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Homogenization of iterated singular integrals with applications to random quasiconformal maps

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Abstract. We study homogenization of iterated randomized singular integrals and homeomorphic solutions to the Beltrami differential equation with a random Beltrami coefficient. More precisely, let $(F_j)_{j\geq 1}$ be a sequence of normalized homeomorphic solutions to the planar Beltrami equation $\partial_{\bar z} F_j(z) = \mu_j(z,\omega)\,\partial_z F_j(z)$, where the random dilatation satisfies $|\mu_j| \leq k < 1$ and has locally periodic statistics, for example of the type

$$\mu_j(z,\omega) = \phi(z) \sum_{n\in\mathbb{Z}^2} g(2^j z - n, X_n(\omega)), \tag{0.1}$$

where $g(z,\omega)$ decays rapidly in z, the random variables X_n are i.i.d., and $\phi \in C_0^\infty$. We establish the almost sure and local uniform convergence as $j \to \infty$ of the maps F_j to a deterministic quasiconformal limit F_∞.

This result is obtained as an application of our main theorem, which deals with homogenization of iterated randomized singular integrals. As a special case of our theorem, let $T_1, \ldots, T_m$ be translation and dilation invariant singular integrals on $\mathbb{R}^d$, and consider a d-dimensional version of μ_j, e.g., as defined above or within a more general setting, see Definition 3.4 below. We then prove that there is a deterministic function f such that almost surely, $\mu_j T_m \mu_j \ldots T_1 \mu_j \to f$ as $j \to \infty$ weakly in L^p, for $1 < p < \infty$.

We dedicate this paper to our friend and colleague 'Josechu' Fernández.

1. Introduction and statement of results

1.1. Background and motivation – a bird's eye view

The purpose of this paper is twofold: we initiate a systematic study of random quasiconformal homeomorphisms, and we develop a framework for homogenization of iterated singular integrals. Our main results regarding the former topic will be obtained as consequences of our results regarding the latter, which are of independent interest. Since the

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precise statements of our results require some preparation, in this section we give a brief and informal description of our work.

Recall that quasiconformal maps of $\mathbb{C}$ are homeomorphic $W^{1,2}_{\mathrm{loc}}$-solutions of the Beltrami equation

$$\partial_{\bar z} F(z) = \mu(z)\, \partial_z F(z), \quad z \in \mathbb{C}, \tag{1.1}$$

and that for any measurable function $\mu\colon \mathbb{C} \to \mathbb{C}$ with $\|\mu\|_\infty < 1$ there is an essentially unique quasiconformal solution. Recent developments have shown an emerging need for a theory of random quasiconformal maps. For example, simple closed planar curves can be described via their welding homeomorphism, and random loops such as those associated with the Schramm–Loewner evolution SLE lead to random circle homeomorphisms. Beginning with the work of Sheffield, these welding homeomorphisms can be described in terms of Liouville quantum gravity. It is still open to analytically solve the "welding problem" of re-constructing the loops from these homeomorphisms. The standard approach of solving welding problems is via the Beltrami equation (1.1), leading to random Beltrami coefficients μ in the case of random weldings. Progress towards solving this problem has been made in [6].

There are also other cases in random geometry where quasiconformal mappings arise naturally. For instance, certain scaling limits of domino tilings [10], and more generally of dimer models [16], exhibit different limiting phases. Quasiconformal mappings appear particularly useful in describing their geometry [3]. Moreover, there is a connection to homogenization of random conductance models, which in turn can be thought of as a special case of Brownian motion in a random environment. Here we refer to the review [8].

In another direction, in material sciences it is important to understand random materials structures, modelled by elliptic PDE's, and look for global or homogenised properties of the material. From the vast literature on homogenization of random PDE's, we mention as examples [17], [12], and [2], where the last mentioned monograph contains an extensive bibliography.

In the present paper we will approach the Beltrami equation (1.1), with a random coefficient μ, via the method of singular integral operators. We will mostly work with solutions normalized by

$$F(w) = w \quad \text{for } w \in \{0, 1, \infty\}. \tag{1.2}$$

However, in the special deterministic case where μ happens to be compactly supported, it is often more convenient to work with the unique homeomorphic solution to (1.1) that has the *hydrodynamic normalization*

$$F(z) - z = o(1) \quad \text{as } z \to \infty. \tag{1.3}$$

This so-called *principal solution* to (1.1) can be obtained from the Neumann series[1]

$$\partial_{\bar z} F = \mu + \mu T\mu + \mu T\mu T\mu + \cdots,$$

with T a specific singular integral operator, the Beurling transform, see (1.16) below.

[1] Operators and multipliers in this paper are always applied from right to left unless otherwise specified, thus for instance $\mu T\mu T\mu = \mu T(\mu T\mu)$.

Therefore, we are naturally led to the study of homogenisation phenomena for iterated singular integral operators. Here it is useful to consider the problem from a broader point of view. Our main result on homogenised iterated singular integrals shows that this can be carried out in surprising generality, allowing for flexibility and a wide range of potential applications:

Theorem 1.1. *For each $1 \le k \le m-1$, let T_k be a translation and dilation invariant singular integral. Further, let $\mu^{(1)} = \mu_\delta^{(1)}, \dots, \mu^{(m)} = \mu_\delta^{(m)}$ be stochastic multiscale functions. Then for any $p \in (1,\infty)$, the iterated singular integral*

$$h_\delta := \mu_\delta^{(m)} T_{m-1} \mu_\delta^{(m-1)} \dots \mu_\delta^{(2)} T_1 \mu_\delta^{(1)}$$

converges weakly in $L^p(\mathbb{R}^d)$ to a deterministic limit function as $\delta \to 0$ (convergence in probability). For the subsequence $h_{2^{-k}}$, the weak convergence takes place almost surely.

The stochastic multiscale functions above are a large class of random functions with δ-periodic statistical structure. Their precise definition is given in Section 3, and Section 4 is devoted to the proof of Theorem 1.1. In general, the multiscale functions need not be bounded or compactly supported. An example of such function is provided by (0.1).

In the next subsection we present a variety of natural and specific random Beltrami equations $\partial_{\bar z} F_\delta = \mu_\delta \partial_z F_\delta$, where the coefficients μ_δ are stochastic multiscale functions with $\|\mu\|_\infty$ bounded by some $k < 1$. To complete the picture, we then need methods more specifically related to quasiconformal mappings to show that the corresponding random solutions F_δ have almost surely a unique deterministic normalised quasiconformal limit F_∞, see e.g. Theorem 1.6 below.

Finally, we mention that Theorem 1.1 also applies to many basic homogenization problems of random partial differential operators, see Example 1.13 in Section 1.3.

1.2. Quasiconformal homogenization

In this subsection we state our main results on quasiconformal homogenization and illustrate them by means of several model examples of coefficients μ_δ. We consider both *deterministic* and *random* quasiconformal maps, though our main emphasis is on the latter case. It will be convenient to adopt the following rescaling notation.

Definition 1.2 (Rescaling notation). If $\delta > 0$, $n \in \mathbb{Z}^d$, and $g\colon \mathbb{R}^d \to \mathbb{C}$ is a function, we define the rescaled function $g_{[n,\delta]}\colon \mathbb{R}^d \to \mathbb{C}$ by the formula

$$g_{[n,\delta]}(x) := g\Big(\frac{x}{\delta} - n\Big).$$

For instance, we will apply this convention to the weight

$$\langle x \rangle := (1 + |x|^2)^{1/2},$$

so that

$$\langle x \rangle_{[n,\delta]} = \Big(1 + \Big|\frac{x}{\delta} - n\Big|^2\Big)^{1/2}$$

for any $x \in \mathbb{R}^d$, $\delta > 0$, and $n \in \mathbb{Z}^d$.

More generally, if $g\colon \mathbb{R}^d \times \Omega \to \mathbb{C}$ is a function of a spatial variable $x \in \mathbb{R}^d$ and a supplementary variable $\omega \in \Omega$, we define $g_{[n,\delta]}\colon \mathbb{R}^d \times \Omega \to \mathbb{C}$ by the formula

$$g_{[n,\delta]}(x,\omega) := g\Big(\frac{x}{\delta} - n, \omega\Big).$$

We extend this convention to the complex plane $\mathbb{C}$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$ (and $\mathbb{Z}^2$ with the Gaussian integers $\mathbb{Z}[i]$.

We will typically apply this convention with functions g that are concentrated near the unit ball $B(0,1)$, in which case the rescaled function $g_{[n,\delta]}$ will be concentrated near the ball $B(n\delta, \delta)$. Conversely, the weight $\langle \cdot \rangle_{[n,\delta]}$ is small in $B(n\delta,\delta)$ and large elsewhere.

1.2.1. Models of complex dilatations. We next consider basic examples of random dilatations we study in this paper. The first example illustrates that our general results apply and give new information even in the deterministic case.

Model 1. The deterministic function

$$\mu_\delta(z) := \varphi(z) \sum_{n\in\mathbb{Z}^2} a_{[n,\delta]}(z), \tag{1.4}$$

where $\varphi \in C_0^\infty(\mathbb{C})$ is a test function and $a\colon \mathbb{C} \to \mathbb{C}$ is a smooth non-constant function supported on $[0,1]^2$, and the rescaling $a_{[n,\delta]}$ is defined by Definition 1.2. One assumes that $\|\varphi\|_\infty \|a\|_\infty < 1$. Hence in this case the $\mu_\delta(x) = \mu_1(\delta^{-1}x)$, and we are dealing with deterministic homogenization of the Beltrami equation.

Model 2. Next assume that μ_δ is a random function given either by

$$\mu_\delta(z) := a\,\mathbb{1}_{Q_0}(z) \sum_{n\in\mathbb{Z}^2} \varepsilon_n (\mathbb{1}_{Q_0})_{[n,\delta]}(z) = a\mathbb{1}_{Q_0}(z) \sum_{n\in\mathbb{Z}^2} \varepsilon_n\, \mathbb{1}_{n\delta+[0,\delta]^2}(z), \tag{1.5}$$

or

$$\mu_\delta(z) := a \sum_{n\in\mathbb{Z}^2} \varepsilon_n\, \mathbb{1}_{n\delta+[0,\delta]^2}(z), \tag{1.6}$$

where $a \in \mathbb{C}$ satisfies $|a| < 1$, $Q_0 := [0,1]^2$ is the unit square with corners $0, 1, i, 1+i$, the $\varepsilon_n \in \{-1,+1\}$ are i.i.d. random signs, and $n\delta + [0,\delta]^2$ is the square of sidelength δ and bottom left corner equal to $n\delta$, $n \in \mathbb{Z}^2$. See Figure 1.

Above one could as well allow the ε_n to be arbitrary i.i.d. random variables with $|\varepsilon_n| \le 1$, but the above case gives the simplest example of stochastic homogenization as now the dilatation is a random function whose *law is δ-periodic* in each coordinate axis direction.

Model 3. A more general model is obtained by allowing the independent 'bumps' to have non-compact support and adding an envelope factor that varies the size of μ locally, and is independent of the scaling δ. Thus, let g be a rapidly decreasing function and define the random 'bump field'

$$B_\delta = \sum_{n\in\mathbb{Z}^2} \varepsilon_n\, g_{[n,\delta]}, \tag{1.7}$$

where ε_n are any i.i.d random variables, the $g_{[n,\delta]}$ are defined by Definition 1.2, and we assume the pointwise bound $|B_\delta| \le 1$. Then set

$$\mu_\delta := \phi \, \mathbb{1}_U \, B_\delta, \tag{1.8}$$

where the 'envelope function' ϕ satisfies the pointwise bound $|\phi| \le k$ for some $k < 1$ and is Hölder continuous with some exponent $\alpha > 0$, and $U \subset \mathbb{C}$ is a domain with piecewise Hölder-boundary (e.g., U could as well be the whole plane).

If we specialize to the case $\phi \equiv a$, where a is a complex constant with $|a| < 1$, then μ_δ becomes a constant multiple of the random bump field (1.7):

$$\mu_\delta(z) := a B_\delta(z). \tag{1.9}$$

In each of the above model cases, let F_δ be the unique solution to the (random or deterministic) Beltrami equation

$$\partial_{\bar z} F_\delta = \mu_\delta \, \partial_z F_\delta \tag{1.10}$$

with 3-point normalization (1.2). The basic question of quasiconformal homogenization then asks if the sequence $F_{2^{-k}}$ converges as $k \to \infty$. We soon answer this question by showing that there is almost sure convergence to a deterministic limit homeomorphism.

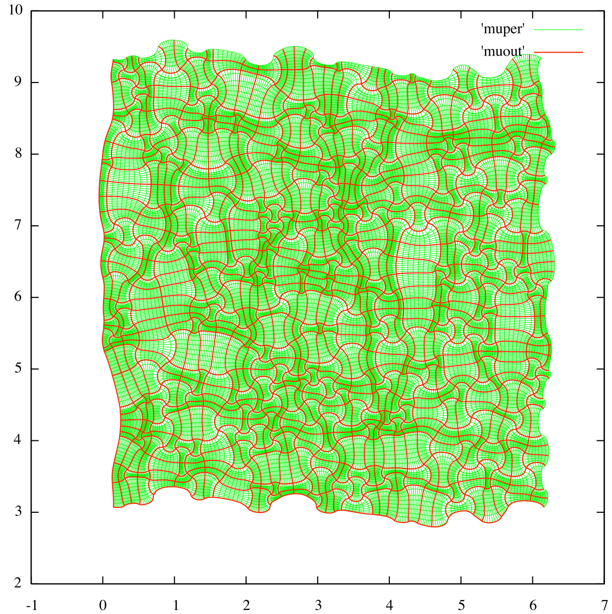

Figure 1. A random qc-map obtained by Model 2, with $a = 1/2$. We thank David White for help in producing the picture.

1.2.2. Bump fields and envelope functions. In order to state our results, we need to define properly the admissible envelope functions and random bump fields already alluded to above when discussing Model 3.

Definition 1.3 (Random bump fields). We define *random bump data* to be a pair (g, X), where X is a random variable taking values in[2] $\mathbb{R}$, and $g: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is a measurable function with rapid decrease in the first variable,

$$|g(z,y)| \le C_M \langle z \rangle^{-M} \quad \text{for all } M \ge 1 \text{ and } z \in \mathbb{C}, y \in \mathbb{R}, \tag{1.11}$$

which obeys the pointwise bound

$$\Big| \sum_{n \in \mathbb{Z}^2} g(z-n, y_n) \Big| \le 1$$

for all $z \in \mathbb{C}$ and all real sequences $(y_n)_{n \in \mathbb{Z}^2}$. We define a *random bump field* with data (g, X) and scaling parameter $\delta > 0$ to be a random field of the form

$$B_\delta(z) := \sum_{n \in \mathbb{Z}^2} g_{[n,\delta]}(z, X_n), \tag{1.12}$$

where the rescaling $g_{[n,\delta]}$ is defined by Definition 1.2, and X_n, $n \in \mathbb{Z}^2$, are independent copies of the random variable X.

In turn, the admissible envelope functions are as follows.

Definition 1.4 (Beltrami envelope functions). A measurable function $\phi: \mathbb{C} \to \mathbb{C}$ is a *Beltrami envelope function* if there is $k \in (0,1)$ such that $|\phi(z)| \le k$ for almost every $z \in \mathbb{C}$ and ϕ is locally Hölder-continuous in L^1-norm: there is $\alpha > 0$ such that, for any $R > 0$, there is $C_R < \infty$ with

$$\|\Delta_h(\mathbb{1}_{B(0,R)}\phi)\|_{L^1(\mathbb{C})} \le C_R |h|^\alpha, \quad \text{for } |h| \le 1,$$

where the difference operator Δ_h is defined by

$$\Delta_h f(x) := f(x+h) - f(x). \tag{1.13}$$

Example 1.5. Assume that $\phi: \mathbb{C} \to \mathbb{C}$ is α-Hölder continuous and satisfies $|\phi| \le k < 1$. Assume also that $U \subset \mathbb{C}$ is a domain with locally Hölder-regular boundary. Then it is easy to verify that $\mathbb{1}_U \phi$ is a Beltrami envelope function. This holds also true if (locally) the Minkowski dimension of ∂U is strictly less than 2.

1.2.3. Results on quasiconformal homogenization. In each of the Models 1-3 discussed above, the random dilatation can be written in the form $\mu_\delta = \phi\, B_\delta$, where ϕ is a Beltrami envelope function and B_δ a random bump field. Hence our results on quasiconformal homogenization, to be stated next, cover all these cases.

[2] We place our random parameter in the space $\mathbb{R}$ for sake of concreteness, but this space could be replaced by a more general measurable space, e.g., $\mathbb{R}^d$ for any d, if one wished.

Theorem 1.6. *Let (g, X) be random bump data, and let ϕ be a Beltrami envelope function. For $\delta > 0$, let*

$$\mu_\delta(z) = \phi(z) B_\delta(z),$$

where B_δ is the random bump field (1.12) determined by (g, X). Denote by F_j, $j \geq 1$, the 3-point normalized solution to the random Beltrami equation

$$\partial_{\bar{z}} F_j = \mu_{2^{-j}} \, \partial_z F_j \, . \tag{1.14}$$

There is a unique deterministic limit function F_∞ such that $F_\infty\colon \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism and, as $j \to \infty$, almost surely,

$$F_j \to F_\infty \quad \textit{locally uniformly.}$$

If in the previous theorem the envelope function ϕ is constant in the whole plane, it follows that the limit function has constant dilatation, and hence is linear. This follows from Lemma 5.5 below. More generally, we have following result.

Theorem 1.7. *Assume that the envelope function ϕ is continuous at z_0. Then the dilatation μ_{F_∞} of the limit function F_∞ is continuous at z_0, and $\mu_{F_\infty}(z_0)$ depends only on the random bump data (g, X) and on the value $\phi(z_0)$. More precisely, one has*

$$\mu_{F_\infty}(z_0) = h_{(g,X)}(\phi(z_0)),$$

where the function $h_{(g,X)}\colon \{|z| < 1\} \to \{|z| < 1\}$ is continuous.

Finally, we note that in some natural situations the limit function reduces to the identity map. We refer to Lemma 5.6 below for a more general condition in this direction.

Theorem 1.8. *If the random variables ε_n are symmetric, the limit F_∞ in both cases of Model 2, (1.5) and (1.6), is given by the identity map, $F_\infty(z) = z$ for all z. This is not necessarily the case in the more general setting of (1.9).*

The proofs of all the above three theorems are contained in Section 5, which also contains other related results and remarks. In particular, the above theorem applies to the deterministic homogenization problem as well. We also stress that the coefficient μ need not be compactly supported, in spite of the fact that the proof is based on the Neumann series.

Remark 1.9. One should note that in the above result there is no need for the stochastic bump fields corresponding to different δ's to be independent. Indeed, their stochastic relation can be arbitrary. This can be understood by writing the dilatation of F_j in the form

$$\mu_{F_j}(z) = \phi(z)\Big(\sum_{n \in \mathbb{Z}^2} g_{[n,2^{-j}]}(z, X_{n,j})\Big),$$

where $X_{n,j} \sim X$, for each n, j, and only for each fixed j the random variables $X_{n,j}$, $n \in \mathbb{Z}^2$, are assumed to be independent. Thus there can be arbitrary stochastic relations between the different layers $(X_{n,j})_{n \in \mathbb{Z}^2}$ and $(X_{n,j'})_{n \in \mathbb{Z}^2}$ for $j \neq j'$. In particular, this possible dependence structure between different scales does not affect the deterministic limit function F_∞, which depends only on the triplet (ϕ, g, X). The main reason for this is that the failure probability in our main estimate (Theorem 3.9) decays polynomially in δ (and hence exponentially in j if $\delta = 2^{-j}$).

Let us also point out that for the sake of simplicity we leave out many considerations that would be possible via the techniques of the present paper. For example, one may relax the speed of convergence to zero in the subsequence $B_{2^{-j}}$, and it is possible to consider quasiconformal maps between arbitrary domains.

In Section 5 we will also present a homogenization result for mappings of finite distortion, i.e., we consider random dilatations μ_δ with $\|\mu_\delta\|_{L^\infty(\mathbb{R}^2)} = 1$. An example of this kind of dilatation is given by

Model 4. A random function as in the model example (1.5),

$$\mu_\delta(z) := \mathbb{1}_{Q_0}(z) \sum_{n\in\mathbb{Z}^2} \varepsilon_n (\mathbb{1}_{[0,1]^2})_{[n,\delta]}(z) = \mathbb{1}_{Q_0}(z) \sum_{n\in\mathbb{Z}^2} \varepsilon_n \mathbb{1}_{n\delta+[0,\delta]^2}(z), \tag{1.15}$$

but now with random i.i.d. variables ε_n such that $|\varepsilon_n| < 1$ and the tail of $(1-|\varepsilon_n|)^{-1}$ has sufficiently fast exponential decay. Theorems 5.8, 5.13 and 5.14 in Section 5 generalize Theorem 1.6 to the degenerate case (1.15) and beyond.

It is tempting to try to prove almost sure convergence of F_δ in the above examples solely using weak convergence of μ_δ. However, it is important to note that this is impossible, as the following example illustrates. Some deeper properties of μ_δ and their interaction with singular integrals are involved here.

Example 1.10. Let $a \in (-1,1)$, and define the Beltrami coefficient $\nu(z)$ that is 2-periodic in the x-variable and constant in the y-variable by setting

$$\nu(z) := \begin{cases} a & \text{if } x \in [2n, 2n+1),\ n \in \mathbb{Z}, \\ -a & \text{if } x \in [2n+1, 2n+2),\ n \in \mathbb{Z}. \end{cases}$$

Write $b = (1-a)/(1+a)$ and observe that the function

$$g(x+iy) := \begin{cases} (x-2n)+n(1+b^2)+iby & \text{if } x\in[2n,2n+1),\ n \in \mathbb{Z}, \\ b^2(x-(2n+1))+n(1+b^2)+1+iby & \text{if } x\in[2n+1,2n+2),\ n \in \mathbb{Z}, \end{cases}$$

solves $g_{\bar z} = \nu g_z$. Now consider the homogenized dilatation $\mu_j(z) := \nu(2jz)$ for any integer $j \geq 1$, and let F_j satisfy

$$\partial_{\bar z} F_j = \mu_j \,\partial_z F_j$$

with the three-point normalization, so that

$$F_j(z) = \frac{g(2jz)}{j(1+b^2)}.$$

As $j \to \infty$, it is clear that μ_j converges locally weakly to zero. However, by the above formulas we see that there is the uniform convergence $F_j \to F_\infty$, where

$$F_\infty(x+iy) = x + \frac{1-a^2}{1+a^2}\, iy,$$

and $\mu_{F_\infty} \equiv a^2$ identically. By considering the sequence $\widetilde{\mu}_j$ given by $\widetilde{\mu}_{2j} = \mu_j$ and $\widetilde{\mu}_{2j+1} = 0$, we obtain a locally weakly null sequence of dilatations for which the homogenization limit does not exist. Finally, we observe that its is easy to localize this observation and obtain the same phenomenon for compactly supported dilatations (see Lemma 5.4 below).

Remark 1.11. In a recent interesting work [14], Ivrii and Markovic provide a more elementary geometric proof of some special cases of our results on quasiconformal homogenization, also allowing for non-uniform ellipticity, and give an application to random Delaunay triangulations.

1.3. Further remarks on Theorem 1.1

As explained in Section 1.1, the application of Theorem 1.1 to quasiconformal homogenization and solutions to Beltrami equation (1.1) comes via the *Beurling transform*

$$Tg(z) := -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{g(w)}{(z-w)^2}\,dw. \tag{1.16}$$

Namely, since $T \circ \partial_{\bar z} = \partial_z$ on $W^{1,2}(\mathbb{C})$, finding a solution to $\partial_{\bar z} f_\delta = \mu_\delta \partial_z f_\delta$, with the hydrodynamic normalization $f_\delta(z) - z = o(1)$ at $z \to \infty$, is equivalent to solving the integral equation $(1 - \mu_\delta T)\partial_{\bar z} f_\delta = \mu_\delta$. One then finds the solution via the L^2-Neumann series representation

$$\partial_{\bar z} f_\delta = \mu_\delta + \mu_\delta T\mu_\delta + \mu_\delta T\mu_\delta T\mu_\delta + \cdots,$$

and the theorem allows us to deduce the weak convergence of each single summand in the above formula.

The Beurling transform extends to all of $L^2(\mathbb{C})$ as an isometric isomorphism. Moreover, it commutes with dilatations and translations. The class of singular integrals in $\mathbb{R}^d$ allowed in Theorem 1.1 shares these two basic symmetries:

Definition 1.12 (Singular integral operator). A dilation and translation invariant *singular integral operator* is any bounded linear operator $T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ of the form

$$Tf(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega((x-y)/|x-y|)}{|x-y|^d}\, f(y)\,dy, \quad \text{for } f \in C_0^\infty(\mathbb{R}^d),$$

where $\Omega\colon S^{d-1} \to \mathbb{C}$ is smooth and has mean zero.

The definition of the general class of random multipliers considered in Theorem 1.1, *the stochastic multifunctions*, is slightly opaque as it employs the notion of *stochastic tensor products*. Both these notions will be explained in Section 3 below. However, to give a perhaps more intuitive idea of these notions, we describe here in detail a special class of multifunctions which fits well with the notions of bump-fields and Beltrami envelope functions discussed in the previous Section 1.2. Thus, working in arbitrary dimension $d \geq 1$, fix $m \geq 1$ and consider for each index $1 \leq \ell \leq m$ the random function

$$\mu_\delta^{(\ell)}(x) = \phi_\ell(x) \sum_{n \in \mathbb{Z}^d} (g_\ell)_{[n,\delta]}(x, X_n), \tag{1.17}$$

where we assume for each fixed ℓ that:

- The '*envelope function*' ϕ_ℓ does not depend on δ. Moreover, $\phi_\ell \in L^p(\mathbb{R}^d)$ for every $p \in (1,\infty)$ with the Hölder bound

$$\|\Delta_h \phi_\ell\|_{L^p(\mathbb{R}^d)} \le C(p)|h|^{\alpha_p}, \quad \text{for } |h| \le 1, \text{ where } \alpha_p > 0.$$

- The random variables $\{X_n\}_{n\in\mathbb{Z}^d}$ are independent and identically distributed, $X_n \sim X$ for all $n \in \mathbb{Z}^d$.
- The '*bump function*' $g_\ell(\cdot,\cdot)$ satisfies an d-dimensional analogue of condition (1.11).

Lemma 3.11 below implies that such $\mu_\delta^{(\ell)}$ is a stochastic multifunction, covered by Theorem 1.1.

As a last aspect, Theorem 1.1 applies easily to homogenization of many random differential operators:

Example 1.13. For each $\ell = 1, \dots, L$, let $P_\ell(D)$ be a constant coefficient second order differential operator on $\mathbb{R}^d$. Also let $\mu_\delta^{(\ell)}$ be random multipliers as in Theorem 1.1. For simplicity, we assume that $d \ge 3$, and that the $\mu_\delta^{(\ell)}$ are supported on a fixed ball $B \subset \mathbb{R}^d$. Our basic ellipticity assumption is that they satisfy, almost surely,

$$\sum_{\ell=1}^{L} a_\ell \|\mu_\delta^{(\ell)}\|_{L^\infty(B)} \le k < 1 \quad \text{for all } \delta \in (0,1), \tag{1.18}$$

where the constants a_j will be soon defined. We consider the following PDE on $\mathbb{R}^d$ with random coefficient functions:

$$\Delta u_\delta + \sum_{\ell=1}^{L} \mu_\delta^{(\ell)} P_\ell(D) u_\delta = h. \tag{1.19}$$

Here the right-hand side $h \in L^2(\mathbb{R}^d)$ is fixed, and is also supported in the ball B. We normalize the solutions u_δ of (1.19) by demanding that $u_\delta(x) \to 0$ as $x \to \infty$.

We claim that this problem has a unique solution $u_\delta \in \dot W^{2,2}(\mathbb{R}^d)$ that converges strongly (in probability) in $\dot W^{s,2}(\mathbb{R}^d)$ for every $s < 2$ towards a deterministic function $u_0 \in \dot W^{2,2}(\mathbb{R}^d)$ as $\delta \to 0$. Thus the present homogenization problem is solvable with a deterministic limit.

In order to sketch the argument, let us denote by T_ℓ the homogeneous Fourier multiplier $T_\ell := \Delta^{-1} P_\ell(D)$, and note that it is a scaling and translation invariant singular integral[3] on $\mathbb{R}^d$. We choose $a_\ell := \sup_{|\xi|=1} |P_\ell(\xi)|$ in condition (1.18), i.e., a_ℓ is the L^2-norm of the operator T_ℓ. Then $f_\delta := \Delta u_\delta \in L^2(B)$ satisfies the equation

$$f_\delta + \sum_{\ell=1}^{L} \mu_\delta^{(\ell)} T f_\delta = h,$$

[3] Strictly speaking, the T_ℓ might not be precisely of the form in Definition 1.12 because there may be an identity component in addition to a principal value integral; however it is a not difficult to extend the analysis of this paper to this more general setting. More precisely, we note, leaving the details to the reader, that the arguments of the proofs in Sections 2-4 remain valid and even simplify if some of the integral operators T_j is replaced by the identity operator.

which can be uniquely solved in $L^2(\mathbb{R}^d)$ by the Neumann series and, via condition (1.18), we obtain an L^2-convergent series

$$f_\delta = h + \sum_{\substack{1\le \ell_1,\ldots,\ell_m\le L\\ m\ge 1}} (-1)^m \mu_\delta^{(\ell_1)} T_{\ell_1} \mu_\delta^{(\ell_2)} T_{\ell_2} \cdots \mu_\delta^{(\ell_m)} T_{\ell_m} h. \tag{1.20}$$

By applying the fundamental solution of the Laplacian, we see that $u_\delta = c_d|\cdot|^{2-d} * f_\delta$ solves (1.19) with the right behaviour in the infinity, as $d \ge 3$. Since any other solution has the same Laplacian, they must differ by a harmonic function that vanishes at infinity, and hence their difference is zero.

Theorem 1.1 applies to each term in the sum (1.20), and together with the uniform convergence in δ of the series in L^2 we deduce that $f_\delta \to f_0$ weakly in $L^2(\mathbb{R}^d)$, where f_0 is also supported in B. The rest of the claim follows from the standard properties of the fundamental solution $c_d|\cdot|^{2-d}$.

We finally note that above the operators P_ℓ may well have lower order terms since those produce compact Fourier multipliers between functions on fixed compact subsets of $\mathbb{R}^d$. Hence the terms in the Neumann-series containing them can be taken care of by multiple application of Theorem 1.1. Actually, we could instead of differential operators P_ℓ with constant coefficients consider as well classical pseudodifferential operators of order 2 whose principal part is a homogeneous Fourier multiplier.

Similarly, the technique applies to fractional Laplacians, and in many other type of homogenization problems. In order to spell out one more specific example – completely without details –, consider the homogenization of the general conductivity equation in the plane.

$$\nabla \cdot \big(A(x)\nabla u(x)\big) = 0,$$

where the 2×2 matrix $A(x) = (\delta_{j,k} + \mu_{j,k}(x))_{j,k=1,2}$ is measurable and uniformly elliptic, and each $\mu_{j,k}$ is a stochastic multifunction. One may reduce this to the study of the generalized Beltrami equation

$$\partial_z f = \eta_1 \partial_{\bar z} f + \eta_2 \overline{\partial_{\bar z} f},$$

where the coefficients η_j's are expressed in terms of in the matrix coefficients $\mu_{j,k}$, see e.g. Theorem 16.1.6 in [5]. The structure of the η_j's allows them be approximated in a suitable sense by multifunctions (see footnote 3 in this connection). The generalized Beltrami equation may be solved by a 2×2-matrix valued Neumann series, and the analysis can be then carried out analogously to the case of the classical Beltrami equation.

For classical treatments of homogenization of the above PDE's (however, not including the case of more general case of Fourier multipliers we allow for), we refer to [17], [18], and [1].

1.4. Structure of the paper

Section 2 develops the homogenization of deterministic iterated singular integrals. This is much easier than the random setting, but has its own interest, and it will provide a handy tool in treating the stochastic case later on. The admissible class of deterministic multipliers will be called called *multiscale functions* (see Definition 2.12). They are defined using the notion of 'multiscale tensor product' (Definition 2.10), which generalizes the product of an envelope function and a bump field. Our deterministic homogenization result is stated as Corollary 2.25.

Section 3 first defines the probabilistic analogues of the deterministic notions, especially the 'stochastic multiscale tensor product' (Definition 3.2) is used to define *stochastic multiscale functions* (Definition 3.1), which are quite a bit more general than the multipliers we discussed in Section 1.2. The general form of our main result on homogenization of randomized iterated singular integrals is formulated in Theorem 3.9 and Corollary 3.10. Lemma 3.11 then verifies that the random multipliers (1.17) are particular instances of a stochastic multiscale tensor product.

The proof of Theorem 1.1 is carried out in Section 4, where it is obtained as a consequence of Theorem 3.9 and Corollary 3.10. Somewhat surprisingly, a considerable effort needs to be spent in establishing the convergence of the expectation of the iterated randomized integrals.

Finally, Section 5 applies Theorem 1.1 to quasiconformal homogenization. There we combine Theorem 1.1 with methods from the theory of planar quasiconformal mappings in order to show that the corresponding random solutions F_δ almost surely have a unique normalised deterministic quasiconformal limit F_∞, see e.g. Theorem 1.6.

2. Deterministic multiscale functions

In the sequel we use extensively the notations $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $|X| \leq CY$, where C is an absolute constant. If we need the constant C to depend on some parameters, we shall indicate this by subscripts, or else indicate the dependence in the text. For instance, $X \lesssim_p Y$ or $X = O_p(Y)$ means that $|X| \leq C_p Y$ for some constant C depending on p.

Our arguments in the following three sections are not specific to the Beurling transform in the plane, and so we shall work in the more general context of singular integral operators in a Euclidean space. Accordingly, we fix a dimension $d \geq 1$; in the application to the Beltrami equation, we will have $d = 2$. We shall work with the standard Euclidean space $\mathbb{R}^d$, the standard lattice $\mathbb{Z}^d$, and the standard torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We also have a scale parameter $0 < \delta < 1$, which we shall think of as being small; several of our functions shall depend on this parameter, and we shall indicate this by including δ as a subscript.

Before we can state the main result, it will be convenient to introduce a certain calculus regarding various classes of functions (namely, envelope functions, localized functions, negligible functions, and multiscale functions; we will define these classes later in this section). To set up this calculus, we shall need a certain amount of notation and basic theory.

Definition 2.1 (Hölder space). If $1 < p < \infty$ and $\alpha \in (0, 1)$, we let $\Lambda^{\alpha,p}(\mathbb{R}^d)$ denote the space of functions f whose norm

$$\|f\|_{\Lambda^{\alpha,p}(\mathbb{R}^d)} := \|f\|_{L^p(\mathbb{R}^d)} + \sup_{0<|h|<1} \frac{\|\Delta_h f\|_{L^p(\mathbb{R}^d)}}{|h|^\alpha}$$

is finite, where Δ_h was defined in (1.13).

Remark. In what follows, one could also use Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$ instead of Hölder spaces $\Lambda^{\alpha,p}(\mathbb{R}^d)$, but we have elected to use Hölder spaces as they are slightly more elementary. Also note that we usually use the symbol ϕ for a Beltrami envelope function, cf. Definition 1.4.

We recall the following definition from page 2294.

Definition 2.2 (Envelope function). An *envelope function* is a function $f:\mathbb{R}^d \to \mathbb{C}$ (not depending on the scale parameter δ) such that for every $1 < p < \infty$ there exists $\alpha > 0$ such that $f \in \Lambda^{\alpha,p}(\mathbb{R}^d)$. Thus the space of all envelope functions is

$$\bigcap_{1<p<\infty} \bigcup_{\alpha\in(0,1)} \Lambda^{\alpha,p}(\mathbb{R}^d).$$

Example 2.3. If Q is a cube in $\mathbb{R}^d$, then one checks that $\mathbb{1}_Q \in \Lambda^{\alpha,p}(\mathbb{R}^d)$ for $\alpha \in (0, 1/p)$, so that the indicator function $\mathbb{1}_Q$ is an envelope function. Any function in the Schwartz class is an envelope function.

Lemma 2.4. *The product of two envelope functions is again an envelope function.*

Proof. From Hölder's inequality, one quickly sees that the product of two functions in $\Lambda^{\alpha,p}(\mathbb{R}^d)$ lies in $\Lambda^{\alpha,p/2}(\mathbb{R}^d)$. The claim follows. ∎

Definition 2.5 (Localized function). A (deterministic) *localized function* is a function $g:\mathbb{R}^d \to \mathbb{C}$ (not depending on the scale parameter δ) such that for every $1 < p < \infty$ and $N > 0$, the function $\langle\cdot\rangle^N g$ lies in $L^p(\mathbb{R}^d)$, where $\langle\cdot\rangle$ is as in Definition 1.2. Thus the space of all localized functions is

$$\bigcap_{1<p<\infty} \bigcap_{N>0} \langle\cdot\rangle^{-N} L^p(\mathbb{R}^d).$$

Example 2.6. The indicator function $\mathbb{1}_Q$ of a cube is a localized function, as is any function in the Schwartz class.

We shall often exploit the ability of localized functions to absorb arbitrary powers of $\langle\cdot\rangle_{[n,\delta]}$ via the following lemma, which improves upon the triangle inequality in L^p at the cost of inserting different localizing weights $\langle\cdot\rangle_{[n,\delta]}$ on each summand.

Lemma 2.7 (Localization lemma). *Let $\delta > 0$ and let $1 < p < \infty$. Then we have the estimate*

$$\Big\| \sum_{n\in\mathbb{Z}^d} f_n \Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \Big(\sum_{n\in\mathbb{Z}^d} \|\langle\cdot\rangle^d_{[n,\delta]} f_n\|^p_{L^p(\mathbb{R}^d)} \Big)^{1/p}$$

for any sequence $f_n \in L^p(\mathbb{R}^d)$ of functions.

Proof. We can rescale $\delta = 1$. It will suffice to prove the pointwise inequality

$$\Big| \sum_{n\in\mathbb{Z}^d} f_n \Big| \lesssim_{p,d} \Big(\sum_{n\in\mathbb{Z}^d} \langle\cdot\rangle^{pd}_{[n,1]} |f_n|^p \Big)^{1/p}.$$

By Hölder's inequality, it is enough to show that pointwise,

$$\Big(\sum_{n\in\mathbb{Z}^d} \langle\cdot\rangle_{[n,1]}^{-p'd}\Big)^{1/p'} \lesssim_{p,d} 1,$$

where $p' = p/(p-1)$ is the dual exponent of p. But this can be established by direct calculation. ■

Let us record a couple of elementary properties of localized functions.

Lemma 2.8.

(i) *The product of two localized functions is a localized function.*

(ii) *For any localized function g and any sequence (a_n), it holds that*

$$\Big\|\sum_{n\in\mathbb{Z}^d} a_n\, g_{[n,\delta]}\Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,g} \delta^{d/p}\|(a_n)\|_{\ell^p}, \quad 1 < p < \infty,$$

where $g_{[n,\delta]}$ is given by Definition 1.2.

Proof. Claim (i) follows from Hölder's inequality, and claim (ii) is an immediate consequence of Lemma 2.7. ■

Definition 2.9 (Discretization). Let f be an envelope function, and let $0 < \delta < 1$. We define the *discretization* $[f]_\delta\colon \mathbb{Z}^d \to \mathbb{C}$ of f at scale δ to be the function

$$[f]_\delta(n) := \frac{1}{\delta^d}\int_{n\delta+[0,\delta]^d} f,$$

thus $[f]_\delta(n)$ is the average value of f on the cube $n\delta + [0,\delta]^d$.

Definition 2.10 (Multiscale tensor product). Let f be an envelope function and let g be a localized function. We define the *multiscale tensor product* $f \otimes_\delta g\colon \mathbb{R}^d \to \mathbb{C}$ of f and g to be the function

$$f \otimes_\delta g := \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\, g_{[n,\delta]},$$

where $g_{[n,\delta]}$ is defined by Definition 1.2.

Definition 2.11 (Negligible function). A function $F = F_\delta\colon \mathbb{R}^d \to \mathbb{C}$ depending on the parameter $0 < \delta < 1$ is said to be *negligible* if for every $1 < p < \infty$ there exists $\varepsilon_p > 0$ and $C_p > 0$ such that

$$\|F_\delta\|_{L^p(\mathbb{R}^d)} \le C_p\, \delta^{\varepsilon_p}$$

for all $0 < \delta < 1$.

Remark. Note in particular that if F is negligible, then F_δ converges to zero in the L^p norm as $\delta \to 0$ for every $1 < p < \infty$, and furthermore the same is true even if one multiplies F_δ by an arbitrary power of $(\log \frac{1}{\delta})$. This freedom to absorb logarithmic factors in δ will be useful for technical reasons later in this paper.

Definition 2.12 (Multiscale function). A function $F = F_\delta\colon \mathbb{R}^d \to \mathbb{C}$ depending on the parameter $0 < \delta < 1$ is said to be a (deterministic) *multiscale function* if it has an expansion

$$F_\delta = \sum_{j=1}^{J} f_j \otimes_\delta g_j + G_\delta,$$

where $J \geq 1$ is an integer, $f_1, \ldots, f_J$ are envelope functions, $g_1, \ldots, g_J$ are localized functions, and G_δ is a negligible function. If F_δ and $\widetilde{F}_\delta$ are two multiscale functions such that the difference $F_\delta - \widetilde{F}_\delta$ is negligible, we say that F_δ and $\widetilde{F}_\delta$ are *equivalent*.

Example 2.13. If Q is a cube, and g is a localized function, then the function

$$F_\delta(x) := \sum_{n \in \mathbb{Z}^d} \mathbb{1}_Q(n\delta)\, g_{[n,\delta]}(x)$$

can be easily verified to be a multiscale function. To this end we use Lemma 2.8 (ii) to estimate

$$\|F_\delta - \mathbb{1}_Q \otimes_\delta g\|_{L^p(\mathbb{R}^d)} \leq \Big\| \sum_{n:\, d(n\delta, \partial Q) \leq 2\sqrt{d}\,\delta} |g_{[n,\delta]}| \Big\|_{L^p(\mathbb{R}^d)} \lesssim (\delta^{(1-d)})^{1/p} \delta^{d/p} = \delta^{1/p}.$$

Hence the difference $F_\delta - \mathbb{1}_Q \otimes_\delta g$ is negligible. A similar statement is true if $\mathbb{1}_Q$ is replaced by a Schwartz function.

Example 2.14. The function μ_δ defined in (1.4) is a multiscale function. Indeed, μ_δ is equivalent to $\varphi \otimes_\delta a$. More generally, we will prove in Lemma 3.11 below that if g is bounded and quickly decaying,

$$|g(x)| \leq C_N \langle x \rangle^{-N} \quad \text{for all } N \geq 1 \text{ and } x \in \mathbb{R}^d,$$

then for any envelope function f the stochastic multiscale tensor product $f \otimes_\delta g$ is equivalent to the function $f \sum_{n \in \mathbb{Z}^d} g_{[n,\delta]}$.

We continue with basic discretization estimates for envelope functions, encoded in the following two lemmas.

Lemma 2.15. *Let f be an envelope function.*

(i) *One has $\|[f]_\delta\|_{\ell^p(\mathbb{Z}^d)} \lesssim_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \delta^{-d/p}$ for all $0 < \delta < 1$ and $1 < p < \infty$.*

(ii) *For any $r \in \mathbb{Z}^d$, we have*

$$\|\Delta_r [f]_\delta\|_{\ell^p(\mathbb{Z}^d)} \lesssim_{p,f,d} (r\delta)^{\varepsilon_p}\, \delta^{-d/p}$$

for some $\varepsilon_p > 0$ independent of δ or r.

Proof. From Hölder's inequality followed by Fubini's theorem, we have

$$\|[f]_\delta\|_{\ell^p(\mathbb{Z}^d)} \leq \Big\| \Big(\frac{1}{\delta^d} \int_{n\delta + [0,\delta]^d} |f|^p \Big)^{1/p}_{n \in \mathbb{Z}^d} \Big\|_{\ell^p(\mathbb{Z}^d)} = \delta^{-d/p} \|f\|_{L^p(\mathbb{R}^d)}$$

and (i) follows since $f \in L^p(\mathbb{R}^d)$. For (ii), observe that

$$\Delta_r [f]_\delta = [\Delta_{\delta r} f]_\delta.$$

The claim now follows from (i) as $f \in \Lambda^{\varepsilon_p, p}(\mathbb{R}^d)$ for some $\varepsilon_p > 0$. ∎

Also, discretization and multiplication almost commute:

Lemma 2.16. *Let f and F be envelope functions. Then for every $1 < p < \infty$ there exists $\varepsilon_p > 0$ such that*

$$\| [fF]_\delta - [f]_\delta [F]_\delta \|_{\ell^p(\mathbb{Z}^d)} \lesssim_{p,f,F,d} \delta^{-d/p+\varepsilon_p}.$$

Proof. From Fubini's theorem, we have

$$[fF]_\delta(n) - [f]_\delta(n)[F]_\delta(n) = \frac{1}{\delta^{2d}} \int_{\delta n + [0,\delta]^d} \int_{\delta n + [0,\delta]^d} f(x)(F(x) - F(y))\, dx\, dy.$$

Writing $y = x + r$, we can thus estimate

$$|[fF]_\delta(n) - [f]_\delta(n)[F]_\delta(n)| \le \frac{1}{\delta^{2d}} \int_{[-\delta,\delta]^d} \int_{\delta n + [0,\delta]^d} |f(x) \Delta_r F(x)|\, dx\, dr,$$

and hence by Minkowski's inequality,

$$\| [fF]_\delta - [f]_\delta [F]_\delta \|_{\ell^p(\mathbb{Z}^d)} \le \frac{1}{\delta^d} \int_{[-\delta,\delta]^d} \left\| \frac{1}{\delta^d} \int_{\delta n + [0,\delta]^d} |f(x) \Delta_r F(x)|\, dx \right\|_{\ell^p_n(\mathbb{Z}^d)} dr.$$

Lemma 2.15 (i) allows us to conclude

$$\| [fF]_\delta - [f]_\delta [F]_\delta \|_{\ell^p(\mathbb{Z}^d)} \le \frac{\delta^{-d/p}}{\delta^d} \int_{[-\delta,\delta]^d} \| f \Delta_r F \|_{L^p(\mathbb{R}^d)}\, dr,$$

and finally, by Hölder's inequality and the fact that $f \in L^{2p}(\mathbb{R}^d)$ and $F \in \Lambda^{\varepsilon_{2p}, 2p}(\mathbb{R}^d)$ for some $\varepsilon_{2p} > 0$, we see that for $r \in [0,\delta]^d$,

$$\| f \Delta_r F \|_{L^p(\mathbb{R}^d)} \lesssim_{p,f,F} \delta^{\varepsilon_{2p}}$$

and the claim follows. ∎

Next we consider the basic properties of multiscale functions. For this purpose we need a couple of useful lemmas.

Lemma 2.17. *Assume that f is an envelope function and g a localized function. Then for any $\delta > 0$,*

$$\| f \otimes_\delta g \|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \| f \|_{L^p(\mathbb{R}^d)} \| \langle \cdot \rangle^d g \|_{L^p(\mathbb{R}^d)}. \tag{2.1}$$

Proof. We apply the localization lemma (Lemma 2.7) to estimate

$$\|f \otimes_\delta g\|_{L^p(\mathbb{R}^d)} = \Big\| \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\, g_{[n,\delta]} \Big\|_{L^p(\mathbb{R}^d)}$$

$$\lesssim_p \Big(\sum_{n\in\mathbb{Z}^d} [f]_\delta^p(n)\, \|\langle\cdot\rangle_{[n,\delta]}^d\, g_{[n,\delta]}\|_{L^p(\mathbb{R}^d)}^p \Big)^{1/p} \lesssim_{p,d} \delta^{-d/p} \|f\|_{L^p(\mathbb{R}^d)} \delta^{d/p} \|\langle\cdot\rangle^d g\|_{L^p(\mathbb{R}^d)},$$

where we have applied Lemma 2.15 (i) and the fact that, for all n, $\|\langle\cdot\rangle_{[n,\delta]} g_{[n,\delta]}\|_{L^p(\mathbb{R}^d)}^p = \delta^d \|\langle\cdot\rangle g\|_{L^p(\mathbb{R}^d)}^p$. ∎

In particular, if $\operatorname{supp}(g) \subset [0,1]^d$, then the above lemma yields the simple estimate

$$\|f \otimes_\delta g\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|f\|_{L^p(\mathbb{R}^d)}\, \|g\|_{L^p(\mathbb{R}^d)}.$$

The following lemma reduces us to considering multiscale tensor products $f \otimes_\delta g$ with g supported in $[0,1]^d$.

Lemma 2.18. *Assume that f is an envelope function and that g is either a localized function, or (more generally) that it satisfies for each $p \in (0,\infty)$,*

$$\|g \mathbb{1}_{k+[0,1]^d}\|_{L^p(\mathbb{R}^d)} \lesssim_{g,p} \langle k\rangle^{-a}, \quad k \in \mathbb{Z}^d, \tag{2.2}$$

with some $a > d$. Then $f \otimes_\delta g$ is a multiscale function that is equivalent to $f \otimes_\delta \widetilde{g}$, where $\widetilde{g}$ is supported in $[0,1]^d$ and given explicitly by the formula

$$\widetilde{g}(x) := \mathbb{1}_{[0,1]^d}(x) \sum_{k\in\mathbb{Z}^d} g(x+k). \tag{2.3}$$

Proof. Observe first that any localized function satisfies (2.2). The idea of the proof is to use the Hölder type continuity of f to show that one may actually treat f locally as a constant in the relevant scales. To show this, fix $p \in (1,\infty)$ and observe that by Lemma 2.17 we have

$$\|f \otimes_\delta \mathbb{1}_{[0,1]^d} g\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}\, \|\mathbb{1}_{[0,1]^d} g\|_{L^p(\mathbb{R}^d)}. \tag{2.4}$$

From Definition 2.10, for any $\delta > 0$ we may decompose

$$f \otimes_\delta g(x) = \sum_{k\in\mathbb{Z}^d} \big(f(\cdot + k\delta) \otimes_\delta (\mathbb{1}_{[0,1]^d} g(\cdot - k)) \big)(x).$$

Hence

$$H_\delta := f \otimes_\delta (g - \widetilde{g}) = \sum_{k\in\mathbb{Z}^d} (\Delta_{k\delta} f) \otimes_\delta \mathbb{1}_{[0,1]^d} g(\cdot - k).$$

By the envelope property of f, there is $\varepsilon \in (0, a-d)$ so that $\|\Delta_{k\delta} f\|_{L^p(\mathbb{R}^d)} \lesssim (|k|\delta)^\varepsilon$. Hence an application of (2.4) and our assumption on g yield that

$$\|H_\delta\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{k\in\mathbb{Z}^d} (|k|\delta)^\varepsilon\, \|\mathbb{1}_{[0,1]^d} g(\cdot - k)\|_{L^p(\mathbb{R}^d)} \lesssim \delta^\varepsilon,$$

and the neglibility of H_δ follows. ∎

We remark that later on, when we deal with stochastic multiscale functions, then the natural analogue of the above lemma is no longer valid, causing some additional technical complications.

We now describe the weak convergence of multiscale functions in the limit $\delta \to 0$.

Lemma 2.19. *Let $F = F_\delta$ be a multiscale function and let $1 < p < \infty$. Then $\|F_\delta\|_{L^p(\mathbb{R}^d)}$ is bounded uniformly in δ. Furthermore, there exists $F_0 \in L^p(\mathbb{R}^d)$ such that F_δ converges weakly in $L^p(\mathbb{R}^d)$ to F_0. Actually, there is $\varepsilon > 0$ such that for any test function $\phi \in C_0^\infty(\mathbb{R}^d)$,*

$$\Big| \int_{\mathbb{R}^d} (F_0(x) - F_\delta(x))\, \phi(x)\, dx \Big| \lesssim_{\phi, F_\delta} \delta^\varepsilon.$$

Proof. By linearity, it suffices to treat the cases when F is either a multiscale tensor product or a negligible function. The claims are trivial in the latter case, so assume that $F_\delta = f \otimes_\delta g$ for some envelope function f and localized function g.

The uniform boundedness of $\|F_\delta\|_{L^p(\mathbb{R}^d)}$ follows immediately from Lemma 2.17. In order to establish the weak convergence, let us first consider the model case in which $g = \mathbb{1}_{[0,1]^d}$. Then for a.e. x we have $F_\delta(x) = [f]_\delta(n)$, where n is the integer part of x/δ. We thus see that

$$F_\delta(x) - f(x) = \frac{1}{\delta^d} \int_{n\delta + [0,\delta]^d} (f(y) - f(x))\, dy,$$

and so, by the triangle inequality,

$$|F_\delta(x) - f(x)| \le \frac{1}{\delta^d} \int_{[-\delta,\delta]^d} |\Delta_r f(x)|\, dr.$$

Taking L^p norms, applying Minkowski's inequality, and using the fact that $f \in \Lambda^{\varepsilon_p, p}(\mathbb{R}^d)$ for some $\varepsilon_p > 0$, we conclude that F_δ converges strongly in $L^p(\mathbb{R}^d)$ to f, which certainly suffices.

By subtracting a constant multiple of this model case, we may assume in general that g has mean zero. We claim that F_δ now converges weakly to zero. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a test function. We need to show that

$$\int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^d} [f]_\delta(n)\, g_{[n,\delta]}(x)\, \phi(x)\, dx \to 0$$

as $\delta \to 0$. Using the mean zero nature of g, we can rewrite the left-hand side as

$$\delta^d \sum_{n \in \mathbb{Z}^d} [f]_\delta(n) \int_{\mathbb{R}^d} g(r)\, \Delta_{r\delta}\phi(n\delta)\, dr.$$

Since ϕ is a test function and g is localized, the inner integral has magnitude $O(\delta)$, and furthermore vanishes unless $n = O_\phi(1/\delta)$. Thus the whole expression is bounded by $\delta \int_{|x| < c(\phi)/\delta} |f|$, and the lemma follows. ∎

Parts (i) and (iii) of the the following corollary follow immediately from the above proof, and (ii) is a consequence of (i).

Corollary 2.20.

(i) *If f is an envelope function, then $f - f \otimes_\delta \mathbb{1}_{[0,1]^d}$ is negligible.*

(ii) *Every envelope function is a multiscale function.*

(iii) *If F_δ is a multiscale function with expansion $F_\delta = \sum_{j=1}^J f_j \otimes_\delta g_j + G_\delta$ (where G_δ is negligible), then for any $p \in (1, \infty)$,*

$$F_\delta \underset{\delta \to 0}{\longrightarrow} \sum_{j=1}^{J} c_j f_j \quad \text{weakly in } L^p, \text{ with } c_j := \int_{\mathbb{R}^d} g_j, \quad j = 1, \dots, J.$$

We remark that conclusion (iii) makes precise the intuitively obvious statement that a multiscale tensor product approximates in some natural sense (a multiple of) the envelope function as $\delta \to 0$.

The sum of two multiscale functions is clearly a multiscale function. We proceed to give other closure properties of multiscale functions, the first one being the closure under multiplication.

Proposition 2.21. *If $F = F_\delta$ and $G = G_\delta$ are multiscale functions, then $FG = F_\delta G_\delta$ is also a multiscale function.*

Proof. If either F or G is negligible, then by Lemma 2.19 and Hölder's inequality we see that FG is also negligible. Hence by linearity we may assume that F, G are multiscale tensor products, e.g., $F = f \otimes_\delta g$ and $G = f' \otimes_\delta g'$. By Lemma 2.18 we may assume in addition that $\mathrm{supp}(g) \subset [0,1]^d$ and $\mathrm{supp}(g') \subset [0,1]^d$. Then, by observing that $g_{[n,\delta]}$ and $g'_{n',\delta}$ have disjoint supports if $n \neq n'$, we get

$$FG(x) = \sum_{n \in \mathbb{Z}^d} [f]_\delta(n)\, [f']_\delta(n)\, g''_{[n,\delta]},$$

where $g'' := gg'$ is a localized function. From Lemma 2.16 we see that

$$\sum_{n \in \mathbb{Z}^d} ([f]_\delta(n)\, [f']_\delta(n) - [ff']_\delta(n))\, g''_{[n,\delta]}$$

is negligible. Hence FG is equivalent to

$$\sum_{n \in \mathbb{Z}^d} [ff']_\delta(n)\, g''_{[n,\delta]},$$

which equals $(ff') \otimes_\delta g''(x)$. Since g'' is localized, and (by Lemma 2.4) ff' is an envelope function, the claim follows. ∎

Corollary 2.22. *Assume that f, f' are envelope functions and g, g' are localized functions. Then the product $(f \otimes_\delta g)(f' \otimes_\delta g')$ is a multiscale function equivalent to either of the multiscale tensor products $ff' \otimes \tilde{g}_1$, $ff' \otimes \tilde{g}_2$, where*

$$\tilde{g}_1(x) := \mathbb{1}_{[0,1]^d}(x) \sum_{n,m \in \mathbb{Z}^d} g(n + x)\, g'(m + x)$$

and

$$\widetilde{g}_2(x) := \sum_{n\in\mathbb{Z}^d} g(n+x)\,g'(x).$$

Proof. The statement concerning the function $\widetilde{g}_1$ follows directly from examining the proofs of Lemma 2.18 and Proposition 2.21. The second statement in turn follows from Lemma 2.18 by observing that $\widetilde{g}_2$ is localized and $\widetilde{g}_1(x) = \mathbb{1}_{[0,1]^d}(x)\sum_{k\in\mathbb{Z}^d}\widetilde{g}_2(x+k)$. ∎

Interestingly enough, the multiscale property is also preserved under (translation and scaling invariant) singular integrals. This is not at all evident a priori since for a singular integral operator T, the function Tg is usually not even integrable when g is a localized function. On the other hand, recall from standard Calderón–Zygmund theory (see e.g. [19]) that T extends to a bounded linear operator on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

Proposition 2.23. *If $F = F_\delta$ is a multiscale function, and T is a (translation and dilation invariant) singular integral operator (independent of δ), then $TF = TF_\delta$ is also a multiscale function.*

Proof. If F is negligible, then TF is negligible also since T is bounded on every $L^p(\mathbb{R}^d)$ space. So we may assume that $F_\delta = f \otimes_\delta g$ for some envelope function f and some localized function g. To simplify the notation, we now allow all implicit constants to depend on f, g, T and d.

First suppose that $g = \mathbb{1}_{[0,1]^d}$. By Corollary 2.20, F_δ differs from f by a negligible function, thus TF_δ differs from Tf by a negligible function. Since f is an envelope function, and T is translation-invariant and bounded on every $L^p(\mathbb{R}^d)$, we conclude that Tf is an envelope function, and thus a multiscale function again by Corollary 2.20, and the claim follows.

By linearity, it now suffices to treat the case when g has mean zero. Using the translation and dilation invariance of T, we observe that

$$TF = \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\,T(g_{[n,\delta]}) = \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\,(Tg)_{[n,\delta]}.$$

If Tg were a localized function, we would now be done, but this clearly not true in general. However, $Tg(x)$ decays roughly like $\langle x\rangle^{-d-1}$ or, more precisely, we have for any $x \in \mathbb{R}^d$ and $1 < p < \infty$ that

$$\|Tg\|_{L^p(B(x,1))} \lesssim_p \langle x\rangle^{-d-1}, \tag{2.5}$$

whence Lemma 2.18 applies and the desired conclusion follows. In order to verify (2.5), observe first from the L^p boundedness of T that the claim is easy for $|x| \le 4$, so we may assume $|x| > 4$. We then use the localized mean zero nature of g to decompose g into a piece g_1 of L^p norm $O_p(\langle x\rangle^{-d-1})$ supported in $\mathbb{R}^d \setminus B(0,|x|/8)$ and mean zero, and a mean zero localized function (with quantitative bounds independent of x) g_2 supported in $B(0,|x|/4)$. The contribution of g_1 is acceptable by the L^p boundedness of T; the contribution of g_2 is acceptable by using the mean zero nature of g_2 to note that, for $x' \in B(x,1)$,

$$Tg_2(x') = \int_{B(0,|x|/4)} (K(x',y) - K(x',0))\, g_2(y)\, dy,$$

where

$$K(x,y) = \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^d}$$

is the singular kernel of T, and then using the triangle inequality, the Calderón–Zygmund type bounds on K, i.e.,

$$|K(x',y) - K(x',0)| \le C|y|\,|x'|^{-d-1} \quad \text{for } |y| < 2|x'|,$$

and the localized nature of $y \mapsto g_2(y)|y|$. ∎

Corollary 2.24. *For any envelope function f, localized function g and singular integral T, the application $T(f \otimes_\delta g)$ is equivalent to the multiscale function $ATf + f \otimes_\delta g'$, where*

$$A := \int_{\mathbb{R}^d} g \quad \textit{and} \quad g'(x) := T(g - A\mathbb{1}_{[0,1]^d})(x).$$

Iterating Proposition 2.21 and Proposition 2.23, we obtain our deterministic homogenization result for iterated singular integrals:

Corollary 2.25. *Let $\mu = \mu_\delta$ be a multiscale function, and let T be a singular integral operator. Let $m \ge 1$, let $1 < p < \infty$, and define $\mu_m = \mu_{m,\delta}$ recursively by $\mu_{1,\delta} := \mu_\delta$ and $\mu_{m,\delta} := \mu_\delta T\mu_{m-1,\delta}$ for $m > 1$. Then $\mu_{m,\delta}$ is bounded in $L^p(\mathbb{R}^d)$ uniformly in δ, and is weakly convergent to a limit $\mu_{m,0} \in L^p(\mathbb{R}^d)$.*

Remark. In principle it is possible to deduce a formula for the limit $\mu_{m,0}$ by a repeated application of Lemma 2.19 and Corollaries 2.22 and 2.25.

3. Stochastic multiscale functions

We now turn to a generalization of the above theory, in which the localized functions g are allowed to be stochastic. We fix a probability space Ω, and then define a product probability space

$$\tilde{\Omega} := \Omega^{\mathbb{Z}^d} := \{(\omega_n)_{n\in\mathbb{Z}^d} : \omega_n \in \Omega \text{ for all } n \in \mathbb{Z}^d\}.$$

We often write $\tilde{\omega} := (\omega_n)_{n\in\mathbb{Z}^d}$ for an element of $\tilde{\Omega}$.

Definition 3.1 (Stochastic localized function). A *stochastic localized function* is a function $g\colon \mathbb{R}^d \times \Omega \to \mathbb{C}$ (not depending on the scale parameter δ), where Ω is a probability space, such that for every $1 < p < \infty$ and $k > 0$, the function $(x,\omega) \mapsto \langle x\rangle^k g(x,\omega)$ lies in $L^p(\mathbb{R}^d \times \Omega)$. We view g as a random function from $\mathbb{R}^d$ to $\mathbb{C}$.

Definition 3.2 (Stochastic multiscale tensor product). Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be an envelope function and let $g\colon \mathbb{R}^d \times \Omega \to \mathbb{C}$ be a localized function. We define the *multiscale tensor product* $f \otimes_\delta g\colon \mathbb{R}^d \times \widetilde{\Omega} \to \mathbb{C}$ of f and g to be the function

$$f \otimes_\delta g(x, \widetilde{\omega}) := \sum_{n \in \mathbb{Z}^d} [f]_\delta(n)\, g_{[n,\delta]}(x, \omega_n), \tag{3.1}$$

where $g_{[n,\delta]}$ is defined by Definition 1.2. One can view $f \otimes_\delta g$ as a random function from $\mathbb{R}^d$ to $\mathbb{C}$.

Remark. In the above definition, we of course could have instead spoken of independent copies of random functions $g(\cdot,\cdot)$ without introducing the product space $\widetilde{\Omega}$. However, the product space perhaps makes things slightly more transparent, at least for readers with little background in probability.

Definition 3.3 (Stochastic negligible function). A function $F = F_\delta\colon \mathbb{R}^d \times \widetilde{\Omega} \to \mathbb{C}$ depending on the parameter $0 < \delta < 1$ is said to be *negligible* if for every $1 < p < \infty$ there exists $\varepsilon_p > 0$ and $C_p > 0$ such that

$$\|F_\delta\|_{L^p(\mathbb{R}^d \times \widetilde{\Omega})} \le C_p\, \delta^{\varepsilon_p}$$

for all $0 < \delta < 1$. We view F_δ as a random function from $\mathbb{R}^d$ to $\mathbb{C}$.

Definition 3.4 (Stochastic multiscale function). A function $F = F_\delta\colon \mathbb{R}^d \times \tilde{\Omega} \to \mathbb{C}$ depending on the parameter $0 < \delta < 1$ is said to be a *stochastic multiscale function* if we can write

$$F_\delta = \sum_{j=1}^{J} f_j \otimes_\delta g_j + G_\delta,$$

where $J \ge 1$ is an integer, $f_1, \ldots, f_J$ are envelope functions, $g_1, \ldots, g_J$ are stochastic localized functions, and $G = G_\delta$ is a stochastic negligible function.

As in the previous section, we say that functions F_δ and F'_δ are *equivalent* if their difference is a stochastic negligible function.

Example 3.5. For each $n \in \mathbb{Z}^d$, let $\varepsilon_n \in \{-1, 1\}$ be an i.i.d. collection of signs, and let Q be a cube in $\mathbb{R}^d$. Then the random function

$$F_\delta(x) := \mathbb{1}_Q(x) \sum_{n \in \mathbb{Z}^d} \varepsilon_n\, \mathbb{1}_{n\delta + [0,\delta)^d}(x)$$

is a stochastic multiscale function (setting $\Omega = \{-1, 1\}$ with the uniform distribution, and ε_n to be the n^{th} coordinate function of $\tilde{\Omega} = \Omega^{\mathbb{Z}^d}$).

Remark. In the case when the probability space Ω is trivial, stochastic multiscale functions are essentially the same as deterministic multiscale functions.

Lemma 2.7 and its proof immediately generalize, so that we have

$$\Big\| \sum_{n \in \mathbb{Z}^d} f_n(x, \omega_n) \Big\|_{L^p(\mathbb{R}^d \times \widetilde{\Omega})} \lesssim_{p,d,N} \Big(\sum_{n \in \mathbb{Z}^d} \|\langle x \rangle_{[n,\delta]}^d\, f_n(x, \omega)\|_{L^p(\mathbb{R}^d \times \Omega)}^p \Big)^{1/p}. \tag{3.2}$$

In a similar vein, Lemma 2.17 also generalizes to the stochastic setup, one just replaces $\|\langle\cdot\rangle^d g\|_{L^p(\mathbb{R}^d)}$ by $\|\langle\cdot\rangle^d g\|_{L^p(\mathbb{R}^d\times\Omega)}$ on the right-hand side. In particular, we obtain:

Lemma 3.6. *Let $F = F_\delta\colon \mathbb{R}^d \times \tilde{\Omega} \to \mathbb{C}$ be a stochastic multiscale function, and let $1 < p < \infty$. Then $\|F_\delta\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})}$ is bounded uniformly in $0 < \delta < 1$.*

Remark 3.7. Exactly the same proof actually shows that $\|F_\delta\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})}$ stays bounded in δ for more general functions of the form

$$F_\delta(x,\tilde{\omega}) = \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\,(g_n)_{[n,\delta]}(x,\tilde{\omega})$$

assuming only the uniform localization $\sup_{n\in\mathbb{Z}^d} \|\langle\cdot\rangle^N g_n(\cdot,\cdot)\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})} < \infty$ for $N \geq 1$ and $p \in (1,\infty)$.

In turn, Proposition 2.21 has the following counterpart.

Proposition 3.8. *If $F' = F'_\delta$ is a deterministic multiscale function and $F = F_\delta$ is a stochastic multiscale function, then $F'F = F'_\delta F_\delta$ is a stochastic multiscale function.*

Proof. It is enough to treat the case $F = f \otimes_\delta g$ and $F' = f' \otimes_\delta g'$, where f and f' are envelope functions, g is a stochastic localized function and g' is a deterministic localized function. Furthermore, by Lemma 2.18, we may assume that g' has support in $[0,1]^d$. Fix $p > 1$ and observe that by the definition of localized functions and Hölder's inequality we have

$$\|\langle\cdot\rangle^d g'(\cdot - m)g(\cdot,\cdot)\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})} \lesssim_{g,g',p,a} \langle m\rangle^{-a}, \quad m \in \mathbb{Z}^d, \tag{3.3}$$

for any $a > 1$, in particular for $a = d + 1$. It follows that $\tilde{g}$ is a stochastic localized function, where $\tilde{g}$ is defined by $\tilde{g}(x,\omega) := \sum_{m\in\mathbb{Z}^d} g'(x-m)g(x,\omega)$. By Lemma 2.4, the proposition follows as soon as we show that $F'_\delta F_\delta$ is stochastically equivalent to $(ff') \otimes_\delta \tilde{g}$. Note that by Lemma 2.16 and the stochastic counterpart of Lemma 2.17, the latter function is equivalent to $H_\delta := \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\,[f']_\delta(n)\,\tilde{g}_{[n,\delta]}(\cdot,\omega_n)$, so it suffices to show that the difference $F'_\delta F_\delta - H_\delta$ is negligible. To that end, we compute

$$(F_\delta F'_\delta - H_\delta)(x,\tilde{\omega}) = \sum_{m\in\mathbb{Z}^d} \Big(\sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\,\big(\Delta_m [f']_\delta(n)\big)\, g'_{[m+n,\delta](x)}\, g_{[n,\delta]}(x,\omega_n)\Big).$$

Using (3.3), first applying the stochastic version of Lemma 2.7, and then Lemma 2.15 together with Hölder's inequality, we obtain

$$\begin{aligned}\|F_\delta F'_\delta - H_\delta\|_{L^p(\mathbb{R}^d\times\tilde{\omega})} &\leq \delta^{d/p} \sum_{m\in\mathbb{Z}^d} \|[f]_\delta\,\Delta_m[f']_\delta\|_{\ell^p(\mathbb{Z}^d)}\,\langle m\rangle^{-d-1} \\ &\lesssim_{f,f',p,g,g'} \delta^{d/p} \sum_{m\in\mathbb{Z}^d} \langle m\rangle^{-d-1} |m\delta|^{\varepsilon_{2p}}\,\delta^{-d/2p}\,\delta^{-d/2p} \lesssim \delta^{\varepsilon_{2p}},\end{aligned}$$

which verifies that the product $F_\delta F'_\delta$ is a stochastic multiscale function. ∎

We can now state our main technical result about stochastic multiscale functions.

Theorem 3.9 (Main estimate on stochastic multiscale functions). *Let $m \geq 1$, let $\mu^{(1)} = \mu_\delta^{(1)}, \dots, \mu^{(m)} = \mu_\delta^{(m)}$ be stochastic multiscale functions, and let $T_1, \dots, T_{m-1}$ be singular integral operators. Define $\mu_m = \mu_{m,\delta}\colon \mathbb{R}^d \times \tilde{\Omega} \to \mathbb{C}$ recursively by*

$$\mu_{1,\delta} := \mu_\delta^{(1)}; \quad \mu_{m,\delta} := \mu_\delta^{(m)} T_{m-1}\, \mu_{m-1,\delta} \quad \text{for } m > 1. \tag{3.4}$$

Then $\mu_{m,\delta}$ is bounded in $L^p(\mathbb{R}^d \times \tilde{\Omega})$ uniformly in $0 < \delta < 1$, for any $p \in (1,\infty)$. Furthermore, there exists a (deterministic) limit function $\mu_{m,0} \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$ with the property that, given any test function $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\mathbf{P}\Big(\Big|\int_{\mathbb{R}^d} [\mu_{m,\delta}(x,\omega) - \mu_{m,0}(x)]\,\phi(x)\,dx\Big| \geq \delta^\varepsilon\Big) \lesssim_{m,d,\varepsilon,\mu^{(1)},\dots,\mu^{(m)},T_1,\dots,T_{m-1},\phi} \delta^\varepsilon, \tag{3.5}$$

where $\mathbf{P}$ denotes the probability measure on $\tilde{\Omega}$.

The proof of this theorem is lengthy and shall occupy the next section. Later, in Section 5 we shall give applications to random Beltrami equations through the following immediate consequence.

Corollary 3.10 (Almost sure convergence). *Let $\mu_{m,\delta}$ be as in Theorem 3.9 and assume that $a \in (0,1)$. Then almost surely $\mu_{m,a^j} \to \mu_{m,0}$ weakly in $L^p(\mathbb{R}^d)$ as $j \to \infty$, for all $p \in (1,\infty)$.*

Proof. We recall that the sequence μ_{m,a^j} is uniformly bounded in each $L^p(\mathbb{R}^d)$ and that one may pick a countable set of test functions in $C_0^\infty(\mathbb{R}^d)$ that is dense in every $L^p(\mathbb{R}^d)$, with $1 < p < \infty$. Hence it is enough to prove almost sure convergence when tested against a fixed test function. However, this follows immediately from Theorem 3.9 and the Borel–Cantelli lemma. ∎

Theorem 1.1 now follows from Theorem 3.9 and Corollary 3.10.

We finish this section by observing that in the case where g is bounded with quick decay as $x \to \infty$, our definition of a deterministic multiscale function is equivalent to the product of the envelope function and the 'bump field defined via g'.

Lemma 3.11. *Assume that g is bounded and has quick decay in the first variable:*

$$|g(x,\omega)| \leq C_N \langle x\rangle^{-N} \quad \text{for all } N \geq 1 \text{ and } x \in \mathbb{R}^d, \omega \in \Omega.$$

Then, for any envelope function f, the stochastic multiscale tensor product $f \otimes_\delta g$ is equivalent to the function $f(x)\big(\sum_{n\in\mathbb{Z}^d} g_{[n,\delta]}(x,\omega_n)\big)$.

Proof. Observe first that, by the decay assumption, g is a stochastic localized function and the bump field is uniformly bounded:

$$\Big|\sum_{n\in\mathbb{Z}^d} g_{[n,\delta]}(x,\omega_n)\Big| \leq C \quad \text{for all } x \in \mathbb{R}^d.$$

Combining this with Corollary 2.20(i), we see that the product $f(x)(\sum_{n\in\mathbb{Z}^d} g_{[n,\delta]}(x,\omega_n))$ is equivalent to $f_\delta(x)(\sum_{n\in\mathbb{Z}^d} g_{[n,\delta]}(x,\omega_n))$, where

$$f_\delta := f \otimes_\delta \mathbb{1}_{[0,1)^d} = \sum_{n\in\mathbb{Z}^d} [f]_\delta(n)\, \mathbb{1}_{n\delta+[0,\delta)^d}.$$

By using for each $n \in \mathbb{Z}^d$ the trivial identities

$$[f]_\delta(n) = \sum_{k\in\mathbb{Z}^d} [f]_\delta(n)\, \mathbb{1}_{(n+k)\delta+[0,\delta)^d} \quad \text{and} \quad f_\delta = \sum_{k\in\mathbb{Z}^d} [f]_\delta(n+k)\, \mathbb{1}_{(n+k)\delta+[0,\delta)^d},$$

we may use the inequality (3.2) to estimate, for any $p \in (1,\infty)$, as follows:

$$\begin{aligned} Q_p &:= \Big\| f\otimes_\delta g - f_\delta\Big(\sum_{n\in\mathbb{Z}^d} g_{[n,\delta]}(x,\omega_n)\Big)\Big\|_{L^p(\mathbb{R}^d\times\tilde{\omega})} \\ &= \Big\| \sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d} \Delta_k[f]_\delta(n)\, \mathbb{1}_{(n+k)\delta+[0,\delta)^d}(x)\, g_{[n,\delta]}(x,\omega_n)\Big)\Big\|_{L^p(\mathbb{R}^d\times\tilde{\omega})} \\ &\lesssim \sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d} |\Delta_k[f]_\delta(n)|^p\, \|\mathbb{1}_{(n+k)\delta+[0,\delta)^d}(x)\,\langle x\rangle^d_{[n,\delta]}\, g_{[n,\delta]}(x,\omega_n)\|^p_{L^p(\mathbb{R}^d\times\tilde{\omega})}\Big)^{1/p} \\ &\lesssim \sum_{k\in\mathbb{Z}^d} \|\Delta_k[f]_\delta\|_{\ell^p(\mathbb{Z}^d)}\, \|\mathbb{1}_{k\delta+[0,\delta)^d}(x)\,\langle x\rangle^d_{[0,\delta]}\, g_{[0,\delta]}(x,\omega)\|_{L^p(\mathbb{R}^d\times\Omega)}. \end{aligned}$$

By assumption,

$$\|\mathbb{1}_{k\delta+[0,\delta)^d}(x)\,\langle x\rangle^d_{[0,\delta]}\, g_{[0,\delta]}(x,\omega)\|_{L^p(\mathbb{R}^d\times\Omega)} \lesssim \langle k\rangle^{-2d}\delta^{d/p},$$

and hence an application of Lemma 2.15 yields that

$$Q_p \lesssim \sum_{k\in\mathbb{Z}^d} \frac{(k\delta)^{\varepsilon_p}\,\delta^{-d/p}\,\delta^{d/p}}{\langle k\rangle^{2d}} \lesssim \delta^{\varepsilon_p},$$

which completes the proof. ∎

4. Proof of Theorem 3.9

In this section we establish Theorem 3.9. For that purpose, the first subsection below introduces some useful notation, and we find that for our purposes it is enough to estimate the size of the second moment and convergence of the first moment of suitable quantities, see (4.8) and (4.9) below.

4.1. Notation and reduction to first and second moment estimates

We first settle the claim that $\mu_{m,\delta}$ is uniformly bounded in $L^p(\mathbb{R}^d\times\tilde{\omega})$, stating this result as a separate lemma.

Lemma 4.1. *Define $\mu_{m,\delta}(\cdot,\omega)$ as in Theorem 3.9. Then $\|\mu_{m,\delta}\|_{L^p(\mathbb{R}^d\times\tilde{\omega})}$ is uniformly bounded in $\delta > 0$.*

Proof. Fix $\omega \in \tilde{\Omega}$. Since $T_1, \ldots, T_{m-1}$ are bounded on $L^q(\mathbb{R}^d)$ for every $1 < q < \infty$, we see from Hölder's inequality and induction on m that

$$\|\mu_{m,\delta}(\cdot,\omega)\|_{L^p(\mathbb{R}^d)} \lesssim_p \prod_{j=1}^{m} \|\mu_\delta^{(j)}(\cdot,\omega)\|_{L^{mp}(\mathbb{R}^d)}. \tag{4.1}$$

Namely, if this is true for the value $m-1$, we choose $q > 1$ with $1/q + 1/(mp) = 1/p$ and use the L^q-boundedness of T_{m-1} to estimate

$$\begin{aligned}\|\mu_{m,\delta}(\cdot,\omega)\|_{L^p(\mathbb{R}^d)} &\le \|\mu_\delta^{(m)}(\cdot,\omega)\|_{L^{mp}(\mathbb{R}^d)}\, \|T_{m-1}\mu_{m-1,\delta}(\cdot,\omega)\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \|\mu_\delta^{(m)}(\cdot,\omega)\|_{L^{mp}(\mathbb{R}^d)}\, \|\mu_{m-1,\delta}(\cdot,\omega)\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \|\mu_\delta^{(m)}(\cdot,\omega)\|_{L^{mp}(\mathbb{R}^d)} \prod_{j=1}^{m-1} \|\mu_\delta^{(j)}(\cdot,\omega)\|_{L^{(m-1)q}(\mathbb{R}^d)},\end{aligned}$$

and as $(m-1)q = mp$, the desired inequality (4.1) with index m follows. Finally, Fubini's theorem and another application of Hölder's inequality yield

$$\|\mu_{m,\delta}\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})} \lesssim_p \prod_{j=1}^{m} \|\mu_\delta^{(j)}\|_{L^{mp}(\mathbb{R}^d\times\tilde{\Omega})}. \tag{4.2}$$

The claim then follows from Lemma 3.6. ∎

Remark 4.2. The above bound also holds if the stochastic multiscale functions $\mu_\delta^{(j)}$ are replaced by more general ones described in Remark 3.7.

We then proceed with some reductions towards the proof of Theorem 3.9. First of all, by multilinearity, we may assume that each of the stochastic multiscale functions $\mu_\delta^{(j)}$ is either a stochastic negligible function, or is a stochastic multiscale tensor product. If at least one of the functions $\mu_\delta^{(j)}$ is stochastically negligible, we see by (4.2) that for every $1 < p < \infty$ there exists $\varepsilon > 0$ such that

$$\|\mu_{m,\delta}\|_{L^p(\mathbb{R}^d\times\tilde{\Omega})} \lesssim_{p,\varepsilon} \delta^\varepsilon.$$

By applying $p = 2$ (say) and setting $\mu_{m,0} :\equiv 0$, we easily obtain the claim (3.5).

We may thus assume that for each $1 \le j \le m$ we have

$$\mu_\delta^{(j)} = f_j \otimes_\delta g_j \tag{4.3}$$

for some envelope functions f_j and some stochastic localized functions g_j. We allow all implied constants to depend on f_j and g_j.

Next, we shall make the qualitative assumption that the envelope functions f_j are compactly supported in $\mathbb{R}^d$. This is purely in order to justify certain interchanges of summation, as now all sums in the multiscale functions are finite for a fixed $\delta > 0$. At the very end of Section 4.3 we describe how this assumption can be dispensed with by a standard limit argument.

As the next step, we observe that for Theorem 3.9 it suffices to show for each $\phi \in C_0^\infty(\mathbb{R}^d)$ that there exists a limit $z = z_\phi \in \mathbb{C}$ such that

$$\mathbf{P}\Big(\Big| \int_{\mathbb{R}^d} \mu_{m,\delta}\, \phi(x)\, dx - z \Big| \geq \delta^\varepsilon\Big) \lesssim_\varepsilon \delta^\varepsilon$$

for some $\varepsilon > 0$ independent of ϕ, δ. Indeed, the map $\phi \mapsto z_\phi$ is then a continuous (by the uniform boundedness of $\|\mu_{m,\delta}(\cdot, \omega)\|_{L^p(\mathbb{R}^d)}$), linear, and densely defined functional on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$, and can then be used to reconstruct $\mu_{m,0}$ by duality.

By (4.3) and (3.1), we can write

$$\mu_\delta^{(j)}(x, \omega) = \sum_{n_j \in \mathbb{Z}^d} \mu_{n_j,\delta}^{(j)}(x, \omega_{n_j}),$$

where

$$\mu_{n_j,\delta}^{(j)}(x, \omega_{n_j}) := [f_j]_\delta(n_j)\, (g_j)_{[n_j,\delta]}(x, \omega_{n_j}). \tag{4.4}$$

We can therefore expand out the expression

$$\int_{\mathbb{R}^d} \mu_{m,\delta}(x, \omega)\, \phi(x)\, dx \tag{4.5}$$

using (3.4) as

$$\sum_{\vec{n} \in (\mathbb{Z}^d)^m} X_{\delta,\vec{n}}, \tag{4.6}$$

where $\vec{n} := (n_1, \ldots, n_m)$, and $X_{\delta,\vec{n}}$ is the complex-valued random variable

$$X_{\delta,\vec{n}} := \int_{\mathbb{R}^d} [\mu_{n_m,\delta}^{(m)}(\cdot, \omega_{n_m})\, T_{m-1} \ldots T_1\, \mu_{n_1,\delta}^{(1)}(\cdot, \omega_{n_1})](x)\, \phi(x)\, dx. \tag{4.7}$$

Note that our qualitative hypotheses ensure that for each fixed δ, only finitely many of the $X_{\delta,\vec{n}}$ are non-zero, and that each of the random variables $X_{\delta,\vec{n}}$ is bounded.

To obtain the concentration result (3.5), we use Chebyshev's inequality. From that inequality we see that it suffices to show a first moment estimate

$$\Big| \sum_{\vec{n} \in (\mathbb{Z}^d)^m} \mathbf{E}(X_{\delta,\vec{n}}) - z \Big| \lesssim_\varepsilon \delta^\varepsilon \tag{4.8}$$

together with a second moment estimate of the form

$$\mathbf{E}\Big| \sum_{\vec{n} \in (\mathbb{Z}^d)^m} X_{\delta,\vec{n}} - \mathbf{E}(X_{\delta,\vec{n}}) \Big|^2 \lesssim_\varepsilon \delta^\varepsilon \tag{4.9}$$

for some $\varepsilon > 0$ (independent of δ and ϕ). (One can also control higher moments, but the second moment will suffice for our application.) The rest of Section 4 is devoted to proving the key estimates (4.8) and (4.9).

4.2. The second moment estimate

In this section our aim is to settle the second moment estimate (4.9). To start with, we expand the left-hand side as

$$\sum_{\vec{n},\vec{n}'\in(\mathbb{Z}^d)^m} \mathbf{E}(X_{\delta,\vec{n}}\overline{X_{\delta,\vec{n}'}}) - \mathbf{E}(X_{\delta,\vec{n}})\,\overline{\mathbf{E}(X_{\delta,\vec{n}'})}.$$

Now observe from (4.7) that $X_{\delta,\vec{n}}$ and $X_{\delta,\vec{n}'}$ are independent and the corresponding term in the above sum vanishes, unless we have $n_j = n'_{j'}$ for some $1 \le j, j' \le m$. Thus by the triangle inequality and Cauchy–Schwarz we can estimate the previous expression by

$$2\sum_{1\le j,j'\le m}\ \sum_{\vec{n},\vec{n}'\in(\mathbb{Z}^d)^m:n_j=n'_{j'}} \mathbf{E}(|X_{\delta,\vec{n}}|^2)^{1/2}\,\mathbf{E}(|X_{\delta,\vec{n}'}|^2)^{1/2}.$$

It therefore suffices to establish an estimate of the form

$$\sum_{\vec{n},\vec{n}'\in(\mathbb{Z}^d)^m:n_j=n'_{j'}} \mathbf{E}(|X_{\delta,\vec{n}}|^2)^{1/2}\,\mathbf{E}(|X_{\delta,\vec{n}'}|^2)^{1/2} \lesssim_\varepsilon \delta^\varepsilon \tag{4.10}$$

for all $1 \le j, j' \le m$.

Fix j, j'. We now pause to give a basic estimate on the size of each of the $X_{\delta,\vec{n}}$. Define the kernel K_0 by setting

$$K_0(n) := \frac{1}{\langle n\rangle^d}.$$

Proposition 4.3 (Size estimate). *If $1 < p < \infty$ and $\vec{n} \in (\mathbb{Z}^d)^m$, then*

$$\mathbf{E}(|X_{\delta,\vec{n}}|^p)^{1/p} \lesssim_p \delta^d \langle \delta|n_m|\rangle^{-2d}\Big(\prod_{i=1}^m \|[f_i]_\delta(n_i)|\Big)\prod_{i=1}^{m-1} K_0(n_{i+1}-n_i).$$

For the proof of the proposition, we shall need the following weighted version of the L^p bounds for singular integrals.

Lemma 4.4 (Localized singular integral bounds). *Let T be a singular integral operator. If $n, n' \in \mathbb{Z}^d$, $\delta > 0$, $1 < p < \infty$, and $N > d$, then we have the bound*

$$\|\langle\cdot\rangle_{[n',\delta]}^{-N} Tf\|_{L^p(\mathbb{R}^d)} \lesssim_{T,p,d,N} K_0(n-n')\,\|\langle\cdot\rangle_{[n,\delta]}^{N} f\|_{L^p(\mathbb{R}^d)},$$

where f is any function for which the right-hand side is finite.

Proof. By scaling, we may set $\delta = 1$. We have

$$\|Tf\|_{L^p(B(n',1))} \lesssim_{T,p,d} K_0(n-n')\,\|f\|_{L^p(B(n,1))}$$

for all $n, n' \in \mathbb{Z}^d$ and all $f \in L^p(B(n,1))$ (extending f by zero outside of this ball). Namely, if $|n - n'| \ge 2$, then the claim follows simply by using the integral representation of T and the triangle inequality (and Hölder's inequality). If $|n-n'|<2$, the claim instead follows by using the boundedness of T on $L^p(\mathbb{R}^d)$. It then follows that

$$\begin{aligned}\|\langle\cdot\rangle_{[n',1]}^{-N}Tf\|_{L^p(\mathbb{R}^d)} &\lesssim \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-N}\|Tf\|_{L^p(B(n'+k,1))}\\ &\lesssim \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-N}\Big(\sum_{\ell\in\mathbb{Z}^d}\langle n'+k-n-\ell\rangle^{-d}\|f\|_{L^p(B(n+\ell,1))}\Big)\\ &\lesssim \Big(\sum_{k,\ell\in\mathbb{Z}^d}\langle k\rangle^{-N}\langle n'+k-n-\ell\rangle^{-d}\langle\ell\rangle^{-N}\Big)\|\langle\cdot\rangle_{[n,1]}^{N}f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

This yields the stated estimate, since the last written sum is easily estimated to be less than $O(\langle n-n'\rangle^{-d})$ by considering separately the case $\max(|k|,|\ell|)\le|n-n'|/4$ and its complement. ∎

Proof of Proposition 4.3. Pick $p\in(1,\infty)$ and fix $\omega\in\widetilde{\Omega}$. Denote

$$g(\cdot,\omega) := \langle x\rangle_{[n_m,\delta]}^{3d}\,[\mu_{\delta,n_m}^{(m)}(\cdot,\omega_{n_m})T_{m-1}\dots T_1\mu_{\delta,n_1}^{(1)}(\cdot,\omega_{n_1})](x)$$

so that we may write

$$|X_{\delta,\vec{n}}| = \Big|\int_{\mathbb{R}^d} g(x,\omega)\,\langle x\rangle_{[n_m,\delta]}^{-N}\phi(x)\,dx\Big| \le \|g(\cdot,\omega)\|_{L^p(\mathbb{R}^d)}\,\|\langle\cdot\rangle_{[n_m,\delta]}^{-3d}\phi\|_{L^{p'}(\mathbb{R}^d)}. \tag{4.11}$$

By an inductive application of Lemma 4.4 and Hölder's inequality as in the proof of (4.1), we see that

$$\|g(\cdot,\omega)\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{i=1}^{m-1}K_0(n_{i+1}-n_i)\prod_{i=1}^{m}\|\langle\cdot\rangle_{[n_i,\delta]}^{6d}\,\mu_{\delta,n_i}^{(i)}(\cdot,\omega_{n_i})\|_{L^{pm}(\mathbb{R}^d)}. \tag{4.12}$$

Since ϕ is a Schwartz function, one easily verifies that

$$\|\langle\cdot\rangle_{[n_m,\delta]}^{-3d}\,\phi\|_{L^{p'}(\mathbb{R}^d)} \lesssim \frac{\delta^{d/p'}}{\langle\delta n_m\rangle^{2d}}.$$

Moreover, (4.4) and the localized nature of g_i yield that

$$\big(\mathbf{E}\|\langle\cdot\rangle_{[n_i,\delta]}^{6d}\,\mu_{n_i,\delta}^{(i)}(\cdot,\omega_{n_i})\|_{L^{pm}(\mathbb{R}^d)}^{pm}\big)^{1/pm} \lesssim \delta^{d/mp}|[f_i]_\delta(n_i)|.$$

By combining these estimates with (4.11), the desired estimate (4.10) follows by Hölder's inequality and the relation $1/p+1/p'=1$. ∎

In order to utilize the above proposition, we need to introduce discrete fractional integrals. To that end, given any real number $\alpha\in[0,d)$, define the more general kernels $K_\alpha\colon\mathbb{Z}^d\to\mathbb{R}^+$ on the integer lattice by

$$K_\alpha(n) := \frac{1}{\langle n\rangle^{d-\alpha}}. \tag{4.13}$$

The convolution of functions defined on the lattice $\mathbb{Z}^d$ is defined in the usual manner:

$$F * G(n) := \sum_{m\in\mathbb{Z}^d} F(m)G(n-m).$$

By direct computation, we have the convolution estimate

$$K_\alpha * K_\beta \lesssim_{\alpha,\beta,n} K_{\alpha+\beta} \tag{4.14}$$

whenever $\alpha, \beta > 0$ and $\alpha + \beta < d$. These estimates are unfortunately not true at the endpoints $\alpha = 0$ or $\beta = 0$, due to the logarithmic failure of summability of K_0. However, from Young's inequality we easily see that

$$\|K_0 * f\|_{\ell^q(\mathbb{Z}^d)} \lesssim_{d,p,q} \|f\|_{\ell^p(\mathbb{Z}^d)} \tag{4.15}$$

for all $1 \le p < q \le \infty$ and all $f \in \ell^p(\mathbb{Z}^d)$.

Finally, we are ready to estimate the left-hand side of (4.10) by

$$\begin{aligned} \lesssim S := \delta^{2d} \sum_{\vec n,\vec n'\in(\mathbb{Z}^d)^m : n_j = n'_{j'}} \langle \delta n_m\rangle^{-2d} \Big(\prod_{i=1}^m |[f_i]_\delta(n_i)|\Big) \prod_{i=1}^{m-1} K_0(n_{i+1}-n_i) \\ \times \langle \delta n'_m\rangle^{-2d} \Big(\prod_{i'=1}^m |[f_{i'}]_\delta(n'_{i'})|\Big) \prod_{i'=1}^{m-1} K_0(n'_{i'+1}-n'_{i'}). \end{aligned} \tag{4.16}$$

Writing $n_j = n'_{j'} = n$, we can rewrite this expression using the convolution operator $T_{K_0} f := f * K_0$ and by denoting $\Phi_\delta(n) := \langle \delta n\rangle^{-2d}$ as

$$\sum_{n\in\mathbb{Z}^d} |[f_j]_\delta(n)|\,|[f_{j'}]_\delta(n)|\, H_{1,\delta}(n)\, H_{2,\delta}(n)\, G_{1,\delta}(n)\, G_{2,\delta}(n), \tag{4.17}$$

with[4]

$$\begin{aligned} H_{1,\delta}(n) &:= \big(T_{K_0}(|[f_{j+1}]_\delta|)\dots T_{K_0}(|[f_m]_\delta|\Phi_\delta)\big)(n),\\ H_{2,\delta}(n) &:= \big(T_{K_0}(|[f_{j'+1}]_\delta|)\dots T_{K_0}(|[f_m]_\delta|\Phi_\delta)\big)(n),\\ G_{1,\delta}(n) &:= \big(T_{K_0}(|[f_{j-1}]_\delta|)\dots T_{K_0}(|[f_1]_\delta|)\big)(n),\\ G_{2,\delta}(n) &:= \big(T_{K_0}(|[f_{j'-1}]_\delta|)\dots T_{K_0}(|[f_1]_\delta|)\big)(n). \end{aligned}$$

In order to bound these functions, observe first that for given $p > 1$, for any $\tilde\varepsilon > 0$, and for an arbitrary sequence $(a(n))_{n\in\mathbb{Z}^d}$,

$$\|a[f_i]_\delta\|_{\ell^p(\mathbb{Z}^d)} \le \|a\|_{\ell^p(\mathbb{Z}^d)} \|[f_i]_\delta\|_{\ell^\infty(\mathbb{Z}^d)} \lesssim_{f_i,p,\tilde\varepsilon} \delta^{-\tilde\varepsilon}\|a\|_{\ell^p(\mathbb{Z}^d)}, \tag{4.18}$$

[4] Below the definitions of $H_{1,\delta}$ and its analogues are to be interpreted as follows: starting from the right, one alternatively performs either a pointwise multiplication by a sequence $|[f_k]_\delta|$ or an application by the operator T_{K_0}.

since Lemma 2.15 yields that $\|[f_i]_\delta\|_{\ell^\infty(\mathbb{Z}^d)} \le \|[f_i]_\delta\|_{\ell^q(\mathbb{Z}^d)} \lesssim \delta^{-d/q}$ for all $q > 1$ and we just take q large enough. Fix $\varepsilon > 0$. Using alternately the above estimate (with a very small value of $\tilde{\varepsilon}$) and the boundedness of $T_{K_0}: \ell^p(\mathbb{Z}^d) \to \ell^q(\mathbb{Z}^d)$ for any $1 < p < q < \infty$ we obtain that

$$\|H_{k,\delta}\|_{\ell^{2+\varepsilon}(\mathbb{Z}^d)} \lesssim \delta^{-\varepsilon} \|\Phi_\delta\|_{\ell^2(\mathbb{Z}^d)} \lesssim \delta^{-\varepsilon-d/2}, \quad k = 1,2. \tag{4.19}$$

Set $q = q(\varepsilon) = 4\varepsilon^{-1}(2+\varepsilon)$ so that $2/q + 1/(2+\varepsilon) = 1/2$, and use (4.18) to similarly obtain the estimate

$$\|G_{k,\delta}\|_{\ell^q(\mathbb{Z}^d)} \lesssim \delta^{-\varepsilon}\|[f_1]_\delta\|_{\ell^{q-\varepsilon'}(\mathbb{Z}^d)} \lesssim \delta^{-\varepsilon-d/(q-\varepsilon')} \lesssim \delta^{-(d+1)\varepsilon}, \quad k = 1,2, \tag{4.20}$$

where we just picked $\varepsilon' > 0$ small enough. Finally, plugging the above bounds in (4.17), using $2/(1+\varepsilon) + 4/q = 1$ and the fact that $\|[f_1]_\delta\|_{\ell^q(\mathbb{Z}^d)} \lesssim \delta^{-d\varepsilon}$, we obtain via Hölder's inequality that

$$\begin{aligned} S &\le \delta^{2d} \|H_{1,\delta}\|_{\ell^{2+\varepsilon}(\mathbb{Z}^d)} \|H_{2,\delta}\|_{\ell^{2+\varepsilon}(\mathbb{Z}^d)} \|G_{1,\delta}\|_{\ell^q(\mathbb{Z}^d)} \|G_{2,\delta}\|_{\ell^q(\mathbb{Z}^d)} \\ &\quad \times \|[f_j]_\delta\|_{\ell^q(\mathbb{Z}^d)} \|[f_{j'}]_\delta\|_{\ell^q(\mathbb{Z}^d)} \\ &\lesssim \delta^{2d}\, \delta^{-\varepsilon-d/2}\, \delta^{-\varepsilon-d/2}\, \delta^{-2(d+1)\varepsilon}\, \delta^{-2d\varepsilon} \lesssim \delta^{d-O(\varepsilon)}. \end{aligned}$$

If $j = 1$ (respectively, $j' = 1$), the term $G_{1,\delta}$ (respectively, $G_{2,\delta}$) is not present in (4.17), and the above argument goes through with obvious modifications. The desired estimate follows as $\varepsilon > 0$ is arbitrary.

Remark. One way to understand the obtained bound for the second moment is to observe that a computation analogous to the above one could also be used, e.g., to estimate the quantity $\mathbf{E}\|X_{\delta,\vec{n},c}\|^2$, which we know to be bounded. However, direct implementation of the above method would give us a divergent upper bound of the form $\delta^{-O(\varepsilon)}$, due to the logarithmic non-boundedness of the kernel K_0 on ℓ^p-spaces, as we are ignoring nontrivial cancellations that are behind Proposition 4.3. Roughly speaking, what saves us above is that the condition $n_j = n'_{j'}$, due to independence, reduces the number of terms by a factor δ^d.

4.3. The first moment estimate

It now remains to establish the first moment estimate (4.8), whose proof is more combinatorial in nature. We can split the left-hand side into finitely many components, depending on the equivalence class that $n_1, \dots, n_m$ generates. Given any surjective coloring function $c: \{1,\dots,m\} \to \{1,\dots,k\}$ which assigns a "color" in some finite set of integers $\{1,\dots,k\}$ to every integer $\{1,\dots,m\}$, let $(\mathbb{Z}^d)^m_c$ denote the set of all $\vec{n} \in (\mathbb{Z}^d)^m$ such that $n_j = n_{j'}$ if and only if $c(j) = c(j')$. Clearly we can partition $(\mathbb{Z}^d)^m$ into finitely many of the $(\mathbb{Z}^d)^m_c$. Thus it will suffice to show that for each coloring function c there exists a complex number z_c (independent of δ, but depending on all other parameters) for which we have

$$\Big| \sum_{\vec{n}\in(\mathbb{Z}^d)^m_c} \mathbf{E}(X_{\delta,\vec{n}}) - z_c \Big| \lesssim_\varepsilon \delta^\varepsilon.$$

Fix c. We can reparameterise this as

$$\Big|\sum_{\vec{n}\in(\mathbb{Z}^d)^k_{\neq}} \mathbf{E}(X_{\delta,\vec{n},c}) - z_c\Big| \lesssim_\varepsilon \delta^\varepsilon,$$

where $(\mathbb{Z}^d)^k_{\neq}$ is the space of all k-tuples $(n_1,\dots,n_k)\in(\mathbb{Z}^d)^k$ with $n_1,\dots,n_k$ distinct, and $X_{\delta,\vec{n},c}\colon \Omega^k\to\mathbb{C}$ is the complex-valued random variable

$$X_{\delta,\vec{n},c} := \int_{\mathbb{R}^d} \big[\mu^{(m)}_{\delta,n_{c(m)}}(\cdot,\omega_{n_{c(m)}})T_{m-1}\dots T_1\mu^{(1)}_{\delta,n_{c(1)}}(\cdot,\omega_{n_{c(1)}})\big](x)\,\phi(x)\,dx.$$

Observe from the inclusion-exclusion principle that the sum $\sum_{\vec{n}\in(\mathbb{Z}^d)^k\setminus(\mathbb{Z}^d)^k_{\neq}}\mathbf{E}(X_{\vec{n},c})$ can be expressed as a finite linear combination of expressions of the form $\sum_{\vec{n}\in(\mathbb{Z}^d)^{k'}}\mathbf{E}_c(X_{\vec{n},c'})$, where $k'<k$ and $c'\colon\{1,\dots,n\}\to\{1,\dots,k'\}$ is a surjective coloring, and c is a refinement of c' (i.e., $c(j_1)=c(j_2)$ implies that $c'(j_1)=c'(j_2)$) (for the readers benefit, this is illustrated by a detailed example in Remark 4.6 below). Thus, by induction on k, it in fact suffices to show that for every pair of colorings (c,c') with c finer than c' there exists a complex number $z'_{c,c'}$ for which we have

$$\Big|\sum_{\vec{n}\in(\mathbb{Z}^d)^k} \mathbf{E}_c(X_{\delta,\vec{n},c'}) - z'_{c,c'}\Big| \lesssim_\varepsilon \delta^\varepsilon,$$

where we used the notation

$$\mathbf{E}_c X_{\delta,\vec{n},c'} := \int_{\mathbb{R}^d}\mathbf{E}\big[\mu^{(m)}_{\delta,n_{c'(m)}}(\cdot,\omega_{c(m)})T_{m-1}\dots T_1\mu^{(1)}_{\delta,n_{c'(1)}}(\cdot,\omega_{c(1)})\big](x)\,\phi(x)\,dx.$$

Let us now use Fubini's theorem to write

$$\sum_{\vec{n}\in(\mathbb{Z}^d)^k}\mathbf{E}_c(X_{\vec{n},c'}) = \int_{\mathbb{R}^d} T_{c,c',\delta}(1)(x)\,\phi(x)\,dx,$$

where $T_{c,c',\delta}$ is the (deterministic) operator

$$T_{c,c',\delta}h(x) := \sum_{\vec{n}\in(\mathbb{Z}^d)^k}\mathbf{E}\big[\mu^{(m)}_{\delta,n_{c'(m)}}(\cdot,\omega_{c(m)})T_{m-1}\dots T_1\mu^{(1)}_{\delta,n_{c'(1)}}(\cdot,\omega_{c(1)})h\big](x).$$

We next verify the uniform boundedness of our 'colored' sum.

Lemma 4.5. *Assume that $h=h_\delta$ is a deterministic multiscale function. Then*

$$\|T_{c,c',\delta}h_\delta\|_{L^p(\mathbb{R}^d)} \le C < \infty \quad \textit{for } \delta>0. \tag{4.21}$$

Proof. We double the number of coordinates in our probability space and consider the product (probability) space $\widetilde{\Omega}\times\widetilde{\Omega}'$ whose elements we can write as sequences $(\widetilde{\omega},\widetilde{\omega}')=(\omega_n,\omega'_n)_{n\in\mathbb{Z}^d}$, and choose unimodular random variables $Y_{k,j}\colon\widetilde{\Omega}'\to\{1,-1\}$ for $k=1,\dots,m$ and $j\in\mathbb{Z}^d$ such that $\mathbf{E}\,Y_{1,n_1}\cdot Y_{m,n_m}$ is equal to 1 if $(n_1,\dots,n_m)$ respects the coloring c' (i.e., $n_\ell=n_{\ell'}$ for those $\ell,\ell'\in\{1,\dots,m\}$ that have the same color with respect to c'), and otherwise this expectation is zero. For example, in the case $m = 2$ and one color (i.e., $c'(1) = c'(2) = 1$), one may take $Y_{1,j} = Y_{2,j} = \Theta_j$, where (Θ_j) is a Bernoulli sequence. In the general case, one associates independent copies of such sequences for all the pairs (k, k') that have the same color. More explicitly, one can set $\widetilde{\Omega}' := \{-1, 1\}^A$ with the Bernoulli measure, where A is the set of triples (n, ℓ, ℓ') with $n \in \mathbb{Z}^d$ and $\ell, \ell' \in \{1, \dots, m\}$ with $c'(\ell) = c'(\ell')$, and set

$$Y_{r,n}(\widetilde{\omega}') = \prod_{(n,\ell,\ell')\in A:\, r\in\{\ell,\ell'\}} \widetilde{\omega}'_{n,\ell,\ell'}$$

for any $\widetilde{\omega}' = \widetilde{\omega}'_{(n,\ell,\ell')\in A}$, $r \in \{1, \dots, m\}$, and $n \in \mathbb{Z}^d$.

We may then write

(4.22)
$$T_{c,c',\delta} h_\delta(x) = \mathbf{E}_{\widetilde{\Omega}\times\widetilde{\Omega}'}\Big(\sum_{\vec{n}\in(\mathbb{Z}^d)^m} [\widetilde{\mu}^{(m)}_{\delta,n_m}(\cdot, \omega_{c(m)}, \widetilde{\omega}') T_{m-1} \dots T_1 \widetilde{\mu}^{(1)}_{\delta,n_1}(\cdot, \omega_{c(1)}, \widetilde{\omega}') h](x)\Big),$$

where for $k \in \{1, \dots, m\}$ and $n \in \mathbb{Z}^d$ we set

$$\widetilde{\mu}^{(k)}_{\delta,n}(x, \widetilde{\omega}, \widetilde{\omega}') := \mu^{(k)}_{\delta,n}(x, \widetilde{\omega})\, Y_{k,n}(\widetilde{\omega}').$$

In particular, we may write

$$T_{c,c',\delta} h = \mathbf{E}_{\widetilde{\Omega}\times\widetilde{\Omega}'} H^{(m)}_\delta T_{m-1} \dots H^{(1)}_\delta h_\delta, \tag{4.23}$$

with

$$H^{(k)}_\delta(x, \widetilde{\omega}, \widetilde{\omega}') = \sum_{n\in\mathbb{Z}^d} [f_k]_\delta(n)\, (g_k)_{[n,\delta]}(x, \omega_{c(k)})\, Y_{k,n}(\widetilde{\omega}).$$

Recalling Remark 3.7, the argument of Lemma 3.6 applies as before since the additional factors $Y_{k,n}$ or having the variable $\omega_{c(k)}$ instead of ω_n do not affect our old estimates, whence

$$\|H^{(k)}_\delta\|_{L^p(\mathbb{R}^d\times\widetilde{\Omega}\times\widetilde{\Omega}')} \lesssim_p C \qquad \text{for all } \delta > 0,\ p \in (1, \infty).$$

Finally, Lemma 4.1 (together with Remark 4.2) and Hölder's inequality yield the desired result (4.21). ∎

We pause to clarify by an example the role of colorings introduced above.

Remark 4.6. In order to illustrate the use of the colorings and the division to cases 'split and 'non-split' (the latter notions will introduced shortly below in the proof of Proposition 4.7), let us consider in the case $m = 3$ the expectation

$$S := \mathbf{E}\Big(\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} X_{n_1} T Y_{n_2} T Z_{n_3}\Big),$$

that is of the type we have to handle. Here, the $X_n = X_n(x, U_n)$, $Y_j = Y_n(x, U_n)$, $Z_n = Z_n(x, U_n)$ $(n \in \mathbb{Z})$ are (say bounded) random functions, and the U_j are i.i.d random variables. The linear operator T could be, e.g., a singular integral operator. In the first step, one uses independence and Fubini to write the above sum in the form (the extra subindex $\neq$ indicates that one sums only over triples or tuples consisting of *unequal* indices)

$$\begin{aligned}
S = & \sum_{n_1,n_2,n_3,\neq} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_2})T(\mathbf{E}Z_{n_3}) + \sum_{n_1,n_2,\neq} (\mathbf{E}(X_{n_1}TY_{n_1}))T\mathbf{E}Z_{n_2} \\
& + \sum_{n_1,n_2,\neq} \mathbf{E}(X_{n_1}T(\mathbf{E}Y_{n_2})TZ_{n_1}) + \sum_{n_1,n_2,\neq} (\mathbf{E}X_{n_1})T\mathbf{E}(Y_{n_2}TZ_{n_2}) \\
& + \sum_{n_1} \mathbf{E}(X_{n_1}TY_{n_1}TZ_{n_1}) \\
=: & S_1 + S_2 + S_3 + S_4 + S_5.
\end{aligned} \tag{4.24}$$

The next step uses the inclusion-exclusion principle to rewrite the sums so that one sums over all indices. For example, we obtain

$$\begin{aligned}
S_1 = & \sum_{n_1,n_2,n_3} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_2})T(\mathbf{E}Z_{n_3}) - \sum_{n_1,n_2} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_1})T(\mathbf{E}Z_{n_2}) \\
& - \sum_{n_1,n_2} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_2})T(\mathbf{E}Z_{n_1}) - \sum_{n_1,n_2} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_2})T(\mathbf{E}Z_{n_2}) \\
& + 2\sum_{n_1} (\mathbf{E}X_{n_1})T(\mathbf{E}Y_{n_1})T(\mathbf{E}Z_{n_1}) \\
:= & S_{11} - S_{12} - S_{13} - S_{14} + 2S_{15}.
\end{aligned}$$

Each of these terms can be expressed via a pair of colourings (c, c'). Let $c_{\ell k}$ and $c'_{\ell k}$ stand for the colours of the term $S_{\ell k}$. At most three colours are needed. We have $c_{1\ell} = (1, 2, 3)$ for each $\ell \in \{1, \dots, 5\}$. In turn, $c'_{11} = (1, 2, 3)$, $c'_{12} = (1, 1, 2)$,$c'_{13} = (1, 2, 1)$, $c'_{11} = (1, 2, 2)$, and $c'_{15} = (1, 1, 1)$.

In a similar vein, the term S_2 can be rewritten as

$$S_2 = \sum_{n_1,n_2} (\mathbf{E}(X_{n_1}TY_{n_1}))T\mathbf{E}Z_{n_2} - \sum_{n_1} (\mathbf{E}(X_{n_1}TY_{n_1}))T\mathbf{E}Z_{n_1} =: S_{21} - S_{22}.$$

Now the colourings are $c_{21} = c_{22} = (1, 1, 2)$, $c'_{21} = (1, 1, 2)$ and $c'_{22} = (1, 1, 1)$. The terms S_3 and S_4 are analogous, and finally the term S_5 needs no further subdivision and one has $c_5 = c'_5 = (1, 1, 1)$.

Among the terms S_{11}, S_{12}, S_{13}, S_{14}, S_{15}, S_{21}, S_{22} and S_5, the terms S_{11}, S_{12}, S_{14} and S_{21} will later on be designated as *split*, and the remaining ones as *nonsplit*. This means the following: for a split term, one can concretely divide the defining sum to independent left-hand and right-hand side summations, and also the expectations split accordingly. For instance, we may write

$$S_{21} = fTg, \quad \text{with} \quad f := \sum_{j_1}^{n} \mathbf{E}(X_{j_1}TY_{j_1}) \quad \text{and} \quad g := \sum_{j_2}^{n} \mathbf{E}Z_{j_2}.$$

We return to the main course of the argument towards the first moment estimate, and note that, in view of Lemma 2.19, it suffices to show the following.

Proposition 4.7 (Main proposition). *If* $c\colon \{1,\dots,m\} \to \{1,\dots,k\}$ *is surjective, and if* $h = h_\delta$ *is a* (*deterministic*) *multiscale function, then* $T_{c,c'}(h) = T_{c,c',\delta}(h_\delta)$ *is also a* (*deterministic*) *multiscale function.*

The remainder of this section is devoted to the proof of this proposition.

We first observe that one proves easily (e.g., compare the proof of Proposition 3.8) that if we know the claim (for a given colouring c') in the special case $h_\delta = 1$, then it is true (for the given colouring c') in the general case. Namely, the proof of Proposition 3.8 applies as such to the product term $H_\delta^{(1)} h_\delta$ in the representation (4.23) verifying that it can be replaced by $\widetilde{H}^{(1)}$, which is of the same form as $H^{(1)}$, and by decoupling the representation we obtain an expression with 1 in place of h_δ.

We induct on k, i.e., the number of colors in c'. If there is only one color in c', then

$$T_{c,c',\delta}(1)(x) := \sum_{n\in(\mathbb{Z}^d)} \mathbf{E}_c\big[\mu_{\delta,n}^{(m)}(\cdot,\omega_{c(m)})T_{m-1}\dots T_1\mu_{\delta,n}^{(1)}(\cdot,\omega_{c(1)})\big](x)$$
$$=\Big[\sum_{n\in\mathbb{Z}^d}\Big(\prod_{j=1}^m [f_j]_\delta(n)\Big) g_{[n,\delta]}(x)\Big],$$

where

$$g(\cdot,\omega) := \mathbf{E}_c g_m(\cdot,\omega_{c(j)})T_{m-1}\dots T_1 g_1(\cdot,\omega_{c(j)}).$$

Obviously, g is a localized function, and hence Lemma 2.16 and Proposition 2.21 verify that $T_{c,c',\delta}(h)$ is a multiscale function. Now we suppose inductively that $k > 1$, and that the claim has already been proven for all smaller values of k.

We begin by disposing of the *split* case, in which there exists a non-trivial partition $\{1,\dots,m\} = \{1,\dots,j\} \cup \{j+1,\dots,m\}$ with $1 \le j < m$ such that $c'(\{1,\dots,j\})$ and $c'(\{j+1,\dots,m\})$ are disjoint. By relabeling colors if necessary, we may assume that $c'(\{1,\dots,m\}) = \{1,\dots,k'\}$ for some $1 \le k' < k$. Then, we let $c_1'\colon\{1,\dots,j\} \to \{1,\dots,k'\}$ be the restriction of c to $\{1,\dots,j\}$, and $c_2'\colon \{1,\dots,m-j\} \to \{1,\dots,k-k'\}$ be the function $c_2'(i) := c'(i+j) - k'$. The restrictions c_1, c_2 are defined analogously using the fact that c refines c'. Observe that, by the definition of $\mathbf{E}_c$,

$$T_{c,c',\delta}(1)(x)$$
$$:= \sum_{\vec{n}\in(\mathbb{Z}^d)^{k-k'}} \mathbf{E}_{c_2}\big[\mu^{(m)}_{\delta,n_{c_2'(m-j)}}(\cdot,\omega_{c_2(m-j)})T_{m-1}\dots T_{j+1}\mu^{(j+1)}_{\delta,n_{c_2'(1)}}(\cdot,\omega_{c_2(1)})T_j T_{c_1,c_1',\delta}(1)\big](x).$$

By the induction hypothesis, $T_{c_1,c_1',\delta}(1)$ is a deterministic multiscale function, and then by Proposition 2.23, $T_j T_{c_1,c_1',\delta}(1)$ is also. The claim then follows by another application of the inductive hypothesis.

Finally, we deal with the more difficult *non-split* case, in which no non-trivial partition of the above type exists. In other words, we need to show that

$$T_{c,c'\delta}(1)(x) = \sum_{\vec{n}\in(\mathbb{Z}^d)^k} \mathbf{E}\big[\mu^{(m)}_{\delta,n_{c'(m)}}(\cdot,\omega_{c(m)})T_{m-1}\dots T_1\mu^{(1)}_{\delta,n_{c'(1)}}(\cdot,\omega_{c(1)})\big](x)$$

is a multiscale function. Using (4.4) and the fact that all the $T_1, \dots, T_{m-1}$ commute with dilations, we can rewrite $T_{c,c',\delta}1(x)$ as

$$T_{c,c'\delta}1(x) := \sum_{\vec{n} \in (\mathbb{Z}^d)^k} \Big(\prod_{i=1}^m [f_i]_\delta(n_{c'(i)}) \Big) (G_{\vec{n}})_{[0,\delta]}(x),$$

where

$$G_{\vec{n}}(x) := \mathbf{E}\big[g_m(\cdot - n_{c'(m)}, \omega_{n_{c(m)}}) T_{m-1} \dots T_1 g_1(\cdot - n_{c'(1)}, \omega_{n_{c(1)}}) \big](x).$$

Using the translation-invariance of the $T_1, \dots, T_{m-1}$, we can further rewrite this as

$$T_{c,c',\delta}1(x) := \sum_{n \in \mathbb{Z}^d} \sum_{\vec{r} \in (\mathbb{Z}^d)^k : r_{c(m)}=0} \Big(\prod_{i=1}^m [f_i]_\delta(n + r_{c(i)}) \Big) (G_{\vec{r}})_{[n,\delta]}(x).$$

To estimate this expression, we observe that, exactly as in (4.12), we have for any $\vec{r} \in (\mathbb{Z}^d)^k$, $N > 0$, and $1 < p < \infty$ the estimate

$$\|\langle \cdot \rangle^N G_{\vec{r}}\|_{L^p(\mathbb{R}^d)} \lesssim_{N,p} \prod_{i=1}^{m-1} K_0(r_{c(i+1)} - r_{c(i)}). \tag{4.25}$$

We combine this lemma with the non-split nature of c to obtain the following.

Lemma 4.8. *For any $N > 0$ and $1 < p < \infty$, there exists $\alpha > 0$ such that*

$$\Big\| \langle \cdot \rangle^N \sum_{\vec{r} \in (\mathbb{Z}^d)^k : r_{c(m)}=0 : R \le \langle \vec{r} \rangle < 2R} |G_{\vec{r}}| \Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,N,\alpha} R^{-\alpha}$$

for all $R \geq 1$.

Proof. In view of (4.25) and the triangle inequality, it suffices to show that

$$\sum_{\vec{r} \in (\mathbb{Z}^d)^k : r_{c(m)}=0, R \le \langle \vec{r} \rangle < 2R} \prod_{i=1}^{m-1} K_0(r_{c(i+1)} - r_{c(i)}) \lesssim_\alpha R^{-\alpha}$$

for α sufficiently small. Now recall the kernels K_α defined in (4.13). From the triangle inequality (and the surjectivity of c), we see that

$$\prod_{i=1}^{m-1} K_0(r_{c(i+1)} - r_{c(i)}) \lesssim_\alpha R^\alpha \prod_{i=1}^{m-1} K_\alpha(r_{c(i+1)} - r_{c(i)})$$

whenever $R \leq \langle \vec{r} \rangle$. Thus it will suffice to show that

$$S_\alpha(c) := \sum_{\vec{r} \in (\mathbb{Z}^d)^k : r_{c(m)}=0} \prod_{i=1}^{m-1} K_\alpha(r_{c(i+1)} - r_{c(i)}) \lesssim_\alpha 1 \tag{4.26}$$

for all $\alpha \leq \alpha_0(m) > 0$.

In order to prove this, we need a simple lemma on colorings, that needs some terminology. Let $c\colon \{1,\dots,m\} \to \{1,\dots k\}$ be a (surjective) coloring. Fix $k' \in \{1,\dots k\}$, and denote $\ell = \#c^{-1}(k')$. One defines in an obvious way the coloring $c'\colon \{1,\dots,m-\ell\} \to \{1,\dots k-1\}$ that is obtained by removing color k' from c. More precisely, if c is thought as a sequence of length m containing integers from $\{1,\dots k\}$, the sequence c' is obtained by taking of all occurrences k from c, keeping the order of the remaining elements, and replacing every $j > k'$ by $j-1$.

Lemma 4.9. *Let c be a non-split coloring with at least 3 colors. Then we may remove from c a color (different from $c(m)$) so that the remaining coloring is also non-split.*

Proof. We begin by defining the convex support of a color k' as the interval $\{j, j+1,\dots,j'\}$, where $j = \min\{i \in \{1,\dots,k\} : c(i) = k'\}$ and $j' = \max\{i \in \{1,\dots,k\} : c(i) = k'\}$. To prove the lemma, note first that in case $c(1) = c(m)$, we may remove any other color and what remains is non-split. In case $c(m) \neq c(1)$, we first try to remove the color $c(1)$. If the outcome is non-split, we are done. If the outcome is split, it means that there must be a color k' whose convex support is contained in the convex support of $c(1)$, in particular that color is different from $c(m)$. When color k' is removed, it is clear that remaining coloring is non-split. ∎

We return to the proof of (4.26) and induct on the number of colours in c. If there is only one color, the statement is obviously true. Assume then that c contains k different colors with $k \geq 2$ and the statement is true if the number of colors does nor exceed $k-1$. Now, if $k \geq 3$, according to the previous lemma, there is a color k' that can be removed from c so that the remaining coloring c' is non-split. If $k = 2$, we just pick k' to be the color different from $c(m)$. Then, since c is non-split, we may pick $1 \leq j < j' \leq k$ so that $j \leq j'-2$ and $c(i) = k'$ for all i with $j' < i < j'$, but $c(i) \neq k'$ for $i = j, j'$. We obtain

$$\sum_{r_{k'} \in \mathbb{Z}^d} \prod_{i=j}^{j'-1} K_\alpha(r_{c(i+1)} - r_{c(i)}) = \sum_{m \in \mathbb{Z}^d} K_\alpha(m - r_{c(j)}) K_\alpha(r_{c(j')} - m) \lesssim_\alpha K_{2\alpha}(r_{c(j)} - r_{c(j')}).$$

We thus obtain

$$S_\alpha(c) \leq S_{3\alpha}(c') \lesssim 1,$$

and by induction the claim follows if we take (say) $\alpha \leq \alpha_0 := 3^{-(m+1)}$ initially. ∎

Now we can finally show that $T_{c,\delta}1$ is a multi-scale function. Fix $1 < p < \infty$, let $N > d$ be large, and let $\varepsilon_0 > 0$ be a small number to be chosen later. Let us first consider the "non-local" contribution when $\langle \vec{r} \rangle \geq R := \delta^{-\varepsilon_0}$. From Lemma 2.15 (applied with p close to infinity), we see that

$$\|[f_i]_\delta\|_{l^\infty(\mathbb{Z}^d)} \lesssim_\varepsilon \delta^{-\varepsilon}$$

for all $\varepsilon > 0$. From Lemma 4.8 and the triangle inequality, we thus see that

$$\Big\| \langle\cdot\rangle^N \sum_{\vec{r} \in (\mathbb{Z}^d)^k : r_{c(m)} = 0, \langle \vec{r} \rangle \geq R} \Big(\prod_{i=1}^m [f_i]_\delta(n + r_{c(i)})\Big) G_{\vec{r}} \Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,N,\varepsilon} \delta^{-\varepsilon} R^{-\alpha} |[f_m]_\delta(n)|$$

for some $\alpha > 0$, and so

$$\Big\| \langle\cdot\rangle_{[n,\delta]}^N \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle \geq R} \Big(\prod_{i=1}^m [f_i]_\delta(n + r_{c(i)})\Big) (G_{\vec r})_{[n,\delta]} \Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,N,\varepsilon} \delta^{-\varepsilon}\, \delta^{d/p}\, R^{-\alpha}\, |[f_m]_\delta(n)|$$

for all $n \in \mathbb{Z}^d$. Taking $l^p(\mathbb{Z}^d)$ norms of both sides and using Hölder and Lemma 2.15, we obtain (if N is large enough)

$$\Big\| \sum_{n\in\mathbb{Z}^d} \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle \geq R} \Big(\prod_{i=1}^m [f_i]_\delta(n + r_{c(i)})\Big) (G_{\vec r})_{[n,\delta]} \Big\|_{L^p(\mathbb{R}^d)} \lesssim_{p,N,\varepsilon} \delta^{-\varepsilon} R^{-\alpha},$$

which is negligible by the choice of R if we let ε be sufficiently small. Thus we only need to consider the "local" contribution when $\langle \vec r\rangle < R$. We split this local contribution into three pieces: the main term

$$\sum_{n\in\mathbb{Z}^d} \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle < R} \Big[\prod_{i=1}^m f_i \Big]_\delta (n)\, (G_{\vec r})_{[n,\delta]}(x), \tag{4.27}$$

a first error term

$$\sum_{n\in\mathbb{Z}^d} \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle < R} \Big(\prod_{i=1}^m [f_i]_\delta(n) - \Big[\prod_{i=1}^m f_i \Big]_\delta (n) \Big) (G_{\vec r})_{[n,\delta]}(x), \tag{4.28}$$

and a second error term

$$\sum_{n\in\mathbb{Z}^d} \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle < R} \Big[\prod_{i=1}^m [f_i]_\delta(n + r_{c(i)}) - \prod_{i=1}^m [f_i]_\delta(n) \Big] (G_{\vec r})_{[n,\delta]}(x). \tag{4.29}$$

Let us first consider the main term (4.27). By Lemma 2.4, $\prod_{i=1}^m f_i$ is an envelope function. From Lemma 4.8, we see that the function

$$\sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle < R} G_{\vec r}$$

is a localized function. By Definition 2.10, we thus see that (4.27) is a multiscale tensor product of an envelope function and a localized function, and is thus a multiscale function.

To conclude the proof of Proposition 4.7, and hence the proof of Theorem 3.9, it suffices to show that the expressions (4.28) and (4.29) are negligible. For this we shall just use (4.25) rather than the more sophisticated estimate in Lemma 4.8 (in particular, we do not need the non-split hypothesis).

Now we turn to (4.28). Let $1 < p < \infty$, and pick any $N > d$. Using the triangle inequality, followed by Lemma 2.7, we can estimate the $L^p(\mathbb{R}^d)$ norm of (4.28) by

$$\lesssim_{p,N} \sum_{\vec r \in (\mathbb{Z}^d)^k : r_{c(m)}=0, \langle \vec r\rangle < R} \Big(\sum_{n\in\mathbb{Z}^d} \Big(\Big| \prod_{i=1}^m [f_i]_\delta(n) - \Big[\prod_{i=1}^m f_i \Big]_\delta (n) \Big| \times \| \langle\cdot\rangle_{[n,\delta]}^N (G_{\vec r})_{[n,\delta]}(x) \|_{L^p(\mathbb{R}^d)} \Big)^p \Big)^{1/p}.$$

Applying a rescaled version of (4.25), we can estimate this by

$$\lesssim_{p,N} \delta^{d/p} \sum_{\vec{r}\in(\mathbb{Z}^d)^k: r_{c(m)}=0, \langle\vec{r}\rangle<R} \prod_{i=1}^{m-1} K_0(r_{c(i+1)}-r_{c(i)}) \Big\| \prod_{i=1}^{m}[f_i]_\delta(\cdot) - \Big[\prod_{i=1}^{m} f_i\Big]_\delta(\cdot)\Big\|_{\ell^p(\mathbb{Z}^d)}.$$

Observe that on the ball of radius R, K_0 has an ℓ^1 norm of $O_\varepsilon(\delta^{-\varepsilon})$ for any ε. Thus we can estimate the previous expression by

$$\lesssim_{p,N,\varepsilon} \delta^{d/p-\varepsilon} \Big\| \prod_{i=1}^{m}[f_i]_\delta(\cdot) - \Big[\prod_{i=1}^{m} f_i\Big]_\delta(\cdot)\Big\|_{\ell^p(\mathbb{Z}^d)}$$

for any $\varepsilon > 0$. Applying Lemma 2.16 and Hölder's inequality repeatedly, we can thus estimate this expression by

$$\lesssim_{p,N,\varepsilon} \delta^{\varepsilon_p-\varepsilon}$$

for some $\varepsilon_p > 0$ depending on p. Setting $\varepsilon := \varepsilon_p/2$ (say), we see that (4.28) is negligible as desired.

Finally, we estimate (4.29). Again let $1 < p < \infty$, and pick any $N > d$. Arguing as before, especially using the ℓ^1 norms of K_0 on the ball of radius R, we can estimate the $L^p(\mathbb{R}^d)$ norm of (4.29) by

$$\lesssim_{p,N,\varepsilon} \delta^{d/p-\varepsilon} \Big\| \prod_{i=1}^{m}[f_i]_\delta(\cdot + r_{c(i)}) - \prod_{i=1}^{m}[f_i]_\delta(\cdot)\Big\|_{\ell^p(\mathbb{Z}^d)}.$$

Using the crude estimate

$$\Big|\prod_{i=1}^{m} a_i - \prod_{i=1}^{m} b_i\Big| \lesssim \sum_{i=1}^{m} |a_i - b_i| \prod_{j\neq i}(|a_i| + |b_i|),$$

the triangle inequality, and the already familiar estimate

$$\|[f_i]_\delta(\cdot + r_{c(i)}) - [f_i]_\delta(\cdot)\|_{\ell^q(\mathbb{Z}^d)} \lesssim \delta^{-d/q+\varepsilon_q},$$

we get by Hölder that the $L^p(\mathbb{R}^d)$-norm of (4.29) has the upper bound $\lesssim_{p,N,\varepsilon} (R\delta)^{\varepsilon_{mp}}\delta^{-\varepsilon}$, where $\varepsilon_{mp} > 0$. By the choice of R and choosing ε sufficiently small, we see that (4.29) is negligible, as required. This proves Proposition 4.7.

The only thing that remains to be done to complete the proof of Theorem 3.9 is to get rid of the assumption that the envelope functions are compactly supported. Recall (4.5) and denote, in the general case, $Z_\delta := \int_{\mathbb{R}^d} \mu_{m,\delta}(x,\omega)\phi(x)\,dx$, and for $R > 0$, set $Z_{\delta,R} := \int_{\mathbb{R}^d} \mu_{R,m,\delta}(x,\omega)\phi(x)\,dx$, where $\mu_{R,m,\delta}$ is obtained from $\mu_{m,\delta}$ by replacing each envelope function f_j in its definition by $f_j \mathbb{1}_{B(0,R)}$. Then, for a suitably chosen sequence $R_k \uparrow \infty$, we have $\|Z_{\delta,R_k} - Z_\delta\|_{L^2(\mathbb{R}^d\times\tilde{\Omega})} \le 2^{-k}$ as $k \to \infty$, according to (4.2), and combined with Hölder's inequality this easily implies that $Z_{\delta,R_k} \to Z_\delta$ almost surely as $k \to \infty$. We know that there are complex numbers z_k and $c, \varepsilon > 0$ so that

$$\mathbf{P}(|Z_{\delta,R_k} - z_k| > \delta^\varepsilon) \le c\delta^\varepsilon, \tag{4.30}$$

and the argument in the present section verifies that c is independent of $k \geq 1$. Now, as $\mathbf{E}|Z_{\delta,R_k}|^2$ is uniformly bounded in δ and k, we deduce that the sequence (z_k) is uniformly bounded, and by moving to a subsequence, we may assume that $z_k \to z$ as $k \to \infty$. One obtains the desired inequality simply by letting $k \to \infty$ in (4.30). The proof is complete.

5. Quasiconformal homogenization

Our remaining task here is to apply Theorem 3.9 with Corollary 3.10 to homogenization of quasiconformal maps. Here it turns out convenient to proceed via the principal solutions, cf. Subsection 1.1. This will require us to first make use of the results we have already proven in the setting of compactly supported envelope functions. Once that is done, the application to general quasiconformal homogenization poses no substantial difficulties. This latter fact will be not come as a surprise for a specialist in quasiconformal maps, but we present rather complete details for the reader's convenience.

We refer to, e.g., Section 1 of [9] for a quick account of basic facts about planar quasiconformal maps, and to [5] for a comprehensive exposition on the topic. Throughout this section, T stands for the Beurling operator (1.16). We denote by $\mathbb{D} := \{|w| < 1\}$ the unit disc in the complex plane. Recall from the introduction that a (quasiconformal) complex dilatation μ is a complex valued measurable function on the plane whose sup-norm is strictly less than 1, that a 3-point normalized homeomorphism of the extended plane $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixes points $0, 1$ and ∞, and that the measurable Riemann mapping theorem guarantees existence and uniqueness of a 3-point normalized homeomorphic $W^{1,2}_{\text{loc}}$-solution to the Beltrami equation $\partial_{\bar{z}} f = \mu \partial_z f$ for any quasiconformal dilatation.

5.1. Proof of Theorems 1.6, 1.7, and 1.8

In preparation for the proof ot Theorem 1.6, we begin with a few simple deterministic lemmas, which are modifications of well-known methods in the theory of planar quasiconformal mappings. Our first lemma shows that weak convergence of each individual term in the Neumann series is enough to guarantee uniform convergence of the corresponding principal solutions and locally uniform convergence of the 3-point normalized solutions.

Lemma 5.1. *Let us assume that for any $j \geq 1$ the dilatation μ_j satisfies $\|\mu_j\|_\infty \leq k < 1$ and* $\operatorname{supp}(\mu_j) \subset B$*, where $B \subset \mathbb{C}$ is a ball. Denote the m-th term in the Neumann series for μ_j by*

$$\psi_{m,j} := \mu_j T\mu_j \dots T\mu_j,$$

where μ_j appears m times, $m \geq 1$. Assume also that for every fixed m there is the weak convergence in $L^p(\mathbb{C})$,

$$\psi_{m,j} \overset{\mathrm{w}}{\to} \psi_m \quad \text{as } j \to \infty,$$

for all $1 < p < \infty$. Then the solution F_j of the Beltrami equation $\partial_{\bar{z}} F_j = \mu_j \partial_z F_j$, normalized by the 3-point condition, converges locally uniformly to a k-quasiconformal limit $F_\infty : \mathbb{C} \to \mathbb{C}$.

Proof. Let first f_j be the principal solution that has the representation

$$f_j = z + \sum_{m=1}^{\infty} C\psi_{m,j},$$

where C is the Cauchy transform. All the functions $\psi_{m,j}$ are supported in the ball B, and by the standard properties of T (see Section 4.5.1 in [5]), we have $\|\psi_{m,j}\|_{L^p(B)} \leq ca^m$ for all j, where $a = a(p,k) < 1$ as soon as if we fix $p > 2$ close enough to 2.

It is well known that for $p > 2$ the map $C: L^p(B) \to C^\alpha(\mathbb{C})$ is bounded and compact for $\alpha \in (0, 1 - 2/p)$, see, e.g., Theorems 4.3.11 and 4.3.14 in [5]. Here clearly the homogeneous norm for C^α used in [5] can be replaced by the non-homogenous norm

$$\|f\|_{C^\alpha(\mathbb{C})} := \|f\|_{L^\infty(\mathbb{C})} + \sup_{z,w} |f(z) - f(w)|\,|z-w|^{-\alpha}$$

by the good decay of the Cauchy transforms of compactly supported functions. We may thus deduce from the weak convergence of $\psi_{m,j}$ in $L^p(B)$ that for each $m \geq 1$ the term $C\psi_{m,j}$ converges in the $C^\alpha(\mathbb{C})$-norm to an element $g_m \in C^\alpha(\mathbb{C})$. Moreover, we have the uniform bounds $\|C\psi_{m,j}\|_{C^\alpha(\mathbb{C})} \leq Ca^m$ and $\|g_m\|_{C^\alpha(\mathbb{C})} \leq Ca^m$ for all $m, j \geq 1$. This clearly yields the uniform convergence of the principal solutions:

$$f_j \to f_\infty = z + \sum_{m=1}^{\infty} C\psi_m \quad \text{as } j \to \infty. \tag{5.1}$$

The limit f_∞ is k-quasiconformal from the normal family property of hydrodynamically normalized k-quasiconformal maps with dilatations supported in a fixed ball.

Finally, to treat the 3-point normalized solutions F_j, simply observe we may write them in terms of the principal solution as

$$F_j(z) = (f_j(1) - f_j(0))^{-1}(f_j(z) - f_j(0)).$$

Thus (F_j) converges uniformly to the k-quasiconformal map

$$F_\infty(z) := (f_\infty(1) - f_\infty(0))^{-1}(f_\infty(z) - f_\infty(0)).$$ ∎

Our second auxiliary result verifies that normalized k-quasiconformal maps whose dilatations agree in a large ball are close to each other near the center of the ball.

Lemma 5.2. *Let $k < 1$ and assume that both $f: \mathbb{C} \to \mathbb{C}$ and $g: \mathbb{C} \to \mathbb{C}$ are k-quasiconformal homeomorphisms that satisfy the 3-point normalization and, moreover, that*

$$\mu_g = \mu_f \quad \text{in } B(0, L),$$

where $L \geq 1$. Then for any $R < L$ we have

$$\sup_{|z| \leq R} |g(z) - f(z)| \leq \varepsilon(L, k, R),$$

where $\lim_{L\to\infty} \varepsilon(L, k, R) = 0$ for any fixed k, R.

Proof. First of all, quasisymmetry (see Definition 3.2.1 and Theorem 3.5.3 in [5]) and the normalization of g imply that $g(B(0,R)) \subset B(0, r_1)$ and that $g(B(0,L)) \supset B(0, r_2)$, with

$r_1 = r_1(R,k)$ and $r_2 = r_2(L,k) \to \infty$ as $L \to \infty$. Writing $f = h \circ g$, it follows that h is analytic in $B(0,r_2)$ with $h(0) = 0$ and $h(1) = 1$. Then the function

$$H(z) := r_2^{-1} h(r_2 z)$$

is analytic and univalent in $B(0,1)$ and satisfies the normalization $H(0) = 0$ and $H(1/r_2) = 1/r_2$. By the Koebe type estimates (see (2.74) in [5]), it is clear that $H'(0) \to 1$ as $L \to \infty$. Since the second derivative of H has a universal bound on, say, $B(0,1/2)$ (see Theorem 1.8 in [9]), we deduce that for any given $\varepsilon > 0$ we have, for large enough L,

$$|H(z) - z| \le \varepsilon |z| \le \varepsilon r_1/r_2 \quad \text{for } |z| < r_1/r_2.$$

This implies that $|f(z) - g(z)| < \varepsilon r_1$ for $|z| < R$, proving the lemma. ∎

Next we have a global variant of Lemma 5.1.

Lemma 5.3. *Let the dilatations μ_j satisfy $|\mu_j| \le k < 1$ for $j = 1,2,\ldots$ For any $L > 1$, we write $\mu_{j,L} := \mu_j \mathbb{1}_{B(0,L)}$ and set $\psi_{m,j,L} := \mu_{j,L} T \mu_{j,L} \ldots T \mu_{j,L}$, where $\mu_{j,L}$ appears m times. Assume that for every $m \ge 1$ and $L > 1$ there is the weak convergence*

$$\psi_{m,j,L} \xrightarrow{\text{w}} \psi_{m,L} \quad \text{as } j \to \infty$$

in $L^p(\mathbb{C})$ for all $1 < p < \infty$. Then the 3-point normalized solution F_j of the Beltrami equation $\partial_{\bar z} F_j = \mu_j \partial_z F_j$ converges locally uniformly on $\mathbb{C}$ to a k-quasiconformal homeomorphism F.

Proof. Fix $R > 0$. For any $L = 1,2,3,\ldots$, let $F_{j,L}$ be the 3-point-normalized solution to the Beltrami equation

$$\partial_{\bar z} F_{j,L} = \mu_{j,L} \partial_z F_{j,L}.$$

By Lemma 5.1, for every $L \ge 1$ we have uniform convergence $F_{j,L} \to F_{\infty,L}$ as $j \to \infty$, where $F_{\infty,L}$ is a k-quasiconformal homeomorphism. Given $\varepsilon > 0$, Lemma 5.2 shows that we may choose $L_0 := L_0(k,\varepsilon,R)$ so that

$$|F_{j,L} - F_{j,L'}| \le \varepsilon \quad \text{in } z \in B(0,R), \text{ for } L, L' \ge L_0.$$

A fortiori,

$$|F_{\infty,L} - F_{\infty,L'}| \le \varepsilon \quad \text{in } z \in B(0,R), \text{ for } L, L' \ge L_0.$$

We deduce that the sequence $(F_{\infty,L})_{L \ge 1}$ is Cauchy in $C(B(0,R))$, so that $F_{\infty,L} \to F_\infty$ uniformly on $B(0,R)$. Since R was arbitrary, we see that F_∞ is a 3-point normalized k-quasiconformal homeomorphism of the plane.

It remains to check that also $F_j \to F_\infty$ uniformly on $B(0,R)$ for any given $R \ge 1$. To this end, take $L \ge L_0$ and estimate

$$\begin{aligned}
&\limsup_{j\to\infty} \|F_j - F_\infty\|_{C(B(0,R))} \\
&\qquad \le \limsup_{j\to\infty} \|F_j - F_{j,L}\|_{C(B(0,R))} + \limsup_{j\to\infty} \|F_{j,L} - F_{\infty,L}\|_{C(B(0,R))} \\
&\qquad\quad + \|F_{\infty,L} - F_\infty\|_{C(B(0,R))} \\
&\qquad \le \varepsilon + 0 + \varepsilon \;=\; 2\varepsilon,
\end{aligned}$$

where we used Lemma 5.2 again to estimate the first term. ∎

We are ready to establish Theorem 1.6.

Proof of Theorem 1.6. Let us first assume that the Beltrami envelope function ϕ in the statement of Theorem 1.6(i) (see Definition 1.4) is compactly supported. Observe that in this case ϕ is an envelope function in the sense of Section 2 (Definition 2.2), since taking R large enough in Definition 1.4 we may apply the bound $|\phi| \leq 1$ to obtain for any $1 < p < \infty$,

$$\|\Delta_h \phi\|_{L^p(\mathbb{C})} \leq 2^{1-1/p} \,|\mathrm{supp}(\phi)|^{1/p}\, \|\Delta_h \phi\|_{L^1(\mathbb{C})}^{1/p} \leq C' \,|h|^{\alpha/p} \quad \text{for } |h| \leq 1.$$

Lemma 3.11 shows that $\phi\, B_\delta$ is a stochastic multiscale function. By Corollary 3.10, for each $m \geq 1$ there exists a (deterministic) limit function ψ_m such that, with probability one, $\psi_{m,j} := \mu_{2^{-j}} T \mu_{2^{-j}} \ldots T \mu_{2^{-j}}$ converges weakly to ψ_m in $L^p(\mathbb{C})$ for each $1 < p < \infty$ and each m. The statement of part (i) then follows from Lemma 5.1.

In the case where the envelope ϕ is not compactly supported, we use Lemma 5.3 to reduce to the compactly supported case. For this reduction, it is enough to note that the function $\phi\, \mathbb{1}_{B(0,R)}$ is an envelope function if ϕ is a Beltrami envelope function, by essentially the same argument as above – one uses additionally the observation that a characteristic function of a ball is an envelope function. ∎

We next turn to the proof of Theorem 1.7. For that end, we first establish a couple of auxiliary results.

Lemma 5.4. *Assume that* $k \in [0, 1)$ *and let* (f_j) *and* (g_j) *be sequences of locally uniformly convergent* k*-quasiconformal maps in a domain* $\Omega \subset \mathbb{C}$ *such that the limit functions* $f = \lim_{j\to\infty} f_j$ *and* $g = \lim_{j\to\infty} g_j$ *are non-constant. Assume also that* $|\mu_{f_j} - \mu_{g_j}| \leq \varepsilon$ *in* Ω *for all* $j \geq 1$. *Then*

$$|\mu_f - \mu_g| \leq \varepsilon\, \frac{1+k^2}{1-k^2} \quad \text{in } \Omega.$$

Proof. Take any ball $B(z_0, R) \subset \Omega$. By considering $f_j(z) - f_j(z_0)$ and $g_j(z) - g_j(z_0)$ instead, we may assume that $g_j(z_0) = f_j(z_0) = 0$ for all j. The assumptions together with the quasisymmetry property of the maps imply that if $r > 0$ is taken small enough, then $B(0, r) \subset g_j(B(z_0, R))$ for all $j \geq j_0$, and hence the map $f_j \circ g_j^{-1}$ is well-defined in $B(0, r)$ for $j \geq j_0$. We may compute (see (13.37) in [5])

$$\mu_{f_j \circ g_j^{-1}}(w) = \Big(\frac{\mu_{f_j} - \mu_{g_j}}{1 - \mu_{f_j}\overline{\mu_{g_j}}}\, \frac{\partial_z g_j}{\overline{\partial_z g_j}}\Big) \circ g^{-1}(w), \quad \text{for a.e. } w \in B(0, r). \tag{5.2}$$

In particular, $|\mu_{f_j \circ g_j^{-1}}| \leq \varepsilon(1-k^2)^{-1}$, and letting $k \to \infty$ we infer by the local uniform convergence that $|\mu_{f \circ g^{-1}}| \leq \varepsilon(1-k^2)^{-1}$ in the neighbourhood of z_0. In particular, applying formula (5.2) to f and g, we obtain

$$|\mu_f - \mu_g| \leq (1+k^2)\left|\frac{\mu_f - \mu_g}{1 - \mu_f \overline{\mu_g}}\right| \leq (1+k^2)\,|\mu_{f\circ g^{-1}}| \leq \varepsilon\, \frac{1+k^2}{1-k^2}. \qquad ∎$$

Our next auxiliary result is quite specialized to our situation. Note that the existence of the deterministic homogenization limit F_∞ is guaranteed by Theorem 1.6 that we already verified.

Lemma 5.5. *Suppose that in Theorem 1.6 the Beltrami envelope function ϕ is constant on the complex plane. Then the dilatation μ of the homogenization limit $F_\infty : \mathbb{C} \to \mathbb{C}$ is constant on $\mathbb{C}$, and therefore F_∞ is linear*:

$$F_\infty(z) = \frac{1}{1+A}\, z + \frac{A}{1+A}\, \bar{z},$$

where the constant $A = \mu_{F_\infty}$ satisfies $|A| < 1$.

Proof. Let F_j be defined via (1.14), and let $B_{2^{-j}}$ be the random bump field defined by (1.12). Denote by $\mathbb{Q}_d^2$ the set of dyadic rational points in $\mathbb{C}$, i.e., numbers of the form $(n + mi)2^{-\ell}$, where m, n and $\ell \geq 1$ are integers. Since now $\mu_{F_j} = aB_{2^{-j}}$, where a is a constant with $|a| < 1$, we have for any $b \in \mathbb{Q}_d^2$,

$$\mu_{F_j(\cdot + b)} \sim \mu_{F_j(\cdot)} \quad \text{for } j \geq j_0(b),$$

where $\sim$ stands for equivalence in distribution. As a consequence of the 3-point normalization, we may write for $j \geq j_0(b)$,

$$F_j(z) \sim a_j F_j(z + b) + c_j,$$

where $a_j = (F_j(b+1)) - F_j(b))^{-1}$ and $c_j = -a_j F_j(b)$. In the limit $j \to \infty$, we thus obtain

$$F_\infty(z) = aF_\infty(z + b) + c$$

with constants $a \neq 0$ and c that depend only on b. This implies that

$$\mu_{F_\infty}(z) = \mu_{F_\infty}(z + b),$$

where the equality is in the sense of L^∞-functions.

Therefore, as an element of $L^\infty(\mathbb{C})$, the dilatation μ is periodic on $\mathbb{C}$ with dyadic rational periods, and this easily implies that μ is constant. Finally, for any $A \in \mathbb{D}$, the linear map $z \mapsto \frac{1}{1+A} z + \frac{A}{1+A}\bar{z}$ satisfies the 3-point normalization and has dilatation A, whence it is the unique quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ with these properties. ∎

After these preparations, we prove Theorem 1.7.

Proof of Theorem 1.7. We first define the function $h_{(g,X)}$ with the help of a reference homogenization limit. For any $a \in \{|w| < 1\}$, let F_a be the unique deterministic limit map of the homogenization problem

$$\partial_{\bar{z}} F_{a,j}(z) = aB_{2^{-j}}(z)\partial_z F_{a,j}.$$

By Lemma 5.5, F_a has constant dilatation in the whole plane; let us denote by $h_{(g,X)}(a)$ its value. Theorem 1.6 and Lemma 5.4 yield immediately that the map $a \to h_{(g,X)}(a)$ is continuous.

Assume next that the envelope function ϕ is continuous in a neighbourhood of z_0, with $\phi(z_0) = a$. Then the dilatations of the sequences $F_{a,j}$ and F_j (where F_j is as in the Theorem, see (1.14)) are ε-close in a small enough neighbourhood U of z_0. Thus Lemma 5.4 shows that the dilatation of the homogenization limit F_∞ differs from $h_{(g,X)}(a)$ by less than $\varepsilon(1 + k^2)(1 - k^2)^{-1}$ in a small enough neighbourhood U, and we deduce the continuity of μ_{F_∞} and the equality $\mu_{F_\infty}(z_0) = h_{(g,X)}(a) = h_{(g,X)}(\phi(z_0))$. ∎

We state one more auxiliary result, which actually contains a more general statement than what is needed for Theorem 1.8.

Lemma 5.6. *Assume that g is invariant under rotation by the angle $\pi/2$:*

$$g(z,t) = g(iz,t) \quad \text{for all } z \in \mathbb{C}, t \in \mathbb{R}.$$

Moreover, assume that X is such that the random field $g(\cdot, X)$ is symmetric, i.e., $g(\cdot, X) \sim -g(\cdot, X)$. Then $h_{(g,X)}(a) = 0$ for every $a \in \mathbb{D}$.

Proof. Let B_δ be the random bump field defined by (1.12). The symmetry of g together with the independence of the X_n imply the symmetry of B_δ. Fix $a \in \mathbb{D}$. For $j \geq 1$, let F_j solve the random Beltrami equation

$$\partial_{\bar{z}} F_j = a B_{2^{-j}}(z) \partial_z F_k, \tag{5.3}$$

and denote $\widetilde{F}_j(z) = (F_j(i))^{-1} F_j(iz)$. One computes that $\mu_{\widetilde{F}_j}(z) = -\mu_{F_j}(iz)$. The assumptions of the lemma thus verify that

$$\mu_{\widetilde{F}_j} \sim \mu_{F_j},$$

whence in the limit $j \to \infty$ we deduce that $F_\infty(z) = cF_\infty(iz)$ with a constant $c \neq 0$. By Lemma 5.5, we obtain the identity

$$\frac{1}{1+A} z + \frac{A}{1+A} \bar{z} = c \Big(\frac{1}{1+A} iz - \frac{Ai}{1+A} \bar{z} \Big) \quad \text{for all } z \in \mathbb{C}.$$

The above identity is possible only if $c = -i$ and $A = 0$. Thus $h_{g,X}(a) = A = 0$, as was to be shown. ∎

Proof of Theorem 1.8. The statement that for both of the models (1.5) and (1.6), the deterministic limit map is the identity map follows immediately from Lemma 5.6 and Theorem 1.7.

Finally, we show that, in the generic case, the limit map is not the identity or, equivalently, that the Beltrami coefficient of the limiting map is not zero. To this end, we consider a very simple case of the general model. Fix a bump function $g \in C_0^\infty((0,1)^2)$ with $\|g\|_\infty \leq 1$ and consider the sequence of random dilatations $\mu_{j,a}$, that depend on the complex parameter $a \in \mathbb{D}$,

$$\mu_{j,a}(z) = a \mathbb{1}_{[0,1]^2}(z) \sum_{n \in \mathbb{Z}^2} \varepsilon_n g(2^j z - n),$$

where the ε_n are an independent sequence of random signs ± 1. Let $f_{j,a}$ be the principal solution of the corresponding Beltrami equation, and denote by f_a the almost sure deterministic limit function $f_a = \lim_{j \to \infty} f_{j,a}$. Using notation as in Lemma 5.1 (with $\mu_j = \mu_{j,a}$), we see from (5.1) that f_a has the (power series) representation

$$f_a(z) = z + \sum_{m=1}^{\infty} (C\psi_m)(z) = z + \sum_{m=1}^{\infty} a^m (C\widetilde{\psi}_m)(z),$$

with

$$\widetilde{\psi}_m = \lim_{j\to\infty} \widetilde{\psi}_{m,j}, \quad \text{where} \quad \widetilde{\psi}_{m,j} := \mu_{j,1} T\mu_{j,1} \dots T\mu_{j,1},$$

and where the almost sure weak convergence to the (deterministic) limit $\widetilde{\psi}_m$ in $L^p(\mathbb{C})$ for each $p > 1$ again follows from Corollary 3.10. We claim that f_a is non-linear (equivalently, that the 3-point normalized limit is not the identity) for all but countably many values of $a \in \mathbb{D}$, unless $\widetilde{\psi}_m$ is identically 0 for all m. To see this, notice that $f_a(z) - z \to 0$ as $z \to \infty$, so that f_a cannot be linear unless $f_a(z) - z$ is independent of z. By interpreting $(C\widetilde{\psi}_m)(z)$ as the Taylor coefficients in the power series representation of $a \mapsto f_a(z) - z$ above, we see that f_a is non-linear for all but countably many values of a unless $C\widetilde{\psi}_m(z)$ is independent of z for all m, or equivalently, $\widetilde{\psi}_m \equiv 0$.

It thus suffices to give an example with $\widetilde{\psi}_2 \not\equiv 0$. Let $h \in C_0^\infty(\mathbb{C})$ be a compactly supported test function that equals 1 on $[0,1]^2$. For $j \geq 1$, set

$$Y_j := \int_{\mathbb{C}} h\widetilde{\psi}_{2,j} = \int_{[0,1]^2} \mu_{j,1} T\mu_{j,1} \quad \text{and} \quad Y := \int_{\mathbb{C}} h\widetilde{\psi}_2 = \int_{[0,1]^2} \widetilde{\psi}_2.$$

Then almost surely $Y = \lim_{j\to\infty} Y_j$ and Y is a deterministic constant. We note that in this special case, the convergence is not difficult to prove directly without resorting to our general theory. In any case, we claim that the limit is non-zero for a suitable choice of g. As the random variables Y_j are uniformly bounded, we actually have $Y = \lim_{j\to\infty} \mathbf{E}Y_j$. Since the supports of $g(2^j z - n)$ are disjoint for different values of n, and $\mathbf{E}\varepsilon_n \varepsilon_{n'} = \delta_{n,n'}$, we may compute

$$\mathbf{E}Y_j = \sum_{n\in\mathbb{Z}^2:\, 2^{-j}n\in[0,1)^2} \int_{\mathbb{C}} g(2^j z - n)\, Tg(2^j z - n)\, dz = \int_{\mathbb{C}} g(z)\, Tg(z)\, dz, \tag{5.4}$$

where in the last step we used the translation and scaling invariance of T.

It remains to verify that $g \in C_0^\infty((0,1)^2)$ can be chosen so that the last integral in (5.4) is not identically zero. The following example can be generalized to all kernels that are not odd. Fix any $\varphi \in C_0^\infty(\mathbb{D})$ with $0 \leq \varphi \leq 1$ and $\varphi \not\equiv 0$. If $\int_{\mathbb{C}} \varphi T\varphi = 0$, then setting

$$\varphi_A := \varphi(\cdot - A) + \varphi(\cdot + A),$$

we have

$$\int_{\mathbb{C}} \varphi_A T\varphi_A \sim 2\frac{-1}{\pi}\Big(\int \varphi\Big)^2 (2A)^{-2} \neq 0 \quad \text{as } A \to \infty.$$

By scaling and translating, the support may be taken to be in $(0,1)^2$, and the choice $g = \varphi_A$ for large enough A completes the proof of Theorem 1.6. ∎

5.2. Convergence in probability

Here we sketch an alternative statement of the solution to the homogenization problem, replacing 'almost sure convergence' by 'convergence in probability'. Then there is no need to restrict to subsequences of $\delta \to 0$. In order to rephrase Theorem 1.6 in this manner, consider the principal solution f_δ of the homogenization problem

$$\partial_{\bar{z}} F_\delta = \phi\, B_\delta\, \partial_z F_\delta. \tag{5.5}$$

In the case where the envelope function ϕ is compactly supported, we know that the terms $\psi_{m,\delta}$ in the corresponding Neumann-series are all supported in a ball $B(0, R)$, where R is independent of δ. Each term in the series converges weakly in probability in $L^p(B(0, R))$ as $\delta \to 0$, i.e., for any $h \in L^{p'}$ there is the convergence in probability

$$\int_{\mathbb{C}} h\,\psi_{m,\delta} \to \int_{\mathbb{C}} h\,\psi_m.$$

Moreover, the Neumann series converges $L^p(\mathbb{C})$, with an exponentially decaying remainder term, uniformly with respect to $\delta > 0$. All this easily implies a norm convergence in C^α (compare the proof of Lemma 5.1), i.e.,

$$\mathbf{P}(\|f_\delta - f\|_{C^\alpha(\mathbb{C})} > t) < t$$

for all $t > 0$ as soon as $\delta < \delta_0(t)$. In particular, $f_\delta \to f$ locally uniformly in probability. Finally, we may argue exactly as in the proof of Theorem 1.6 and dispense with the assumption that the envelope has compact support. Let us record our conclusion as a theorem:

Theorem 5.7. *Let μ_δ be as in Theorem 1.6, and denote by F_δ the 3-point normalized solution to the Beltrami equation (5.5). Then $F_\delta \to F_\infty$ locally uniformly in probability as $\delta \to 0$, where F_∞ is the deterministic limit map given by Theorem 1.6. In other words, for any $R > 0$ and $\varepsilon > 0$ one has for $\delta < \delta_0(\varepsilon, R)$ that*

$$\mathbf{P}(\|F_\delta - F\|_{L^\infty(B(0,R))} > \varepsilon) < \varepsilon.$$

5.3. Mappings of finite distortion

As our final application to quasiconformal homogenization, we consider some random mappings of finite distortion, i.e., homeomorphisms for which the assumption $\|\mu\|_\infty \le a < 1$ is relaxed. This leads to the study of solutions to the Beltrami equation $\partial_{\bar z} f = \mu \partial_z f$ where we only have $|\mu(z)| < 1$ almost everywhere. From the general theory of quasiconformal mappings and mappings of finite distortion, one knows that in order to have a viable theory one needs some control on the size of the set where $|\mu(z)|$ is close to 1. For basic properties of planar maps of finite distortion we refer to Chapter 20 in [5] or [4].

There is a well-established theory for mappings of G. David type, i.e., maps whose distortion function

$$K(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

is exponentially integrable, namely $\exp(aK(z)) \in L^1_{\mathrm{loc}}$ for some $a > 0$. With this theory in mind, a natural model for degenerate random Beltrami coefficients is

$$\mu_j(z) := \sum_{n\in\mathbb{Z}^2:\ 2^{-j}n\in[0,1)^2} \varepsilon_{j,n}\, g(2^j z - n), \tag{5.6}$$

where $\|g\|_{L^\infty(\mathbb{C})} = 1$, one has $\mathrm{supp}(g) \subset [0, 1]^2$, and for each $j \ge 1$ we assume that $\varepsilon_{j,n}$ ($n \in \mathbb{Z}^2$) are complex valued i.i.d. random variables taking values in $\mathbb{D}$. Their common distribution is assumed to be independent of j. In this situation, we have the following result.

Theorem 5.8. *Assume the uniform tail estimate*

$$\mathbf{P}\Big(\frac{1+|\varepsilon_{j,n}|}{1-|\varepsilon_{j,n}|} > t\Big) \le e^{-\gamma t} \tag{5.7}$$

for some $\gamma > 2$. Define the (possibly degenerate) Beltrami coefficients μ_j as in (5.6). Then the 3-point normalized solutions F_j of the Beltrami equation $\partial_{\bar z} F_j = \mu_j \partial_z F_j$ converge almost surely locally uniformly to a deterministic limit homeomorphism $F\colon \mathbb{C} \to \mathbb{C}$.

We precede the proof with a couple of auxiliary observations.

First of all, we again use that convergence of the 3-point normalizations is equivalent to convergence of the hydrodynamically normalized ones. Thus, we again consider the principal solution

$$f_j(z) := z + C\Big(\sum_{m=1}^{\infty} \psi_{m,j}\Big), \tag{5.8}$$

of the Beltrami equation, where as before $\psi_{m,j} = \mu_j T\mu_j \ldots T\mu_j$, with μ_j occurring m times. This series is well-defined since almost surely each μ_j satisfies

$$\|\mu_j\|_{L^\infty(\mathbb{C})} \le \max\{|\varepsilon_{j,n}| : n \in \mathbb{Z}^2,\ 2^{-j} n \in [0,1)^2\} < 1.$$

By Corollary 3.10, almost surely each of the terms $\psi_{m,j}$ converges weakly to a limit ψ_m in L^p for every $1 < p < \infty$, and $C(\psi_{m,j})(z)$ converges locally uniformly on $\mathbb{C}$. Therefore we expect that the limit map can be written again as

$$f_\infty = z + C\Big(\sum_{m=1}^{\infty} \psi_m\Big), \tag{5.9}$$

and in proving the convergence one only needs to control the tail of this series. Our main tool will be the following statement:

Lemma 5.9. *In the situation just described, we have*

$$\lim_{M\to\infty} \sup_{j\ge 1} \sum_{m=M}^{\infty} \|\psi_{m,j}\|_{L^2(\mathbb{C})} = 0 \quad \textit{almost surely}. \tag{5.10}$$

We will base our proof of Lemma 5.9 on the following basic estimate on the decay of the L^2-norm of the terms in the Neumann series.

Lemma 5.10 (Theorem 3.1 in [4], see also [11]). *Assume that the dilatation μ is compactly supported,* $\operatorname{supp}(\mu) \subset B(0,R)$. *If for some $p > 0$ we have*

$$A := \int_{B(0,R)} e^{pK(z)}\, dz < \infty, \tag{5.11}$$

where

$$K := \frac{1+|\mu|}{1-|\mu|},$$

then for any $q \in (0, p/2)$, the m-th term in the Neumann-series satisfies the bound

$$\|\psi_m\|_{L^2(\mathbb{C})} \le C_{R,q,A}\, m^{-q}. \tag{5.12}$$

Proof of Lemma 5.9. Denote the distortion function of f_j by

$$K_j(z) := \frac{1+|\mu_j(z)|}{1-|\mu_j(z)|},$$

where μ_j is as in (5.6). In view of the above lemma, (5.10) follows as soon as we verify that there is $p > 2$ such that

$$\sup_{j\geq 1} \int_{[0,1]^2} e^{pK_j(z)}\, dz < \infty \quad \text{almost surely.} \tag{5.13}$$

To this end, choose $q \in (1,2)$ and $p > 2$ so that $pq < \gamma$, where $\gamma > 2$ is from condition (5.7). Denote by Y a random variable with the distribution

$$Y \sim \exp\Big(p\Big(\frac{1+|\varepsilon|}{1-|\varepsilon|}\Big)\Big) - M \quad \text{with} \quad M := \mathbf{E}\exp\Big(p\Big(\frac{1+|\varepsilon|}{1-|\varepsilon|}\Big)\Big),$$

where ε has the same distribution as all of the variables $\varepsilon_{j,n}$. The expectation M above is finite according to our assumption (5.7), in fact $\mathbf{E}Y^q < \infty$. The very definition of μ_j yields that

$$\int_{[0,1]^2} e^{pK_j(z)}\, dz \leq M + Z_j,$$

with

$$Z_j \sim 2^{-2j} \sum_{\ell=1}^{2^{2j}} Y_{j,\ell},$$

where for each $j \geq 1$ the random variables $Y_{j,\ell}$ are identically distributed copies of Y. In order to estimate the tail of Z_j, we recall the von Bahr and Esseen estimate [7], that states for centered i.i.d. random variables $X_1, \ldots, X_N$, the inequality

$$\mathbf{E}|X_1 + \cdots + X_N|^q \leq C_q \sum_{s=1}^{N} \mathbf{E}|X_s|^q, \quad 1 \leq q \leq 2.$$

We obtain

$$\mathbf{P}(Z_j > 1) \leq \mathbf{E}Z_j^q \leq 2^{-2jq}\, C_q\, 2^{2j}\, \mathbf{E}Y^q = O(2^{-2(q-1)j}),$$

and the Borel–Cantelli lemma then yields that almost surely eventually $Z_j \leq 1$. This proves (5.13), and we have finished the verification of Lemma 5.9. ∎

Below we will prove Theorem 5.8 using the Arzelà–Ascoli theorem. To this end we need uniform modulus of continuity estimates for both sequences (f_j) and (f_j^{-1}). Here note first that (5.10) implies the uniform bounds (with a random constant C)

$$\|\partial_{\bar z} f_j\|_{L^2(\mathbb{C})} = \|\partial_z f_j - 1\|_{L^2(\mathbb{C})} \leq C, \quad \text{for all } j \geq 1. \tag{5.14}$$

Since the support of each μ_j is contained in $2\mathbb{D}$, this estimate together with the properties of the Cauchy transform shows that, outside $3\mathbb{D}$, the functions f_j are uniformly equicontinuous and $f_j(z) - z$ is uniformly bounded. Thus uniform equicontinuity in all of $\mathbb{C}$ follows from the following useful result (see [13] and [5], Theorem 20.1.6).

Lemma 5.11 (Gehring, Goldstein and Vodopyanov). *Assume that $f \in W^{1,2}(4\mathbb{D})$ is a homeomorphism. Then, if $z_1, z_2 \in 4\mathbb{D}$, one has*

$$|f(z_1) - f(z_2)|^2 \leq \frac{9\pi \int_{2\mathbb{D}} |\nabla f|^2}{\log(e + 1/|z_1 - z_2|)}.$$

Next, the equicontinuity of the inverse maps is dealt with by another lemma (whose proof actually reduces the situation to Lemma 5.11, see [15], [5], Lemma 20.2.3).

Lemma 5.12 (Iwaniec and Sverak). *Assume that f is a (homeomorphic) principal solution of the Beltrami equation with distortion function K, and with μ supported in $B(0, R')$. Then, for z_1, z_2 in the disc $B(0, R)$, the inverse map $g := f^{-1}$ satisfies*

$$|g(z_1)) - g(z_2)|^2 \leq \frac{C(R, R')}{\log(e + 1/|z_1 - z_1|)} \int_{B(0,R')} K(z)\, dz.$$

The original version assumes that μ is supported in $\mathbb{D}$, but the more general statement follows again by scaling. Now (5.13) entails that in our case $\int_{B(0,R)} K_j(z)\, dz$ is uniformly bounded, and we obtain a (locally) uniform modulus of continuity for the inverse maps f_j^{-1}.

After all these preparations, the proof of Theorem 5.8 can be done quickly.

Proof of Theorem 5.8. Almost surely, we have local uniform equicontinuity for both sequences (f_j) and (f_j^{-1}), uniform boundedness of $f_j(z)$ at every point z outside $3\mathbb{D}$, and thus locally uniform subsequential convergence to a homeomorphism by Arzelà–Ascoli.

Moreover, as in the proof of Lemma 5.1, almost surely each term in the series (5.8) converges locally uniformly. Also, (5.10) implies that the VMO-norm of the remainder in (5.8) converges uniformly to zero ([5], Theorem 4.3.9). Put together, we deduce the convergence in VMO($3\mathbb{D}$) of the whole sequence f_j. Since the f_j are analytic outside $2\mathbb{D}$, this implies the uniqueness of the subsequential limit in $\mathbb{C}$ and establishes almost sure locally uniform converge $f_j \to f_\infty$, where the limit f_∞ is a self-homeomorphism of the plane given by (5.9). ■

Let us finally observe that the above proof actually yields the following more general results, stated both for the deterministic and random homogenization problem. We assume that the complex dilatations μ_δ are supported on a fixed ball $B \subset \mathbb{C}$.

Theorem 5.13. *Let $\mu = \mu_\delta$ be a compactly supported deterministic multiscale function such that for every $0 < \delta < 1$, we have $|\mu_\delta(x)| < 1$ for almost all x, and furthermore the dilatation*

$$K_{\mu_\delta}(x) := \frac{1 + |\mu_\delta(x)|}{1 - |\mu_\delta(x)|}$$

is such that $\int_B \exp(pK_{\mu_\delta})$ is bounded uniformly in δ for some $p > 2$. Then the associated normalized solutions F_δ with dilatation μ_δ converge locally uniformly in distribution to a homeomorphism $F_\infty\colon \mathbb{C} \to \mathbb{C}$ as $\delta \to 0$.

Theorem 5.14. *Let $\mu = \mu_\delta$ be a stochastic multiscale function such that, for $\delta > 0$, we have almost surely $|\mu_\delta(x)| < 1$ for almost all x, and furthermore for some $p > 2$ almost surely $\sup_{j\geq 1} \int_B \exp(pK_{\mu_{2^{-j}}}) < \infty$. Then the associated normalized solutions $F_{\mu_{2^{-j}}}$ are almost surely locally uniformly convergent as $j \to \infty$.*

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