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The large sieve with prime moduli

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Abstract. The large sieve type estimates of true order of magnitude for character sums to prime moduli are established. The main result holds for coefficients supported on numbers which have no small prime divisors.

Dedicated to Professor Antonio Córdoba Barba, with admiration and friendship.

1. Introduction

The classical large sieve deals with general exponential sums

$$F(\alpha) = \sum_n a_n\, e(\alpha n), \quad e(x) = e^{2\pi i x},$$

at real points α (mod 1) in various sets which are well-spaced, but not necessarily regularly spaced. In 1966, H. Davenport and H. Halberstam proved that if $\|\alpha_r - \alpha_s\| > \delta$, with $0 < \delta < 1$ for $r \neq s$, then

$$\sum_r \Big| \sum_{n \leqslant N} a_n\, e(\alpha_r n) \Big|^2 \ll (\delta^{-1} + N) \sum_{n \leqslant N} |a_n|^2$$

for any complex numbers a_n, where the implied constant is absolute. This covers the case of all the rational points $\alpha = a/q$ with $1 \leqslant q \leqslant Q$, $a \pmod q$, $(a,q) = 1$, giving

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{n \leqslant N} a_n\, e\Big(\frac{an}{q}\Big) \Big|^2 \ll (Q^2 + N) \sum_{n \leqslant N} |a_n|^2. \tag{1.1}$$

Expanding the primitive Dirichlet characters $\chi \pmod q$ into the additive characters $e(an/q)$ by means of Gauss sums, one derives from (1.1) the following inequality (see [2]):

$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi \pmod q} \Big| \sum_{M < n \leqslant M+N} a_n\, \chi(n) \Big|^2 \ll (Q^2 + N) \sum_{M < n \leqslant M+N} |a_n|^2. \tag{1.2}$$

Here and thereafter, the $\sum^*$ denotes restricted summation over primitive characters.

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The large sieve inequality (1.2) is powerful enough to produce results of the Riemann hypothesis quality. Actually, LSI is robust not only due to arbitrary coefficients a_n but also that it holds true for character sums over short segments. A great panorama and nice refined large sieve inequalities were presented at the Harish–Chandra Research Institute in February 2005 by Olivier Ramaré [3]. Dieter Wolke (1969–1971) established a few significant large sieve type estimates with fractions a/q of prime denominators. His best results in [4] and [5] show the following.

Theorem 1 (D. Wolke). *For any complex numbers a_n, $1 \leqslant n \leqslant N$, with $N \leqslant Q^2(\log Q)^{-8}$, we have*

$$\sum_{\substack{q\leqslant Q\\ q\text{ prime}}} \sideset{}{^*}\sum_{a \pmod q} \Big|\sum_{n\leqslant N} a_n\, e\Big(\frac{an}{q}\Big)\Big|^2 \ll \frac{Q^2\log\log Q}{\log(Q/\sqrt{N})}\sum_{n\leqslant N}|a_n|^2. \tag{1.3}$$

Our goal is to remove the factor $\log\log Q$ in (1.3) subject to some usable restrictions for the coefficients a_n. We do not know if the bound (1.3) holds true without the factor $\log\log Q$ for general coefficients. Throughout we assume

$$L = \log Q, \quad N \leqslant Q^2L^{-8} \quad \text{and} \quad D = Q/\sqrt{N}. \tag{1.4}$$

Theorem 2. *For any complex numbers a_n, $1 \leqslant n \leqslant N$, we have*

$$\sum_{\substack{q\leqslant Q\\ q\text{ prime}}} \sideset{}{^*}\sum_{a \pmod q} \Big|\sum_{\substack{n\leqslant N\\ (n,P)=1}} a_n\, e\Big(\frac{an}{q}\Big)\Big|^2 \ll \frac{(1+\delta)Q^2}{\log D}\sum_{\substack{n\leqslant N\\ (n,P)=1}}|a_n|^2, \tag{1.5}$$

where P is any positive integer and

$$\delta = (\log\log Q)^{5/2}\prod_{\substack{p\mid P\\ p<D}}\Big(1-\frac{1}{p}\Big)^{1/2}. \tag{1.6}$$

The implied constant in (1.5) is absolute.

Note that one needs a few small prime divisors of P to kill the factor $(\log\log Q)^{5/2}$ in (1.6). For example, if $P = P(z)$ is the product of all primes $p \leqslant z$ and $N \leqslant Q^2z^{-2}$ with $z = \exp(\log\log Q)^5$, then (1.5) holds with $\delta = 1$.

Remarks. Some sequences a_n are naturally supported on numbers free of prime divisors in a set $\mathcal{P}$ of positive density κ. For example, the numbers of type $n = u^2 + v^2$ with $(u,v) = 1$ have no prime divisors $p \equiv 3 \pmod 4$, so $\kappa = 1/2$. Letting P be the product of $p \in \mathcal{P}$, $p < z$, we get

$$\prod_{p\mid P}\Big(1-\frac{1}{p}\Big) \asymp (\log z)^{-\kappa},$$

which kills $(\log\log Q)^{5/2}$ in (1.6) if $z = \exp(\log\log Q)^{5/\kappa}$. Therefore (1.5) holds for $N \leqslant Q^2z^{-2}$ with $\delta = 1$, and the restriction $(n,P) = 1$ for these numbers a_n is redundant.

Note that for every complex numbers a_n we have the formula

$$\sum_{a \pmod q} \Big| \sum_{n \leqslant N} a_n \, e\Big(\frac{an}{q}\Big)\Big|^2 = (q + O(N)) \sum_{n \leqslant N} |a_n|^2,$$

which yields more precise estimates than (1.3) and (1.5) if $N \leqslant Q$.

In our applications (a work in progress), the coefficients a_n satisfy

$$a_n \ll \frac{1}{\sqrt{n}} \frac{\log p}{\log N}, \quad \text{if } p \mid n, \ 1 \leqslant n \leqslant N. \tag{1.7}$$

Hence

$$\sum_{\substack{n \leqslant N \\ (n, P(z)) \neq 1}} |a_n|^2 \ll \sum_{p < z} \frac{1}{p} \Big(\frac{\log p}{\log N}\Big)^2 \sum_{\substack{n < N \\ (n, P(p)) = 1}} \frac{1}{n} \ll \frac{\log z}{\log N}.$$

We do not need (1.7), but only the following weaker condition:

$$\sum_{\substack{n \leqslant N \\ (n, P(z)) \neq 1}} |a_n|^2 \leqslant A \frac{\log z}{\log N}, \tag{1.8}$$

where A is a positive constant. Now applying (1.5) twice with $P = 1$ and $P = P(z)$, we derive the following.

Theorem 3. *Let $N \leqslant Q^2 z^{-2}$ with $z = \exp(\log\log Q)^5$. Suppose (1.8) holds with some $A > 0$. Then*

$$\sum_{\substack{q \leqslant Q \\ q \text{ prime}}} \ \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_{n \leqslant N} a_n \, e\Big(\frac{an}{q}\Big)\Big|^2 \ll \frac{Q^2}{\log D} \Big(\sum_{n \leqslant N} |a_n|^2 + A\varepsilon(Q)\Big), \tag{1.9}$$

where $\varepsilon(Q) = (\log\log Q)^8 / \log Q$ and the implied constant is absolute.

Remarks. A typical sequence, but not the only one, which satisfies (1.8) consists of the coefficients of a mollified L-function. For example, one can apply Theorem 3 to show that the mollified Dirichlet L-functions

$$\mathcal{L}(s, \chi) = L(s, \chi) \sum_{m < M} \mu(m) \Big(1 - \frac{\log m}{\log M}\Big) \frac{\chi(m)}{m^s},$$

with $M = Q^{1/8}$ and $\operatorname{Re} s = 1/2$, satisfy

$$\sum_{\substack{q \leqslant Q \\ q \text{ prime}}} \sum_{\chi \pmod q} |\mathcal{L}(s, \chi)|^6 \ll \frac{|s|^2 Q^2}{\log Q},$$

where the implied constant is absolute.

In our applications, we can reduce the coefficient a_n by the slowly vanishing crop factor $1 - \log n / \log N$ for $1 \leqslant n \leqslant N = Q^2$. We take advantage of this factor to replace $\log D$ by $\log Q$. Precisely, we deduce from Theorem 2 the following.

Theorem 4. *Let a_n be complex numbers for $1 \leqslant n \leqslant N = Q^2$ which satisfy (1.8) with $z = \exp(\log\log Q)^8$. Then*

$$\sum_{\substack{q\leqslant Q\\ q \text{ prime}}} \sideset{}{^*}\sum_{a \ (\mathrm{mod}\ q)} \Big|\sum_{n\leqslant N} a_n\Big(1-\frac{\log n}{\log N}\Big) e\Big(\frac{an}{q}\Big)\Big|^2 \ll \frac{Q^2}{\log Q}\Big(\sum_{n\leqslant N} |a_n|^2 + A\varepsilon(Q)\Big), \tag{1.10}$$

$$\sum_{\substack{q\leqslant Q\\ q \text{ prime}}} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} \Big|\sum_{n\leqslant N} a_n\Big(1-\frac{\log n}{\log N}\Big) \chi(n)\Big|^2 \ll \frac{Q^2}{\log Q}\Big(\sum_{n\leqslant N} |a_n|^2 + A\varepsilon(Q)\Big), \tag{1.11}$$

where $\varepsilon(Q) = (\log\log Q)^8/\log Q$ and the implied constant is absolute.

Proof. The contribution of terms $a_n(1-\log n/\log N)$ in (1.10) with $M < n < N$, $M = Q^2 z^{-2}$ and $N = Q^2$, is bounded by

$$\Big(Q\frac{\log z}{\log Q}\Big)^2 \sum_{n\leqslant N} |a_n|^2$$

on using (1.1). For $1 \leqslant n \leqslant M$, we write

$$\log\frac{N}{n} = \int_n^N \frac{du}{u},$$

so the corresponding part of (1.10) is estimated as follows:

$$\begin{aligned}
&(\log N)^{-2}\sum_q \sideset{}{^*}\sum_a \Big|\int_1^N \sum_{n<\min(u,M)} a_n\, e\Big(\frac{an}{q}\Big)\frac{du}{u}\Big|^2 \\
&\quad\leqslant (\log N)^{-2}\Big(\int_1^N \Big(\log\frac{N}{u}\Big)^{-1/2}\frac{du}{u}\Big) \\
&\qquad\times \int_1^N \Big(\log\frac{N}{u}\Big)^{1/2} \sum_q \sideset{}{^*}\sum_a \Big|\sum_{n<\min(u,M)} a_n\, e\Big(\frac{an}{q}\Big)\Big|^2 \frac{du}{u} \\
&\quad\ll \Big(\frac{Q}{\log N}\int_1^N \Big(\log\frac{N}{u}\Big)^{-1/2}\frac{du}{u}\Big)^2 \Big(\sum_n |a_n|^2 + A\varepsilon(Q)\Big) \\
&\quad= \frac{2Q^2}{\log Q}\Big(\sum_n |a_n|^2 + A\varepsilon(Q)\Big)
\end{aligned}$$

on using (1.9) for sums of length $\min(u, M)$. Adding the above estimates, we get (1.10). Then (1.11) follows from (1.10) by means of Gauss sums. ∎

Remarks. For technical simplifications, we replace the range of moduli $1 \leqslant q \leqslant Q$ in every statement above by the dyadic segment $Q < q \leqslant 2Q$. It is clear that the results so modified are sufficient for deriving the primary ones.

2. First estimation of $S(Q,N)$ and of $S^\flat(Q,N)$

For the proof of Theorem 1 and Theorem 2, we need to estimate

$$S(Q,N) = \sum_q \sideset{}{^*}\sum_{a \pmod q} \Big| \sum_n a_n\, e\Big(\frac{an}{q}\Big)\Big|^2. \tag{2.1}$$

Here and thereafter, q runs over primes in the segment $Q < q \leqslant 2Q$, a_n are complex numbers supported on $n \leqslant N$, $(n,P) = 1$, with

$$Q \leqslant N \leqslant Q^2 L^{-8}, \quad L = \log Q.$$

Recall that $D = Q/\sqrt{N}$, so $L^4 \leqslant D \leqslant \sqrt{Q}$. By the duality principle it suffices to estimate

$$S^*(Q,N) = \sum_n f(n) \Big| \sum_q \sideset{}{^*}\sum_{a \pmod q} \lambda(q,a)\, e\Big(\frac{an}{q}\Big)\Big|^2$$

for any complex numbers $\lambda(q,a)$ with q prime, $Q < q \leqslant 2Q$ and $a \pmod q$, $(a,q) = 1$. Here $f(x)$ is the majorizing function given by (A.3). Squaring out, we write

$$\begin{aligned} S^*(Q,N) &= \sum_q \sideset{}{^*}\sum_{a \pmod q} \sum_{q_1} \sideset{}{^*}\sum_{a_1 \pmod{q_1}} \lambda(q,a)\, \bar\lambda(q_1,a_1) \sum_n f(n)\, e\Big(\Big(\frac{a}{q} - \frac{a_1}{q_1}\Big)n\Big) \\ &\leqslant \sum_q \sideset{}{^*}\sum_{a \pmod q} |\lambda(q,a)|^2 \sum_{q_1} \sideset{}{^*}\sum_{a_1 \pmod{q_1}} \sum_n f(n)\, e\Big(\Big(\frac{a}{q} - \frac{a_1}{q_1}\Big)n\Big), \end{aligned}$$

where (see (A.2) and (A.3))

$$\sum_n f(n)\, e(\alpha n) = 6N \max(1 - 2\|\alpha\| N, 0).$$

Hence

$$S^*(Q,N) \leqslant 6N \sum_q \sideset{}{^*}\sum_{a \pmod q} |\lambda(q,a)|^2\, \nu(q,a), \tag{2.2}$$

where $\nu(q,a)$ is the number of fractions $a_1/q_1 \pmod 1$ such that

$$\Big\| \frac{a}{q} - \frac{a_1}{q_1} \Big\| < \frac{1}{2N}. \tag{2.3}$$

To estimate $\nu(q,a)$, we approximate a/q by a fraction b/c with $1 \leqslant c \leqslant C$, $(b,c) = 1$, such that

$$\frac{a}{q} \equiv \frac{b}{c} + \theta \pmod 1 \quad \text{with } |\theta| \leqslant \frac{1}{cC},$$

where C is at our disposal, $C < Q$. Then (2.3) becomes

$$\Big\| \frac{b}{c} - \frac{a_1}{q_1} + \theta \Big\| < \frac{1}{2N}. \tag{2.4}$$

Take $h \equiv bq_1 - ca_1 \pmod{cq_1}$, $|h| \leqslant cq_1/2$ so (2.4) becomes

$$\left|\frac{h}{cq_1} + \theta\right| < \frac{1}{2N}. \tag{2.5}$$

Note that $\theta = \ell/cq$, with $1 \leqslant |\ell| \leqslant q/C$, $(\ell, c) = 1$. If $h = 0$, then $q_1 \mid c$, $c \mid q_1$ and $q_1 = c \leqslant C < Q$, contradiction. If $(h, c) \neq 1$, then $(h, c) = q_1 \leqslant c$, contradiction. Therefore we have

$$(h, c) = 1, \quad 1 \leqslant |h| \leqslant H = (1 + 2|\theta|N)cQN^{-1} \tag{2.6}$$

by (2.5). Given h as above, the prime number q_1 lies in the intersection of the segments

$$\left|q_1 + \frac{h}{\theta c}\right| < \frac{Q}{|\theta|N}, \quad Q < q_1 \leqslant 2Q,$$

so q_1 is in a segment of length

$$Y = 5Q/(1 + 2|\theta|N)$$

and in the residue class $q_1 \equiv h\overline{b} \pmod{c}$. Now we choose

$$C = \sqrt{N}$$

so that $Y \geqslant 5Q(1 + 2c/\sqrt{N})^{-1} \geqslant cQ/\sqrt{N} = cD$. By the Brun–Titchmarsh theorem (see (A.9)), we find that the number of primes q_1 is $\ll Y/\varphi(c)\log D$. Multiplying this by $2H$, we get

$$\nu(q, a) \ll \frac{c}{\varphi(c)} \frac{Q^2}{N\log D} \ll \frac{Q^2}{N} \frac{\log\log Q}{\log D}. \tag{2.7}$$

Finally, introducing (2.7) into (2.2), we obtain

$$S^*(Q, N) \ll \Delta \sum_q \sideset{}{^*}\sum_{a \pmod{q}} |\lambda(q, a)|^2,$$

with $\Delta \ll Q^2(\log\log Q)(\log D)^{-1}$. This completes the proof of Theorem 1 by duality.

Remarks. We lost the factor $\log\log Q$ in (2.7) after applying the maximum bound for $c/\varphi(c)$. Although the average value of $c/\varphi(c)$ is bounded, it does not apply here. However, we have not exploited fully the properties (2.6). If H is relatively large, we can save the factor $\varphi(c)/c$ from the condition $(h, c) = 1$, see Lemma A.3. Unfortunately, if H is small, then this coprimality condition does not help to recover $\varphi(c)/c$. This case occurs when many rationals a/q are very close to the same point b/c. Recall that

$$\frac{a}{q} - \frac{b}{c} \equiv \frac{\ell}{cq} \quad \text{with } 1 \leqslant |\ell| \leqslant \frac{q}{C}. \tag{2.8}$$

Hence

$$H = \frac{cQ}{N} + |\ell|\frac{2Q}{q}.$$

If $c \geqslant LNQ^{-1}$ or $|\ell| \geqslant L$, then $H \geqslant L$, and Lemma A.3 is applicable saving the factor $\varphi(c)/c$. Accordingly, we split (2.1) into two sums,

$$S(Q,N) = S^\flat(Q,N) + S^\sharp(Q,N).$$

The first partial sum $S^\flat(Q,N)$ runs over the fractions a/q which admit the approximations (2.8) with either $c \geqslant LNQ^{-1}$ or $|\ell| \geqslant L$. We just proved by duality that

$$S^\flat(Q,N) \ll \frac{Q^2}{\log D} \sum_n |a_n|^2 \tag{2.9}$$

for any complex numbers a_n, $1 \leqslant n \leqslant N \leqslant Q^2 L^{-8}$. We have not used the restriction $(n,P) = 1$ to get (2.9).

It remains to estimate the second partial sum $S^\sharp(Q,N)$, which runs over the fractions a/q very close to b/c such that

$$\frac{a}{q} - \frac{b}{c} \equiv \frac{\ell}{cq} \pmod 1, \quad (b,c) = 1, (\ell,c) = 1,$$

with

$$1 \leqslant |\ell| \leqslant L \quad \text{and} \quad 1 \leqslant c \leqslant LNQ^{-1} = C_0, \text{ say.}$$

Recall our choice (1.4) and note that $C_0 = CLD^{-1} \leqslant CL^{-1}$. Hence

$$S^\sharp(Q,N) = 2 \sum_{1 \leqslant \ell \leqslant L} S_\ell(Q,N), \tag{2.10}$$

$$S_\ell(Q,N) = \sum_{\substack{1 \leqslant c \leqslant C_0 \\ (c,\ell)=1}} \sum_{\substack{Q<q\leqslant 2Q \\ q \text{ prime}}} \Big| \sum_{\substack{n \leqslant N \\ (n,P)=1}} a_n \, e\Big(n\Big(\frac{b}{c} + \frac{\ell}{cq}\Big)\Big)\Big|^2, \tag{2.11}$$

where $b \pmod c$ is determined by $bq \equiv -\ell \pmod c$.

3. Applying sieve to $S_\ell(Q,N)$

First we attach to q an upper-bound sieve of level D, that is, the factor

$$\theta(q) = \sum_{m|q} \lambda(m),$$

which is non-negative for every positive integer q, see (A.5). Here $\lambda(m)$ are sieve weights supported on squarefree numbers $m \leqslant D$, $(m,c) = 1$. If q is prime, $q > D$, these factors are redundant; $\theta(q) = 1$ because $m = 1$ is the only possible divisor of q.

Next we extend the summation (2.11) to all integers q weighted by a smooth function $\phi(q/Q)$ with $0 \leqslant \phi(x) \leqslant 1$, $\phi(x) = 1$ if $1/2 < x < 1$, and $\phi(x) = 0$ if x is not in the segment $1/3 < x < 3$. We also assume the symmetry $\phi(x) = \phi(1/x)$ for elegant writing. We get the upper bound

$$S_\ell(Q,N) \leqslant S^+(Q,N),$$

where

$$S^+(Q,N) = \sum_{\substack{1\leqslant c\leqslant C_0 \\ (c,b\ell)=1}} \sum_{b \pmod c} S\Big(\frac{b}{c}\Big)$$

and

$$S\Big(\frac{b}{c}\Big) = \sum_{q\equiv -\ell\bar{b} \pmod c} \theta(q)\,\phi\Big(\frac{q}{Q}\Big) \Big| \sum_{\substack{n\leqslant N \\ (n,P)=1}} a_n\, e\Big(n\Big(\frac{b}{c}+\frac{\ell}{cq}\Big)\Big)\Big|^2. \tag{3.1}$$

We dropped the subscript ℓ in the notation of $S^+(Q,N)$ and $S(b/c)$ for simplicity. We open the square and change the order of summation to write (3.1) in the bilinear form

$$S\Big(\frac{b}{c}\Big) = \sum\sum_{\substack{n_1,n_2\leqslant N \\ (n_1n_2,P)=1}} a_{n_1}\bar{a}_{n_2}\, A(n_1-n_2)\, e\Big((n_1-n_2)\frac{b}{c}\Big), \tag{3.2}$$

with

$$A(n) = \sum_{q\equiv -\ell\bar{b} \pmod c} \theta(q)\,\phi\Big(\frac{q}{Q}\Big)\, e\Big(\frac{\ell n}{cq}\Big) = \sum_{(m,c)=1} \lambda(m)\, A(m,n), \tag{3.3}$$

$$A(m,n) = \sum_{u\equiv -\ell\overline{bm} \pmod c} \phi\Big(\frac{mu}{Q}\Big)\, e\Big(\frac{\ell n}{cmu}\Big). \tag{3.4}$$

We evaluate $A(m,n)$ by the Euler–Maclaurin formula (see (A.1)), giving

$$A(m,n) = A^\infty(m,n) + A'(m,n),$$

with

$$A^\infty(m,n) = \frac{1}{c}\int \phi\Big(\frac{mx}{Q}\Big)\, e\Big(\frac{\ell n}{cmx}\Big)\, dx, \tag{3.5}$$

$$A'(m,n) = \int \psi\Big(\frac{x+\ell\overline{bm}}{c}\Big)\, d\Big(\phi\Big(\frac{mx}{Q}\Big) e\Big(\frac{\ell n}{cmx}\Big)\Big). \tag{3.6}$$

Accordingly, we split

$$A(n) = A^\infty(n) + A'(n),$$
$$S\Big(\frac{b}{c}\Big) = S^\infty\Big(\frac{b}{c}\Big) + S'\Big(\frac{b}{c}\Big),$$
$$S^+(Q,N) = S^\infty(Q,N) + S'(Q,N).$$

4. Estimation of $S'(Q,N)$

We treat $S'(Q,N)$ first because it is simpler than $S^\infty(Q,N)$. A crude bound for $S'(Q,N)$ will be good enough. The restriction $(n,P)=1$ on the coefficients a_n is still not needed.

We change x in (3.6) into $y = \ell/cmx$, getting

$$
\begin{aligned}
A'(m,n) &= \int \psi\Big(\frac{\ell}{c}\Big(\frac{1}{cmy} + \overline{bm}\Big)\Big)\, d\Big(\phi\Big(\frac{\ell}{cyQ}\Big)\, e(ny)\Big) \\
&= \int \psi\Big(\frac{\ell}{c}\Big(\frac{1}{cmy} + \overline{bm}\Big)\Big)\Big(\frac{-\ell}{cy^2Q}\,\phi'\Big(\frac{\ell}{cyQ}\Big) + 2\pi i n \phi\Big(\frac{\ell}{cyQ}\Big)\Big)\, e(ny)\, dy,
\end{aligned}
$$

where y is in the segment $\ell/3cQ < y < 3\ell/cQ$ due to the support of $\phi(x)$. Hence

$$
\begin{aligned}
S'\Big(\frac{b}{c}\Big) \ll D \int_{\ell/3cQ}^{3\ell/cQ} \Big\{ &\frac{cQ}{\ell}\Big|\sum_n a_n\, e\Big(n\Big(\frac{b}{c}+y\Big)\Big)\Big|^2 \\
&+ \Big|\sum_n n a_n\, e\Big(n\Big(\frac{b}{c}+y\Big)\Big)\Big|\,\Big|\sum_n a_n\, e\Big(n\Big(\frac{b}{c}+y\Big)\Big)\Big|\Big\}\, dy,
\end{aligned}
$$

where D is the trivial bound for $\sum|\lambda(m)|$. Note that $3\ell/cQ < 1/c(c+C_0)$, where $C_0 = LNQ^{-1}$, so we are in the range of Lemma A.1, giving

$$
S'(Q,N) \ll D\Big(\frac{C_0 Q}{\ell} + N\Big)\sum_n |a_n|^2 \leqslant 2DLN \sum_n |a_n|^2.
$$

Summing over $1 \leqslant \ell \leqslant L = \log Q$, we see that $S'(Q,N)$ contributes to $S^{\sharp}(Q,N)$ at most $DL^2N\sum|a_n|^2$ and $DL^2N \leqslant Q^2L^{-2}$, which is stronger than the desired bound $Q^2/\log D$.

5. Estimation of $S^{\infty}(Q,N)$

Finally, we come to the main part of this work:

$$
S^{\infty}(Q,N) = \sum_{\substack{1\leqslant c\leqslant C_0 \\ (c,b\ell)=1}} \sum_{b \pmod c} S^{\infty}\Big(\frac{b}{c}\Big).
$$

Changing x into $y = \ell/cmx$, we write (3.5) as (by the property $\phi(x) = \phi(1/x)$)

$$
A^{\infty}(m,n) = \frac{\ell}{c^2 m}\int \phi\Big(\frac{cyQ}{\ell}\Big)\, e(ny)\,\frac{dy}{y^2}
$$

for $n = n_1 - n_2$. Inserting this into (3.3) and then into (3.2), we get

$$
S^{\infty}\Big(\frac{b}{c}\Big) = \frac{\ell}{c^2}V(c)\int \phi\Big(\frac{cyQ}{\ell}\Big)\Big|F\Big(\frac{b}{c}+y\Big)\Big|^2\frac{dy}{y^2},
$$

where

$$
V(c) = \sum_{(m,c)=1} \lambda(m)m^{-1} \ll \frac{c}{\varphi(c)}(\log D)^{-1}, \tag{5.1}
$$

(see (A.6), (A.7)), and

$$F(\alpha) = \sum_{\substack{n \leqslant N \\ (n,P)=1}} a_n e(\alpha n).$$

We lost the factor $c/\varphi(c)$ in (5.1), which will be compensated with saving from the restriction of the coefficients a_n by $(n, P) = 1$. We get

$$S^\infty\Big(\frac{b}{c}\Big) \leqslant 9Q^2 \ell^{-1} V(c) \int_{\ell/3cQ}^{3\ell/cQ} \Big|F\Big(\frac{b}{c}+y\Big)\Big|^2\, dy.$$

Hence

$$S^\infty(Q,N) \ll \frac{Q^2}{\ell \log D} T(N), \tag{5.2}$$

with

$$T(N) = \sum_{1 \leqslant c \leqslant C_0} \frac{c}{\varphi(c)} \sideset{}{^*}\sum_{b \,(\mathrm{mod}\, c)} \int_{\ell/3cQ}^{3\ell/cQ} \Big|F\Big(\frac{b}{c}+y\Big)\Big|^2\, dy.$$

Remarks. We have $3\ell/cQ \leqslant 1/c(c + C_0)$ and $c/\varphi(c) \ll \log\log Q$, so Lemma A.1 implies

$$T(N) \ll (\log\log Q) \sum_{\substack{n \leqslant N \\ (n,P)=1}} |a_n|^2,$$

but for the proof of Theorem 2 we need

$$T(N) \ll (\log\log Q)^{3/2} \prod_{\substack{p \mid P \\ p<D}} \Big(1 - \frac{1}{p}\Big)^{1/2} \sum_{\substack{n \leqslant N \\ (n,P)=1}} |a_n|^2. \tag{5.3}$$

To this end, we establish a suitable hybrid large sieve estimate in Theorem 5.

6. A hybrid large sieve

Let P be a positive integer and let (a_n) be a sequence of complex numbers supported on $1 \leqslant n \leqslant N$, $(n, P) = 1$.

Theorem 5. *Let $C \geqslant 3$, $D \geqslant 4$, $DC^2 \leqslant N$. Then*

$$\sum_{c \leqslant C} \sideset{}{^*}\sum_{b \,(\mathrm{mod}\, c)} \int_{1/9cCD}^{1/cCD} \Big|\sum_n a_n\, e\Big(n\Big(\frac{b}{c}+y\Big)\Big)\Big|^2\, dy \ll \Delta \sum_n |a_n|^2, \tag{6.1}$$

where

$$\Delta^2 = (\log\log C) \prod_{\substack{p \mid P \\ p<D}} \Big(1 - \frac{1}{p}\Big)$$

and the implied constant is absolute.

Proof. Let

$$F(\alpha) = \sum_n a_n \, e(\alpha n).$$

We write the left side of (6.1) in the following fashion (the first step of the duality operation):

$$\begin{aligned}\mathcal{A} &= \sum_c \sideset{}{^*}\sum_b \int \Big| F\Big(\frac{b}{c} + y\Big)\Big|^2 \, dy \\ &= \sum_n a_n \sum_c \sideset{}{^*}\sum_b \int e\Big(n\Big(\frac{b}{c} + y\Big)\Big) \overline{F}\Big(\frac{b}{c} + y\Big) \, dy.\end{aligned}$$

We introduce here the redundant factors of type (A.11) where $\rho(d)$ are supported on squarefree divisors of P,

$$\rho(1) = 1, \quad |\rho(d)| \leqslant 1, \quad \rho(d) = 0 \ \text{ if } d > z.$$

Actually, we choose (A.10) with $z = \sqrt{D}/2$. We get

$$\mathcal{A} = \sum_n a_n \sum_c \Big(\sum_{\substack{d \mid n \\ (d,c)=1}} \rho(d) \Big) \sideset{}{^*}\sum_b \int e\Big(n\Big(\frac{b}{c} + y\Big)\Big) \, \overline{F}\Big(\frac{b}{c} + y\Big) \, dy.$$

Note that the condition $(d, c) = 1$ is redundant because $d = 1$ is the only possible divisor of n and P. After that, we apply Cauchy's inequality (the second step of the duality operation), getting

$$\mathcal{A}^2 \leqslant \mathcal{B} \sum_n |a_n|^2, \tag{6.2}$$

where

$$\mathcal{B} = \sum_n f(n) \Big| \sum_c \sum_{\substack{d \mid n \\ (c,d)=1}} \rho(d) \sideset{}{^*}\sum_b \int e\Big(n\Big(\frac{b}{c} + y\Big)\Big) \overline{F}\Big(\frac{b}{c} + y\Big) \, dy \Big|^2$$

and $f(x)$ is given by (A.3). Squaring out, we get the sum

$$\mathcal{C} = \sum_{n \equiv 0 \pmod{[d,d_1]}} f(n) \, e\Big(n\Big(\frac{b}{c} - \frac{b_1}{c_1} + y - y_1\Big)\Big)$$

with $1/9 < ycCD \leqslant 1$ and $1/9 \leqslant y_1 c_1 CD < 1$. By Poisson's formula (A.2), we get

$$\mathcal{C} = \frac{1}{[d, d_1]} \sum_k \hat{f}\Big(\frac{k}{[d, d_1]} + \frac{b}{c} - \frac{b_1}{c_1} + y - y_1\Big). \tag{6.3}$$

Since $[d, d_1] \leqslant D/4 < N$, all terms in (6.3) vanish except for at most one integer k such that

$$\Big| k + [d, d_1]\Big(\frac{b}{c} - \frac{b_1}{c_1} + y - y_1\Big)\Big| < \frac{[d, d_1]}{2N}.$$

Take $g \equiv [d, d_1](bc_1 - b_1 c) \pmod{cc_1}$, $|g| \leqslant cc_1/2$ and move k so that

$$\left|k + \frac{g}{cc_1} + [d, d_1](y - y_1)\right| < \frac{[d, d_1]}{2N}.$$

Here we have

$$\frac{[d, d_1]}{2N} \leqslant \frac{D}{8N} \leqslant \frac{1}{8cc_1} \quad \text{and} \quad [d, d_1]|y - y_1| < \frac{1}{4C} \max\left(\frac{1}{c}, \frac{1}{c_1}\right) \leqslant \frac{1}{4cc_1}.$$

Hence $k = 0$, $g = 0$, $\mathcal{C} = \hat{f}(y - y_1)/[d, d_1]$,

$$[d, d_1](bc_1 - b_1 c) \equiv 0 \pmod{cc_1} \tag{6.4}$$

and

$$\mathcal{B} = \sum_d \sum_{d_1} \frac{\rho(d)\rho(d_1)}{[d, d_1]}$$

$$\times \sum_c \sum_b \sum_{c_1} \sum_{\substack{b_1 \\ (c,bd)=(c_1,b_1d_1)=1 \\ [d,d_1](bc_1-b_1c)\equiv 0 \ (\mathrm{mod}\ cc_1)}} \int_{1/9cCD}^{1/cCD} \int_{1/9c_1CD}^{1/c_1CD} \hat{f}(y-y_1) F\left(\frac{b}{c}+y\right) \overline{F}\left(\frac{b_1}{c_1}+y_1\right) dy\, dy_1. \tag{6.5}$$

Recall that $\hat{f}(y - y_1) = 6N(1 - 2|y - y_1|N)$ if

$$|y - y_1| < \frac{1}{2N} \tag{6.6}$$

and $\hat{f}(y - y_1) = 0$ otherwise. Now we solve the congruence (6.4). Let $\gamma = (c, c_1)$, so $c = \gamma s$, $c_1 = \gamma s_1$ with $(s, s_1) = 1$. We have $(b, \gamma s) = (b_1, \gamma s_1) = 1$, $(d, s) = (d_1, s_1) = 1$ and $(dd_1, \gamma) = 1$. Then (6.4) becomes

$$[d, d_1](bs_1 - b_1 s) \equiv 0 \pmod{\gamma s s_1}.$$

This is equivalent to $s \mid d_1$, $s_1 \mid d$ and

$$bs_1 - b_1 s \equiv 0 \pmod{\gamma}. \tag{6.7}$$

Putting

$$d = es_1 \quad \text{and} \quad d_1 = e_1 s,$$

we have $(ee_1ss_1, \gamma) = 1$, $(e, s) = (e_1, s_1) = 1$ and $[d, d_1] = ss_1[e, e_1]$. Since d and d_1 are squarefree, we have the extra property $(ee_1, ss_1) = 1$. By the periodicity of $F(\alpha)$ in $\alpha \pmod 1$, we can change b (mod$^*\gamma s$) and b_1 (mod$^*\gamma s_1$) into $b \equiv \alpha s + \beta\gamma$ and $b_1 \equiv \alpha s_1 + \beta_1\gamma$ with α (mod$^*\gamma$), β (mod$^*s$), β_1 (mod$^*s_1$). Note that we have the same α in both b, b_1 due to (6.7). Hence

$$\begin{aligned}
\sum_b \sum_{b_1} F\left(\frac{b}{c} + y\right) \overline{F}\left(\frac{b_1}{c_1} + y_1\right) \\
&= \sideset{}{^*}\sum_{\alpha \ (\mathrm{mod}\ \gamma)} \left(\sum_\beta F\left(\frac{\alpha}{\gamma} + \frac{\beta}{s} + y\right)\right)\left(\sum_{\beta_1} \overline{F}\left(\frac{\alpha}{\gamma} + \frac{\beta_1}{s_1} + y_1\right)\right) \\
&= \mu(ss_1) \sideset{}{^*}\sum_{\alpha \ (\mathrm{mod}\ \gamma)} F\left(\frac{\alpha}{\gamma} + y\right)\overline{F}\left(\frac{\alpha}{\gamma} + y_1\right),
\end{aligned}$$

because the sum over $\beta \pmod{^*s}$ of $F(\alpha/\gamma + \beta/s + y)$ results in twisting the coefficients by the Ramanujan sums

$$\sideset{}{^*}\sum_{\beta \pmod{s}} e\Big(n\frac{\beta}{s}\Big) = \mu(s).$$

This sum is the Möbius function because $(n, P) = 1$ and $[d, d_1] = ss_1[e, e_1]$ divides P. Now (6.5) becomes

$$\mathcal{B} = \sum_{\gamma} \sideset{}{^*}\sum_{\alpha \pmod{\gamma}} \sum\sum_{(ss_1,\gamma)=1} \frac{\mu(ss_1)}{ss_1} \times \sum\sum_{(ee_1,\gamma ss_1)=1} \frac{\rho(es_1)\rho(e_1 s)}{[e, e_1]} \iint \hat{f}(y - y_1) F\Big(\frac{\alpha}{\gamma} + y\Big) \overline{F}\Big(\frac{\alpha}{\gamma} + y_1\Big)\, dy\, dy_1, \tag{6.8}$$

where $1 \leqslant \gamma \leqslant C$ and the integrals are over the segments

$$\frac{1}{9} \leqslant y\gamma sCD \leqslant 1, \quad \frac{1}{9} \leqslant y_1\gamma s_1 CD \leqslant 1.$$

Hence $y \geqslant 1/9N$, $y_1 \geqslant 1/9N$, $y \asymp y_1$ by (6.6), and $s \asymp s_1$. Before handling the integrals, we are going to estimate the above double sum over e, e_1. For $\rho(d)$ given by (A.10), we have

$$\rho(es_1)\,\rho(e_1 s) = \mu(ss_1)\,\mu(e)\,\mu(e_1)\,(\log z/es_1)\,(\log z/e_1 s)\,(\log z)^{-2},$$

with $es_1 < z$ and $e_1 s < z$, or else the terms vanish. Hence the sum over e, e_1 in (6.8) agrees with (A.17) for z replaced by z/s, z_1 replaced by z/s_1, and P having its prime divisors of $q = \gamma ss_1$ removed, up to the factor $\mu(ss_1)(\log z)^{-2}$. Therefore (A.17) gives the bound

$$\frac{\gamma ss_1}{\varphi(\gamma ss_1)} \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p}\Big).$$

Inserting this bound into (6.8), we obtain (note that $\hat{f} \geqslant 0$)

$$\begin{aligned} \mathcal{B} &\ll \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p}\Big) \sum_{\gamma \leqslant C} \frac{\gamma}{\varphi(\gamma)} \sideset{}{^*}\sum_{\alpha \pmod{\gamma}} \int_0^{1/\gamma CD} \int_0^{1/\gamma CD} \hat{f}(y - y_1) \\ &\qquad \times \Big|F\Big(\frac{\alpha}{\gamma} + y\Big) F\Big(\frac{\alpha}{\gamma} + y_1\Big)\Big| \Big(\sum_{s \asymp 1/y\gamma CD} \frac{1}{\varphi(s)}\Big)\Big(\sum_{s_1 \asymp 1/y_1\gamma CD} \frac{1}{\varphi(s_1)}\Big)\, dy\, dy_1 \\ &\ll \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p}\Big) \sum_{\gamma \leqslant C} \frac{\gamma}{\varphi(\gamma)} \sideset{}{^*}\sum_{\alpha \pmod{\gamma}} \int_0^{1/\gamma CD} \Big|F\Big(\frac{\alpha}{\gamma} + y\Big)\Big|^2 dy \end{aligned}$$

by $\int \hat{f}(y - y_1)\, dy_1 = f(0) = 3$. Finally, by (A.8) and (A.4), we get

$$\mathcal{B} \ll \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p}\Big) (\log\log C) \sum_n |a_n|^2.$$

Then by (6.2) we complete the proof of Theorem 5. ∎

To get (5.3), we use (6.1) with C replaced by $C_0 = LNQ^{-1}$ and $D = Q^2/3L^2N$ (note that $C_0 D = Q/3L$). Introducing this into (5.2) and summing $1/\ell$ over $1 \leqslant \ell \leqslant L = \log Q$ (see (2.10)), we get a contribution bounded by (1.5). This completes the proof of Theorem 2.

A. Appendix

We list results with some comments and some proofs which are known in the literature in different forms.

The Euler–Maclaurin and Poisson summation formulas assert that

$$\sum_{n\equiv a \pmod q} f(n) = \frac{1}{q}\int f(x) + \int \psi\Big(\frac{x-a}{q}\Big) f'(x)\,dx, \tag{A.1}$$

$$\sum_{n\equiv a \pmod q} f(n)e(\alpha n) = \frac{1}{q}\sum_k \widehat{f}\Big(\frac{k}{q}+\alpha\Big) e\Big(\frac{ak}{q}\Big), \tag{A.2}$$

where $\psi(x) = x - [x] - 1/2$, $f'(x)$ is the derivative of $f(x)$ and $\widehat{f}(y)$ is the Fourier transform of $f(x)$ provided $f(x)$ and $\widehat{f}(y)$ satisfy suitable growth and smoothness conditions.

There are Fourier pairs $f(x)$, $\widehat{f}(y)$ such that

$$0 \leqslant f(x) \leqslant 3, \quad f(x) \geqslant 1 \text{ if } |x| \leqslant N,$$
$$0 \leqslant \widehat{f}(y) \leqslant 6N, \quad \widehat{f}(y) = 0 \text{ if } |y| > 1/2N.$$

For example,

$$f(x) = 3\Big(\frac{\sin \pi x/2N}{\pi x/2N}\Big)^2, \quad \widehat{f}(y) = 6N\max(1-2|y|N, 0). \tag{A.3}$$

The Farey points of order $C \geqslant 1$ are the fractions a/c with $1 \leqslant c \leqslant C$, $(a,c) = 1$. If

$$\frac{a'}{c'} < \frac{a}{c} < \frac{a''}{c''}, \quad 1 \leqslant c', c, c'' \leqslant C,$$

are consecutive points, then $C < c + c' \leqslant c + C$, $C < c + c'' \leqslant c + C$ and their Farey mediants are

$$\frac{a+a'}{c+c'} = \frac{a}{c} - \frac{1}{c(c+c')} \quad\text{and}\quad \frac{a+a''}{c+c''} = \frac{a}{c} + \frac{1}{c(c+c'')}\cdot$$

Hence, given $C \geqslant 1$, every real α can be written as

$$\alpha = \frac{a}{c} + \theta \quad \text{with } 1 \leqslant c \leqslant C, (a,c) = 1, |\theta| \leqslant \frac{1}{cC}\cdot$$

The unit segment $\alpha \pmod 1$ is covered exactly by the segments with end-points being the Farey mediants. Hence, the Parseval formula

$$\int_0^1 \Big|\sum_n a_n e(\alpha n)\Big|^2 d\alpha = \sum_n |a_n|^2$$

implies the following hybrid large sieve inequality.

Lemma A.1. *For any complex numbers a_n, we have*

$$\sum_{1\leqslant c\leqslant C}\ \sideset{}{^*}\sum_{a \pmod c}\int_{-1/c(c+C)}^{1/c(c+C)}\Big|\sum_n a_n\, e\Big(n\Big(\frac{a}{c}+\alpha\Big)\Big)\Big|^2\,d\alpha \leqslant \sum_n |a_n|^2. \tag{A.4}$$

Next we show some elementary estimates related to sieve theory. We use an upper-bound sieve (λ_d) of level $D \geqslant 2$. This is a sequence supported on squarefree numbers $d < D$ with $\lambda_1 = 1$, $-1 \leqslant \lambda_d \leqslant 1$, such that

$$\sum_{d\mid n}\lambda_d \geqslant 0 \quad \text{for every positive } n. \tag{A.5}$$

Applying this property for numbers n coprime with P, it follows that the sequence (λ_d) restricted by $(d,P)=1$ is also an upper-bound sieve. The sieve methods (see the Fundamental Lemma in [1]) yield

$$0 < \sum_d \frac{\lambda_d}{d} \ll \prod_{p<D}\Big(1-\frac{1}{p}\Big) \asymp (\log D)^{-1}. \tag{A.6}$$

If $P \geqslant 1$ is fixed, then

$$\sum_{\substack{n\leqslant X\\ (n,P)=1}}\Big(\sum_{d\mid n}\lambda_d\Big) \sim X\,\frac{\varphi(P)}{P}\Big(\sum_{(d,P)=1}\frac{\lambda_d}{d}\Big)$$

as $X \to \infty$. Hence we get the clear inequalities

$$0 \leqslant \sum_{(d,P)=1}\frac{\lambda_d}{d} \leqslant \frac{P}{\varphi(P)}\Big(\sum_d \frac{\lambda_d}{d}\Big). \tag{A.7}$$

Using (A.7), one can generalize (A.6) as follows:

$$0 \leqslant \sum_{d\mid c}\frac{\lambda_d}{d} \ll \prod_{\substack{p\mid c\\ p<D}}\Big(1-\frac{1}{p}\Big),$$

where the implied constant is absolute. To see this, apply (A.7) with P being the product of primes $p < D$, $p \nmid c$. Then (A.6) yields

$$\frac{P}{\varphi(P)}\Big(\sum_d \frac{\lambda_d}{d}\Big) \ll \prod_{\substack{p\nmid c\\ p<D}}\Big(1+\frac{1}{p}\Big)(\log D)^{-1} \asymp \prod_{\substack{p\mid c\\ p<D}}\Big(1-\frac{1}{p}\Big).$$

Lemma A.2. *For every $c \geqslant 1$, we have*

$$\frac{c}{\varphi(c)} \ll \log\log(3c). \tag{A.8}$$

Proof. If the number of distinct prime divisors of c is r, then the rth prime number is $p_r \ll \log(3c)$. Hence $c/\varphi(c)$ is bounded by the product of $(1-1/p)^{-1}$ over $p \leqslant p_r$, which is $\ll \log p_r \ll \log\log(3c)$. ∎

Lemma A.3. *Let $A \geqslant 1$ and $H^A \geqslant \log(3c)$. Then the number of integers h with $(h,c) = 1$ in every interval of length H is bounded by $AH\varphi(c)/c$ up to an absolute constant factor.*

Proof. By an upper-bound sieve of level $D = H\varphi(c)/c$, the number in question is bounded by

$$H \prod_{\substack{p|c \\ p<D}} \Big(1 - \frac{1}{p}\Big) = H\,\frac{\varphi(c)}{c} \prod_{\substack{p|c \\ p\geqslant D}} \Big(1 - \frac{1}{p}\Big)^{-1},$$

and the last product is bounded by

$$A \prod_{\substack{p|c \\ p>H^A}} \Big(1 - \frac{1}{p}\Big)^{-1} \ll A \exp(H^{-A}\log c) < 3A. \qquad \blacksquare$$

Lemma A.4 (Brun–Titchmarsh theorem). *Let $q \geqslant 1$ and $(a,q) = 1$. The number of primes $p \equiv a \pmod q$ in an interval of length $y \geqslant 4q$ is*

$$\pi(x+y;q,a) - \pi(x;q,a) \ll \frac{y}{\varphi(q)}\Big(\log\frac{y}{q}\Big)^{-1}, \tag{A.9}$$

where the implied constant is absolute.

The particular sequence of numbers

$$\rho(d) = \mu(d)\max\Big(1 - \frac{\log d}{\log z}, 0\Big) \tag{A.10}$$

is often used for mollification of L-functions. This sequence is not an upper-bound sieve, because it fails (A.5). Nevertheless, we built

$$\sum_{d|n} \rho(d) \tag{A.11}$$

as redundant factors into our architecture of dual forms. We come up with the sequence of numbers

$$\eta(d) = \sum_{[d_1,d_2]=d} \rho(d_1)\rho(d_2),$$

which satisfies (A.5); it is an upper-bound Selberg sieve. We need estimates for various sums involving $\rho(d)$ and $\eta(d)$. First we give two easy lemmas.

Lemma A.5. *For $M \geqslant 1$, we have*

$$\sum_{m\leqslant M} \frac{\mu(m)}{m}\log\frac{M}{m} \ll 1. \tag{A.12}$$

Proof. This can be seen between the following elementary lines:

$$1 = \sum_{\ell m\leqslant M} \frac{\mu(m)}{\ell m} = \sum_{m\leqslant M} \frac{\mu(m)}{m}\Big(\log\frac{M}{m} + \gamma + O\Big(\frac{m}{M}\Big)\Big),$$

$$1 = \sum_{\ell m\leqslant M} \mu(m) = \sum_{m\leqslant M} \mu(m)\Big(\frac{M}{m} + O(1)\Big). \qquad \blacksquare$$

Lemma A.6. *For any $d \geqslant 1$ and $M \geqslant 1$, we have*

$$\sum_{\substack{m \leqslant M \\ (m,d)=1}} \frac{\mu(m)}{m} \log \frac{M}{m} \ll \frac{d}{\varphi(d)}. \tag{A.13}$$

Proof. Let δ run over numbers with prime divisors in d. We have

$$\sum_{\delta n = m} \mu(n) = \begin{cases} \mu(m), & \text{if } (m,d)=1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence (A.13) follows from (A.12) and the fact that $\sum \delta^{-1} = d/\varphi(d)$. ∎

If (ρ_d) and (ξ_k) are finite sequences of real numbers, let

$$G_q(\rho,\xi) = \sum\sum_{(dk,q)=1} \rho_d \, \xi_k / [d,k],$$

and let $G(\rho,\xi)$ be the case of $q=1$. By the asymptotic formula

$$\sum_{\substack{n \leqslant X \\ (n,q)=1}} \Big(\sum_{d \mid n} \rho_d\Big)\Big(\sum_{k \mid n} \xi_k\Big) \sim \frac{\varphi(q)}{q} G_q(\rho,\xi)\, X$$

as $X \to \infty$, it follows by Cauchy's inequality that

$$\Big(\frac{\varphi(q)}{q} G_q(\rho,\xi)\Big)^2 \leqslant G(\rho,\rho)\, G(\xi,\xi).$$

In particular,

$$G_q(\rho,\rho) \leqslant \frac{q}{\varphi(q)} G(\rho,\rho).$$

In general, we have the expression (see the lines between (7.12)–(7.13) of [1] for $g(p) = p^{-1}, h(p) = (p-1)^{-1}$)

$$G(\rho,\rho) = \sum_d \varphi(d) \Big(\sum_{m \equiv 0 \pmod d} \rho(m) m^{-1}\Big)^2.$$

For our choice (A.10), one derives from (A.13) with $M = z/d$ that

$$G(\rho,\rho) \ll (\log z)^{-2} \sum_{d<z} \frac{\mu^2(d)}{\varphi(d)} \ll (\log z)^{-1}. \tag{A.14}$$

From the above observations, one derives the following.

Lemma A.7. *Let $z > 1$, $z_1 > 1$ and*

$$V_q(z,z_1) = \sum\sum_{\substack{d<z,\, d_1<z_1 \\ (dd_1,q)=1}} \frac{\mu(d)\mu(d_1)}{[d,d_1]} \Big(\log\frac{z}{d}\Big)\Big(\log\frac{z_1}{d_1}\Big). \tag{A.15}$$

For any $q \geqslant 1$, *we have*

$$V_q(z, z_1) \ll \frac{q}{\varphi(q)} (\log z)^{1/2} (\log z_1)^{1/2}, \tag{A.16}$$

where the implied constant is absolute.

Lemma A.8. *Let* $1 < z_1 \leqslant z$ *and* $P \geqslant 1$. *Then*

$$W_P(z, z_1) = \sum_{\substack{d<z,\, d_1<z_1 \\ d|P,\, d_1|P}} \sum \frac{\mu(d)\mu(d_1)}{[d, d_1]} \Big(\log \frac{z}{d} \Big) \Big(\log \frac{z_1}{d_1} \Big) \ll (\log z)^2 \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p} \Big), \tag{A.17}$$

where the implied constant is absolute.

Proof. Apply (A.16) for q being the product of primes $p < z$, $p \nmid P$. Then (A.17) is equal to (A.15), so it is bounded by

$$(\log z) \prod_{\substack{p<z \\ p\nmid P}} \Big(1 + \frac{1}{p} \Big) \asymp (\log z)^2 \prod_{\substack{p<z \\ p|P}} \Big(1 - \frac{1}{p} \Big). \qquad \blacksquare$$

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