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Unique continuation on convex domains

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Abstract. In this paper, we obtain estimates on the quantitative strata of the critical set of non-trivial harmonic functions u which vanish continuously on $V \subset \partial\Omega$, a relatively open subset of the boundary of a convex domain $\Omega \subset \mathbb{R}^n$. In particular, these estimates improve dimensional estimates on $\{|\nabla u| = 0\}$ both in $V \subset \partial\Omega$ and as it *approaches* $V \cap \overline{\Omega}$. These estimates are not obtainable by naively combining interior and boundary estimates, and represent a significant improvement upon existing results for boundary analytic continuation in the convex case.

1. Introduction

Unique continuation is a fundamental property for functions which solve the Laplace and related linear equations. A closely related problem is that of boundary unique continuation: given a domain $\Omega \subset \mathbb{R}^n$ and a function u which is harmonic in Ω and vanishes continuously on $V \subset \partial\Omega$, how large can the set $\{Q \in V : |\nabla u| = 0\}$ be if $u \not\equiv 0$? Boundary unique continuation is closely tied to the Cauchy problem and questions of well-posedness and stability of solutions to boundary value problems (see, for instance, [21] and [4]). In this paper, we address two questions. First, we address the question of boundary unique continuation for harmonic functions on convex domains. Second, we also address the question of how the critical set $\{|\nabla u| = 0\} \cap \Omega$ approaches $V \subset \partial\Omega$. We follow the approach of Garofalo and Lin [11] insofar as we make essential use of the Almgren frequency function. And, because we want to obtain results on the full critical set $\{|\nabla u| = 0\}$, we use packing estimates inspired by Cheeger, Naber, and Valtorta [8]. These tools allow us to obtain results about the strata of the critical set $\{|\nabla u| = 0\}$ *as it approaches* $V \subset \partial\Omega$.

1.1. Background on boundary unique continuation for harmonic functions

For dimensions $n \geq 3$, Bourgain and Wolff [6] have constructed an example of a function $u: \mathbb{R}^n_+ \to \mathbb{R}$ which is harmonic in $\mathbb{R}^n_+$, C^1 up to the boundary $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, and for which both u and ∇u vanish on a set of positive surface measure. This result has been generalized

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by Wang [23] to $C^{1,\alpha}$ domains, $\Omega \subset \mathbb{R}^n$, for $n \geq 3$. However, the sets of positive measure for which these functions vanish are *not* open.

In general, the following question posed by Lin in [18] is still open.

Question 1.1. *Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an open, connected Lipschitz domain. If u is a harmonic function which vanishes continuously on a relatively open set $V \subset \partial\Omega$, does*

$$\mathcal{H}^{n-1}(\{x \in V : |\nabla u| = 0\}) > 0$$

imply that u is the zero function?

If u is non-negative, the techniques of PDEs on non-tangentially accessible (NTA) domains give a comparison principle [9] which allows us to say that the norm of the normal derivative is point-wise comparable to the density of the harmonic measure with respect to the surface measure $d\sigma$. Additionally, for Lipschitz domains it is well known that the harmonic measure is mutually absolutely continuous with respect to $d\sigma$. These two facts then imply that if the normal derivative vanishes on a set of positive (surface) measure, then u must be identically 0.

The challenge is for harmonic functions u which change sign. For such functions, the aforementioned techniques fail completely because we cannot apply the Harnack principle. Authors have therefore approached this problem by asking for additional regularity. In [18], Lin proves that for $C^{1,1}$ domains, $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, if u is a non-constant harmonic function which vanishes on a relatively open set $V \subset \partial\Omega$, then $\dim_{\mathcal{H}}(\{x \in V : |\nabla u| = 0\}) \leq n-2$. Similar results were later shown by Adolfsson and Escauriaza for domains with locally $C^{1,\alpha}$ boundary, [1]. Relatedly, Kukavica and Nystöm showed that $\mathcal{H}^{n-1}(\{x \in V : |\nabla u| = 0\}) > 0$ implies that $u \equiv 0$ if $\partial\Omega$ is C^1 Dini, [17]. Recently, this result has been greatly improved. Kenig and Zhou [16] employed powerful techniques from [19] and [10] to show that for C^1 Dini domains, the $(n-2)$-generalized singular set $\{u = 0 = |\nabla u|\}$ has finite $(n-2)$-dimensional upper Minkowski content.

For merely convex domains $\Omega \subset \mathbb{R}^n$, Adolfsson, Escauriaza, and Kenig showed that if u is a harmonic function in Ω which vanishes continuously on a relatively open set $V \subset \partial\Omega$, then if $\{x \in V : |\nabla u| = 0\}$ has positive surface measure, u must be a constant function [2]. The method of attack pursued in [2] (and [1, 17, 18]) was centered on showing that the harmonic function is "doubling" on the boundary in the following sense. If $\Omega \subset \mathbb{R}^n$, then there exists an absolute constant $M < \infty$ such that, for all $B_{2r}(Q) \cap \partial\Omega \subset V$,

$$\int_{B_{2r}(Q)\cap\Omega} u^2\, dx \leq M \int_{B_r(Q)\cap\Omega} u^2\, dx.$$

This doubling property allows the authors to show that the normal derivative is an A_2-Muckenhoupt weight with respect to surface measure, a kind of quantified version of mutual absolute continuity. It is well known that if u vanishes in a surface ball $\Delta_r(Q)$ and the normal derivative of u is a 2-weight with respect to surface measure, then either $\{Q' \in \Delta_r(Q) : |\nabla u| = 0\}$ has measure zero or $\{Q' \in \Delta_r(Q) : |\nabla u| > 0\}$ has measure zero. The improvement from measure to dimension bounds in [1] and [18] comes from applying an additional Federer dimension-reduction type argument.

Recently, Tolsa has answered Question 1.1 in the affirmative for C^1 domains and Lipschitz domains with small Lipschitz constant, [22].

In this paper, we restrict our investigation to convex domains Ω and obtain bounds upon the *full* generalized critical set $\{|\nabla u| = 0\}$. We note that this includes $\{x \in \partial\Omega : |\nabla u| = 0\}$ and $\{x \in \Omega : |\nabla u| = 0\}$.

Theorem 1.2. *Let Ω be a convex domain and let $u \in C^0(\overline{\Omega})$ be a non-constant function such that $\Delta u = 0$ in Ω. Let $V \subset \partial\Omega$ be a relatively open set. If $u = 0$ on V, then for any compact subset $K \subset V$, there exists a radius $0 < r(K)$ such that*

$$\dim_{\mathcal{H}}(B_r(K) \cap \{|\nabla u| = 0\} \setminus \mathrm{sing}(\partial\Omega)) \leq \overline{\dim}_{\mathcal{M}}(B_r(K) \cap \{|\nabla u| = 0\} \setminus \mathrm{sing}(\partial\Omega)) \leq n-2.$$

Furthermore,

$$\overline{\dim}_{\mathcal{M}}\big(B_r(K) \cap \overline{\{|\nabla u| = 0\} \cap \Omega}\big) \leq n-2.$$

The content of Theorem 1.2 is two-fold. First, consider the results restricted to the boundary $\{|\nabla u| = 0\} \cap V \subset \partial\Omega$. Alberti [3] proved (among other things) that the singular set of a convex function is a C^2 $(n-2)$-rectifiable set, which implies that the geometric singular set of a convex body satisfies $\dim_{\mathcal{H}}(\mathrm{sing}(\partial\Omega)) \leq n-2$. Thus, Theorem 1.2 combined with [3] implies

$$\dim_{\mathcal{H}}(V \cap \{|\nabla u| = 0\}) \leq n-2,$$

which gives a strong improvement on the results of [2], which proved that in this situation $\mathcal{H}^{n-1}(V \cap \{|\nabla u| = 0\}) = 0$.

Second, Theorem 1.2 provides new insight into how the critical set interacts with $\partial\Omega$. Returning to [3], Alberti also proved that the singular set of a convex function may be prescribed to be *any* C^2 $(n-2)$-rectifiable set. Thus, it can happen that $\overline{\dim}_{\mathcal{M}}(\mathrm{sing}(\partial\Omega)) = n-1$. On the other hand, from the interior perspective, [20] proved finite $(n-2)$-dimensional upper Minkowski content bounds on $\{|\nabla u| = 0\}$ in the interior. However, the naive application of these estimates degenerate as one approaches the boundary because upper Minkowski dimension is *not* stable under countable unions. Considering $\{|\nabla u| = 0\} \cap \Omega$ as it approaches $\partial\Omega$, it was unknown whether or not $\{|\nabla u| = 0\} \cap \Omega$ could oscillate wildly and have positive $(n-1)$-upper Minkowski dimension like $\mathrm{sing}(\partial\Omega)$, would remain $(n-2)$-upper Minkowski dimensional like the interior, or if something in between these two held. Theorem 1.2 proves that the set $\{|\nabla v| = 0\} \cap \Omega$ cannot oscillate too wildly as it approaches $\partial\Omega$, and that $\overline{\{|\nabla u| = 0\} \cap \Omega} \cap K$ inherits its upper Minkowski dimension bounds from the interior rather than the boundary.

It is still an open question whether or not the $(n-2)$-upper Minkowski content of $\{|\nabla u| = 0\} \cap \partial\Omega \setminus \mathrm{sing}(\partial\Omega)$ is finite.

2. Definitions and main results

Theorem 1.2 is a corollary to Theorem 2.11 below and a containment result (Lemma 2.12). In order to state these results, we need the following definitions.

We start by defining a certain class of domains.

Definition 2.1 (A normalized class of convex domains). Let $\mathcal{D}(n)$ be the collection of connected, open domains $\Omega \subset \mathbb{R}^n$ which satisfy the following conditions:

(1) $0 \in \partial\Omega$.

(2) $\Omega \cap B_2(0)$ is convex.

(3) $\Omega \cap (B_2(0))^c \neq \emptyset$.

One of the key tools of this paper will be an Almgren frequency function, introduced by Almgren in [5].

Definition 2.2. (Almgren frequency function) Let $r > 0$, $\Omega \subset \mathbb{R}^n$, $u\colon \mathbb{R}^n \to \mathbb{R}$ such that $u \in C(B_{2r}(p)) \cap W^{1,2}(B_{2r}(p))$ and $p \in \overline{\Omega}$. We define the following quantities:

$$H_\Omega(p,r,u) := \int_{\partial B_r(p) \cap \overline{\Omega}} |u - u(p)|^2 \, d\sigma,$$

$$D_\Omega(p,r,u) := \int_{B_r(p)} |\nabla u|^2 \, dx,$$

$$N_\Omega(p,r,u) := r \, \frac{D_\Omega(p,r,u)}{H_\Omega(p,r,u)}.$$

Remark 2.3 (Invariances of the Almgren frequency function). This normalized version of the Almgren frequency function is invariant in the following senses. Let $a, b, c \in \mathbb{R}$ with $a, r \neq 0$. If $w(x) = au(bx + p) + c$ and $T_{p,b}\Omega = \frac{1}{b}(\Omega - p)$, then

$$N_\Omega(p,r,u) = N_{T_{p,b}\Omega}(0, b^{-1}r, w).$$

We now define the class of functions in which we will work in this paper.

Definition 2.4 (A class of functions). Let $\mathcal{A}(n,\Lambda)$ be the set of functions $u\colon \mathbb{R}^n \to \mathbb{R}$ which have the following properties:

(1) $u\colon \mathbb{R}^n \to \mathbb{R}$ is harmonic in a convex domain, $\Omega \in \mathcal{D}(n)$.

(2) $u \in C(\overline{B_2(0)})$, $u = 0$ on $\Omega^c \cap B_2(0)$, and u is non-constant.

(3) $N_\Omega(0,2,u) \leq \Lambda$.

We shall use rescalings which are adapted to the quantitative stratification methods introduced by Cheeger and Naber in [7] for studying the regularity of stationary harmonic maps and minimal currents.

Definition 2.5 (Rescalings). For a set $\Omega \subset \mathbb{R}^n$, let $T_{p,r}\Omega := (\Omega - p)/r$. For $u \in \mathcal{A}(n,\Lambda)$, let Ω be its associated domain. We define the rescaled function $T_{x,r}u$ of u at a point $x \in B_1(0)$ at scale $0 < r < 1$ by

$$T_{x,r}u(y) := \frac{u(x + ry) - u(x)}{\left(\int_{\partial B_1(0) \cap \overline{T_{x,r}\Omega}} (u(x + rz) - u(x))^2 \, d\sigma(z)\right)^{1/2}}.$$

In the case that the denominator is zero, we define $T_{x,r}u = \infty$.

The geometry we wish to capture with the rescalings $T_{x,r}f$ is encoded in their translational symmetries.

Definition 2.6 (The class of blow-up profiles). Let $u \in C(\mathbb{R}^n)$. We say u is 0-*symmetric* if u satisfies one of the following conditions:

(1) u is a homogeneous harmonic polynomial,

(2) u is homogeneous and harmonic in a convex cone $\Omega' \in \mathcal{D}(n)$ and vanishes in Ω'^{c}.

We will say that u is k-*symmetric* if u is 0-symmetric and there exists a k-dimensional subspace V such that $u(x+y) = u(x)$ for all $x \in \mathbb{R}^n$ and all $y \in V$.

We now define the quantitative version of symmetry which describes how close to being k-symmetric a function is in a ball, $B_r(x) \subset \mathbb{R}^n$.

Definition 2.7 (Quantitative symmetry). Let $u \in \mathcal{A}(n,\Lambda)$, with associated domain Ω. The function u will be called (k,ε,r,p)-*symmetric* if there exists a k-symmetric function P with degree of homogeneity bounded by $C(n,\Lambda)$ such that

(1) $\int_{\partial B_1(0)} |P|^2 \, d\sigma = 1$,

(2) $\int_{B_1(0)\cap \overline{T_{p,r}\Omega}} |T_{p,r}u - P|^2 \, dV < \varepsilon$.

We shall say that u is (k,δ_0,r,p)-symmetric *with respect to a k-dimensional subspace V* if there is a k-symmetric function P which satisfies that u is (k,δ_0,r,p)-symmetric such that $P(x+y) = P(x)$ for all $y \in V$.

Remark 2.8. The value of the homogeneity bound $C(n,\Lambda)$ in Definition 2.7 will be taken to be $2C_2(n,\Lambda)$ for C_2 as in Lemma 6.2, below.

Definition 2.9 (Quantitative generalized critical strata). Let $u \in \mathcal{A}(n,\Lambda)$, with $\Omega \in \mathcal{D}(n)$ its associated domain. For $0 < \varepsilon$, $0 < r \leq 1$, and integer $0 \leq k \leq n-1$, we denote the (k,ε,r)-*generalized critical strata* of u by $\mathcal{C}^k_{\varepsilon,r}(u)$, and we define it by

$$\mathcal{C}^k_{\varepsilon,r}(u) := \{x \in \overline{\Omega} : u \text{ is not } (k+1,\varepsilon,s,x)\text{-symmetric for all } r \leq s \leq 1\}.$$

We shall also use the notation $\mathcal{C}^k_{\varepsilon}(u)$ for $\mathcal{C}^k_{\varepsilon,0}(u)$.

It is immediate from the definitions that $\mathcal{C}^k_{\varepsilon,r}(u) \subset \mathcal{C}^{k'}_{\varepsilon',r'}(u)$ if $k \leq k'$, $\varepsilon' \leq \varepsilon$ and $r \leq r'$. Moreover, if $u \in \mathcal{A}(n,\Lambda)$ is (k,ε,r,x)-symmetric, then by the continuity of $g(h) = \int_{B_r(x+h)} |u|^2 \, dx$, there exists a radius $0 < r_x$ such that u is (k,ε,r,x)-symmetric in $B_{r_x}(x)$. As a special case, this implies that the $\mathcal{C}^k_{\varepsilon,r}(u)$ are closed.

Definition 2.10 (Qualitative generalized critical set). Let $u \in \mathcal{A}(n,\Lambda)$, with $\Omega \in \mathcal{D}(n)$ its associated domain. Using the quantitative generalized critical strata, we define the *generalized critical set* of u as $\mathcal{C}^{n-2}(u) := \cup_\eta \cap_r \mathcal{C}^{n-2}_{\eta,r}(u)$. In turn, we define the strata of the generalized critical set as follows: $\mathcal{C}^k(u) := \cup_\eta \cap_r \mathcal{C}^k_{\eta,r}(u)$.

We shall use the convention that for any $A \subset \mathbb{R}^n$, $B_r(A) = \{x \in \mathbb{R}^n : d(A,x) < r\}$. Recall that we can define the upper Minkowski s-content by

$$\mathcal{M}^{*,s}(A) = \limsup_{r\to 0} \frac{\mathrm{Vol}(B_r(A))}{\omega_{n-s} r^{n-s}}, \tag{2.1}$$

and the upper Minkowski dimension as

$$\overline{\dim}_{\mathcal{M}}(A) := \inf\{s : \mathcal{M}^{*,s}(A) = 0\} = \sup\{s : \mathcal{M}^{*,s}(A) > 0\}.$$

It is a basic result that the s-dimensional Hausdorff measure satisfies

$$\mathcal{H}^s(A) : \lim_{\delta\to 0} \inf\left\{\sum \operatorname{diam}(U_i)^s : A \subset \cup_i U_i \text{ and } \operatorname{diam}(U_i) \leq \delta\right\} \leq \mathcal{M}^{*,s}(A),$$

and hence the Hausdorff dimension

$$\dim_{\mathcal{H}}(A) := \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) > 0\}$$

satisfies $\dim_{\mathcal{H}}(A) \leq \overline{\dim_{\mathcal{M}}}(A)$ for all sets $A \subset \mathbb{R}^n$. Now, we can state the main technical results.

Theorem 2.11 (Technical theorem). *Let $u \in \mathcal{A}(n, \Lambda)$. Then, for any $r_0 > 0$ and for all $0 < r_0 < r$,*

$$\operatorname{Vol}\big(B_r(\mathcal{C}^k_{\varepsilon,r_0}(u) \cap B_{1/4}(0))\big) \leq C(n, \Lambda, k, \varepsilon)\, r^{n-k-\varepsilon}. \tag{2.2}$$

In particular, letting $r_0 \to 0$,

$$\mathcal{H}^{k+\varepsilon}\big(\mathcal{C}^k_\varepsilon(u) \cap B_{1/4}(0)\big) \leq \mathcal{M}^{*,k+\varepsilon}\big(\mathcal{C}^{k+\varepsilon}_\varepsilon(u) \cap B_{1/4}(0)\big) \leq C(n, \Lambda, k, \varepsilon).$$

Lemma 2.12 (Containment). *There exists an $0 < \varepsilon = \varepsilon(n, \Lambda)$ such that*

$$\big(\overline{\mathcal{C}^{n-2}(u) \cap \Omega}\big) \cap B_{1/8}(0) \subset \mathcal{C}^{n-2}_\varepsilon(u) \tag{2.3}$$

and

$$\big(\mathcal{C}^{n-2}(u) \cap \partial\Omega \setminus \operatorname{sing}(\partial\Omega)\big) \cap B_{1/8}(0) \subset \mathcal{C}^{n-2}_\varepsilon(u). \tag{2.4}$$

Proof of Theorem 1.2 assuming Lemma 2.12. For each point $x \in K \subset V$, there is a radius $0 < r$ such that $B_{4r}(x) \cap \partial\Omega \subset V$. Since K is compact, we may find a finite subcover $\{B_{r_i}(x_i)\}_i$. Thus, Lemma 2.12 implies that in each $B_{r_i}(x_i)$,

$$\inf\{s : \mathcal{M}^{*,s}(B_{r_i}(x_i) \cap \{|\nabla u| = 0\} \setminus \operatorname{sing}(\partial\Omega)) < \infty\} \leq n - 2.$$

Since upper Minkowski dimension is stable under finite unions, the first claim of Theorem 1.2 holds. The second follows from an identical argument using Lemma 2.12. ∎

2.1. Outline of the paper

The structure of this paper is roughly in four parts. Sections 3 and 4 use the geometric techniques of [11, 12] (and many, many others) to establish that the Almgren frequency is monotonically non-decreasing and bounded on $\{u = 0\}$. Section 5 uses these results to establish compactness of $\{T_{p,r}u\}$ for $u(p) = 0$. Section 6 extends these results to $p \in \Omega$ such that $u(p) \neq 0$.

The second part of this paper is devoted to obtaining geometric control upon $\mathcal{C}^k_{\varepsilon,r}(u)$. The general idea is to employ the usual "frequency pinching" (Lemma 7.2) and cone-splitting results (Lemma 7.6). However, as we are considering $N_\Omega(p, r, u)$ at points p such that $u(p) \neq 0$, the Almgren frequency is *not* monotonic. Thus the usual frequency pinching argument must be modified. This is overcome by proving that

$$\operatorname{dist}(p, \{u = 0\}) < \gamma \quad \text{and} \quad N_\Omega(p, 1, u) - N_\Omega(p, 1/10, u) \leq \gamma,$$

for $\gamma > 0$ sufficiently small implies that u is $(0, \varepsilon, 1, p)$-symmetric (see Lemma 7.2). In Corollary 7.7 we prove that if $x \in \mathcal{C}^k_\varepsilon(u)$, then $\{p \in \overline{\Omega} : u \text{ is } (0, \delta, r, p)\text{-symmetric}\} \cap B_r(x)$ is contained in a tubular neighborhood of a k-plane L^k.

The third part of this paper is devoted to obtaining packing estimates to prove Theorem 2.11. To do so, we use the tools of [8], which do not require restricting to a level set or the delicate machinery which powers the finer estimates of [10]. The fact that we do not control the tilt of approximating L^k at different scales accounts for the $(k + \varepsilon)$-dimensional results.

The fourth part of this paper is devoted to proving the containment results which prove Lemma 2.12.

Throughout this paper, the constant C will by ubiquitous and represent different constants even within the same string of inequalities. A constant written $C(n, \Lambda)$ will only depend upon n and Λ, but each instantiation may represent a distinct constant.

3. The Almgren frequency function

In this section, we develop crucial properties of the Almgren frequency function. The main result of this section is the monotonicity of the Almgren frequency on $p \in \{u = 0\} \cap \overline{\Omega}$ (Lemma 3.4).

We now register some of the elementary properties of $H_\Omega(p, r, u)$, $D_\Omega(p, r, u)$, and $N_\Omega(p, r, u)$, and of their derivatives.

Lemma 3.1. *Let $u \in \mathcal{A}(n, \Lambda)$ and $p \in \overline{\Omega} \cap B_1(0)$ and all $0 < r < 1$. Then,*

$$\frac{d}{dr} H_\Omega(p, r, u) = \frac{n-1}{r} H_\Omega(p, r, u) + 2D_\Omega(p, r, u) + 2u(p) \int_{\partial\Omega \cap B_r(p)} \nabla u \cdot \vec{\eta}\, d\sigma, \tag{3.1}$$

$$\begin{aligned}\frac{d}{dr} D_\Omega(p, r, u) = \frac{n-2}{r} D_\Omega(p, r, u) &+ 2 \int_{\partial B_r(p)} (\nabla u \cdot \vec{\eta})^2\, d\sigma \\ &+ \int_{\partial\Omega \cap B_r(p)} (Q - p) \cdot \vec{\eta}\, (\nabla u \cdot \vec{\eta})^2\, d\sigma(Q),\end{aligned} \tag{3.2}$$

$$\frac{d}{dr} \ln\left(\frac{H_\Omega(p, r, u)}{r^{n-1}}\right) = \frac{2}{r} N_\Omega(p, r, u) + 2\, \frac{u(p) \int_{\partial\Omega \cap B_r(p)} \nabla u \cdot \vec{\eta}\, d\sigma}{H_\Omega(p, r, u)}, \tag{3.3}$$

$$\frac{d}{dr} \ln(H_\Omega(p, r, u)) = \frac{n-1}{r} + \frac{2}{r} N_\Omega(p, r, u) + 2\, \frac{u(p) \int_{\partial\Omega \cap B_r(p)} \nabla u \cdot \vec{\eta}\, d\sigma}{H_\Omega(p, r, u)}, \tag{3.4}$$

where $\vec{\eta}$ is the unit outer normal of the relevant domains.

Proof. In the interior setting, for $\overline{B_r(p)} \subset \Omega$, these identities follow from straightforward computation. The identity (3.1) follows from the change of variables $y \to rx + p$ and the divergence theorem, while (3.2) relies upon the Rellich–Necas identity,

$$\operatorname{div}(X|\nabla u|^2) = 2\operatorname{div}((X \cdot \nabla u)\nabla u) + (n-2)|\nabla u|^2, \tag{3.5}$$

the divergence theorem, and the fact that u vanishes on the boundary. The last two equations follow immediately from (3.1). Without exception, the standard interior computations go through identically for radii for which $B_r(p) \cap \partial\Omega \neq \emptyset$, where we also use the identity

$$\int_{B_r(p)} (u - u(p))\Delta u = u(p) \int_{\partial\Omega\cap B_r(p)} \nabla u \cdot \vec{\eta}\, d\sigma,$$

where $\vec{\eta}$ is the *outer* unit normal vector to Ω. See Theorem 2.2.3 and Corollary 2.2.5 in [12], and the proof of the doubling property in [2] for details. ∎

The following lemma records a useful identity which follows from the previous lemma by straightforward computation.

Lemma 3.2. *For $u \in \mathcal{A}(n, \Lambda)$, $p \in \overline{\Omega} \cap B_1(0)$, and all $0 < r < 1$, $\frac{d}{dr} N_\Omega(p, r, u)$ may be decomposed into four terms:*

$$\frac{d}{dr} N_\Omega(p, r, u) = N_1'(r) + N_2'(r) + N_3'(r) + N_4'(r), \tag{3.6}$$

where

$$N_1'(r) := \frac{2r\big[H_\Omega(p,r,u) \int_{\partial B_r(p)\cap\overline{\Omega}} (\nabla u\cdot\vec{\eta})^2\, d\sigma - \big(\int_{\partial B_r(p)\cap\overline{\Omega}} (u-u(p))(\nabla u\cdot\vec{\eta})\, d\sigma\big)^2\big]}{H_\Omega(p,r,u)^2},$$

$$N_2'(r) := \frac{1}{H_\Omega(p,r,u)}\, r \int_{\partial\Omega\cap B_r(p)} (Q - p)\cdot\vec{\eta}\,(\nabla u\cdot\vec{\eta})^2\, d\sigma(Q),$$

$$N_3'(r) := 2N_\Omega(p,r,u)\, \frac{u(p)}{H_\Omega(p,r,u)} \int_{\partial\Omega\cap B_r(p)} \nabla u\cdot\vec{\eta}\, d\sigma,$$

$$N_4'(r) := \frac{2r}{H_\Omega(p,r,u)^2} \Big(u(p) \int_{\partial\Omega\cap B_r(p)} \nabla u\cdot\vec{\eta}\, d\sigma\Big)^2,$$

and $\vec{\eta}$ is the unit outer normal.

Lemma 3.3. *Let $u \in \mathcal{A}(n, \Lambda)$ and $p \in \overline{\Omega} \cap B_1(0) \cap \{u = 0\}$. For all $0 < r < 1$,*

$$\begin{aligned} N_1'(r) &= \frac{2r\big[H_\Omega(p,r,u) \int_{\partial B_r(p)} (\nabla u\cdot\vec{\eta})^2\, d\sigma - \big(\int_{\partial B_r(p)} (u-u(p))(\nabla u\cdot\vec{\eta})\, d\sigma\big)^2\big]}{H_\Omega(p,r,u)^2} \\ &= \frac{2}{rH_\Omega(p,r,u)} \Big(\int_{\partial B_r(p)\cap\overline{\Omega}} |\nabla u\cdot(y-p) - N_\Omega(p,r,u)(u(y)-u(p))|^2\, d\sigma(y)\Big). \end{aligned}$$

Proof. Recall that by the Cauchy–Schwarz inequality, we have that for $\lambda = \langle w, v\rangle/\|v\|^2$,

$$\|v\|^2\|w - \lambda v\|^2 = \|w\|^2\|v\|^2 - |\langle w, v\rangle|^2.$$

Choosing $w = \nabla u\cdot(y - p)$ and $v = u - u(p)$, we have

$$\begin{aligned} N_1'(r) &= H_\Omega(p,r,u)^{-1}\, 2r \Big(\int_{\partial B_r(p)\cap\overline{\Omega}} \Big|(u)_\nu - \frac{1}{r}\lambda(p,r,u)(u-u(p))\Big|^2 d\sigma\Big) \\ &= \frac{2}{rH_\Omega(p,r,u)} \Big(\int_{\partial B_r(p)\cap\overline{\Omega}} |\nabla u\cdot(y-p) - \lambda(p,r,u)(u(y)-u(p))|^2\, d\sigma(y)\Big), \end{aligned}$$

where

$$\lambda(p,r,u) := \frac{\int_{\partial B_r(p)\cap\overline{\Omega}}(u(y)-u(p))\nabla u\cdot(y-p)\,d\sigma(y)}{H_\Omega(p,r,u)}. \tag{3.7}$$

The divergence theorem then implies that for $u(p)=0$, $\lambda(p,r,u)=N_\Omega(p,r,u)$. ∎

Lemma 3.4 (Monotonicity). *Let $u\in\mathcal{A}(n,\Lambda)$ and let $p\in\{u=0\}\cap\overline{\Omega}\cap B_1(0)$. Then $N_\Omega(p,r,u)$ is monotonically non-decreasing in $0<r<1$.*

Proof. Recall Lemma 3.2. Note that $N_1'(r)$ is non-negative, by the Cauchy–Schwartz inequality. Furthermore, $N_2'(r)$ is non-negative because $\partial\Omega$ is a convex surface and $\vec{\eta}$ is the outer normal, implying $(Q-p)\cdot\vec{\eta}\geq 0$ for all $p\in\Omega\cap B_2(0)$ and all $Q\in\partial\Omega\cap B_2(0)$. Observe that $N_3'(r)=N_4'(r)=0$ because $u(p)=0$. Therefore, $\frac{d}{dr}N_\Omega(p,r,u)$ is non-negative. ∎

4. The zero set: uniform frequency bounds

The main result in this section is Lemma 4.3, which gives a uniform bound on the Almgren frequency function for all $p\in\overline{\Omega}\cap B_{1/4}(0)$ for which $u(p)=0$ and all $0<r\leq 1/2$. We begin with a few basic results.

Lemma 4.1 ($H_\Omega(p,r,u)$ is doubling). *Let $u\in\mathcal{A}(n,\Lambda)$, with $p\in B_1(0)\cap\{u=0\}=\overline{\Omega}$. For any $0<s<S\leq 1$,*

$$H_\Omega(p,S,u)\leq\left(\frac{S}{s}\right)^{(n-1)+2N_\Omega(p,S,u)}H_\Omega(p,s,u). \tag{4.1}$$

Proof. Recalling equations (3.1) and (3.3),

$$\ln\left(\frac{H_\Omega(p,S,u)}{H_\Omega(p,s,u)}\right)=\ln(H_\Omega(p,S,u))-\ln(H_\Omega(p,s,u))=\int_s^S\frac{H_\Omega'(p,r,u)}{H_\Omega(p,r,u)}\,dr$$
$$=\int_s^S\left(\frac{n-1}{r}+\frac{2}{r}N_\Omega(p,r,u)\right)dr.$$

We bound $N_\Omega(p,r,u)$ by $N_\Omega(p,S,u)$ using Lemma 3.4. Plugging in these bounds, we have that for $r\in[s,S]$,

$$\ln\left(\frac{H_\Omega(p,S,u)}{H_\Omega(p,s,u)}\right)\leq[(n-1)+2N_\Omega(p,S,u)]\ln(r)|_s^S.$$

Evaluating and exponentiating gives the desired result. ∎

Remark 4.2. Because $N_\Omega(p,r,u)$ is monotonic for $p\in B_1(0)\cap\{u=0\}\cap\overline{\Omega}$, we can also extract the inequality

$$\ln\left(\frac{H_\Omega(p,S,u)}{H_\Omega(p,s,u)}\right)\geq[(n-1)+2N_\Omega(p,s,u)]\ln(r)|_s^S,$$

which leads to

$$H_\Omega(p,s,u) \le \left(\frac{s}{S}\right)^{(n-1)+2N_\Omega(p,s,u)} H_\Omega(p,S,u)\cdot \tag{4.2}$$

If $S = 1$ and $u = T_{p,r}u$, then we have that for all $1 > s > 0$,

$$H_\Omega(0,s,T_{p,r}u) \le s^{(n-1)+2N_\Omega(Q,0,T_{p,r}u)}. \tag{4.3}$$

We are now ready for the main result of this section.

Lemma 4.3 (Uniform bound on frequency). *Let $u \in \mathcal{A}(n,\Lambda)$, as above. There is a constant $C_1(n,\Lambda)$ such that for all $p \in \{u=0\} \cap \overline{\Omega} \cap B_{1/4}(0)$ and all $r \in (0,1/2]$,*

$$N_\Omega(p,r,u) \le C_1(n,\Lambda). \tag{4.4}$$

Proof. Recall that $0 \in \partial\Omega$ and that the Almgren frequency function is invariant under rescalings. Therefore, we normalize our function u by the rescaling $v = T_{0,1}u$.

Therefore, applying Lemma 4.1 to $Q = 0$, letting $r = cR$, and integrating both sides with respect to R from 0 to S, we have that for any $c \in (0,1)$,

$$\begin{aligned}\int_{B_S(0)} |v|^2\, dV &\le \int_0^S \left(\frac{1}{c}\right)^{(n-1)+2N_\Omega(0,R,v)} \int_{\partial B_{cR}(0)} |v|^2\, d\sigma dR\\ &\le \left(\frac{1}{c}\right)^{(n-1)+2N_\Omega(0,S,v)} \int_0^S \int_{\partial B_{cR}(0)} |v|^2\, d\sigma dR.\end{aligned}$$

Thus, letting $S = 1$ and $c = 1/16$ and dividing by ω_n, we obtain

$$⨍_{B_1(0)} |v|^2\, dV \le 16^{2N_\Omega(0,1,v)} ⨍_{B_{1/16}(0)} |v|^2\, dV. \tag{4.5}$$

Thus, for any $p \in \{v=0\} \cap \overline{\Omega} \cap B_{1/4}(0)$, by inclusion,

$$\int_{B_1(0)} |v|^2\, dV \ge \int_{B_{3/4}(p)} |v|^2\, dV \quad \text{and} \quad \int_{B_{1/16}(0)} |v|^2\, dV \le \int_{B_{9/16}(p)} |v|^2\, dV.$$

Therefore, substituting these bounds into (4.5),

$$⨍_{B_{3/4}(p)} |v|^2\, dV \le 16^{2N_\Omega(0,1,v)}\Big(⨍_{B_{9/16}(p)} |v|^2\, dV\Big). \tag{4.6}$$

Now, we wish to bound $⨍_{B_{3/4}(p)} |v|^2\, dV$ from below and $⨍_{B_{9/16}(p)} |v|^2\, dV$ from above. By (3.1), if $v(p) = 0$, $\frac{d}{dr}\int_{\partial B_r(p)} |v|^2\, d\sigma \ge 0$. Thus, for all $p \in \{v=0\} \cap \overline{\Omega} \cap B_{1/4}(0)$, we bound

$$\int_{B_{3/4}(p)} |v|^2\, dV \ge \int_{5/8}^{3/4} \int_{\partial B_r(p)} |v|^2\, d\sigma dr \ge c \int_{\partial B_{5/8}(p)} |v|^2\, d\sigma,$$

$$\int_{B_{9/16}(p)} |v|^2\, dV \le \int_0^{9/16} \int_{\partial B_{9/16}(p)} |v|^2\, d\sigma dr \le c \int_{\partial B_{9/16}(p)} |v|^2\, d\sigma.$$

Plugging the above bounds into (4.6) and dividing, we obtain

$$\frac{\fint_{\partial B_{5/8}(p)} |v|^2 \, d\sigma}{\fint_{\partial B_{9/16}(p)} |v|^2 \, d\sigma} \leq C(n)\, 16^{2N_\Omega(0,1,v)} \tag{4.7}$$

for all $p \in \{v = 0\} \cap \overline{\Omega} \cap B_{1/4}(0)$. Recalling (3.3) and Lemma 3.4, we see that

$$\ln\Big(\fint_{\partial B_{5/8}(p)} v^2 \, d\sigma\Big) - \ln\Big(\fint_{\partial B_{9/16}(p)} v^2 \, d\sigma\Big) = \int_{9/16}^{5/8} \frac{d}{dr} \ln\Big(\frac{1}{r^{n-1}} H_\Omega(p,r,v)\Big) dr$$
$$= \int_{9/16}^{5/8} \frac{2}{r} N_\Omega(p,r,v)\, dr \geq 2N_\Omega(p,1/2,v)\Big(\ln\Big(\frac{5}{8}\Big) - \ln\Big(\frac{9}{16}\Big)\Big) \geq 2cN_\Omega(p,1/2,v).$$

Thus, (4.7) gives us that

$$2cN_\Omega(p,1/2,v) \leq \ln\Bigg(\frac{\fint_{\partial B_{5/8}(p)} |v|^2 \, d\sigma}{\fint_{\partial B_{9/16}(p)} |v|^2 \, d\sigma}\Bigg) \leq \ln\big(C(n)\, 16^{2N_\Omega(0,1,v)}\big)$$
$$= 2N_\Omega(0,1,v)\ln(16) + C(n) \leq 2\Lambda \ln(16) + C(n).$$

Now, Lemma 3.4 gives, for $1/2 > s > 0$, that $N_\Omega(p,1/2,v) \geq N_\Omega(p,s,v)$. Since $N_\Omega(p,r,v) = N_\Omega(p,r,u)$, we have the desired claim. ■

5. The zero set: compactness

The uniform bounds on the Almgren frequency function allow us to prove compactness results on the collection of rescaling $\{T_{p,r}u\}$. The main results of this section are weak compactness (Lemma 5.4), and the geometric non-degeneracy of the domains Ω (Corollary 5.2).

We now state a sequence of preliminary corollaries to Lemma 4.3. We shall denote the $C^{0,\gamma}(B_1(0))$-norm by

$$\|u\|_{C^{0,\gamma}(B_1(0))} := \|u\|_{C^0(B_1(0))} + \sup_{\substack{x,y \in B_1(0)\\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

Lemma 5.1 (Uniform Hölder continuity). *Let $u \in \mathcal{A}(n,\Lambda)$, $Q \in B_{1/4}(0) \cap \partial\Omega$, and $r \in (0,1/2]$. Then*

$$\|T_{Q,r}u\|_{C^{0,\gamma}(B_1(0))} \leq C(n,\Lambda).$$

We defer the proof of this statement to the Appendix A. The techniques are standard.

Corollary 5.2 (Non-degeneracy of domains). *Let $u \in \mathcal{A}(n,\Lambda)$ and let $\Omega \in \mathcal{D}(n)$ be its associated convex domain. There exists a constant $0 < c = c(\Lambda,n)$ such that, for all $Q \in \partial\Omega \cap B_{1/4}(0)$ and $0 < r \leq 1/2$, $\partial B_1(0) \cap T_{Q,r}\Omega$ is a relatively open convex surface with*

$$\mathcal{H}^{n-1}(\partial B_1(0) \cap T_{Q,r}\Omega) > c.$$

Proof. That $\partial B_r(Q) \cap \Omega$ is relatively open and relatively convex is immediate from the definition of Ω. By Lemma 5.1, we see that $\max_{B_1(0)} |T_{Q,r}u(x)| \le C(n,\Lambda)$. And by definition, $H_{T_{Q,r}\Omega}(0,1,T_{Q,r}u) = 1$. Furthermore, we have that

$$H_{T_{Q,r}\Omega}(0,1,T_{Q,r}u) \le \mathcal{H}^{n-1}(\partial B_1(0) \cap T_{Q,r}\Omega)\, C^2.$$

Therefore, $\mathcal{H}^{n-1}(\partial B_1(0) \cap T_{Q,r}\Omega) \ge C^{-2} = c$. ■

Corollary 5.3. *For all $u \in \mathcal{A}(n,\Lambda)$, $Q_0 \in \partial\Omega \cap B_{1/4}(0)$, and $0 < r \le 1/2$, the following estimate holds. Let $Q \in T_{Q_0,r}\partial\Omega \cap B_{1/2}(0)$ and let L_Q be a supporting hyperplane to $Q \in T_{Q_0,r}\partial\Omega$. Then for all $p \in \overline{T_{Q_0,r}\Omega} \cap B_{1/4}(Q)$,*

$$|T_{Q_0,r}u(p)| \le C(n,\Lambda)\,\mathrm{dist}(p, L_Q).$$

In particular, $|\nabla T_{Q_0,r}u(Q)| \le C(n,\Lambda)$ for all $Q \in \partial T_{Q_0,r}\Omega \cap B_{1/2}(0)$.

Proof. Let $\mathbb{H}_Q$ be the half-space with boundary L_Q which contains $T_{Q_0,r}\Omega$. Consider the Dirichlet problem

$$\begin{aligned} \Delta\phi = 0 \qquad & \text{in } \mathbb{H}_Q \cap B_{1/2}(Q), \\ \phi = \begin{cases} C(n,\Lambda) & \text{on } \partial B_{1/2}(Q) \cap T_{Q_0,r}\Omega, \\ 0 & \text{on } \partial(B_1(Q) \cap \mathbb{H}_Q) \setminus (\partial B_{1/2}(Q) \cap T_{Q_0,r}\Omega), \end{cases} \end{aligned}$$

where we choose $C(n,\Lambda)$ to be the same constant in Lemma 5.1 for which we have $\sup_{\partial B_1(0)} |T_{Q_0,r}u| \le C(n,\Lambda)$. Note that for any $Q \in \partial\Omega$, $\mathbb{H}_Q \cap B_{1/2}(Q)$ is a Wiener regular domain and the boundary data is piecewise continuous, so a unique solution ϕ must exist.

By the maximum principle, $T_{Q_0,r}u \le \phi$ in $T_{Q_0,r}\Omega \cap B_{1/2}(Q)$. We now argue that ϕ is comparable to a linear function in $B_{1/4}(Q) \cap \mathbb{H}_Q$. Let L be the affine linear function with $\{L = 0\} = L_Q$ such that

$$\max_{\partial B_{1/2}(Q)} L = \max_{\partial B_{1/2}(Q)} \phi = C(n,\Lambda).$$

By Theorem 5.1 in [14], there is a constant C such that, for all $x \in B_{1/4}(Q) \cap \mathbb{H}_Q$,

$$\phi(x) \le CL(x),$$

where C depends only upon the geometry of $B_{1/4}(Q) \cap \mathbb{H}_Q$. Since this geometry is always a half-ball, this constant is uniform. So we have that for all $x \in B_{1/4}(Q) \cap \mathbb{H}_Q$,

$$\phi(x) \le CL(x) \le C2C(n,\Lambda)\,\mathrm{dist}(x, L_Q).$$

Thus, for $p \in \overline{T_{Q_0,r}\Omega} \cap B_{1/4}(Q)$, we have

$$T_{Q_0,r}u(p) \le \phi(p) \le C(n,\Lambda)\,\mathrm{dist}(x, L_Q).$$

Applying this argument to $\pm T_{Q_0,r}u$, we obtain the desired estimate. ■

Lemma 5.4 (Preliminary compactness). *Let $u_i \in \mathcal{A}(n,\Lambda)$, $Q_i \in \partial\Omega_i \cap B_{1/4}(0)$, and $0 < r_i \leq 1/4$. Then there exists a subsequence (also indexed by i) such that*

(1) *$T_{Q_i,r_i}u_i \to u_\infty$ in $C^{0,\gamma}(\overline{B_1(0)})$.*

(2) *If we define $\Omega_\infty = \mathrm{Int}(\overline{\{|u_\infty| > 0\}})$, then $\overline{T_{Q_i,r_i}\Omega_i \cap B_1(0)} \to \overline{\Omega_\infty \cap B_1(0)}$ in the Hausdorff metric on compact subsets, and Ω_∞ is a non-degenerate convex domain with $0 \in \partial\Omega_\infty$ which satisfies the same non-degeneracy as that of Corollary 5.2.*

(3) *u_∞ is harmonic in Ω_∞.*

Proof. By definition, $T_{Q_i,r_i}u_i(0) = 0$. Therefore, Lemma 5.1 implies the first convergence result by Arzelà–Ascoli. Note that since $H_{T_{Q_i,r_i}\Omega_i}(0,1,T_{Q_i,r_i}u_i) = 1$ for all i, $H_{\Omega_\infty}(0,1,u_\infty) = 1$.

By taking a further subsequence, we may assume that $\lim_i T_{Q_i,r_i}\Omega_i = \Omega'$ exists in a set theoretic sense. The uniform convergence in (1) implies that, for all $0 < \varepsilon$, $\{|u_\infty| > \varepsilon\} \subset \bigcap_{N=1}^{\infty}\bigcup_{i=N}^{\infty}\{|T_{Q_i,r_i}u_i| \geq \varepsilon - 1/N\}$. Since $\{|T_{Q_i,r_i}u_i| > 1/2\}$ is non-empty, and as $|T_{Q_i,r_i}u_i| > 1/4$ in $B_{(2C(n,\Lambda))^{-1/\gamma}}(\{|T_{Q_i,r_i}u_i| > 1/2\})$, Ω_∞ is non-degenerate. For all $y \in \Omega_\infty$ and all $0 < \delta$, there exists a point $x \in \Omega_\infty$ such that $|x - y| \leq \delta$ and $|u_\infty(x)| > \varepsilon$ for some $\varepsilon > 0$. Since $x \in \overline{\lim}_i T_{Q_i,r_i}\Omega_i$ and $0 < \delta$ was arbitrary, $\Omega_\infty \subset \lim_i B_\delta(T_{Q_i,r_i}\Omega_i)$ for all $\delta > 0$.

To see the converse containment, we observe that for any $x_0 \in \mathrm{Int}(\Omega')$, if $B_\delta(x_0) \subset \Omega'$ then, for all sufficiently large i, $B_{\delta/2}(x) \subset T_{Q_i,r_i}\Omega_i$ and $T_{Q_i,r_i}u_i$ is uniformly bounded in $W^{1,2}(B_\delta(x_0))$. Therefore, $T_{Q_i,r_i}u_i$ converge to a harmonic function in $\mathrm{Int}(\Omega')$. By the convexity of $T_{Q_i,r_i}\Omega_i$ for all i, $\mathrm{Int}(\Omega')$ is connected.

Now, let $0 < \delta$ and suppose that for all sufficiently large i there exists $x_i \in T_{Q_i,r_i}\Omega_i \setminus B_\delta(\Omega_\infty) \cap B_1(0)$. By passing to a subsequence, we may assume $x_i \to x_\infty \in B_1(0) \cap \overline{\Omega'} \setminus B_{\delta/2}(\Omega_\infty)$ Since $u_\infty = 0$ on $B_{\delta/2}(x_\infty)$, it must be that $\lim_{i\to\infty}\sup_{B_{\delta/2}(x_\infty)}|T_{Q_i,r_i}u_i| = 0$. Since $\mathrm{Int}(\Omega')$ is connected, by unique continuation $T_{Q_i,r_i}u_i \to 0$ in Ω'. But this contradicts u_∞ being non-trivial. Thus, there must be a subsequence such that $\overline{T_{Q_i,r_i}\Omega_i \cap B_1(0)} \to \overline{\Omega_\infty \cap B_1(0)}$ in the Hausdorff metric on compact subsets. Convexity and the conclusion of Corollary 5.2 are preserved under this mode of convergence, and so (2) is proved.

Now that we know that $\Omega' = \Omega_\infty$, the previous argument proves (3), as well. ∎

6. Estimates off the zero set

In this section we prove that analogs of the results of Sections 4 and 5 hold for all $p \in B_{1/8}(0) \cap \overline{\Omega}$ and $0 < r \leq 1/8$. The key to obtaining estimates off $\{u = 0\}$ is the following technical lemma.

Lemma 6.1 (Technical lemma). *For any $u \in \mathcal{A}(n,\Lambda)$, $Q \in B_{1/4}(0)$, and $0 < r < 1/4$, there is a constant $0 < c(n,\Lambda)$ such that for all $y \in B_{1/2}(0)$,*

$$c(n,\Lambda) < H_{T_{Q,2r}\Omega}(y, 1/4, T_{Q,2r}u) < C(n,\Lambda).$$

Proof. Note that the upper bound follows directly from Lemma 5.1. To show the lower bound, we argue by compactness. Suppose that there is a sequence of functions $u_i \in$

$\mathcal{A}(n,\Lambda)$, points $Q_i \in \partial\Omega_i \cap B_{1/4}(0)$ and radii $0 < r_i < 1/4$, such that there exist points $y_i \in B_{1/2}(0) \cap \overline{\Omega}$ for which

$$H_{T_{Q_i,2r_i}\Omega_i}(y_i, 1/4, T_{Q_i,2r_i}u_i) \leq 2^{-i}.$$

Letting $i \to \infty$, by Lemma 5.4, there exists a subsequence $T_{Q_j,2r_j}u_j$ which converges to a Hölder continuous function, u_∞, which is harmonic in a non-degenerate convex domain, Ω_∞. Note that u_∞ vanishes on the boundary of $\partial\Omega_\infty \cap B_8(0)$. Similarly, we may take subsequences such that $y_i \to y_\infty$. Note that Hölder convergence implies that $H_{\Omega_\infty}(0,1,u_\infty) = 1$. Since we have that $H_{\Omega_\infty}(y, 1/4, u_\infty) = 0$, it must be that $u_\infty = u_\infty(y_\infty)$ on $\partial B_{1/4}(y) \cap \Omega_\infty$. If $\partial B_{1/4}(y) \subset \Omega_\infty$, then $u_\infty \equiv u_\infty(y_\infty)$ in Ω_∞. This contradicts $u_\infty(0) = 0$ and $H_{\Omega_\infty}(0,1,u_\infty) = 1$. If $\partial B_{1/4}(y)$ intersects $\partial\Omega_\infty$, then $u_\infty(y_\infty) = 0$, since u_∞ must vanish continuously on $\partial\Omega_\infty$. However, this forces $u_\infty \equiv 0$, which contradicts $H_{\Omega_\infty}(0,1,u_\infty) = 1$. ■

Lemma 6.2 (Bounding the Almgren frequency). *Let $u \in \mathcal{A}(n,\Lambda)$, $p \in B_{1/8}(0) \cap \overline{\Omega}$ and $0 < r \leq 1/8$. Then, there is a constant $C_2 = C_2(n,\Lambda) < \infty$ such that*

$$N_\Omega(p,r,u) \leq C_2.$$

Proof. For $p \in \{u = 0\} \cap \overline{\Omega}$, Lemma 4.3 proves the desired inequality. Let $p \in \Omega$ be such that $u(p) \neq 0$. Let $0 < \delta = \text{dist}(p, \partial\Omega)$ and $Q \in \partial\Omega$ be such that $|p - Q| = \delta$. If $B_r(p) \subset \Omega$, we use the monotonicity of the Almgren frequency function to reduce to considering $B_\delta(p)$. We let $\tau = \max\{r,\delta\} \leq 1/8$. Consider $T_{Q,4\tau}u$. We note that

$$\begin{aligned} C_1(n,\Lambda) &\geq N_{T_{Q,4\tau}\Omega}(0,1,T_{Q,4\tau}u) \\ &= \frac{\int_{B_1(0)} |\nabla T_{Q,4\tau}u|^2 dV}{\int_{\partial B_1(0)} T_{Q,4\tau}u^2\, d\sigma} \geq \int_{B_{1/4}(T_{Q,4\tau}p)} |\nabla T_{Q,4\tau}u|^2\, dV. \end{aligned}$$

On the other hand, Lemma 6.1 implies that

$$H_{T_{Q,4\tau}\Omega}(T_{Q,4\tau}p, 1/4, T_{Q,4\tau}u) \geq c(n,\Lambda).$$

Therefore, we have

$$N_\Omega(p,r,u) = \frac{D_{T_{Q,4\tau}\Omega}(T_{Q,4\tau}p, 1/4, T_{Q,4\tau}u)}{4H_{T_{Q,4\tau}\Omega}(T_{Q,4\tau}p, 1/2, T_{Q,4\tau}u)} \leq C(n,\Lambda).$$ ■

Lemma 6.3 (Lipschitz bounds). *For $u \in \mathcal{A}(n,\Lambda)$, for all $Q \in \partial\Omega \cap B_{1/4}(0)$, and all radii $0 < r \leq 1/8$, $T_{Q,r}u \in \text{Lip}(B_1(0))$ with uniform Lipschitz constant $\text{Lip}(T_{Q,r}u) \leq C(n,\Lambda)$.*

Proof. Since $T_{Q,r}u$ is continuous and constant outside of $T_{Q,r}\Omega$, we reduce to bounding $\nabla T_{Q,r}u$ at interior points $y \in T_{Q,r}\Omega \cap B_1(0)$. Note that by our definition of the rescalings (Definition 2.5) and Lemma 4.1,

$$\begin{aligned} |T_{Q,r}u(y)| &= \left(\frac{\frac{1}{(4r)^{n-1}} H_\Omega(Q,4r,u)}{\frac{1}{r^{n-1}} H_\Omega(Q,r,u)} \right)^{1/2} |T_{Q,4r}u(y')| \\ &\leq 4^{2C_1(n,\Lambda)} |T_{Q,4r}u(y')| \leq C(n,\Lambda)\, |T_{Q,4r}u(y')|, \end{aligned}$$

where $y' = y/4$. Note that $y' \in B_{1/4}(0) \cap T_{Q,4r}\Omega$. Let $\delta = \mathrm{dist}(y', T_{Q,4r}\partial\Omega)$. Therefore, $\nabla T_{Q,4r}\, u(y') = \fint_{B_\delta(y')} \nabla T_{Q,4r}\, u \, dV$. Recall that $|\nabla u|$ is subharmonic, and therefore by Lemma 6.2,

$$|\nabla T_{Q,4r}\, u(y')| \leq \fint_{B_\delta(y')} |\nabla T_{Q,4r}\, u| \, dV \leq \Big(\fint_{B_\delta(y')} |\nabla T_{Q,4r}\, u|^2 \, dV \Big)^{1/2}$$
$$\leq \Big(C_2(n,\Lambda)\, \delta^{-2} \fint_{\partial B_\delta(y')} (T_{Q,4r}\, u - T_{Q,4r}\, u(y'))^2 \, d\sigma \Big)^{1/2}.$$

Now, let $Q' \in T_{Q,4r}\partial\Omega$ be a point such that $\delta = |y' - Q'|$ and let $y' = y'' + Q'$. Now, we translate the domain by Q:

$$\fint_{\partial B_\delta(y')} (T_{Q,4r}u - T_{Q,4r}u(y'))^2 \, d\sigma = \fint_{\partial B_\delta(y'')} (T_{Q,4r}u(x+Q) - T_{Q,4r}u(y''+Q))^2 \, d\sigma.$$

Note that $T_{Q,4r}\, u(x + Q') \in \mathcal{A}(n, C_1(n,\Lambda))$.

Now, by Corollary 5.3 applied to $T_{Q,4r}\, u(x+Q) \in \mathcal{A}(n, C_1(n,\Lambda))$ with $Q_0 = 0$, we bound

$$\fint_{\partial B_\delta(y')} (T_{Q,4r}\, u - T_{Q,4r}\, u(y'))^2 \, d\sigma$$
$$= \fint_{\partial B_\delta(y'')} (T_{Q,4r}\, u(x+Q) - T_{Q,4r}\, u(y''+Q))^2 \, d\sigma(x)$$
$$\leq \fint_{\partial B_\delta(y'')} (4C(n, C_1(n,\Lambda))\,\delta)^2 \, d\sigma = (4C(n, C_1(n,\Lambda))\,\delta)^2.$$

Thus, we have that

$$|\nabla T_{Q,r}\, u(y)| \leq C(n,\Lambda)\, |T_{Q,4r}\, u(y')|$$
$$\leq C(n,\Lambda) \Big(C_1(n,\Lambda)\, \delta^{-2} \Big(\fint_{\partial B_\delta(y)} (T_{Q,r}\, u - T_{Q,r}\, u(y))^2 \, d\sigma \Big) \Big)^{1/2}$$
$$\leq C(n,\Lambda)\, C_1(n,\Lambda)^{1/2} \frac{1}{\delta} (4C(n, C_1(n,\Lambda))\,\delta) \leq C(n,\Lambda). \qquad \blacksquare$$

We now prove that the Almgren frequency is a function of uniformly bounded variation.

Lemma 6.4 (Bounded variation). *Let $u \in \mathcal{A}(n,\Lambda)$ and $p \in B_{1/8}(0) \cap \overline{\Omega}$. Then, there is a constant $C_3 = C_3(n,\Lambda) < \infty$ such that for all $0 < r \leq 1/8$,*

$$\mathrm{var}(N_\Omega(p,r,u), [0, 1/8]) \leq C_3.$$

Proof. We estimate the variation by a "rays of the sun" argument. Since $N_\Omega(p,r,u)$ is monotone increasing and bounded for $p \in \{u = 0\} \cap \overline{\Omega}$, we argue for $u(p) \neq 0$. Again, we let $\delta = \mathrm{dist}(p, \partial\Omega)$. Then

$$\mathrm{var}(N_\Omega(p,r,u), [0,1/8]) \leq 2 \int_0^{1/8} N_3'(r)\, dr + |N_\Omega(p, 0^+, u) - N_\Omega(p, 1/8, u)|$$
$$\leq 2 \int_\delta^{1/8} 2N_\Omega(p,r,u) \frac{1}{H_\Omega(p,r,u)} \int_{B_r(p)\cap\partial\Omega} u(p)\, \nabla u \cdot \vec{\eta}\, d\sigma dr + 2C_2.$$

Now, if we let $Q_0 \in \partial\Omega$ be a point such that $|p - Q_0| = \delta$, we may calculate by Lemma 6.1, Lemma 4.1, Lemma 5.3, and Lemma 6.3,

$$\begin{aligned}
\frac{1}{H_\Omega(p,r,u)} \int_{B_r(p)\cap\partial\Omega} & u(p)\nabla u\cdot\vec{\eta}\,d\sigma \\
&\le C(n,\Lambda)\int_{B_1(T_{Q_0,r}p)\cap T_{Q_0,r}\partial\Omega} T_{Q_0,r}u(y)\,\frac{1}{r}\,\nabla T_{Q_0,r}u\cdot\vec{\eta}_{T_{Q_0,r}\Omega}\,d\sigma \\
&\le C(n,\Lambda)\int_{B_1(T_{Q_0,r}p)\cap T_{Q_0,r}\partial\Omega} \frac{\delta}{r^2}\,d\sigma \le C(n,\Lambda)\,\frac{\delta}{r^2}.
\end{aligned}$$

Thus, by Lemma 6.2, we may bound

$$\begin{aligned}
\mathrm{var}(N_\Omega(p,r,u),[0,1/8]) &\le 8C_2(n,\Lambda)\,C(n,\Lambda)\int_\delta^{1/8}\frac{\delta}{r^2}\,dr + 2C_2(n,\Lambda) \\
&\le C(n,\Lambda)\,\delta(8+1/\delta) + 2C_2(n,\Lambda) \le C_3(n,\Lambda).
\end{aligned}$$

This proves the lemma. ∎

Lemma 6.5 (Compactness). *Let $u_i \in \mathcal{A}(n,\Lambda)$, $p_i \in \overline{\Omega_i} \cap B_{1/8}(0)$, and $r_i \in (0,1/8]$. Then, there exist a subsequence and a function, $u_\infty \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, such that $T_{p_j,r_j}u_j$ converges to u_∞ in the following senses:*

1. *$T_{p_i,r_i}u_i \to u_\infty$ in $C^0(B_1(0))$.*
2. *$T_{p_i,r_i}u \to u_\infty$ in $L^2(B_1(0))$.*
3. *$\overline{T_{p_i,r_i}\Omega_i} \cap B_1(0) \to \overline{\Omega_\infty} \cap B_1(0)$ in the Hausdorff metric on compact subset and $\overline{\Omega_\infty} \cap B_1(0) = \mathrm{supp}(u_\infty) \cap B_1(0)$ is a non-degenerate, convex set.*
4. *$\nabla T_{p_i,r_i}u \to \nabla u_\infty$ in $L^2(B_1(0);\mathbb{R}^n)$.*

Proof. To see (1), we observe that $T_{p_i,r_i}u_i(0) = 0$ and $\{T_{p_i,r_i}u_i\}$ are uniformly Lipschitz. Therefore, by Arzelà–Ascoli, there exists a subsequence which converges in $C^0(B_1(0))$. Since $C^0(B_1(0)) \subset L^2(B_1(0))$, this also proves (2).

(3) follows analogously as in the proof of Lemma 5.4(2). That is, if $\overline{T_{p_i,r_i}\Omega_i} \cap B_1(0) \neq B_1(0)$ for a subsequence of i, then $\overline{T_{p_i,r_i}\Omega_i}$ is a translation of $\overline{T_{Q_i,r_i}\Omega_i}$ for some $Q_i \in \partial T_{p_i,r_i}\Omega_i \cap B_1(0)$. Thus, after possibly passing to a subsequence so that the $\lim_i Q_i$ exists, the argument of Lemma 5.4(2) applies, and (3) follows immediately.

Our choice of rescaling $T_{p_j,r_j}u$ gives $N_\Omega(0,1,T_{p_j,r_j}u_j) = \int_{B_1(0)}|\nabla T_{p_j,r_j}u_j|^2\,dV$. Therefore, Lemma 6.2 yields that the $\nabla T_{p_j,r_j}u_j$ are uniformly bounded in $L^2(B_1(0);\mathbb{R}^n)$. Therefore, Rellich compactness gives weak convergence.

The only thing remaining to show is that $\nabla T_{p_j,r_j}u_j \to \nabla u_\infty$. By (3), we may choose a subsequence such that $\partial\Omega_j$ have a convergent subsequence and $T_{p_i,r_i}\partial\Omega_i \to \partial\Omega_\infty$ locally in the Hausdorff metric to a non-degenerate convex domain. Since the boundary of a convex domain is locally the graph of a Lipschitz function, $\overline{\dim}_{\mathcal{M}}(\partial\Omega_\infty \cap B_1(0)) = n-1$. Thus, by continuity of measures and Lemma 6.3, for all $\varepsilon > 0$ we can find a $\tau(\Lambda,n,\varepsilon)$ independent of $T_{p_j,r_j}u_j$, such that

$$\int_{B_1(0)\cap B_\tau(\partial\Omega_\infty)} |\nabla T_{p_j,r_j}u_j|^2\,dV \le \varepsilon.$$

Therefore, using the notation $\partial\Omega_{j,\tau} = B_\tau(T_{p_j,r_j}\partial\Omega_j)$,

$$\begin{aligned}\lim_{j\to\infty} D_{\Omega_i}(1,0,T_{p,r_j}u_j) &= \lim_{j\to\infty}\int_{B_1(0)} |\nabla T_{p_j,r_j}u_j|^2\,dV \\ &= \int_{B_1(0)\setminus\partial\Omega_{j,\tau}} |\nabla T_{p_j,r_j}u_j|^2\,dV + \lim_{j\to\infty}\int_{B_1(0)\cap\partial\Omega_{j,\tau}} |\nabla T_{p,r_j}u_j|^2\,dV \\ &\le \lim_{j\to\infty}\int_{B_1(0)\setminus B_{\tau/2}(\partial\Omega_\infty)} |\nabla T_{p_j,r_j}u_j|^2\,dV + \varepsilon \le D_{\Omega_\infty}(1,0,u_\infty) + \varepsilon,\end{aligned}$$

where the last inequality follows from the $W^{1,2}$-convergence of harmonic functions in the region $B_1(0)\setminus B_\tau(\partial\Omega_\infty)$. Since $\varepsilon > 0$ was arbitrary, we have $\lim_{j\to\infty} D_{\Omega_i}(1,0,T_{p_j,r_j}u_j) \le D_{\Omega_\infty}(1,0,u_\infty)$. The other inequality follows from the same trick or from lower semicontinuity. Thus, $\lim_{j\to\infty} D(1,0,T_{Q_j,r_j}u_j) = D_{\Omega_\infty}(1,0,u_\infty)$. This implies strong convergence. ∎

Corollary 6.6 (Convergence of the Almgren frequency). *For $u_j \in \mathcal{A}(n,\Lambda)$, $p_j \in B_{1/8}(0) \cap \overline{\Omega}_i$, and $r_j \in (0,1/8]$, there exist a subsequence and a limit function such that*

$$N_{T_{p_j,r_j}\Omega_i}(0,1,T_{p_j,r_j}u_j) \to N_{\Omega_\infty}(0,1,u_\infty). \tag{6.1}$$

Proof. The continuous convergence of $T_{p_j,2r_j}u_j$ in $B_1(0)$ and the strong convergence $\nabla T_{p_j,2r_j}u_j$ in $B_1(0)$ give the desired convergence of $H_{T_{p_j,2r_j}\Omega_i}(0,1/2,T_{p_j,2r_j}u_j)$ and of $D_{T_{p_j,2r_j}\Omega_j}(0,1/2,T_{p_j,2r_j}u_j)$, respectively. Recall that by Definitions 2.2 and 2.5,

$$N_{T_{p,2r}\Omega}(0,1/2,T_{p,2r}u) = \frac{1}{2}\,\frac{D_{T_{p,2r}\Omega}(0,1/2,T_{p,2r}u)}{H_{T_{p,2r}\Omega}(0,1/2,T_{p,2r}u)} = \frac{D_{T_{Q,r}\Omega}(0,1,T_{Q,r}u)}{H_{T_{Q,r}\Omega}(0,1,T_{Q,r}u)}. \quad ∎$$

Corollary 6.7 (Limit functions are harmonic in the limit domain). *Let the sequence of functions $T_{p_j,r_j}u_j$ converge to the function u_∞ in the senses of Lemma 6.5. Then, u_∞ is harmonic in Ω_∞.*

Proof. Recall that, up to a subsequence, the boundaries $T_{Q,r}\partial\Omega_j$ converge to the boundary $\partial\Omega_\infty$ of a convex domain in the Hausdorff distance on compact subsets. Therefore, for any $0 < \varepsilon$, for j large enough, every $T_{Q,r}u$ will be harmonic in the region $B_1(0)\setminus B_\varepsilon(\partial\Omega_\infty)$. By the $C^{0,\gamma}(B_1(0))$ convergence of harmonic functions, u_∞ is therefore harmonic in $B_1(0)\setminus B_\varepsilon(\partial\Omega_\infty)$. Letting $\varepsilon \to 0$ gives the desired statement. ∎

7. Geometric control

The main results of this section are two "quantitative rigidity" results about homogeneous harmonic functions. Both are essentially consequences of the compactness obtained in Lemma 6.5.

Lemma 7.1. *Let $u \in \mathcal{A}(n,\Lambda)$. Let $p \in \overline{\Omega} \cap B_{1/8}(0)$ and $0 < r \le 1/8$. If*

$$N_\Omega(p,r,u) = N_\Omega(p,r/10,u),$$

and either $u(p) = 0$ or $B_r(p) \subset \Omega$, then u is $(0,0,s,p)$-symmetric for all $0 < s \le 1$.

Proof. The hypotheses imply that $N_\Omega(p,s,u)$ is a constant for all $r/10 \leq s \leq r$. Furthermore, using the notation in Lemma 3.2, $N_1'(s) = N_2'(s) = N_3'(s) = N_4'(s) = 0$ for all $r/10 \leq s \leq r$. Thus, by Lemma 3.3 we have that for all $y \in \partial B_s(p)$,

$$\nabla u \cdot (y-p) = \partial_r u = N_\Omega(p,s,u)(u - u(p)).$$

Since $N_\Omega(p,s,u)$ is a constant for all $r/10 \leq s \leq r$, this becomes a separable ODE in polar coordinates, and $u - u(p) = v(s,\theta) = s^{N_\Omega(p,r,u)} v(\theta)$. Since $\Omega \cap B_r(p)$ is open, unique continuation implies that $u - u(p)$ is a homogeneous function of degree $N_\Omega(p,r,u)$ in $\Omega \cap B_1(p)$. In particular, u is $(0,0,s,p)$-symmetric for all $0 < s \leq 1$. ■

Standard quantitative rigidity results usually prove that if $N_\Omega(p,r,u)$ is *almost* constant ($N_\Omega(p,1,u) - N_\Omega(p,r,u) \leq \delta(\varepsilon)$), then u is *almost* a homogeneous harmonic polynomial ($\|T_{p,1}u - P\|_{L^2(B_1(0))} \leq \sqrt{\varepsilon}$). However, these results rely essentially upon the monotonicity of $N_\Omega(p,r,u)$. If $N_\Omega(p,r,u)$ is not monotonic, then $N_\Omega(p,1,u) = N_\Omega(p,r,u)$ does not imply that the Almgren frequency is constant. In fact, even when $N_\Omega(p,r,u)$ is constant for $1/10 < r < 1$, if $u(p) \neq 0$, it is not clear that u would be homogeneous. To overcome this technical issue, we consider p which are merely very close to $\{u = 0\}$.

Lemma 7.2 (Quantitative rigidity). *Let $u \in \mathcal{A}(n,\Lambda)$, as above. Let $p \in B_{1/8}(0) \cap \overline{\Omega}$ and $0 < r \leq 1/8$. For every $\delta > 0$, there is an $0 < \gamma_0 = \gamma_0(n,\Lambda,\delta)$ such that for any $0 < \gamma \leq \gamma_0$, if*

$$|N_\Omega(0,1,T_{p,r}u) - N_\Omega(0,1/10,T_{p,r}u)| \leq \gamma$$

and either $\text{dist}(p,\{u=0\}) \leq \gamma r$ *or $B_1(0) \subset T_{p,r}\Omega$, then $T_{p,r}u$ is $(0,\delta,1,0)$-symmetric.*

Proof. We argue by contradiction. Assume that there exists a $\delta > 0$ such that there is a sequence of functions $u_i \in \mathcal{A}(n,\Lambda)$, points $p_i \in B_{1/8}(0) \cap \overline{\Omega}_i$, and radii $0 < r_i < 1/8$, such that $\text{dist}(p_i,\{u_i = 0\}) \leq r_i 2^{-i}$ and

$$|N_{\Omega_i}(0,1,T_{p_i,r_i}u_i) - N_{\Omega_i}(0,1/10,T_{p_i,r_i}u_i)| \leq 2^{-i},$$

but that no $T_{p_i,r_i}u_i$ is $(0,\delta,1,0)$-symmetric.

By Lemma 6.5, we have that there exists a subsequence such that $T_{p_j,r_j}u_j$ converges strongly in $W^{1,2}(B_1(0))$ to a function u_∞. By Corollary 6.7, we know that u_∞ is harmonic in a convex domain Ω_∞. Furthermore, by Lemma 6.5 (1), $u_\infty(0) = 0$ and $u_\infty = 0$ on $\partial\Omega_\infty$. By Lemma 6.2 and the proof of Corollary 6.6 applied to $T_{p_j,r_j}u_j$, we have that $\lim_{j\to\infty} N_{\Omega_j}(0,r,T_{p_j,r_j}u_j) = N_{\Omega_\infty}(0,r,u_\infty) \in [0, C_2(n,\Lambda)]$, and that $N_{\Omega_\infty}(0,r,u_\infty)$ is constant for $1/10 \leq r \leq 1$. Therefore, by Lemma 7.1, u_∞ is $(0,0,1,0)$-symmetric. This contradicts our assumption that that no $T_{p_i,r_i}u_i$ is $(0,\delta,1,0)$-symmetric.

For the case that $B_1(0) \subset T_{p_i,r_i}\Omega_i$, repeat the argument to obtain the same contradiction. ■

Remark 7.3. By the scale invariance of the Almgren frequency, Lemma 7.2 implies that for all $0 < \delta$, if $0 < \gamma \leq \gamma_0(n,\Lambda,\delta)$, then

$$|N_\Omega(p,r,u) - N_\Omega(p,r/10,u)| \leq \gamma$$

and either $\text{dist}(p,\{u=0\}) \leq \gamma r$ or $B_r(p) \subset \Omega$ implies u is $(0,\delta,r,p)$-symmetric.

Next, we obtain a "cone-splitting" result. The prototypical example of a result like this is the following proposition. See Theorem 4.1.3 in [12] for the proof of similar results.

Proposition 7.4. *Let $P\colon \mathbb{R}^n \to \mathbb{R}$ be a 0-symmetric function. Let $k \leq n-2$. If P is symmetric with respect to some k-dimensional subspace V and P is homogeneous with respect to some point $x \notin V$, then P is $(k+1)$-symmetric with respect to* span$\{x, V\}$.

In order to prove a similar result for our almost-symmetric functions $u \in \mathcal{A}(n, \Lambda)$, we need the following preliminary observation.

Lemma 7.5. *Let $0 \leq k \leq n-1$ be an integer and let $0 < \varepsilon_i$ satisfy $\varepsilon_i \to 0$. If $u_i \in \mathcal{A}(n, \Lambda)$ with associated domain Ω_i, $p_i \in \overline{\Omega_i} \cap B_1(0)$, $0 < r_i \leq 1$, and u_i is $(k, \varepsilon_i, r_i, p_i)$-symmetric, then there is a function $u_\infty \in L^2(B_1(0))$ such that there exists a subsequence such that*

$$T_{p_j, r_j} u_j \to u_\infty \quad \textit{in the senses of Lemma 6.5,}$$

and u_∞ is $(k, 0, 1, 0)$-symmetric.

Proof. The existence of u_∞ is proven in Lemma 6.5. In order to prove the lemma, it suffices to prove that if P_j is a k-symmetric function which satisfies that u_j is $(k, \varepsilon_j, r_j, p_j)$-symmetric, then there is a subsequence such that $P_j \to P_\infty$ for some k-symmetric function P_∞ with order of homogeneity less than or equal to $2C_2(n, \Lambda)$. Since by assumption $N_{\mathbb{R}}(0, 1, P_j) = \|\nabla P_j\|_{L^2(B_1(0);\mathbb{R}^n)} \leq 2C_2(n, \Lambda)$, the sequence P_j is bounded in $W^{1,2}(B_1(0))$. Hence, by Rellich–Kondrakov, there is a subsequence, which we continue to label P_j, such that

$$P_j \rightharpoonup P_\infty \quad \text{in } W^{1,2}(B_1(0))$$

for some $P_\infty \in L^2(B_1(0))$ satisfying $N_{\mathbb{R}}(0, 1, P_\infty) = \|\nabla P_j\|_{L^2(B_1(0);\mathbb{R}^n)} \leq 2C_2(n, \Lambda)$.

If there is a subsequence for which P_j are homogeneous harmonic polynomials, then we may reduce to a subsequence of the same order. Since convergence in $L^2(B_1(0))$ implies $C^\infty(B_1(0))$ convergence, this implies that, in particular, the coefficients converge. Hence, P_∞ is a homogeneous harmonic polynomial.

If there is a subsequence for which P_j are homogeneous harmonic polynomials, then we may extract a subsequence such that P_j are defined upon convex cones Ω_j in $\mathcal{D}(n)$. By following the arguments of Lemma 6.5 and Corollary 6.7, we have that P_∞ is harmonic and defined upon a non-degenerate convex cone Ω_∞.

To see that in either case P_∞ has k symmetries, we precompose with rotations O_j sending the k-planes $V_j \to \mathbb{R}^k \subset \mathbb{R}^n$. Thus the $P_j \circ O_j$ only rely upon (the same) $(n-k)$ variables. Since O_j converges as $j \to \infty$, we see that $P_\infty \circ O_j$ converges to $P_\infty \circ O_\infty$, which itself only depends upon those $(n-k)$ variables. Thus, P_∞ is a k-symmetric function. ∎

Lemma 7.6 (Cone-splitting). *Let $0 \leq k \leq n-2$ be an integer and let $u \in \mathcal{A}(n, \Lambda)$. For any fixed $\varepsilon, \tau > 0$ and any $0 < r \leq 1/8$, there is a $0 < \delta_0 = \delta_0(n, \tau, \varepsilon, \Lambda)$ such that for $0 < \delta < \delta_0$, the following holds. If $p \in \overline{\Omega} \cap B_{1/8}(0)$ and u is (k, δ, r, p)-symmetric with respect to a k-dimensional subspace V and $(0, \delta, r, x)$-symmetric for some $x \in B_r(p) \setminus B_{\tau r}(V + p)$, then v is $(k+1, \varepsilon, r, p)$-symmetric.*

Proof. Assume that there exists a $\delta, \tau > 0$ for which there exist a sequence of $0 < r_i \leq 1$, functions $u_i \in \mathcal{A}(n, \Lambda)$ and points $\{p_i\}$ for which u_i is (k, i^{-1}, r_i, p_i)-symmetric with respect to some V_i and $(0, i^{-1}, r_i, x_i)$-symmetric for some $x_i \in B_{r_i}(p_i) \setminus B_\tau(V_i + p_i)$, but that all u_i are not $(k+1, \delta, r_i, p_i)$-symmetric.

By considering $T_{p_i, r_i} u_i$ and applying Lemma 6.5 and Lemma 6.7, there exists a harmonic function $u_\infty \in L^2(B_1(0))$ such that a subsequence $T_{p_j, r_j} u_j \to u_\infty$ in the senses of the lemma. Note that u_∞ is non-degenerate. Taking further subsequences, we may reduce to a sequence for which

$$T_{p_j, r_i} V_j \to V, \quad x_j \to x \in \overline{B_1(0)} \setminus B_\tau(V).$$

Now, by Lemma 7.5, u_∞ is $(k, 0, 1, 0)$-symmetric with respect to V and $(0, 0, 1, x)$-symmetric. Then, by Proposition 7.4, u_∞ is $(k+1, 0, 1, 0)$-symmetric. Since $u_i \to u$ in $L^2(B_2(0))$, we have our contradiction. Thus the lemma follows by taking the smallest $0 < \delta_0$ for $0 \leq k \leq n-2$, eliminating the dependence upon k. ■

Corollary 7.7. *Let $k \leq n-2$ and $u \in \mathcal{A}(n, \Lambda)$. For any fixed $\varepsilon, \tau > 0$ and any $0 < r \leq 1$, there is a $0 < \delta_0 = \delta_0(n, \tau, \varepsilon, \Lambda)$ such that for $0 < \delta < \delta_0$, the following holds. If $p \in \overline{\Omega} \cap B_{1/4}(0)$ and u is $(0, \delta, r, p)$-symmetric but not $(k+1, \varepsilon, r, p)$-symmetric, then there exists an affine k-plane V such that*

$$\{x \in B_r(p) : u \text{ is } (0, \delta, r, x)\text{-symmetric}\} \subset B_{\tau r}(V). \tag{7.1}$$

Proof. Let $0 < \varepsilon_1, \ldots, \varepsilon_{n-1}$ be small parameters to be chosen later. To prove the lemma, we inductively apply Lemma 7.6. That is, suppose that there is another point $x_1 \in B_r(p) \setminus B_{\tau r}(p)$ such that u is $(0, \delta, r, x)$-symmetric; then Lemma 7.6 implies that for any $0 < \varepsilon_1$ there is a $0 < \delta_0(n, \tau, \varepsilon_1, \Lambda)$ such that if $\delta \leq \delta_0(n, \tau, \varepsilon_1, \Lambda)$, then u is $(1, \varepsilon_1, r, p)$-symmetric with respect to some affine 1-plane V_1. Inducting up, assume that u is (i, ε_i, r, p)-symmetric with respect to V_i. Then, for any $0 < \varepsilon_{i+1}$ there exists a $\delta_0(n, \tau, \varepsilon_{i+1}, \Lambda)$ such that if we can find an $x_{i+1} \in B_r(p) \setminus B_{\tau r}(V_i)$ such that u is $(0, \delta, r, x)$-symmetric and $0 < \delta, \varepsilon_i \leq \delta_0(n, \tau, \varepsilon_{i+1}, \Lambda)$, then u is $(i+1, \varepsilon_{i+1}, r, p)$-symmetric with respect to V_i.

Therefore, choosing $0 < \varepsilon_{n-1} \leq \delta_0(n, \tau, \varepsilon, \Lambda)$ and $0 < \varepsilon_{i-1} \leq \delta_0(n, \tau, \varepsilon_i, \Lambda)$ for all $i = 1, \ldots, n-1$, we obtain $0 < \delta_0(n, \tau, \varepsilon, \Lambda) = \delta_0(n, \tau, \varepsilon_1, \Lambda)$, as desired. Since it is assumed that such that u is not $(k+1, \varepsilon, r, p)$-symmetric, this procedure must terminate before step $k+1$, and therefore there must exist an affine k-plane such that the claim of the corollary holds. ■

8. The covering and its properties

The lemmata in the previous section allow us to inductively define a covering with the right packing conditions. Quantitative rigidity allows us to prove a "quantitative differentiation" lemma that bounds the number of scales across which the frequency can change by more than some threshold $\gamma > 0$. Cone splitting, on the other hand, will give us good geometric control of the singular set at scales for which v is close to a homogeneous harmonic polynomial. Together, these things will give us the necessary packing conditions. First, we describe the covering.

8.1. The general construction

Let $u \in \mathcal{A}(n, \Lambda)$, and let $\varepsilon, r > 0$, $k \leq n-2$, and $N \in \mathbb{N}$ be given. We use the notation $\rho_i = 10^{-i}$. In this section we describe a general procedure which will produce a cover of $\mathcal{C}^k_{\varepsilon,r}(u) \cap B_{1/10}(0)$ by balls of radius ρ_N.

We begin by defining an auxiliary quantity. Let

$$\mathcal{D}(u, x, r) = \inf\{\delta' > 0 : u \text{ is } (0, \delta', r, x)\text{-symmetric}\}. \tag{8.1}$$

Let $0 < \delta_0$. We shall refer to δ_0 as the *sorting threshold*. For any $i \in \mathbb{N}$, we can assign to each $x \in \mathcal{C}^k_{\varepsilon,r}(u) \cap B_{1/8}(0)$ an i-tuple $T^i(x)$ according to the rule

$$(T^i(x))_j = 1 \quad \text{if } \mathcal{D}(u, x, \rho_j) \geq \delta_0,$$
$$(T^i(x))_j = 0 \quad \text{if } \mathcal{D}(u, x, \rho_j) < \delta_0.$$

For any T_i we shall use $|T^i|$ to denote the sum of the entries. Note that there is a partial ordering on the set of these i-tuples. That is, if $k < i$, we can say that $T^k < T^i$ if $(T^k)_j = (T^i)_j$ for all $j \in \{1, 2, \ldots, k\}$.

Now, we partition our set according to these i-tuples. For any given i-tuple $T^i \in \{0,1\}^i$, we define

$$E(T^i) = \{x \in \mathcal{C}^k_{\varepsilon,r}(u) \cap B_{1/10}(0) : T^i(x) = T^i\}.$$

It follows immediately from the definitions that $E(T^i) \subset E(T^k)$ if and only if $T^k < T^i$.

We now define our covering inductively. For $i = 1$, we let $C^k_{\varepsilon,r}(T^i) = B_{1/10}(0)$ for both 1-tuples $T^i \in \{0,1\}^1$. Now, assume that $i \in \mathbb{N}$, $i < N$, and $C^k_{\varepsilon,r}(T^i)$ has been defined and consists of balls of radius ρ_i. Within each ball $B_{\rho_i}(y) \in C^k_{\varepsilon,r}(T^i)$ partition the set $B_{\rho_i}(y) \cap E(T^i)$ into the sets $E(T^{i+1})$ for T^{i+1} such that $T^i < T^{i+1}$. For either such T^{i+1}, take a minimal covering of $B_{\rho_i}(y) \cap E(T^{i+1})$ by balls of radius ρ^{i+1} centered at points in $B_{\rho_i}(y) \cap E(T^{i+1})$. The union of these balls is $C^k_{\varepsilon,r}(T^{i+1})$. For some i-tuples, the set $E(T^i)$ may be empty. In this case, we simply allow the corresponding collection of balls $C^k_{\varepsilon,r}(T^i)$ be empty.

If $i = N$, we terminate the procedure. Note that for any sorting threshold $0 < \delta_0$ and $N \in \mathbb{N}$, this procedure defines a sequence of collections such that

$$\mathcal{C}^k_{\varepsilon,r}(u) \cap B_{1/10}(0) \subset \bigcup_{T^N} \bigcup_{B_{\rho_N}(y) \in C^k_{\varepsilon,r}(T^N)} B_{\rho_N}(y).$$

8.2. Properties of the construction

Now, we argue that there is a choice of sorting threshold $0 < \delta_0$ with the desired properties.

Lemma 8.1. *Let $u \in \mathcal{A}(n, \Lambda)$. Let $0 < \delta'$ be the sorting threshold in the construction above. There is a constant $D(n, \Lambda, \delta') < \infty$ such that for any $N \in \mathbb{N}$ there are at most $N^{D(n,\Lambda,\delta')}$ sets $E(T^N)$ such that $E(T^N)$ is non-empty.*

Proof. Let $0 < \delta'$ be given. Let $0 < \gamma_0(n, \Lambda, \delta') \ll 1$ as in Lemma 7.2 and Remark 7.3. Now, decompose $\Omega = \cup_{j=0}^{\infty} A_j(\Omega)$, where

$$A_j(\Omega) = \Omega \cap B_{\gamma_0^j}(\partial\Omega) \setminus B_{\gamma_0^{j+1}}(\partial\Omega).$$

We shall argue that there is a $D = D(n, \Lambda, \delta')$ such that $|T^N(p)| \le D$ for all $p \in \overline{\Omega} \cap B_{10}(0)$. If the claim is true, then if $N \le D$ there are at most $2^N \le N^D$ N-tuples with $|T^N| \le D$. And, if $N \ge D$ there are at most $\binom{N}{D}$ many N-tuples with $|T^N| \le D$. Since $\binom{N}{D} \le N^D$, we have the desired claim: $\mathcal{C}^k_{\varepsilon,r}(u) \cap B_{1/10}(0)$ is contained in the union of at most N^D nonempty sets $E(T^N)$ and covered by at most N^D collections $C^k_{\varepsilon,r}(T^N)$.

To prove the claim, we argue by cases. Let $p \in E(T^N)$. If $p \in \partial\Omega$, then $u(p) = 0$ and

$$|T^N(p)| \le |\{i \in \mathbb{N} : |N_\Omega(p, \rho_i, u) - N_\Omega(p, \rho_{i+1}, u)| \ge \gamma_0(n, \Lambda, \delta')\}| \le C_3(n, \Lambda)\gamma_0^{-1},$$

where $C_3(n, \Lambda)$ is the bounded variation bound from Lemma 6.4 and $\gamma_0(n, \Lambda, \delta')$ is as in Remark 7.3.

Now, assume that $p \in A_j(\Omega)$. Note that, by Remark 7.3,

$$|N_\Omega(p, \rho_i, u) - N_\Omega(p, \rho_{i+1}, u)| \le \gamma_0(n, \Lambda\delta')$$

still implies $(T^N(p))_i = 0$ if $B_{\rho_i}(p) \subset \Omega$ or if $\mathrm{dist}(p, \partial\Omega) \le \gamma_0(n, \Lambda, \delta')\rho_i$. But the conditions $B_{\rho_i}(p) \subset \Omega$ or $\mathrm{dist}(p, \partial\Omega) \le \gamma_0(n, \Lambda, \delta')\rho_i$ only fail for $i \in [(j-1)\frac{\ln(\gamma_0)}{\ln(\rho_0)}, (j+1)\frac{\ln(\gamma_0)}{\ln(\rho_0)}]$. Therefore,

$$|T^N(p)| \le \frac{C_3(n, \Lambda)}{\gamma_0(n, \Lambda, \delta')} + 3\frac{\ln(\gamma_0(n, \Lambda, \delta'))}{\ln(\rho_0)} =: D(n, \Lambda, \delta')$$

independently of j. This proves the claim. ∎

Remark 8.2. If, later, we choose $0 < \gamma \le \gamma_0(n, \Lambda, \delta')$ so that $\gamma = \gamma(n, \Lambda, \varepsilon)$, then the statement of Lemma 8.1 holds with a new $D = D(n, \Lambda, \varepsilon)$.

We now prove that this construction satisfies the claimed packing condition.

Lemma 8.3. *Let $u \in \mathcal{A}(n, \Lambda)$, $0 < \varepsilon, r < 1$, and $k \le n-2$. Let $\delta' = \delta_0(n, 1/10, \varepsilon, \Lambda)$ be as in Corollary 7.7. Let $\gamma_0(n, \Lambda, \delta')$ be as in Lemma 7.2.*

Then for all $N \in \mathbb{N}$ with $\rho_N \ge r$, there exist constants $0 < c_1, c_2$, depending only on the ambient dimension n and a constant $D(n, \Lambda, \varepsilon)$, such that each collection $C^k_{\varepsilon,r}(T^N)$ consists of at most $(c_1\rho_1^{-n})^D(c_2\rho_1^{-k})^{N-D}$ balls of radius ρ_N.

Proof. For any given N-tuple T^N for which $E(T^N)$ is non-empty, let $T^i < T^N$. Now let $B_{\rho_i}(x) \in C^k_{\varepsilon,r}(T^i)$. Consider the set

$$A := B_{\rho_i}(x) \cap E(T^N).$$

We argue by cases.

Case 1. $(T^N)_i = 0$. In this case, u is $(0, \delta_0, \rho_i, x)$-symmetric. By applying Corollary 7.7, we see that $A \subset B_{\rho_i}(x) \cap B_{\rho_{i+1}}(V^k)$ for some k-dimensional plane V^k. Thus, the minimal covering from the construction can cover $A \subset B_{\rho_i}(x) \cap B_{\rho_{i+1}}(V^k)$ by at most $c_1(n)\rho_1^{-k}$ balls of radius ρ_{i+1}.

Case 2. $(T^N)_i = 1$. In this case, we have no control. Therefore, the minimal covering of A described in the construction could consist of at most $c_2(n)\rho_1^{-n}$ balls of radius ρ_{i+1}.

Carrying this process through, by the proof of Lemma 8.1, Case 2 can only happen at most $D(n, \Lambda, \varepsilon)$ times. Thus, $C^k_{\varepsilon,r}(T^N)$ is a collection of at most $(c_1\rho_1^{-n})^D(c_2\rho_1^{-k})^{N-D}$ balls of radius ρ^N, as claimed. ∎

8.3. Proof of Theorem 2.11

Proof. Let $0 < \varepsilon, r < 1$, and $k \leq n-2$. Let $\delta_0 = \delta_0(n, 1/10, \varepsilon, \Lambda)$ be as in Corollary 7.7. Let $0 < \gamma_0(n, \Lambda, \delta_0)$ be as in Lemma 7.2. Recalling $c_2(n)$ from Lemma 8.3, let

$$0 < \gamma \leq \min\{\gamma_0(n, \Lambda, \delta_0), c_2^{-2/\varepsilon}\} < 1.$$

The construction given in Section 8.1 gives a covering of $B_{1/10}(0) \cap \mathcal{C}^k_{\varepsilon,r}(u)$ by balls of radius ρ_N. Doubling the radius of these balls is sufficient, then, to cover $B_{1/10}(0) \cap B_{\rho^N}(\mathcal{C}^k_{\varepsilon,r}(u))$.

Thus, by Remark 8.2 for $D = D(n, \Lambda, \varepsilon)$,

$$\begin{aligned}\mathrm{Vol}(B_{1/10}(0) \cap B_{\rho_N}(\mathcal{C}^k_{\varepsilon,r}(u))) &\leq N^D (c_1 \rho_1^{-n})^D (c_2 \rho_1^{-k})^{N-D} (\omega_n 2\rho^N)^n \\ &\leq c(n)\, N^D c_1^D c_2^{N-D} (\omega_n 2)^n \rho_N^{n-k} \rho_1^{-D(n-k)}.\end{aligned}$$

Estimating $\rho_1^{-D(n-k)} \leq C(n, \Lambda, \varepsilon)$ and $N^D \leq C(n, \Lambda, \varepsilon) c_2(n)^N$ for all $N \in \mathbb{N}$ and recalling from our choice of $0 < \gamma$ that $c_2(n) \leq \gamma^{-\varepsilon/2}$, we obtain

$$\begin{aligned}\mathrm{Vol}(B_{1/8}(0) \cap B_{\rho_N}(\mathcal{C}^k_{\varepsilon,r}(u))) &\leq N^D c_1^D c_2^{N-D} (\omega_n 2)^n \rho_N^{n-k} \rho_1^{-D(n-k)} \\ &\leq C(n, \Lambda, \varepsilon)\, c_2(n)^{2N-D} c_1(n)^D (\omega_n 2)^n \rho_N^{n-k} \\ &\leq C(n, \Lambda, \varepsilon)\, \gamma^{-N\varepsilon} \rho_N^{n-k} \leq C(n, \Lambda, \varepsilon)\, \rho_N^{n-k-\varepsilon}.\end{aligned}$$

Thus, for any $0 < r$ we may find an $N \in \mathbb{N}$ such that $\rho_{N+1} \leq r < \rho_N$. Thus, we may estimate

$$\mathrm{Vol}(B_{1/10}(0) \cap B_r(\mathcal{C}^k_{\varepsilon,r}(u))) \leq C(n, \Lambda, \varepsilon)\, \rho_N^{n-k-\varepsilon} = C'(n, \Lambda, \varepsilon)\, r^{n-k-\varepsilon}.$$

Covering $B_{1/4}(0)$ by at most $C(n)$ many such balls of radius $1/10$ and repeating the argument produces (2.2). ∎

9. Containment

We now turn to proving Lemma 2.12, which follows from the rigidity and continuity of the Almgren frequency function.

Lemma 9.1. *Let $u \in \mathcal{A}(n, \Lambda)$. Suppose that $Q \in \partial\Omega \cap B_{1/4}(0)$ and that there exists a $p \in \mathcal{C}^{n-2}(u)$ such that $B_r(p) \subset \Omega \cap B_{1/4}(0)$ and $p \subset B_{2r}(Q)$. Then there is an $0 < \delta(n, \Lambda)$ such that $N_\Omega(Q, 4r, u) \geq 1 + \delta$.*

Proof. Suppose that the statement is false. Then, there exists a sequence of u_i, Q_i, p_i and $0 < r_i$ such that $B_{r_i}(p_i) \subset \Omega_i \cap B_{1/4}(0)$ for which

$$N_{T_{Q_i,4r_i}\Omega_i}(0, 1, T_{Q_i,4r_i} u_i) \leq 1 + 2^{-i}.$$

By Lemma 6.5, we may extract a subsequence $T_{Q_j,4r_j} u_j$ which converges strongly in $W^{1,2}(B_1(0))$ to a function u_∞. Similarly, we may assume that $T_{Q_j,4r_j} p_j \to p$ and that

$B_{1/4}(p) \subset \overline{\Omega_\infty \cap B_{1/2}(0)}$. Observe that by Lemma 6.6, $N_{\Omega_\infty}(0, 1, u_\infty) = 1$. Therefore, since Ω_∞ is convex, $N_{\Omega_\infty}(0, 0^+, u_\infty) \geq 1$, which implies $N_{\Omega_\infty}(0, r, u_\infty)$ is constant for $0 < r \leq 1$. By Lemma 7.1, u_∞ must be a piecewise linear function. Therefore, $N_{\Omega_\infty}(p, 1/4, u_\infty) = 1$. However, $N_{\Omega_j}(p_j, r_j, u_j) \geq 2$ for all j, which implies by Lemma 6.6 that $N_{\Omega_\infty}(p, 1/4, u_\infty) \geq 2$. This is a contradiction. ∎

Lemma 9.2. *Let $u \in \mathcal{A}(n, \Lambda)$. Suppose that $p \in \overline{\Omega} \cap B_{1/4}(0)$, $0 < r \leq 1/4$, and that $N_\Omega(p, r, u) \geq 2$. If either $p \in \partial\Omega$ or $B_r(p) \subset \Omega$, then there is an $0 < \varepsilon = \varepsilon(n, \Lambda)$ such that u is not $(n-1, \varepsilon, r, p)$-symmetric.*

Proof. Suppose that the statement is false. Then there exists a sequence of u_i, p_i, and $0 < r_i$ such that $N_{T_{p_i,r_i}\Omega_i}(0, 1, T_{p_i,r_i}u_i) \geq 2$ and u_i is $(n-1, 2^{-i}, r, p)$-symmetric. Applying Lemma 6.5, we may extract a subsequence $T_{p_j,r_j}u_j$ which converges strongly in $W^{1,2}(B_1(0))$ to a function u_∞. By Lemma 6.6, $N_{\Omega_\infty}(0, 1, u_\infty) \geq 2$. But, by Lemma 7.5, the function u_∞ is $(n-1, 0, 1, 0)$-symmetric and hence piecewise linear.

If either $B_1(0) \subset \Omega_\infty$ or $0 \in \partial\Omega$, then, as u_∞ is piecewise linear, $N_{\Omega_\infty}(0, 1, u_\infty) = 1$. This is a contradiction. ∎

9.1. Proof of Lemma 2.12

Proof. First, we prove (2.3). Because for all integers k and all $0 < \varepsilon$ the set $\mathcal{C}^k_\varepsilon(u)$ is closed, we reduce to proving that there is an $0 < \varepsilon(n, \Lambda)$ such that $\mathcal{C}^{n-2}(u) \cap B_{1/8}(0) \subset \mathcal{C}^{n-2}_\varepsilon(u)$. Suppose that this containment is false. Then there would exist a sequence of functions $u_i \in \mathcal{A}(n, \Lambda)$, points $p_i \in \mathcal{C}^{n-2}(u_i) \cap B_{1/8}(0)$, and scales $0 < r_i \leq 1$, such that u_i is $(n-1, 2^{-i}, p_i, r_i)$-symmetric. We rescale to the functions $T_{p_i,r_i}u_i$. By Lemmas 6.5 and 7.5, there exist a subsequence (also indexed by i) and a $(n-1)$-symmetric function u_∞ such that $T_{p_i,r_i}u_i \to u_\infty$ strongly in $W^{1,2}(B_1(0))$ and $C^0(B_1(0))$. Note that for all $Q \in \partial\Omega_\infty$ and all $B_r(p) \subset \Omega_\infty$, $N_{\Omega_\infty}(Q, r, u_\infty) = N_{\Omega_\infty}(p, r, u_\infty) = 1$.

If $B_\delta(0) \in \Omega_\infty$, then for all sufficiently large $i \in \mathbb{N}$, $N_{\Omega_i}(p_i, \delta/2r_i, u_i) \geq 2$. Letting $i \to \infty$, by Lemma 6.6, $N_{\Omega_\infty}(0, \delta/2, u_\infty) \geq 2$, which contradicts u_∞ being piecewise linear.

If $0 \in \partial\Omega_\infty$, then for each p_i we let $Q_i \in \partial\Omega_i$ be such that $|p_i - Q_i| = \mathrm{dist}(p_i, \partial\Omega_i)$. By Lemma 9.1, there exists a $0 < \delta(n, \Lambda)$ such that $N_{\Omega_i}(Q_i, 4|Q_i - p_i|, u_i) \geq 1 + \delta$ for all $i \in \mathbb{N}$. For sufficiently large i, $4|Q_i - p_i| \leq r_i$, and so by Lemma 3.4, for sufficiently large i, $N_{\Omega_i}(Q_i, r_i, u_i) \geq 1 + \delta$. Letting $i \to \infty$, we obtain by Lemma 6.6 that $N_{\Omega_\infty}(0, 1, u_\infty) > 1 + \delta$. This is a contradiction. Thus, there exists an $0 < \varepsilon(n, \Lambda)$ such that $\mathcal{C}^{n-2}(u) \cap \Omega \cap B_{1/8}(0) \subset \mathcal{C}^{n-2}_\varepsilon(u)$.

For proving (2.4), we note that if $Q \in \mathcal{C}^{n-2}(u) \cap \partial\Omega \setminus \mathrm{sing}(\partial\Omega)$, then letting $r \to 0$, we may extract a subsequence such that $T_{Q,r_j}u \to u_\infty$ in the sense of Lemma 5.4 and Lemma 6.5. By the monotonicity of the Almgren frequency, Lemma 6.6, and by considering $T_{Q,cr_j}u$ for any $0 < c < 1$, we see that $N_{\Omega_\infty}(0, r, u_\infty) \equiv \lim_{r\to 0^+} N_\Omega(Q, r, u)$. Thus, Lemma 7.1 implies that u_∞ is a homogeneous function which is harmonic in Ω_∞. Since $Q \notin \mathrm{sing}(\partial\Omega)$, Ω_∞ is a half-space and we may extend u_∞ to an entire, homogeneous harmonic function by reflection. Since $Q \in \mathcal{C}^{n-2}(u)$, this polynomial must be a non-linear homogeneous harmonic polynomial and $\lim_{r\to 0^+} N_\Omega(Q, r, u) \geq 2$. By Lemma 3.4, then $N_\Omega(Q, r, u) \geq 2$ for all $0 < r \leq 1$, and Lemma 9.2 gives the claim. ∎

A. Hölder continuity

In this appendix, we provide a proof of Lemma 5.1. First, some standard results.

Definition A.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be of class S if there exist numbers $0 < c_0 \le 1$ and $0 < r_0$ such that for all $Q \in \partial\Omega$ and all $0 < r \le r_0$,

$$\mathcal{H}^n(B_r(Q) \cap \Omega^c) \ge c_0 \mathcal{H}^n(B_r(Q)).$$

Lemma A.2 (Bounding the supremum, Lemma 1.1.22 in [15]). *Let Ω be a domain of class S. Let $Q \in \partial\Omega \cap B_1(0)$ and $0 < r \le 1/2$. Let u be a function which is harmonic in Ω such that $u \in C(\overline{B_{2r}(Q) \cap \Omega})$ and $u \equiv 0$ on $B_{2r}(Q) \cap \partial\Omega$. There exists a $c(n)$ such that for any $p \in B_r(Q) \cap \Omega$,*

$$\max_{B_r(Q)\cap\Omega} |u| \le c(n)\Big(\fint_{B_{2r}(Q)\cap\Omega} u^2\,dx\Big)^{1/2}.$$

Lemma A.3 (Hölder continuity up to the boundary, Corollary 1.1.24 in [15]). *Let Ω be a domain of class S. Let $Q \in \partial\Omega \cap B_1(0)$ and $0 < r \le 1/2$. Let u be a function which is harmonic in Ω, $u \in C(\overline{B_{2r}(Q) \cap \Omega})$, $u \equiv 0$ on $B_{2r}(Q) \cap \partial\Omega$, and $u \ge 0$. There exist a $c(n)$ and an exponent $0 < \alpha(n) \le 1$ such that for any $p \in B_r(Q) \cap \Omega$,*

$$u(p) \le c(n)\Big(\frac{|p-Q|}{r}\Big)^{\alpha} \sup\{u(y) : y \in B_{2r}(Q)\}.$$

Theorem A.4 (Oscillation in the interior, Theorem 6.6 in [13]). *Suppose that u is harmonic in $\tilde{\Omega}$. If $0 < r < R < \infty$ are such that $B_r(x) \subset B_R(x_0) \subset \tilde{\Omega}$, then*

$$\operatorname{osc}(u, B_r(x_0)) \le 2^{\alpha}\Big(\frac{r}{R}\Big)^{\alpha} \operatorname{osc}(u, B_R(x_0)),$$

where $\alpha = \alpha(n) \in (0,1]$ only depends on n.

Theorem A.5 (Hölder continuity in $B_2(0)$, Theorem 6.44 in [13]). *Suppose that Ω_1 is of class S with constant $c_0 > 0$ and $0 < r_0 \le 1$. Let $h \in C^0(\overline{\Omega_1})$ be a harmonic function in Ω_1. If there are constants $M \ge 0$ and $0 < \alpha \le 1$ such that*

$$|h(x) - h(y)| \le M|x-y|^{\alpha}$$

for all $x, y \in \partial\Omega_1$, then

$$|h(x) - h(y)| \le M_1|x-y|^{\gamma}$$

for all $x, y \in \overline{\Omega_1}$. Moreover, $\gamma = \gamma(n, \alpha, c_0) > 0$ and one can choose the constant M_1 to be $M_1 = 80Mr_0^{-2}\max\{1, 2\operatorname{diam}(\Omega_1)\}$.

A.1. Proof of Lemma 5.1

Let $u \in \mathcal{A}(n, \Lambda)$ with associated domain $\Omega \in \mathcal{D}(n)$, and let $Q_0 \in \partial\Omega \cap B_1(0)$ and $0 < r_0 \le 1/2$. First, we claim that

$$\Big(\fint_{B_2(0)\cap\Omega} (T_{Q_0,r_0}u)^2\,dx\Big)^{1/2} \le C(n, \Lambda).$$

By the Poincaré inequality,

$$\fint_{B_2(0)} |T_{Q,r}u - \mathrm{Avg}_{B_2(0)}(T_{Q,r}u)|^2\, dx \le C(n)|B_2(0)|^{2/n}\Big(\fint_{B_2(0)} |\nabla T_{Q,r}u|^2\, dx\Big)$$
$$\le C(n)C(\Lambda, n).$$

Furthermore, since $\mathcal{H}^n(\Omega^c \cap B_2(0)) \ge \frac{1}{10}\mathcal{H}^n(B_2(0))$, we estimate

$$|\mathrm{Avg}_{B_2(0)}(T_{Q,r}u)|^2 \frac{1}{10}\mathcal{H}^n(B_2(0)) \le C(n)C(\Lambda, n).$$

Thus, $|\fint_{B_2(0)} T_{Q,r}u\, dx| \le C'(n)\sqrt{C(n,\Lambda)}$.

Next, we claim that there exist a $c(n)$ and an exponent $0 < \alpha(n) \le 1$ such that, for $Q \in T_{Q_0,r_0}\partial\Omega \cap B_1(0)$ and $p \in T_{Q_0,r_0}\Omega \cap B_{1/2}(Q)$,

$$|T_{Q_0,r_0}u(p)| \le C(n,\Lambda)\Big(\frac{|p-Q|}{r}\Big)^\alpha.$$

For any $T_{Q_0,r_0}u$ which changes sign in $T_{Q_0,r_0}\Omega \cap B_1(Q)$, we decompose $T_{Q_0,r_0}u = T_{Q_0,r_0}u^+ - T_{Q_0,r_0}u^-$. Note that both $T_{Q_0,r_0}u^\pm$ are subharmonic. Let $h_\pm$ be the harmonic extension of $T_{Q_0,r_0}u^\pm$ to $B_1(Q) \cap T_{Q_0,r_0}\Omega$. Note that $B_1(Q) \cap T_{Q_0,r_0}\Omega$ is convex, and so is of class S. Then, by Lemma A.3, and the maximum principle,

$$h_\pm(p) \le c(n)\Big(\frac{|p-Q|}{r}\Big)^\alpha \sup\{h_\pm(y) : y \in \partial(B_1(Q) \cap T_{Q_0,r_0}\Omega)\}.$$

By subharmonicity, $T_{Q_0,r_0}u^\pm \le h_\pm$, respectively. By construction, $h_\pm = T_{Q_0,r_0}u^\pm$ on $\partial(B_1(Q) \cap T_{Q_0,r_0}\Omega)$ and by our first claim and Lemma A.2,

$$\sup\{h_\pm(y) : y \in \partial(B_1(Q) \cap T_{Q_0,r_0}\Omega)\} \le C(n,\Lambda).$$

Note that this gives uniform control on the oscillation in $T_{Q_0,r_0}\Omega \cap B_2(0)$. This uniform control, together with Theorem A.4, implies that $T_{Q_0,r_0}u$ is locally Hölder on $\partial B_1(0) \cap T_{Q_0,r_0}\Omega$.

Now, we claim that for all $x, y \in \partial(T_{Q_0,r_0}\Omega \cap B_1(0))$,

$$|T_{Q_0,r_0}u(x) - T_{Q_0,r_0}u(y)| \le C(n,\Lambda)|x-y|^\alpha.$$

We argue by cases. Suppose that $|x-y| < \max\{\mathrm{dist}(x, T_{Q_0,r_0}\partial\Omega), \mathrm{dist}(y, T_{Q_0,r_0}\partial\Omega)\}$. Then, there is a ball $B_r(z) \subset T_{Q_0,r_0}\Omega$ with $|x-y| < r \le 2|x-y|$ which contains both x and y. By Theorem A.4 and the preceding paragraph, then we have the desired statement.

Suppose $|x-y| \ge \max\{\mathrm{dist}(x, T_{Q_0,r_0}\partial\Omega), \mathrm{dist}(y, T_{Q_0,r_0}\partial\Omega)\}$. Let $x_0, y_0 \in T_{Q_0,r_0}\partial\Omega$ be points such that $|x-x_0| = \mathrm{dist}(x,\partial\Omega)$ and $|y-y_0| = \mathrm{dist}(y,\partial\Omega)$. Then

$$\begin{aligned}|T_{0,1}u(x) - T_{0,1}u(y)| &\le |T_{0,1}u(x) - T_{0,1}u(x_0)| + |T_{0,1}u(y) - T_{0,1}u(y_0)|\\ &\le C(n,\Lambda)\,2^\alpha|x-x_0|^\alpha + C(n,\Lambda)\,2^\alpha|y-y_0|^\alpha\\ &\le C(n,\Lambda)\,2^{\alpha+1}(\max\{\mathrm{dist}(x, T_{Q_0,r_0}\partial\Omega), \mathrm{dist}(y, T_{Q_0,r_0}\partial\Omega)\})^\alpha\\ &\le C(n,\Lambda)|x-y|^\alpha.\end{aligned}$$

This proves the claim. To obtain uniform interior Hölder continuity on the interior of $T_{Q_0,r_0}\Omega \cap B_1(0)$, we invoke Theorem A.5 with $\Omega_1 = T_{Q_0,r_0}\Omega \cap B_1(0)$.

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