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Expansion of harmonic functions near the boundary of Dini domains

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Abstract. Let u be a harmonic function in a C^1-Dini domain such that u vanishes on an open set of the boundary. We show that near every point in the open set, u can be written uniquely as the sum of a non-trivial homogeneous harmonic polynomial and an error term of higher degree (depending on the Dini parameter). In particular, this implies that u has a unique tangent function at every such point, and that the convergence rate to the tangent function can be estimated. We also study the relationship of tangent functions at nearby points in a special case.

Dedicated to Antonio Córdoba and Josechu Fernández for their great work as editors of the Revista.

1. Introduction and main results

A harmonic function can be decomposed into the summation of homogeneous harmonic polynomials of integer degrees. In particular, it can be written as a homogeneous harmonic polynomial plus a higher-order error term. In [5], the author proved that a similar expansion holds for solutions to elliptic operators whose coefficients are Lipschitz. This is optimal: there are examples of elliptic operators with Hölder coefficients for which the solution does not have finite order of vanishing (see [11] for an example of non-divergence form operator, and [10] for an example of divergence form operator), so one cannot expect an expansion in homogeneous harmonic polynomials of finite degrees. On the other hand, if a solution of an elliptic operator with Hölder-continuous coefficient does have a finite order of vanishing at a point, Han's argument works and he gets a similar expansion near that point.

In a C^1-Dini domain, consider a non-trivial harmonic function u which vanishes on an open set of the boundary $\partial D \cap B_{5R}(0)$. Then u has a finite order of vanishing in $B_R(0)$, which follows from a doubling property proven in [1] and later in [8] using a different method. Moreover, in a previous paper [9], we proved a more precise decay rate for such

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function (see Lemma 2.4); more importantly, we gave an estimate of the size of the singular set

$$\mathcal{S}(u) := \{X \in \overline{D} \cap B_R(0) : u(X) = 0 = |\nabla u(X)|\}.$$

Combining the arguments in [9] and [5], we are able to show that u has a similar expansion at the boundary of a Dini domain:

Theorem 1.1. *Let $D \subset \mathbb{R}^d$ be a Dini domain with parameter θ (see Definition 2.1) and $\partial D \ni 0$. Let $R_0, \Lambda > 0$ be finite. Suppose that u is a non-trivial harmonic function in $D \cap B_{5R_0}(0)$, $u = 0$ on $\partial D \cap B_{5R_0}(0)$, and the (modified) frequency function (defined in Sections 4 and 3 of [9], and restated in (2.6)) at the origin satisfies $N_0(4R_0) \le \Lambda$.*

Then for any boundary point $X_0 \in \partial D \cap B_{R_0}(0)$, there exists $R > 0$ such that u has a unique expansion

$$u(X) = P_N(X - X_0) + \tilde{\psi}(X - X_0) \quad \text{in } B_R(X_0), \tag{1.1}$$

where P_N is a non-trivial homogeneous harmonic polynomial of degree $N \in \mathbb{N}$, and the error term $\tilde{\psi}$ satisfies

$$|\tilde{\psi}(Y)| \le C|Y|^N \, \tilde{\theta}(2|Y|), \tag{1.2}$$

and

$$|\nabla\tilde{\psi}(Y)| \le C|Y|^{N-1} \, \mathring{\theta}(2|Y|), \tag{1.3}$$

Here,

- *N agrees with the vanishing order of u at X_0, i.e., $N = N_{X_0} = \lim_{r\to 0} N_{X_0}(r)$, where $N_{X_0}(\cdot)$ is the (modified) frequency function of u centered at X_0;*
- *the radius R is determined by the frequency function at X_0 and the Dini parameter θ (see (4.7));*
- *$\tilde{\theta}$ is determined by the Dini parameter θ as in (5.19), and satisfies $\tilde{\theta}(r) \to 0$ as $r \to 0$;*
- *$\mathring{\theta}$ is determined by θ as in (6.13) and (6.10), and satisfies $\mathring{\theta}(r) \to 0$ as $r \to 0$.*

Remark 1.2. We remark that when D is a $C^{1,\alpha}$ domain with $\alpha \in (0, 1)$ (that is, when $\theta(r) \approx r^\alpha$), the upper bounds of the error term satisfy that $\tilde{\theta}(r), \mathring{\theta}(r) \lesssim r^\alpha$.

The significance of the above theorem is that we get a higher-order expansion of u even though u only has regularity up to C^1 at the boundary. Moreover, it is more difficult to estimate the gradient of the error term compared to [5]. This is not only because of difficulties at the boundary, but also due to regularity issues. Recall that (because of a different regularity and structure of the coefficient matrix) the solutions in the setting of [5] are in the Sobolev space $W^{2,p}$ for any $p > 1$, i.e., they are strong solutions. So the L^p estimates of $\nabla\tilde{\psi}$ as well as $\nabla^2\tilde{\psi}$ follow directly from the estimate of $\tilde{\psi}$ in (1.2), using interior L^p estimates for strong solutions, see Theorem 9.11 in [4]. But more work is needed here to obtain the gradient estimate in (1.3).

We also remind the readers that for an interior point $X_0 \in D$, we can simply use the decomposition of u (into homogeneous harmonic polynomials of integer degrees) near X_0 to obtain the expansion

$$u(X) - u(X_0) = P_N(X - X_0) + \tilde{\psi}(X - X_0)$$

for any $X \in D$ such that $|X - X_0| < \operatorname{dist}(X_0, \partial D)$, where the error term $\tilde{\psi}$ satisfies $|\tilde{\psi}(Y)| \le C|Y|^{N+1}$ as well as higher regularity estimates.

Recall that in [9] we have studied the blow up of the function u at a boundary point as follows. For any $X_0 \in \partial D \cap B_{R_0}(0)$ and $r > 0$, let

$$T_{X_0,r}u(Z) := \frac{u(X_0 + rZ)}{\left(\frac{1}{r^d} \iint_{B_r(X_0)\cap D} u^2 dY\right)^{1/2}}, \quad \text{for any } Z \in \frac{D - X_0}{r}. \tag{1.4}$$

Since D is a C^1 domain (i.e., the domain above the graph of a C^1 function $\varphi: \mathbb{R}^{d-1} \to \mathbb{R}$), clearly $(D - X_0)/r$ converges locally graphically to a half space, above the hyperplane determined by $\nabla\varphi(x_0)$, where $x_0 \in \mathbb{R}^{d-1}$ is such that $X_0 = (x_0, \varphi(x_0)) \in \partial D$. Assuming without loss of generality that $\nabla\varphi(x_0) = 0$, then $(D - X_0)/r$ converges graphically to the upper half space $\mathbb{R}^d_+$. Then for any sequence $r_j \to 0$, there exists a homogeneous harmonic polynomial P in $\mathbb{R}^d_+$ (possibly depending on the sequence $\{r_j\}$) of degree N_{X_0}, such that modulo passing to a subsequence,

$$T_{X_0,r_j}u \to P \quad \text{locally uniformly and locally strongly in } L^2, \text{ and weakly in } W^{1,2},$$

and

$$\iint_{B_1^+(0)} |P(Z)|^2 \, dZ = 1,$$

where we denote $B_1^+(0) := B_1(0) \cap \mathbb{R}^d_+$. We say that P is a tangent function of u at the point X_0. A priori for different sequences $\{r_j\}$, we may get different tangent functions. However, using the expansion in (1.1), we can prove the following corollary.

Corollary 1.3. *For any $X_0 \in \partial D \cap B_{R_0}(0)$, we have*

$$T_{X_0,r}u(Z) = cP_N(Z) + O(\tilde{\theta}(r)), \tag{1.5}$$

where P_N is the homogeneous harmonic polynomial as in (1.1), c is a normalizing constant so that $P = cP_N$ has unit L^2 norm in $B_1^+(0)$, and $\tilde{\theta}$ is as in Theorem 1.1. In particular, the polynomial cP_N is the unique tangent function of u at X_0, and the convergence rate to the tangent function is bounded by a constant multiple of $\tilde{\theta}(r)$.

We remark that the global estimate we obtained in Theorem 1.1 of [9] does not imply the above result. In Corollary 1.3, not only do we know that there is a unique tangent function at every point, we also know the convergence rate. The result in the current paper complements the main theorem in [9] and uses the frequency function and purely PDE arguments.

By the monotonicity of the frequency function $N_{X_0}(\cdot)$ and the fact that its limit $N_{X_0} = \lim_{r\to 0} N_{X_0}(r)$ is integer-valued, we can show that

$$X_0 \in \partial D \cap B_{R_0}(0) \mapsto N_{X_0} \in \mathbb{N}$$

is upper semi-continuous. The proof uses a standard argument adapted to the modified frequency function we introduced in [9]. Since this fact is tangential to the main topic of this paper, we defer the proof to the appendix. In general, the vanishing order could jump up, and we give a simple example in the footnote.[1] But in the particular case where a sequence $X_j \in \partial D \cap B_{R_0}(0)$ converging to X_0 is such that $N_{X_j} \to N_{X_0}$, since the vanishing order is integer-valued, we have $N_{X_j} \equiv N_{X_0}$ for j sufficiently large. We can then show that the leading order polynomials in the expansion also converge:

Proposition 1.4. *Let $\{X_j\}$, X_0 be points in $\partial D \cap B_{R_0}(0)$ satisfying $X_j \to X_0$. Suppose that $N_{X_j} = N_{X_0}$ for each j. Let P_{X_j} and P_{X_0} denote the homogeneous harmonic polynomials in the expansions (1.1) near X_j and X_0, respectively. Then P_{X_j} converges to P_{X_0} in the C^k-topology for any $k \in \mathbb{N}$.*

The paper is organized as follows. In Section 2 we introduce some notation, recall how we defined the modified frequency function in [9], and use that to estimate the ratio of the L^2 norm of u in two concentric balls of different radii. In Sections 3 and 4 we reduce the problem from a harmonic function u in a C^1-Dini domain to a solution v in the upper half-space to a divergence-form elliptic operator, whose coefficient matrix is the identity matrix at the center point and is Dini-continuous everywhere. Then, in Section 5, we write down the expansion of v, estimate the error term using Dini-continuity of the coefficient matrix, and show the leading-order homogeneous harmonic polynomial is nontrivial. Moreover, in Section 6 we estimate the gradient of the error term in L^p and L^∞. These are combined to give us the expansion of the original function u (i.e., Theorem 1.1) in Section 7. The convergence rate to the (unique) tangent function is just a simple corollary of that expansion. Finally, in Section 8 we prove Proposition 1.4, namely the tangent functions are continuous at the boundary point where the vanishing orders do not jump up.

2. Preliminaries

Definition 2.1 (Dini domains). Let $\theta : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function satisfying[2]

$$\int_0^* \frac{\theta(r)}{r} < \infty. \tag{2.1}$$

In particular, (2.1) implies that $\theta(r) \to 0$ as $r \to 0$. A connected domain D in $\mathbb{R}^d$ is a *C^1-Dini domain* with parameter θ if for each point X_0 on the boundary of D there is a coordinate system $X = (x, x_d)$, $x \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$, such that with respect to this coordinate system $X_0 = (0, 0)$, and there are a ball B centered at X_0 and a continuously differentiable function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying the following:

(1) $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{d-1})} \le C_0$ for some $C_0 > 0$;

(2) $|\nabla\varphi(x) - \nabla\varphi(y)| \le \theta(|x - y|)$ for all $x, y \in \mathbb{R}^{d-1}$;

(3) $D \cap B = \{(x, x_d) \in B : x_d > \varphi(x)\}$.

[1] Consider the upper-half space $\mathbb{R}^3_+ = \{(x_1, x_2, t) : x_1, x_2 \in \mathbb{R}, t > 0\}$. The function $u : \mathbb{R}^3_+ \to \mathbb{R}$ defined as $u(x_1, x_2, t) = (x_1 + x_2) \cdot t$ is harmonic. Let $L := \{(x_1, x_2, 0) : x_1 + x_2 = 0\}$ be a subset of $\partial\mathbb{R}^3_+$. It is an easy exercise to show that for any $X_0 \in L$, the vanishing order N_{X_0} is 2; and for any $X_0 \in \partial\mathbb{R}^3_+ \setminus L$, the vanishing order N_{X_0} is 1.

[2] In particular, we can choose $R_0 > 0$ so that $\theta(8R_0) < 1/72$ and $\int_0^{16R_0} \frac{\theta(s)}{s}\, ds \le 1$.

Remark 2.2. By shrinking the ball B if necessary, we may modify the coordinate system so that $\nabla\varphi(0) = 0$.

Under the assumptions of Theorem 1.1, we have $u \in C^1(\overline{D \cap B_{4R_0}(0)})$ by the work of [3]. Note that in [3], the Dini parameter is required to be doubling, in the sense that there exists a constant $C > 1$ such that

$$\theta(2r) \leq C\theta(r) \quad \text{for all } r, \tag{2.2}$$

see (1.4) in [3]. This is not necessarily satisfied by all θ above fulfilling (2.1), in which case we just replace $\theta(r)$ by

$$\alpha(r) := \sup_{\substack{x,y \in \mathbb{R}^{d-1} \\ |x-y| \leq r}} |\nabla\varphi(x) - \nabla\varphi(y)|.$$

(In general, a bounded Dini domain D is characterized by finitely many coordinate systems and C^1 functions φ_i's, as in Definition 2.1. In that case, we take $\alpha(r)$ to be the maximum of the above value for all φ_i's in their respective domains.) We claim that $\alpha(\cdot)$ is doubling. In fact, assume that $\alpha(2r) = |\nabla\varphi(x) - \nabla\varphi(y)|$ for some $x, y \in \mathbb{R}^{d-1}$ with $|x - y| \leq 2r$. Let z be the middle point on the line segment $[x, y]$. Clearly $|x - z|, |z - y| \leq r$. Thus

$$\alpha(2r) = |\nabla\varphi(x) - \nabla\varphi(y)| \leq |\nabla\varphi(x) - \nabla\varphi(z)| + |\nabla\varphi(z) - \nabla\varphi(y)| \leq 2\alpha(r).$$

Besides, α also satisfies the Dini condition (2.1), since $\alpha(r) \leq \theta(r)$ by the property of $\nabla\varphi$. Therefore, without loss of generality, we assume the above Dini parameter θ satisfies (2.2). Moreover, we remark that an example in [6] seems to indicate that Dini regularity is the optimal condition to guarantee continuous differentiability of u.[3]

When D is not a convex domain, the standard Almgren frequency function for u centered at a boundary point X, defined as

$$r \mapsto N(u, X, r) := \frac{r \iint_{B_r(X)} |\nabla u|^2 \, dX}{\int_{\partial B_r(X)} u^2 \, d\mathcal{H}^{d-1}}, \tag{2.3}$$

may not be monotone. (In the above definition, we assume we have extended u by zero across the boundary, to simplify the notation.) However, in [9], for a Dini domain D and for every boundary point $X_0 \in B_{R_0}(0) \cap \partial D$, we were able to define a modified frequency function for u, denoted by $N_{X_0}(\cdot)$, using a special transformation Ψ_{X_0}, and prove that the map $r \mapsto N_{X_0}(r)$ is monotone. More precisely, using the notation in Sections 3 and 4 of [9], we recall the definition of the transformation

$$\Psi_{X_0} : X = (x, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto X_0 + (x, x_d + 3|X|\,\hat{\theta}(|X|)) \in \mathbb{R}^d, \tag{2.4}$$

[3] In [6], the authors give a divergence-form elliptic operator $L = -\operatorname{div}(A(\cdot)\nabla)$, where the coefficient matrix $A(\cdot)$ is continuous but its modulus of continuity fails the Dini condition (2.1), and a solution u to L which satisfies $u \in W^{1,p}_{\mathrm{loc}}$ for every $p > 1$ but $u \notin W^{1,\infty}_{\mathrm{loc}}$.

where

$$\hat{\theta}(r) = \frac{1}{\log^2 2} \int_r^{2r} \frac{1}{t} \int_t^{2t} \frac{\theta(s)}{s}\, ds\, dt. \tag{2.5}$$

is a smoothed version of the Dini parameter θ, and it satisfies $\theta(r) \le \hat{\theta}(r) \le \theta(4r)$. Then $u \circ \Psi_{X_0}$ satisfies a divergence-form elliptic equation in the domain $\Psi_{X_0}^{-1}(D)$, see Section 4 in [9]. As in Section 3 of [9], we may define the frequency function

$$N(u \circ \Psi_{X_0}, r) := \frac{r\, D(u \circ \Psi_{X_0}, r)}{H(u \circ \Psi_{X_0}, r)}$$

for non-homogeneous elliptic operator satisfying certain assumptions, see (3.8) in [9] or the proof of Lemma A.2 for the details. Finally, in Proposition 3.10 of [9] we proved the following.

Proposition 2.3. *The map*

$$r \mapsto N_X(r) := N(u \circ \Psi_X, r) \exp\left(C \int_0^r \frac{\theta(s)}{s}\, ds\right) \tag{2.6}$$

is monotone nondecreasing.

The following doubling property essentially follows from Corollary 3.28 in [9] and the monotonicity of the (modified) frequency function for $u \circ \Psi_{X_0}$. (Recall that u is extended by zero outside of D.)

Lemma 2.4 (L^2-doubling property). *Let $X_0 = (x_0, \varphi(x_0)) \in B_{R_0}(0)$ be a boundary point of D. Then for any pair of radii $0 < s < r$ sufficiently small, we have*

$$\left(\frac{s}{r}\right)^{d+2N_{X_0}(2r)} \lesssim \frac{\iint_{B_s(X_0)} u^2\, dX}{\iint_{B_r(X_0)} u^2\, dX} \lesssim \left(\frac{s}{r}\right)^{d+2N_{X_0} \exp(-C \int_0^{4r} \frac{\theta(s)}{s}\, ds)}, \tag{2.7}$$

where $N_{X_0}(\cdot) = \widetilde{N}(X_0, \cdot)$ is the monotone frequency function centered at X_0, as is defined in Sections 3 and 4 of [9], and $N_{X_0} = \lim_{r \to 0} N_{X_0}(r) \in \mathbb{N}$.

Remark 2.5. We follow the convention in [9] and call the above a *doubling property*. But we point out that it is actually a misnomer. In fact, under the same assumption it has already been proven in Theorem 0.4 of [1] and Theorem 2.2 of [8] that there exists a constant $C > 0$ such that

$$\iint_{B_{2r}(X_0)} u^2\, dX \le C \iint_{B_r(X_0)} u^2\, dX \tag{2.8}$$

for every $X_0 \in \partial D \cap B_{R_0}(0)$ and r sufficiently small. And (2.8) is what is usually referred to as an *L^2-doubling property*. In Lemma 2.4, not only do we compare the L^2-norm of u for a pair of balls of any radii $0 < s < r$, we also get a precise estimate on the decay rate, which is very close to $d + 2N_{X_0}$, the decay rate for homogeneous harmonic polynomials of degree N_{X_0}. So Lemma 2.4 is much stronger than a doubling property.

Proof. By (8.16) in [9], for r sufficiently small we have

$$B_r(X_0) \subset B_{2r}(X_0 + 6r\hat{\theta}(2r)e_d) = \Psi_{X_0}(B_{2r}),$$

and similarly,

$$B_r(X_0) \supset B_{r/2}\Big(X_0 + \frac{3r}{2}\hat{\theta}\Big(\frac{r}{2}\Big)e_d\Big) = \Psi_{X_0}(B_{r/2}),$$

where $\hat{\theta}$ is defined as in (2.5). Hence

$$\iint_{B_r(X_0)} u^2\,dX \lesssim \iint_{B_{2r}} |u\circ\Psi_{X_0}|^2\,dY \lesssim \int_0^{2r} H(u\circ\Psi_{X_0},\rho)\,d\rho, \tag{2.9}$$

and

$$\iint_{B_r(X_0)} u^2\,dX \gtrsim \iint_{B_{r/2}} |u\circ\Psi_{X_0}|^2\,dY \gtrsim \int_0^{r/2} H(u\circ\Psi_{X_0},\rho)\,d\rho, \tag{2.10}$$

where $H(u\circ\Psi_{X_0},\rho)$ is defined as in (3.8) of [9], and it is essentially the L^2-surface integral of $u\circ\Psi_{X_0}$ in $B_\rho(0)$, adapted to a certain elliptic coefficient matrix. By (2.9), (2.10), (3.30) in [9], (2.1) and the monotonicity of $N_{X_0}(\cdot)$, we have

$$\begin{aligned}\frac{\iint_{B_r(X_0)} u^2\,dX}{\iint_{B_s(X_0)} u^2\,dX} &\lesssim \frac{\int_0^{2r} H(u\circ\Psi_{X_0},\rho)\,d\rho}{\int_0^{s/2} H(u\circ\Psi_{X_0},\rho)\,d\rho}\\ &= \frac{4r}{s}\cdot\frac{\int_0^{s/2} H(u\circ\Psi_{X_0},\frac{4r}{s}\rho)\,d\rho}{\int_0^{s/2} H(u\circ\Psi_{X_0},\rho)\,d\rho} \lesssim \Big(\frac{4r}{s}\Big)^{d+2N_{X_0}(2r)}.\end{aligned}$$

Since $N_{X_0}(2r)$ is uniformly bounded depending on Λ (see Lemma 5.1 in [9]), in particular it follows that

$$\iint_{B_{4r}(X_0)} u^2\,dX \lesssim_\Lambda \iint_{B_r(X_0)} u^2\,dX. \tag{2.11}$$

On the other hand, by (3.29) in [9] and the monotonicity of $N_{X_0}(\cdot)$, we have

$$\begin{aligned}\frac{\iint_{B_r(X_0)} u^2\,dX}{\iint_{B_s(X_0)} u^2\,dX} &\gtrsim \frac{\iint_{B_{4r}(X_0)} u^2\,dX}{\iint_{B_s(X_0)} u^2\,dX} \gtrsim \frac{\int_0^{2r} H(u\circ\Psi_{X_0},\rho)d\rho}{\int_0^{2s} H(u\circ\Psi_{X_0},\rho)d\rho}\\ &= \frac{r}{s}\cdot\frac{\int_0^{2s} H(u\circ\Psi_{X_0},\frac{r}{s}\rho)d\rho}{\int_0^{2s} H(u\circ\Psi_{X_0},\rho)d\rho} \gtrsim \Big(\frac{r}{s}\Big)^{d+2N_{X_0}\exp(-C\int_0^{4r}\frac{\theta(s)}{s}ds)},\end{aligned}$$

where we have used (2.11) in the first inequality. This finishes the proof of the lemma. ■

The next lemma about matrices will be needed in Section 8. If not specified otherwise, for any $n\times n$ matrix M we always use the matrix norm

$$|M| := \sup_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{|Mx|}{|x|}, \tag{2.12}$$

that is compatible with the ℓ^2 vector norms in $\mathbb{R}^n$.

Lemma 2.6. *Suppose S_1 and S_2 are $n \times n$ symmetric matrices such that $|S_1|, |S_2| \ll 1$. Suppose that A and B are $n \times n$ symmetric, positive semi-definite matrices*[4] *such that*

$$A^2 = \mathrm{Id}_n + S_1, \quad B^2 = \mathrm{Id}_n + S_2. \tag{2.13}$$

Then A and B are invertible, and moreover,

$$|A^{-1} - B^{-1}| \lesssim |A^2 - B^2| = |S_1 - S_2|. \tag{2.14}$$

Proof. For each $i = 1, 2$, since $|S_i| \ll 1$, all the eigenvalues of the symmetric matrix $\mathrm{Id}_n + S_i$ are real-valued and close to 1. By the diagonalization of symmetric matrices and (2.13), it follows that all the eigenvalues of A and B are real-valued and close to 1. More precisely, by choosing $|S_i|$ small, we can guarantee that all the eigenvalues of A and B lie in the interval $[1/2, 2]$. The same statements hold for the eigenvalues of their inverse matrices A^{-1} and B^{-1}. Hence by the sub-multiplicativity of matrix norms, we have

$$|A^{-1} - B^{-1}| = |A^{-1}(A - B)B^{-1}| \lesssim |A - B|.$$

Therefore to prove (2.14) it suffices to show that

$$|A - B| \lesssim |A^2 - B^2|.$$

Suppose that $e_1, \dots, e_d$ are eigenvectors of the matrix A (with respective eigenvalues $\lambda_1, \dots, \lambda_d \in [1/2, 2]$) which form an orthonormal basis of $\mathbb{R}^d$. Let $|\cdot|_*$ denote the maximum norm of matrices, i.e., $|M|_* := \max_{i,j} |m_{ij}|$ for any matrix $M = (m_{ij})$. Suppose that under the orthonormal basis $\{e_1, \dots, e_d\}$, the matrix $A - B$ is written as (m_{ij}), and that $|A - B|_* = |m_{ij}|$ for some $i, j \in \{1, \dots, d\}$.

Since

$$A^2 - B^2 = (A - B)A + B(A - B),$$

when we multiply both matrices above by the vector e_j we get that

$$\begin{aligned}(A^2 - B^2)e_j &= (A - B)Ae_j + B(A - B)e_j = \lambda_j(A - B)e_j + B(A - B)e_j \\ &= (\lambda_j \mathrm{Id}_d + B)(A - B)e_j.\end{aligned} \tag{2.15}$$

Consider the diagonalization of the symmetric matrix B in the form

$$B = UDU^{-1}, \quad \text{where } U \text{ is an orthogonal matrix, and } D = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_d \end{pmatrix}.$$

Notice that each μ_k is an eigenvalue of B, and thus $\mu_k \in [1/2, 2]$. It follows that $\lambda_j + \mu_k \in [1, 4]$ for all $k \in \{1, \dots, d\}$. Because orthogonal matrices do not change vector norms, we have that

$$\begin{aligned}&|(\lambda_j \mathrm{Id}_d + B)(A - B)e_j| \\ &\quad = |U(\lambda_j \mathrm{Id}_d + D)U^{-1} \cdot (A - B)e_j| = |(\lambda_j \mathrm{Id}_d + D)U^{-1} \cdot (A - B)e_j| \\ &\quad = \left| \begin{pmatrix} \lambda_j + \mu_1 & & \\ & \ddots & \\ & & \lambda_j + \mu_d \end{pmatrix} \cdot U^{-1}(A - B)e_j \right| \geq |U^{-1}(A - B)e_j| = |(A - B)e_j|.\end{aligned} \tag{2.16}$$

[4] The *positive* square root of the matrix $\mathrm{Id}_n + S_i$ always exists, by considering the diagonalization of the symmetric matrix $\mathrm{Id}_n + S_i$ whose eigenvalues are all strictly positive (since $|S_i| \ll 1$).

Because $A - B = (m_{ij})$ under the orthonormal basis $\{e_1, \dots, e_d\}$, we have by the choice of e_j that

$$|(A - B)e_j| = |(m_{1j}, \dots, m_{dj})| \geq |m_{ij}| = |A - B|_* \approx |A - B|, \tag{2.17}$$

where we used the equivalence of all matrix norms in the last inequality. Combining (2.17), (2.16) and (2.15), we have that

$$|A - B| \lesssim |(A - B)e_j| \leq |(\lambda_j \operatorname{Id}_d + B)(A - B)e_j| = |(A^2 - B^2)e_j| \leq |A^2 - B^2|,$$

which finishes the proof of (2.14). ∎

3. Orthogonal transformation

Let $(x_0, \varphi(x_0))$ be a boundary point such that $\nabla\varphi(x_0) \neq 0$. We want to find an orthogonal transformation $O = O_{x_0} : \mathbb{R}^d \to \mathbb{R}^d$ and a function $\tilde{\varphi} = \tilde{\varphi}_{x_0} : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$O \text{ maps } \operatorname{graph}(\varphi) - (x_0, \varphi(x_0)) \text{ to } \operatorname{graph}(\tilde{\varphi}), \text{ and } \tilde{\varphi}(0) = 0, \nabla\tilde{\varphi}(0) = 0.$$

We first determine the orthogonal matrix O. We write O in the form of a block matrix,

$$O = \begin{pmatrix} \tilde{O} & b \\ d^\intercal & c \end{pmatrix},$$

where $\tilde{O}$ is a $(d-1) \times (d-1)$ matrix, $b, d \in \mathbb{R}^{d-1}$ and $c \in \mathbb{R}$. Since O should be an orthogonal matrix, the block matrices ought to satisfy

$$\begin{cases} \tilde{O}\tilde{O}^\intercal + bb^\intercal = \operatorname{Id}_{d-1}, \\ \tilde{O}d + cb = 0, \\ |d|^2 + c^2 = 1. \end{cases} \tag{3.1}$$

Moreover, in order to guarantee that

$$O\Big(\begin{pmatrix} x \\ \varphi(x) \end{pmatrix} - \begin{pmatrix} x_0 \\ \varphi(x_0) \end{pmatrix}\Big) = \begin{pmatrix} y \\ \tilde{\varphi}(y) \end{pmatrix},$$

or equivalently,

$$\begin{cases} \tilde{O}(x - x_0) + (\varphi(x) - \varphi(x_0))b = y, \\ d \cdot (x - x_0) + c(\varphi(x) - \varphi(x_0)) = \tilde{\varphi}(y), \end{cases} \tag{3.2}$$

and the property that $\nabla\tilde{\varphi}(0) = 0$, the matrix should satisfy

$$d + c\,\nabla\varphi(x_0) = \tilde{O}^\intercal\,\nabla\tilde{\varphi}(0) + (b \cdot \nabla\tilde{\varphi}(0))\,\nabla\varphi(x_0) = 0.$$

Combined with (3.1), we just need

$$\begin{cases} c^2 = (1 + |\nabla\varphi(x_0)|^2)^{-1} \neq 0, \\ d = -c\,\nabla\varphi(x_0), \\ b = \tilde{O}\,\nabla\varphi(x_0), \\ \tilde{O}(\operatorname{Id}_{d-1} + \nabla\varphi(x_0)\,\nabla\varphi(x_0)^\intercal)\,\tilde{O}^\intercal = \operatorname{Id}_{d-1}. \end{cases} \tag{3.3}$$

Modulo the sign, $c \in \mathbb{R}$ is uniquely determined. Since for any non-zero vector $z \in \mathbb{R}^{d-1}$,

$$z^{\mathsf{T}}(\mathrm{Id}_{d-1} + \nabla\varphi(x_0)\nabla\varphi(x_0)^{\mathsf{T}})z = |z|^2 + (z \cdot \nabla\varphi(x_0))^2 \geq |z|^2 > 0, \tag{3.4}$$

we have that the matrix $\mathrm{Id}_{d-1} + \nabla\varphi(x_0)\nabla\varphi(x_0)^{\mathsf{T}}$ is symmetric and positive semi-definite. We can find a solution to the last equation in (3.3), for example, by letting $\tilde{O}$ be a symmetric, positive semi-definite matrix whose inverse matrix $\tilde{O}^{-1}$ is the square root of $\mathrm{Id}_{d-1} + \nabla\varphi(x_0)\nabla\varphi(x_0)^{\mathsf{T}}$. In particular,

$$|\det \tilde{O}|^2 = \frac{1}{\det(\mathrm{Id}_{d-1} + \nabla\varphi(x_0)\nabla\varphi(x_0)^{\mathsf{T}})} = \frac{1}{1 + |\nabla\varphi(x_0)|^2};$$

besides, by (3.4) and by choosing x_0 sufficiently close to the origin so that $|\nabla\varphi(x_0)| \leq 1$, we can guarantee that the eigenvalues of $\tilde{O}$ are bounded from below and above (and the bounds are uniform for all x_0 near the origin). To sum up, the orthogonal matrix is of the form

$$O = \begin{pmatrix} \tilde{O} & \tilde{O}\nabla\varphi(x_0) \\ (-c\,\nabla\varphi(x_0))^{\mathsf{T}} & c \end{pmatrix}, \tag{3.5}$$

where $c \in \mathbb{R}$ and the block matrix $\tilde{O}$ satisfies (3.3).

Next we show that the image of $\mathrm{graph}(\varphi) - (x_0, \varphi(x_0))$ under O is indeed graphical. First, considering (3.2) we look at the map

$$g : x \in \mathbb{R}^{d-1} \mapsto \tilde{O}(x - x_0) + (\varphi(x) - \varphi(x_0))\,b =: y \in \mathbb{R}^{d-1}. \tag{3.6}$$

Clearly $g(x_0) = 0$. We compute

$$Dg(x) = \tilde{O} + b\,(\nabla\varphi(x))^{\mathsf{T}} = \tilde{O} + \tilde{O}\,\nabla\varphi(x_0)(\nabla\varphi(x))^{\mathsf{T}}. \tag{3.7}$$

Hence, in particular,

$$\begin{aligned} Dg(x)|_{x=x_0} &= \tilde{O} + \tilde{O}\,\nabla\varphi(x_0)\big(\nabla\varphi(x_0)\big)^{\mathsf{T}} \\ &= \tilde{O}(\mathrm{Id}_{d-1} + \nabla\varphi(x_0)(\nabla\varphi(x_0))^{\mathsf{T}}) = (\tilde{O}^{\mathsf{T}})^{-1} = \tilde{O}^{-1}, \end{aligned}$$

where we use (3.3) and the symmetry of $\tilde{O}$ in the second to last and last equalities, respectively. By the inverse function theorem, near x_0 the function g has an inverse function g^{-1}, which is defined in a neighborhood of the origin and satisfies

$$Dg^{-1}(y) = \big(Dg(g^{-1}(y))\big)^{-1}. \tag{3.8}$$

Therefore, by defining

$$\tilde{\varphi}(y) = -c\,\nabla\varphi(x_0) \cdot (g^{-1}(y) - x_0) + c\,(\varphi(g^{-1}(y)) - \varphi(x_0)) \tag{3.9}$$

in a neighborhood of the origin, it satisfies the equality (3.2). Moreover,

$$\partial_j \tilde{\varphi}(y) = \sum_i c(\partial_i\varphi(g^{-1}(y)) - \partial_i\varphi(x_0))\,\partial_j(g^{-1}(y))_i,$$

or equivalently,

$$\nabla\tilde{\varphi}(y) = c(Dg^{-1}(y))^{\mathsf{T}}(\nabla\varphi(g^{-1}(y)) - \nabla\varphi(x_0)). \tag{3.10}$$

Moreover, we claim that for y, y' sufficiently close to the origin, we have

$$|\nabla\tilde{\varphi}(y) - \nabla\tilde{\varphi}(y')| \lesssim \theta(2|y - y'|), \tag{3.11}$$

where θ is the modulus of continuity for $\nabla\varphi$. In fact, (3.7) implies that

$$\begin{aligned} |Dg(x) - Dg(x')| &= \left|\tilde{O}\nabla\varphi(x_0)(\nabla\varphi(x) - \nabla\varphi(x'))^{\intercal}\right| \\ &\lesssim |\nabla\varphi(x) - \nabla\varphi(x')| \le \theta(|x - x'|). \end{aligned} \tag{3.12}$$

Here we use the boundedness of $|\nabla\varphi(x_0)|$ and the maximum norm $|\tilde{O}|$. In particular, since the eigenvalues of the matrix $Dg(x_0) = \tilde{O}^{-1}$ are bounded from above and below, it follows from (3.12) that for x sufficiently close to x_0, the eigenvalues of $Dg(x)$ are also uniformly bounded from above and below, and thus the same holds for its inverse $Dg^{-1}(y)$ (by (3.8)), for any y sufficiently close to $0 = g(x_0)$. Moreover, let $x = g^{-1}(y)$, $x' = g^{-1}(y')$. Since

$$\left|Dg(x)\big[(Dg(x))^{-1} - (Dg(x'))^{-1}\big]Dg(x')\right| = |Dg(x') - Dg(x)| \lesssim \theta(|x - x'|),$$

we get

$$\begin{aligned} |Dg^{-1}(y) - Dg^{-1}(y')| &= |(Dg(x))^{-1} - (Dg(x'))^{-1}| \\ &\lesssim |Dg^{-1}(y)| \cdot \theta(|x - x'|) \cdot |Dg^{-1}(y')| \lesssim \theta(|x - x'|). \end{aligned} \tag{3.13}$$

Additionally,

$$|x - x'| = |g^{-1}(y) - g^{-1}(y')| \le \|Dg^{-1}\|_\infty |y - y'| \le 2|y - y'|. \tag{3.14}$$

Therefore, by combining (3.10), (3.13) and (3.14), we get

$$\begin{aligned} &|\nabla\tilde{\varphi}(y) - \nabla\tilde{\varphi}(y')| \\ &= \left|c(Dg^{-1}(y))^{\intercal}(\nabla\varphi(g^{-1}(y)) - \nabla\varphi(x_0)) - c(Dg^{-1}(y'))^{\intercal}(\nabla\varphi(g^{-1}(y')) - \nabla\varphi(x_0))\right| \\ &\le c\left|(Dg^{-1}(y))^{\intercal}[\nabla\varphi(g^{-1}(y)) - \nabla\varphi(g^{-1}(y'))]\right| \\ &\quad + c\left|\big[Dg^{-1}(y) - Dg^{-1}(y')\big]^{\intercal}(\nabla\varphi(g^{-1}(y')) - \nabla\varphi(x_0))\right| \\ &\lesssim \theta(|g^{-1}(y) - g^{-1}(y')|) + |Dg^{-1}(y) - Dg^{-1}(y')| \cdot \theta(|g^{-1}(y') - x_0|) \\ &\lesssim \theta(2|y - y'|), \end{aligned}$$

which finishes the proof of the claim (3.11).

4. Flattening and extension of *u* across the boundary

By the previous section, we may assume that near any boundary point $(x_0, \varphi(x_0)) \in \partial D \cap B_{R_0}(0)$, we have $\nabla\varphi(x_0) = 0$. If not, we just apply the orthogonal transformation O_{x_0}, under which the domain D (locally) becomes the region above the graph $\tilde{\varphi} = \tilde{\varphi}_{x_0}$, which satisfies $\tilde{\varphi}(0) = 0$, $\nabla\tilde{\varphi}(0) = 0$ and the modulus of continuity becomes $\theta(2\cdot)$ (modulo uniform constants). Hence it suffices to consider D near the boundary point $X_0 = (0, 0)$ with a flat tangent, i.e., $\nabla\varphi(0) = 0$.

Let u be a harmonic function in D. We consider the map

$$\Phi : (y,s) \in \mathbb{R}^d_+ \mapsto (y, s+\varphi(y)) =: (x,t) \in D, \tag{4.1}$$

and $v\colon \mathbb{R}^d_+ \to \mathbb{R}$ defined by $v(y,s) := u \circ \Phi(y,s)$. A simple computation shows that v is the solution to the elliptic operator $-\operatorname{div}(A(y,s)\nabla v) = 0$ in $\mathbb{R}^d_+$, where the coefficient matrix $A(y,s)$ is given by

(4.2)

$$A(y,s) = (\det D\Phi)\cdot (D\Phi(y,s))^{-1}(D\Phi^{\intercal}(y,s))^{-1} = \begin{pmatrix} \mathrm{Id}_{d-1} & -\nabla\varphi(y) \\ (-\nabla\varphi(y))^{\intercal} & 1+|\nabla\varphi(y)|^2 \end{pmatrix}.$$

In particular, $A(y,s)$ is independent of the s-variable, so we will denote it by $A(y)$. By the properties of φ, we know that $A(0) = \mathrm{Id}$ and $|A(y) - A(y')| \lesssim \theta(2|y-y'|)$.

Since $v = u\circ\Phi$ vanishes on $B_{4R_0}(0) \cap \partial\mathbb{R}^d_+$, we can extend v by odd reflection, i.e., we let

$$\tilde{v}(y,s) = \begin{cases} v(y,s), & (y,s) \in \mathbb{R}^d_+\,, \\ -v(y,-s), & (y,s)\in \mathbb{R}^d_-\,. \end{cases} \tag{4.3}$$

We also define

$$\tilde{A}(y,s) := \begin{cases} A(y,s) = \begin{pmatrix} \mathrm{Id}_{d-1} & -\nabla\varphi(y) \\ (-\nabla\varphi(y))^{\intercal} & 1+|\nabla\varphi(y)|^2 \end{pmatrix}, & (y,s)\in\mathbb{R}^d_+, \\ \begin{pmatrix} \mathrm{Id}_{d-1} & \nabla\varphi(y) \\ (\nabla\varphi(y))^{\intercal} & 1+|\nabla\varphi(y)|^2 \end{pmatrix}, & (y,s)\in\mathbb{R}^d_-. \end{cases}$$

A simple computation shows that the co-normal derivatives of $\tilde{v}(y,s)$ from above (that is, $\mathbb{R}^d_+$) and below (that is, $\mathbb{R}^d_-$) cancel each other out, or more precisely,

$$\lim_{s\to 0+} A(y,s)\nabla v(y,s)\cdot(0,-1) + \lim_{s\to 0-} \tilde{A}(y,s)\nabla\tilde{v}(y,s)\cdot(0,1) = 0.$$

Using integration by parts, the newly-defined function $\tilde{v}$ satisfies $-\operatorname{div}(\tilde{A}(y,s)\nabla\tilde{v}) = 0$ in $B_{4R}(0)$. For simplicity we still denote $\tilde{v}$ as v.

To summarize, by an orthogonal transformation (in Section 3), flattening the domain, and an odd reflection, we have modified the original harmonic function u near any boundary point $X_0 \in B_{R_0}(0)\cap\partial D$ into a solution v to a divergence-form elliptic operator $Lv := -\operatorname{div}(\tilde{A}(y,s)\nabla v)$ in an entire ball $B_{4R_0}(0)$, where the coefficient matrix $\tilde{A}$ is the identity matrix at the origin, and it is Dini continuous in the upper and lower half space, respectively. We emphasize that $\tilde{A}$ is not even continuous across $\partial\mathbb{R}^d_+$. In general, solutions to operators of the form L may not have finite vanishing order at an interior point. In fact, even if the coefficient matrix is Hölder continuous with exponent less than 1, the corresponding solution may still have infinite vanishing order, for example see [10]. However, since v comes from the harmonic function u in a C^1-Dini domain, with vanishing boundary data, we can show v does have finite vanishing order.

By the doubling property of u in Lemma 2.4, we can easily show the following doubling property of v.

Lemma 4.1. *For any pair of radii* $0 < r_1 < r_2$ *sufficiently small, we have*

$$\left(\frac{r_1}{r_2}\right)^{d+2N_{X_0}(2r_2)} \lesssim \frac{\iint_{B_{r_1}(0)} v^2 dy ds}{\iint_{B_{r_2}(0)} v^2 dy ds} \lesssim \left(\frac{r_1}{r_2}\right)^{d+2N_{X_0}\exp(-C\int_0^{4r_2}\frac{\theta(s)}{s}ds)}. \tag{4.4}$$

Proof. Recall we defined v by $v = u \circ \Phi$ and reflection across $\partial\mathbb{R}^d_+$. Hence

$$\iint_{B_r(0)} v^2\, dy\, ds = 2\iint_{B_r^+(0)} |u \circ \Phi(y,s)|^2\, dy\, ds \approx \iint_{\Phi(B_r^+(0))} u^2\, dx\, dt.$$

For any $(y,s) \in B_r^+(0)$, since $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$, it follows that

$$|\varphi(y)| = |\varphi(y) - \varphi(0)| \le \sup_{\xi\in[0,y]} |\nabla\varphi(\xi)| \cdot |\xi| \lesssim r\,\theta(2r).$$

Hence

$$|\Phi(y,s)| = |(y, s + \varphi(y))| < r(1 + C\theta(2r))$$

and

$$\iint_{B_r(0)} v^2\, dy\, ds \approx \iint_{\Phi(B_r^+(0))} u^2\, dx\, dt \le \iint_{B_{2r}(X_0)} u^2\, dX.$$

Similarly,

$$\iint_{B_r(0)} v^2\, dy\, ds \approx \iint_{\Phi(B_r^+(0))} u^2\, dx\, dt \ge \iint_{B_{\frac{r}{2}}(X_0)} u^2\, dX.$$

On the other hand, in Lemma 2.4 we have shown that

$$\iint_{B_{2r}(X_0)} u^2\, dX \approx \iint_{B_r(X_0)} u^2\, dX \approx \iint_{B_{\frac{r}{2}}(X_0)} u^2\, dX,$$

with constants depending on Λ. Therefore we conclude that

$$\iint_{B_r(0)} v^2\, dy\, ds \approx \iint_{B_r(X_0)} u^2\, dX,$$

and the estimates (4.4) follows from (2.7). ■

Corollary 4.2. *For any pair of radii* $0 < r_1 < r_2$ *sufficiently small, we have*

$$\left(\frac{r_1}{r_2}\right)^{N_{X_0}(2r_2)} \lesssim \frac{\sup_{B_{r_1}(0)} |v|}{\sup_{B_{r_2}(0)} |v|} \lesssim \left(\frac{r_1}{r_2}\right)^{N_{X_0}\exp(-C\int_0^{4r_2}\frac{\theta(s)}{s}ds)}. \tag{4.5}$$

Proof. Recall that solutions to elliptic PDEs with vanishing boundary data satisfy the following boundary L^∞ bound:

$$\sup_{B_r(0)} |v| \lesssim \left(\frac{1}{r^d}\iint_{B_{2r}(0)} v^2\, dy\, ds\right)^{1/2},$$

see for instance Lemma 1.1.22 in [7]. Combined with the doubling property (4.4), we have that

$$\sup_{B_r(0)} |v| \lesssim \left(\frac{1}{r^d}\iint_{B_{2r}(0)} v^2\,dy\,ds\right)^{1/2} \lesssim \left(\frac{1}{r^d}\iint_{B_r(0)} v^2\,dy\,ds\right)^{1/2}.$$

On the other hand,

$$\left(\frac{1}{r^d}\iint_{B_r(0)} v^2\,dy\,ds\right)^{1/2} \lesssim \sup_{B_r(0)} |v|.$$

Therefore,

$$\sup_{B_r(0)} |v| \approx \left(\frac{1}{r^d}\iint_{B_r(0)} v^2\,dy\,ds\right)^{1/2},$$

and the L^∞-doubling property follows from the L^2-doubling property in Corollary 4.1. ∎

Let $R \in (0, R_0)$ be sufficiently small such that Corollary 4.2 holds up to scale $2R$. Then for any $0 < r < 2R$ we have

$$C_1(R)\cdot r^{N_{X_0}(4R)} \le \sup_{B_r(0)} |v| \le C_2(R)\cdot r^{N_{X_0}\exp(-C\int_0^{8R}\frac{\theta(s)}{s}\,ds)}. \tag{4.6}$$

For any $\alpha \in (0,1)$ sufficiently small, we may choose R small enough such that

$$\exp\left(C\int_0^{8R}\frac{\theta(s)}{s}\,ds\right) \le \frac{N_{X_0}}{N_{X_0}-\alpha}, \quad N_{X_0}(4R) \le N_{X_0}+\alpha. \tag{4.7}$$

Note that in order to satisfy the second inequality, the choice of R is X_0-dependent. It then follows from (4.6) that

$$r^{N_{X_0}+\alpha} \lesssim \sup_{B_r(0)} |v| \lesssim r^{N_{X_0}-\alpha}. \tag{4.8}$$

In particular, since $\alpha < 1$, it follows that

$$\limsup_{Y\to 0}\frac{|v(Y)|}{|Y|^{N_{X_0}+1}} = +\infty.$$

On the other hand, by the boundary gradient estimate with Dini-continuous coefficient in $\mathbb{R}^d_+$ and in $\mathbb{R}^d_-$ (see Proposition 2.7 in [3], or more precisely, Lemma 2.11 in [3]), for any $Y \in B_R(0)$ we have

$$\frac{|v(Y)|}{|Y|} \lesssim \sup_{B_R(0)} |\nabla v| \lesssim \frac{1}{R}\left(\frac{1}{R^d}\iint_{B_{2R}(0)} v^2\,dy\,ds\right)^{1/2} < +\infty.$$

Hence

$$\frac{\sup_{B_r(0)}|v|}{r} \le C(R) < +\infty.$$

This estimate, combined with (4.8), implies that for any $k = 1,\dots,N_{X_0}-1$ (or for $k=1$ when $N_{X_0}=1$) we have

$$|v(Y)| \le C_k|Y|^k \quad \text{for any } Y \in B_R(0).$$

We consider two cases:

$$\text{either} \quad \limsup_{Y\to 0} \frac{|v(Y)|}{|Y|^{N_{X_0}}} = +\infty, \quad \text{or} \quad \limsup_{Y\to 0} \frac{|v(Y)|}{|Y|^{N_{X_0}}} < +\infty.$$

(When $N_{X_0} = 1$, we can only have the second case.) In both cases, there exists $N \in \mathbb{N}$ such that

$$|v(Y)| \le C_N |Y|^N \quad \text{for any } Y \in B_{2R}(0), \tag{4.9}$$

and

$$\limsup_{Y\to 0} \frac{|v(Y)|}{|Y|^{N+1}} = +\infty. \tag{4.10}$$

We call N the *vanishing order* of v (at the origin). Notice that the integer $N = N_{X_0} - 1$ in the first case, and $N = N_{X_0}$ in the second case. A priori we can not rule out the first case, but at the end of the paper we will show it is impossible and v does have vanishing order exactly N_{X_0}.

We remark that a priori we only know there exist $R' = R'(X_0)$, possibly smaller than the R chosen in (4.7), and $C'_N > 0$, such that

$$|v(Y)| \le C'_N |Y|^N \quad \text{for any } Y \in B_{2R'}(0), \tag{4.11}$$

i.e., the inequality (4.9) holds in a smaller ball. When $Y \in B_{2R}(0) \setminus B_{2R'}(0)$, by the upper bound in (4.8), we have

$$|v(Y)| \le C|Y|^{N_{X_0}-\alpha} \le C' R^{N_{X_0}-\alpha} \le C(R', R, N_{X_0})(2R')^N \le C_N |Y|^N.$$

Therefore (4.11) holds for all $Y \in B_{2R}(0)$, possibly with a bigger constant $C_N \ge C'_N$.

5. Proof of the expansion

In this section, we will prove that there exists a non-trivial homogeneous harmonic polynomial P_N of degree N such that, in $B_{R/2}(0)$, v has the expansion

$$v(y,s) = P_N(y,s) + \psi(y,s), \tag{5.1}$$

where

$$|\psi(y,s)| \le C\, C_N |(y,s)|^N\, \tilde{\theta}(|(y,s)|), \tag{5.2}$$

and $\tilde{\theta}(\cdot)$ is defined in (5.19) and it satisfies that $\tilde{\theta}(r) \to 0$ as $r \to 0$.

For simplicity, we denote $r := |(y,s)|$. Assume that $0 < r \le R/2$. We rewrite the equation $-\operatorname{div}(\tilde{A}(y,s)\nabla v) = 0$ as

$$-\Delta v = \operatorname{div}\big((\tilde{A}(y,s) - \mathrm{Id})\nabla v\big).$$

Note that the coefficient matrix satisfies $\tilde{A}(0) = A(0) = \mathrm{Id}$. We denote

$$\vec{f}(y,s) := (\tilde{A}(y,s) - \mathrm{Id})\nabla v(y,s).$$

In Proposition 2.7 of [3] (or more precisely, in Lemma 2.11 of [3]), the authors proved that a solution to an elliptic operator with Dini-continuous coefficients and which vanishes on an open set of the boundary satisfies the boundary gradient estimate. Applying it to $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ respectively, we get

$$|\nabla v(y,s)| \lesssim \frac{1}{r}\Big(\frac{1}{r^d}\iint_{B_{2r}(0)} v^2\, dZ\Big)^{1/2} \lesssim \frac{1}{r}\sup_{B_{2r}(0)} |v|. \tag{5.3}$$

Combined with the estimate (4.9), we get

$$|\vec{f}(y,s)| \leq |\tilde{A}(y,s) - \mathrm{Id}\,| \cdot |\nabla v(y,s)| \lesssim \frac{\theta(2r)}{r} \cdot \sup_{B_{2r}(0)} |v| \lesssim \theta(2r) r^{N-1}, \tag{5.4}$$

where the constant is just a constant multiple of the constant C_N in the estimate (4.9).

Let ζ be a smooth cut-off function such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{R/2}(0)$, and ζ is compactly supported in $B_R(0)$. Let $\Gamma(\xi) = c_d|\xi|^{2-d}$ be the fundamental solution of the Laplacian in $\mathbb{R}^d$ with $d \geq 3$. (The proof for the planar case $d = 2$ with $\Gamma(\xi) = c \log|\xi|$ is similar.) In the ball $B_R(0)$, we define

$$w(Y) := \iint_{\{|Z|<R\}} \Gamma(Y - Z)\, \mathrm{div}(\vec{f}\zeta)(Z)\, dZ. \tag{5.5}$$

By the divergence theorem, we have

$$\begin{aligned} w(Y) &= \iint_{\{|Z|<R\}} \Gamma(Y - Z)\, \mathrm{div}(\vec{f}\zeta)(Z)\, dZ \\ &= -\iint_{\{|Z|<R\}} \nabla_Z(\Gamma(Y - Z)) \cdot \vec{f}\zeta(Z)\, dZ = \iint_{\{|Z|<R\}} \nabla\Gamma(Y - Z) \cdot \vec{f}\zeta(Z)\, dZ. \end{aligned}$$

By considering the above integral in the regions $\{|Z| < 2|Y|\}$ and $\{2|Y| \leq |Z| < R\}$, one can show it is well defined, and hence $w(Y)$ is well defined. Moreover, it satisfies

$$-\Delta w(Y) = \mathrm{div}(\vec{f}\zeta)(Y) = -\Delta v(Y), \quad \text{for } Y \in B_{R/2}(0),$$

i.e., $v - w$ is a harmonic function in $B_{R/2}(0)$. Hence $v - w(Y)$ can be written as the infinite sum of homogeneous harmonic polynomials. In particular, we have

$$v - w(Y) = P_1(Y) + \psi_1(Y), \tag{5.6}$$

where P_1 is a harmonic polynomial of degree at most N, and the error term ψ_1 satisfies $|\psi_1(Y)| \leq C_1|Y|^{N+1}$, where C_1 only depends on the radius R and the constant C_N in (4.9).

Next, we consider the Taylor expansion of $\nabla\Gamma(\cdot - Z)$ near the origin. We let $\beta = (\beta_1, \dots, \beta_d)$ denote a d-index. For each $k \in \{0, \dots, N\}$, we define an $\mathbb{R}^d$-valued function as follows:

$$\widetilde{\Gamma}_k(Y, Z) := \sum_{|\beta|=k} D^\beta \nabla\Gamma(-Z) \frac{Y^\beta}{\beta!}.$$

For fixed $Z \in \mathbb{R}^d \setminus \{0\}$, the function $\widetilde{\Gamma}_k(\cdot, Z)$ is a harmonic homogeneous polynomial of degree k. Besides, since

$$|D^\beta \nabla\Gamma(-Z)| \lesssim |Z|^{1-d-|\beta|},$$

we have

$$|\widetilde{\Gamma}_k(Y, Z)| \le C_k |Z|^{1-d-k} |Y|^k, \tag{5.7}$$

where the constant C_k depends on k as well as the dimension d.

Let

$$P_2(Y) := \iint_{\{|Z|<R\}} \sum_{k=0}^N \widetilde{\Gamma}_k(Y, Z) \cdot \vec{f}\,\zeta(Z)\, dZ. \tag{5.8}$$

Since $\widetilde{\Gamma}_k$ is not well defined at $Z = 0$, we first need to justify that the above integral is well defined. In fact, for any $\delta \in (0, R)$, let

$$f_\delta(Y) := \iint_{\{\delta\le|Z|<R\}} \sum_{k=0}^N \widetilde{\Gamma}_k(Y, Z) \cdot \vec{f}\,\zeta(Z)\, dZ.$$

By (5.7) and (5.4), we have

$$\begin{aligned} |f_\delta(Y)| &\le \sum_{k=0}^N \iint_{\{\delta\le|Z|<R\}} |\widetilde{\Gamma}_k(Y, Z)| \cdot |\vec{f}(Z)|\, dZ \lesssim \sum_{k=0}^N |Y|^k \int_\delta^R s^{N-k-1}\theta(2s)\, ds \\ &\le \sum_{k=0}^{N-1} |Y|^k R^{N-k}\theta(2R) + |Y|^N \int_{2\delta}^{2R} \frac{\theta(s)}{s}\, ds, \end{aligned}$$

which is uniformly bounded as $\delta \to 0$. Moreover, let $\gamma = (\gamma_1, \dots, \gamma_d)$ be a d-index such that $|\gamma| = j \in \{0, 1, \dots, N\}$. Notice that when we take the Y-derivative of $\widetilde{\Gamma}_k$, it does not affect the coefficients which just depend on Z. Then similarly we obtain

$$|D^\gamma f_\delta(Y)| \lesssim \sum_{k=j}^{N-1} |Y|^{k-j} R^{N-k}\theta(2R) + |Y|^{N-j} \int_{2\delta}^{2R} \frac{\theta(s)}{s}\, ds,$$

which is also uniformly bounded as $\delta \to 0$. (When $j = N$, the first term on the right-hand side does not appear.) Since $f_\delta(Y)$ is a polynomial of degree at most N, it is completely determined by $D^\gamma f_\delta(0)$ with indices $|\gamma| \in \{0, 1, \dots, N\}$. Therefore as $\delta \to 0$ (modulo passing to a subsequence), the sequence $f_\delta(Y)$ converges to $P_2(Y)$ in $C^j_{\mathrm{loc}}(\mathbb{R}^d)$, for any $j \in \mathbb{N}$. Therefore P_2 is well defined. Moreover, since $f_\delta(Y)$ is a harmonic function for any $\delta > 0$, the limit function $P_2(Y)$ is a harmonic polynomial of degree less than or equal to N.

We will estimate the error

$$\psi_2(Y) := w(Y) - P_2(Y) = \iint_{\{|Z|<R\}} \Big(\nabla\Gamma(Y-Z) - \sum_{k=0}^{N} \widetilde{\Gamma}_k(Y,Z) \Big) \cdot \vec{f}\,\zeta(Z)\, dZ. \tag{5.9}$$

For each $\tau > 0$, we denote

$$\bar{v}(\tau) := \sup_{B_\tau(0)} |v|. \tag{5.10}$$

By the estimate (4.9), we know that $\bar{v}(\tau) \lesssim \tau^N$ whenever $0 < \tau \le 2R$. Denote $r = |Y| < R/2$. We split the integral in (5.9) into three parts:

$$\mathrm{I} := \iint_{\{|Z|<2r\}} \nabla\Gamma(Y-Z) \cdot \vec{f}\,\zeta(Z)\, dZ,$$

$$\mathrm{II} := \iint_{\{|Z|<2r\}} \sum_{k=0}^{N} \widetilde{\Gamma}_k(Y,Z) \cdot \vec{f}\,\zeta(Z)\, dZ,$$

$$\mathrm{III} := \iint_{\{2r\le|Z|<R\}} \Big(\nabla\Gamma(Y-Z) - \sum_{k=0}^{N} \widetilde{\Gamma}_k(Y,Z) \Big) \cdot \vec{f}\,\zeta(Z)\, dZ.$$

By (5.4) and the bound on the fundamental solution Γ, we can easily estimate

$$|\mathrm{I}| \lesssim \iint_{\{|Z|<2r\}} |Y-Z|^{1-d} \cdot |\vec{f}(Z)|\, dZ \lesssim \frac{\theta(4r)}{r} \cdot \bar{v}(4r) \cdot \iint_{\{|X|<3r\}} |X|^{1-d}\, dX$$

$$\lesssim \theta(4r) \cdot \bar{v}(4r) \tag{5.11}$$

$$\lesssim \theta(4r)\, r^N. \tag{5.12}$$

Combining (5.7) and (5.4), we get

$$|\mathrm{II}| \lesssim \sum_{k=0}^{N} r^k \cdot \iint_{\{|Z|<2r\}} |\vec{f}(Z)||Z|^{1-d-k}\, dZ$$

$$\lesssim \sum_{k=0}^{N} r^k \int_0^{2r} \tau^{1-d-k} \cdot \frac{\theta(2\tau)}{\tau} \cdot \bar{v}(2\tau) \cdot \tau^{d-1}\, d\tau$$

$$\lesssim r^N \int_0^{2r} \frac{\theta(2\tau)}{\tau} \cdot \frac{\bar{v}(2\tau)}{\tau^N}\, d\tau \tag{5.13}$$

$$\lesssim \Big(\int_0^{4r} \frac{\theta(s)}{s}\, ds \Big) r^N. \tag{5.14}$$

Lastly, since $\nabla\Gamma(\cdot)$ is smooth away from the origin, on the set $\{|Z|\geq 2r\}$ we have the expansion

$$\nabla\Gamma(Y-Z)-\sum_{k=0}^{N}\widetilde{\Gamma}_k(Y,Z)=\sum_{|\beta|=N+1}D^\beta\nabla\Gamma(\theta Y-Z)\frac{Y^\beta}{\beta!}$$

for some $\theta\in[0,1]$. Hence by the decay of the fundamental solution, we have

$$\Big|\nabla\Gamma(Y-Z)-\sum_{k=0}^{N}\widetilde{\Gamma}_k(Y,Z)\Big|\leq\sum_{|\beta|=N+1}|\theta Y-Z|^{1-d-|\beta|}\cdot\frac{r^{|\beta|}}{\beta!}\lesssim\frac{r^{N+1}}{|Z|^{d+N}},$$

where in the last inequality the constant multiple also depends on N. Therefore

$$\begin{aligned}|\,\mathrm{III}\,|&\lesssim\iint_{\{2r\leq|Z|<R\}}\frac{r^{N+1}}{|Z|^{d+N}}\cdot|\vec{f}(Z)|\,dZ\\&\lesssim r^{N+1}\int_{2r}^{R}\frac{1}{\tau^{d+N}}\cdot\frac{\theta(2\tau)}{\tau}\cdot\bar{v}(2\tau)\cdot\tau^{d-1}\,d\tau\\&\lesssim r^{N+1}\int_{2r}^{R}\frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau\qquad(5.15)\\&\lesssim r^N\Big(r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds\Big).\qquad(5.16)\end{aligned}$$

We claim that

$$r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds\to 0\quad\text{as } r\to 0.\tag{5.17}$$

In fact, we split into two cases: either $\int_0^{2R}\frac{\theta(s)}{s^2}\,ds<+\infty$ (which happens if $\theta(s)$ decays faster than s), or $\int_r^{2R}\frac{\theta(s)}{s^2}\,ds\to+\infty$ as $r\to 0+$. In the first case,

$$r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds\leq\Big(\int_0^{2R}\frac{\theta(s)}{s^2}\,ds\Big)r\to 0\quad\text{as } r\to 0;$$

and in the second case, applying L'Hospital rule, we get

$$\lim_{r\to0+}r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds=\lim_{r\to0+}\frac{-\frac{\theta(4r)}{4r^2}}{-\frac{1}{r^2}}=\lim_{r\to0+}\frac{\theta(4r)}{4}=0,$$

which also proves the claim (5.17). Combining (5.9), (5.12), (5.14), (5.16) and (5.17), we conclude that

$$|\psi_2(Y)|\lesssim r^N\Big(\theta(4r)+\int_0^{4r}\frac{\theta(s)}{s}\,ds+r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds\Big)=:r^N\tilde{\theta}(r),\tag{5.18}$$

with

$$\tilde{\theta}(r)=\theta(4r)+\int_0^{4r}\frac{\theta(s)}{s}\,ds+r\int_{4r}^{2R}\frac{\theta(s)}{s^2}\,ds\to 0\quad\text{as } r\to 0.\tag{5.19}$$

Finally, combining (5.6) and (5.9), we have the following expansion in $B_{R/2}(0)$:

$$v(Y) = P_1(Y) + P_2(Y) + \psi_1(Y) + \psi_2(Y), \tag{5.20}$$

where $P_1 + P_2$ is a harmonic polynomial of degree less or equal to N, $|\psi_1(Y)| \le C_1 r^{N+1}$ and $|\psi_2(Y)| \le C_2 r^N \tilde{\theta}(r)$. In the special case when $\theta(s) \sim s^\alpha$ with $\alpha \in (0,1)$, it is easy to see that $\tilde{\theta}(r) \sim r^\alpha$. By the estimate (4.9), we know that either $P_1 + P_2 \equiv 0$, or it is nontrivial and homogeneous of degree exactly N. In the special case when $\theta(s) \sim s^\alpha$, it is easy to rule out the first case, as is shown in [5]. However, the proof is more delicate for general Dini parameters.

Assume for the sake of contradiction that $P_1 + P_2 \equiv 0$. Then (5.20) implies that

$$v(Y) = \psi_1(Y) + \psi_2(Y), \tag{5.21}$$

where $|\psi_1(Y)| \le C_1 r^{N+1}$. Recall that we split $\psi_2(Y)$ into three terms, I, II and III. Combining (5.21) and (5.11), (5.13), (5.15), we get

$$\begin{aligned} |v(Y)| &\le |\,\mathrm{I}\,| + |\,\mathrm{II}\,| + |\,\mathrm{III}\,| + |\psi_1(Y)| \\ &\lesssim \theta(4r)\cdot \bar{v}(4r) + r^N \int_0^{2r} \frac{\theta(2\tau)}{\tau}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau \\ &\quad + r^{N+1}\int_{2r}^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau + C_1 r^{N+1}. \end{aligned} \tag{5.22}$$

Now let $\rho \in (0, R/2)$ be fixed, and we let Y vary in the annulus $B_\rho(0) \setminus B_{\rho/2}(0)$. Then $r = |Y| \in [\rho/2, \rho)$, and (5.22) implies

$$\begin{aligned} |v(Y)| &\lesssim \theta(4\rho)\cdot\bar{v}(4\rho) + \rho^N \int_0^{2\rho} \frac{\theta(2\tau)}{\tau}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau \\ &\quad + \rho^{N+1}\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau + C_1\rho^{N+1}. \end{aligned} \tag{5.23}$$

Similarly to (5.10), we define

$$\bar{\bar{v}}(\tau) := \sup_{B_\tau(0)\setminus B_{\tau/2}(0)} |v|, \quad \text{for any } \tau > 0.$$

We claim that

$$\bar{v}(\tau) \lesssim \bar{\bar{v}}(\tau) \le \bar{v}(\tau). \tag{5.24}$$

The second equality is simply because of the inclusion $B_\tau(0) \setminus B_{\tau/2}(0) \subset B_\tau(0)$. To prove the first inequality, we note that

$$\bar{v}(\tau) = \sup_{k\in\mathbb{N}_0} \bar{\bar{v}}(2^{-k}\tau). \tag{5.25}$$

For each $k \in \mathbb{N}$, as in the proof of Corollary 4.2 and Lemma 4.1 (applying the same argument to annuli instead of solid balls), we get

$$\frac{\bar{\bar{v}}(2^{-k}\tau)}{\bar{\bar{v}}(\tau)} \lesssim (2^{-k})^{N\chi_0-\alpha} \le (2^{-k})^{1-\alpha}, \quad \text{for any } 0 < \tau < 2R.$$

To get the exponent $N_{X_0} - \alpha$ in the first inequality, we have used the choice of R in (4.7). Combined with (5.25), we get

$$\bar{v}(\tau) = \sup_{k\in\mathbb{N}_0} \bar{\bar{v}}(2^{-k}\tau) \lesssim \bar{\bar{v}}(\tau),$$

with a constant depending on α. This finishes the proof of the claim. Applying the doubling property in Corollary 4.2 and (5.24) to the estimate (5.23), we get

$$\begin{aligned} \bar{\bar{v}}(\rho) &\lesssim \theta(4\rho)\cdot\bar{v}(4\rho) + \rho^N \int_0^{2\rho} \frac{\theta(2\tau)}{\tau}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau \\ &\quad + \rho^{N+1}\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{v}(2\tau)}{\tau^N}\,d\tau + C_1\rho^{N+1} \\ &\lesssim \theta(4\rho)\cdot\bar{\bar{v}}(\rho) + \rho^N \int_0^{2\rho} \frac{\theta(2\tau)}{\tau}\cdot\frac{\bar{\bar{v}}(\tau/2)}{\tau^N}\,d\tau \\ &\quad + \rho^{N+1}\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{\bar{v}}(\tau)}{\tau^N}\,d\tau + C_1\rho^{N+1} \\ &\lesssim \theta(4\rho)\cdot\bar{\bar{v}}(\rho) + \rho^N \int_0^{\rho} \frac{\theta(4\tau)}{\tau}\cdot\frac{\bar{\bar{v}}(\tau)}{\tau^N}\,d\tau \\ &\quad + \rho^{N+1}\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{\bar{v}}(\tau)}{\tau^N}\,d\tau + C_1\rho^{N+1}. \end{aligned} \tag{5.26}$$

We choose R sufficiently small so that

$$C\cdot\theta(2R) < 1/2,$$

where $C > 0$ denotes the constant in front of the first term in (5.26). This way, we can move the first term to the left-hand side for any $\rho < R/2$, and (5.26) becomes

$$\bar{\bar{v}}(\rho) \lesssim \rho^N \int_0^{\rho} \frac{\theta(4\tau)}{\tau}\cdot\frac{\bar{\bar{v}}(\tau)}{\tau^N}\,d\tau + \rho^{N+1}\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot\frac{\bar{\bar{v}}(\tau)}{\tau^N}\,d\tau + C_1\rho^{N+1} \tag{5.27}$$

By setting

$$h(\tau) := \frac{\bar{\bar{v}}(\tau)}{\tau^N}$$

and dividing both sides of (5.27) by ρ^N, we get

$$h(\rho) \lesssim \int_0^{\rho} \frac{\theta(4\tau)}{\tau}\cdot h(\tau)\,d\tau + \rho\int_\rho^R \frac{\theta(2\tau)}{\tau^2}\cdot h(\tau)\,d\tau + C_1\rho. \tag{5.28}$$

For every $\varepsilon > 0$, let

$$g_\varepsilon(\tau) := \frac{h(\tau)}{\tau+\varepsilon} = \frac{\bar{\bar{v}}(\tau)}{\tau^N}\cdot\frac{1}{\tau+\varepsilon} > 0.$$

By (4.9), each $g_\varepsilon(\cdot)$ is bounded from above (with a constant depending on ε):

$$g_\varepsilon(\tau) \le \frac{C_N}{\tau+\varepsilon} < \frac{C_N}{\varepsilon} < +\infty.$$

Let $\rho_\varepsilon := \rho + \varepsilon$ and $\tau_\varepsilon := \tau + \varepsilon$. Dividing both sides of (5.28) by ρ_ε and plugging in $g_\varepsilon(\cdot)$, the inequality becomes

$$g_\varepsilon(\rho) = \frac{h(\rho)}{\rho_\varepsilon} \lesssim \frac{1}{\rho_\varepsilon}\int_0^\rho \frac{\theta(4\tau)}{\tau}\cdot g_\varepsilon(\tau)\,\tau_\varepsilon\,d\tau + \frac{\rho}{\rho_\varepsilon}\int_\rho^R \frac{\theta(2\tau)}{\tau}\cdot g_\varepsilon(\tau)\,\frac{\tau_\varepsilon}{\tau}\,d\tau + C_1.$$

Notice that $\tau_\varepsilon < \rho_\varepsilon$ when $\tau < \rho$, and

$$\frac{\rho}{\rho_\varepsilon}\cdot\frac{\tau_\varepsilon}{\tau} < 1 \quad \text{when } \tau > \rho.$$

It then follows that

$$g_\varepsilon(\rho) \lesssim \int_0^\rho \frac{\theta(4\tau)}{\tau}\cdot g_\varepsilon(\tau)\,d\tau + \int_\rho^R \frac{\theta(2\tau)}{\tau}\cdot g_\varepsilon(\tau)\,d\tau + C_1 \tag{5.29}$$

$$\le C_2\int_0^R \frac{\theta(4\tau)}{\tau}\cdot g_\varepsilon(\tau)\,d\tau + C_1'$$

$$\le C_2 \sup_{\tau\in[0,R]} g_\varepsilon \cdot \int_0^R \frac{\theta(4\tau)}{\tau}\,d\tau + C_1', \tag{5.30}$$

where C_2 is chosen to be the larger constants in front of the first two terms in the right-hand side of (5.29). Since (5.30) holds for any $\rho < R/2$, we can take the supremum of $\rho \in [0, R/2]$ and obtain

$$\sup_{\tau\in[0,R/2]} g_\varepsilon \le C_2 \sup_{\tau\in[0,R]} g_\varepsilon \cdot \int_0^R \frac{\theta(4\tau)}{\tau}\,d\tau + C_1'. \tag{5.31}$$

For any τ satisfying $R/2 \le \tau \le R$, by the doubling property of $\bar{\bar{v}}$ we have

$$g_\varepsilon(\tau) = \frac{\bar{\bar{v}}(\tau)}{\tau^N(\tau+\varepsilon)} \le C_3\,\frac{\bar{\bar{v}}(R/2)}{(R/2)^N(R/2+\varepsilon)} = C_3\cdot g_\varepsilon\Big(\frac{R}{2}\Big) \le C_3 \sup_{\tau\in[0,R/2]} g_\varepsilon.$$

Hence (5.31) can be rewritten as

$$\sup_{\tau\in[0,R/2]} g_\varepsilon \le C_2C_3 \sup_{\tau\in[0,R/2]} g_\varepsilon \cdot \int_0^R \frac{\theta(4\tau)}{\tau}\,d\tau + C_1'. \tag{5.32}$$

We can choose R sufficiently small so that

$$C_2C_3\int_0^R \frac{\theta(4\tau)}{\tau}\,d\tau < \frac{1}{2},$$

and thus (5.32) implies that

$$\sup_{\tau\in[0,R/2]} g_\varepsilon \le 2C_1' < +\infty.$$

Since each g_ε has a uniform upper bound independent of the parameter ε, we conclude that

$$\frac{\bar{\bar{v}}(\tau)}{\tau^{N+1}} = \lim_{\varepsilon\to 0} g_\varepsilon(\tau) \le 2C_1' < +\infty.$$

In particular,

$$\limsup_{\rho\to 0} \frac{\bar{\bar{v}}(\rho)}{\rho^{N+1}} \le 2C_1' < +\infty.$$

On the other hand, by (5.24) and (4.10) we also know

$$\limsup_{\rho\to 0} \frac{\bar{\bar{v}}(\rho)}{\rho^{N+1}} \approx \limsup_{\rho\to 0} \frac{\bar{v}(\rho)}{\rho^{N+1}} = \limsup_{Y\to 0} \frac{|v(Y)|}{|Y|^{N+1}} = +\infty.$$

This is a contradiction. Therefore we have shown that in the expansion (5.20), it is impossible that $P_1 + P_2$ is trivial, and thus it must be a non-trivial homogeneous harmonic polynomial of degree exactly N. This finishes the proof of (5.1) with the desired decay.

We remark that if $N = N_{X_0} - 1$, by the expansion (5.1) it is impossible that

$$\sup_{B_r(0)} |v| \lesssim r^{N_{X_0}-\alpha},$$

as is shown in (4.8). Therefore we must have that the degree N is exactly N_{X_0}, and in particular,

$$|v(Y)| \le C_N |Y|^{N_{X_0}} \quad \text{for any } Y \in B_{R/2}(0), \tag{5.33}$$

and

$$\limsup_{Y\to 0} \frac{|v(Y)|}{|Y|^{N_{X_0}+1}} = 0.$$

6. Gradient estimate for the error term

In this section we estimate the gradient of the error term ψ. We first remark that ψ also satisfies

$$\psi \equiv 0 \quad \text{on } B_R(0) \cap \partial\mathbb{R}^d_+.$$

Since v vanishes on the boundary, it suffices to show that P_N vanishes as well on $\partial\mathbb{R}^d_+$. If not, since P_N is a homogeneous function, there exists a unit vector $\vec{e} \in \partial\mathbb{R}^d_+$ such that $P_N(\vec{e}) \neq 0$. Moreover,

$$P_N(r\vec{e}) = r^N P_N(\vec{e}) \quad \text{for any } r > 0. \tag{6.1}$$

On the other hand, by the estimate (5.2) we have

$$|\psi(r\vec{e})| \le C\, C_N\, r^N \tilde{\theta}(r).$$

Hence for any $0 < r < R/2$ we always have

$$|P_N(r\vec{e})| = |v - \psi(r\vec{e})| = |\psi(r\vec{e})| \le C' r^N \tilde{\theta}(r). \tag{6.2}$$

Combining (6.1) and (6.2), and letting $r \to 0$, we get $P_N(\vec{e}) = 0$, which is a contradiction. Therefore $P_N \equiv 0$ on $\partial\mathbb{R}^d_+$, and hence $\psi \equiv 0$ on $\partial\mathbb{R}^d_+ \cap B_R(0)$.

Since v satisfies $-\operatorname{div}(A(\cdot)\nabla v) = 0$ and P_N is a harmonic function, we have

$$\begin{aligned} -\operatorname{div}(A(\cdot)\nabla\psi) &= -\operatorname{div}(A(\cdot)\nabla(v - P_N)) \\ &= -\operatorname{div}(A(\cdot)\nabla v) + \Delta P_N + \operatorname{div}((A(\cdot) - \mathrm{Id})\nabla P_N) = \operatorname{div}((A(\cdot) - \mathrm{Id})\nabla P_N). \end{aligned}$$

That is to say, the error term ψ satisfies

$$\begin{cases} -\operatorname{div}(A(\cdot)\nabla\psi) = \operatorname{div}\vec{g} & \text{in } B_R^+(0) := B_R(0) \cap \mathbb{R}^d_+, \\ \psi = 0 & \text{on } B_R(0) \cap \partial\mathbb{R}^d_+, \end{cases} \tag{6.3}$$

where $\vec{g}$ is defined by

$$\vec{g}(Z) = (A(Z) - \mathrm{Id})\,\nabla P_N(Z)$$

in the upper half space. Notice that when $N = 1$, P_N must be a linear function and thus ∇P_N is a constant vector; when $N \geq 2$, ∇P_N is (at least) Lipschitz continuous. In both cases, it follows that $\vec{g}$ is Dini continuous. Recall that the coefficient matrix $A(\cdot)$ in the equation (6.3) is also Dini continuous in the upper half space. We will use the arguments in Section 2 of [3] (more precisely, Lemma 2.11 in [3]) to estimate $\nabla\psi$.

Let $r \in (0, R/6)$ be fixed, and denote $\psi_r(Y) := \psi(rY)$ in $B_1^+(0)$. Then it satisfies the rescaled equation

$$\begin{cases} -\operatorname{div}(A_r(\cdot)\psi_r) = \operatorname{div}\vec{g}_r & \text{in } B_2^+(0), \\ \psi_r \equiv 0 & \text{on } B_2(0) \cap \partial\mathbb{R}^d_+, \end{cases}$$

where we denote

$$A_r(Y) := A(rY) \quad \text{and} \quad \vec{g}_r(Y) := r\vec{g}(rY). \tag{6.4}$$

For each $Y \in B_1^+(0)$ and $0 < t \leq 2$, we denote

$$\omega_{A_r}(t) := \sup_{\substack{Y,Y' \in B_2^+(0) \\ |Y'-Y| \leq t}} |A_r(Y') - A_r(Y)| = \sup_{\substack{Z',Z \in B_{2r}^+(0) \\ |Z'-Z| \leq tr}} |A(Z') - A(Z)|,$$

and

$$\omega_{\vec{g}_r}(t) := \sup_{\substack{Y,Y' \in B_2^+(0) \\ |Y'-Y| \leq t}} |\vec{g}_r(Y) - \vec{g}_r(Y')| = r \sup_{\substack{Y,Y' \in B_2^+(0) \\ |Y'-Y| \leq t}} |\vec{g}(rY') - \vec{g}(rY)|.$$

Since the modulus of continuity of $A(\cdot)$ is bounded above by $\theta(2\cdot)$ (by (4.2)), it follows that

$$\omega_{A_r}(t) \lesssim \theta(2tr). \tag{6.5}$$

On the other hand, since P_N is a homogeneous harmonic polynomial of degree N, its derivative of any order is uniformly bounded in $B_2^+(0)$ by a constant multiple of $\|P_N\|_{L^\infty(B_1^+(0))}$. Moreover,

$$\begin{aligned} &\left|\vec{g}(rY') - \vec{g}(rY)\right| \\ &\quad \leq \left|(A(rY') - \mathrm{Id})(\nabla P_N(rY') - \nabla P_N(rY))\right| + |(A(rY') - A(rY))\nabla P_N(rY)| \\ &\quad \lesssim \theta(2r|Y'|) \cdot r^{N-1}|\nabla P_N(Y') - \nabla P_N(Y)| + \theta(2r|Y' - Y|) \cdot r^{N-1}|\nabla P_N(Y)| \\ &\quad \lesssim r^{N-1}\theta(4r) \cdot |Y' - Y| + r^{N-1} \cdot \theta(2r|Y' - Y|), \end{aligned}$$

where the constant depends on $\|P_N\|_{L^\infty(B_1^+(0))}$. Hence

$$\omega_{\vec{g}_r}(t) \lesssim r^N \theta(4r) \cdot t + r^N \cdot \theta(2tr). \tag{6.6}$$

In particular $\omega_{\vec{g}_r}(\cdot)$ is Dini continuous. Therefore Lemma 2.11 in [3] implies that for any $Y \in B_1^+(0)$,

$$|\nabla \psi_r(Y)| \lesssim \|\nabla \psi_r\|_{L^1(B_2^+(0))} + \int_0^{1/2} \frac{\hat{\omega}_{\vec{g}_r}(t)}{t}\, dt, \tag{6.7}$$

where the constant depends on d, the ellipticity constants and ω_{A_r}, which we have shown in (6.5) to be uniformly bounded. Moreover, following the notation in [3], $\hat{\omega}_\bullet(t)$ is determined by $\omega_\bullet(t)$ as follows: let $\beta \in (0,1)$, we define[5]

$$\hat{\omega}_\bullet(t) := \omega_\bullet(t) + \omega_\bullet(4t) + \omega_\bullet^\sharp(4t), \tag{6.8}$$

with

$$\omega_\bullet^\sharp(t) := \sup_{s\in[t,1]} \left(\frac{t}{s}\right)^\beta \omega_\bullet(s). \tag{6.9}$$

It is also proven in [3] that if $\omega_\bullet(\cdot)$ satisfies (2.1) and is doubling (i.e., (2.2)), then $\omega_\bullet^\sharp(\cdot)$ also satisfies (2.1). By the above definitions (6.8) and (6.9), it is not hard to see if $\omega(t) \le \lambda_1\omega_1(t) + \lambda_2\omega_2(t)$, then $\hat{\omega}(t) \le \lambda_1\hat{\omega}_1(t) + \lambda_2\hat{\omega}_2(t)$. Besides, when $\omega_\bullet(t)$ is taken to be $\theta(2tr)$, we have that

$$\omega_\bullet^\sharp(t) := \sup_{s\in[t,1]} \left(\frac{t}{s}\right)^\beta \theta(2rs) = \sup_{s'\in[2tr,2r]} \left(\frac{2tr}{s'}\right)^\beta \theta(s') \le \sup_{s'\in[2tr,R]} \left(\frac{2tr}{s'}\right)^\beta \theta(s') = \theta^\sharp(2tr),$$

where, as in (6.9), we define

$$\theta^\sharp(t) := \sup_{s\in[t,R]} \left(\frac{t}{s}\right)^\beta \theta(s). \tag{6.10}$$

Hence

$$\hat{\omega}_\bullet(t) = \theta(2tr) + \theta(8tr) + \omega^\sharp(4t) \le 2\theta(8tr) + \theta^\sharp(8tr).$$

When $\omega_\bullet(t)$ is taken to be t, we have that

$$\hat{\omega}_\bullet(t) = t + 4t + \omega_\bullet^\sharp(4t) \lesssim t^\beta.$$

Therefore (6.6) implies that

$$\hat{\omega}_{\vec{g}_r}(t) \lesssim r^N \theta(4r) \cdot t^\beta + r^N \cdot [\theta(8tr) + \theta^\sharp(8tr)],$$

and thus

$$\int_0^{1/2} \frac{\hat{\omega}_{\vec{g}_r}(t)}{t}\, dt \lesssim r^N \theta(4r) + r^N \cdot \Big[\int_0^{4r} \frac{\theta(s)}{s}\, ds + \int_0^{4r} \frac{\theta^\sharp(s)}{s}\, ds\Big]. \tag{6.11}$$

[5] In [3] they need the additional parameter $\tilde{\omega}_\bullet(\cdot)$ because they work with Dini continuous functions in the average sense, i.e., functions with *Dini-mean oscillation*. When one works with uniform Dini function, which is our case here, $\tilde{\omega}_\bullet(\cdot)$ can be simply taken the same as $\omega_\bullet(\cdot)$.

On the other hand, by Hölder's inequality and the energy estimate with vanishing boundary data (see, for example, Lemma 1.41 in [2]), we have

$$\iint_{B_2^+(0)} |\nabla\psi_r|\,dY \lesssim \Big(\iint_{B_2^+(0)} |\nabla\psi_r|^2 dY\Big)^{1/2} \lesssim \Big(\iint_{B_3^+(0)} |\psi_r|^2 dY + \iint_{B_3^+(0)} |\vec{g}_r|^2 dY\Big)^{1/2}$$

$$\lesssim \sup_{B_{3r}^+(0)} |\psi| + r\cdot \sup_{B_{3r}^+(0)} |\vec{g}| \lesssim r^N \tilde{\theta}(3r), \tag{6.12}$$

where we recall that $\tilde{\theta}(\cdot)$ is defined in (5.19). Inserting (6.11) and (6.12) back into (6.7), we obtain,

$$|\nabla\psi_r(Y)| \lesssim \iint_{B_2^+(0)} |\nabla\psi_r|\,dY + \int_0^{1/2} \frac{\hat{\omega}_{\vec{g}_r}(t)}{t}\,dt$$

$$\lesssim r^N\tilde{\theta}(3r) + r^N\theta(4r) + r^N\cdot\Big[\int_0^{4r}\frac{\theta(s)}{s}\,ds + \int_0^{4r}\frac{\theta^\sharp(s)}{s}\,ds\Big].$$

Or equivalently,

$$|\nabla\psi(rY)| \lesssim r^{N-1}\cdot\Big[\tilde{\theta}(3r) + \theta(4r) + \int_0^{4r}\frac{\theta(s)}{s}\,ds + \int_0^{4r}\frac{\theta^\sharp(s)}{s}\,ds\Big].$$

Finally, let

$$\mathring{\theta}(r) := \tilde{\theta}(3r) + \theta(4r) + \int_0^{4r}\frac{\theta(s)}{s}\,ds + \int_0^{4r}\frac{\theta^\sharp(s)}{s}\,ds, \tag{6.13}$$

where we recall that $\theta^\sharp(\cdot)$ is defined in (6.10) and it satisfies the Dini condition (2.1). We conclude that

$$|\nabla\psi(Y)| \le C\,|Y|^{N-1}\,\mathring{\theta}(|Y|) \quad \text{for any } Y \in B_{R/6}^+(0), \tag{6.14}$$

where

$$\mathring{\theta}(r) \to 0 \quad \text{as } r \to 0.$$

We remark that exactly the same proof as above yields the gradient estimate of $\nabla\psi$ on the lower half space. Moreover $\nabla\psi = \nabla v - \nabla P_N$ is continuous up to the boundary from above and below, by Proposition 2.7 in [3]. Therefore (6.14) holds in the entire ball $B_{R/6}(0)$.

7. Proof of Theorem 1.1 and Corollary 1.3

Now we are ready to prove the expansion of u by the expansion (5.1) for v which is proven in the previous section. By the definition of v in Section 4, we have

$$u(x,t) = v(x,t-\varphi(x)) = P_N(x,t-\varphi(x)) + \psi(x,t-\varphi(x)). \tag{7.1}$$

Let $r = |(x,t)|$; then $|\varphi(x)| \le \theta(2r)r$. Hence for r sufficiently small, we have

$$\frac{r}{2} < |(x,t-\varphi(x))| < \frac{3r}{2}\cdot$$

By the error estimate (5.2), we have

$$|\psi(x,t-\varphi(x))| \le C' C_N r^N \tilde{\theta}(2r). \tag{7.2}$$

On the other hand,

$$P_N(t,x-\varphi(x)) = P_N(x,t) - \varphi(x) \cdot \int_0^1 \partial_d P_N(x,t-\tau\varphi(x))\,d\tau. \tag{7.3}$$

By (5.1), (5.33) and (5.2), we can estimate

$$\frac{1}{r^d}\iint_{B_{2r}(0)} |P_N|^2\,dX = \frac{1}{r^d}\iint_{B_{2r}(0)} |v-\psi|^2\,dX$$
$$\lesssim \frac{1}{r^d}\iint_{B_{2r}(0)} v^2\,dX + \frac{1}{r^d}\iint_{B_{2r}(0)} \psi^2\,dX \lesssim C_N^2 r^{2N} + C_N^2 r^{2N}\tilde{\theta}(2r)^2 \lesssim C_N^2 r^{2N},$$

with a uniform constant (which only depends on the dimension d and the ellipticity). (The r^{2N}-decay clearly just follows from the homogeneity of P_N. But here we want to emphasize how the constant in front depends on the constant C_N from (4.9).) Since P_N is a harmonic function in $\mathbb{R}^d$, we have

$$\sup_{B_{\frac{3r}{2}}(0)} |\nabla P_N| \lesssim \frac{1}{r}\Big(\frac{1}{r^d}\iint_{B_{2r}(0)} |P_N|^2\,dX\Big)^{1/2} \lesssim C_N r^{N-1}. \tag{7.4}$$

Moreover,

$$\|\nabla^2 P_N\|_{L^\infty(B_r)} \lesssim \frac{1}{r^2}\|P_N\|_{L^\infty(B_{2r})} \lesssim C_N r^{N-2}. \tag{7.5}$$

Therefore

$$\Big|\varphi(x)\cdot\int_0^1 \partial_d P_N(x,t-\tau\varphi(x))\,d\tau\Big| \le \sup_{B_{3r/2}(0)} |\nabla P_N|\cdot|\varphi(x)|$$
$$\lesssim C_N r^{N-1}\cdot r\,\theta(2r) = C_N r^N \theta(2r).$$

Combined with (7.1), (7.2) and (7.3), we conclude that in $B_{R/3}(0)$, u has the expansion

$$u(x,t) = P_N(x,t) + \tilde{\psi}(x,t), \tag{7.6}$$

where the error term

$$\tilde{\psi}(x,t) = \psi(x,t-\varphi(x)) - \varphi(x)\cdot\int_0^1 \partial_d P_N(x,t-\tau\varphi(x))\,d\tau$$

satisfies

$$|\tilde{\psi}(x,t)| \le C\,C_N |(x,t)|^N\,\tilde{\theta}(2|(x,t)|).$$

(For our purpose, the expansion (7.6) is meaningful only inside $B_{R/3}(0)\cap D$, i.e., when $t>\varphi(x)$, but the expansion holds in the entire ball if we consider an extension of u

across the boundary by the odd reflection of v in (4.3) and the transformation as in (7.1).) Moreover, by the gradient estimates in (6.14) and (7.5), we have

$$\left|\nabla\tilde{\psi}(x,t)\right| \leq C\, C_N |(x,t)|^{N-1}\, \mathring{\theta}(2|(x,t)|).$$

Recall that for any $X_0 = (x_0, \varphi(x_0)) \in \partial D$, we can apply a translation and orthogonal transformation O_{x_0} as in Section 3 so that X_0 becomes the origin and the tangent plane to ∂D at X_0 is flat (i.e., $\nabla\varphi(x_0) = 0$). Taking into account the orthogonal transformation, we in fact get

$$\begin{aligned} u(x,t) &= P_N\big(O_{x_0}((x,t)-X_0)\big) + \tilde{\psi}\big(O_{x_0}((x,t)-X_0)\big) \\ &= \tilde{P}_N((x,t)-X_0) + \tilde{\tilde{\psi}}((x,t)-X_0), \end{aligned}$$

where $\tilde{P}_N$ is still a non-trivial homogeneous harmonic polynomial of degree $N = N_{X_0}$. For simplicity, we still denote it as P_N, and simply write

$$u(x,t) = P_N((x,t)-X_0) + \tilde{\psi}((x,t)-X_0), \quad \text{in } B_{R/3}(X_0) \cap D. \tag{7.7}$$

In order to prove the *uniqueness of the expansion*, we assume that u has two such expansions

$$u(X) = P_N(X-X_0) + \tilde{\psi}(X-X_0)$$

and

$$u(X) = P_N'(X-X_0) + \tilde{\psi}'(X-X_0),$$

such that

$$|\tilde{\psi}(Y)| \leq C_1 |Y|^N\, \tilde{\theta}(|Y|) \quad \text{and} \quad |\tilde{\psi}'(Y)| \leq C_2 |Y|^N\, \tilde{\theta}(|Y|). \tag{7.8}$$

Notice that the degree of the homogeneous harmonic polynomial is uniquely determined by N_{X_0}. It follows that

$$P_N(Y) - P_N'(Y) = \tilde{\psi}'(Y) - \tilde{\psi}(Y) \quad \text{for } Y \in B_{R/3}(0).$$

Let $\tilde{P}_N := P_N - P_N'$. Then it is also a homogeneous harmonic polynomial of degree N. Assuming that $\tilde{P}_N \not\equiv 0$, then there exists a unit vector $\vec{e} \in \mathbb{S}^{d-1}$ such that $\tilde{P}_N(e) \neq 0$. In particular, $\tilde{P}_N(r\vec{e}) = r^N \tilde{P}_N(e) \neq 0$. On the other hand, by the estimates (7.8), we have

$$\left|\tilde{P}_N(r\vec{e})\right| = \left|\tilde{\psi}'(r\vec{e}) - \tilde{\psi}(r\vec{e})\right| \leq (C_1 + C_2) r^N\, \tilde{\theta}(r).$$

Hence it follows that

$$\left|\tilde{P}_N(\vec{e})\right| \leq (C_1 + C_2)\, \tilde{\theta}(r) \to 0 \quad \text{as } r \to 0,$$

which contradicts the assumption that $\tilde{P}_N(\vec{e}) \neq 0$. Therefore it must be the case that $\tilde{P}_N \equiv 0$. As a result, $P_N \equiv P_N'$ and $\tilde{\psi} \equiv \tilde{\psi}'$, i.e., the expansion is unique. This finishes the proof of Theorem 1.1.

Now we set out to prove Corollary 1.3, or more precisely, to prove (1.5). Denote $X_0 = (x_0, \varphi(x_0))$. We recall that $T_{X_0,r}u$ is defined in the domain $(D - X_0)/r$, which is the region above the graph of the function

$$\varphi_r : y \in \mathbb{R}^{d-1} \mapsto \frac{\varphi(x_0 + ry) - \varphi(x_0)}{r}.$$

Assuming without loss of generality that $\nabla\varphi(x_0) = 0$, we have that $(D - X_0)/r$ converges to the upper half space $\mathbb{R}^d_+$. Moreover, the Lebesgue measure of the set difference between $(D - X_0)/r$ and $\mathbb{R}^d_+$ can be estimated as

$$\left|B_1(0) \cap \Big(\frac{D - X_0}{r} \,\Delta\, \mathbb{R}^d_+\Big)\right| \le \int_{B_1^{d-1}(0)} |\varphi_r(y)|\, dy \lesssim \sup_{B_r^{d-1}(x_0)} |\nabla\varphi - \nabla\varphi(x_0)| \le \theta(r). \tag{7.9}$$

Since P_N is homogeneous of degree N, we have

$$\frac{1}{r^d} \iint_{B_r^+(0)} |P_N|^2\, dY = \iint_{B_1^+(0)} |P_N(rZ)|^2\, dZ = r^{2N} \iint_{B_1^+(0)} |P_N|^2\, dZ. \tag{7.10}$$

Combined with the estimate of $\tilde{\psi}$, we have

$$\frac{1}{r^d} \iint_{B_r^+(0)} \left|P_N(Y) + \tilde{\psi}(Y)\right|^2 dY = \frac{1}{r^d} \iint_{B_r^+(0)} |P_N|^2\, dY + O(r^{2N}\tilde{\theta}(r)). \tag{7.11}$$

By a change of variable, the pointwise bounds of P_N, $\tilde{\psi}$ and the estimate (7.9), we have

$$\begin{aligned}
&\left|\frac{1}{r^d} \iint_{B_r(0)\cap(D-X_0)} |P_N(Y) + \tilde{\psi}(Y)|^2\, dY - \frac{1}{r^d} \iint_{B_r^+(0)} |P_N(Y) + \tilde{\psi}(Y)|^2\, dY\right| \\
&\qquad \le \iint_{B_1(0)\cap(\frac{D-X_0}{r}\,\Delta\,\mathbb{R}^d_+)} \left|P_N(rZ) + \tilde{\psi}(rZ)\right|^2 dZ \\
&\qquad \le \sup_{B_r(0)} \big(|P_N| + |\tilde{\psi}|\big)^2 \cdot \left|B_1(0) \cap \Big(\frac{D - X_0}{r} \,\Delta\, \mathbb{R}^d_+\Big)\right| \lesssim r^{2N}\theta(r).
\end{aligned} \tag{7.12}$$

Therefore by combining (7.12), (7.11) and (7.10), we conclude

$$\begin{aligned}
\frac{1}{r^d} \iint_{B_r(X_0)\cap D} u^2\, dY &= \frac{1}{r^d} \iint_{B_r(0)\cap(D-X_0)} \left|P_N(Y) + \tilde{\psi}(Y)\right|^2 dY \\
&= \frac{1}{r^d} \iint_{B_r^+(0)} |P_N + \tilde{\psi}|^2 dY + O(r^{2N}\theta(r)) \\
&= \frac{1}{r^d} \iint_{B_r^+(0)} |P_N|^2\, dY + O(r^{2N}\tilde{\theta}(r)) \\
&= \frac{1}{r^d} \iint_{B_r^+(0)} |P_N|^2\, dY \cdot (1 + O(\tilde{\theta}(r))).
\end{aligned}$$

Hence

$$
\begin{aligned}
T_{X_0,r}u(Z) &= \frac{u(X_0+rZ)}{\left(\frac{1}{r^d}\iint_{B_r^+(0)}|P_N|^2\,dY\right)^{1/2}\cdot(1+O(\tilde\theta(r)))^{1/2}}\\
&= \frac{P_N(rZ)+\tilde\psi(rZ)}{\left(\frac{1}{r^d}\iint_{B_r^+(0)}|P_N|^2\,dY\right)^{1/2}}(1+O(\tilde\theta(r)))\\
&= \left[\frac{P_N(rZ)}{\left(\frac{1}{r^d}\iint_{B_r^+(0)}|P_N|^2\,dY\right)^{1/2}}+\tilde\psi(rZ)\cdot O\Big(\frac{1}{r^N}\Big)\right](1+O(\tilde\theta(r)))\\
&= cP_N(Z)+O(\tilde\theta(r)),
\end{aligned}
$$

where

$$c=\Big(\iint_{B_1^+(0)}|P_N|^2dZ\Big)^{-1/2}.$$

This finishes the proof of the claim (1.5).

8. Proof of Proposition 1.4

We denote $X_j=(x_j,\varphi(x_j))$ for each $j\in\mathbb{N}_0$. Recall that in Section 3 we found an orthogonal transformation O_{x_j}, which locally maps the domain $D-X_j$ to a domain D_{x_j}, defined as the region above the graph of a function $\tilde\varphi_{x_j}$. Under this transformation, the harmonic function u in D becomes a harmonic function $\tilde u$ in D_{x_j}: for any $Y\in D_{x_j}$ sufficiently close to the origin, we have

$$\tilde u(Y):=u(X_j+O_{x_j}^{\mathsf T}Y). \tag{8.1}$$

Recall that in Section 4, we were able to study the harmonic function $\tilde u$ using the flattening map

$$\Phi_{x_j}:(y,s)\in\mathbb{R}^d_+\mapsto(y,s+\tilde\varphi_{x_j}(y))\in D_{x_j}$$

and

$$v(y,s)=\tilde u\circ\Phi_{x_j}(y,s). \tag{8.2}$$

Combining (8.1) and (8.2), we get a function $v_j:\mathbb{R}^d_+\to\mathbb{R}$ defined as

$$v_j(y,s)=\tilde u(y,s+\tilde\varphi_{x_j}(y))=u\big(X_j+O_{x_j}^{\mathsf T}(y,s+\tilde\varphi_{x_j}(y))\big). \tag{8.3}$$

To study how the functions v_j's are related, we need to study how the map O_{x_j} and $\tilde\varphi_{x_j}$ depend on the sub-index x_j.

Recall that for any $(x,\varphi(x))\in\partial D$, the orthogonal matrix O_x is explicitly determined by $\nabla\varphi(x)$, as in (3.5), where c_x satisfies $c_x=(1+|\nabla\varphi(x)|^2)^{-1/2}$ and the block matrix $\tilde O_x$ is symmetric, positive semi-definite and satisfies that $\tilde O_x^{-1}$ is the square root of $\mathrm{Id}_{d-1}+\nabla\varphi(x)\nabla\varphi(x)^{\mathsf T}$. Hence

$$|c_x-c_{x'}|\lesssim\big||\nabla\varphi(x)|-|\nabla\varphi(x')|\big|\le|\nabla\varphi(x)-\nabla\varphi(x')|\le\theta(|x-x'|); \tag{8.4}$$

and the block matrices $\tilde{O}_x^{-1}$ and $\tilde{O}_{x'}^{-1}$ satisfy the assumptions of Lemma 2.6. Therefore we have that

$$\begin{aligned}|\tilde{O}_x - \tilde{O}_{x'}| &\lesssim \left|((\tilde{O}_{x'})^{-1})^2 - ((\tilde{O}_x)^{-1})^2\right| = \left|\nabla\varphi(x')\nabla\varphi(x')^\mathsf{T} - \nabla\varphi(x)\nabla\varphi(x)^\mathsf{T}\right| \\ &\lesssim |\nabla\varphi(x')| \cdot \left|\nabla\varphi(x')^\mathsf{T} - \nabla\varphi(x)\right| + \left|\nabla\varphi(x') - \nabla\varphi(x)\right| \cdot |\nabla\varphi(x)^\mathsf{T}| \\ &\lesssim |\nabla\varphi(x') - \nabla\varphi(x)| \le \theta(|x - x'|). \end{aligned} \tag{8.5}$$

Combining (3.5), (8.4) and (8.5), we get

$$|O_x - O_{x'}| \lesssim \theta(|x - x'|). \tag{8.6}$$

On the other hand, the map $\tilde{\varphi}$ is defined as in (3.9), where the function g is defined as in (3.6): that is, for any $(x, \varphi(x)) \in \partial D$,

$$g_x : z \in \mathbb{R}^{d-1} \mapsto \tilde{O}_x(z - x) - (\varphi(z) - \varphi(x))\tilde{O}_x\nabla\varphi(x) = y \in \mathbb{R}^{d-1}.$$

It follows that

$$\begin{aligned} g_x(z) - g_{x'}(z) = \tilde{O}_x \big[(x' - x) &+ \varphi(z)(\nabla\varphi(x') - \nabla\varphi(x)) + (\varphi(x) - \varphi(x'))\nabla\varphi(x) \\ &+ \varphi(x')(\nabla\varphi(x) - \nabla\varphi(x'))\big] + (\tilde{O}_x - \tilde{O}_{x'})\big[(z - x') + \big(\varphi(x') - \varphi(z)\big)\nabla\varphi(x')\big]. \end{aligned}$$

Hence by (8.5), we get

$$\|g_x - g_{x'}\|_{L^\infty(B_1^{d-1}(0))} \lesssim \theta(|x - x'|).$$

Similarly, by (3.7), we obtain

$$\|Dg_x - Dg_{x'}\|_{L^\infty(B_1^{d-1}(0))} \lesssim |\tilde{O}_x - \tilde{O}_{x'}| + |\tilde{O}_x\nabla\varphi(x) - \tilde{O}_{x'}\nabla\varphi(x')| \lesssim \theta(|x - x'|).$$

In the same fashion (and using (3.14)), we conclude that

$$\|\tilde{\varphi}_x - \tilde{\varphi}_{x'}\|_{L^\infty(B_{1/2}^{d-1}(0))} \lesssim \theta(|x - x'|), \quad \|\nabla\tilde{\varphi}_x - \nabla\tilde{\varphi}_{x'}\|_{L^\infty(B_{1/2}^{d-1}(0))} \lesssim \theta(|x - x'|). \tag{8.7}$$

Recall that u is continuously differentiable near the boundary of the Dini domain (by the work of [3]). Therefore combining (8.3), (8.6), (8.7) and $X_j \to X_0$, we conclude that $v_j \to v_0$ (locally uniformly) in the C^1-topology.

Let $N = N_{X_0} = N_{X_j} \in \mathbb{N}$. By Section 5, each v_j has the expansion

$$v_j(Y) = P_j(Y) + \psi_j(Y)$$

in some ball $B_{R_j}(0)$, where P_j is a non-trivial homogeneous harmonic polynomial of degree N, and the error term ψ_j satisfies $|\psi_j(Y)| \le C_j|Y|^N\tilde{\theta}(|Y|)$. By the proof in Section 7, it suffices to show that P_j converges to P_0 in the C^N-topology. By the definitions of w_j and $P_{j,2}$ in (5.5) and (5.8), respectively, and the fact that $\nabla v_j \to \nabla v_0$ locally uniformly[6], we get that

$$w_j \to w_0 \quad \text{and} \quad P_{j,2} \to P_{0,2} \quad \text{uniformly.}$$

[6]In fact, it suffices to know that $\nabla v_j \rightharpoonup \nabla v_0$ weakly in L^p for some $p > d$.

On the other hand, since $v_j \to v_0$ uniformly, the harmonic functions $v_j - w_j$ also converge uniformly to $v_0 - w_0$. By the expansions of these harmonic functions to degree N as in (5.6), the polynomials $P_{j,1}$ also converge uniformly to $P_{0,1}$. Thus

$$P_j = P_{j,1} + P_{j,2} \to P_{0,1} + P_{0,2} = P_0$$

locally uniformly. Since P_j, P_0 are homogeneous harmonic polynomials of the same degree N, they also converge in the C^N-topology. This finishes the proof of Proposition 1.4.

A. Appendix. Proof of upper semi-continuity of the vanishing order

The goal of this appendix is to prove the upper semi-continuity of the vanishing order.

Lemma A.1. *Let D and u be as in Theorem 1.1. The map*

$$X \in \partial D \cap B_{R_0}(0) \mapsto N_X \in \mathbb{N}$$

is upper semi-continuous. That is,

$$\limsup_{\substack{X \in \partial D \cap B_{R_0}(0) \\ X \to X_0}} N_X \leq N_{X_0}.$$

Recall that in Section 4 of [9], we define the modified frequency functions at different boundary points by applying different transformation maps. To compare them, we need to understand what the modified frequency function at each boundary point means in the original domain D.

Lemma A.2. *Let D and u be as in Theorem 1.1. For any $X \in \partial D \cap B_{2R_0}(0)$ and $r > 0$ small (so that $\theta(4r) < 1/26$), we have*

$$N(u \circ \Psi_X, r) = [1 + O(\theta(4r))] \cdot N(u, X + 3r\hat{\theta}(r)e_d, r), \tag{A.1}$$

where Ψ_X and $\hat{\theta}$ are defined in (2.4) and (2.5), respectively; $N(u, Y, r)$ denotes the standard Almgren frequency function of u centered at $Y \in D$ and at scale r, see (2.3); and $N(u \circ \Psi_X, r)$ denotes the frequency function for an elliptic equation (satisfied by $u \circ \Psi_X$) in the domain $\Psi_X^{-1}(D)$, see Section 3 of [9].

Remark A.3. The formula (A.1) is related to an observation pointed out in [8]: the Dini domain is star-shaped near the boundary. To be more precise, let $X \in \partial D$ and let $r > 0$ be sufficiently small. Then the domain $D \cap B_r(X)$ is star-shaped with respect to some $Y_r \in D$. (See the proof of Lemma 3.2 in [8].)

Proof. Recall that in (3.8) of [9], we define

$$D(u \circ \Psi_X, r) = \iint_{B_r \cap \Omega_X} \mu |\nabla_g (u \circ \Psi_X)|_g^2 \, dV_g = \iint_{\Psi_X(B_r) \cap D} |\nabla u|^2 \, dZ =: \widehat{D}(X, r)$$

and

$$H(u\circ\Psi_X,r)=\int_{\partial B_r\cap\Omega_X}\mu(u\circ\Psi_X)^2\,dV_{\partial B_r}=\int_{\partial B_r\cap\Omega_X}\tilde\eta(u\circ\Psi_X)^2\,d\mathcal{H}^{d-1}$$
$$=(1+O(\theta(4r)))\int_{\Psi_X(\partial B_r)\cap D}u^2\,d\mathcal{H}^{d-1}=(1+O(\theta(4r)))\,\widehat H(X,r),$$

where we introduce the notation

$$\widehat H(X,r):=\int_{\Psi_X(\partial B_r)\cap D}u^2\,d\mathcal{H}^{d-1}.$$

Let

$$\widehat N(X,r):=\frac{r\,\widehat D(X,r)}{\widehat H(X,r)}.$$

Then the frequency function satisfies

$$N(u\circ\Psi_X,r)=\frac{r\,D(u\circ\Psi_X,r)}{H(u\circ\Psi_X,r)}=(1+O(\theta(4r)))\,\frac{r\,\widehat D(X,r)}{\widehat H(X,r)}=(1+O(\theta(4r)))\,\widehat N(X,r). \tag{A.2}$$

By the definition of Ψ_X in (2.4), it is clear that it can be written as $X+\Psi(\cdot)$ for a map Ψ independent of $X\in\partial D$. Besides, we have

$$\Psi(\partial B_r)=\partial B_r+3r\,\hat\theta(r)\,e_d=\partial B_r(3r\,\hat\theta(r)\,e_d).$$

To understand what the set $\partial\Psi(B_r)$ is, we first study the set $\Psi(B_r)$. Clearly,

$$\Psi(B_r)=\bigcup_{\rho\in[0,r)}\Psi(\partial B_\rho)=\bigcup_{\rho\in[0,r)}\partial B_\rho\big(3\rho\,\hat\theta(\rho)\,e_d\big).$$

Consider the function

$$f:\rho\in[0,r)\mapsto-\rho+3\rho\,\hat\theta(\rho),$$

which corresponds to the height of the lowest point of the (shifted) ball $\partial B_\rho(3\rho\,\hat\theta(\rho)e_d)$. A simple computation shows that f is a continuous function, and that

$$f'(\rho)=-1+3\hat\theta(\rho)+3\rho\,\hat\theta'(\rho)=-1+3\hat\theta(\rho)+\frac{3}{\log^2 2}\int_\rho^{2\rho}\frac{\theta(2s)-\theta(s)}{s}\,ds$$
$$\le-1+3\theta(4\rho)+\frac{3}{\log 2}\,\theta(4\rho)\le-1+13\,\theta(4r).$$

By choosing r sufficiently small so that $\theta(4r)<1/26$, we can guarantee that f is decreasing. In particular, this implies that the balls $\Psi(\partial B_\rho)=\partial B_\rho(3\rho\,\hat\theta(\rho)\,e_d)$ with $\rho\in[0,r)$ are nested, i.e.,

$$B_\rho(3\rho\,\hat\theta(\rho)\,e_d)\subset B_{\rho'}(3\rho'\,\hat\theta(\rho')\,e_d),\quad\text{if }\rho\le\rho'.$$

In fact, let $Y \in B_\rho(3\rho\hat{\theta}(\rho)e_d)$ be arbitrary. Then

$$|Y - 3\rho'\hat{\theta}(\rho')e_d| \leq |Y - 3\rho\hat{\theta}(\rho)e_d| + (3\rho'\hat{\theta}(\rho') - 3\rho\hat{\theta}(\rho)) < \rho + f(\rho') + \rho' - (f(\rho) + \rho)) = \rho' + (f(\rho') - f(\rho)) \leq \rho'.$$

Hence $Y \in B_{\rho'}(3\rho'\hat{\theta}(\rho')e_d)$. Moreover, by the intermediate value theorem, $f(\rho)$ assumes all the values between $\lim_{\rho\to r-} f(\rho) = -r + 3r\hat{\theta}(r)$ and $\lim_{\rho\to 0+} f(\rho) = 0$. Therefore we have that

$$\Psi(B_r) = B_r(3r\hat{\theta}(r)e_d)$$

and

$$\partial\Psi(B_r) = \partial B_r(3r\hat{\theta}(r)e_d) = \Psi(\partial B_r). \tag{A.3}$$

Therefore,

$$\widehat{H}(X,r) = \int_{\Psi_X(\partial B_r)\cap D} u^2\, d\mathcal{H}^{d-1} = \int_{\partial B_r(X+3r\hat{\theta}(r)e_d)\cap D} u^2\, d\mathcal{H}^{d-1},$$

$$\widehat{D}(X,r) = \iint_{\Psi_X(B_r)\cap D} |\nabla u|^2\, dZ = \iint_{B_r(X+3r\hat{\theta}(r)e_d)} |\nabla u|^2\, dZ,$$

and the proof is finished. ∎

Recall that in Proposition 3.10 of [9] we have shown that

$$r \mapsto N_X(r) := N(u\circ\Psi_X, r)\exp\left(C\int_0^r \frac{\theta(s)}{s}\,ds\right) \tag{A.4}$$

is monotone nondecreasing. Since $N_{X_0} = \lim_{r\to 0} N_{X_0}(r)$, for r sufficiently small we have

$$N_{X_0}(r) \leq N_{X_0} + \frac{1}{5}. \tag{A.5}$$

By Lemma A.2 and (A.4), we have

$$\begin{aligned} N_{X_0}(r) &= N(u\circ\Psi_{X_0}, r)\exp\left(C\int_0^r \frac{\theta(s)}{s}\,ds\right) \\ &= \left[1 + O(\theta(4r))\right] N(u, X_0 + 3r\hat{\theta}(r)e_d, r)\exp\left(C\int_0^r \frac{\theta(s)}{s}\,ds\right). \end{aligned} \tag{A.6}$$

Let r be sufficiently small so that

$$\theta(4r) \lesssim \frac{N_{X_0} + 1/4}{N_{X_0} + 1/5}.$$

Then by (A.5) and (A.6) we get

$$N\left(u, X_0 + 3r\hat{\theta}(r)e_d, r\right)\exp\left(C\int_0^r \frac{\theta(s)}{s}\,ds\right) \leq N_{X_0} + \frac{1}{4}. \tag{A.7}$$

Suppose $\hat{X}_j, \hat{X}_0 \in \overline{D}$ satisfy $\hat{X}_j \to \hat{X}_0$. Then the standard Almgren frequency function (see (2.3)) satisfies

$$N(u, \hat{X}_j, r) \to N(u, \hat{X}_0, r) \quad \text{as } j \to \infty.$$

In fact, clearly the map

$$X \mapsto \iint_{B_r(X)} |\nabla u|^2 \, dY$$

is continuous, since $u \in W^{1,2}$. By a change of variable, it is also easy to see the map

$$X \mapsto \int_{B_r(X)} u^2 \, d\mathcal{H}^{d-1}$$

is differentiable (and strictly positive for non-trivial harmonic functions u). Therefore,

$$N(u, \hat{X}_j, r) = \frac{r \iint_{B_r(\hat{X}_j)} |\nabla u|^2 \, dY}{\int_{B_r(\hat{X}_j)} u^2 \, d\mathcal{H}^{d-1}} \to \frac{r \iint_{B_r(\hat{X}_0)} |\nabla u|^2 \, dY}{\int_{B_r(\hat{X}_0)} u^2 \, d\mathcal{H}^{d-1}} = N(u, \hat{X}_0, r).$$

In particular, this combined with (A.7) and $X_j \to X_0$ gives

$$N\big(u, X_j + 3r\hat{\theta}(r)e_d, r\big) \exp\Big(C \int_0^r \frac{\theta(s)}{s}\, ds\Big) \le N_{X_0} + \frac{1}{3}, \tag{A.8}$$

for j sufficiently large. Again by Lemma A.2 and by taking r sufficiently small, we have

$$N_{X_j}(r) = \big[1 + O(\theta(4r))\big] N(u, X_j + 3r\hat{\theta}(r)e_d, r) \exp\Big(C \int_0^r \frac{\theta(s)}{s}\, ds\Big) \le N_{X_0} + \frac{1}{2}\cdot$$

By the monotonicity of the frequency function $r \mapsto N_{X_j}(r)$, we finally conclude that

$$N_{X_j} \le N_{X_j}(r) \le N_{X_0} + \frac{1}{2}\cdot$$

Since N_X take integer values, we have $N_{X_j} \le N_{X_0}$. This finishes the proof of Lemma A.1.

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References

[1] Adolfsson, V. and Escauriaza, L.: $C^{1,\alpha}$ domains and unique continuation at the boundary. *Commun. Pur. Appl. Math.* **50** (1997), no. 10, 935–969.

[2] Caffarelli, L. and Kenig, C.: Gradient estimates for variable coefficient parabolic equations and singular perturbation problems. *Amer. J. Math.* **120** (1998), no. 2, 391–439.

[3] Dong, H., Escauriaza, L. and Kim, S.: On C^1, C^2, and weak type-$(1, 1)$ estimates for linear elliptic operators: part II. *Math. Ann.* **370** (2018), no. 1-2, 447–489.

[4] Gilbarg, D. and Trudinger, N.: *Elliptic partial differential equations of second order*. Second edition. Grundlehren der mathematischen Wissenschaften 224, Springer, Berlin, 1983.

[5] Han, Q.: Singular sets of solutions to elliptic equations. *Indiana Univ. Math. J.* **43** (1994), no. 3, 983–1002.

[6] Jin, T., Maz'ya V. and Schaftingen, J. V.: Pathological solutions to elliptic problems in divergence form with continuous coefficients. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 13-14, 773–778.

[7] Kenig, C.: *Harmonic analysis techniques for second order elliptic boundary value problems*. CBMS Regional Conference Series in Mathematics 83, American Mathematical Society, Providence, RI, 1994.

[8] Kukavica, I. and Nyström, K.: Unique continuation on the boundary for Dini domains. *Proc. Amer. Math. Soc.* **126** (1998), no. 2, 441–446.

[9] Kenig, C. and Zhao, Z.: Boundary unique continuation on C^1-Dini domains and the size of the singular set. *Arch. Ration. Mech. Anal.* **245** (2022), no. 1, 1–88.

[10] Miller, K.: Nonunique continuation for uniformly parabolic and elliptic equations in selfadjoint divergence form with Hölder continuous coefficients. *Bull. Amer. Math. Soc.* **79** (1973), no. 2, 350–354.

[11] Plis, A.: On non-uniqueness in Cauchy problem for an elliptic second order differential equation. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **11** (1963), 95–100.

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