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Strongly Hyperbolic Systems of Maximal Rank

Dedicated to Professor Yujiro Ohya on his sixtieth birthday

By

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§ 1. Introduction

Let L be a first order differential operator defined in an open set Ω in $\mathbf{R}^{n+1}$

$$L(x, D) = D_0 I_m + \sum_{j=1}^{n} A_j(x) D_j$$

where $A_j(x)$ are real analytic $m \times m$ matrices defined in Ω and $x = (x_0, x') = (x_0, x_1, \ldots, x_n)$, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \ldots, \xi_n)$. Let us denote by $h(x, \xi)$ and $M(x, \xi)$ the determinant and the cofactor matrix of $L(x, \xi)$ respectively. Let $\Sigma = \{z = (x, \xi) \mid h(z) = \cdots = d^{m-1}h(z) = 0\}$ be the set of characteristics of order m of h. We assume that Σ is a real analytic manifold near a reference point $\hat{z} = (\hat{x}, \hat{\xi})$. Without restrictions we may suppose that $0 \in \Omega$ and $\hat{x} = 0$. Let Σ be given by

$$\phi_0(x, \xi) = \xi_0 = 0, \quad \phi_j(x, \xi') = 0, \ 1 \le j \le k$$

where $\phi_j(x, \xi')$ are real analytic, homogeneous of degree 0 in ξ' with linearly independent differentials at $\hat{z}$. Since we are interested in strongly hyperbolic systems, we assume that $L(x, \xi)$ satisfies a necessary condition for strong hyperbolicity obtained in [5], that is

$$M(x, \xi) \text{ vanishes of order } m-2 \text{ on } \Sigma$$

which implies in particular $(L|_\Sigma)^2 = O$ where $L|_\Sigma$ is the restriction of L to Σ. Thus we have

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$$0\leq \operatorname{rank}(L|_\Sigma)\leq [m/2].$$

In our previous papers [6], [7] we studied the extreme case that $\operatorname{rank}(L|_\Sigma)=0$, the case closest to symmetric systems. In this note we study the other extreme case when $\operatorname{rank}(L|_\Sigma)=[m/2]$, which is, in a sense, farthest away from symmetric systems. Our aim is to show that if the localization h_z of $h(x, \xi)$ at $z\in\Sigma$, the first non-trivial term in the Taylor expansion of h at z which is a polynomial on $T_z(T^*\Omega)/T_z\Sigma$, is strictly hyperbolic and the propagation cone of h_z is transversal to Σ at every $z\in\Sigma$ then $L(x, D)$ is strongly hyperbolic (Theorem 1.2). Here the propagation cone is defined as the dual cone of the hyperbolic cone of h_z with respect to the canonical symplectic structure on $T_z(T^*\Omega)$.

We remark that L is not symmetrizable and h, as a scalar operator, is not strongly hyperbolic if $m\geq 3$. In fact in order that h is strongly hyperbolic then every characteristic must be at most double ([1]).

The idea of the proof of strong hyperbolicity is very simple. Let $S^m(=S^m_{1,0})$ denote the space of symbols of order m and denote by Ψ^m the space of pseudo-differential operators with symbol in S^m (for the definition, see for example [2]). Then we can find $M_i\in\Psi^{m-1-i}$ so that for any lower order $B(x)$ we can apply our previous results in [4], [3], on the well posedness of the Cauchy problem for scalar operators or rather its proof to $(L+B)(M+M_1+M_2)$.

Theorem 1.1. *Let $\hat{z}\in\Sigma$ and on Σ near $\hat{z}$ we assume that* $\operatorname{rank} L(x, \xi)=[m/2]$ *and $M(x, \xi)$ vanishes of order $m-2$. Then one can find $M_i\in\Psi^{m-1-i}$, $i=1$, 2 defined near $\hat{z}$ such that with*

$$(L+B)(M+M_1+M_2)=hI_m+H_{m-1}+\cdots+H_{m-j}+\cdots$$

where $H_{m-j}\in\Psi^{m-j}$ near $\hat{z}$, we have either (i) *or* (ii):

(i) *for every $B(x)\in C^\infty(\Omega; M(m, \mathbb{C}))$, $H_{m-j}(x, \xi)$ vanishes of order $m-2j$ on Σ near $\hat{z}$,*

(ii) *for every $B(x)\in C^\infty(\Omega; M(m, \mathbb{C}))$ all elements of $H_{m-j}(x, \xi)$ vanish of order $m-2j$ on Σ near $\hat{z}$ except for the last row and column which vanishes of order $m-2j-1$ and $m-2j+1$ on Σ near $\hat{z}$ respectively.*

Remark. We can find $M_1\in\Psi^{m-2}$ so that either $(L+B)(M+M_1)$ or $(M+M_1)(L+B)$ verifies the assertion (i) of Theorem 1.1. We give the proof at the end of Section 3.

In virtue of Theorem 1.1 we can apply our previous results or rather its proof in [4], [3] to get

Theorem 1.2. *Assume the same assumptions as in* Theorem 1.1. *Suppose that the localization $h_{\hat{z}}$ is strictly hyperblic on $T_{\hat{z}}(T^*\Omega)/T_{\hat{z}}\Sigma$ and the propagation cone of $h_{\hat{z}}$ is transversal to Σ at $\hat{z}$. Then L is microlocally strongly hyperbolic near $\hat{z}$.*

With $A(x, \xi')=\sum_{j=1}^{n}A_j(x)\xi_j$ we can write

$$A(x, \xi')=\sum_{j=1}^{k}B_j(x, \xi')\phi_j(x, \xi')+P(x, \xi')$$

with $P(x, \xi')=A(x, \xi')|_{\Sigma}$. Then we have $\operatorname{rank} P(x, \xi')=[m/2]$ near $\hat{z}$ by assumption. Let us write

$$J(r)=\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\in M(r, \mathbf{R})$$

and denote $J(r_1, \ldots, r_s)=\oplus_{j=1}^{s}J(r_j)$. The next lemma is well known.

Lemma 1.3. *Suppose that* $\operatorname{rank} P(x, \xi')=[m/2]$ *near* $\hat{z}$. *Then there is a real analytic* $N(x, \xi')\in S^0$ *defined near* $\hat{z}$ *satisfying*

$$N(x, \xi')^{-1}P(x, \xi')N(x, \xi')=J|\xi'|$$

where $J=\oplus_1^{[m/2]}J(2)$ *if* m *is even and* $J=\oplus_1^{[m/2]}J(2)\oplus\{0\}$ *if* m *is odd.*

Since the existence of a parametrix of the Cauchy problem with finite propagation speed, which assures the well posedness of the Cauchy problem, is independent of changes of basis for $\mathbb{C}^m$, one can assume that

$$A(x, \xi')=\sum_{j=1}^{k}B_j(x, \xi')\phi_j(x, \xi')+J|\xi'|$$

if $\operatorname{rank} A(x, \xi')=[m/2]$ near $\hat{z}$ where $B_j\in S^1$ is real analytic near $\hat{z}$ and J is given in Lemma 1.3.

§ 2. Lemmas

Let

$$(2.1)\qquad L(x, \xi)=(a_{ij}(x, \xi))_{1\leq i,j\leq m}=\xi_0 I_m+\sum_{j=1}^{k}B_j(x, \xi')\phi_j(x, \xi')+J|\xi'|$$

where B_j are real analytic near $\hat{z}$. Let $J=\oplus_1^{s}J(2)\oplus_{s+1}^{t}\{0\}$ with $m=2s+t$. We actually interested in the case $t=s+1$. Denote by $\tilde{a}_{ij}$ the cofactor of a_{ij} in $L(x, \xi)$ so that $M=(\tilde{a}_{ji})$, the transposed of $(\tilde{a}_{ij})$. We denote $C(x, \xi)=O(s)$ if $C(x, \xi)$ vanishes of order s on Σ near $\hat{z}$ and we write $C(x, \xi)=O_w(s)$ if $C(x, \xi)$ vanishes of order s at $w\in\Sigma$. Note that

$$a_{ij}=O(1) \text{ unless } (i, j)=(2k-1, 2k),\quad 1\leq k\leq s.$$

In this section we show

Lemma 2.1. *Assume that* $h=O(m)$, $M=O(m-2)$. *Then we have*

$$a_{2i,2j-1}=O(2),\quad 1\le i,j\le s.$$

We first observe that

Lemma 2.2. *Assume that* $M(x,\xi)=O(m-2)$. *Then we have*

$$a_{2k-2l+2,2k-1}=O(2),\quad 1\le k\le s,\quad 2\le l,$$
$$a_{2k,2k-2l+1}=O(2),\quad 1\le k\le s,\quad 2\le l.$$

In particular, $a_{2i,2j-1}=O(2)$, $1\le i,j\le s$, $i\neq j$.

Proof. Let us denote by S_m the set of all permutations on $\{1,2,\dots,m\}$. Note that

$$\tilde{a}_{2k,2k-2l+1}=\cdots+\sum_{\sigma\in T}a^{\sigma}\xi_0^{m-4}+\cdots=O(m-2),\ a^{\sigma}=\prod_{i\neq 2k,\sigma(i)\neq i}a_{i\sigma(i)}$$

where $T=\{\sigma\in S_m|\ \#\{i|\sigma(i)=i\}=m-4,\ \sigma(2k)=2k-2l+1\}$. We show that $a^{\sigma}=O(2)$ if $\sigma\in T$ unless $\sigma(2k-2l+1)=2k-2l+2$, $\sigma(2k-1)=2k$ and hence $a^{\sigma}=\pm a_{2k-2l+2,2k-1}+O(2)$ which proves the assertion. Let $1\le k\le s$ and $\sigma\in T$. Assume that $\sigma(2k-2l+1)=p\neq 2k-2l+2$. If $p=2k$ there is i with $\sigma(i)<i$ and then $a_{i\sigma(i)}a_{2k-2l+1,2k}=O(2)$. If $p\neq 2k$, with $\sigma(p)=q$, $\sigma(q)=r$ we have $q\neq 2k$ if $\sigma\in T$. Thus $p\neq q$, $q\neq r$ hence $a_{p,q}a_{q,r}=O(1)$ and therefore $a_{2k-2l+1,p}a_{p,q}a_{q,r}=O(2)$. Thus we have $a^{\sigma}=O(2)$.

We next assume that $\sigma(2k-2l+1)=2k-2l+2$, $\sigma(2k-1)=p\neq 2k$. With $\sigma(p)=q$, $\sigma(q)=r$ we see $p\neq 2k-1$, $q\neq 2k$ if $\sigma\in T$. Thus $p\neq q$, $q\neq r$ and hence $a_{p,q}a_{q,r}=O(1)$. Then as above we have $a_{2k-1,p}a_{p,q}a_{q,r}=O(2)$ and hence the assertion. The second assertion can be proved similarly considering $\tilde{a}_{2k-2l+2,2k-1}$.

To complete the proof of Lemma 2.1 it is enough to show

Lemma 2.3. *Assume that* $h=O(m)$ *and* $a_{2i,2j-1}=O(2)$, $1\le i,j\le s$, $i\neq j$. *Then we have*

$$a_{2i,2j-1}=O(2),\quad 1\le i,j\le s.$$

Proof. It is enough to prove $a_{2i,2i-1}=O_w(2)$ for every $w\in\Sigma$ near $\hat{z}$, $1\le i\le s$. Suppose that the assertion does not hold and hence the differentials of $a_{2i_p,2i_p-1}$ at $w\in\Sigma$ were different from zero for $p=1,\dots,l$, $1\le i_p\le s$ and $a_{2i,2i-1}=O_w(2)$ if $i\notin\{i_1,\dots,i_l\}$. Set $T=\{\sigma\in S_m|\ \#\{i|\sigma(i)=i\}=m-2l\}$ and $J^{\sigma}=\{1,\dots,m\}\setminus\{i|\sigma(i)=i\}$ for $\sigma\in T$. We recall that

$$h=\cdots+\sum_{\sigma\in T}a^{\sigma}\xi_0^{m-2l}+\cdots=O(m),\quad a^{\sigma}=\prod_{i,\sigma(i)\neq i}a_{i\sigma(i)}\tag{2.2}$$

by assumption. Let $\nu \in T$ be such that $\nu(2i_p-1)=2i_p$, $\nu(2i_p)=2i_p-1$ and note that a^ν vanishes at w exactly of order l. We show that $a^\sigma=O_w(l+1)$ unless $\sigma=\nu$. If $\#\{i|\sigma(2i-1)=2i\}<l$ then it is clear that $a^\sigma=O_w(l+1)$. Thus it suffices to consider the case $J^\sigma=\{2j_p-1, 2j_p\}_{p=1}^l$. Suppose that $J^\sigma \neq J^\nu$ and hence there were $2q-1$ such that $2q-1 \in J^\sigma$, $2q-1 \notin J^\nu$. Since $a_{2q,2q-1}=O_w(2)$ by assumption we get $a^\sigma=O_w(l+1)$ and hence the assertion. Therefore from (2.2) we would have $a^\nu=O_w(l+1)$ which is a contradiction.

§ 3. Proof of Theorem 1.1

In this section we work near $\hat{z}$ without mention it. We denote by $\sigma(M)$ the symbol of an operator M and by $Op(M)$ the operator with symbol M. But we frequently use M to denote both an operator and its symbol if this leads no confusion.

We first assume that m is even and we write $2m$ instead of m.

Proposition 3.1. *Assume that $h(x, \xi)$ and $M(x, \xi)$ vanishes of order $2m$ and $2m$-2 on $\sum$ respectively. Then there is $M_1 \in \Psi^{2m-2}$ with $\sigma(M_1)=O(2m-3)$ such that for any $B(x)$ we have*

$$(L+B)(M+M_1)=hI_{2m}+H_{2m-1}+\cdots+H_{2m-j}+\cdots$$

where $H_{2m-j} \in \Psi^{2m-j}$ and $\sigma(H_{2m-j})=O(2m-2j)$.

For the proof we remark that

Lemma 3.2. *Under the same assumptions as in* Proposition 3.1 *every even row and odd column of M is $O(2m-1)$.*

Proof. Since $LM=hI_{2m}$ it follows that $\{J|\xi'|+O(|\phi|)\}M=hI_{2m}$ that is $J|\xi'|M=O(2m-1)$. This implies clearly that every even row of M is $O(2m-1)$. Considering $ML=hI_{2m}$ the second assertion follows similarly.

Proof of Proposition 3.1. Recall that $\sigma(LM)=hI_{2m}+\sum_\alpha L^{(\alpha)}M_{(\alpha)}/\alpha!$ and note that

$$S^{2m-j} \ni \sum_{|\alpha|=j} \frac{1}{\alpha!} L^{(\alpha)}M_{(\alpha)}=O(2m-2-j).$$

Since $2m-2j \leq 2m-2-j$ for $j \geq 2$, it is clear that $LM-Op(\sum_{|\alpha|=1}L^{(\alpha)}M_{(\alpha)})$ verifies the desired properties. Set

$$\sum_{|\alpha|=1} L^{(\alpha)}M_{(\alpha)}=T_e+T_o$$

where T_e and T_o consists of even and odd rows of $\sum_{|\alpha|=1}L^{(\alpha)}M_{(\alpha)}$ respectively. Set

$$M_1=-{}^tJT_o|\xi'|^{-1}$$

so that $J|\xi'|M_1=-T_o$ and $M_1=O(2m-3)$. Then it follows that $L(M+M_1)=T_e+$ desired form, because $L=J|\xi'|+O(|\phi|)$. It is also clear that $B(M+M_1)$ has the desired form for every B. Thus it suffices to study T_e. Note that the $2i$-th row of T_e is given by a sum of the following terms over $|\alpha|=1$

$$\sum_{k=1}^{2m} a_{2i,k}^{(\alpha)}\tilde{a}_{jk(\alpha)}=\sum_{k\,:\,\text{even}} a_{2i,k}^{(\alpha)}\tilde{a}_{jk(\alpha)}+\sum_{k\,:\,\text{odd}} a_{2i,k}^{(\alpha)}\tilde{a}_{jk(\alpha)}.$$

By Lemma 3.2 we see that $\tilde{a}_{jk}=O(2m-1)$ if k is even and by Lemma 2.1 we have $a_{2i,k}=O(2)$ if k is odd. Then it follows that $T_e=O(2m-2)$. This proves the assertion.

We turn to the odd m case. We write $2m+1$ instead of m. Our aim is to prove that

Proposition 3.3. *Assume that $h(x, \xi)$ and $M(x, \xi)$ vanishes of order $2m+1$ and $2m-1$ on Σ respectively. Then we have either (i) or (ii):*

(i) *there is $M_1\in\Psi^{2m-1}$ with $\sigma(M_1)=O(2m-2)$ such that*

$$(L+B)(M+M_1)=hI_{2m+1}+H_{2m}+\cdots+H_{2m+1-j}+\cdots$$

where $H_{2m-j}\in\Psi^{2m+1-j}$ and $\sigma(H_{2m+1-j})=O(2m+1-2j)$,

(ii) *there are $M_i\in\Psi^{2m-i}$, $i=1, 2$ such that*

$$(L+B)(M+M_1+M_2)=hI_{2m+1}+H_{2m}+\cdots+H_{2m+1-j}+\cdots$$

where every element of $\sigma(H_{2m+1-j})$ is $O(2m+1-2j)$ except for the last row and column which is $O(2m-2j)$ and $O(2m+2-2j)$ respectively.

To prove the proposition we start with

Lemma 3.4. *Assume that $h(x, \xi)$ and $M(x, \xi)$ vanishes of order $2m+1$ and $2m-1$ respectively. Then every even row of M is $O(2m)$ and every odd column except for the last one is $O(2m)$.*

Proof. The proof is a repetition of that of Lemma 3.2.

Lemma 3.5. *Assume that $M(x, \xi)$ vanishes of order $2m-1$ on Σ. If there is i such that $da_{2m+1,2i-1}(w)\neq 0$ with some $w\in\Sigma$ near $\hat{z}$ then we have $\tilde{a}_{2m+1,j}=O(2m)$ for every j, $\tilde{a}_{2m+1,2i}=O(2m+1)$ for $i\leq m$ and $a_{2j,2m+1}=O(2)$ for $j\leq m$.*

Proof. Note that

$$\sum_{k=1}^{2m+1}\tilde{a}_{kj}a_{k,2i-1}=\sum_{k\,:\,\text{odd},k\neq 2m+1}\tilde{a}_{kj}a_{k,2i-1}+\sum_{k\,:\,\text{even}}\tilde{a}_{kj}a_{k,2i-1}+\tilde{a}_{2m+1,j}a_{2m+1,2i-1}=O(2m+1). \tag{3.1}$$

Recall that $a_{k,2i-1}=O(2)$ if k is even and $i\leq m$ by Lemma 2.1 and $\tilde{a}_{kj}=O(2m)$ if $k\neq 2m+1$ is odd by Lemma 3.4. From (3.1) we get $\tilde{a}_{2m+1,j}\,a_{2m+1,2i-1}=O(2m+1)$ for every j. Since $\tilde{a}_{2m+1,j}$ is, up to term $O(|\phi|^{2m})$, a polynomial in ϕ of degree $2m-1$ with coefficients which are real analytic on Σ we conclude from (3.1) that the coefficient vanishes near w and so does near $\hat{z}$. Thus we get the first assertion. To prove the second assertion we note that

$$\sum_{k=1}^{2m+1} a_{2i-1,k}\,\tilde{a}_{2m+1,k}=0 \tag{3.2}$$

and $a_{2i-1,2i}=|\xi'|$, $a_{2i-1,k}=O(1)$ if $k\neq 2i$. Since $\tilde{a}_{2m+1,k}=O(2m)$ by the first assertion we get the second assertion from (3.2).

We turn to the third assertion. Consider

$$\sum_{k\,:\,\mathrm{odd},\,k\neq 2m+1} a_{2j,k}\tilde{a}_{2m+1,k}+\sum_{k\,:\,\mathrm{even}} a_{2j,k}\tilde{a}_{2m+1,k} + a_{2j,2m+1}\tilde{a}_{2m+1,2m+1}=0. \tag{3.3}$$

If k is even it follows from the second assertion that $\tilde{a}_{2m+1,k}=O(2m+1)$. On the other hand we have $\tilde{a}_{2m+1,k}=O(2m)$ from the first assertion and $a_{2j,k}=O(2)$ if $k\neq 2m+1$ is odd by Lemma 2.1. Since $\tilde{a}_{2m+1,2m+1}$ vanishes at $\hat{z}$ exactly of order $2m$ the third assertion follows from (3,3).

Lemma 3.6. *Assume that $a_{2m+1,2i-1}=O(2)$ for every $i\leq m$. Then we have $\tilde{a}_{j,2m+1}=O(2m)$ for every j.*

Proof. Note that

$$\sum_{k=1}^{2m+1} a_{2m+1,k}\tilde{a}_{jk}=\sum_{k\,:\,\mathrm{odd},\,k\neq 2m+1} a_{2m+1,k}\tilde{a}_{jk}+\sum_{k\,:\,\mathrm{even}} a_{2m+1,k}\tilde{a}_{jk} + a_{2m+1,2m+1}\tilde{a}_{j,2m+1}=O(2m+1). \tag{3.4}$$

If $k\neq 2m+1$ is odd then $a_{2m+1,k}=O(2)$ by assumption. If k is even then $\tilde{a}_{jk}=O(2m)$ by Lemma 3.4. Since $a_{2m+1,2m+1}$ vanishes at $\hat{z}$ exactly of order 1, the assertion follows from (3.4).

Proof of Proposition 3.3. We divide the proof into two cases.

(a) Assume that $a_{2m+1,2i-1}=O(2)$, $i\leq m$. Set

$$S_j=\sum_{|\alpha|=j} L^{(\alpha)}M_{(\alpha)}/\alpha!=T_{je}+T_{jo}$$

where T_{je} consists of even rows and $(2m+1)$-th row of S_j while T_{jo} consists of $(2i-1)$-th rows of S_j, $1\leq i\leq m$. Consider the $(2m+1)$-th row of S_1 which is a sum of the following terms over $|\alpha|=1$

$$\sum_{k=1}^{2m+1} a^{(\alpha)}_{2m+1,k}\tilde{a}_{jk(\alpha)} = \sum_{k:\,\text{odd},k\neq 2m+1} a^{(\alpha)}_{2m+1,k}\tilde{a}_{jk(\alpha)} + \sum_{k:\,\text{even}} a^{(\alpha)}_{2m+1,k}\tilde{a}_{jk(\alpha)}$$
$$+a^{(\alpha)}_{2m+1,2m+1}\tilde{a}_{j,2m+1(\alpha)}.$$

From the assumption it follows that $a^{(\alpha)}_{2m+1,k}=O(1)$ if $k\neq 2m+1$ is odd. On the other hand from Lemma 3.4 we get $\tilde{a}_{jk(\alpha)}=O(2m-1)$ if k is even and by Lemma 3.6 we have $\tilde{a}_{j,2m+1(\alpha)}=O(2m-1)$ for every j. This proves $T_{1e}=O(2m-1)$. As in the proof of Proposition 3.1 we set

$$M_1=-{}^tJT_{1o}|\xi'|^{-1}.$$

Then we have $M_1\in\Psi^{2m-1}$ with $M_1=O(2m-2)$ and setting

$$(L+B)(M+M_1)=hI_{2m+1}+H_{2m}+\cdots+H_{2m+1-j}+\cdots$$

it is easy to see that $H_{2m+1-j}\in\Psi^{2m+1-j}$ and $H_{2m+1-j}=O(2m+1-2j)$ since L has the form $J|\xi'|+O(|\phi|)$.

(b) Assume that there is i such that $da_{2m+1,2i-1}(w)\neq 0$ with some $w\in\sum$ near $\hat{z}$. Let S_j, T_{je} and T_{jo} be as in (a) again. From Lemmas 2.1, 3.4 and 3.5 we see that $a_{2i,k}=O(2)$ for odd k and $\tilde{a}_{jk}=O(2m)$ for even k. This proves that

$$\text{even row of } S_j=O(2m+1-2j),\ j=1,2$$

while the last row of S_j is $O(2m-1-j)$, $j=1,2$. Moreover by Lemma 3.5 we have $\tilde{a}_{2m+1,k}=O(2m)$ for every k, $\tilde{a}_{2m+1,k}=O(2m+1)$ for even k and $a_{2i,k}=O(2)$ for odd k by Lemmas 2.1 and 3.5. This proves that

$$\sum_{k=1}^{2m+1} a^{(\alpha)}_{2i,k}\tilde{a}_{2m+1,k(\alpha)}=O(2\text{m}+2-2|\alpha|)$$

and hence the last column of T_{je} is $O(2m+2-2j)$, $j=1,2$. Let us set

$$M_j=-{}^tJT_{jo}|\xi'|^{-1},\quad j=1,2$$

so that $M_j\in\Psi^{2m-j}$ and $M_j=O(2m-1-j)$. We remark that the last column of M_j is $O(2m-j)$ for $j=0,1,2$ with $M_0=M$ because $\tilde{a}_{2m+1,k}=O(2m)$.

Consider $(L+B)(M+M_1+M_2)$. Note that $S_j=O(2m+2-2j)$ if $j\geq 3$ and hence $LM=T_1+T_2+$desired form (ii). Set

$$(L+B)(M_1+M_2)=-T_{1o}-T_{2o}+F_{2m}+\cdots+F_{2m+1-j}+\cdots$$

where $F_{2m+1-j}\in\Psi^{2m+1-j}$. It is not difficult to see that $F_{2m+1-j}=O(2m+1-2j)$ and the last column of F_{2m+1-j} is $O(2m+2-2j)$. This proves the assertion (ii).

Finally we give a proof of Remark in Section 1. Since the last column of M_j is $O(2m-j)$ and $L=J|\xi'|+O(|\phi|)$ it is enough to prove that, in case (b) above, one can find $M_1\in\Psi^{2m}$ so that $(M+M_1)(L+B)$ verifies the assertion (i) of Proposition 3.1. From Lemma 3.5 it follows that $\tilde{a}_{2m+1,j}$ $O(2m)$ and $a_{2j,2m+1}=O(2)$. Let

$$S_1=\sum_{|\alpha|=1} M^{(\alpha)}L_{(\alpha)}=T_e+T_o$$

where T_e and T_o consists of even and odd columns of S_1 respectively. Set

$$M_1=-T_e^tJ|\xi'|^{-1}$$

so that $M_1J|\xi'|=-T_e$ and $M_1=O(2m-2)$. Then it is clear that $(M+M_1)(L+B)$ $=T_o+$desired term (:). Thus it is enough to study T_o. Consider

$$(3.5)\qquad \sum_{k\,:\,\mathrm{odd},k\neq 2m+1} \tilde{a}_{kj}^{(\alpha)}a_{k,2i+1(\alpha)}+\sum_{k\,:\,\mathrm{even}} \tilde{a}_{kj}^{(\alpha)}a_{k,2i+1(\alpha)}+\tilde{a}_{2m+1,j}^{(\alpha)}a_{2m+1,2i+1(\alpha)}.$$

From Lemma 3.4 we have $\tilde{a}_{kj}=O(2m)$ if $k\neq 2m+1$ is odd while $a_{k,2i-1}=O(2)$ if k is even and $i\leq m$ which follows from Lemma 2.1. Then (3.5) with $|\alpha|=1$ shows that $T_o=O(2m-1)$ except for the last column. We treat the last column which is a sum of (3.5) with $i=m$ over $|\alpha|=1$. Since $a_{k,2m+1}=O(2)$ by virtue of Lemma 3.5 the same arguments as above show $T_o=O(2m-1)$ and hence the result.

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