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A remark on weak-strong uniqueness for suitable weak solutions of the Navier–Stokes equations

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Abstract. We extend Barker's weak-strong uniqueness results for the Navier–Stokes equations and consider a criterion involving Besov spaces and weighted Lebesgue spaces.

Since my first paper in the Revista in 1986, I always enjoyed being published in this journal which performs a wonderful job. I enjoyed as well reading in the Revista such a nice collection of papers written by a nice (*harmonious*) *community of* (*harmonic*) *analysts. So many thanks to Antonio and to Josechu!*

1. The Prodi–Serrin criterion for weak-strong uniqueness

In this paper, we are interested in extensions of the Prodi–Serrin weak-strong uniqueness for (suitable) weak Leray solutions of the Navier–Stokes equations. We consider solutions of the Navier–Stokes equations

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p, \\ \operatorname{div} \vec{u} = 0, \\ \vec{u}(0, \cdot) = \vec{u}_0, \end{cases}$$

where $\vec{u}_0$ is a square-integrable divergence-free vector field on the space $\mathbb{R}^3$.

Looking for weak solutions, where the derivatives are taken in the sense of distributions, it is better to write the first line of the system as

$$\partial_t \vec{u} + \operatorname{div} (\vec{u} \otimes \vec{u}) = \Delta \vec{u} - \vec{\nabla} p.$$

If $\vec{u}$ is a solution on $(0, T) \times \mathbb{R}^3$ such that $\vec{u} \in L^\infty((0, T), L^2)$, then the pressure p can be eliminated through the formula

$$\operatorname{div}(\vec{u} \otimes \vec{u}) + \vec{\nabla} p = \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u})),$$

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where $\mathbb{P}$ is the Leray projection operator on solenoidal vector fields:

$$\mathbb{P}\vec{f} = -\frac{1}{\Delta}\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{f}).$$

Moreover, $\vec{u}$ can be represented as a distribution which depends continuously on the time t (see [19]) as

$$\vec{u} = \vec{u}_0 + \int_0^t \Delta\vec{u} - \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u}))\, ds.$$

Leray [21] proved the existence of solutions $\vec{u}$ on $(0, +\infty) \times \mathbb{R}^3$ such that:

- $\vec{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$,
- $\lim_{t\to 0^+} \|\vec{u}(t, \cdot) - \vec{u}_0\|_2 = 0$,
- we have the Leray energy inequality

$$\|\vec{u}(t, \cdot)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2\, ds \le \|\vec{u}_0\|_2^2. \tag{1.1}$$

Such solutions are called *Leray solutions*.[1] His proof is based on a compactness criterion; and provides no clue on the uniqueness of the solution to the Cauchy initial value problem.

A classical case of uniqueness of Leray weak solutions is the weak-strong uniqueness criterion described by Prodi and Serrin [22, 23]: If $\vec{u}_0 \in L^2$ and the Navier–Stokes equations have a solution $\vec{u}$ on $(0, T)$ such that

$$\vec{u} \in L^p_t L^q_x, \quad \text{with } \frac{2}{p} + \frac{3}{q} \le 1 \text{ and } 2 \le p \le +\infty,$$

then, if $\vec{v}$ is a Leray solution with the same initial value $\vec{u}_0$, we have $\vec{u} = \vec{v}$ on $(0, T)$. Let us remark that the existence of such a solution $\vec{u}$ restricts the range of the initial value $\vec{u}_0$. As a matter of fact, when $2 < p < +\infty$, existence of a time $T > 0$ and of a solution $\vec{u} \in L^p_t L^q_x$ is equivalent to the fact that $\vec{u}_0$ belongs to the Besov space $B^{-2/p}_{q,p}$ (see Theorem 2.7 below).

We will see that a corollary of Barker's theorem [1] shows the following extension of the criterion: If $\vec{u}_0 \in L^2$ and the Navier–Stokes equations have a solution $\vec{u}$ on $(0, T)$ such that

$$\sup_{0<t<T} t^{1/p}\|\vec{u}\|_q < +\infty, \quad \text{with } \frac{2}{p} + \frac{3}{q} \le 1 \text{ and } 2 < p < +\infty,$$

and

$$\lim_{t\to 0} t^{1/p}\|\vec{u}\|_q = 0 \quad \text{if } \frac{2}{p} + \frac{3}{q} \le 1,$$

then, if $\vec{v}$ is a Leray solution with the same initial value $\vec{u}_0$, we have $\vec{u} = \vec{v}$ on $(0, T)$. Let us remark again that the existence of such a time T and such a solution $\vec{u}$ is equivalent to the fact that $\vec{u}_0$ belongs to the Besov space $B^{-2/p}_{q,\infty} \cap \mathrm{bmo}^{-1}_0$ (see Definition 1.4 and Theorem 2.8 below).

The space bmo^{-1} was introduced in 2001 by Koch and Tataru [14] for the study of mild solutions to the Navier–Stokes problem. Let us recall the characterization of bmo^{-1} through the heat kernel [14, 15].

[1] Note that the continuity at $t = 0$ of $t \mapsto \vec{u}(t, \cdot)$ in L^2 norm is a consequence of the Leray inequality (1.1).

Proposition 1.1. *For $0 < T < \infty$, define*

$$\|\vec{u}\|_{X_T} = \sup_{0<t<T} \sqrt{t}\, \|\vec{u}(t,\cdot)\|_\infty + \sup_{\substack{0<t<T \\ x_0\in\mathbb{R}^3}} \Big(t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |\vec{u}(s,y)|^2 \,dy\, ds\Big)^{1/2}.$$

Then $\vec{u}_0 \in \mathrm{bmo}^{-1}$ if and only if $(e^{t\Delta}\vec{u}_0)_{0<t<T} \in X_T$ (with equivalence of the norms $\|\vec{u}_0\|_{\mathrm{bmo}^{-1}}$ and $\|e^{t\Delta}\vec{u}_0\|_{X_T}$).

Recall that the differential Cauchy problem for Navier–Stokes equations reads as

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} = \Delta\vec{u} - \vec{\nabla} p, \\ \operatorname{div} \vec{u} = 0, \\ \vec{u}(0,\cdot) = \vec{u}_0. \end{cases}$$

Under reasonable assumptions, the problem is equivalent to the following integro-differential problem:

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})(t,x),$$

where

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\, \mathbb{P} \operatorname{div}(\vec{u}\otimes\vec{v})\, ds, \tag{1.2}$$

and $\mathbb{P}$ is the Leray projection operator (see [15, 19] for details).

Theorem 1.2 (Koch and Tataru's theorem). *There exists C_0 (which does not depend on T) such that if $\vec{u}$ and $\vec{v}$ are defined on $(0,T)\times\mathbb{R}^3$, then*

$$\|B(\vec{u},\vec{v})\|_{X_T} \le C_0 \|\vec{u}\|_{X_T} \|\vec{v}\|_{X_T}.$$

Corollary 1.3. *Let $\vec{u}_0 \in \mathrm{bmo}^{-1}$ with $\operatorname{div}\vec{u}_0 = 0$. If $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0,T)$ such that $\|\vec{u}\|_{X_T} \le 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$. This is the unique solution such that $\|\vec{u}\|_{X_T} \le \frac{1}{2C_0}$.*

The solution $\vec{u}$ can be computed through Picard iteration as the limit of $\vec{U}_n$, where $\vec{U}_0 = e^{t\Delta}\vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta}\vec{u}_0 - B(\vec{U}_n,\vec{U}_n)$. In particular, we have, by induction,

$$\|\vec{U}_{n+1} - \vec{U}_n\|_{X_T} \le (4C_0\|e^{t\Delta}\vec{u}_0\|_{X_T})^{n+1} \|e^{t\Delta}\vec{u}_0\|_{X_T}.$$

Thus, Corollary 1.3 grants local existence of a solution for the Navier–Stokes equations when the initial value belongs to the space bmo_0^{-1}.

Definition 1.4. $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$ if $\vec{u} \in \mathrm{bmo}^{-1}$ and $\lim_{T\to 0}\|e^{t\Delta}\vec{u}_0\|_{X_T} = 0$.

Theorem 1.5 (Barker's theorem, [1]). *Let $\vec{u}_0 \in L^2$ be a divergence-free vector field. Assume, moreover,*

$$\vec{u}_0 \in \mathrm{bmo}_0^{-1} \cap B^{-s}_{q,\infty}, \quad \text{with } 3 < q < +\infty \text{ and } s < 1 - \frac{2}{q},$$

and let $\vec{u}$ be the mild solution of the Navier–Stokes equations with initial value $\vec{u}_0$ such that $\|\vec{u}\|_{X_T} \le \frac{1}{2C_0}$. If $\vec{v}$ is a weak Leray solution of the Navier–Stokes equations with the same initial value $\vec{u}_0$, then $\vec{u} = \vec{v}$ on $(0,T)$.

Again, we remark that if $0 < s < 1 - 2/q$, $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$ and $\vec{u}$ is the mild solution with $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$, then $\vec{u}_0 \in B^{-s}_{q,\infty}$ is equivalent to

$$\sup_{0<t<T} t^{s/2}\|\vec{u}(t,\cdot)\|_q < +\infty.$$

In the following theorems, we shall state the assumptions in terms of the mild solution $\vec{u}$ instead of the initial value $\vec{u}_0$. In Theorem 2.4, we shall give the equivalence between the assumption on the solution $\vec{u}$ and the assumption on the initial value $\vec{u}_0$.

We aim to generalize Barker's result to a larger class of mild solutions. Barker's result is based on an interpolation lemma which states that, if $\vec{u}_0 \in \mathrm{bmo}_0^{-1} \cap L^2 \cap B^{-s}_{q,\infty}$, with $3 < q < +\infty$ and $-s > -1 + 2/q$, then $\vec{u}_0 \in [L^2, B^{-\delta}_{\infty,\infty}]_{\theta,\infty}$ for some $\theta \in (0,1)$ and some $\delta \in (0,1)$. (Those conditions are in a way equivalent, as we shall see in Corollary 3.3.) Then the comparison between the Leray solution $\vec{v}$ and the mild solution $\vec{u}$ is performed through an estimation of both $\|\vec{u} - \vec{w}_\varepsilon\|_2$ and $\|\vec{v} - \vec{w}_\varepsilon\|_2$, where $\vec{w}_\varepsilon$ is the solution of the Navier–Stokes problem with initial value $\vec{w}_{0,\varepsilon}$ such that $\|\vec{w}_{0,\varepsilon} - \vec{u}_0\|_2 \leq C_1\varepsilon^\theta$ and $\|\vec{w}_\varepsilon\|_{B^{-\delta}_{\infty,\infty}} < C_1\varepsilon^{\theta-1}$ (with C_1 depending on $\vec{u}_0$ but not on ε).

Our idea is to replace the space L^2 by the larger space $L^2_w = L^2(w\,dx)$ with $w(x) = (1+|x|)^{-2}$, and use the interpolation space $[L^2_w, B^{-\delta}_{\infty,\infty}]_{\theta,\infty}$ for some $\theta \in (0,1)$ and some $\delta \in (0,1)$. As we shall no longer deal with the L^2 norm, the Leray inequality on $\|\vec{v}\|_2$ will not be sufficient. Instead, we shall consider a stricter class of weak solutions, namely, the suitable weak Leray solutions [3].

Definition 1.6. A Leray solution is suitable on $(0,T)$ if it fulfills the local energy inequality: there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that we have

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \mathrm{div}((2p + |\vec{u}|^2)\vec{u}) - \mu. \tag{1.3}$$

We may now state our main results. The first one (stated in [20]) weakens the integrability requirement on the solution $\vec{u}$ from the Lebesgue space L^q to the Morrey space $M^{p,q}$. Recall that the Morrey space $M^{p,q}$, $1 < p \leq q < +\infty$, is defined by

$$\|f\|_{M^{p,q}} = \sup_{x_0\in\mathbb{R}^3} \sup_{0<r\leq 1} r^{3/q-3/p}\left(\int_{B(x_0,r)} |f(x)|^p\,dx\right)^{1/p} < +\infty.$$

For $p = 1$, one replaces the requirement $f \in L^p_{\mathrm{loc}}$ by the assumption that f is a locally finite Borel measure μ with

$$\|f\|_{M^{1,q}} = \sup_{x_0\in\mathbb{R}^3} \sup_{0<r\leq 1} r^{3/q-3} \int_{B(x_0,r)} d|\mu|(x) < +\infty.$$

For $1 < p \leq +\infty$, we have the continuous embeddings

$$L^q \subset M^{q,q} \subset M^{p,q} \subset M^{1,q}.$$

The idea of considering Morrey spaces instead of Lebesgue spaces is quite natural. Indeed, in the direct proof of the Prodi–Serrin criterion, a key estimate is the inequality

$$\int |uv||\vec{\nabla} w|\,dx \leq C\|u\|_q\|v\|_2^{1-\theta}\,\|\vec{\nabla} v\|_2^\theta\,\|\vec{\nabla} w\|_2$$

for $0 \leq \theta \leq 1$ and $1/q = \theta/3$. This inequality still holds when the L^q norm is replaced by the norm in the homogeneous Morrey space $\dot{M}^{2,q}$ with $0 < \theta < 1$ and $1/q = \theta/3$, see [16].

Theorem 1.7. *Let $\vec{u}_0$ be a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \mathrm{bmo}_0^{-1}$. Assume moreover that the mild solution $\vec{u}$ of the Navier–Stokes equations with initial value $\vec{u}_0$ such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$ is such that*

$$\sup_{0<t<T} t^{s/2} \|\vec{u}(t,\cdot)\|_{\dot{M}^{p,q}} < +\infty, \quad \text{with } 2 < p \leq q < +\infty \text{ and } 0 \leq s < 1 - \frac{2}{p}\cdot$$

If $\vec{v}$ is a suitable weak Leray solution of the Navier–Stokes equations with the same initial value $\vec{u}_0$, then $\vec{u} = \vec{v}$ on $(0, T)$.

Let us remark that the statement and proof of Theorem 1.7 we gave in [20] was false (we assumed only that $s < 1 - 2/q$).[2]

The second one weakens the integrability requirement on the solution $\vec{u}$ from the Lebesgue space L^q to the weighted Lebesgue space $L^q((1+|x|)^{-N}\,dx)$ for some $N \geq 0$.

Theorem 1.8. *Let $\vec{u}_0$ be a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \mathrm{bmo}_0^{-1}$. Assume moreover that the mild solution $\vec{u}$ of the Navier–Stokes equations with initial value $\vec{u}_0$ such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$ is such that*

$$\sup_{0<t<T} t^{s/2} \|\vec{u}\|_{L^q((1+|x|)^{-N}\,dx)} < +\infty, \quad \text{with } N \geq 0,\ 2 < q < +\infty \text{ and } 0 \leq s < 1 - \frac{2}{q}\cdot$$

If $\vec{v}$ is a suitable weak Leray solution of the Navier–Stokes equations with the same initial value $\vec{u}_0$, then $\vec{u} = \vec{v}$ on $(0, T)$.

Of course, Theorem 1.7 is a corollary of Theorem 1.8, as $\dot{M}^{p,q} \subset L^p((1+|x|)^{-N}\,dx)$ for $N > 3 - 3p/q$.

The paper is then organized in the following manner. In Section 2, we define stable spaces and collect some technical results on generalized Besov spaces based on stable spaces. In Section 3, we define potential spaces based on stable spaces and prove some interpolation estimates. In Section 4, we give some remarks on the Koch and Tataru solutions for the Navier–Stokes problem. In Section 5, we study stability estimates for suitable weak Leray solutions with initial data in $L^2 \cap [L^2((1+|x|)^{-2}\,dx), B^{-\delta}_{\infty,\infty}]_{\theta,\infty}$ (see Theorem 5.2). In Section 6, we prove the uniqueness theorem (Theorem 1.8). In Section 7, we give some further comments on Barker's conjecture on the uniqueness problem.

2. Stable spaces and Besov spaces

We define the convolutor space $\mathbb{K}$ by the following convention:

- a *suitable kernel* is a function $K \in L^1(\mathbb{R}^3)$ such that K is radial and radially non-increasing (in particular, K is non-negative); this is noted as $K \in \mathbb{K}_0$;

[2] The mistake was due to an incorrect equality $\rho = \eta\gamma$, while it should have been $\gamma = \eta\rho$; as $\eta < 1$, the equality turned to be incorrect.

- f is a convolutor if $f \in L^1$ and there exists $K \in \mathbb{K}_0$ such that $|f| \leq K$ almost everywhere;
- the norm of f in $\mathbb{K}$ is defined as

$$\|f\|_{\mathbb{K}} = \inf\{\|K\|_1 \mid K \in \mathbb{K}_0 \text{ and } |f| \leq K \text{ a.e.}\}.$$

One easily checks that $\|\cdot\|_{\mathbb{K}}$ is a norm and that $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ is a Banach space.

Definition 2.1. A *stable space* of measurable functions on $\mathbb{R}^3$ is a Banach space E such that

- $E \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)$,
- if $f \in E$ and $g \in L^\infty$, $fg \in E$ and $\|fg\|_E \leq C\|f\|_E\|g\|_\infty$ (where C does not depend on f nor g),
- if $f \in E$ and $g \in \mathbb{K}$, $f * g \in E$ and $\|f * g\|_E \leq C\|f\|_E\|g\|_{\mathbb{K}}$ (where C does not depend on f nor g).

Examples of stable spaces:

(a) $E = L^p$, $1 \leq p \leq +\infty$.

(b) $E = L^p(w\,dx)$, where w belongs to the Muckenhoupt class $\mathcal{A}_p$ for some $1 < p < +\infty$. If $g \in \mathbb{K}_0$, then

$$|f * g(x)| \leq \|g\|_1 \mathcal{M}_f(x),$$

where $\mathcal{M}_f$ is the Hardy–Littlewood maximal function of f; recall that the Hardy–Littlewood maximal function is a bounded sublinear operator on $L^p(w\,dx)$ when $w \in \mathcal{A}_p$, see [25].

(c) $E = L^p_{\mathrm{uloc}}$ for some $1 \leq p \leq +\infty$, where

$$\|f\|_{L^p_{\mathrm{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \Big(\int_{B(x_0,1)} |f(x)|^p\,dx \Big)^{1/p}.$$

By Minkowski's inequality, we have

$$\|f * g\|_E \leq \int |g(y)|\|f(\cdot - y)\|_{L^p_{\mathrm{uloc}}}\,dy = \|g\|_1\|f\|_{L^p_{\mathrm{uloc}}}.$$

(d) This example can be generalized to other shift-invariant spaces (for which the norms $\|f\|_E$ and $\|f(\cdot - y)\|_E$ are equal). For instance, we may take E as the Morrey space $M^{p,q}$, $1 < p \leq q < +\infty$.

Our next step is to introduce Besov-like Banach spaces based on stable spaces and to describe the regularity of Koch–Tataru solutions when the initial value belongs additionally to the Besov space.

Definition 2.2. Let $T \in (0, +\infty)$. Let E be a stable space of measurable functions on $\mathbb{R}^3$. For $s > 0$ and $1 \leq q \leq +\infty$, we define the Besov-like Banach space $B^{-s}_{E,q}$ as the space of tempered distributions such that

$$t^{s/2}\|e^{t\Delta} f\|_E \in L^q((0, T), dt/t).$$

Proposition 2.3. *The norms* $\|t^{s/2}\|e^{t\Delta}f\|_E\|_{L^q((0,T),dt/t)}$ *are all equivalent, so that* $B^{-s}_{E,q}$ *does not depend on* T.

Proof. Assume that $t^{s/2}\|e^{t\Delta}f\|_E \in L^q((0,T),dt/t)$ for some $T>0$ and consider $t \geq T$. We have

$$e^{t\Delta}f = \frac{2}{T}\int_{T/2}^{T} e^{(t-\theta)\Delta}e^{\theta\Delta}f\,d\theta,$$

so that

$$\begin{aligned}\|e^{t\Delta}f\|_E &\leq C\,\frac{2}{T}\int_{T/2}^{T}\|e^{\theta\Delta}f\|_E\,d\theta\\ &\leq C\,\frac{2}{T}\left\|\theta^{s/2}\|e^{\theta\Delta}f\|_E\right\|_{L^q((0,T),d\theta/\theta)}\|\mathbb{1}_{T/2<\theta}\,\theta^{1-s/2}\|_{L^{q/(q-1)}((0,T),d\theta/\theta)}.\end{aligned}$$

The equivalence of the norms is proved. ∎

We remark that this proof shows as well that if $1 \leq q \leq r \leq +\infty$, then $B^{-s}_{E,q} \subset B^{-s}_{E,r}$. Another obvious property of Besov spaces is that if $0 < s < \sigma$, then $B^{-s}_{E,\infty} \subset B^{-\sigma}_{E,1}$.

The main result in this section is the following theorem.

Theorem 2.4. *Let* E *be a stable space of measurable functions on* $\mathbb{R}^3$. *Let* $0 < T < +\infty$, *and let* $\vec{u}_0 \in \mathrm{bmo}^{-1}$, *with* $\operatorname{div}\vec{u}_0 = 0$ *and* $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$. *Let* $\vec{u}$ *be the solution of the integral Navier–Stokes equations on* $(0,T)$ *such that* $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$. *Then the following assertions are equivalent for* $0<\sigma<1$ *and* $2 < q \leq +\infty$:

(A) $\vec{u}_0 \in B^{-\sigma}_{E,q}$,

(B) $t^{\sigma/2}\|\vec{u}\|_E \in L^q((0,T),dt/t)$.

Proof. Let us remark that the operator $e^{(t-s)\Delta}\,\mathbb{P}\operatorname{div}$ is a matrix of convolution operators whose kernels are bounded by $C(\sqrt{t-s}+|x-y|)^{-4}$, hence are controlled in the convolutor norm $\|\cdot\|_{\mathbb{K}}$ by $C\,\frac{1}{\sqrt{t-s}}$. We thus have the inequality

$$\begin{aligned}\|B(\vec{u},\vec{v})\|_E &\leq C\int_0^t \frac{1}{\sqrt{t-s}}\,\|\vec{u}\otimes\vec{v}\|_E\,ds\\ &\leq C'\sup_{0<s<t}\sqrt{s}\,\|\vec{u}(s,\cdot)\|_\infty\int_0^t\frac{1}{\sqrt{t-s}}\,\frac{1}{\sqrt{s}}\,\|\vec{v}(s,\cdot)\|_E\,ds\end{aligned}$$

(and we get a similar estimate by interchanging $\vec{u}$ and $\vec{v}$ in the last line). We thus want to estimate

$$J(t) = t^{-1/q+\sigma/2}\int_0^t\frac{1}{\sqrt{t-s}}\,\frac{1}{\sqrt{s}}\,s^{1/q-\sigma/2}L(s)\,ds, \quad \text{with } L \in L^q((0,T),dt).$$

- If $q=+\infty$, we easily check that $\|J\|_\infty \leq C_\sigma\|L\|_\infty$ (since $\sigma<1$).
- If $\sigma \leq 2/q$, we have $s^{1/q-\sigma/2} \leq t^{1/q-\sigma/2}$, so that $J(t) \leq \int_0^t \frac{1}{\sqrt{t-s}}\,\frac{1}{\sqrt{s}}\,L(s)\,ds$. If $2 < q < +\infty$, as $1/\sqrt{s}$ belongs to the Lorentz space $L^{2,\infty}$, we use the product laws and convolution laws in Lorentz spaces to get that, if $L \in L^q$, $\frac{1}{\sqrt{s}}L \in L^{r,q}$ with $1/r = 1/q+1/2$ and $\frac{1}{\sqrt{s}} * (\frac{1}{\sqrt{s}}L) \in L^{q,q} = L^q$. Thus, $\|J\|_q \leq C\|L\|_q$.

- If $\sigma > 2/q$, we write

$$J(t) \leq C\Big(\int_0^t \frac{(t-s)^{-1/q+\sigma/2}}{\sqrt{t-s}} \frac{1}{\sqrt{s}} s^{1/q-\sigma/2} L(s)\, ds + \int_0^t \frac{1}{\sqrt{t-s}} \frac{s^{-1/q+\sigma/2}}{\sqrt{s}} s^{1/q-\sigma/2} L(s)\, ds\Big),$$

and we use again the product laws and convolution laws in Lorentz spaces to get, if $L \in L^q$, that $\frac{1}{s^{(1+\sigma)/2-1/q}} L \in L^{r,q}$ with $\frac{1}{r} = \frac{1+\sigma}{2}$ and $\frac{1}{s^{(1-\sigma)/2+1/q}} * (\frac{1}{s^{(1+\sigma)/2-1/q}} L) \in L^{q,q} = L^q$. We find again $\|J\|_q \leq C\|L\|_q$.

We may now easily check that (B) $\Rightarrow$ (A): we just write $e^{t\Delta}\vec{u}_0 = \vec{u} + B(\vec{u}, \vec{u})$ and

$$\big\|t^{\sigma/2}\|B(\vec{u}, \vec{u})\|_E\big\|_{L^q((0,T),\, dt/t)} \leq C \sup_{0<t<T} \sqrt{t}\, \|\vec{u}(t, \cdot)\|_\infty \big\|t^{\sigma/2}\|\vec{u}\|_E\big\|_{L^q((0,T),\, dt/t)}.$$

In order to prove (A) $\Rightarrow$ (B), we write $\vec{u}$ as the limit of $\vec{U}_n$, where $\vec{U}_0 = e^{t\Delta}\vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta}\vec{u}_0 - B(\vec{U}_n, \vec{U}_n)$. By induction, $\vec{U}_n$ satisfies

$$\big\|t^{\sigma/2}\,\|\vec{U}_n\|_E\big\|_{L^q((0,T),\, dt/t)} < +\infty$$

and

$$\big\|t^{\sigma/2}\|\vec{U}_{n+1} - \vec{U}_n\|_E\big\|_{L^q((0,T),\, dt/t)} \leq C \sup_{0<t<T} \sqrt{t}\, \|\vec{U}_n - \vec{U}_{n-1}\|_\infty \times \big(\big\|t^{\sigma/2}\|\vec{U}_n\|_E\big\|_{L^q((0,T),\, dt/t)} + \big\|t^{\sigma/2}\|\vec{U}_{n-1}\|_E\big\|_{L^q((0,T),\, dt/t)}\big).$$

If

$$A_N = \big\|t^{\sigma/2}\,\|\vec{U}_0\|_E\big\|_{L^q((0,T),\, dt/t)} + \sum_{n=0}^{N-1} \big\|t^{\sigma/2}\,\|\vec{U}_{n+1} - \vec{U}_n\|_E\big\|_{L^q((0,T),\, dt/t)}$$

and $\varepsilon = 4C_0\|\vec{U}_0\|_{X_T}$, we have

$$\big\|t^{\sigma/2}\,\|\vec{U}_N\|_E\big\|_{L^q((0,T),\, dt/t)} \leq A_N$$

and

$$A_{N+1} \leq A_N(1 + 2C\varepsilon^{N+1}) \leq A_0 \prod_{j=1}^{N+1} (1 + 2C\varepsilon^j).$$

This proves that $\|t^{\sigma/2}\|\vec{u}\|_E\|_{L^q((0,T),\, dt/t)} < +\infty$. ∎

Let us remark that the assumption $\vec{u}_0 \in \mathrm{bmo}^{-1}$ can be dropped in some cases, as for example the solutions $\vec{u}$ in the Serrin class $L^q((0,T), L^r)$ with $2/q + 3/r \leq 1$ and $3 < r < +\infty$. In analogy with L^r, we define r-stable spaces in the following way.

Definition 2.5. For $2 < r < +\infty$, an *r-stable space* of measurable functions on $\mathbb{R}^3$ is a stable space E such that

- E is contained in $B^{-3/r}_{\infty,\infty}$ and, for $f \in E$, $\|f\|_{B^{-3/r}_{\infty,\infty}} \leq C\|f\|_E$.

- E is contained in L^2_{loc}.
- If $f, g \in E$, then $fg \in B^{-3/r}_{E,\infty}$ and $\|fg\|_{B^{-3/r}_{E,\infty}} \le C\|f\|_E\|g\|_E$.

The Morrey space $M^{2,r}$ is a r-stable space; more precisely, it is the largest r-stable space.

Lemma 2.6. *Let E be a r-stable space of measurable functions on $\mathbb{R}^3$, where $r \in (2, +\infty)$. Then $E \subset M^{2,r}$ and $\|f\|_{M^{2,r}} \le C\|f\|_E$.*

Proof. Let $\rho < 1$ and $x_0 \in \mathbb{R}^3$. We have

$$e^{\rho^2\Delta}(f^2)(x_0) \ge \int_{B(x_0,\rho)} f^2(y)\,dy \inf_{y\in B(x_0,\rho)} W_{\rho^2}(x_0 - y) = \frac{e^{-1/4}}{(4\pi\rho^2)^{3/2}} \int_{B(x_0,\rho)} f^2(y)\,dy,$$

where $W_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-x^2/(4t)}$. On the other hand, we have

$$e^{\rho^2\Delta}(f^2)(x_0) \le C\rho^{-3/r}\|e^{\rho^2\Delta}(f^2)\|_{B^{-3/r}_{\infty,\infty}} \le C'\rho^{-3/r}\|e^{\rho^2\Delta/2}(f^2)\|_E \le C''\rho^{-6/r}\|f\|^2_E.$$

This gives

$$\int_{B(x_0,\rho)} f^2(y)\,dy \le C\rho^{3-6/r}\|f\|^2_E,$$

and thus $f \in M^{2,r}$. ∎

Theorem 2.7. *Let E be a r-stable space of measurable functions on $\mathbb{R}^3$. Let $\vec{u}_0 \in E$ with $\operatorname{div}\vec{u}_0 = 0$. Let $0 < \sigma < 1$ and $2 < q < +\infty$, with*

$$\frac{2}{q} \le \sigma \le 1 - \frac{3}{r}$$

and $q < +\infty$ if $\sigma = 1 - 3/r$. Then the following assertions are equivalent:

(A) $\vec{u}_0 \in B^{-\sigma}_{E,q}$,

(B) *there exist $T > 0$ and a solution $\vec{u}$ of the integral Navier–Stokes equations on $(0, T)$ with initial value $\vec{u}_0$ such that $t^{\sigma/2}\|\vec{u}\|_E \in L^q((0, T), dt/t)$.*

(This theorem thus holds for solutions $\vec{u} \in L^q((0, T), E)$ under the Serrin condition $2/q + 3/r \le 1$.)

Proof. (A) ⇒ (B) is a direct consequence of Theorem 2.4 and of the embedding $B^{-\sigma}_{M^{2,r},q} \subset \mathrm{bmo}^{-1}_0$ for $\sigma \le 1 - 3/r$ and $(\sigma, q) \ne (1 - 3/r, \infty)$. Indeed, for $0 < t < 1$, we have

$$\begin{aligned}
\|e^{t\Delta}f\|_\infty &\le \frac{2}{t}\int_{t/2}^{t} \|e^{\theta\Delta}f\|_\infty\,d\theta \\
&\le C\,\frac{2}{t}\,t^{-3/(2r)}\int_{t/2}^{t} \|e^{\theta/2\Delta}f\|_{M^{2,r}}\,d\theta \\
&\le C't^{-1-3/(2r)}\big\|\theta^{\sigma/2}\|e^{\theta\Delta}f\|_{M^{2,r}}\big\|_{L^q((0,t),\,d\theta)}\|\theta^{-\sigma/2}\|_{L^{q/(q-1)}((t/2,t),\,d\theta)} \\
&\le C''t^{-1-3/(2r)}\,t^{1/q}\big\|\theta^{\sigma/2}\|e^{\theta\Delta}f\|_{M^{2,r}}\big\|_{L^q((0,t),\,d\theta/\theta)}\,t^{1-1/q}\,t^{-\sigma/2} \\
&\le C'''t^{-1/2}\,t^{(1-\sigma-3/r)/2}\Big(\int_0^t (\theta^{\sigma/2}\|e^{\theta\Delta}f\|_{M^{2,r}})^q\,\frac{d\theta}{\theta}\Big)^{1/q}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_{B(x_0,\sqrt{t})} |e^{s\Delta} f|^2 \, dy \, ds \\
&\quad \le C \int_0^t \|e^{s\Delta} f\|_{M^{2,r}}^2 \, t^{3/2-3/r} \, ds \\
&\quad \le C' t^{3/2-3/r} \left\| s^{\sigma/2} \|e^{s\Delta} f\|_{M^{2,r}} \right\|_{L^q((0,t),\, ds/s^\sigma)}^2 \|\mathbb{1}\|_{L^{q/q-2}((0,t),\, ds/s^\sigma)} \\
&\quad \le C'' t^{3/2-3/r} \, t^{(1-\sigma)2/q} \left\| s^{\sigma/2} \|e^{s\Delta} f\|_{M^{2,r}} \right\|_{L^q((0,t),\, ds/s)}^2 \, t^{(1-\sigma)(1-2/q)} \\
&\quad \le C'' t^{3/2} \, t^{1-\sigma-3/r} \Big(\int_0^t (s^{\sigma/2} \|e^{s\Delta} f\|_{M^{2,r}})^q \, \frac{ds}{s} \Big)^{2/q}.
\end{aligned}$$

We now prove (B) $\Rightarrow$ (A). We use again the identity

$$e^{t\Delta} \vec{u}_0 = \frac{2}{t} \int_{t/2}^t e^{(t-s)\Delta} \, e^{s\Delta} \vec{u}_0 \, ds$$

and get

$$e^{2t\Delta} \vec{u}_0 = \frac{2}{t} \int_{t/2}^t e^{(2t-s)\Delta} \vec{u}(s,\cdot) \, ds + \frac{2}{t} \int_{t/2}^t e^{(2t-s)\Delta} B(\vec{u},\vec{u}) \, ds = \vec{v}(t,\cdot) + \vec{w}(t,\cdot).$$

We want to estimate $\|t^{\sigma/2} \|e^{2t\Delta} \vec{u}_0\|_E \|_{L^q((0,T),\, dt/t)} = \|t^{\sigma/2-1/q} \|e^{2t\Delta} \vec{u}_0\|_E \|_{L^q((0,T),dt)}$.

We have

$$\begin{aligned}
t^{\sigma/2-1/q} \|\vec{v}(t,\cdot)\|_E &\le C t^{\sigma/2-1/q} \, \frac{2}{t} \int_{t/2}^t \|\vec{u}\|_E \, ds \\
&\le C \, \frac{2}{t} \int_{t/2}^t s^{\sigma/2-1/q} \|\vec{u}\|_E \, ds \le 4C \, \mathcal{M}_{s^{\sigma/2-1/q} \|\vec{u}\|_E}(t)
\end{aligned}$$

and thus $t^{\sigma/2-1/q} \|\vec{v}(t,\cdot)\|_E \in L^q((0,T),dt)$.

On the other hand, we have

$$\begin{aligned}
\|\vec{w}(t,\cdot)\|_E &\le \sup_{t/2 \le s \le t} \left\| \int_0^s e^{(3t/2-\tau)\Delta} \, \mathbb{P} \operatorname{div} e^{t\Delta/2} (\vec{u} \otimes \vec{u}) \, d\tau \right\|_E \\
&\le C \int_0^t \frac{1}{\sqrt{3t/2-\tau}} \|e^{t\Delta/2} (\vec{u} \otimes \vec{u})\|_E \, d\tau \\
&\le C' \int_0^t \frac{t^{1/2-\sigma+1/q}}{(t-\tau)^{1-\sigma+1/q}} \|e^{t\Delta} |\vec{u}|^2\|_E \, d\tau \\
&\le C'' t^{1/2-\sigma+1/q-3/(2r)} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+1/q}} \|\vec{u}\|_E^2 \, d\tau,
\end{aligned}$$

and thus

$$\begin{aligned}
t^{\sigma/2-1/q} \|\vec{w}(t,\cdot)\|_E &\le C T^{1/2-\sigma/2-3/(2r)} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+1/q}} \|\vec{u}\|_E^2 \, d\tau \\
&= C T^{1/2-\sigma/2-3/(2r)} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+1/q}} \, \tau^{-\sigma+2/q} (\tau^{\sigma/2-1/q} \|\vec{u}\|_E)^2 \, d\tau.
\end{aligned}$$

If $J(\tau) = \tau^{\sigma/2-1/q}\|\vec{u}\|_E$, we have $J(\tau) \in L^q((0,T), d\tau)$, hence $J^2 \in L^{q/2}((0,T), d\tau)$, $\tau^{-\sigma+2/q} J^2 \in L^{p_0,q/2}((0,T),\ dt)$, with $1/p_0 = 2/q + \sigma - 2/q = \sigma$, and $\frac{1}{\tau^{1-\sigma+1/q}} * (\tau^{-\sigma+2/q} J^2) \in L^{p_1,q/2}((0,T),\ dt)$, with $1/p_1 = 1/p_0 + 1 - 1/\sigma + 1/q - 1 = 1/q$.

Thus, $t^{\sigma/2-1/q}\|e^{2t\Delta}\vec{u}_0(t,\cdot)\|_E \in L^q((0,T),\ dt)$ and $\vec{u}_0 \in B^{-\sigma}_{E,q}$. ■

The case $(\sigma, q) = (1 - 3/r, +\infty)$ can be treated in a similar way.

Theorem 2.8. *Let E be a r-stable space of measurable functions on $\mathbb{R}^3$ with $3 < r < +\infty$. Let $\vec{u}_0 \in E$ with* $\operatorname{div} \vec{u}_0 = 0$. *Then the following assertions are equivalent*:

(A) $\vec{u}_0 \in B^{-1+3/r}_{E,\infty}$ *and* $\lim_{t\to 0} t^{1/2-3/(2r)}\|e^{t\Delta}\vec{u}_0\|_E = 0$,

(B) *there exist $T > 0$ and a solution $\vec{u}$ of the integral Navier–Stokes equations on* $(0,T)$, *with initial value $\vec{u}_0$, such that* $\sup_{0<t<T} t^{1/2-3/(2r)}\|\vec{u}\|_E < +\infty$, *and such that* $\lim_{t\to 0} t^{1/2-3/(2r)}\|\vec{u}\|_E = 0$.

Remark. We have the embedding $B^{-1+3/r}_{E,\infty} \subset \mathrm{bmo}^{-1}$, but this does not grant existence of a solution. The extra condition $\lim_{t\to 0} t^{1/2-3/(2r)}\|e^{t\Delta}\vec{u}_0\|_E = 0$ is used to get $\vec{u}_0 \in \mathrm{bmo}^{-1}_0$, and thus to have the existence of a local solution.

3. Potential spaces and interpolation

If E is a stable space, we define, for $s \in \mathbb{R}$, the potential space H^s_E as $H^s_E = (\mathrm{Id} - \Delta)^{-s/2} E$, normed with $\|f\|_{H^s_E} = \|(\mathrm{Id} - \Delta)^{s/2} f\|_E$. For positive s, we have an obvious comparison of the potential space H^{-s}_E with the Besov spaces.

Lemma 3.1. *Let E be a stable space and $s > 0$. Then*

$$B^{-s}_{E,1} \subset H^{-s}_E \subset B^{-s}_{E,\infty}.$$

Proof. Indeed, we have

$$(\mathrm{Id} - \Delta)^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-t}\, e^{t\Delta}\, t^{s/2}\, \frac{dt}{t}.$$

If f belongs to $B^{-s}_{E,1}$, then $t^{s/2}\|e^{t\Delta} f\|_E \in L^1((0,1), dt/t)$ while $\|e^{\Delta} f\|_1 \le \|f\|_{B^{-s}_{E,\infty}} \le C\|f\|_{B^{-s}_{E,1}}$, so that

$$\|f\|_{H^{-s}_E} \le \frac{1}{\Gamma(s/2)}\Big(\int_0^1 t^{s/2}\|e^{t\Delta} f\|_E\, \frac{dt}{t} + C\|e^{\Delta} f\|_E \int_1^{+\infty} e^{-t}\, t^{s/2}\, \frac{dt}{t}\Big) \le C'\|f\|_{B^{-s}_{E,1}}.$$

Conversely, if $f \in H^{-s}_E$, $f = (\mathrm{Id} - \Delta)^{s/2} g$, where $g \in E$, and if $0 < \theta < 1$, then we pick $N \in \mathbb{N}$ with $N > s/2$ and write

$$\begin{aligned} e^{\theta\Delta} f &= e^{\theta\Delta}(\mathrm{Id} - \Delta)^N (\mathrm{Id} - \Delta)^{s/2-N} g \\ &= \frac{1}{\Gamma(N - s/2)} \int_0^{+\infty} e^{-t}(\mathrm{Id} - \Delta)^N e^{(t+\theta)\Delta} g\, t^{N-s/2}\, \frac{dt}{t}. \end{aligned}$$

For $\alpha \in \mathbb{N}^3$, with $0 \le |\alpha| \le 2N$, we have

$$\|\partial^\alpha e^{(t+\theta)\Delta} g\|_E \le C_\alpha (t+\theta)^{-|\alpha|/2}\|g\|_E \le C_\alpha(1 + (t+\theta)^{-N})\|g\|_E,$$

so that

$$\begin{aligned}\|e^{\theta\Delta} f\|_E &\leq C\|g\|_E \int_0^{+\infty} e^{-t}(1+(t+\theta)^{-N})\, t^{N-s/2}\,\frac{dt}{t}\\ &\leq C\|g\|_E\Big(\Gamma(N-s/2)+\int_0^\theta t^{N-s/2}\,\frac{dt}{t}+\int_\theta^{+\infty}\frac{dt}{t^{1+s/2}}\Big)\\ &\leq C'\|g\|_E\,\theta^{-s/2}.\end{aligned}$$

The lemma is proved. ∎

Let us recall the definition of Calderón's interpolation spaces $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$, see [4]. We assume that A_0 and A_1 are subspaces of $\mathcal{S}'$, so that $A_0 \cap A_1$ and $A_0 + A_1$ are well defined.

We begin with the definition of the first interpolate $[A_0, A_1]_\theta$. Let Ω be the open complex strip $\Omega = \{z \in \mathbb{C} \mid 0 < \Re z < 1\}$. We let $\mathcal{F}(A_0, A_1)$ be the space of functions F defined on the closed complex strip $\overline{\Omega}$ such that:

- F is continuous and bounded from $\overline{\Omega}$ to $A_0 + A_1$,
- F is analytic from Ω to $A_0 + A_1$,
- $t \mapsto F(it)$ is continuous from $\mathbb{R}$ to A_0, and $\lim_{|t|\to+\infty}\|F(it)\|_{A_0} = 0$,
- $t \mapsto F(1+it)$ is continuous from $\mathbb{R}$ to A_1, and $\lim_{|t|\to+\infty}\|F(1+it)\|_{A_0} = 0$.

Then

$$f \in [A_0, A_1]_\theta \iff \text{there exists } F \in \mathcal{F}(A_0, A_1) \text{ such that } f = F(\theta)$$

and

$$\|f\|_{[A_0,A_1]_\theta} = \inf_{f=F(\theta)} \max\big(\sup_{t\in\mathbb{R}}\|F(it)\|_{A_0}, \sup_{t\in\mathbb{R}}\|F(1+it)\|_{A_1}\big).$$

Now, let us recall the definition of the second interpolate $[A_0, A_1]^\theta$. Let $\mathcal{G}(A_0, A_1)$ be the space of functions G defined on the closed complex strip $\overline{\Omega}$ such that:

- $\frac{1}{1+|z|} G$ is continuous and bounded from $\overline{\Omega}$ to $A_0 + A_1$,
- G is analytic from Ω to $A_0 + A_1$,
- $t \mapsto G(it) - G(0)$ is Lipschitz from $\mathbb{R}$ to A_0,
- $t \mapsto G(1+it) - G(1)$ is Lipschitz from $\mathbb{R}$ to A_1.

Then

$$f \in [A_0, A_1]^\theta \iff \text{there exists } G \in \mathcal{G}(A_0, A_1) \text{ such that } f = G'(\theta)$$

and

$$\begin{aligned}&\|f\|_{[A_0,A_1]^\theta}\\ &= \inf_{f=G'(\theta)} \max\Big(\sup_{t_1,t_2\in\mathbb{R}}\Big\|\frac{G(it_2)-G(it_1)}{t_2-t_1}\Big\|_{A_0}, \sup_{t_1,t_2\in\mathbb{R}}\Big\|\frac{G(1+it_2)-G(1+it_1)}{t_2-t_1}\Big\|_{A_1}\Big).\end{aligned}$$

Three important properties of those complex interpolation functors are as follows:

- The equivalence theorem: if A_0 (or A_1) is reflexive, then $[A_0, A_1]^\theta = [A_0, A_1]_\theta$ for $0 < \theta < 1$.

- The duality theorem: if $A_0 \cap A_1$ is dense in A_0 and A_1, then $([A_0, A_1]_\theta)' = [A_0', A_1']^\theta$ for $0 < \theta < 1$.
- The density theorem: $A_0 \cap A_1$ is dense in $[A_0, A_1]_\theta$.

An easy classical example of interpolation concerns the Lebesgue spaces L^p on a measured space (X, μ): $[L^{p_0}, L^{p_1}]_\theta = L^p$, with $1 < p_0 < +\infty$, $1 < p_1 < +\infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Indeed, if $f \in L^p$, we write $f = F_\theta$, where $F_z(x) = |f(x)|^{(1-z)p/p_0 + zp/p_1} f(x)/|f(x)|$. If $p_0 \le p_1$, we have $|F_z(x)| \le |f(x)|^{p/p_0}$ if $|f(x)| \ge 1$. and $|F_z(x)| \le |f(x)|^{p/p_1}$ if $|f(x)| < 1$. By dominated convergence, this gives the continuity of F from $\bar{\Omega}$ to $L^{p_0} + L^{p_1}$. For the holomorphy, we use the equivalence between (strong) holomorphy and weak-* holomorphy; thus, it is enough to check that $z \in \Omega \mapsto \int F_z(x) g(x)\, d\mu$ is holomorphic if $g \in L^{q_0} \cap L^{q_1}$, where $1/q_i + 1/p_i = 1$. Thus, we obtain that $L^p \subset [L^{p_0}, L^{p_1}]_\theta$. As

$$[L^{p_0}, L^{p_1}]_\theta = [L^{p_0}, L^{p_1}]^\theta = ([L^{q_0}, L^{q_1}]_\theta)'$$

and as L^q is dense in $[L^{q_0}, L^{q_1}]_\theta$ (where $1/q + 1/p = 1$), we obtain from the embedding $L^q \subset [L^{q_0}, L^{q_1}]_\theta$ that $[L^{p_0}, L^{p_1}]_\theta \subset L^p$.

A similar result holds for weighted Lebesgue spaces $L^p(w\, d\mu)$:

$$[L^{p_0}(w_0\, d\mu), L^{p_1}(w_1\, d\mu)]_\theta = L^p(w\, d\mu),$$

with $1 < p_0 < +\infty$, $1 < p_1 < +\infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, and $w = w_0^{1-\theta} w_1^\theta$. If $f \in L^p(w\, d\mu)$, one defines

$$F_z(x) = \Big(\frac{w(x)}{w_0(x)}\Big)^{(1-z)/p_0} \Big(\frac{w(x)}{w_1(x)}\Big)^{z/p_1} |f(x)|^{(1-z)p/p_0 + zp/p_1} \frac{f(x)}{|f(x)|}.$$

We have

$$|F_z(x)| \le \max\Big(\Big(\frac{w(x)}{w_0(x)}\Big)^{1/p_0} |f(x)|^{p/p_0}, \Big(\frac{w(x)}{w_1(x)}\Big)^{1/p_1} |f(x)|^{p/p_1}\Big).$$

The proof is then similar to the case of Lebesgue spaces.

If we want to interpolate Morrey spaces $M^{p_0,q_0}(\mathbb{R}^3)$ and $M^{p_1,q_1}(\mathbb{R}^3)$ and obtain a Morrey space, then it is necessary to assume that $p_0/q_0 = p_1/q_1$, see [17, 18]. We then obtain

$$[M^{p_0,q_0}, M^{p_1,q_1}]^\theta = M^{p,q}$$

when $1 < p_0 \le q_0 < +\infty$, $1 < p_1 \le q_1 < +\infty$, $p_0/q_0 = p_1/q_1$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. As $M^{p_0,q_0} \cap M^{p_1,q_1}$ is not dense in $M^{p,q}$ and is dense in $[M^{p_0,q_0}, M^{p_1,q_1}]_\theta$, we can see that we must use the second interpolation functor. The embedding $[M^{p_0,q_0}, M^{p_1,q_1}]^\theta \subset M^{p,q}$ is obvious: for a ball B with radius $r \le 1$, we have that the map $f \mapsto f \mathbb{1}_B$ is bounded from M^{p_0,q_0} to L^{p_0} with norm less or equal to $r^{3(1/p_0 - 1/q_0)}$, and from M^{p_1,q_1} to L^{p_1} with norm less or equal to $r^{3(1/p_0 - 1/q_0)}$, hence from $[M^{p_0,q_0}, M^{p_1,q_1}]^\theta$ to $[L^{p_0}, L^{p_1}]^\theta$ with norm less or equal to $r^{3(1/p - 1/q)}$. As $[L^{p_0}, L^{p_1}]^\theta = L^p$, we obtain the desired estimates.

If f belongs to $M^{p,q}$, we define $F_z(x) = |f(x)|^{(1-z)p/p_0 + zp/p_1} f(x)/|f(x)|$. Now since $|F_z(x)| \le \max((|f(x)|^{p/p_0} |f(x)|^{p/p_1})$, we deduce that $z \mapsto F_z$ is bounded from $\bar{\Omega}$ to $M^{p_0,q_0} + M^{p_1,q_1}$ and holomorphic on the open strip Ω (again by equivalence between analyticity and weak-* analyticity). But it is no longer continuous, and we cannot apply the first functor of Calderón. Instead, we follow Cwikel and Janson [9] and define $G_z = \int_{1/2}^{z} F_w\, dw$. We may then apply the definition of the second functor and find that $f \in [M^{p_0,q_0}, M^{p_1,q_1}]^\theta$. Thus, $[M^{p_0,q_0}, M^{p_1,q_1}]^\theta = M^{p,q}$.

Now, we are going to describe complex interpolation of potential spaces on weighted Lebesgue spaces when varying both the regularity exponents and the weights.[3]

Proposition 3.2. *Let $\theta \in (0,1)$, let s_0, s_1 be real numbers, and let $1 < p_0, p_1 < +\infty$, $s = (1-\theta)s_0 + \theta s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. If w_0 is a weight in the Muckenhoupt class $\mathcal{A}_{p_0}$ and w_1 is a weight in the Muckenhoupt class $\mathcal{A}_{p_1}$, then*

$$(\mathrm{Id}-\Delta)^{-s} L^p(w_0^{1-\theta} w_1^\theta\, dx) = [(\mathrm{Id}-\Delta)^{-s_0} L^{p_0}(w_0\, dx), (\mathrm{Id}-\Delta)^{-s_1} L^{p_1}(w_1\, dx)]_\theta.$$

Proof. Let $f = (\mathrm{Id}-\Delta)^{-s} g$, where $g \in L^p(w\, dx)$. We define

$$H_z(x) = \Big(\frac{w(x)}{w_0(x)}\Big)^{(1-z)/p_0} \Big(\frac{w(x)}{w_1(x)}\Big)^{z/p_1} |f(x)|^{(1-z)p/p_0 + zp/p_1} \frac{f(x)}{|f(x)|}$$

and

$$F_{z,\varepsilon}(\cdot) = \Big(\frac{2-\theta}{2-z}\Big)^4 e^{\varepsilon\Delta} (\mathrm{Id}-\Delta)^{-(1-z)s_0 - zs_1} H_z.$$

We first remark that, for $\varepsilon > 0$ fixed, the operators $e^{\varepsilon\Delta}(\mathrm{Id}-\Delta))^{-\tau}$ with $\tau \in [s_0, s_1]$ are equicontinuous from $L^{p_i}(w_i\, dx)$ to $(\mathrm{Id}-\Delta)^{-s_i} L^{p_i}(w_i\, dx)$ (it is enough to check that the norms of the convolutors $e^{\varepsilon\Delta}(\mathrm{Id}-\Delta)^{s_i-\tau}$ in $\mathbb{K}$ are uniformly bounded.

Moreover, the operators $(\frac{2-\theta}{2-it})^4 (\mathrm{Id}-\Delta)^{-it}$, with $t \in \mathbb{R}$, are uniformly bounded on $L^{p_i}(w_i\, dx)$. Let us recall the definition of Calderón–Zygmund convolutors. A Calderón–Zygmund convolutor is a distribution $K \in \mathcal{S}'(\mathbb{R}^3)$ such that $\hat{K} \in L^\infty$ (so that the convolution with K is a bounded operator on L^2) and, when restricted to $\mathbb{R}^3 \setminus \{0\}$, K is defined by a locally Lipschitz function such that $\sup_{x\neq 0} |x|^3 |K(x)| + |x|^4 |\vec{\nabla} x| < +\infty$. The space CZ of Calderón–Zygmund convolutors is normed by

$$\|K\|_{\mathrm{CZ}} = \|\hat{K}\|_\infty + \sup_{x\neq 0} |x|^3 |K(x)| + |x|^4 |\vec{\nabla} x|.$$

If $1 < p < +\infty$, $w \in \mathcal{A}_p$ and $K \in \mathrm{CZ}$, then $\|f * K\|_{L^p(w\, dx)} \le C_{w,p} \|f\|_{L^p(w\, dx)} \|K\|_{\mathrm{CZ}}$. Since we have

$$\|K\|_{\mathrm{CZ}} \le C \sum_{|\alpha|\le 4} \| |\xi|^{|\alpha|} \partial_\xi^\alpha \hat{K} \|_\infty,$$

it is clear that $(\frac{2-\theta}{2-it})^4 (\mathrm{Id}-\Delta)^{-it} f = K_t * f$ with $\sup_{t\in\mathbb{R}} \|K_t\|_{\mathrm{CZ}} < +\infty$.

Now we may apply the second interpolation functor and find that $e^{\varepsilon\Delta} f = F_{\theta,\varepsilon} \in [(\mathrm{Id}-\Delta)^{-s_0} L^{p_0}(w_0\, dx), (\mathrm{Id}-\Delta)^{-s_1} L^{p_1}(w_1\, dx)]_\theta$ if $g \in L^p(w\, dx)$. Moreover, its norm is controlled independently from $\varepsilon > 0$, as, for $\alpha = 0$ or $\alpha = 1$, we have that the functions $H_{\alpha+it}$ are bounded in $L^{p_\alpha}(w_\alpha\, dx)$, the operators $(\frac{2-\theta}{2-it})^4 (\mathrm{Id}-\Delta)^{-it}$ are equicontinuous

[3] This can be seen as a variation on Stein's interpolation theorem [9, 24].

on $L^{p_\alpha}(w_\alpha\, dx)$ and the operators $e^{\varepsilon\Delta}$ are equicontinuous on $L^{p_\alpha}(w_\alpha\, dx)$. One then writes

$$F_{\alpha+it,\varepsilon} = (\mathrm{Id}-\Delta)^{-(1-\alpha)s_0-\alpha s_1}\Big(e^{\varepsilon\Delta}\Big(\frac{2-\theta}{2-it}\Big)^4(\mathrm{Id}-\Delta)^{-it}H_{\alpha+it}\Big).$$

To conclude, we remark that $L^{p_i}(w_i\, dx)$ is the dual of $L^{q_i}(w^{-q_i/p_i}\, dx)$ and that $\mathcal{S}$ is dense in this predual. Thus, $e^{\varepsilon\Delta} f$ is bounded in

$$\begin{aligned}&[H^{s_0}_{L^{p_0}(w_0\, dx)}, H^{s_1}_{L^{p_1}(w_1\, dx)}]^\theta\\ &\qquad = \big([(\mathrm{Id}-\Delta)^{s_0}L^{q_0}(w_0^{-q_0/p_0}\, dx), (\mathrm{Id}-\Delta)^{s_1}L^{q_1}(w_1^{-q_1/p_1}\, dx)]_\theta\big)'\end{aligned}$$

if $(\mathrm{Id}-\Delta)^s f \in L^p(w\, dx)$. As ε goes to 0, $e^{\varepsilon\Delta} f$ is weak-$*$ convergent to f. Thus,

$$(\mathrm{Id}-\Delta)^{-s}L^p(w\, dx) \subset [(\mathrm{Id}-\Delta)^{-s_0}L^{p_0}(w_0\, dx), (\mathrm{Id}-\Delta)^{-s_1}L^{p_1}(w_1\, dx)]^\theta,$$

and we can interchange the second and the first interpolation functors as $H^{s_0}_{L^{p_0}(w_0\, dx)}$ is reflexive.

Conversely, assume that $f \in [(\mathrm{Id}-\Delta)^{-s_0}L^{p_0}(w_0\, dx), (\mathrm{Id}-\Delta)^{-s_1}L^{p_1}(w_1\, dx)]^\theta$ and pick $F \in \mathcal{F}((\mathrm{Id}-\Delta)^{-s_0}L^{p_0}(w_0\, dx), (\mathrm{Id}-\Delta)^{-s_1}L^{p_1}(w_1\, dx)))$ such that $f = F(\theta)$. Set

$$H_{z,\varepsilon} = \Big(\frac{2-\theta}{2-z}\Big)^4 e^{\varepsilon\Delta}(\mathrm{Id}-\Delta)^{(1-z)s_0+zs_1}F_z.$$

We easily check that $H_{z,\varepsilon} \in \mathcal{A}(L^{p_0}(w_0\, dx), L^{p_1}(w_1\, dx))$ with $H_{\theta,\varepsilon} = e^{\varepsilon\Delta}(\mathrm{Id}-\Delta)^s f$. Thus, we find that $e^{\varepsilon\Delta}(\mathrm{Id}-\Delta)^s f$ is bounded in $[L^{p_0}(w_0\, dx), L^{p_1}(w_1\, dx)]_\theta = L^p(w\, dx)$, and finally $f \in (\mathrm{Id}-\Delta)^{-s}L^p(w\, dx)$. ∎

Corollary 3.3. *Let $2 < q < +\infty$ and $s < 1 - 2/q$.*

(a) *There exist $\gamma > 0$ and $2 < r < +\infty$ such that $\gamma + 3/r < 1$, and $\theta \in (0,1)$ such that*

$$B^{-s}_{q,\infty} \subset [L^2, H^{-\gamma}_r]_{\theta,\infty} \subset [L^2, B^{-\gamma-3/r}_{\infty,\infty}]_{\theta,\infty}.$$

(b) *For $0 \le N < 4/q$, there exist $\gamma > 0$ and $2 < r < +\infty$ such that $\gamma + 3/r < 1$, and $\theta \in (0,1)$ such that*

$$B^{-s}_{L^q((1+|x|)^{-N}dx),\infty} \subset \Big[L^2\Big(\frac{dx}{(1+|x|)^2}\Big), H^{-\gamma}_r\Big]_{\theta,\infty} \subset \Big[L^2\Big(\frac{dx}{(1+|x|)^2}\Big), B^{-\gamma-3/r}_{\infty,\infty}\Big]_{\theta,\infty}.$$

Proof. If $s < \sigma < 1-2/q$, then $B^{-s}_{q,\infty} \subset H^{-\sigma}_{L^q}$ and $B^{-s}_{L^q((1+|x|)^{-N}dx),\infty} \subset H^{-\sigma}_{L^q((1+|x|)^{-N}dx)}$. Thus, if $r > q$, then, for $\theta \in (0,1)$ and $\gamma > \sigma$ such that $(1-\theta)/2 + \theta/r = 1/q$ and $\theta\gamma = \sigma$, we have

$$B^{-s}_{q,\infty} \subset [L^2, H^{-\gamma}_r]_\theta \subset [L^2, H^{-\gamma}_r]_{\theta,\infty} \subset [L^2, B^{-\gamma-3/r}_{\infty,\infty}]_{\theta,\infty}.$$

As $\gamma + \frac{3}{r} = (1-\frac{2}{r})\frac{\sigma}{1-2/q} + \frac{3}{r} = \frac{\sigma}{1-2/q} + O(\frac{1}{r})$, we have $\gamma + \frac{3}{r} < 1$ for r large enough. Similarly, if $(1-\theta)M = N$ and $M < 2$ (so that, in particular, $(1+|x|)^{-M} \in \mathcal{A}_2$), we have

$$\begin{aligned}B^{-s}_{L^q((1+|x|)^{-N}dx),\infty} &\subset [L^2((1+|x|)^{-M}dx), H^{-\gamma}_r]_\theta \subset [L^2((1+|x|)^{-2}dx), H^{-\gamma}_r]_{\theta,\infty}\\ &\subset [L^2((1+|x|)^{-2}dx), B^{-\gamma-3/r}_{\infty,\infty}]_{\theta,\infty}.\end{aligned}$$

As $M = \frac{N}{1-\theta} = N\frac{1/2-1/r}{1/q-1/r} = \frac{qN}{2} + O(\frac{1}{r})$, we have $M < 2$ for r large enough. ∎

4. Mild solutions for the Navier–Stokes equation

In this section, we develop some remarks on the solutions provided by Koch and Tataru's theorem (Theorem 1.2 and Corollary 1.3).

Let $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$, with $\operatorname{div}\vec{u}_0 = 0$. If $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0,T)$ such that $\|\vec{u}\|_{X_T} \le 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$. This solution is computed through Picard iteration as the limit of $\vec{U}_n$, where $\vec{U}_0 = e^{t\Delta}\vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta}\vec{u}_0 - B(\vec{U}_n, \vec{U}_n)$. In particular, by induction, we have

$$\|\vec{U}_{n+1} - \vec{U}_n\|_{X_T} \le (4C_0\|e^{t\Delta}\vec{u}_0\|_{X_T})^{n+1}\|e^{t\Delta}\vec{u}_0\|_{X_T}$$

and

$$\|\vec{U}_n\|_{X_T} \le 2\|e^{t\Delta}\vec{u}_0\|_{X_T}.$$

It is easy to check that $\vec{u}$ is smooth. For $X_\alpha = L^\infty$ if $\alpha = 0$ and $\dot{B}^\alpha_{\infty,\infty}$ if $\alpha > 0$, we have

$$\|uv\|_{X_\alpha} \le C_\alpha(\|u\|_\infty\|v\|_{X_\alpha} + \|v\|_\infty\|u\|_{X_\alpha}),$$

and $\|\vec{u}\|_\infty \le 2\|e^{t\Delta}\vec{u}_0\|_{X_T}/\sqrt{t}$ for $0 < t < T$, while

$$\vec{u}(t,\cdot) = e^{t\Delta/2}\vec{u}(t/2,\cdot) - \int_0^{t/2} e^{(t/2-s)\Delta}\,\mathbb{P}\operatorname{div}(\vec{u}(t/2+s,\cdot)\otimes\vec{u}(t/2+s,\cdot))\,ds,$$

so that

$$\begin{aligned}\|\vec{u}(t,\cdot)\|_{X_{(n+1)/2}} \le{}& C\frac{1}{t^{1/4}}\|\vec{u}(t/2,\cdot)\|_{X_{n/2}} \\ &+ C\|e^{t\Delta}\vec{u}_0\|_{X_T}\int_0^{t/2}\frac{1}{(t/2-s)^{3/4}}\frac{1}{\sqrt{s}}\|\vec{u}(t/2+s,\cdot)\|_{X_{n/2}}\,ds,\end{aligned}$$

and, by induction on n,

$$\|\vec{u}(t,\cdot)\|_{X_{n/2}} \le C_n t^{-1/2-n/4}.$$

Thus, for $0 < t < T$, $\vec{u}$ is smooth with respect to the space variable x. So is $\vec{\nabla} p$, by hypoellipticity of the Laplacian (as $\Delta p = -\sum_{i=1}^3\sum_{j=1}^3 \partial_i u_j \partial_j u_i$). Then we have smoothness with respect to the time variable by controlling the time derivatives through the Navier–Stokes equations.

Proposition 4.1. *Let $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$ with $\operatorname{div}\vec{u}_0 = 0$. Let $E \subset \mathcal{S}'$ be a stable space. If, moreover, $\vec{u}_0$ belongs to E, then the small solution $\vec{u}$ to the integral Navier–Stokes equations with initial value $\vec{u}_0$, i.e., the solution on $(0,T)$ such that $\|\vec{u}\|_{X_T} \le 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$, satisfies $\sup_{0<t<T}\|\vec{u}(t,\cdot)\|_E < +\infty$ and $\lim_{t\to 0}\|\vec{u}(t,\cdot) - e^{t\Delta}\vec{u}_0\|_E = 0$. In particular, if $\mathcal{S}$ is dense in E, then $\lim_{t\to 0}\|\vec{u}(t,\cdot) - \vec{u}_0\|_E = 0$.*

Moreover, if $E \subset \mathcal{S}'$ is the dual of a space E_0 where $\mathcal{S}$ is dense,

$$\sup_{0<t<T}\sqrt{t}\,\|\vec{\nabla}\otimes\vec{u}\|_E < +\infty.$$

Proof. We have

$$\|B(\vec{u},\vec{v})(t,\cdot)\|_E \le C_E\int_0^t \frac{1}{\sqrt{t-s}}\min(\|\vec{u}\|_\infty\|\vec{v}\|_E, \|\vec{u}\|_E\|\vec{v}\|_\infty)\,ds.$$

By induction, we have $\vec{U}_n \in L^\infty((0,T),E)$ with, for $n \geq 0$ (and $\vec{U}_{-1} = 0$),

$$\begin{aligned}\|\vec{U}_{n+1}(t,\cdot) - \vec{U}_n(t,\cdot)\|_E \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} \sqrt{s} \|\vec{U}_n(s,\cdot) - \vec{U}_{n-1}(s,\cdot)\|_\infty (\|\vec{U}_n(s,\cdot)\|_E + \|\vec{U}_{n-1}(s,\cdot)\|_E)\, ds \\ &\leq C'(4C_0\|e^{t\Delta}\vec{u}_0\|_{X_T})^n \sum_{k=0}^n \|\vec{U}_k - \vec{U}_{k-1}\|_{L^\infty((0,T),E)}.\end{aligned}$$

Thus, we have

$$\sum_{k=0}^{+\infty} \|\vec{U}_k - \vec{U}_{k-1}\|_{L^\infty((0,T),E)} \leq \|\vec{U}_0\|_{L^\infty((0,T),E)} \prod_{n=0}^{\infty} (1 + C(4C_0\|e^{t\Delta}\vec{u}_0\|_{X_T})^n).$$

Thus, $\sup_{0<t<T} \|\vec{u}(t,\cdot)\|_E < +\infty$.

We have that $\sup_{t>0} \sqrt{t}\, \|\vec{\nabla} \otimes \vec{U}_0\|_E < +\infty$, and we will show by induction that $\sup_{t>0} \sqrt{t}\, \|\vec{\nabla} \otimes \vec{U}_n\|_E < +\infty$. Indeed, for $\eta \in (0,1)$ and $0 < t < T$, we have

$$\begin{aligned}\vec{U}_{n+1}(t,\cdot) &= e^{\eta t\Delta}\vec{U}_{n+1}((1-\eta)t,\cdot) \\ &\quad - \int_0^{\eta t} e^{(\eta t - s)\Delta} \mathbb{P} \operatorname{div}(\vec{U}_n((1-\eta)t+s,\cdot) \otimes \vec{U}_n((1-\eta)t+s,\cdot))\, ds\end{aligned}$$

and, since $\operatorname{div}(\vec{u} \otimes \vec{v}) = \vec{u} \cdot \vec{\nabla}\vec{v}$,

$$\begin{aligned}\partial_j \vec{U}_{n+1}(t,\cdot) &= e^{\eta t\Delta}\vec{U}_{n+1}((1-\eta)t,\cdot) \\ &\quad - \int_0^{\eta t} e^{(\eta t - s)\Delta} \mathbb{P}\, \partial_j (\vec{U}_n((1-\eta)t+s,\cdot) \cdot \vec{\nabla}\vec{U}_n((1-\eta)t+s,\cdot))\, ds.\end{aligned}$$

This gives

$$\begin{aligned}\|\vec{\nabla}\vec{U}_{n+1}(t,\cdot)\|_E &\leq C \frac{1}{\sqrt{\eta t}} \|\vec{U}_{n+1}\|_{L^\infty((0,T),E)} + C \int_0^{\eta t} \frac{1}{\sqrt{\eta t - s}} \frac{1}{(1-\eta)t + s}\, ds \\ &\quad \times \sup_{0<s<T} \sqrt{s}\, \|\vec{\nabla} \otimes \vec{U}_n(s,\cdot)\|_E \sqrt{s}\, \|\vec{U}_n(s,\cdot)\|_\infty \\ &\leq C_1 \frac{1}{\sqrt{\eta t}} + C_1 \frac{\sqrt{\eta}}{1-\eta} \frac{1}{\sqrt{t}} \sup_{0<s<T} \sqrt{s}\, \|\vec{\nabla} \otimes \vec{U}_n(s,\cdot)\|_E,\end{aligned}$$

where C_1 does not depend on n nor on η. For η small enough, we have $C_1\sqrt{\eta}/(1-\eta) < 1/4$ and $\sup_{0<s<T} \sqrt{s}\|\vec{\nabla} \otimes \vec{U}_0(s,\cdot)\|_E \leq 2C_1/\sqrt{\eta}$. By induction, we get $\sup_{0<s<T} \sqrt{s}\|\vec{\nabla} \otimes \vec{U}_n(s,\cdot)\|_E \leq 2C_1/\sqrt{\eta}$ for every $n \in \mathbb{N}$. If $E \subset \mathcal{S}'$ is the dual of a space E_0, where $\mathcal{S}$ is dense, we conclude that $\sup_{0<s<T} \sqrt{s}\|\vec{\nabla} \otimes \vec{u}(s,\cdot)\|_E < +\infty$. ■

Proposition 4.2. *Let* $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$ *with* $\operatorname{div}\vec{u}_0 = 0$. *Let* $w = (1+|x|)^{-N}$, *where* $0 \leq N < 3$. *If, moreover,* $\vec{u}_0$ *belongs to* $L^2(w\,dx)$, *then the small solution* $\vec{u}$ *to the integral Navier–Stokes equations with initial value* $\vec{u}_0$, *i.e., the solution on* $(0,T)$ *such that* $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$, *satisfies* $\vec{u} \in L^\infty((0,T), L^2(w\,dx))$ *and* $\vec{\nabla} \otimes \vec{u} \in L^2((0,T), L^2(w\,dx))$.

Proof. Let

$$\phi_R = \theta\Big(\frac{x}{R}\Big)\,\frac{1}{((1+|x|)^2)^{N/2}},$$

where $\theta \in \mathcal{D}$ is equal to 1 on a neighborhood of 0. We know that $\vec u$ is smooth, so that, for $0 < t_0 \le t < T$,

$$\partial_t(|\vec u|^2) + 2|\vec\nabla \otimes \vec u|^2 = \Delta(|\vec u|^2) - \operatorname{div}((2p + |\vec u|^2)\vec u),$$

and thus

$$\begin{aligned}\int \phi_R(x)|\vec u(t,x)|^2\,dx &+ 2\int_{t_0}^t \int \phi_R(x)|\vec\nabla \otimes \vec u(s,x)|^2\,dx\,ds\\ &= \int \phi_R(x)|\vec u(t_0,x)|^2\,dx + \int_{t_0}^t \int \Delta(\phi_R(x))|\vec u(t,x)|^2\,dx\,ds\\ &\quad + \int_{t_0}^t \int (2p + |\vec u|^2)\vec u \cdot \vec\nabla(\phi_R(x))\,dx\,ds.\end{aligned}$$

For $|\alpha| \le 2$, we have $|\partial^\alpha(\phi_R)| \le C\,w$. On the other hand, we know that $\vec u \in L^\infty(L^2(w\,dx))$ and $\sqrt{t}u_i u_j \in L^\infty(L^2(w\,dx))$, and thus $\sqrt{t}(2p + |\vec u|^2) \in L^\infty(L^2(w\,dx))$ (as $w \in \mathcal{A}_2$ and $p = -\sum_{1\le i\le 3}\sum_{j=1}^3 \frac{\partial_i\partial_j}{\Delta}(u_i u_j)$). Therefore, we get that

$$\begin{aligned}\int \phi_R(x)\,|\vec u(t,x)|^2\,dx &+ 2\int_{t_0}^t \int \phi_R(x)\,|\vec\nabla \otimes \vec u(s,x)|^2\,dx\,ds\\ &\le C \sup_{0<s<T} \int |\vec u(s,x)|^2\,w(x)\,dx + C\int_0^T \int |\vec u(s,x)|^2\,w(x)\,dx\,ds\\ &\quad + \int_0^T \int \sqrt{s}\,|2p + |\vec u|^2|\,|\vec u|\,w(x)\,dx\,\frac{ds}{\sqrt{s}} < +\infty.\end{aligned}$$

We then let R go to $+\infty$ and t_0 go to 0. ∎

5. Barker's stability theorem

In this section, we extend a lemma of Barker on Leray weak solutions with initial values in $L^2 \cap [L^2, \dot B^{-\delta}_{\infty,\infty}]_{\theta,\infty}$ (for some $\delta < 1$ and $\theta \in (0,1)$) to the case of some solutions with initial values in $L^2(w\,dx) \cap [L^2(w\,dx), H_r^{-\gamma}]_{\theta,\infty}$, where $w = (1+|x|)^{-N}$, $0 \le N \le 2$, and $\gamma + 3/r < 1$.

Definition 5.1. A weighted Leray weak solution for the Navier–Stokes equations with divergence-free initial value $\vec u_0 \in L^2(w\,dx)$, where $w = (1+|x|)^{-N}$, $0 \le N \le 2$, is a divergence-free vector field $\vec u$ defined on $(0,T) \times \mathbb{R}^3$ such that

- $\vec u \in L^\infty((0,T), L^2(w\,dx))$ and $\vec\nabla \otimes \vec u \in L^2((0,T), L^2(w\,dx))$,
- there exists $p \in \mathcal{D}'((0,T) \times \mathbb{R}^3$ such that

$$\partial_t \vec u = \Delta \vec u - \vec u \cdot \vec\nabla \vec u - \vec\nabla p,$$

- $\lim_{t\to 0}\|\vec u(t,\cdot) - \vec u_0\|_{L^2(w\,dx)} = 0$,

- $\vec{u}$ fulfills the weighted Leray inequality: for $0 < t < T$,

$$\int |\vec{u}(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla} \otimes \vec{u}(s,x)|^2\, w(x)\, dx\, ds$$
$$\leq \int |\vec{u}_0(t,x)|^2\, w(x)\, dx - 2\sum_{i=1}^{3} \int_0^t \int \partial_i w(s,x)\vec{u}(s,x) \cdot \partial_i \vec{u}(s,x)\, dx\, ds$$
$$+ \int_0^t \int (|\vec{u}(s,x)|^2 + 2p(s,x))\vec{u}(s,x) \cdot \vec{\nabla} w(x)\, dx\, ds.$$

The Navier–Stokes problem in $L^2(w\,dx)$ has recently been studied by Bradshaw, Kukavica and Tsai [2], and Fernández-Dalgo and Lemarié-Rieusset [11]. As $|\vec{\nabla} w| \leq Nw$, we find that $\sqrt{w}\,\vec{u} \in L^2((0,T), H^1)$. In particular, we have $wu_iu_j \in L^4((0,T), L^{6/5})$. The pressure p is determined by the equation $\Delta p = -\sum_{i=1}^3 \sum_{j=1}^3 u_iu_j$ (see [12]) and, as $w^{6/5} \in \mathcal{A}_{6/5}$, we have $p \in L^4((0,T), L^{6/5}(w^{6/5}\,dx))$. As $|\vec{\nabla} w| \leq Nw^{3/2}$, we see that the right-hand side of the weighted Leray inequality is well defined. As in the case of Leray solutions, the strong continuity at $t = 0$ of $t \in [0,T) \mapsto \vec{u}(t,\cdot) \in L^2(w\,dx)$ (which is only weakly continuous for $t > 0$) is a consequence of the weighted Leray inequality.

Theorem 5.2. *Let $\vec{u}_0$ be a divergence-free vector field such that $\vec{u}_0 \in L^2(w\,dx)$, where $w = (1+|x|)^{-N}$, $0 \leq N \leq 2$. Let $\vec{u}_1$, $\vec{u}_2$ be two weighted Leray weak solutions for the Navier–Stokes equations with initial value $\vec{u}_0$. If, moreover, $\vec{u}_0 \in [L^2(w\,dx), H_r^{-\gamma}]_{\theta,\infty}$ for some $\gamma > 0$, $2 < r < +\infty$ with $\gamma + 3/r < 1$ and $\theta \in (0,1)$, then there exist $T_0 > 0$, $C \geq 0$ and $\eta > 0$ such that, for $0 \leq t \leq T_0$,*

$$\|\vec{u}_1(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^2(w\,dx)} \leq Ct^{\eta}.$$

Proof. This theorem was proved by Barker [1] in the case $N = 0$. Our proof will follow the same lines as Barker's proof.

As $\vec{u}_0 \in [L^2(w\,dx), H_r^{-\gamma}]_{\theta,\infty}$, for every $\varepsilon \in (0,1)$, we may split $\vec{u}_0$ in $\vec{u}_0 = \vec{v}_{0,\varepsilon} + \vec{w}_{0,\varepsilon}$ with $\|\vec{v}_{0,\varepsilon}\|_{H_r^{-\gamma}} \leq C_1\varepsilon^{\theta-1}$ and $\|\vec{w}_{0,\varepsilon}\|_{L^2(w\,dx)} \leq C_1\varepsilon^{\theta}$, where C_1 depends only on $\vec{u}_0$. As $\vec{u}_0 = \mathbb{P}\vec{u}_0$ and as $\mathbb{P}$ is continuous on $H_r^{-\gamma}$ and on $L^2(w\,dx)$, we may assume (changing the value of the constant C_1) that $\vec{v}_{0,\varepsilon}$ and $\vec{w}_{0,\varepsilon}$ are divergence free. Let $\delta = \gamma + 3/r < 1$. Since $H_r^{-\gamma} \subset B^{-\delta}_{\infty,\infty}$, for $0 < t \leq 1$, we have $\|e^{t\Delta}\vec{v}_{0,\varepsilon}\|_\infty \leq C_2 t^{-\delta/2}\varepsilon^{\theta-1}$. If $0 < T_1 < 1$, we have

$$\sup_{0<t<T_1} \sqrt{t}\,\|e^{t\Delta}\vec{v}_{0,\varepsilon}\|_\infty \leq C_2\,\varepsilon^{\theta-1}\,T_1^{1-\delta/2}$$

and

$$\sup_{0<t<T_1, x\in\mathbb{R}^3} \sqrt{\frac{1}{t^{3/2}} \int_0^t \int_{B(x,\sqrt{t})} |e^{t\Delta}\vec{v}_{0,\varepsilon}|^2\, dx} \leq C_3\,\varepsilon^{\theta-1}\,T_1^{1-\delta/2},$$

so that

$$\|e^{t\Delta}\vec{v}_{0,\varepsilon}\|_{X_{T_1}} \leq (C_2 + C_3)\,\varepsilon^{\theta-1}T_1^{1-\delta/2} < \frac{1}{8C_0} \quad \text{if } T_1 < \min(1, C_4\,\varepsilon^{2/1-\delta(1-\theta)}).$$

By (the proof of) Theorem 2.4, we know that the Navier–Stokes equations with initial value $\vec{v}_{0,\varepsilon}$ will have a solution $\vec{v}_\varepsilon$ on $(0,T_1)$ such that $\|\vec{v}_\varepsilon(t,\cdot)\|_\infty \leq C_5 t^{-\delta/2}\,\varepsilon^{\theta-1}$. Moreover, by Proposition 4.1, $\vec{v}_\varepsilon$ is a weighted Leray weak solution.

Let $\vec{u}$ be a weighted Leray solution on $(0, T)$ for the Navier–Stokes equations with initial value $\vec{u}_0$. We are going to compare $\vec{u}$ and $\vec{v}_\varepsilon$. We know that $\vec{v}_\varepsilon$ is smooth, so that

$$\partial_t(\vec{u}\cdot\vec{v}_\varepsilon) = \vec{u}\cdot\partial_t\vec{v}_\varepsilon + \vec{v}_\varepsilon\cdot\partial_t\vec{u}.$$

If p_ε is the pressure associated to $\vec{v}_\varepsilon$, then on $(0, T_2)$, where $T_2 = \min(T, T_1)$, we have

$$\begin{aligned}\partial_t(\vec{u}\cdot\vec{v}_\varepsilon) &= \vec{u}\cdot\Delta\vec{v}_\varepsilon + \vec{v}_\varepsilon\cdot\Delta\vec{u} - \operatorname{div}(p_\varepsilon\vec{u} + p\vec{v}_\varepsilon) - \vec{u}\cdot(\vec{v}_\varepsilon\cdot\vec{\nabla}\vec{v}_\varepsilon) - \vec{v}_\varepsilon\cdot(\vec{u}\cdot\vec{\nabla}\vec{u})\\ &= \vec{u}\cdot\Delta\vec{v}_\varepsilon + \vec{v}_\varepsilon\cdot\Delta\vec{u} - \operatorname{div}(p_\varepsilon\vec{u} + p\vec{v}_\varepsilon) - (\vec{u}-\vec{v}_\varepsilon)\cdot(\vec{v}_\varepsilon\cdot\vec{\nabla}\vec{v}_\varepsilon)\\ &\quad - \vec{v}_\varepsilon\cdot(\vec{u}\cdot\vec{\nabla}(\vec{u}-\vec{v}_\varepsilon)) - \operatorname{div}\Big(\frac{|\vec{v}_\varepsilon|^2}{2}(\vec{u}+\vec{v}_\varepsilon)\Big)\\ &= \vec{u}\cdot\Delta\vec{v}_\varepsilon + \vec{v}_\varepsilon\cdot\Delta\vec{u} - \vec{v}_\varepsilon\cdot((\vec{u}-\vec{v}_\varepsilon)\cdot\vec{\nabla}(\vec{u}-\vec{v}_\varepsilon))\\ &\quad - \operatorname{div}\Big(p_\varepsilon\vec{u} + p\vec{v}_\varepsilon + \frac{|\vec{v}_\varepsilon|^2}{2}(\vec{u}+\vec{v}_\varepsilon) + (\vec{v}_\varepsilon\cdot(\vec{u}-\vec{v}_\varepsilon))\vec{v}_\varepsilon\Big).\end{aligned}$$

Since $\vec{v}_\varepsilon \in L^2((0, T_2), L^\infty)$, this can be integrated on $(0, t)\times\mathbb{R}^3$ against the measure $w(x)\,dx\,ds$, giving

$$\begin{aligned}&\int \vec{u}\cdot\vec{v}_\varepsilon\, w(x)\,dx - \int \vec{u}_0\cdot\vec{v}_{0,\varepsilon}\, w(x)\,dx\\ &\quad= -\int_0^t\int\sum_{i=1}^3 \partial_i w(x)(\vec{u}(s,x)\cdot\partial_i\vec{v}_\varepsilon(s,x) + \vec{v}_\varepsilon(s,x)\cdot\partial_i\vec{u}(s,x))\,dx\,ds\\ &\quad\quad -2\int_0^t\int(\vec{\nabla}\otimes\vec{u}(s,x)\cdot\vec{\nabla}\otimes\vec{v}_\varepsilon(s,x))\,w(x)\,dx\,ds\\ &\quad\quad -\int_0^t\int \vec{v}_\varepsilon(s,x)\cdot((\vec{u}(s,x)-\vec{v}_\varepsilon(s,x))\cdot\vec{\nabla}(\vec{u}(s,x)-\vec{v}_\varepsilon(s,x)))\,w(x)\,dx\,ds\\ &\quad\quad +\int_0^t\int p(s,x)\,\vec{v}_\varepsilon(s,x)\cdot\vec{\nabla}w(x) + p_\varepsilon(s,x)\vec{u}(s,x)\cdot\vec{\nabla}w(x)\,dx\,ds\\ &\quad\quad +\int_0^t\int\frac{|\vec{v}_\varepsilon(s,x)|^2}{2}\,(\vec{u}(s,x)-\vec{v}_\varepsilon(s,x))\cdot\vec{\nabla}w(x)\\ &\quad\quad +(\vec{v}_\varepsilon(s,x)\cdot\vec{u}(s,x))\vec{v}_\varepsilon(s,x)\cdot\vec{\nabla}w(x)\,dx\,ds.\end{aligned}$$

Together with

$$\begin{aligned}&\int|\vec{u}(t,x)|^2\,w(x)\,dx + 2\int_0^t\int|\vec{\nabla}\otimes\vec{u}(s,x)|^2\,w(x)\,dx\,ds\\ &\quad\le\int|\vec{u}_0(t,x)|^2\,w(x)\,dx - 2\sum_{i=1}^3\int_0^t\int\partial_i w(s,x)\vec{u}(s,x)\cdot\partial_i\vec{u}(s,x)\,dx\,ds\\ &\quad\quad+\int_0^t\int(|\vec{u}(s,x)|^2 + 2p(s,x))\,\vec{u}(s,x)\cdot\vec{\nabla}w(x)\,dx\,ds\end{aligned}$$

and

$$\int |\vec{v}_\varepsilon(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla}\otimes \vec{v}_\varepsilon(s,x)|^2\, w(x)\, dx\, ds$$
$$= \int |\vec{v}_{0,\varepsilon}(t,x)|^2\, w(x)\, dx - 2\sum_{i=1}^3 \int_0^t \int \partial_i w(s,x)\vec{v}_\varepsilon((s,x)\cdot \partial_i \vec{v}_\varepsilon(s,x)\, dx\, ds$$
$$+ \int_0^t \int (|\vec{v}_\varepsilon((s,x)|^2 + 2p_\varepsilon(s,x))\, \vec{v}_\varepsilon(s,x)\cdot \vec{\nabla} w(x)\, dx\, ds,$$

this gives

$$\int |\vec{v}_\varepsilon(t,x) - \vec{u}(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla}\otimes (\vec{v}_\varepsilon - \vec{u})|^2\, w(x)\, dx\, ds$$
$$\leq \int |\vec{v}_{0,\varepsilon} - \vec{u}_0|^2\, w(x)\, dx - 2\sum_{i=1}^3 \int_0^t \int \partial_i w(\vec{v}_\varepsilon - \vec{u})\cdot \partial_i(\vec{v}_\varepsilon - \vec{u})\, dx\, ds$$
$$+ 2\int_0^t \int (p_\varepsilon - p)(\vec{v}_\varepsilon - \vec{u})\cdot \vec{\nabla} w\, dx\, ds$$
$$- 2\int_0^t \int \vec{v}_\varepsilon \cdot ((\vec{u} - \vec{v}_\varepsilon)\cdot \vec{\nabla}(\vec{u} - \vec{v}_\varepsilon))\, w\, dx\, ds$$
$$+ \int_0^t \int |\vec{v}_\varepsilon - \vec{u}|^2 \vec{v}_\varepsilon \cdot \vec{\nabla} w + (|\vec{u}|^2 - |\vec{v}_\varepsilon|^2)(\vec{u} - \vec{v}_\varepsilon)\cdot \vec{\nabla} w\, dx\, ds.$$

Thus, we have

$$\int |\vec{v}_\varepsilon(t,x) - \vec{u}(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla}\otimes (\vec{v}_\varepsilon - \vec{u})|^2\, w(x)\, dx\, ds$$
$$\leq \int |\vec{v}_{0,\varepsilon} - \vec{u}_0|^2\, w(x)\, dx + C_6 \int_0^t \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_2\, \|\sqrt{w}\,\vec{\nabla}(\vec{u} - \vec{v}_\varepsilon)\|_2\, ds$$
$$+ C_6 \int_0^t \|(p - p_\varepsilon)\, w\|_{L^{6/5}}\, \|\sqrt{w}\,(\vec{v}_\varepsilon - \vec{u})\|_6\, ds$$
$$+ C_6 \int_0^t \|\vec{v}_\varepsilon\|_\infty\, \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_2\, \|\sqrt{w}\,\vec{\nabla}\otimes (\vec{u} - \vec{v}_\varepsilon)\|_2\, ds$$
$$+ C_6 \int_0^t \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_3^2\, (\|\sqrt{w}\,\vec{u}\|_3 + \|\sqrt{w}\,\vec{v}_\varepsilon\|_3)\, ds.$$

We have

$$\|w(p - p_\varepsilon)\|_{6/5} \leq C_7\, \|w(\vec{u}\otimes \vec{u} - \vec{v}_\varepsilon \otimes \vec{v}_\varepsilon\|_{6/5}$$
$$\leq C_7\, \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_2\, (\|\sqrt{w}\,\vec{u}\|_3 + \|\sqrt{w}\,\vec{v}_\varepsilon\|_3),$$
$$\|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_3^2 \leq \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_2\, \|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_6$$

and

$$\|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_6 \leq C_8\big(\|\sqrt{w}\,(\vec{u} - \vec{v}_\varepsilon)\|_2 + \|\sqrt{w}\,\vec{\nabla}\otimes (\vec{u} - \vec{v}_\varepsilon)\|_2\big),$$

so that

$$\|\sqrt{w}\,(\vec u(t,\cdot)-\vec v_\varepsilon(t,\cdot))\|_2^2+2\int_0^t\int\|\sqrt{w}\,\vec\nabla\otimes(\vec v_\varepsilon-\vec u)\|_2^2\,ds$$

$$\begin{aligned}
&\le\|\sqrt{w}\,(\vec v_{0,\varepsilon}-\vec u_0)\|_2^2+C_9\int_0^t\|\sqrt{w}\,(\vec u-\vec v_\varepsilon)\|_2\,\|\sqrt{w}\,\vec\nabla(\vec u-\vec v_\varepsilon)\|_2\,ds\\
&\quad+C_9\int_0^t(\|\sqrt{w}\,(\vec u-\vec v_\varepsilon)\|_2+\|\sqrt{w}\,\vec\nabla\otimes(\vec u-\vec v_\varepsilon)\|_2)\,\|\sqrt{w}\,(\vec u-\vec v_\varepsilon)\|_2\\
&\qquad\times(\|\sqrt{w}\,\vec u\|_3+\|\sqrt{w}\,\vec v_\varepsilon\|_3)\,ds\\
&\quad+C_9\int_0^t\|\vec v_\varepsilon\|_\infty\,\|\sqrt{w}\,(\vec u-\vec v_\varepsilon)\|_2\,\|\sqrt{w}\,\vec\nabla\otimes(\vec u-\vec v_\varepsilon)\|_2\,ds\\
&\le\|\sqrt{w}\,(\vec v_{0,\varepsilon}-\vec u_0)\|_2^2+\int_0^t\|\sqrt{w}\,\vec\nabla\otimes(\vec v_\varepsilon-\vec u)\|_2^2\,ds\\
&\quad+C_{10}\int_0^t\|\sqrt{w}\,(\vec u-\vec v_\varepsilon)\|_2^2\,(1+\|\sqrt{w}\,\vec u\|_3^2+\|\sqrt{w}\,\vec v_\varepsilon\|_3^2+\|\vec v_\varepsilon\|_\infty^2)\,ds.
\end{aligned}$$

By Gronwall's lemma, for $0<t<T_2$, we have

$$\begin{aligned}
&\|\sqrt{w}\,(\vec u(t,\cdot)-\vec v_\varepsilon(t,\cdot))\|_2^2\\
&\qquad\le\|\sqrt{w}\,(\vec v_{0,\varepsilon}-\vec u_0)\|_2^2\,\exp\Big(\int_0^{T_2}C_{10}(1+\|\sqrt{w}\,\vec u\|_3^2+\|\sqrt{w}\,\vec v_\varepsilon\|_3^2+\|\vec v_\varepsilon\|_\infty^2)\,ds\Big).
\end{aligned}$$

Since $T_2\le T$,

$$\int_0^{T_2}\|\sqrt{w}\,\vec u\|_3^2\,ds\le\int_0^T\|\sqrt{w}\,\vec u\|_3^2\,ds<+\infty,$$

and, by Propositions 4.1 and 4.2,

$$\int_0^{T_2}\|\sqrt{w}\,\vec v_\varepsilon\|_3^2\,ds\le\int_0^{T_1}\|\sqrt{w}\,\vec v_\varepsilon\|_3^2\,ds\le C_{11}\|\vec u_0\|_{L^2(w\,dx)}^2.$$

Finally, we have

$$\int_0^{T_2}\|\vec v_\varepsilon\|_\infty^2\,ds\le C_{12}\int_0^{T_1}t^{-\delta}\|\vec v_{0,\varepsilon}\|_{B^{-\delta}_{\infty,\infty}}^2\,dt\le C_{13}\,T_1^{1-\delta}\,\varepsilon^{2(\theta-1)}\le C_{14}.$$

Thus, we have

$$\|\sqrt{w}\,(\vec u(t,\cdot)-\vec v_\varepsilon(t,\cdot))\|_2^2\le C_{15}\,\varepsilon^{2\theta},$$

where C_{15} depends only on $\vec u$ and $\vec u_0$.

We may now estimate $\|\vec u_1(t,\cdot)-\vec u_2(t,\cdot)\|_{L^2(w\,dx)}$ for two weighted Leray weak solutions defined on $(0,T)$. If $t\in(0,T)$, we define $\varepsilon=(\frac{4}{C_4}t)^{1-\delta/(2(1-\theta))}$ and $T_3=\frac12 C_4\,\varepsilon^{2(1-\theta)/(1-\delta)}=2t$. If t is small enough, we have $0<\varepsilon<1$ and $T_3<\min(1,T)$. Thus, we know that, for a constant C that depends only on $\vec u_1$, $\vec u_2$ and $\vec u_0$,

$$\begin{aligned}
\|\vec u_1(t,\cdot)-\vec u_2(t,\cdot)\|_{L^2(w\,dx)}&\le\|\vec u_1(t,\cdot)-\vec v_\varepsilon(t,\cdot)\|_{L^2(w\,dx)}+\|\vec v_\varepsilon(t,\cdot)-\vec u_2(t,\cdot)\|_{L^2(w\,dx)}\\
&\le C\,\varepsilon^\theta=C\Big(\frac{4}{C_4}t\Big)^{\theta\frac{1-\delta}{2(1-\theta)}}.
\end{aligned}$$

The theorem is proved. ∎

6. Weak-strong uniqueness

Proof of Theorem 1.8. Recall that we consider two solutions $\vec{u}$, $\vec{v}$ of the Navier–Stokes equations on $(0, T)$ with the same initial value $\vec{u}_0$ such that:

- $\vec{u}_0$ is a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \mathrm{bmo}_0^{-1}$,
- $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$,
- $\vec{u}$ is the mild solution of the Navier–Stokes equations with initial value $\vec{u}_0$ such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$,
- for some $N \geq 0$, $2 < q < +\infty$ and $0 \leq s < 1 - 2/q$,

$$\sup_{0<t<T} t^{s/2} \|\vec{u}\|_{L^q((1+|x|)^{-N}\,dx)} < +\infty,$$

- $\vec{v}$ is a suitable weak Leray solution of the Navier–Stokes equations.

We know, by Propositions 4.1 and 4.2, that the mild solution $\vec{u}$ is a suitable weak Leray solution. In particular, we have $\sup_{0<t<T} \|\vec{u}(t,\cdot)\|_2 < +\infty$ and $\sup_{0<t<T} t^{1/2}\|\vec{u}(t,\cdot)\|_\infty \leq \|\vec{u}\|_{X_T} < +\infty$. Thus,

$$\sup_{0<s<T} t^{1/2-1/q} \|\vec{u}\|_q < +\infty.$$

If $0 \leq \alpha \leq 1$, we find that

$$\sup_{0<t<T} (\sqrt{t})^{(1-\alpha)(1-2/q)+\alpha s} \|\vec{u}\|_{L^q((1+|x|)^{-\alpha N}\,dx)} < +\infty.$$

By Theorem 2.4, we find that

$$\vec{u}_0 \in B^{-s_\alpha}_{L^q((1+|x|)^{-\alpha N}\,dx),\infty}, \quad \text{with } s_\alpha = (1-\alpha)(1-2/q) + \alpha s.$$

For $0 < \alpha < \min(1, \frac{4}{Nq})$, we have $0 < s_\alpha < 1 - 2/q$ and $\alpha N < 4/q$, so that we may apply Corollary 3.3.

The next step is to check that $\vec{u}$ and $\vec{v}$, which are suitable Leray weak solutions, are weighted Leray weak solutions, for the weight $w(x) = (1 + |x|)^{-2}$. This means that we must check that $\vec{v}$ (and $\vec{u}$) fulfills the weighted Leray energy inequality. We consider a non-negative function $\theta \in \mathcal{D}(\mathbb{R}^3)$ equal to 1 on a neighborhood of 0 and 0 for $|x| \geq 1$, and a function α smooth on $\mathbb{R}$ such that $0 \leq \alpha \leq 1$, with $\alpha(t)$ equal to 0 on $(\infty, 0)$ and 1 on $(1, +\infty)$. For $0 < t_0 < t_1 < T$, $R > 0$ and $0 < \varepsilon < \min(t_1 - t_0, T - t_1)$, we define the test function

$$\begin{aligned} \varphi_{t_0,t_1,\varepsilon,R}(t,x) &= \alpha\Big(\frac{t-t_0}{\varepsilon}\Big)\Big(1 - \alpha\Big(\frac{t-t_1}{\varepsilon}\Big)\Big)\frac{1}{(1+\sqrt{1/R^2+x^2})^2}\,\theta\Big(\frac{x}{R}\Big) \\ &= \alpha_{t_0,t_1,\varepsilon}(t)\,\theta_R(x), \end{aligned}$$

which is non-negative and supported in $[t_0, t_1 + \varepsilon] \times \overline{B(0,R)}$. If q is the pressure associated to the solution $\vec{v}$, by the suitability of $\vec{v}$, we have

$$\iint \varphi_{t_0,t_1,\varepsilon,R}\big(\partial_t(|\vec{v}|^2) + 2|\vec{\nabla} \otimes \vec{v}|^2 - \Delta(|\vec{v}|^2) + \mathrm{div}((2q + |\vec{v}|^2)\vec{v})\big)\,dx\,dt \leq 0.$$

As, for $R \geq 1$, $|\theta_R| \leq C w$ and $|\vec{\nabla}\theta_R| \leq C w^{3/2}$, dominated convergence when $R \to +\infty$ gives us

$$\iint \Big(\frac{1}{\varepsilon}\alpha'\Big(\frac{t-t_1}{\varepsilon}\Big) - \frac{1}{\varepsilon}\alpha'\Big(\frac{t-t_0}{\varepsilon}\Big)\Big)|\vec{v}|^2\, w(x)\,dx\,dt + 2\iint \alpha_{t_0,t_1,\varepsilon}|\vec{\nabla}\otimes\vec{v}|^2\, w(x)\,dx\,dt$$
$$\leq -2\sum_{i=1}^{3}\iint \alpha_{t_0,t_1,\varepsilon}\,\partial_i w(\vec{v}\cdot\partial_i\vec{v})\,dx\,dt + \iint \alpha_{t_0,t_1,\varepsilon}(|\vec{v}|^2+2q)\vec{v}\cdot\vec{\nabla}w\,dx\,dt.$$

If ε goes to 0, we get

$$\limsup_{\varepsilon\to 0}\int \Big(\frac{1}{\varepsilon}\alpha'\Big(\frac{s-t_1}{\varepsilon}\Big) - \frac{1}{\varepsilon}\alpha'\Big(\frac{s-t_0}{\varepsilon}\Big)\Big)\Big(\int |\vec{v}(s,x)|^2\, w(x)\,dx\Big)\,ds$$
$$+ 2\int_{t_0}^{t_1}\int |\vec{\nabla}\otimes\vec{v}|^2\, w(x)\,dx\,ds$$
$$\leq -2\sum_{i=1}^{3}\int_{t_0}^{t_1}\int \partial_i w(\vec{v}\cdot\partial_i\vec{v})\,dx\,ds + \int_{t_0}^{t_1}\int (|\vec{v}|^2+2q)\vec{v}\cdot\vec{\nabla}w\,dx\,ds.$$

For almost every t_0, t_1, we have that t_0 and t_1 are Lebesgue points of the map

$$s \mapsto \int |\vec{v}(s,x)|^2\, w(x)\,dx,$$

so that

$$\lim_{\varepsilon\to 0}\int \Big(\frac{1}{\varepsilon}\alpha'\Big(\frac{s-t_1}{\varepsilon}\Big) - \frac{1}{\varepsilon}\alpha'\Big(\frac{s-t_0}{\varepsilon}\Big)\Big)\Big(\int |\vec{v}(s,x)|^2\, w(x)\,dx\Big)\,ds$$
$$= \int |\vec{v}(t_1,x)|^2\, w(x)\,dx - \int |\vec{v}(t_0,x)|^2\, w(x)\,dx.$$

If t_0 goes to 0 and t_1 goes to t, we have

$$\|\vec{v}(t_0,\cdot) - \vec{u}_0\|_{L^2(w\,dx)} \leq \|\vec{v}(t_0,\cdot) - \vec{u}_0\|_2 \to 0,$$

so that

$$\lim_{t_0\to 0}\int |\vec{v}(t_0,x)|^2\, w(x)\,dx = \int |\vec{u}_0(x)|^2\, w(x)\,dx,$$

while $\vec{v}(t_1,\cdot)$ is weakly convergent to $\vec{v}(t,\cdot)$, so that

$$\int |\vec{v}(t,x)|^2\, w(x)\,dx \leq \liminf_{t_1\to t}\int |\vec{v}(t_1,x)|^2\, w(x)\,dx.$$

Thus, we get the weighted Leray energy inequality.

By Theorem 5.2, we then know that there exists $T_0 > 0$, $C \geq 0$ and $\eta > 0$ such that, for $0 \leq t \leq T_0$,

$$\|\vec{u}(t,\cdot) - \vec{v}(t,\cdot)\|_{L^2(w\,dx)} \leq C t^{\eta}.$$

Moreover, we can do the same computations as in the proof of Theorem 5.2 in order to estimate $\partial_t(\vec{u}\cdot\vec{v})$ (since $\vec{u}$ is smooth) and write, if p is the pressure associated to $\vec{u}$ and q

the pressure associated to $\vec{v}$,

$$\partial_t(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\Delta\vec{v} + \vec{v}\cdot\Delta\vec{u} - \vec{u}\cdot((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v})) - \operatorname{div}\Big(q\vec{u} + p\vec{v} + \frac{|\vec{u}|^2}{2}(\vec{u}+\vec{v}) + (\vec{u}\cdot(\vec{v}-\vec{u}))\vec{u}\Big).$$

As $\vec{u}\in L^2((\varepsilon,T),L^\infty)$ for every $\varepsilon>0$, this can be integrated on $(\varepsilon,t)\times\mathbb{R}^3$ against the measure $w(x)\,dx\,ds$ and gives

$$\begin{aligned}\int \vec{u}(t,x)\cdot\vec{v}(t,x)\,w(x)\,dx - \int \vec{u}(\varepsilon,x)\cdot\vec{v}(\varepsilon,x))\,w(x)\,dx\\ = -\int_\varepsilon^t\int\sum_{i=1}^3 \partial_i w(\vec{u}\cdot\partial_i\vec{v} + \vec{v}\cdot\partial_i\vec{u})\,dx\,ds\\ -2\int_\varepsilon^t\int(\vec{\nabla}\otimes\vec{u}\cdot\vec{\nabla}\otimes\vec{v})\,w(x)\,dx\,ds\\ -\int_\varepsilon^t\int\vec{u}\cdot((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v}))\,w(x)\,dx\,ds\\ +\int_\varepsilon^t\int p\vec{v}\cdot\vec{\nabla}w + q\vec{u}\cdot\vec{\nabla}w\,dx\,ds\\ +\int_\varepsilon^t\int\frac{|\vec{u}|^2}{2}(\vec{v}-\vec{u})\cdot\vec{\nabla}w + (\vec{v}\cdot\vec{u})\vec{u}\cdot\vec{\nabla}w(x)\,dx\,ds.\end{aligned}$$

As $\vec{u}(\varepsilon,\cdot)$ and $\vec{v}(\varepsilon,\cdot)$ are strongly convergent to $\vec{u}_0$ in $L^2(w\,dx)$, we find

$$\begin{aligned}\int \vec{u}(t,x)\cdot\vec{v}(t,x)\,w(x)\,dx - \int \vec{u}_0\cdot\vec{u}_0\,w(x)\,dx\\ = -\int_0^t\int\sum_{i=1}^3 \partial_i w(\vec{u}\cdot\partial_i\vec{v} + \vec{v}\cdot\partial_i\vec{u})\,dx\,ds\\ -2\int_0^t\int(\vec{\nabla}\otimes\vec{u}\cdot\vec{\nabla}\otimes\vec{v})\,w\,dx\,ds\\ -\lim_{\varepsilon\to0}\int_\varepsilon^t\int\vec{u}\cdot((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v}))w\,dx\,ds\\ +\int_0^t\int p\vec{v}\cdot\vec{\nabla}w + q\vec{u}\cdot\vec{\nabla}w\,dx\,ds\\ +\int_0^t\int\frac{|\vec{u}|^2}{2}(\vec{v}-\vec{u})\cdot\vec{\nabla}w + (\vec{v}\cdot\vec{u})\vec{u}\cdot\vec{\nabla}w\,dx\,ds.\end{aligned}$$

We have

$$\begin{aligned}\lim_{\varepsilon\to0}\int_\varepsilon^t\int\vec{u}\cdot((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v}))w\,dx\,ds\\ = \int_0^t\int s^\eta\vec{u}\cdot s^{-\eta}((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v}))w\,dx\,ds,\end{aligned}$$

as $s^\eta\vec{u}\in L^2L^\infty$, $s^{-\eta}(\vec{u}-\vec{v})\in L^\infty(L^2(w\,dx))$ and $\vec{\nabla}\otimes(\vec{u}-\vec{v})\in L^2(L^2(w\,dx))$.

Using now the weighted Leray inequalities on $\vec{v}$ and on $\vec{u}$, we get

$$\begin{aligned}\int |\vec{v}(t,x)-\vec{u}(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\, w\, dx\, ds \\ \leq -2\sum_{i=1}^{3}\int_0^t \int \partial_i w(\vec{v}-\vec{u})\cdot\partial_i(\vec{v}-\vec{u})\, dx\, ds \\ + 2\int_0^t \int (q-p)(\vec{v}-\vec{u})\cdot\vec{\nabla} w\, dx\, ds - 2\int_0^t \int \vec{u}\cdot((\vec{u}-\vec{v})\cdot\vec{\nabla}(\vec{u}-\vec{v}))w\, dx\, ds \\ + \int_0^t \int |\vec{v}-\vec{u}|^2\vec{u}\cdot\vec{\nabla} w + (|\vec{u}|^2-|\vec{v}|^2)(\vec{u}-\vec{v})\cdot\vec{\nabla} w\, dx\, ds,\end{aligned}$$

and thus

$$\begin{aligned}\int |\vec{v}(t,x)-\vec{u}(t,x)|^2\, w(x)\, dx + 2\int_0^t \int |\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\, w\, dx\, ds \\ \leq C\int_0^t \|\sqrt{w}\,(\vec{u}-\vec{v})\|_2 \|\sqrt{w}\,\vec{\nabla}(\vec{u}-\vec{v})\|_2\, ds \\ + C\int_0^t \|(p-q)\, w\|_{L^{6/5}} \|\sqrt{w}\,(\vec{v}-\vec{u})\|_6\, ds \\ + C\int_0^t \|\vec{u}\|_\infty \|\sqrt{w}\,(\vec{u}-\vec{v})\|_2 \|\sqrt{w}\,\vec{\nabla}\otimes(\vec{u}-\vec{v})\|_2\, ds \\ + C\int_0^t \|\sqrt{w}(\vec{u}-\vec{v})\|_3^2(\|\sqrt{w}\,\vec{u}\|_3 + \|\sqrt{w}\,\vec{v}\|_3)\, ds.\end{aligned}$$

At this point, we get

$$\begin{aligned}\|\sqrt{w}\,(\vec{u}(t,\cdot)-\vec{v}(t,\cdot))\|_2^2 + 2\int_0^t \int \|\sqrt{w}\,\vec{\nabla}\otimes(\vec{v}-\vec{u})\|_2^2\, ds \\ \leq \int_0^t \|\sqrt{w}\,\vec{\nabla}\otimes(\vec{v}-\vec{u})\|_2^2\, ds \\ + C\int_0^t \|\sqrt{w}\,(\vec{u}-\vec{v})\|_2^2(1 + \|\sqrt{w}\,\vec{u}\|_3^2 + \|\sqrt{w}\,\vec{v}\|_3^2 + \|\vec{u}\|_\infty^2)\, ds.\end{aligned}$$

Let

$$A(t) = t^{-2\eta}\|\sqrt{w}\,(\vec{u}(t,\cdot)-\vec{v}(t,\cdot))\|_2^2$$

and

$$B(t) = 1 + \|\sqrt{w}\,\vec{u}\|_3^2 + \|\sqrt{w}\,\vec{v}\|_3^2.$$

We have

$$A(t) \leq C\int_0^t A(s)B(s)\, ds + C t^{-2\eta}\int_0^t A(s) s^{2\eta}\,\|\vec{u}\|_\infty^2\, ds.$$

Thus, for $0 < t < \tau < T$,

$$A(t) \leq C\sup_{0<s<\tau} A(s)\Big(\int_0^\tau B(s)\, ds + \frac{1}{2\eta}\sup_{0<s<\tau} s\|\vec{u}(s,\cdot)\|_\infty^2\Big).$$

For τ small enough, we have

$$C\Big(\int_0^\tau B(s)\,ds + \frac{1}{2\eta}\sup_{0<s<\tau} s\|\vec{u}(s,\cdot)\|_\infty^2\Big) < 1,$$

and thus $\sup_{0<t<\tau} A(t) = 0$. We conclude that $\vec{u} = \vec{v}$ on $[0, \tau]$. Since $\vec{u}$ is bounded on $[\tau, T]$, the uniqueness is easily extended to the whole interval $[0, T]$. ■

7. Further comments on Barker's conjecture

In his paper [1], Barker raised the following question.

Question 7.1. If $\vec{u}_0$ belongs to $L^2 \cap \mathrm{bmo}_0^{-1}$, does there exists a positive time T such that every weak Leray solution of the Cauchy problem for the Navier–Stokes equations with $\vec{u}_0$ as initial value coincide with the mild solution in X_T?

This can be seen as the endpoint case of the Prodi–Serrin weak-strong uniqueness criterion, as the assumption of Prodi–Serrin's criterion, i.e., existence of a solution $\vec{u}$ such that

$$\vec{u} \in L_t^p L_x^q, \quad \text{with } \frac{2}{p} + \frac{3}{q} \le 1 \text{ and } 2 \le p \le +\infty,$$

is equivalent, if $2 < p < +\infty$, to the fact that $\vec{u}_0$ belongs to $B_{q,p}^{-1+3/q} \subset \mathrm{bmo}_0^{-1}$. Existence of a mild solution when $\vec{u}_0$ belongs to $B_{q,p}^{-1+3/q}$ goes back to the paper of Fabes, Jones and Rivière [10]. Existence of mild solutions has been extended by Cannone [5] to the case of $B_{q,\infty}^{-1+3/q} \cap \mathrm{bmo}_0^{-1}$, and Koch and Tataru's theorem [14] can be seen as the endpoint case of the theory for existence of mild solutions.

Barker [1] extended weak-strong uniqueness to the case $B_{q,\infty}^{-1+3/q} \cap \mathrm{bmo}_0^{-1}$, and he could even relax the regularity exponent and consider the case $B_{q,\infty}^{-s} \cap \mathrm{bmo}_0^{-1}$, $s < 1-2/q$. We have shown that the integrability could even be relaxed into $B_{L^q((1+|x|)^{-N}\,dx),\infty}^{-1+3/q} \cap \mathrm{bmo}_0^{-1}$ with $N \ge 0$ and $s < 1 - 2/q$. But under the sole assumption $\vec{u}_0 \in L^2 \cap \mathrm{bmo}_0^{-1}$, weak-strong uniqueness remains an open question.

An alternative way to study the problem is to impose restrictions on the class of solutions, beyond the Leray energy inequality or the local Leray energy inequality. One may for instance consider an approximation process that provides weak Leray solutions when $\vec{u}_0 \in L^2$ and consider whether the solutions provided by this process coincide with the mild solution when, moreover, $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$. There are many processes that pave the way to Leray solutions (and in most cases to suitable weak Leray solutions); in [19], we described fourteen different processes (including α-models, frequency cut-off, damping, artificial viscosity, hyperviscosity, etc.).

The scheme is always the same. One approximates the Navier–Stokes equations (NS) by equations (NS$_\alpha$) depending on a small parameter $\alpha \in (0, 1)$. Equations (NS$_\alpha$) with initial value $\vec{u}_0 \in L^2$ have a unique solution $\vec{u}_\alpha$. One then establishes an energy (in)equality that allows to control $\vec{u}_\alpha$ uniformly on $L^\infty((0,T), L^2) \cap L^2((0,T), H^1)$. Moreover, one proves that $\partial_t \vec{u}_\alpha$ is controlled uniformly in $L^{6/5}((0,T), H^{-3})$. By the Aubin–Lions theorem, there exists a sequence $\alpha_k \to 0$ such that $\vec{u}_{\alpha_k}$ is weakly convergent in $L^2((0,T), H^1)$

and strongly convergent in $(L^2((0,T)\times\mathbb{R}^3))_{\rm loc}$ to a limit $\vec v$. One then checks that $\vec v$ is a weak Leray solution of the Navier–Stokes equations with initial value $\vec u_0$.

Some of those processes behave well for initial values $\vec u_0\in \mathrm{bmo}_0^{-1}$, others do not seem to be well adapted to such initial values. More precisely, if one can prove that, when $\vec u_0$ belongs to $L^2\cap\mathrm{bmo}_0^{-1}$, there exists a time T_0 such that the solutions $\vec u_\alpha$ remain small in X_{T_0} ($\|e^{t\Delta}\vec u_0\|_{X_{T_0}}<\eta<\frac{1}{4C_0}$ and $\sup_{\alpha\in(0,1)}\|\vec u_\alpha\|_{X_{T_0}}\le 2\eta\le\frac{1}{2C_0}$), then the weak limit $\vec v$ will still remain controlled in X_{T_0}. But there is only one weak solution $\vec u$ in X_{T_0} such that $\|\vec u\|_{X_{T_0}}\le\frac{1}{2C_0}$. Thus, the process cannot create a Leray solution that would escape the weak–strong uniqueness.

Such processes can be found in processes that mimick Leray's mollification. Mollification has been introduced by Leray [21] in his seminal paper on weak solutions for the Navier–Stokes equations. The approximated problem he considered is to solve

$$\partial_t\vec u_\alpha+(\varphi_\alpha*\vec u_\alpha).\vec\nabla\vec u_\alpha=\Delta\vec u_\alpha-\vec\nabla p_\alpha,$$

with $\operatorname{div}\vec u_\alpha=0$ and $\vec u_\alpha(0,\cdot)=\vec u_0$. Here, $\varphi\in\mathcal{D}$, $\varphi\ge 0$, $\int\varphi\,dx=1$ and $\varphi_\alpha(x)=\frac{1}{\alpha^3}\varphi(\frac{x}{\alpha})$. Solving the mollified problem amounts to solving the integro-differential problem

$$\vec v=e^{t\Delta}\vec u_0-B(\varphi_\alpha*\vec v,\vec v)(t,x),$$

where

$$B(\vec v,\vec w)=\int_0^t e^{(t-s)\Delta}\,\mathbb{P}\operatorname{div}(\vec v\otimes\vec w)\,ds.$$

Since $g\varphi_\alpha*\vec v(t,\cdot)\|_\infty\le\|\vec v(t,\cdot)\|_\infty$ and

$$\begin{aligned}
&\Big(\int_0^t\int_{B(x_0,\sqrt t)}|\varphi_\alpha*\vec v(s,\cdot)(y)|^2\,dy\,ds\Big)^{1/2}\\
&\quad=\Big(\int_0^t\int_{B(x_0,\sqrt t)}\Big|\int\varphi_\alpha(z)\,\vec v(s,y-z)\,dz\Big|^2\,dy\,ds\Big)^{1/2}\\
&\quad\le\Big(\int_0^t\int_{B(x_0,\sqrt t)}\int\varphi_\alpha(z)\,|\vec v(s,y-z)|^2\,dz\,dy\,ds\Big)^{1/2}\\
&\quad=\Big(\int\varphi_\alpha(z)\Big(\int_0^t\int_{B(x_0+z,\sqrt t)}|\vec v(s,y)|^2\,dy\,ds\Big)\,dz\Big)^{1/2},
\end{aligned}$$

we find that $\|\varphi_\alpha*\vec v\|_{X_T}\le\|\vec v\|_{X_T}$. Thus, the theorem of Koch and Tataru (Theorem 1.2 and Corollary 1.3) still applies:

- For every $\alpha>0$ and every $T>0$, we have
$$\|B(\varphi_\alpha\vec v,\vec w)\|_{X_T}\le C_0\|\vec v\|_{X_T}\|\vec w\|_{X_T}.$$
- If $\|e^{t\Delta}\vec u_0\|_{X_T}<\frac{1}{4C_0}$, then the mollified Navier–Stokes equations have a solution on $(0,T)$ such that $\|\vec u_\alpha\|_{X_T}\le 2\|e^{t\Delta}\vec u_0\|_{X_T}$.

Now, we may consider various other approximations of the Navier–Stokes equations of the form

$$\vec v=e^{t\Delta}\vec u_0-\sum_{i=1}^N\varphi_{i,\alpha}*B_i(\psi_{i,\alpha}*\vec v,\chi_{i,\alpha}*\vec v)(t,x),\tag{7.1}$$

where

- φ_i, ψ_i, χ_i are either the Dirac mass or functions in L^1,
- $f_\alpha(x) = \frac{1}{\alpha^3} f(\frac{x}{\alpha})$ for $f \in \{\varphi_i, \psi_i, \chi_i, i = 1, \dots, N\}$,
- $B_i(\vec{v}, \vec{w}) = \int_0^t e^{(t-s)\Delta} \sigma_i(D)(\vec{v} \otimes \vec{w})\, ds$, where σ_i is given convolutions with smooth Fourier multipliers homogeneous of degree 1, that is, if $\vec{z} = \sigma_i(D)(\vec{v} \otimes \vec{w})$, then $z_k = \sum_{p,q\leq 3} K_{i,k,p,q} * (v_p w_q)$, where the Fourier transform of $K_{i,k,p,q}$ is and homogeneous of degree 1 and is smooth on $\mathbb{R}^3$.

The proof of the Koch and Tataru theorem asserts that operators of the form $B(\vec{v}, \vec{w}) = \int_0^t e^{(t-s)\Delta} \sigma(D)(\vec{v} \otimes \vec{w})\, ds$ are bounded on X_T.

Writing $\|\delta\|_1 = 1$, we have

$$\Big\| \sum_{i=1}^N \varphi_{i,\alpha} * B(\psi_{i,\alpha} * \vec{v}, \chi_{i,\alpha} * \vec{v})(t,x) \Big\|_{X_T} \leq \Big(\sum_{i=1}^N \|B_i\|_{\mathrm{op}} \|\varphi_i\|_1 \|\psi_i\|_1 \|\chi_i\|_1 \Big) \|\vec{v}\|_{X_T} \|\vec{w}\|_{X_T} = C_1 \|\vec{v}\|_{X_T} \|\vec{w}\|_{X_T}$$

If $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_1}$, then the modified equations (7.1) have a solution on $(0, T)$ such that $\|\vec{u}_\alpha\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$.

Note that the equations (7.1) can be written as well as

$$\partial_t \vec{v} = \Delta \vec{v} - \sum_{i=1}^N \varphi_{i,\alpha} * \sigma_i(D)((\psi_{i,\alpha} * \vec{v}) \otimes (\chi_{i,\alpha} * \vec{v}))$$

with initial value $\vec{v}(0, \cdot) = \vec{u}_0$. Among example of such approximations, we have the various α-models studied by Holm and Titi:

The Leray-α model. The Leray-α model has been discussed in 2005 by Cheskidov, Holm, Olson and Titi [8]. The approximated problem is to solve

$$\partial_t \vec{u}_\alpha + ((\mathrm{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \vec{\nabla} p_\alpha,$$

with $\operatorname{div} \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. This is equivalent to write

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P} \operatorname{div}(((\mathrm{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \otimes \vec{u}_\alpha).$$

The Navier–Stokes-α model. The mathematical study of the Navier–Stokes-α model has been done by Foias, Holm and Titi in 2002, see [13]. The approximated problem is to solve

$$\partial_t \vec{u}_\alpha + ((\mathrm{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \sum_{k=1}^3 u_{\alpha,k} \vec{\nabla} (\mathrm{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k} - \vec{\nabla} p_\alpha,$$

with $\operatorname{div} \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. We can rewrite the equation as

$$\partial_t \vec{u}_\alpha + ((\mathrm{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \sum_{k=1}^3 (\alpha^2 \Delta (\mathrm{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k}) \vec{\nabla} (\mathrm{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k} - \vec{\nabla} \Big(p_\alpha + \frac{|(\mathrm{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha|^2}{2} \Big).$$

This is equivalent to write

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P}\,\mathrm{div}(((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \otimes \vec{u}_\alpha)$$
$$- \sum_{j=1}^{3}\sum_{k=1}^{3} \mathbb{P}\partial_j((\alpha\partial_j(\mathrm{Id} - \alpha^2\Delta)^{-1}u_{\alpha,k})(\alpha\vec{\nabla}(\mathrm{Id} - \alpha^2\Delta)^{-1}u_{\alpha,k})).$$

The Clark-α model. The Clark-α model has been discussed in 2005 by Cao, Holm and Titi [6]. The approximated problem is to solve

$$\partial_t \vec{u}_\alpha + (\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha \cdot \vec{\nabla}\,\vec{u}_\alpha = \Delta\vec{u}_\alpha + ((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha - \vec{u}_\alpha) \cdot \vec{\nabla}(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha$$
$$+ \alpha^2 \sum_{k=1}^{3} (\partial_k(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \cdot \vec{\nabla}(\partial_k(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) - \vec{\nabla} p_\alpha,$$

with $\mathrm{div}\,\vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0,\cdot) = \vec{u}_0$. We can rewrite the equation as

$$\partial_t \vec{u}_\alpha + ((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \cdot \vec{\nabla}\vec{u}_\alpha$$
$$= \Delta\vec{u}_\alpha + \sum_{k=1}^{3} \alpha^2 \partial_k\big((\partial_k(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \cdot \vec{\nabla}(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha\big) - \vec{\nabla} \cdot p_\alpha.$$

This is equivalent to write

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P}\,\mathrm{div}(((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \otimes \vec{u}_\alpha)$$
$$- \sum_{k=1}^{3} \mathbb{P}\partial_j\big((\alpha\partial_k(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \cdot (\alpha\vec{\nabla}(\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha)\big).$$

The simplified Bardinal model. The simplified Bardina model is another α-model studied by Cao, Lunasin and Titi in 2006, see [7]. This model is given by

$$\partial_t \vec{u}_\alpha + ((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \cdot \vec{\nabla}\,((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) = \Delta\vec{u}_\alpha - \vec{\nabla} p_\alpha,$$

where we have again $\mathrm{div}\,\vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0,\cdot) = \vec{u}_0$. This is equivalent to write

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P}\,\mathrm{div}\big(((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha) \otimes ((\mathrm{Id} - \alpha^2\Delta)^{-1}\vec{u}_\alpha)\big).$$

Thus, when $\vec{u}_0 \in \mathrm{bmo}_0^{-1}$, all those α-models give back the mild solution $\vec{u} \in X_T$ when α goes to 0.

A. Comments on the weights $w_N(x) = (1 + |x|)^{-N}$

In this paper, we considered a suitable weak Leray solution $\vec{u}$ associated to an initial data $\vec{u}_0 \in L^2 \cap \mathrm{bmo}_0^{-1}$. This solution satisfies on a small time interval $(0, T_0)$ that $\vec{u} \in L^\infty L^2$ and that $\sup_{0<t<T} \sqrt{t}\,\|\vec{u}(t,\cdot)\|_\infty < +\infty$. In particular, for $2 < q < +\infty$,

$$\sup_{0<t<T} t^{1/2-1/q}\|\vec{u}(t,\cdot)\|_q = \sup_{0<t<T} t^{1/2-1/q}\|\vec{u}(t,\cdot)\|_{L^q(W_0\,dx)} < +\infty. \tag{A.1}$$

In Theorem 1.8, the assumption on the solution $\vec{u}$ is

$$\sup_{0<t<T} t^{s/2} \|\vec{u}\|_{L^q((1+|x|)^{-N}\,dx)} = \sup_{0<t<T} t^{s/2} \|\vec{u}\|_{L^q(w_N\,dx)} < +\infty, \tag{A.2}$$

with

$$N \geq 0, \quad 2 < q < +\infty \quad \text{and} \quad 0 \leq s < 1 - \frac{2}{q}\cdot$$

This means that we ask a little more integrability in time but relax a lot integrability in space, as N may be as large as we want (in particular, in order to include Morrey spaces).

However, in order to deal with tools of harmonic analysis, we need to consider N not too large. This is fixed by interpolating between (A.1) and (A.2). If $0 \leq \alpha \leq 1$, we find that

$$\sup_{0<t<T} (\sqrt{t})^{(1-\alpha)(1-2/q)+\alpha s} \|\vec{u}\|_{L^q((1+|x|)^{-\alpha N}\,dx)} = \sup_{0<t<T} t^{s_\alpha/2} \|\vec{u}\|_{L^q(w_{N_\alpha}\,dx)} < +\infty.$$

We still have $s_\alpha < 1 - 2/q$, but (taking small values of α) we may have N_α as small as we want. In particular, for $N_\alpha < 4/q$, we may apply Corollary 3.3.

Thus, $\vec{u}_0$ belongs to an interpolate of $L^2(w_2\,dx)$ with a Besov space $B^{-\delta}_{\infty,\infty}$ with $\delta < 1$. Moreover, since $\vec{u}$ and $\vec{v}$ belong to $L^\infty L^2 \subset L^\infty L^2(w_2\,dx)$, the end of the proof is done by energy estimates in $L^2(w_2\,dx)$. Thus, the proof deals with a lot of change in the weights.

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