1,\ldots,x_{d+1}]$ be the standard graded polynomial ring, and let $\mathfrak{m} \subset R$ be the graded irrelevant ideal $\mathfrak{m} = (x_1,\ldots,x_{d+1})$. Let

$$\pi_1 : \mathbb{R}^{d+1} \to \mathbb{R}^d,\ (\alpha_1,\ldots,\alpha_d,\alpha_{d+1}) \mapsto (\alpha_1,\ldots,\alpha_d)$$

be the projection into the first d factors. Let $\pi\colon \mathbb{R}^{d+1} \to \mathbb{R}$ be the linear map given by $(\alpha_1,\ldots,\alpha_d,\alpha_{d+1}) \mapsto \alpha_1 + \cdots + \alpha_d + \alpha_{d+1}$. Let $(K_1,\ldots,K_r)$ be a sequence of convex bodies in $\mathbb{R}^d_{\ge 0}$.

The notation below introduces a process that we call the homogenization of a convex body.

Notation 5.2. Let $K \subset \mathbb{R}^d_{\ge 0}$ be a convex body in $\mathbb{R}^d_{\ge 0}$. Choose $h_K \in \mathbb{N}$ a positive integer such that $|\mathbf{x}| \le h_K$ for all $\mathbf{x} \in K$. We call h_K a *suitable degree of homogenization of* K. The corresponding *homogenization* of K (with respect to h_K) is defined as the convex body

$$\widetilde{K} = (K \times \mathbb{R}) \cap \pi^{-1}(h_K) \ \subset\ \mathbb{R}^{d+1}_{\ge 0}.$$

Let C_K be the corresponding cone $C_K := \mathrm{Cone}(\widetilde{K})$. Consider the semigroup $S_K \subset \mathbb{N}^{d+1}$ given by

$$S_K := C_K \cap \mathbb{N}^{d+1} \cap \Big(\bigcup_{k=1}^{\infty} \pi^{-1}(kh_K)\Big).$$

For each $1 \le i \le r$, let h_{K_i} be a suitable degree of homogenization for K_i, and let S_{K_i} be the corresponding semigroup in $\mathbb{N}^{d+1}$ determined by the homogenization $\widetilde{K}_i \subset \mathbb{R}^{d+1}_{\ge 0}$.

For each $1 \leq i \leq r$, we consider the (not necessarily Noetherian) graded family of monomial ideals

(5.1) $\quad \mathbb{J}(i) = \{J(i)_n\}_{n\in\mathbb{N}}, \quad$ where $\quad J(i)_n = (\mathbf{x}^{\mathbf{m}} \mid \mathbf{m} \in S_{K_i} \text{ and } |\mathbf{m}| = nh_{K_i}) \subset R.$

Let $K_0 \subset \mathbb{R}^d$ be the convex hull of the points $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ and let $\widetilde{K_0}$ be its homogenization $\widetilde{K_0} = \{\mathbf{x} \in \mathbb{R}^{d+1}_{\geq 0} \mid |\mathbf{x}| = 1\} \subset \mathbb{R}^{d+1}$; notice that $h_{K_0} = 1$ is a suitable degree of homogenization for K_0. We let $\mathbb{M}$ be the graded family $\mathbb{M} = \{\mathfrak{m}^n\}_{n\in\mathbb{N}}$. Denote by $\mathbf{K}$ the sequence of convex bodies $\mathbf{K} = (K_0, K_1, \ldots, K_r)$.

Let $p > 0$ be a positive integer. For each $1 \leq i \leq r$, let $K_i(p)$ be the lattice polytope given by

$$K_i(p) \;:=\; \pi_1\big(\operatorname{conv}(\{\mathbf{m} \in \mathbb{N}^{d+1} \mid \mathbf{x}^{\mathbf{m}} \in [J(i)_p]_{ph_{K_i}}\})\big),$$

where $\operatorname{conv}(-)$ denotes the convex hull of a finite set of points; the polytope defined above corresponds with the generators of the ideal $J(i)_p$. Denote by $\mathbf{K}(p)$ the sequence of lattice polytopes $\mathbf{K}(p) := (K_0, K_1(p), \ldots, K_r(p))$. The next lemma shows that the mixed volumes of $\mathbf{K}$ can be approximated with the ones of $\mathbf{K}(p)$.

Lemma 5.3. *For each $(d_0, \mathbf{d}) \in \mathbb{N}^{r+1}$ with $d_0 + |\mathbf{d}| = d$, we have the equality*

$$\operatorname{MV}_d\left(\mathbf{K}_{(d_0,\mathbf{d})}\right) = \lim_{p\to\infty} \frac{\operatorname{MV}_d\left(\mathbf{K}(p)_{(d_0,\mathbf{d})}\right)}{p^{|\mathbf{d}|}}.$$

Proof. By construction, we have that $\frac{1}{p}K_i(p)$ converges to K_i in the Hausdorff metric (see Definition 2.1 on p. 109 of [10]) when $p \to \infty$. Thus, Lemma 3.8 on p. 119 of [10] yields that

$$\operatorname{MV}_d\left(\left(K_0, \frac{1}{p}\,K_1(p), \ldots, \frac{1}{p}\,K_r(p)\right)_{(d_0,\mathbf{d})}\right)$$

converges to $\operatorname{MV}_d(\mathbf{K}_{(d_0,\mathbf{d})})$ when $p \to \infty$. Therefore, the result follows from the linearity of mixed volumes (see, e.g., Lemma 3.6 on p. 118 of [10]). ∎

Finally, the next theorem expresses the mixed volume of the convex bodies $K_1, \ldots, K_r$ as a mixed multiplicity of the chosen graded families $\mathbb{J}(1), \ldots, \mathbb{J}(r)$. Additionally, we also express the mixed volume of the convex bodies $K_1, \ldots, K_r$ as two types of normalized limits of mixed multiplicities of ideals. This result can be seen as an extension of Theorem 2.4 and Corollary 2.5 in [24].

Remark 5.4. The proof of the following theorem is the combination of three main ingredients. By the result of Trung and Verma (Theorem 2.4 and Corollary 2.5 in [24]), we can express mixed volumes of lattice polytopes as mixed multiplicities of monomial ideals. By Lemma 5.3, mixed volumes of arbitrary convex bodies can be written as the normalized limit of the mixed volumes of certain lattice polytopes. These lattice polytopes induce graded families of monomial ideals in a natural way (see (5.1)). Finally, everything is glued together by Theorem 4.7, which gives our "volume = multiplicity formula" for mixed multiplicities.

Theorem 5.5. *Assume Setup 5.1. Let $\mathbb{J}(1), \ldots, \mathbb{J}(r)$ be the graded families of monomial ideals defined in (5.1). For each $(d_0, \mathbf{d}) \in \mathbb{N}^{r+1}$ with $d_0 + |\mathbf{d}| = d$, we have the equalities*

$$\begin{aligned} \mathrm{MV}_d\left(\mathbf{K}_{(d_0,\mathbf{d})}\right) &= e_{(d_0,\mathbf{d})}(\mathbb{M} \mid \mathbb{J}(1), \ldots, \mathbb{J}(r)) \\ &= \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(\mathfrak{m}^p \mid J(1)_p, \ldots, J(r)_p)}{p^{d+1}} \\ &= \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(\mathfrak{m} \mid J(1)_p, \ldots, J(r)_p)}{p^{|\mathbf{d}|}}. \end{aligned}$$

In particular, when $r = d$, we obtain the equalities

$$\begin{aligned} \mathrm{MV}_d(K_1, \ldots, K_d) &= e_{(0,1,\ldots,1)}(\mathbb{M} \mid \mathbb{J}(1), \ldots, \mathbb{J}(d)) \\ &= \lim_{p\to\infty} \frac{e_{(0,1,\ldots,1)}(\mathfrak{m}^p \mid J(1)_p, \ldots, J(d)_p)}{p^{d+1}} \\ &= \lim_{p\to\infty} \frac{e_{(0,1,\ldots,1)}(\mathfrak{m} \mid J(1)_p, \ldots, J(d)_p)}{p^{d}}. \end{aligned}$$

Proof. By applying Theorem 2.4 and Corollary 2.5 in [24] to the generators of the monomial ideals $J(1)_p, \ldots, J(r)_p$, we obtain the equality $\mathrm{MV}_d(\mathbf{K}(p)_{(d_0,\mathbf{d})}) = e_{(d_0,\mathbf{d})}(\mathfrak{m} \mid J(1)_p, \ldots, J(r)_p)$. Thus, Lemma 5.3 yields the equality

$$\mathrm{MV}_d(\mathbf{K}_{(d_0,\mathbf{d})}) = \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(\mathfrak{m} \mid J(1)_p, \ldots, J(r)_p)}{p^{|\mathbf{d}|}}.$$

Consider

$$G_1(n_0, \mathbf{n}) = \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}(J(1)_p^{mn_1} \cdots J(r)_p^{mn_r} / \mathfrak{m}^{mn_0} J(1)_p^{mn_1} \cdots J(r)_p^{mn_r})}{m^{d+1}}$$

and

$$G_2(n_0, \mathbf{n}) = \lim_{m\to\infty} \frac{\dim_{\mathbb{k}}(J(1)_p^{mn_1} \cdots J(r)_p^{mn_r} / (\mathfrak{m}^p)^{mn_0} J(1)_p^{mn_1} \cdots J(r)_p^{mn_r})}{m^{d+1}}.$$

By Theorem 3.13, the limits in G_1 and G_2 exist and are polynomials. Due to Lemma 4.2, we have that

$$G_1(n_0, \mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d} \frac{1}{(d_0+1)!\,\mathbf{d}!} e_{(d_0,\mathbf{d})}(\mathfrak{m}|J(1)_p, \ldots, J(r)_p)\, n_0^{d_0+1}\mathbf{n}^{\mathbf{d}}.$$

and

$$G_2(n_0, \mathbf{n}) = \sum_{d_0+|\mathbf{d}|=d} \frac{1}{(d_0+1)!\,\mathbf{d}!} e_{(d_0,\mathbf{d})}\left(\mathfrak{m}^p|J(1)_p, \ldots, J(r)_p\right) n_0^{d_0+1}\mathbf{n}^{\mathbf{d}}.$$

Since $G_1(pn_0, \mathbf{n}) = G_2(n_0, \mathbf{n})$, we obtain, by comparing the coefficients of $G_1(pn_0, \mathbf{n})$ and $G_2(n_0, \mathbf{n})$, that $e_{(d_0,\mathbf{d})}(\mathfrak{m}^p|J(1)_p, \ldots, J(r)_p) = p^{d_0+1} e_{(d_0,\mathbf{d})}(\mathfrak{m}|J(1)_p, \ldots, J(r)_p)$. Hence, the equality

$$\lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(\mathfrak{m}^p \mid J(1)_p, \ldots, J(r)_p)}{p^{d+1}} = \lim_{p\to\infty} \frac{e_{(d_0,\mathbf{d})}(\mathfrak{m} \mid J(1)_p, \ldots, J(r)_p)}{p^{|\mathbf{d}|}}$$

is clear. Finally, the proof is concluded by invoking Theorem 4.7. ∎

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