Rev. Mat. Iberoam. 38 (2022), no. 6, 2011–2032
DOI 10.4171/RMI/1323

A non-local inverse problem with boundary response

Tuhin Ghosh

Abstract. The problem of interest in this article is to study the (non-local) inverse problem of recovering a potential based on the boundary measurement associated with the fractional Schrödinger equation. Let $0 < a < 1$, and let u solve

$$\begin{cases} ((-\Delta)^a + q)\, u = 0 & \text{in } \Omega, \\ \operatorname{supp} u \subseteq \overline{\Omega} \cup \overline{W}, \\ \overline{W} \cap \overline{\Omega} = \emptyset. \end{cases}$$

We show that, by making an exterior to boundary measurement as $\big(u|_W, \frac{u(x)}{d(x)^a}\big|_\Sigma\big)$, it is possible to determine q uniquely in Ω, where $\Sigma \subseteq \partial\Omega$ is a non-empty open subset and $d(x) = d(x, \partial\Omega)$ denotes the boundary distance function.

We also discuss a local characterization of large a-harmonic functions in a ball or in the half space and its applications, which include boundary unique continuation and a local density result.

1. Introduction and main results

In this paper, we address the so-called fractional Calderón problem (see [18]) through the fractional Schrödinger equation, and study the global identifiability of the potential based on the boundary response. So far, the fractional Calderón problem has been studied (see [16–18]) based on the data measured in the exterior of the domain. In [18], it has been shown that one can recover the potential q from the exterior measurements of the non-local Cauchy data $(v|_W, (-\Delta)^a v|_{\widetilde{W}})$, where W and $\widetilde{W}$ are non-empty open subsets of the exterior domain $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$, and v solves

$$\begin{cases} ((-\Delta)^a + q)\, v = 0 & \text{in } \Omega, \\ \operatorname{supp} v \subseteq \overline{\Omega} \cup \overline{W}, \\ \overline{W} \cap \overline{\Omega} = \emptyset. \end{cases}$$

In this paper, we address the inverse problem through introducing the exterior to boundary response map. Based on that, we look for a new global uniqueness result of recovering the potential.

2020 Mathematics Subject Classification: 35R11, 35B60, 35B30.
Keywords: Non-local Calderón problem, exterior to boundary map, unique continuation, Runge approximation.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a non-empty smooth bounded domain, and let $W \subset \Omega_e$ be another non-empty smooth domain such that $\overline{W} \cap \overline{\Omega} = \emptyset$. Let us take $q \in C_c^\infty(\Omega)$ and $f \in C_c^\infty(W)$, which can be extended by 0 outside of W. Let $0 < a < 1$ and consider the following Dirichlet problem: $u \in H^a(\mathbb{R}^n)$ satisfies

$$\begin{cases} ((-\Delta)^a + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{1.1}$$

We recall that there is a countable set of Dirichlet eigenvalues. We will assume that q is such that 0 is not an eigenvalue, that is,

$$\text{if } u \in H^a(\mathbb{R}^n) \text{ solves } ((-\Delta)^a + q)u = 0 \text{ in } \Omega,\ u|_{\Omega_e} = 0, \text{then } u \equiv 0. \tag{1.2}$$

This holds, for example, if $q \geq 0$. Then there exists a unique solution to (1.1) in $H^a(\mathbb{R}^n)$.

Let us define

$$\mathcal{E}_a(\overline{\Omega}) = e^+ d^a(x) C^\infty(\overline{\Omega}),$$

where e^+ denotes the extension by zero outside Ω, and d is a C^∞ function in $\overline{\Omega}$ which is positive in Ω and satisfies $d(x) = \mathrm{dist}(x, \partial\Omega)$ near $\partial\Omega$.

It follows from Section 6 in [18] and [21] that $u \in \mathcal{E}_a(\overline{\Omega})$. Then $\frac{u(x)}{d^a(x)}\big|_{x\in\partial\Omega}$ exists as a function in $C^\infty(\partial\Omega)$, see Subsection 2.3.

We define the *exterior to boundary response map* by

$$\mathcal{A}_q : C_c^\infty(W) \to C^\infty(\partial\Omega), \quad \mathcal{A}_q(f) = \frac{u(x)}{d^a(x)}\Big|_{\partial\Omega}, \tag{1.3}$$

where $u \in \mathcal{E}_a(\overline{\Omega})$ solves (1.1). It is a well-defined map. In Subsection 2.3, we explore various properties of $\mathcal{A}_q$.

In this article, we would like to examine whether

$$\text{the map} \quad q \mapsto \mathcal{A}_q \quad \text{is injective or not.}$$

This is the inverse problem we want to study here.

Further, we introduce the partial boundary data problem: let $\Sigma \subset \partial\Omega$ be a non-empty open set, and let the *partial boundary response map* be

$$\mathcal{A}_q^\Sigma : C_c^\infty(W) \to C^\infty(\Sigma), \quad \mathcal{A}_q^\Sigma(f) = \frac{u(x)}{d^a(x)}\Big|_{\Sigma}. \tag{1.4}$$

Here is our main result.

Theorem 1.1. *Let $\Sigma \subseteq \partial\Omega$ be a non-empty open subset. Suppose that, for $q^1, q^2 \in C_c^\infty(\Omega)$ satisfying the eigenvalue condition (1.2), we have*

$$\mathcal{A}_{q^1}^\Sigma(f) = \mathcal{A}_{q^2}^\Sigma(f), \quad \textit{for all } f \in C_c^\infty(W). \tag{1.5}$$

Then $q^1 = q^2$ in Ω.

Theorem 1.1 is a global uniqueness result for the inverse problem of the fractional Schrödinger equation with both partial exterior ($W \subset \Omega_e$) and boundary data ($\Sigma \subset \partial\Omega$).

As an application, we also solve an inverse problem for a Robin boundary value problem. Let Ω, W, $q \in C_c^\infty(\Omega)$ and $f \in C_c^\infty(W)$ be as in Theorem 1.1. Let $0 < a < 1$, and consider the following Robin boundary value problem for the degenerate elliptic equation in the half space $\mathbb{R}^{n+1}_+ = \{(y, x) \in (0, \infty) \times \mathbb{R}^n\}$:

$$\begin{cases} \nabla \cdot \big(y^{1-2a} \nabla U(y, x)\big) = 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\ \lim_{y \to 0} y^{1-2a} \partial_y U(y, x) + q(x) U(0, x) = 0 & \text{on } \Omega, \\ U(0, x) = \begin{cases} f & \text{on } W, \\ 0 & \text{on } \Omega_e \setminus \overline{W}. \end{cases} \end{cases} \tag{1.6}$$

The above local problem is the Caffarelli–Silvestre extension [7] in the half space $\mathbb{R}^{n+1}_+$ of the non-local problem (1.1) on $\mathbb{R}^n$. It has a non-zero solution $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2a})$. The trace $u(x) = U(0, x) \in H^a(\mathbb{R}^n)$ solves (1.1), where $U(y, x) \in H^1(\mathbb{R}^{n+1}_+, y^{1-2a})$ solves the above problem (1.6). Note that the Neumann derivative satisfies, in the sense of distributions, $\lim_{y \to 0} y^{1-2a} \partial_y U(y, x) = (-\Delta)^a u(x)$ in $\mathbb{R}^n$.

Let $W, \widetilde{W} \subset \Omega_e$ be non-empty subsets. We know from [17, 18] that by measuring the partial Cauchy data $\big(U(0, x)|_W, \lim_{y \to 0} y^{1-2a} \partial_y U(y, x)\,|_{\widetilde{W}}\big)$, one can determine q uniquely in Ω. See also the recent article [14] establishing the same result in a more general setup.

Here we show, as a direct consequence of Theorem 1.1, that it is enough to measure $\big(U(0, x)|_W, \frac{U(0,x)}{d^a(x)}|_\Sigma\big)$ to determine q uniquely in Ω, where $d(x) = d(x, \partial\Omega)$, $x \in \Omega$ denotes the distance function to the boundary, and $\Sigma \subseteq \partial\Omega$ is any non-empty open subset.

Corollary 1.2. *Let Ω, W, Σ and $d(x)$ be as in Theorem 1.1, and let $q_1, q_2 \in C_c^\infty(\Omega)$ satisfy the eigenvalue condition (1.2). Further, let $U_1, U_2 \in H^1(\mathbb{R}^{n+1}_+, y^{1-2a})$ be the solutions of the Robin boundary value problem (1.6) corresponding to the potentials q_1, q_2 respectively, having the same partial Dirichlet data $U_1(0, x)|_W = U_2(0, x)|_W = f \in C_c^\infty(W)$. If we have the same partial boundary data also,*

$$\frac{U_1(0, x)}{d^a(x, \partial\Omega)}\Big|_\Sigma = \frac{U_2(0, x)}{d^a(x, \partial\Omega)}\Big|_\Sigma \quad \textit{for all } f \in C_c^\infty(W),$$

then $q_1 = q_2$ in Ω.

The important observation here is that we are measuring on $W \times \Sigma$, where W is an n-dimensional domain and Σ is an $(n-1)$-dimensional domain, to recover the potential q uniquely in Ω, a n-dimensional domain. Whereas, in the previous case, we were measuring on $W \times \widetilde{W}$, where both W and $\widetilde{W}$ are n-dimensional domains, to recover the potential q uniquely in Ω. So our new result has the merit to obtain the same conclusion, but using one-dimensional less information.

Literature

These type of inverse problems are often addressed as generalized Calderón type inverse problems. In the original Calderón problem [8], the objective was to know about the internal conductivity of an object from the static voltage and current measurements at the boundary. The study of the inverse boundary value problems has a long history, in particular, in the context of electrical impedance tomography, on seismic and medical imaging, inverse scattering problems and so on. We refer to [40] and the references therein for a survey on this topic.

The study of fractional and non-local operators and its related inverse problems is a very active field in recent years. These non-local equations appear in modelling various problems from diffusion process [3], finance [39], image processing [19], biology [33] etc. See [6, 14, 25, 34] for further references. The mathematical study of inverse problems for fractional equations (one dimensional time-space) goes back to [11]; while the multidimensional space-fractional equations, in particular the fractional Calderón problem, begins with the article [18]. Subsequent developments in this particular area include results for low regularity and stability [36, 37], matrix coefficients [16], variable coefficient [13], semilinear equations [27, 31], reconstruction from single measurement [17], shape detection [24], local and non-local lower order perturbation [4, 9, 12, 30], other time dependent equations [28, 38], etc.

Qualitative results on large a-harmonic functions in balls

After discussing the inverse problem, we analyse large a-harmonic functions in balls or in the half space. By providing its local characterization, we actually come up with some qualitative results which include some boundary unique continuation, a local density result to say. For instance, we show the following.

Theorem 1.3 (Boundary unique continuation). *For $0 < a < 1$, let $u \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ be a solution of*

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } B(0,1), \\ \operatorname{supp} u \subseteq \overline{B}(0,1). \end{cases} \tag{1.7}$$

Suppose $\Gamma \subset \partial B(0,1)$ is some non-empty connected open subset such that

$$\frac{u(x)}{(1-|x|^2)^{a-1}}\Big|_\Gamma = \frac{u(x)}{(1-|x|^2)^{a}}\Big|_\Gamma = 0.$$

Then $u \equiv 0$.

Note that the above result also holds for harmonic functions, i.e., for $a = 1$. Let us mention a density result which is known for harmonic functions.

Theorem 1.4 (Density result). *Let $n \geq 2$ and let $u_k \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ solve (1.7), for $k = 1, 2$. Then the set $\{u_1 u_2\}$ is dense in $L^1_{\mathrm{loc}}(B(0,1))$.*

Though the above two results are stated for the ball, the author conjectures that they hold for any bounded domain as well. Since our proof explicitly uses the fact that the given domain is a ball, the case of general domains remains open; this could be an interesting question to look at. Let us formalize those questions here.

Conjectures

(a) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, let $0 < a < 1$, and let $u \in \mathcal{E}_{a-1}(\overline{\Omega})$ be a solution of

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } \Omega, \\ \operatorname{supp} u \subseteq \overline{\Omega}. \end{cases} \tag{1.8}$$

If $\Gamma \subset \partial\Omega$ is some non-empty connected open subset such that

$$\frac{u}{d^{a-1}}\Big|_{\Gamma} = \frac{u}{d^a}\Big|_{\Gamma} = 0,$$

then $u \equiv 0$.

(b) Let $n \geq 2$ and let $u_k \in \mathcal{E}_{a-1}(\overline{\Omega})$ solve (1.8), for $k = 1, 2$. The set $\{u_1 u_2\}$ is dense in $L^1_{\text{loc}}(\Omega)$.

This paper is organized as follows. In Section 2, we review the exterior value problem of the fractional Schrödinger equation, and discuss the well-definedness of the exterior to boundary map $\mathcal{A}_q$. Following that, we introduce and discuss the large a-harmonic functions and the local boundary value problem for the fractional Schrödinger equation. In Section 3, we complete the proof of Theorem 1.1, the solution of the inverse problem. Finally, in Section 4, we discuss the local characterization of large a-harmonic functions in balls and in the half space, and complete the proofs of Theorems 1.3 and 1.4.

2. Preliminaries: Direct problem

We begin with recalling some preliminary results.

2.1. Fractional Laplacian and fractional Sobolev space

Let $0 < a < 1$. The fractional Laplacian operator $(-\Delta)^a$ is defined over the space of Schwartz class functions $\mathscr{S}(\mathbb{R}^n)$ as

$$\forall x \in \mathbb{R}^n, \quad (-\Delta)^a u(x) = \mathscr{F}^{-1}\{|\xi|^{2a}\, \hat{u}(\xi)\}, \quad u \in \mathscr{S}(\mathbb{R}^n) \tag{2.1}$$

where $\hat{\cdot}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse, respectively.

There are a number of equivalent definitions of the fractional Laplacian (see [26]). For instance, it can be given by the Cauchy principal value as $(0 < a < 1)$

$$(-\Delta)^a u(x) = C_{n,a}\ \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2a}}\, dy,$$

where $C_{n,a} = \frac{4^a \Gamma(n/2+a)}{\pi^{n/2}\Gamma(-a)}$. Throughout the paper, Γ stands for the usual Gamma function.

The fractional Laplacian extends as a bounded map

$$(-\Delta)^a : H^s(\mathbb{R}^n) \mapsto H^{s-2a}(\mathbb{R}^n)$$

for $s \in \mathbb{R}$ and $a \in (0,1)$. Here,

$$H^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \widehat{u} \in L^2(\mathbb{R}^n)\}$$

is the fractional Sobolev space, with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions in $\mathbb{R}^n$, equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\langle\xi\rangle^s \widehat{u}\|_{L^2(\mathbb{R}^n)}.$$

Let Ω be an open subset of $\mathbb{R}^n$ which is either bounded with smooth boundary or equal to the half space $\mathbb{R}^n_+$. We introduce the notations r^+ for restriction to Ω and e^+ for extension by zero outside Ω; r^- and e^- are the analogous notations for Ω_e. Then we define the following Sobolev spaces:

$$\overline{H}^s(\Omega) := r^+ H^s(\mathbb{R}^n) \quad \text{and} \quad \dot{H}^s(\overline{\Omega}) := \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\},$$

equipped with the norm

$$\|u\|_{\overline{H}^s(\Omega)} = \inf_{v \in H^s(\mathbb{R}^n),\, v|_\Omega = u} \|v\|_{H^s(\mathbb{R}^n)}.$$

We recall the spaces

$$\begin{aligned} L^2(\mathbb{R}^{n+1}_+, y^{1-2a}) &= \big\{U : \mathbb{R}^{n+1}_+ \mapsto \mathbb{R} \text{ measurable},\ y^{(1-2a)/2} U \in L^2(\mathbb{R}^{n+1}_+)\big\},\\ H^1(\mathbb{R}^{n+1}_+, y^{1-2a}) &= \big\{U \in L^2(\mathbb{R}^{n+1}_+, y^{1-2a}),\ \nabla U \in L^2(\mathbb{R}^{n+1}_+, y^{1-2a})\big\}, \end{aligned}$$

and the trace space of $H^1(\mathbb{R}^{n+1}_+, y^{1-2a})$ at $y=0$ as $H^a(\mathbb{R}^n)$, see [32], Paragraph 5.

2.2. Exterior value problem

Probabilistically, the fractional Laplacian operator $(-\Delta)^a$ represents the infinitesimal generator of a symmetric $2a$-stable Lévy process in the entire space. Here we are interested in the restriction of $(-\Delta)^a$ to a bounded domain Ω. For example, one can think of the homogeneous Dirichlet exterior value problem for the fractional Laplacian operator (e.g., $(-\Delta)^a v =$ in Ω, and $v = 0$ in Ω_e) which represents the infinitesimal generator of a symmetric $2a$-stable Lévy process for which particles are killed upon leaving the domain Ω (see [2]).

Existence, uniqueness and stability

Let $q \in L^\infty(\Omega)$. Then, for a given $h \in \widetilde{H}^a(\Omega)^*$ and $f \in H^a(\mathbb{R}^n)$, there exists a (weak) solution $u \in H^a(\mathbb{R}^n)$ solving (see [18])

$$\begin{cases} ((-\Delta)^a + q)u = h & \text{in } \Omega,\\ u = f & \text{in } \Omega_e. \end{cases} \tag{2.2}$$

Eigenvalue condition

Let $q \in L^\infty(\Omega)$ be such that

(2.3) the problem $((-\Delta)^a + q)w = 0$ in Ω, $w = 0$ in Ω_e has a unique solution $w = 0$.

This implies the problem (2.2) has a unique solution. In addition, we have the following H^a stability estimate

$$\|u\|_{H^a(\mathbb{R}^n)} \leq C_{n,s}\big(\|h\|_{\widetilde{H}^a(\Omega)^*} + \|f\|_{H^a(\mathbb{R}^n)}\big), \tag{2.4}$$

for some constant $C_{n,a} > 0$, independent of h and f.

Regularity

We write $\mathcal{E}_\mu(\overline{\Omega})$ for $e^+ d^\mu C^\infty(\overline{\Omega})$, with $\mathscr{R}(\mu) > -1$, and where $d(x)$ is a smooth positive extension into Ω of $\operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. In general, for $\mu \in \mathbb{C}$ with $\mathscr{R}(\mu) > -1$ one has $\mathcal{E}_{\mu-k}(\overline{\Omega}) = \operatorname{span} D^{(k)}\mathcal{E}_\mu(\overline{\Omega})$, where $D^{(k)}$ denotes the smooth differential operator of order $k \in \mathbb{N}$.

Let $q \in C_c^\infty(\Omega)$ and let $u \in \dot{H}^a(\mathbb{R}^n)$ solve the homogeneous Dirichlet problem

$$\begin{cases} ((-\Delta)^a + q)\, u = h & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases} \tag{2.5}$$

Then, due to the results in [35] and [21], we have the following:

- $h \in L^\infty(\Omega) \Longrightarrow u \in d^a\, C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ satisfying

$$\left\| \frac{u(x)}{d^a(x)}\Big|_{\partial\Omega} \right\|_{C^\alpha(\partial\Omega)} \leq C_{n,a}\, \|h\|_{L^\infty(\Omega)}. \tag{2.6}$$

- $h \in C^\infty(\overline{\Omega}) \Longleftrightarrow u \in \mathcal{E}_a(\overline{\Omega})$.

2.3. Well-definedness of the map $\mathcal{A}_q$

Let us now discus the well-definedness of the map $\mathcal{A}_q$, introduced earlier in (1.3):

$$\mathcal{A}_q : C_c^\infty(W) \to C^\infty(\partial\Omega), \quad \mathcal{A}_q(f) = \frac{u(x)}{d^a(x)}\Big|_{\partial\Omega},$$

where $u \in H^a(\mathbb{R}^n)$ uniquely solves

$$\begin{cases} ((-\Delta)^a + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases} \tag{2.7}$$

Here W is a smooth open set in Ω_e satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$, and $f \in C_c^\infty(W)$, extended by 0 outside. Therefore, $w = (u - f) \in \dot{H}^a(\mathbb{R}^n)$ is the solution of

$$\begin{cases} ((-\Delta)^a + q)\, w = -(-\Delta)^a f & \text{in } \Omega, \\ w = 0 & \text{in } \Omega_e. \end{cases}$$

Then, for $h := -(-\Delta)^a f \in C^\infty(\overline{\Omega})$, it follows that $h \in C^\infty(\overline{\Omega})$ since, by definition,

$$h(x) = C_{n,a} \int_W \frac{f(y)}{|x-y|^{n+2a}}\, dy, \quad x \in \overline{\Omega},$$

has no singularity as $\overline{W} \cap \overline{\Omega} = \emptyset$. Therefore, by the above regularity result, $w \in \mathcal{E}_a(\overline{\Omega})$ and $\frac{w(x)}{d^a(x)}\big|_{\partial\Omega} \in C^\infty(\overline{\Omega})$. But since $\frac{w(x)}{d^a(x)}\big|_{\overline{\Omega}} = \frac{u(x)}{d^a(x)}\big|_{\overline{\Omega}}$, we have $u \in \mathcal{E}_a(\overline{\Omega})$ and in particular $\frac{u(x)}{d^a(x)}\big|_{\partial\Omega} \in C^\infty(\partial\Omega)$, where u solves (2.7). This shows that the map $f \mapsto \mathcal{A}_q(f)$ is well-defined from $C_c^\infty(W)$ to $C^\infty(\partial\Omega)$.

Boundedness of $\mathcal{A}_q$

The estimate (2.6) asserts that $\mathcal{A}_q$ can be considered as a bounded map with respect to L^∞-norm, i.e.,

$$\|\mathcal{A}_q(f)\|_{L^\infty(\partial\Omega)} \le C_{n,a} \left(\operatorname{dist}(\overline{\Omega}, \overline{W})\right)^{-2a} \|f\|_{L^\infty(W)},$$

where we use the following equivalence estimate (see for instance Lemma 2.4 in [10]): since $\overline{\Omega} \cap \overline{W} = \emptyset$,

$$(\operatorname{dist}(x, \overline{W}))^{-2a} \sim \int_W \frac{1}{|x-y|^{n+2a}}\, dy, \quad x \in \overline{\Omega}.$$

Range characterization of $\mathcal{A}_q$

Next, let us consider the range of the map $\mathcal{A}_q$,

$$\mathcal{R}_{\partial\Omega} := \left\{ \frac{u(x)}{d^a(x)}\Big|_{\partial\Omega} \,:\, u \text{ solves (2.7)},\ f \in C_c^\infty(W) \right\}. \tag{2.8}$$

We have the following result, proved in [18], Theorem 1.3.

Proposition 2.1. *$\mathcal{R}_{\partial\Omega}$ is dense in $C^\infty(\partial\Omega)$.*

Remark 2.2. As $\mathcal{A}_q$ is a continuous linear map, by the open mapping theorem it is an open map. But it may not be injective. On the other hand, since its range $\mathcal{R}_{\partial\Omega}$ is dense, we can assert that the pre-image set is also dense in the domain of the map, i.e.,

$$\mathcal{Z} = \{0\} \cup \{f \in C_c^\infty(W) : \mathcal{A}_q(f) \neq 0\} \quad \text{is dense in } C_c^\infty(W).$$

2.4. Boundary value problem

In this subsection, we introduce the local boundary value problem for the fractional Schrödinger operator, which plays a key role to solve our inverse problem.

2.4.1. Examples of large a-harmonic functions (cf. [1]). (1) The function

$$u_\sigma(x) = \begin{cases} c_{n,a} \frac{1}{(1-|x|^2)^\sigma} & \text{in } B(0,1), \\ c_{n,a+\sigma} \frac{1}{(|x|^2-1)^\sigma} & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1), \end{cases} \qquad \sigma \in (0, 1-a),$$

solves

$$(-\Delta)^a u_\sigma = 0 \quad \text{in } B(0,1).$$

Clearly, u_σ blows up over the boundary $\partial B(0,1)$.

(2) The function

$$u_{1-a}(x) = \begin{cases} c_{n,a}\, \frac{1}{(1-|x|^2)^{1-a}} & \text{in } B(0,1), \\ 0 & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1), \end{cases} \tag{2.9}$$

solves

$$(-\Delta)^a u_{1-a} = 0 \quad \text{in } B(0,1).$$

Over the boundary $\partial B(0,1)$, the limit

$$\frac{u_{1-a}}{d(x)^{a-1}}\Big|_{\partial B(0,1)} = \lim_{|x|\to 1} \frac{u_{1-a}}{(1-|x|)^{a-1}} = \frac{c_{n,a}}{2^{1-a}}$$

is a non-zero constant, assuming $c_{n,a} \neq 0$.

2.4.2. Fractional Schrödinger equation. Motivated by the above examples, we would like to study the following general problem:

$$\begin{cases} ((-\Delta)^a + q)\, u = 0 & \text{in } \Omega, \\ \operatorname{supp} u \subseteq \overline{\Omega}. \end{cases} \tag{2.10}$$

We will specify the boundary condition later. Let us introduce the order-reducing operators of plus/minus type, see [20–22]. We define

$$\Xi^t_\pm := \operatorname{Op}(\chi^t_\pm) \text{ on } \mathbb{R}^n, \quad \chi^t_\pm = \left(\langle \xi' \rangle \pm i\xi_n\right)^t.$$

These symbols extend analytically in ξ_n to $\operatorname{Im} \xi_n \lessgtr 0$. Hence, by the Paley–Wiener theorem, $\Xi^t_\pm$ preserve supports in $\overline{\mathbb{R}}^n_\pm$. Then, for all $s \in \mathbb{R}$,

$$\Xi^t_+ : \dot{H}^s(\overline{\mathbb{R}}^n_+) \to \dot{H}^{s-t}(\overline{\mathbb{R}}^n_+), \quad r^+ \Xi^t_- e^+ : \overline{H}^s(\mathbb{R}^n_+) \to \overline{H}^{s-t}(\mathbb{R}^n_+).$$

In fact, Ξ^t_+ and $r^+\Xi^t_- e^+$ are disjoint. The corresponding inverses are Ξ^{-t}_+ and $r^+\Xi^{-t}_- e^+$, respectively.

The operator Ξ^t_+ maps $\mathcal{E}_t(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'$ to $e^+ C^\infty(\overline{\mathbb{R}}^n_+)$, with the property that, for $u \in \mathcal{E}_t(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'$,

$$\begin{cases} \gamma_0(\Xi^t_+ u) = \Gamma(t+1)\, \gamma_0(u/x_n^t), & \text{(Dirichlet)}, \\ \gamma_1(\Xi^t_+ u) = \Gamma(t+2)\, \gamma_0(\partial_{x_n}(u/x_n^t)), & \text{(Neumann)}, \end{cases}$$

where γ_0 and γ_1 are the Dirichlet and Neumann trace operator, respectively.

Following that, we define the $a(s)$-transmission spaces, which can be thought of as a generalization of $\mathcal{E}_a$ spaces. We define, as in [21],

$$H^{a(s)}(\overline{\mathbb{R}}^n_+) := \Xi^{-a}_+ \overline{H}^{s-a}(\mathbb{R}^n_+), \quad \text{for } s - a > -\frac{1}{2}.$$

Note that generally $e^+\overline{H}^{s-a}(\mathbb{R}^n_+)$ has a jump at $x_n = 0$; it is mapped by Ξ^{-a}_+ to a singularity of the type x_n^a. We have the following properties:

- The inclusion $H^{a(s)}(\overline{\mathbb{R}}^n_+) \subset \dot{H}^{a-1/2}(\overline{\mathbb{R}}^n_+)$ is continuous.
- The inclusions $\dot{H}^{a-1/2}(\overline{\mathbb{R}}^n_+) \subseteq H^{a(s)}(\overline{\mathbb{R}}^n_+) \subseteq H^t_{\mathrm{loc}}(\mathbb{R}^n_+)$ are continuous, i.e., multiplication by any $\chi \in C_c^\infty(\mathbb{R}^n_+)$ is bounded $H^{a(s)}(\overline{\mathbb{R}}^n_+) \mapsto \overline{H}^a(\mathbb{R}^n_+)$.
- The set $\cap_{s-a-1/2>0} H^{a(s)}(\overline{\mathbb{R}}^n_+) = \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ is dense in $H^{a(s)}(\overline{\mathbb{R}}^n_+)$.

Therefore, one has the following characterization:

$$H^{a(s)}(\overline{\mathbb{R}}^n_+) \begin{cases} = \dot{H}^s(\overline{\mathbb{R}}^n_+) & \text{if } -1/2 < s-a < 1/2, \\ \subseteq e^+ x_n^a \, \overline{H}^{s-a}(\mathbb{R}^n_+) + \dot{H}^s(\overline{\mathbb{R}}^n_+) & \text{if } s-a > 1/2, \end{cases}$$

with $\dot{H}^s(\overline{\mathbb{R}}^n_+)$ replaced by $\dot{H}^{s-\epsilon}(\overline{\mathbb{R}}^n_+)$ if $s-a-1/2 \in \mathbb{N} \cup \{0\}$.

Remark 2.3. It is shown in [21] that in the homogeneous Dirichlet problem (2.5), we actually have $h \in H^{s-2a}(\Omega) \iff u \in H^{a(s)}(\overline{\Omega})$.

Now we study the non-homogeneous boundary value problem.

Dirichlet boundary value problem

Let $d(x)$ be a smooth positive extension of $\mathrm{dist}(x, \partial\Omega)$ into Ω, near the boundary $\partial\Omega$. The trace map

$$\begin{aligned} \gamma_{a-1,0} : \mathcal{E}_{a-1}(\overline{\Omega}) &\to C^\infty(\partial\Omega) \\ u &\mapsto \Gamma(a)\gamma_0\left(\frac{u}{d^{a-1}}\right) \end{aligned}$$

extends to $\gamma_{a-1,0}$: $H^{(a-1)(s)}(\overline{\Omega}) \to H^{s-a+1/2}(\partial\Omega)$ for $s-a+1/2 > 0$. Moreover, it becomes a bijection as

$$\gamma_{a-1,0} : H^{(a-1)(s)}(\overline{\Omega})/H^{a(s)}(\overline{\Omega}) \to H^{s-a+1/2}(\partial\Omega).$$

With that in hand, we address the following non-homogeneous Dirichlet problem:

$$\begin{cases} ((-\Delta)^a + q)\, u = h & \text{in } \Omega, \\ \frac{u(x)}{d(x)^{a-1}} = f(x) & \text{on } \partial\Omega, \\ u = 0 & \text{in } \Omega_e, \end{cases} \tag{2.11}$$

where $h \in H^{s-2a}(\Omega)$ and $f \in H^{s-a+1/2}(\partial\Omega)$. Then we have the following result.

Proposition 2.4 (Non-homogeneous Dirichlet problem, [1, 20]). *Let $q \in C_c^\infty(\Omega)$ and let $s-a+1/2 > 0$. Then*

$$\{r^+ ((-\Delta)^a + q)\, , \gamma_{a-1,0}\} : H^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}^{s-2a}(\Omega) \times H^{s-2a+1/2}(\partial\Omega)$$

is a Fredholm mapping.

We also have the following regularity result:

$$(h, f) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega) \Longrightarrow u \in \mathcal{E}_{a-1}(\overline{\Omega}).$$

Remark 2.5. Under the eigenvalue condition (2.3), the solution of (2.11) turns out to be unique. Indeed, we can write $u = v + w$, where $v \in H^{(a-1)(s)}(\overline{\Omega})$ uniquely (the a-harmonic function in Ω) solves

$$\begin{cases} (-\Delta)^a v = 0 & \text{in } \Omega, \\ \frac{v(x)}{d(x)^{a-1}} = f(x) & \text{on } \partial\Omega, \\ v = 0 & \text{in } \Omega_e, \end{cases}$$

and w solves

$$\begin{cases} ((-\Delta)^a + q)\, w = h - qv & \text{in } \Omega, \\ w = 0 & \text{in } \Omega_e, \end{cases}$$

for $q \in C_c^\infty(\Omega)$. Due to the eigenvalue condition (2.3), $w \in H^{a(s)}(\overline{\Omega})$ is unique. Therefore, the solution $u = v + w \in H^{(a-1)(s)}(\overline{\Omega})$ of (2.11) is also unique.

Neumann boundary value problem

One can also study the following Neumann problem:

$$\begin{cases} ((-\Delta)^a + q)\, u = h & \text{in } \Omega \\ \partial_\nu\left(\frac{u(x)}{d(x)^{a-1}}\right) = f(x) & \text{on } \partial\Omega, \\ u = 0 & \text{in } \Omega_e, \end{cases} \tag{2.12}$$

where $h \in H^{s-2a}(\Omega)$ and $f \in H^{s-a-1/2}(\partial\Omega)$. Note that the function space for f follows from the Taylor expansion of u/d^{a-1} over the boundary (normalized with Gamma coefficients). If we denote

$$\begin{cases} \gamma_{a-1,0}\, u = u_0 := \Gamma(a)\, \gamma_0\left(\frac{u}{d^{a-1}}\right) & \text{(Dirichlet)}, \\ \gamma_{a-1,1}\, u = u_1 := \Gamma(a+1)\, \gamma_0\left(\partial_\nu\left(\frac{u}{d^{a-1}}\right)\right) & \text{(Neumann)}, \end{cases}$$

then

$$\gamma_{a-1,1}\, u = \gamma_{a,0}\, u', \qquad \text{where } u' = u - \frac{1}{\Gamma(a)}\, d^{a-1} u_0. \tag{2.13}$$

In particular, when $u_0 = 0$, then $\gamma_{a-1,1}\, u = \gamma_{a,0}\, u = \Gamma(a+1)\frac{u}{d^a}$.

Proposition 2.6 (Non-homogeneous Neumann problem, [20, 21]). *Let $q \in C_c^\infty(\Omega)$ and let $s - a - 1/2 > 0$. Then*

$$\{r^+\, ((-\Delta)^a + q)\, , \gamma_{a-1,1}\} : H^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}^{s-2a}(\Omega) \times H^{s-2a+1/2}(\partial\Omega)$$

is a Fredholm mapping.

We also have a regularity result similar to that of the previous case:

$$(h, f) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega) \Longrightarrow u \in \mathcal{E}_{a-1}(\overline{\Omega}).$$

Remark 2.7. If $\frac{u(x)}{d(x)^{a-1}}|_{\partial\Omega} = 0$ on $\partial\Omega$ in Proposition 2.4 or in Proposition 2.6, then the solutions of the respective Dirichlet or Neumann problem lie in the space $H^{a(s)}(\overline{\Omega})$.

Integration by parts formula

Let us recall the following result from [23], Theorem 5.1. Let $u, v \in H^{(a-1)(s)}(\overline{\Omega})$. Then, for $s - a - 1/2 > 0$,

$$\int_\Omega ((-\Delta)^a u\, \overline{v} - u\, \overline{(-\Delta)^a v})\, dx = \Gamma(a)\,\Gamma(a+1) \int_{\partial\Omega} (u_1\, \overline{v_0} - u_0\, \overline{v_1})\, dx \tag{2.14}$$

with $u_0 = \frac{u}{d^{a-1}}\big|_{\partial\Omega}$, $u_1 = \partial_\nu\big(\frac{u}{d^{a-1}}\big)\big|_{\partial\Omega}$, $v_0 = \frac{v}{d^{a-1}}\big|_{\partial\Omega}$, $v_1 = \partial_\nu\big(\frac{v}{d^{a-1}}\big)\big|_{\partial\Omega}$.

When $u \in H^{(a-1)(s)}(\overline{\Omega})$, $v \in H^{a(s)}(\overline{\Omega})$ are real valued, then (2.14) becomes

$$\int_\Omega v\, (-\Delta)^a u - \int_\Omega u\, (-\Delta)^a v = -\Gamma(a)\,\Gamma(a+1) \int_{\partial\Omega} \frac{u}{d^{a-1}}\, \frac{v}{d^a}. \tag{2.15}$$

This is because $v_0 = 0$ and $v_1 = \frac{v}{d^a}|_{\partial\Omega}$, as $v \in H^{a(s)}(\overline{\Omega})$.

Observe that the boundary contribution is completely local. This formula (2.15) was obtained in [1] for the fractional Laplacian operator, and later was generalized for any classical pseudo-differential operator of order $2a$ (not necessarily elliptic) in [22].

3. Inverse problem: Uniqueness result

Proof of Theorem 1.1. Let $f \in C_c^\infty(W)$; we extend it by 0 elsewhere. Let $v_f^k = v^k \in H^a(\mathbb{R}^n) \cap \mathcal{E}_a(\overline{\Omega})$, for $k = 1, 2$ solve

$$\begin{cases} \big((-\Delta)^a + q^k\big)\, v^k = 0 & \text{in } \Omega, \\ v^k = f & \text{in } \Omega_e, \end{cases}$$

and let $\Sigma \subset \partial\Omega$ be a non-empty open set. By the hypothesis, we have

$$\frac{v_f^1}{d^a}\Big|_\Sigma = \frac{v_f^2}{d^a}\Big|_\Sigma \quad \text{for all } f \in C_c^\infty(W), \tag{3.1}$$

and we want to show that this implies $q^1 = q^2$ in Ω.

Let $g \in C^\infty(\partial\Omega)$ be some non-zero function such that $\operatorname{supp} g \subseteq \overline{\Sigma}$, and let $u^k \in \mathcal{E}_{a-1}(\overline{\Omega})$, $k = 1, 2$, be the non-zero unique solutions of the inhomogeneous Dirichlet problem

$$\begin{cases} \big((-\Delta)^a + q^k\big)\, u^k = 0 & \text{in } \Omega, \\ \frac{u^k(x)}{d(x)^{a-1}} = g & \text{on } \partial\Omega, \\ u^k = 0 & \text{in } \Omega_e. \end{cases} \tag{3.2}$$

Using the integration by-parts formula (2.15) for $u^k \in \mathcal{E}_{a-1}(\overline{\Omega})$ and $w^k := (v^k - f) \in \mathcal{E}_a(\overline{\Omega})$, we have

$$\int_\Omega w^k\, (-\Delta)^a u^k - \int_\Omega u^k\, (-\Delta)^a w^k = -\Gamma(a)\,\Gamma(a+1) \int_{\partial\Omega} \frac{u^k}{d^{a-1}}\, \frac{w^k}{d^a}. \tag{3.3}$$

Since w^k satisfies

$$\begin{cases} ((-\Delta)^a + q^k)w^k = -(-\Delta)^a f & \text{in } \Omega, \\ \frac{w^k}{d^a}|_{\partial\Omega} = \frac{v^k}{d^a}|_{\partial\Omega}, \end{cases}$$

together with equation (3.2), we rewrite the identity (3.3) as

$$\int_\Omega u^k\,(-\Delta)^a f = -\Gamma(a)\,\Gamma(a+1)\int_\Sigma g\,\frac{v^k}{d^a}, \tag{3.4}$$

since $\operatorname{supp} g \subseteq \Sigma \subseteq \partial\Omega$. Then we can conclude from our hypothesis (3.1) that

$$\int_\Omega (u^1 - u^2)\,(-\Delta)^a f = 0, \quad \text{for all } f \in C_c^\infty(W).$$

But since the difference $(u_1 - u_2) \in H^a(\mathbb{R}^n)$ (cf. Remark 2.7), it follows that

$$\int_W f\,(-\Delta)^a(u^1 - u^2) = 0, \quad \text{for all } f \in C_c^\infty(W),$$

and hence,

$$(-\Delta)^a(u^1 - u^2) = 0 \quad \text{in } W.$$

Since $(u^1 - u^2) = 0$ in W as well (cf. (3.2)), by the unique continuation of the fractional Laplacian (see Theorem 1.2 in [18]), we actually have

$$u^1 = u^2 \quad \text{in } \mathbb{R}^n.$$

This leads to the following identity from (3.2):

$$(q^1 - q^2)u^1 = 0 \quad \text{in } \Omega.$$

Moreover, as u^1 is non-zero in every open set in Ω, thanks to the unique continuation of the fractional Laplacian (see Theorem 1.2 in [18]), and its smoothness in the set containing the support of $q^1, q^2 \in C_c^\infty(\Omega)$, we conclude

$$q^1 \equiv q^2.$$

This completes the proof. ■

4. Local characterization of large *a*-harmonic functions and applications

As our proof relies on the class of solutions of the non-local boundary valued problem, we want to further discuss some new results about them.

Let $u \in \mathcal{E}_{a-1}(\overline{\Omega})$ be the solution of

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } \Omega, \\ \frac{u(x)}{d(x)^{a-1}} = f(x) & \text{on } \partial\Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases}$$

As it turns out, it is possible to give a local characterization of the solution of the above non-local problem, in terms of the solution of a local boundary value problem, whenever the domain Ω is a ball or the half-space $\mathbb{R}^n_+$. For instance, one has the following.

Proposition 4.1 (Local characterization in a ball). *Let $n \geq 1$ and let $u \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ be the solution of the problem*

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } B(0,1), \\ \frac{u(x)}{(1-|x|^2)^{a-1}}\Big|_{\partial B(0,1)} = f \in C^\infty(\partial B(0,1)), & \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1). \end{cases} \tag{4.1}$$

Then $\frac{u(x)}{(1-|x|^2)^{a-1}} \in C^\infty(\overline{B}(0,1))$ solves

$$\begin{cases} (-\Delta)\, \frac{u(x)}{(1-|x|^2)^{a-1}} = 0 & \text{in } B(0,1), \\ \frac{u(x)}{(1-|x|^2)^{a-1}}\Big|_{\partial B(0,1)} = f & \text{on } \partial B(0,1)), \end{cases} \tag{4.2}$$

and vice-versa.

The work of [5] contains such characterization, see Example 1 there. To be self-contained, we present the proof later. We also present a similar result in the half space, appeared in [20]. For that, we need to introduce some related notions.

Dirichlet Green's kernel

Let us recall the Green kernel associated with the fractional Laplacian operator in a bounded domain (cf. [1]).

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then the Green kernel $G^a_\Omega(\cdot,\cdot)$ is defined as

$$G^a_\Omega(x,z) = c_{n,-a}\, \frac{1}{|x-z|^{n-2a}} - H^a_\Omega(x,z), \quad x,z \in \Omega,\ x \neq z,$$

where $H^a_\Omega \in H^a(\mathbb{R}^n) \cap C^a(\mathbb{R}^n)$ solves

$$\begin{cases} (-\Delta)^a H^a_\Omega(x,\cdot) = 0 & \text{in } \Omega, \\ H^a_\Omega(x,z) = c_{n,-a}\, \frac{1}{|x-z|^{n-2a}} & \text{in } \Omega_e. \end{cases}$$

Then $G^a_\Omega(x,\cdot)$ is known as the Dirichlet Green kernel for the bounded domain Ω, solving

$$\begin{cases} (-\Delta)^a_x G^a_\Omega(x,\cdot) = \delta_z(x) & \text{in } \Omega, \\ G^a_\Omega(x,z) = 0 & \text{in } \Omega_e. \end{cases}$$

Here we mention that $G^a_\Omega(x,z) = G^a_\Omega(z,x)$ for all $z, x \in \mathbb{R}^n$, and that the following limit exists (cf. [1], (c) Proposition 2, and Remark 2):

$$\forall\, \omega \in \partial\Omega,\ x \in \Omega, \quad D^a G_\Omega(x,\omega) := \lim_{\Omega \ni z \to \omega} \frac{G^a_\Omega(x,z)}{d^a(z)}. \tag{4.3}$$

Poisson formula

Let $n \geq 1$ and let $u \in \mathcal{E}_{a-1}(\overline{\Omega})$ be the solution of the following problem:

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } \Omega, \\ \frac{u(x)}{(d(x))^{a-1}}\Big|_{\partial\Omega} = f \in C^\infty(\partial\Omega), & \\ u = 0 & \text{in } \Omega_e. \end{cases} \tag{4.4}$$

Using the integration by parts formula (2.15), the solution of (4.4) can be then expressed as

$$u(x) = \int_{\partial\Omega} D^a G_\Omega(x,\omega) f(\omega)\, dS(\omega). \tag{4.5}$$

This formula is also derived in Theorem 1.2 of [1].

Ω is a ball

In the particular case where $\Omega = B(\theta, r)$ is a ball of radius r, centered at $\theta \in \mathbb{R}^n$, we have for $n \geq 2$,

$$G^a_{B(\theta,r)}(x,z) = \begin{cases} \widetilde{c}_{n,a} \frac{1}{|z-x|^{n-2a}} \left(\int_0^{R_0(x,z)} \frac{t^{a-1}}{(1+t)^{n/2}}\, dt \right), & x, z \in B(\theta,r),\ x \neq z, \\ 0 & \text{in } \mathbb{R}^n \setminus B(\theta, r), \end{cases} \tag{4.6}$$

where

$$R_0(x,z) = \frac{(r^2 - |x-\theta|^2)(r^2 - |z-\theta|^2)}{r^2 |x-z|^2} \tag{4.7}$$

and $\widetilde{c}_{n,a}$ is some constant depending only on n and a.

Lemma 4.2. *In the case of a ball, we have, for all $\omega \in \mathbb{S}^{n-1}$, $x \in B(0,1)$,*

$$\lim_{B(0,1) \ni z \to \omega} \frac{G^a_{B(0,1)}(x,z)}{(1-|z|^2)^a} = k_n \frac{(1-|x|^2)^a}{|x-\omega|^n}, \tag{4.8}$$

where $\kappa_n = \frac{1}{n\alpha(n)}$, and $\alpha(n)$ is the volume of the unit ball.

Proof. Let us recall the expression for $G^a_{B(0,1)}$ given in (4.6), where now

$$R_0(x,z) = \frac{(1-|x|^2)(1-|z|^2)}{|x-z|^2}.$$

Note that, for $t \in [0, R_0]$, one has

$$\frac{t^{a-1}}{(1+R_0)^{n/2}} \leq \frac{t^{a-1}}{(1+t)^{n/2}} \leq t^{a-1}.$$

Hence, it follows that

$$\frac{1}{(1+R_0)^{n/2}} \frac{R_0^a}{a} \leq \int_0^{R_0(x,z)} \frac{t^{a-1}}{(1+t)^{n/2}}\, dt \leq \frac{R_0^a}{a}.$$

Then from the expression of $R_0(x;z)$, we obtain, for all $\omega \in \mathbb{S}^{n-1}$, $x \in B(0,1)$,

$$\lim_{B(0;1)\ni z\to\omega} \frac{G^a_{B(0,1)}(x,z)}{(1-|z|^2)^a} = \frac{\widetilde{c}_{n,a}}{a}\,\frac{(1-|x|^2)^a}{|x-\omega|^n}$$

and $\kappa_n := \widetilde{c}_{n,a}/a = 1/(n\alpha(n))$. This proves the lemma. ∎

Now we are in the position to present the proof of Proposition 4.1.

Proof of Proposition 4.1. We use (4.8) for the expression of $D^a G_{B(0,1)}(x,\omega)$ to write the Poisson formula (4.5) as

$$u(x) = \frac{1}{n\alpha(n)}(1-|x|^2)^a \int_{\partial B(0,1)} \frac{f(\omega)}{|x-\omega|^n}\,dS(\omega), \quad x \in B(0,1), \tag{4.9}$$

or

$$\frac{u(x)}{(1-|x|^2)^{a-1}} = \frac{1}{n\alpha(n)}(1-|x|^2) \int_{\partial B(0,1)} \frac{f(\omega)}{|x-\omega|^n}\,dS(\omega), \quad x \in B(0,1).$$

The above expression is nothing but the Poisson integral formula of the harmonic function which solves (4.2),

$$(-\Delta)\frac{u(x)}{(1-|x|^2)^{a-1}} = 0 \quad \text{in } B(0,1), \qquad \frac{u(x)}{(1-|x|^2)^{a-1}} = f \quad \text{on } \partial B(0,1).$$

Conversely, let us define $v \in C^\infty(\overline{B}(0,1))$ by the Poisson integral as

$$v(x) = \frac{1}{n\alpha(n)}(1-|x|^2) \int_{\partial B(0,1)} \frac{f(\omega)}{|x-\omega|^n}\,dS(\omega), \quad x \in B(0,1)$$

which solves

$$(-\Delta)v = 0 \quad \text{in } B(0,1), \qquad v = f \in C^\infty(\partial B(0,1)).$$

Then the function u defined, for $x \in B(0,1)$, as

$$u(x) = (1-|x|^2)^{a-1}v(x) = \frac{1}{n\alpha(n)}(1-|x|^2)^a \int_{\partial B(0;1)} \frac{f(\omega)}{|x-\omega|^n}\,dS(\omega),$$

solves (4.1), due to (4.9). This completes the proof. ∎

Remark 4.3 (Ω is the half space). There is a similar connection between null-solutions for $(1-\Delta)^a$ and $(1-\Delta)$ in the case when $\Omega = \mathbb{R}^n_+$ (here we write $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1})$). Let K_0 be the Poisson operator with symbol $(\langle\xi'\rangle + i\xi_n)^{-1}$ (that is, $K_0\colon f(x') \mapsto \mathcal{F}^{-1}_{\xi\to x}[(\langle\xi'\rangle + i\xi_n)^{-1}\mathcal{F}_{x'\to\xi'} f]$). It solves the problem

$$\begin{cases} r^+(1-\Delta)v = 0 & \text{in } \mathbb{R}^n_+, \\ \gamma_0 v = f & \text{on } \mathbb{R}^{n-1}, \\ v = 0 & \text{in } \mathbb{R}^n_-. \end{cases} \tag{4.10}$$

Grubb shows that the operator $K'_{a-1,0} = x_n^{a-1} K_0$ solves the problem

$$
\begin{cases} r^+(1-\Delta)^a u = 0 & \text{in } \mathbb{R}^n_+, \\ \gamma_0\big(\frac{u}{x_n^{a-1}}\big) = f & \text{on } \mathbb{R}^{n-1}, \\ u = 0 & \text{in } \mathbb{R}^n_-. \end{cases} \tag{4.11}
$$

The operator called $K_{a-1,0}$ in Remark 2.10 of [20] equals to

$$
K_{a-1,0} = \frac{1}{\Gamma(a)} K'_{a-1,0} = \frac{1}{\Gamma(a)} x_n^{a-1} K_0,
$$

normalizing a Gamma-factor (which is denoted as K_0^{a-1} in some later publications). It is shown in [20], (A.13), Appendix A, that

$$
K_{a-1,0} = \Xi_+^{a-1} K_0
$$

which is, by Remark 2.10 in [20],

$$
K_{a-1,0} = \Xi_+^{a-1} K_0 = e^+ c_{a-1} \, x_n^{a-1} K_0,
$$

where c_{a-1} is some non-zero constant and $e^+ K_0$ is understood as mapping of functions in $\mathbb{R}^n_+$.

The operator $K'_{a-1,0}$ maps, for $s > -1/2$,

$$
K'_{a-1,0} : H^{s-1/2}(\mathbb{R}^{n-1}) \mapsto x_n^{a-1} e^+ \overline{H}^s(\mathbb{R}^n_+) \cap H^{(a-1)(s+a-1)}(\overline{\mathbb{R}^n_+})
$$

and maps

$$
K'_{a-1,0} : C_c^\infty(\mathbb{R}^{n-1}) \mapsto \mathcal{E}_{a-1}(\overline{\mathbb{R}^n_+}).
$$

The remarkable fact that this Poisson-like operator for $(1-\Delta)^a$ is just x_n^{a-1} times the Poisson operator for $(1-\Delta)$ allows to reach the conclusion that

$$
u \textit{ is a solution of } (4.11) \textit{ if and only if } v(x) := \frac{u(x)}{x_n^{a-1}} \textit{ solves } (4.10).
$$

As mentioned in the mapping properties, if $f \in C_c^\infty(\mathbb{R}^{n-1})$, then u is in $\mathcal{E}_{a-1}(\overline{\mathbb{R}^n_+})$ solving (4.11); And if $f \in H^{s-1/2}(\mathbb{R}^{n-1})$ for $s > -1/2$, then $u \in x_n^{a-1} e^+ \overline{H}^s(\mathbb{R}^n_+) \cap H^{(a-1)(s+a-1)}(\overline{\mathbb{R}^n_+})$ solving (4.11).

4.1. Applications

Based on the above characterization, we discuss some qualitative results. We present it for balls only; similar claims can be proved for the half space as well.

Unique continuation principle

We begin with the following boundary unique continuation result.

Proposition 4.4 (Boundary UCP). *Let* $u \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ *be a solution of*

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } B(0,1), \\ \operatorname{supp} u \subseteq \overline{B}(0,1), \end{cases} \tag{4.12}$$

and let $\Gamma \subset \partial B(0,1)$ *be some non-empty connected open subset such that*

$$\frac{u(x)}{(1-|x|^2)^{a-1}}\Big|_\Gamma = \frac{u(x)}{(1-|x|^2)^{a}}\Big|_\Gamma = 0.$$

Then $u \equiv 0$.

Proof. Since $\frac{u(x)}{(1-|x|^2)^{a-1}}|_\Gamma = 0$, we have $\partial_\nu\big(\frac{u(x)}{(1-|x|^2)^{a-1}}\big)\big|_\Gamma = \frac{u(x)}{(1-|x|^2)^{a}}\big|_\Gamma$. The rest follows from the boundary unique continuation principle for harmonic functions. As we find that $(-\Delta)\,\frac{u(x)}{(1-|x|^2)^{a-1}} = 0$ in $B(0,1)$ with $\frac{u(x)}{(1-|x|^2)^{a-1}}\big|_\Gamma = \partial_\nu\big(\frac{u(x)}{(1-|x|^2)^{a-1}}\big)\big|_\Gamma = 0$, this implies that $\frac{u(x)}{(1-|x|^2)^{a-1}} = 0$ in $B(0,1)$. This completes the proof. ∎

This settles the proof of Theorem 1.3.

Lack of injectivity

Here we would like to state the following result.

Proposition 4.5. *Let* $0 < a < 1$. *There exists a non-zero function* $v \in \mathcal{E}_a(\overline{B}(0,1))$ *that satisfies the four conditions*

$$\begin{cases} (-\Delta)^a v \in C_c^\infty(B(0,1)), \\ v = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B}(0,1), \\ \frac{v}{(1-|x|^2)^a}\big|_{\partial B(0,1)} = 0, \end{cases}$$

and

$$v|_E \neq 0 \tag{4.13}$$

for every measurable set $E \subset B(0,1)$ *with* $|E| > 0$.

Remark 4.6. For $a = 1$, (4.13) does not hold: the function v is zero near $\partial B(0,1)$; this follows from standard unique continuation results for harmonic functions (cf. [29]).

Proof of Proposition 4.5. We show first that there exists a non-zero $g \in L^2(B(0,1))$ with $\eta := \operatorname{supp} g \Subset B(0,1)$ which solves

$$\begin{cases} (-\Delta)^a v = g & \text{in } B(0,1), \\ v = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1), \\ \eta = \operatorname{supp} g \Subset B(0,1) \end{cases} \tag{4.14}$$

satisfying

$$\frac{v(x)}{(1-|x|^2)^a} = 0 \quad \text{on } \partial B(0,1). \tag{4.15}$$

If $0 < a < 1$, the function v would be non-zero in every positive measure set, i.e.,

$$v|_E \neq 0 \quad \text{for all } E \subset B(0,1),\ |E| > 0. \tag{4.16}$$

Unlike in the $a = 1$ case, we can not expect $v = 0$ near $\partial B(0,1)$. In fact, that would imply $v = 0$ everywhere, since it means $v = (-\Delta)^a v = 0$ in $\Omega \setminus \overline{\eta}$, and hence $v \equiv 0$, by Theorem 1.2 in [18]. Further, Proposition 5.1 in [17] generalizes this result for any measurable set E with $|E| > 0$. It says that, for $v \in H^a(\mathbb{R}^n)$, if $v|_E = (-\Delta)^a v|_E = 0$ for some $E \subset \mathbb{R}^n$ with $|E| > 0$, then $v \equiv 0$.

Therefore, if $g \neq 0$ in (4.14), (4.16) is automatically guaranteed for $0 < a < 1$. Now we need to find g which satisfies (4.15).

Let us multiply equation (4.14) by $u \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ which solves

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } B(0,1), \\ \frac{u(x)}{d(x)^{a-1}} = f & \text{on } \partial B(0,1), \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1). \end{cases} \tag{4.17}$$

By using the integration by parts formula (2.15) and (4.15), we then find

$$\int_\eta gu = 0, \qquad \text{for all } u \text{ solving (4.17)}. \tag{4.18}$$

Since by Proposition 4.1, $u \in \mathcal{E}_{a-1}(\overline{B}(0,1))$ solving (4.17) means that $\frac{u(x)}{(1-|x|^2)^{a-1}} \in C^\infty(\overline{B}(0,1))$ is harmonic. By rewriting (4.18) as

$$\int_\eta g(1-|x|^2)^{a-1} \, \frac{u(x)}{(1-|x|^2)^{a-1}} \, dx = 0 \tag{4.19}$$

for all harmonic functions $\frac{u(x)}{(1-|x|^2)^{a-1}}$ in $B(0,1)$, we find that

$$g(1-|x|^2)^{a-1} \in (H(\eta))^\perp,$$

where $H(\eta)$ is the set of all harmonic functions in η.

In order to prove the existence of v as in Proposition 4.5, we simply choose some $0 \neq h \in (H(\eta))^\perp \cap C_c^\infty(B(0;1))$ and consider $g = h(1-|x|^2)^{1-a}$ in (4.14). We claim that the corresponding solution, say v_g, of (4.14) satisfies (4.15), i.e.,

$$\frac{v_g(x)}{(1-|x|^2)^a(x)} = 0 \quad \text{on } \partial B(0,1). \tag{4.20}$$

In order to show (4.20) for our choice of g, let us multiply (4.14) by u which solves (4.17) with $f \in C^\infty(\partial B(0,1))$. Following integration by parts (cf. (2.15)), we get

$$\int_{\partial B(0,1)} \frac{v_g(x)}{(1-|x|^2)^a} \, f \, d\sigma = \int_\eta g(1-|x|^2)^{a-1} \, \frac{u(x)}{(1-|x|^2)^{a-1}} \, dx.$$

Since by our choice $g = h(1-|x|^2)^{1-a}$ where $h \in (H(\eta))^\perp$, this means

$$\int_\eta g(1-|x|^2)^{a-1} \, \frac{u(x)}{(1-|x|^2)^{a-1}} \, dx = 0.$$

Therefore

$$\int_{\partial B(0,1)} \frac{v_g(x)}{(1-|x|^2)^a} f = 0 \quad \text{for all } f \in C^\infty(\partial B(0,1)),$$

forcing (4.20) as desired. This completes the proof of Proposition 4.5. ∎

A density result

We prove here that the set consisting of product of a-harmonic functions $\{u_1 u_2\}$ in a ball forms a dense set in L^1_{loc}. Let u_1 and u_2 solve

$$\begin{cases} (-\Delta)^a u = 0 & \text{in } B(0,1), \\ \frac{u(x)}{(1-|x|^2)^{a-1}}\big|_{\partial B(0;1)} = f \in C_c^\infty(\Gamma), & \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B}(0,1), \end{cases} \tag{4.21}$$

where $\Gamma \subseteq \partial B(0,1)$ is some non-empty open set.

Proposition 4.7 (Density result). *Let $n \geq 2$. The set $\{u_1 u_2\}$, where u_k solves (4.21), is dense in $L^1_{\mathrm{loc}}(B(0,1))$.*

Proof. It is enough to show that, for $h \in C_c(B(0,1))$,

$$\int_{B(0,1)} h u_1 u_2 = 0 \quad \text{for all } u_1, u_2 \text{ satisfying (4.21) implies } \ h = 0. \tag{4.22}$$

Writing $\widetilde{h} = (1-|x|^2)^{2a-2} h$ and $v_k = \frac{u_k}{(1-|x|^2)^{a-1}}$, the above identity (4.22) becomes

$$\int_{B(0,1)} \widetilde{h} v_1 v_2 \, dx = 0,$$

where the v_k are harmonic functions in $B(0,1)$ with $\operatorname{supp} v_k|_{\partial\Omega} \subseteq \Gamma$, thanks to Proposition 4.1. Then the result of the linearised Calderón problem (see Theorem 1.1 in [15]) concludes that $\widetilde{h} = 0$, and hence $h = 0$. For $\Gamma = \partial B(0,1)$, the L^1-density of the product of the harmonic functions was first observed by A. P. Calderón in his seminal article [8]. ∎

This completes the discussion of the proof of Theorem 1.4.

Funding. This research was supported by the Collaborative Research Center, membership no. 1283, Universität Bielefeld.

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Received April 1, 2021; revised September 30, 2021. Published online December 16, 2021.

Tuhin Ghosh
Faculty of Mathematics, Universität Bielefeld, Universitätsstraße 25, 33501 Bielefeld, Germany;
tghosh@math.uni-bielefeld.de