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Extremal mappings of finite distortion and the Radon–Riesz property

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Abstract. We consider Sobolev mappings $f \in W^{1,q}(\Omega, \mathbb{C})$, $1 < q < \infty$, between planar domains $\Omega \subset \mathbb{C}$. We analyse the Radon–Riesz property for polyconvex functionals of the form

$$f \mapsto \int_{\Omega} \Phi(|Df(z)|, J(z,f))\, dz$$

and show that under certain criteria, which hold in important cases, weak convergence in $W^{1,q}_{\text{loc}}(\Omega)$ of (for instance) a minimising sequence can be improved to strong convergence. This finds important applications in the minimisation problems for mappings of finite distortion and the L^p and Exp-Teichmüller theories.

In honour of Antonio Córdoba and José Luis Fernández.

1. Introduction

Recently, geometric function theory has developed strong connections with the calculus of variations and planar nonlinear elasticity by identifying a facinating interplay between analysis and topology for mappings of finite distortion. As a simple example, if f is a homeomorphism, then $f \in W^{1,1}_{\text{loc}}(\Omega)$ implies $J(z,f) \in L^1_{\text{loc}}(\Omega)$ and f is differentiable almost everywhere. Other examples have led to the solution of the Nitsche conjecture [4, 6], higher regularity of extremal monotone mappings, [7], and other applications in nonlinear elasticity, see for instance [3, 8] and the references therein. Using the method of p-harmonic replacement based on the Radó–Choquet–Kneser theorem, Iwaniec and Onninen [9] have shown (aside from some minor technical issues) that for each $p \geq 1$, given a weakly converging sequence of homeomorphisms $h_j : \Omega \to \Omega'$ in $W^{1,p}(\Omega)$, with $h_j \rightharpoonup h$, then there exist diffeomorphisms $\tilde{h}_j : \Omega \to \Omega'$ with $\tilde{h}_j$ converging to h strongly in $W^{1,p}(\Omega)$ and with $\tilde{h}_j - h \in W^{1,p}_0(\Omega)$. So weak convergence of a sequence is replaced by strong convergence of a "nicer" sequence. As an application, the authors show that in some natural planar models of nonlinear elasticity, the minimizers of the energy are strong limits of homeomorphisms.

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An issue with this result however is that the initial sequence $\{h_j\}_{j=1}^{\infty}$ may carry information that the approximations $\{\tilde{h}_j\}_{j=1}^{\infty}$ do not, simply because the sequence $\{\tilde{h}_j\}_{j=1}^{\infty}$ may not be a minimising sequence even if $\{h_j\}_{j=1}^{\infty}$ is. For instance, uniform or local uniform bounds on the higher integrability of certain convex combinations of minors may not follow from any $W^{1,p}$-bound. Another example where this behaviour might arise is as follows. Suppose we seek a minimiser to a variational problem whose Euler–Lagrange, or other variational equations are sufficiently degenerate that known methods do not provide existence or regularity. Suppose further one can perturb these equations to gain ellipticity and thereby existence and some regularity. The solutions to the perturbed equation may have a weakly convergent subsequence for which local uniform bounds on nonlinear quantities hold and which do not depend on the ellipticity constants. Strong convergence may then imply that a weak limit satisfies the unperturbed equation, from which one might deduce higher regularity. Despite all the suppositions here, we outline some concrete examples later among mean distortion functionals and Ahlfors–Hopf type equations.

1.1. Finite distortion functions and polyconvexity

Let Ω be a planar domain. A mapping $f:\Omega\to\mathbb{C}$ has finite distortion if

(1) $f\in W^{1,1}_{\mathrm{loc}}(\Omega)$, the Sobolev space of functions with locally integrable first derivatives,

(2) the Jacobian determinant $J(z,f)\in L^1_{\mathrm{loc}}(\Omega)$,

(3) and there is a measurable function $K(z,f)\geq 1$, finite almost everywhere, such that

$$|Df(z)|^2\leq K(z,f)\,J(z,f),\qquad \text{almost everywhere in } \Omega. \tag{1.1}$$

We recommend [2], Chapter 20, for the basic theory of mappings of finite distortion and the associated governing equations: degenerate elliptic Beltrami systems.

In (1.1), the operator norm of Df is used, but for variational problems it is more common to use

$$\mathbb{K}(z,f)=\frac{\|Df(z)\|^2}{J(z,f)}, \tag{1.2}$$

where $\|\cdot\|$ is the Hilbert–Schmidt norm.

The notion of polyconvexity was introduced to the theory on nonlinear elasticity by Ball in [5], and has proved to be an important concept in the calculus of variations ever since. A matrix function $\Xi:\mathbb{R}^{n\times n}\to\mathbb{R}$ is polyconvex if it can be expressed as a convex function of minors of $\mathbb{R}^{n\times n}$. See Definition 10.25 in [15], and in the current context, Section 14.2 of [4]. Here we only consider the two-dimensional case and our main interest here is the convexity of the function x^2/y (which the reader should compare with the definition at (1.2)). Namely,

$$\frac{x^2}{y}-\frac{x_0^2}{y_0}\geq\frac{2x_0}{y_0}(x-x_0)-\frac{x_0^2}{y_0^2}(y-y_0),$$

for any $x,x_0\geq 0$ and $y,y_0>0$. In our applications, x will be $|Df(z)|$ or $\|Df(z)\|$, and y will be $J(z,f)$. We state our first lemma in a more general setting, but the proof is essentially the same as Theorem 12.2 in [4]. Also we only write for $|Df(z)|$, but everything follows similarly for $\|Df(z)\|$.

Lemma 1.1. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose that*

(1) $\{f_j\}_{j=1}^{\infty}$ *is a sequence of* $W^{1,q}_{\mathrm{loc}}(\Omega)$ *functions, where* $1 \le q < \infty$,

(2) f *is a weak limit of* f_j *in* $W^{1,q}_{\mathrm{loc}}(\Omega)$,

(3) $J(z, f_j) \rightharpoonup J(z, f)$ *weakly in* $L^1_{\mathrm{loc}}(\Omega)$,

(4) $J(z, f) > 0$ *almost everywhere in* Ω, *and*

(5) $\Phi(x, y)\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a convex function such that for almost every* $z \in \Omega$, $\Phi_x(|Df(z)|, J(z, f))$ *and* $\Phi_y(|Df(z)|, J(z, f))$ *exist, and*

$$0 \le \Phi_x(|Df(z)|, J(z, f)) < \infty, \quad |\Phi_y(|Df(z)|, J(z, f))| < \infty.$$

Then

$$\int_\Omega \Phi(|Df(z)|, J(z, f)) \le \liminf_{j\to\infty} \int_\Omega \Phi(|Df_j(z)|, J(z, f_j)).$$

In Theorem 12.2 of [4], equality is actually proved for certain functionals because f_j is assumed a minimising sequence. This should remind us of the Radon–Riesz property.

1.2. Radon–Riesz property

A Banach space is called a Radon–Riesz space if every weakly convergent sequence $x_j \rightharpoonup x$ with $\|x_j\| \to \|x\|$ is strongly convergent, namely $\|x_j - x\| \to 0$. See e.g. [14].

Lemma 1.2. *Every uniformly convex Banach space is a Radon–Riesz space. In particular, every* L^p *space with* $1 < p < \infty$ *is a Radon–Riesz space.*

Clearly, the map

$$f \mapsto \int_\Omega \Phi(|Df(z)|, J(z, f))\, dz$$

is usually not a norm. Our aim is to prove that under certain criteria, weak convergence of $f_j \to f$ implies convergence strongly in some $W^{1,q}_{\mathrm{loc}}(\Omega)$. This is our main result.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose that*

(1) f_j *is a sequence of* $W^{1,q}_{\mathrm{loc}}(\Omega)$ *functions, for some* $1 < q < \infty$,

(2) f *is a weak limit of* f_j *in* $W^{1,q}_{\mathrm{loc}}(\Omega)$,

(3) $J(z, f_j) \rightharpoonup J(z, f)$ *weakly in* $L^1_{\mathrm{loc}}(\Omega)$,

(4) $J(z, f) > 0$ *almost everywhere in* Ω,

(5) *and* $\Phi, \Phi_j : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a sequence of functions which satisfies the following conditions*:

 (a) *There is a* $p > 1$ *such that* $\Phi_j(|Df_j|, J(z, f_j))$ *are uniformly bounded in* $L^p(\Omega)$, *and*

$$\lim_{j\to\infty} \int_\Omega \Phi_j^p(|Df_j(z)|, J(z, f_j)) = \int_\Omega \Phi^p(|Df(z)|, J(z, f)). \tag{1.3}$$

 (b) *For each point* $(x, y) \in \Omega$, $\Phi_j(x, y)$ *is a non-decreasing sequence and we have* $\Phi_j(x, y) \to \Phi(x, y)$.

(c) *Each* $\Phi_j(x, y)$ *is a convex function satisfying that, for almost every* $z \in \Omega$, $(\Phi_j)_x(|Df(z)|, J(z, f))$ *and* $(\Phi_j)_y(|Df(z)|, J(z, f))$ *exist, and*

$$0 \le (\Phi_j)_x(|Df(z)|, J(z, f)) < \infty, \quad |(\Phi_j)_y(|Df(z)|, J(z, f))| < \infty.$$

(d) *There is an* $s \in (0, 1 - 1/p)$ *such that each* $\Phi_j(x, y)y^s$ *is a convex function.*

(e) *For almost every point* $(x, y) \in \Omega$, *if* $\Phi_j(x_j, y_j) \to \Phi(x, y)$ *and* $y_j \to y$, *then* $x_j \to x$.

Then, the following convergences hold both strongly and pointwise almost everywhere.

(i) $\Phi_j(|Df_j|, J(z, f_j)) \to \Phi(|Df|, J(z, f))$ *in* $L^p(\Omega)$,

(ii) $f_j \to f$ *in* $W^{1,r}_{\mathrm{loc}}(\Omega)$, $0 < r < q$,

(iii) $J(z, f_j) \to J(z, f)$ *in* $L^r_{\mathrm{loc}}(\Omega)$, $0 < r < 1$,

(iv) $\mu_{f_j} \to \mu_f$ *in* $L^r_{\mathrm{loc}}(\Omega)$, $0 < r < \infty$,

where $\mu_f = f_{\bar z}/f_z$ *is the Beltrami coefficient of* f.

The following weighted case will be proved in a similar way.

Theorem 1.4 (Weighted case). *Let* $\Omega \subset \mathbb{C}$ *be a domain. Suppose that*

(1) f_j *is a sequence of* $W^{1,q}_{\mathrm{loc}}(\Omega)$ *functions, for some* $1 < q < \infty$,

(2) f *is a weak limit of* f_j *in* $W^{1,q}_{\mathrm{loc}}(\Omega)$,

(3) $J(z, f_j) \rightharpoonup J(z, f)$ *weakly in* $L^1_{\mathrm{loc}}(\Omega)$,

(4) $J(z, f) > 0$ *almost everywhere in* Ω,

(5) $\eta_j, \eta > 0$ *a.e. in* Ω, *both in* $L^\infty_{\mathrm{loc}}(\Omega)$ *and* $\eta_j \to \eta$ *locally uniformly in* Ω,

(6) $\Phi, \Phi_j : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is a sequence of functions which satisfies the following conditions*:

(a) *There is a* $p > 1$ *such that* $\Phi_j(|Df_j|, J(z, f_j))$ *are uniformly bounded in* $L^p(\Omega)$, *and*

$$\lim_{j\to\infty} \int_\Omega \Phi_j^p(|Df_j(z)|, J(z, f_j))\eta_j(z) = \int_\Omega \Phi^p(|Df(z)|, J(z, f))\eta(z). \tag{1.4}$$

(b) *For each point* $(x, y) \in \Omega$, $\Phi_j(x, y)$ *is a non-decreasing sequence and we have* $\Phi_j(x, y) \to \Phi(x, y)$.

(c) *Each* $\Phi_j(x, y)$ *is a convex function satisfying that for almost every* $z \in \Omega$, $(\Phi_j)_x(|Df(z)|, J(z, f))$ *and* $(\Phi_j)_y(|Df(z)|, J(z, f))$ *exist, and*

$$0 \le (\Phi_j)_x(|Df(z)|, J(z, f)) < \infty, \quad |(\Phi_j)_y(|Df(z)|, J(z, f))| < \infty.$$

(d) *There is an* $s \in (0, 1 - 1/p)$ *such that each* $\Phi_j(x, y)y^s$ *is a convex function.*

(e) *For almost every point* $(x, y) \in \Omega$, *if* $\Phi_j(x_j, y_j) \to \Phi(x, y)$ *and* $y_j \to y$, *then* $x_j \to x$.

Then, the following convergences hold both strongly and pointwise almost everywhere.

(i′) $\Phi_j(|Df_j|, J(z, f_j))\eta_j^{1/p} \to \Phi(|Df|, J(z, f))\eta^{1/p}$ *in* $L^p(\Omega)$, (ii′) $f_j \to f$ *in* $W^{1,r}_{\mathrm{loc}}(\Omega)$, $0 < r < q$,

(iii′) $J(z, f_j) \to J(z, f)$ *in* $L^r_{\mathrm{loc}}(\Omega)$, $0 < r < 1$,

(iv′) $\mu_{f_j} \to \mu_f$ *in* $L^r_{\mathrm{loc}}(\Omega)$, $0 < r < \infty$,

where $\mu_f = f_{\bar z}/f_z$ *is the Beltrami coefficient of* f.

2. Proof of Theorems 1.3 and 1.4

To simplify notation, we write

$$\Phi_{k,j} = \Phi_k(|Df_j(z)|, J(z, f_j)), \quad \Phi_{k,f} = \Phi_k(|Df(z)|, J(z, f)),$$
$$\Phi_f = \Phi(|Df(z)|, J(z, f)), \quad J_j = J(z, f_j), \quad J_f = J(z, f).$$

Lemma 2.1. $\Phi_{j,j} \to \Phi_f$ *strongly in* $L^p(\Omega)$.

Proof. By (1.3), the sequence $\Phi_{j,j}$ is uniformly bounded in $L^p(\Omega)$, so there is a weak limit $\Psi \in L^p(\Omega)$. Note that Lemma 1.1 holds for every fixed k and any measurable subset $\Omega' \subset \Omega$. Thus

$$\int_{\Omega'} \Phi_{k,f} \le \liminf_{j\to\infty} \int_{\Omega'} \Phi_{k,j} \le \liminf_{j\to\infty} \int_{\Omega'} \Phi_{j,j} = \int_{\Omega'} \Psi.$$

This holds for every k, and so by Fatou's lemma,

$$\int_{\Omega'} \Phi_f \le \liminf_{k\to\infty} \int_{\Omega'} \Phi_{k,f} \le \int_{\Omega'} \Psi.$$

From the Lebesgue differentiation theorem, $\Phi_f(z) \le \Psi(z)$ for almost every $z \in \Omega$. Now it follows from (1.3) that

$$\int_{\Omega} \Phi_f^p \le \int_{\Omega} \Psi^p \le \liminf_{j\to\infty} \int_{\Omega} \Phi_{j,j}^p = \int_{\Omega} \Phi_f^p.$$

This forces $\Phi_f = \Psi$, so it is the weak limit of $\Phi_{j,j}$ in $L^p(\Omega)$, and then the claim follows from Lemma 1.2. ∎

Next, as $J_f > 0$ almost everywhere we can choose $\Omega_\varepsilon \Subset \Omega$ on which

$$\varepsilon < J_f < \frac{1}{\varepsilon}, \quad \Phi_f < \frac{1}{\varepsilon},$$

and

$$\Big|\Omega - \bigcup_{\varepsilon>0} \Omega_\varepsilon\Big| = 0.$$

Now let $s \in (0, 1 - 1/p)$ be as in Condition (5d) in Theorem 1.3. We choose a $p' \in (1, p)$ such that $sp' < (s + 1/p)p' < 1$. Then

Lemma 2.2.

$$\lim_{j\to\infty} \int_{\Omega_\varepsilon} \Phi_{j,j}^{p'} J_j^{sp'} = \int_{\Omega_\varepsilon} \Phi_f^{p'} J_f^{sp'}.$$

Proof. One direction comes from the polyconvexity and follows the same argument as above:

$$\int_{\Omega_\varepsilon} \Phi_f^{p'} J_f^{sp'} \le \liminf_{k\to\infty} \int_{\Omega_\varepsilon} \Phi_{k,f}^{p'} J_f^{sp'} \le \liminf_{j\to\infty} \int_{\Omega_\varepsilon} \Phi_{j,j}^{p'} J_j^{sp'}.$$

Lemma 2.1 and the choice of p' give

$$\lim_{j\to\infty} \int_{\Omega_\varepsilon} \left|\Phi_{j,j}^{p'} J_j^{sp'} - \Phi_f^{p'} J_j^{sp'}\right| \le \lim_{j\to\infty} C\|\Phi_{j,j} - \Phi_f\|_{L^p(\Omega_\epsilon)}^{p'} \|J_j\|_{L^1(\Omega_\epsilon)}^{sp'} = 0.$$

So we only need to show

$$\limsup_{j\to\infty} \int_{\Omega_\varepsilon} \Phi_f^{p'} J_j^{sp'} \le \int_{\Omega_\varepsilon} \Phi_f^{p'} J_f^{sp'}. \tag{2.1}$$

Note that as $sp' < 1$, the function $x \mapsto x^{sp'}$ is concave for $x > 0$. So

$$J_j^{sp'} - J_f^{sp'} \le sp' J_f^{sp'-1}(J_j - J_f).$$

It follows that

$$\int_{\Omega_\varepsilon} \Phi_f^{p'} (J_j^{sp'} - J_f^{sp'}) \le sp' \int_{\Omega_\varepsilon} \Phi_f^{p'} J_f^{sp'-1}(J_j - J_f) \to 0,$$

as $J_j \rightharpoonup J_f$ in $L^1(\Omega)$. This proves (2.1) and completes the proof of the lemma. ∎

If we apply Lemma 2.2 and follow the same proof as in Lemma 2.1, we can establish the following.

Lemma 2.3. *$\Phi_{j,j} J_j^s \to \Phi_f J_f^s$ strongly in $L^{p'}(\Omega_\varepsilon)$*

Now by Lemma 2.3, in every Ω_ε, up to a subsequence we have the pointwise almost everywhere convergence $\Phi_{j,j} J_j^s \to \Phi_f J_f^s$. Let $\varepsilon \to 0$, we can choose diagonally and to obtain a subsequence that converges pointwise in Ω. By Lemma 2.1, we also have the pointwise convergence $\Phi_{j,j} \to \Phi_f$. It follows that $J_j \to J_f$ pointwise almost everywhere, then $|Df_j| \to |Df|$ pointwise almost everywhere, and then $K(z, f_j) \to K(z, f)$, $|\mu(z, f_j)| \to |\mu(z, f)|$, $|(f_j)_z| \to |f_z|$ and $|(f_j)_{\bar z}| \to |f_{\bar z}|$, all pointwise almost everywhere.

Next, let $\Omega' \Subset \Omega$. By Vitali's convergence theorem, both $|(f_j)_z| \to |f_z|$ and $|(f_j)_{\bar z}| \to |f_{\bar z}|$ strongly in $L^r(\Omega')$, for any $1 \le r < q$. In particular,

$$\int_{\Omega'} |(f_j)_z|^r \to \int_{\Omega'} |f_z|^r \quad \text{and} \quad \int_{\Omega'} |(f_j)_{\bar z}|^r \to \int_{\Omega'} |f_{\bar z}|^r.$$

On the other hand, $Df_j \rightharpoonup Df$ weakly in $L^r(\Omega')$. So it follows from Lemma 1.2 that $Df_j \to Df$ strongly in $L^r(\Omega')$. We may now apply Vitali's convergence theorem once again to get the remaining claims of Theorem 1.3.

The proof of Theorem 1.4 is essentially same. We only need to replace Φ by $\Phi\eta^{1/p}$ in the proof.

3. Applications

3.1. exp(p) minimising sequence

In Theorem 12.2 of [4] it is proved that the exp(p) mean distortion

$$f \mapsto \int_\Omega \exp[p\mathbb{K}(z,f)]\,dz$$

for homeomorphisms from $\overline{\Omega}$ to $\overline{\Omega}'$ admits a minimiser for suitable boundary data.

In fact, there is minimising sequence f_j which converges weakly to a minimiser f in the Sobolev–Orlicz space $W^{1,P}(\Omega)$, where $P(t) = t^2/\log(e+t)$, and

$$\int_\Omega \exp[p\mathbb{K}(z,f)] = \lim_{j\to\infty} \int_\Omega \exp[p\mathbb{K}(z,f_j)].$$

Thus in Theorem 1.3 we may set $\Phi_j(x,y) = \Phi(x,y) = \exp[\frac{p}{2}\frac{x^2}{y}]$ and any small $s > 0$, recalling that that in our case the domain is always $x^2 \geq y$ and so $\exp[\frac{p}{2}\frac{x^2}{y}]y^s$ is convex there. Thus we obtain the strong convergence of $\exp(p\mathbb{K}(z,f_j))$, Df_j, and so forth. In particular, for the case $p > 2$ we have $f_j \to f$ strongly in $W^{1,2}_{loc}(\Omega)$ (cf. [1]). Furthermore, after changing variables we also have

$$\int_{\Omega'} \exp[p\mathbb{K}(w,h)]\,J(w,h) = \lim_{j\to\infty} \int_{\Omega'} \exp[p\mathbb{K}(w,h_j)]\,J(w,h_j),$$

where $h_j = f_j^{-1}$ and $h = f^{-1}$. Again in Theorem 1.3 we may set $\Phi_j(x,y) = \Phi(x,y) = \exp[\frac{p}{2}\frac{x^2}{y}]y^{1/2}$, and $s > 0$ very small, to obtain the strong convergence of $h_j \to h$ in $W^{1,2}_{loc}(\Omega')$. In fact, all we need is the pointwise inequality

$$\Phi(x,y) - \Phi(x_0,y_0) \geq \Phi_x(x_0,y_0)(x-x_0) + \Phi_y(x_0,y_0)(y-y_0),$$

which proves the lower-semi continuity of the energies, as stated in Lemma 1.1. Further, since $J(z,f) > 0$ a.e., the singularities do not affect the integrals. That proof is essentially same as in §12 of [4].

3.2. L^p minimising sequence

In [11] we considered the boundary value problems for L^p-mean distortion for self homeomorphisms of the unit disk $\mathbb{D}$,

$$f \mapsto \int_{\mathbb{D}} \mathbb{K}^p(z,f)\,dz.$$

Unfortunately, in this case Df is not a priori in a sufficiently regular space, so a minimising sequence f_j might not have weakly convergent $J(z,f_j)$ in $L^1_{loc}(\mathbb{D})$, and further the limit function f might not be a homeomorphism. However, the inverse sequence $h_j \rightharpoonup h$ weakly in $W^{1,2}(\mathbb{D})$, where h is a minimiser in an enlarged space where pseudo-inverses exist. See Section 5 of [11] for the definitions of the enlarged space and pseudo-inverses. In particular,

$$\int_{\mathbb{D}} \mathbb{K}^p(w,h)\, J(w,h) = \lim_{j\to\infty} \int_{\mathbb{D}} \mathbb{K}^p(w,h_j)\, J(w,h_j).$$

Here we can apply Theorem 1.3. Set $\Phi_j(x,y) = \Phi(x,y) = (x^2/y)y^{1/p}$ and any small $s > 0$ to obtain the strong convergence of $h_j \to h$ in $W^{1,2}(\mathbb{D})$. In fact, more is true. In the enlarged space there are pseudo-inverses $f_j = h_j^{-1}$ and $f = h^{-1}$ such that $f_j \rightharpoonup f$ in $W^{1,2p/(p+1)}(\mathbb{D})$, each $f_j(\mathbb{D}) \subset \mathbb{D}$, and

$$\int_{\mathbb{D}} \mathbb{K}^p(z,f) = \lim_{j\to\infty} \int_{\mathbb{D}} \mathbb{K}^p(z,f_j).$$

We consider the sequence $J^{1/2}(z,f_j)$, which is bounded in $L^2(\mathbb{D})$. For all $\varphi \in C_0^\infty(\mathbb{D})$,

$$\begin{aligned}\lim_{j\to\infty} \int_{\mathbb{D}} J^{1/2}(z,f_j)\,\varphi(z) &= \lim_{j\to\infty} \int_{\mathbb{D}} J^{1/2}(w,h_j)\,\varphi(h_j(w))\\ &= \int_{\mathbb{D}} J^{1/2}(w,h)\,\varphi(h(w)) = \int_{\mathbb{D}} J^{1/2}(z,f)\,\varphi(z).\end{aligned}$$

So $J^{1/2}(z,f)$ is the weak limit of $J^{1/2}(z,f_j)$ in $L^2(\mathbb{D})$. Also the minimiser h has the holomorphic Hopf differential

$$\Psi = \mathbb{K}^{p-1}(w,h)h_w \overline{h_{\overline{w}}} = \mathbb{K}^p(w,h)\, J(w,h)\, \frac{\overline{\mu(w,h)}}{1+|\mu(w,h)|^2}.$$

This gives us two cases: firstly, if $\Psi \equiv 0$, then h is a holomorphic function. But we know h is monotone, so actually it is a conformal mapping, thus $f(\mathbb{D}) = \mathbb{D}$. If Ψ is not identically zero, then $J(w,h) > 0$ almost everywhere in $\mathbb{D}$. But we know $J(w,h) = 0$ almost everywhere in $\mathbb{D} - f(\mathbb{D})$. So in either case we have $|f(\mathbb{D})| = \pi$. Now

$$\pi \ge \liminf_{j\to\infty} \int_{\mathbb{D}} J(z,f_j) \ge \int_{\mathbb{D}} J(z,f) = \int_{f(\mathbb{D})} \frac{1}{J(w,h)}\, J(w,h) = \pi.$$

So the inequalities hold with equality, and again the Radon–Riesz Lemma 1.2 applies. As $J^{1/2}(z,f_j) \rightharpoonup J^{1/2}(z,f)$ in $L^2(\mathbb{D})$, we obtain $J^{1/2}(z,f_j) \to J^{1/2}(z,f)$ strongly in $L^2(\mathbb{D})$, which is equivalent to $J(z,f_j) \to J(z,f)$ strongly in $L^1(\mathbb{D})$. Now Theorem 1.3 applies and we also get that $\mathbb{K}(z,f_j) \to \mathbb{K}(z,f)$ strongly in $L^p(\mathbb{D})$, and $f_j \to f$ strongly in $W^{1,q}(\mathbb{D})$ for all $1 \le q < 2p/(p+1)$.

3.3. Truncated exponential minimising sequence

The exponential finite distortion problem is not variational, [12]. Thus in [13], to study the exponential finite distortion problems, we consider the truncated problems and the associated inverse problems:

$$f \mapsto \int_{\mathbb{D}} \sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(z,f)}{n!}, \quad h \mapsto \int_{\mathbb{D}} \sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,h)}{n!}\, J(w,h).$$

As linear combinations of the L^p problems, in the enlarged space there are minimisers h_N which have holomorphic Ahlfors–Hopf differentials

$$\Psi_N = \sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,h_N)}{n!}\,(h_N)_w\,\overline{(h_N)_{\overline{w}}}.$$

In Theorem 1.3 we set

$$\Phi_{N,N} = \sqrt{\sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,h_N)}{n!}\,J(w,h_N)},$$

which is a polyconvex function of Dh_N for sufficiently large N. So Theorem 1.3 gives $h_N \to h$ strongly in $W^{1,2}(\mathbb{D})$. A similar argument as in the last part tells us that their inverses $f_N = h_N^{-1}$ also converge strongly to $f = h^{-1}$, but in $W^{1,q}(\mathbb{D})$ for all $q \in [1,2)$. In fact, we can prove that the limit function f is a homeomorphic minimiser for the $\exp(p)$ problem, and if the sequence $\Psi_N/\|\Psi_N\|_{L^1(\mathbb{D})}$ is nondegenerate, then there is a holomorphic Ψ such that $\Psi_N \to \Psi$ locally uniformly, and then by the strong convergence we have

$$\Psi = \lim_{N\to\infty}\Psi_N = \lim_{N\to\infty}\sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,h_N)}{n!}\,(h_N)_w\,\overline{(h_N)_{\overline{w}}} = \exp[p\mathbb{K}(w,h)]\,h_w\,\overline{h_{\overline{w}}},$$

and we can prove such an h is diffeomorphic in $\mathbb{D}$, and then so is $f = h^{-1}$. See [13] for detailed discussions.

4. Mappings between surfaces

In this section we consider the following minimising problem. Let $f_0 \colon S \to \tilde{S}$ be a quasiconformal mapping, where S and $\tilde{S}$ are compact Riemann surfaces. We seek to study the critical points of

$$f \mapsto \int_S \exp[p\mathbb{K}(z,f)]\,d\sigma(z), \tag{4.1}$$

where f is in the same homotopy class as the datum f_0 and where $d\sigma(z)$ is the hyperbolic area measure on S.

Again, this problem is not variational. However, following the argument of the previous section we can show that the inverse of a minimiser satisfies the variational equations. That is the Ahlfors–Hopf equation.

We lift this problem to the universal cover. Any such mapping f as appears in (4.1) has a lift $\tilde{f} \colon \mathbb{D} \to \mathbb{D}$ which commutes with the fundamental groups Γ, of S, and $\tilde{\Gamma}$, of $\tilde{S}$. That is, $f_{0*} \colon \pi_1(S) \to \pi_1(\tilde{S})$ induces an isomorphism which we simply denote as $\gamma \mapsto \tilde{\gamma}$. Then

$$\tilde{f} \circ \gamma = \tilde{\gamma} \circ \tilde{f} : \mathbb{D} \to \mathbb{D}. \tag{4.2}$$

Under these circumstances, $\tilde{f}$ extends to $\mathbb{S} = \partial\mathbb{D}$ and $\tilde{f}|\mathbb{S}$ is quasisymmetric. Notice too that if $\tilde{f}_j \to \tilde{f}$ locally uniformly, and if $\tilde{f}_j$ satisfies (4.2), then so does the limit $\tilde{f}$.

Next, if $\mathscr{P}$ denotes a (hyperbolically) convex fundamental polyhedron for Γ, then

$$\int_S \exp[p\mathbb{K}(z, f)]\, d\sigma(z) = \int_{\mathscr{P}} \exp[p\mathbb{K}(z, \tilde{f})]\, \eta(z)\, dz, \tag{4.3}$$

where the weight $\eta(z)$ is the hyperbolic metric $1/(1-|z|^2)^2$. See [10] for more details.

Thus we may now we can consider (writing f for $\tilde{f}$)

$$\int_{\mathscr{P}} \exp[p\mathbb{K}(z, f)]\, \eta(z),$$

where we also impose the automorphic condition at (4.2). Again we can consider the inverse problem

$$\int_{\tilde{\mathscr{P}}} \exp[p\mathbb{K}(w, h)]\, J(w, h)\, \eta(h),$$

where $\tilde{\mathscr{P}}$ is a fundamental domain for $\tilde{\Gamma}$ and we impose the automorphy condition

$$\gamma \circ h = h \circ \tilde{\gamma} : \mathbb{D} \to \mathbb{D}. \tag{4.4}$$

The associated truncated problems are to minimise

$$\int_{\tilde{\mathscr{P}}} \sum_{n=0}^{N} \frac{p^n\, \mathbb{K}^n(w, h)}{n!}\, J(w, h)\, \eta(h). \tag{4.5}$$

Then, as we have found earlier, there is a sequence of minimisers h_N (in the enlarged space) with holomorphic Ahlfors–Hopf differentials

$$\Psi_N = \sum_{n=0}^{N} \frac{p^n\, \mathbb{K}^n(w, h_N)}{n!}\, (h_N)_w\, \overline{(h_N)_{\overline{w}}}\, \eta(h_N). \tag{4.6}$$

In fact, this is basically how Ahlfors sets up his approach to the proof of Teichmüller's theorem. However he multiplies through by a "convergence factor" to eliminate the bad term $\eta(h)$ in (4.5) as it will make no difference in his application as he lets $p \to \infty$. Now $\eta(h_N) \to \eta(h)$ locally uniformly in $\mathbb{D}$ and so uniformly on $\tilde{\mathscr{P}}$. So by Theorem 1.4 there is an h such that $h_N \to h$ strongly in $W^{1,2}(\tilde{\mathscr{P}})$. Here the difference is that we know the space of quadratic differentials is finite dimensional by the Riemann–Roch theorem, so $\Psi_N/\|\Psi_N\|_{L^1(\mathscr{P})}$ is nondegenerate, and so there is another holomorphic Ψ such that

$$\Psi = \exp[p\mathbb{K}(w, h)]\, h_w\, \overline{h_{\overline{w}}}\, \eta(h).$$

In fact, this argument works for each domain $\tilde{\gamma}(\tilde{\mathscr{P}})$, where $\tilde{\gamma} \in \tilde{\Gamma}$, so both of the functions h and Ψ extend to $\mathbb{D}$, where Ψ is a holomorphic function in $\mathbb{D}$, and the equation $\Psi = \exp[p\mathbb{K}(w, h)]\, h_w\, \overline{h_{\overline{w}}}\, \eta(h)$ holds over $\mathbb{D}$.

Also, since f is in the exponential class and is automorphic with respect to Fuchsian groups of compact type, both f and h must be self-homeomorphisms of the closed disk $\overline{\mathbb{D}}$. To see this, we may argue as follows. If $\mathbb{D} = h(\Omega)$, $h\colon \Omega \to \mathbb{D}$ smooth, then $h(\Omega) = (\tilde{\gamma} \circ h \circ \gamma)(\Omega)$, $\mathbb{D} = \tilde{\gamma}^{-1}(\mathbb{D}) = h(\gamma(\Omega))$. As h is a diffeomorphism onto, it follows that

$\gamma(\Omega) = \Omega$ for each $\gamma \in \Gamma$. As Ω is simply connected, Ω/Γ is a Riemann surface with fundamental group Γ. This is the fundamental group of a closed surface, so Ω/Γ is a closed surface and a fundamental domain for Γ lies in Ω. This shows $\Omega = \mathbb{D}$.

As noted, the automorphy condition passes to the limit. In the case $\eta(z) = \frac{1}{(1-|z|^2)^2}$, this and (4.6) together gives

$$\begin{aligned}\Psi_N(w) &= \sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,\gamma\circ h_N)}{n!}\,(\gamma\circ h_N)_w\,\overline{(\gamma\circ h_N)_{\overline{w}}}\,\eta(\gamma\circ h_N)\\ &= \sum_{n=0}^{N} \frac{p^n\,\mathbb{K}^n(w,h_N\circ\tilde{\gamma})}{n!}\,(h_N\circ\tilde{\gamma})_w\,\overline{(h_N\circ\tilde{\gamma})_{\overline{w}}}\,\eta(h_N\circ\tilde{\gamma}) = \Psi_N(\tilde{\gamma})\,\tilde{\gamma}'^2.\end{aligned}$$

By the strong convergence, these properties of h_N and Ψ_N persist in the limit for h and Ψ. We conclude as follows.

Theorem 4.1. *Consider the inverse hyperbolic exponential finite distortion problem* $h \mapsto \int_{\mathbb{D}} \exp[p\mathbb{K}(w,h)]J(w,h)\eta(h)$, *where* $\eta(z) = 1/(1-|z|^2)^2$, *and* h *is a self-homeomorphism of* $\overline{\mathbb{D}}$ *and is automorphic with respect to the Fuchsian groups* $(\tilde{\Gamma},\Gamma)$. *Then, there are a critical point* h *and a holomorphic Ahlfors–Hopf differential* Ψ *such that*

$$\Psi = \exp[p\mathbb{K}(w,h)]\,h_w\,\overline{h_{\overline{w}}}\,\eta(h). \tag{4.7}$$

Furthermore, $\Psi = \Psi(\tilde{\gamma})\,\tilde{\gamma}'^2$ *for every* $\tilde{\gamma} \in \tilde{\Gamma}$.

Note that in these circumstances, we prove in [13] that there is a diffeomorphic solution to (4.5). Suitably normalised, we expect this solution to be unique. Further, we also show that any quasiconformal solution to (4.5) is already a diffeomorphism. These two fact strongly suggest the minimiser is a diffeomorphism, but our results to date fall a little short of this as our uniqueness results are not yet strong enough.

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