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Fully nonlinear singularly perturbed models with non-homogeneous degeneracy

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Abstract. This work is devoted to studying non-variational, nonlinear singularly perturbed elliptic models enjoying a double degeneracy character with prescribed boundary value in a domain. In its simplest form, for each $\varepsilon > 0$ fixed, we seek a non-negative function u^{ε} satisfying

$$\begin{cases} [|\nabla u^{\varepsilon}|^{p} + \mathfrak{a}(x)|\nabla u^{\varepsilon}|^{q}]\,\Delta u^{\varepsilon} = \zeta_{\varepsilon}(x, u^{\varepsilon}) & \text{in } \Omega, \\ u^{\varepsilon}(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense for suitable data $p, q \in (0, \infty)$, $\mathfrak{a}$ and g, where ζ_{ε} behaves singularly as $\mathrm{O}(\varepsilon^{-1})$ near ε-level surfaces. In such a context, we establish the existence of certain solutions. We also prove that solutions are locally (uniformly) Lipschitz continuous, and they grow in a linear fashion. Moreover, solutions and their free boundaries possess a sort of measure-theoretic and weak geometric properties. In particular, for a restricted class of nonlinearities, we prove the finiteness of the $(N-1)$-dimensional Hausdorff measure of level sets. We also address a complete and in-deep analysis concerning the asymptotic limit as $\varepsilon \to 0^{+}$, which is related to one-phase solutions of inhomogeneous nonlinear free boundary problems in flame propagation and combustion theory. Finally, we present some fundamental regularity tools in the theory of doubly degenerate fully nonlinear elliptic PDEs, which may have their own mathematical interest.

1. Introduction

In this manuscript we shall develop an approach to study (locally) sharp and geometric estimates of one-phase solutions to singularly perturbed problems having a non-homogeneous double degeneracy, whose mathematical model is given as follows: fixed a parameter $\varepsilon \in (0, 1)$, we would like to find

$$u^{\varepsilon} \geq 0 \ \text{ viscosity solution to } \begin{cases} \mathcal{H}(x, \nabla u^{\varepsilon}) F(x, D^2 u^{\varepsilon}) = \zeta_{\varepsilon}(x, u^{\varepsilon}) & \text{in } \Omega, \\ u^{\varepsilon}(x) = g(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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for a bounded and open set $\Omega \subset \mathbb{R}^N$, where $0 \le g \in C^0(\partial\Omega)$, F is a second order, fully nonlinear (uniformly elliptic) operator, i.e., nonlinear in its highest derivatives, $(x,\zeta) \to \mathcal{H}(x,\zeta)$ is a continuos vector-valued function, satisfying a non-homogeneous degeneracy in the second variable, and ζ_ε behaves singularly as $O(\varepsilon^{-1})$ close to ε-level surfaces.

We will focus our attention to reaction-diffusion models with singular behavior of order $\mathrm{O}(\varepsilon^{-1})$ near ε-level layers, i.e., $\{u^\varepsilon \sim \varepsilon\}$. Furthermore, the diffusion process is assumed to be anisotropic and doubly degenerate, thereby collapsing as $|\nabla u^\varepsilon| \sim 0$.

In a few words, under the appropriated hypothesis on data, we show that, for $\varepsilon \to 0^+$, the family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ to (1.1) are asymptotic approximations to a one-phase solution u_0 of an inhomogeneous nonlinear free boundary problem (for short, FBP), which arises in the mathematical formulation of some issues in flame propagation and combustion theory (stationary setting, cf. [15], [47] and [67]).

1.1. Main assumptions

We will assume the following structural assumptions.

(A0) (**Continuity and normalization condition**)

$$\text{Fixed } \Omega \ni x \mapsto F(x,\cdot) \in C^0(\mathrm{Sym}(N)) \quad \text{and} \quad F(\cdot, \mathrm{O}_{N\times N}) = 0.$$

(A1) (**Uniform ellipticity**) For any pair of matrices $\mathrm{X}, \mathrm{Y} \in \mathrm{Sym}(N)$,

$$\mathcal{M}^-_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \le F(x,\mathrm{X}) - F(x,\mathrm{Y}) \le \mathcal{M}^+_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}),$$

where $\mathcal{M}^\pm_{\lambda,\Lambda}$ stand for the *Pucci extremal operators* given by

$$\mathcal{M}^-_{\lambda,\Lambda}(\mathrm{X}) := \lambda \sum_{e_i>0} e_i(\mathrm{X}) + \Lambda \sum_{e_i<0} e_i(\mathrm{X}) \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(X) := \Lambda \sum_{e_i>0} e_i(\mathrm{X}) + \lambda \sum_{e_i<0} e_i(\mathrm{X})$$

for *ellipticity constants* $0 < \lambda \le \Lambda < \infty$, where $\{e_i(\mathrm{X})\}_i$ are the eigenvalues of X.

Moreover, for our Lipschitz estimates, we must require some sort of continuity assumption on coefficients:

(A2) (ω**-continuity of coefficients**) There exist a uniform modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ and a constant $\mathrm{C_F} > 0$ such that

$$\Omega \ni x, x_0 \mapsto \Theta_{\mathrm{F}}(x,x_0) := \sup_{\substack{\mathrm{X}\in\mathrm{Sym}(N)\\ \mathrm{X}\neq 0}} \frac{|F(x,\mathrm{X}) - F(x_0,\mathrm{X})|}{\|\mathrm{X}\|} \le \mathrm{C_F}\,\omega(|x-x_0|),$$

which measures the oscillation of the coefficients of F around x_0. Finally, we define

$$\|F\|_{C^\omega(\Omega)} := \inf\left\{\mathrm{C_F} > 0 : \frac{\Theta_{\mathrm{F}}(x,x_0)}{\omega(|x-x_0|)} \le \mathrm{C_F}, \quad \forall x, x_0 \in \Omega,\ x \neq x_0\right\}.$$

In our study, the diffusion properties of the model (1.1) degenerate along an a priori unknown set of singular points of existing solutions:

$$\mathcal{S}_0(u,\Omega') := \{x \in \Omega' \Subset \Omega : |\nabla u(x)| = 0\}.$$

For this reason, we will enforce that $\mathcal{H}\colon \Omega \times \mathbb{R}^N \to [0,\infty)$ behaves as

$$L_1 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x,|\xi|) \le \mathcal{H}(x,\xi) \le L_2 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x,|\xi|) \tag{1.2}$$

for constants $0 < L_1 \le L_2 < \infty$, where

(N-HDeg) $$\mathcal{K}_{p,q,\mathfrak{a}}(x,|\xi|) := |\xi|^p + \mathfrak{a}(x)|\xi|^q, \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}^N.$$

In addition, for the non-homogeneous degeneracy (**N-HDeg**), we suppose that the exponents p, q and the modulating function $\mathfrak{a}(\cdot)$ fulfil

$$0 < p \le q < \infty \quad \text{and} \quad \mathfrak{a} \in C^0(\Omega, [0,\infty)). \tag{1.3}$$

Roughly speaking, $\mathcal{H}$ satisfies distinct growths at the origin and at infinity:

$$\lim_{|\xi|\to 0^+} \frac{\mathcal{H}(x,\xi)}{|\xi|^p(1+\mathfrak{a}(x))} \in (0,\infty) \quad \text{and} \quad \lim_{|\xi|\to +\infty} \frac{\mathcal{H}(x,\xi)}{|\xi|^q(1+\mathfrak{a}(x))} \in [0,\infty)$$

uniformly in $x \in \Omega$. Moreover, for $p \ne q$,

$$\lim_{|\xi|\to 0^+} \frac{\mathcal{H}(x,\xi)}{|\xi|^q} = +\infty \quad \text{and} \quad \lim_{|\xi|\to +\infty} \frac{\mathcal{H}(x_0,\xi)}{|\xi|^p} = \begin{cases} \text{finite}, & \text{if } \mathfrak{a}(x_0) = 0, \\ +\infty, & \text{if } \mathfrak{a}(x_0) > 0. \end{cases}$$

In turn, in our research, the reaction term, i.e., $\zeta_\varepsilon\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, represents the singular perturbation of the model. In this point, we are interested in a singular behaviour of order $\mathrm{O}(1/\varepsilon)$ along ε-level layers $\{u^\varepsilon \sim \varepsilon\}$. Hence, we are led to consider reaction terms fulfilling

$$\mathcal{B}_0 \le \zeta_\varepsilon(x,t) \le \frac{\mathcal{A}}{\varepsilon}\chi_{(0,\varepsilon)}(t) + \mathcal{B}, \quad \forall (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.4}$$

for nonnegative constants $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B} \ge 0$. Notice that $\zeta_\varepsilon \equiv 0$ satisfies (1.4). Nevertheless, we shall also impose the following non-degeneracy assumption in order to ensure that such a reaction term enjoys an authentic singular character:

$$\mathfrak{I} := \inf_{\Omega \times [t_0, \mathrm{T}_0]} \varepsilon \zeta_\varepsilon(x, \varepsilon t) > 0, \tag{1.5}$$

for some constants $0 \le t_0 < \mathrm{T}_0 < \infty$, where $\mathfrak{I}$ does not depend on ε. Intuitively, (1.5) means that the singular term behaves as $\sim \frac{1}{\varepsilon}\chi_{(0,\varepsilon)}$ plus a non-negative noise that remains uniformly controlled. Indeed, simpler cases covered by our analysis are singular reaction terms built up as a multiple of the approximation of unity plus a uniform bounded function,

$$\zeta_\varepsilon(x,t) := \mathcal{Q}(x)\frac{1}{\varepsilon}\zeta\left(\frac{t}{\varepsilon}\right) + f_\varepsilon(x). \tag{1.6}$$

For such approximations, $0 < \mathcal{Q} \in C^0(\overline{\Omega})$, $0 \le \zeta \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\zeta = [0,1]$, and f_ε is a non-negative continuous function bounded away from infinity. Finally, it is readily verifiable that the reaction term in (1.6) fulfills (1.4) and (1.5) with $\mathcal{A} = \|\mathcal{Q}\|_{L^\infty(\Omega)}\|\zeta\|_{L^\infty(\mathbb{R}_+)}$, $\mathcal{B}_0 = \inf_\Omega f_\varepsilon(x)$ and $\mathcal{B} = \|f_\varepsilon\|_{L^\infty(\Omega)}$.

1.2. Statement of main results

To formulate our main results, we need to introduce some definitions. We will start with the definition of the viscosity solution to

$$\mathcal{G}(x, \nabla u, D^2 u) := \mathcal{H}(x, \nabla u) F(x, D^2 u). \tag{1.7}$$

Definition 1.1 (Viscosity solution). A function $u \in C^0(\Omega)$ is called a viscosity sub-solution (super-solution) of

$$\mathcal{G}(x, \nabla u(x), D^2 u(x)) = f(x, u(x)) \quad \text{in } \Omega$$

if whenever $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local maximum (minimum) at $x_0 \in \Omega$, there holds

$$\mathcal{G}(x, \nabla \varphi(x_0), D^2 \varphi(x_0)) \geq f(x_0, \varphi(x_0)) \quad (\text{respectively}, \ \leq f(x_0, \varphi(x_0))).$$

Finally, a function u is a viscosity solution when it is a simultaneously a viscosity sub- and super-solution.

In order to prove some key geometric properties of solutions, it is essential to adopt a more appropriate notion of viscosity solution. As a matter of fact, (1.1) has a lack of comparison principle, thus uniqueness assertions might not be true. Therefore, we shall make a particular election of solutions. For this reason, the *least supersolution approach* takes place in our studies by way of *Perron type solutions*.

Definition 1.2 (Perron type solution). Throughout this manuscript we will work with Perron type solutions to the singularly perturbed problem (1.1): given a viscosity sub-solution $u_\star$ and a viscosity super-solution $u^\star$ to (1.1) such that $u_\star \leq u^\star$ in Ω, the Perron solution u^ε is given by

$$u^\varepsilon(x) = \inf \left\{ w(x) \text{ such that } w \text{ is a super-solution to (1.1), and } u_\star \leq w \leq u^\star \right\}. \tag{1.8}$$

It is worth noting that for each $\varepsilon > 0$ fixed, the existence of such a Perron solution follows by sub/supersolutions methods, see e.g. [19], Theorem 4.1. Therefore, from now on, by a solution u^ε to (1.1), we denote a Perron type solution built-up as in (1.8).

We establish the existence of Perron type solutions to (1.1) in our first result.

Theorem 1.3 (Existence of Perron solutions). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $0 \leq g \in C^0(\partial\Omega)$ be a boundary datum. Then, for each fixed $\varepsilon > 0$ there exists a non-negative viscosity solution $u^\varepsilon \in C^0(\overline{\Omega})$ to (1.1).*

We prove uniform gradient estimates, which supply local compactness in the uniform convergence topology, in our next result.

Theorem 1.4 (Optimal Lipschitz estimate). *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of solutions to (1.1). Given $\Omega' \Subset \Omega$, there exists a constant C_0, depending on the dimension, the ellipticity constants and on Ω', but independent of $\varepsilon > 0$, such that*

$$\|\nabla u^\varepsilon\|_{L^\infty(\Omega')} \leq C_0.$$

Additionally, if $\{u^\varepsilon\}_{\varepsilon>0}$ is a uniformly bounded family[1]*, then it is pre-compact in the Lipschitz topology.*

We stress that a key ingredient in order to prove optimal Lipschitz regularity is the $C^{1,\alpha}_{\text{loc}}$ estimates addressed in [24]. Other important pieces of information are the versions of the Harnack inequality and an inhomogeneous Hopf type result adapted to our double degenerate context (see Appendix 8 for more details).

From now on, we will label the distance of a point in the non-coincidence set $x_0 \in \Omega \cap \{u^\varepsilon > 0\}$ to the approximating transition boundary, Γ_ε, by

$$d_\varepsilon(x_0) := \operatorname{dist}(x_0, \{u^\varepsilon \leq \varepsilon\}).$$

Next, we prove that, inside $\{u^\varepsilon > \varepsilon\}$, solutions grow in a linear fashion away from ε-level surfaces.

Theorem 1.5 (Linear growth). *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a Perron solution of* (1.1)*. There exists* c(*universal parameters*) > 0 *such that, for $x_0 \in \{u^\varepsilon > \varepsilon\}$ and $0 < \varepsilon \ll d_\varepsilon(x_0) \ll 1$, there holds*

$$u^\varepsilon(x_0) \geq \mathrm{c} \cdot d_\varepsilon(x_0).$$

The proof of the linear growth consists of combining the construction of an appropriate barrier function with the minimality of Perron solutions. Such an instrumental idea was first introduced in the last author's works [3] and [59] for the fully nonlinear scenario.

From a free boundary point of view, it is important highlighting that viscosity solutions of (1.1) develop two "distinct free boundaries". The first one is the set of singular points of existing solutions $\mathcal{S}_0(u^\varepsilon, \Omega')$, and the second one is the so-named "physical transition", i.e., $\Gamma_\varepsilon = \{u^\varepsilon \sim \varepsilon\}$ (ε-level surfaces). One of most the difficult tasks in our research consists in showing that these two free boundaries do not intersect in measure. As a matter of fact, we are able to obtain a uniform lower/upper control of u^ε in terms of $\operatorname{dist}(\cdot, \Gamma_\varepsilon)$:

$$\operatorname{dist}(x_0, \Gamma_\varepsilon) \lesssim u^\varepsilon(x_0) \lesssim \operatorname{dist}(x_0, \Gamma_\varepsilon).$$

Next, we prove that Perron type solutions are strongly non-degenerate near ε-level surfaces. Summarily, the maximum of u^ε on the boundary of a ball $B_r(x_0)$, centered in $\{u^\varepsilon > \varepsilon\}$, is of the order of r.

Theorem 1.6 (Strong non-degeneracy). *Given $\Omega' \Subset \Omega$, there exists a positive constant* c(*universal*) *such that, for $x_0 \in \{u^\varepsilon > \varepsilon\}$, $\varepsilon \ll \rho \ll 1$, there holds*

$$\mathrm{c} \cdot \rho \leq \sup_{B_\rho(x_0)} u^\varepsilon(x) \leq \mathrm{c}^{-1} \cdot (\rho + u^\varepsilon(x_0)).$$

As a consequence of Theorem 1.4, we get the following result.

Theorem 1.7 (The limiting PDE). *Let u^ε be a solution to* (1.1)*. Then for any sequence $\varepsilon_k \to 0^+$, there exist a subsequence $\varepsilon_{k_j} \to 0^+$ and $u_0 \in C^{0,1}_{\text{loc}}(\Omega)$ such that*

(1) *$u^{\varepsilon_{k_j}} \to u_0$ locally uniformly in Ω;*

[1] Such a bound will be universal, i.e., it will depend only on the data/parameters of the problem, namely, the dimension, the elipticity constants, p, q, $\|\mathfrak{a}\|_{L^\infty(\Omega)}$, $\mathcal{A}$, $\mathcal{B}_0$ and $\mathcal{B}$. Moreover, this uniform bound is obtained via the application of the Alexandroff–Bakelman–Pucci estimate adapted to our context.

(2) $u_0 \in [0, K_0]$ *in* $\overline{\Omega}$ *for some constant* $K_0(universal) > 0$;
(3) $\mathcal{G}(x, \nabla u_0, D^2 u_0) = f_0(x)$ *in* $\{u_0 > 0\}$, *with* $0 \le f_0 \in L^\infty(\Omega) \cap C^0(\Omega)$.

Let us introduce the notation

$$\mathfrak{F}(u_0, \mathcal{O}) := \partial\{u_0 > 0\} \cap \mathcal{O}.$$

Theorem 1.8 (Asymptotic behavior close free boundary). *Let* $\Omega' \Subset \Omega$. *Fix* $x_0 \in \{u_0 > 0\} \cap \Omega'$ *such that* $\operatorname{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \le \frac{1}{2} \operatorname{dist}(\Omega', \partial\Omega)$. *Then there exists a constant* $\mathrm{C} > 0$, *independent of* ε, *such that*

$$\mathrm{C}^{-1} \cdot \operatorname{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \le u_0(x_0) \le \mathrm{C} \cdot \operatorname{dist}(x_0, \mathfrak{F}(u_0, \Omega')). \tag{1.9}$$

Finally, we will prove that the limiting free boundary $\mathfrak{F}(u_0, \Omega')$ has local finite $\mathcal{H}^{N-1}$-Hausdorff measure. To this end, we must restrict our analysis to the class of operators satisfying an asymptotic concavity property, which will be stated precisely in Section 6.

Theorem 1.9 (Hausdorff estimates). *Given* $\Omega' \Subset \Omega$, *there exists a positive constant* $\mathrm{C}(\Omega', \textit{universal parameters})$ *such that, for* $x_0 \in \mathfrak{F}(u_0, \Omega')$,

$$\mathcal{H}^{N-1}(\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)) \le \mathrm{C} \cdot \rho^{N-1}.$$

Additionally, there exists a positive constant $\mathrm{C}_1(\Omega', \textit{universal parameters})$ *such that, for* $\rho \ll 1$ *and* $x_0 \in \mathfrak{F}(u_0, \Omega')$, *there holds*

$$\mathrm{C}_1^{-1} \cdot \rho^{N-1} \le \mathcal{H}^{N-1}(\mathfrak{F}_{\mathrm{red}}(u_0, \Omega') \cap B_\rho(x_0)) \le \mathrm{C}_1 \cdot \rho^{N-1}$$

where $\mathfrak{F}_{\mathrm{red}}(u_0, \Omega') := \partial_{\mathrm{red}}\{u_0 > 0\} \cap \Omega'$ *is the reduced transition boundary*[2]. *In particular,*

$$\mathcal{H}^{N-1}(\mathfrak{F}(u_0, \Omega') \setminus \mathfrak{F}_{\mathrm{red}}(u_0, \Omega')) = 0.$$

In conclusion, our findings extend, regarding non-variational scenarios, former results from [3,56,59,62], and to some extent, those from [20,36,52,54,55,64], concerning degenerate and variational models, by making using of different systematic approaches and techniques adjusted to the framework of fully nonlinear models with non-homogeneous degeneracy. Moreover, they are new even for the toy model

$$[|\nabla u^\varepsilon|^p + \mathfrak{a}(x)|\nabla u^\varepsilon|^q]\, \Delta u^\varepsilon = \mathcal{Q}(x)\, \frac{1}{\varepsilon}\, \zeta\Big(\frac{u^\varepsilon}{\varepsilon}\Big) + f_\varepsilon(x) \quad \text{(with (1.3) and (1.6) in force).}$$

Lastly, it is noteworthy to point out that in order to establish our findings, we have developed pivotal auxiliary tools which, according to our scientific knowledge, were not available in the current literature for our model equations. Thereby they may have their own mathematical interest. Among these, we must quote: weak and Harnack inequalities, local maximum principle, Hölder regularity, ABP estimate and inhomogeneous Hopf type results, just to mention a few (see Appendix 8 for more details).

[2]The reduced free boundary, i.e., $\partial_{\mathrm{red}}\{u_0 > 0\}$ is a subset of $\partial\{u_0 > 0\}$ where there exists, in the measure theoretic sense, the normal vector, see the monograph [38] for a survey concerning geometric measure theory.

1.3. Motivations and state-of-art

The mathematical theory of singular perturbation concerns a wide class of methods employed in several fields of mathematics, physics and their affine areas (see [39] for an introductory essay). As a matter of fact, penalization methods were pivotal in studying certain discontinuous minimization problems in the theory of critical points of non-differentiable functionals, where the Alt–Caffarelli seminal work [1] marks the genesis of such a theory by carrying out the analysis of the minimization problem

$$\min_{\substack{H_0^1(\Omega) \\ v_{|\partial\Omega}=g}} \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + \mathcal{Q}(x)\,\chi_{\{v>0\}}\right) dx \quad \text{for suitable data } g \geq 0 \text{ and } \mathcal{Q} > 0. \tag{1.10}$$

Historically, such a variational problem (1.10) has appeared in the mathematical formulation of a variety of relevant one-phase models: cavity type problems [36], jets problem (see Chapter 1 in [18] and therein references), optimal design problems ([63], Chapter 6), just to mention few of them. Note that the Euler–Lagrange equation associated to (1.10) is

$$\Delta u_0(x) = \mathcal{Q}(x)\,\delta_0(u_0) \quad \text{in } \Omega$$

in an appropriate distributional sense, addressed in [1]. Thereby, minimizers to (1.10) are obtained as the uniform limit when $\varepsilon \to 0^+$ of the problem

$$\Delta u^\varepsilon(x) = \mathcal{Q}^2(x)\,\beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega \quad (\text{for } \beta_\varepsilon \sim \varepsilon^{-1}\chi_{(0,\varepsilon)}).$$

Therefore, the core idea of studying approximating solutions is that small perturbations for certain elliptic problems propagate in a quantifiable fashion. Thus, analysing perturbed solutions can be useful to establish regularity estimates for the desired minimal solution of (1.10) and its free boundary.

Such an influential idea can also be employed in analysing over-determined problems as follows: given a bounded and smooth domain $\Omega \subset \mathbb{R}^N$ and functions $0 \leq f, g \in C^0(\overline{\Omega})$ and $0 < \mathcal{Q} \in C^0(\overline{\Omega})$, we would like to find a "compact hyper-surface" $\Gamma_0 := \partial\Omega' \subset \Omega$ such that the inhomogeneous *one-phase Bernoulli-type problem*

$$\begin{cases} \mathcal{L}\,u(x) = f(x) & \text{in } \Omega\setminus\Omega', \\ u(x) = g(x) & \text{on } \partial\Omega, \\ u(x) = 0 & \text{on } \Omega', \\ \frac{\partial u}{\partial \nu}(x) = \mathcal{Q}(x) & \text{on } \Gamma_0 \quad (\text{in a suitable sense}) \end{cases} \tag{1.11}$$

admits a non-negative solution for a second order elliptic operator $\mathcal{L}$ (in divergence or in non-divergence form) with suitable structure. As above, limiting solutions coming from certain approximating regularized problems are natural profiles to solve (1.11) (in an appropriate sense with $\Gamma_0 := \partial\{u > 0\}$). This will motivate the next paragraph.

1.3.1. Modern developments in singular perturbation theory. Our impetus for current investigations in this work also comes from their intrinsic connections with nonlinear one-phase problems, which arise in the mathematical theory of combustion, as well as in the study of flame propagation problems (stationary setting). Precisely, they appear in the description of laminar flames as an asymptotic limit for the nonlinear formulation of high energy activation models with source terms (cf. [15], [20], [44], [47] and [67]). In a general framework, such models corresponds to the limit as $\varepsilon \to 0$ in (1.1), i.e., a one-phase inhomogeneous FBP, where the reaction-diffusion is driven by a doubly degenerate operator (cf. [3, 56, 59]):

$$\text{(I-FBP-NH)} \quad \begin{cases} \mathscr{G}(x, \nabla u, D^2 u) = f(x) & \text{in } \{u > 0\} \quad (\text{for } f \in C^0(\Omega) \cap L^\infty(\Omega)), \\ u(x) \geq 0 & \text{in } \Omega, \\ \mathrm{H}(x, |\nabla u(x)|) \leq \mathcal{T}(x) & \text{on } \partial\{u > 0\} \quad (\text{for } 0 < \mathcal{T} \in C^0(\overline{\Omega})) \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

The condition that H enforces on u is commonly referred to as *free boundary condition*.

The mathematical development of these regularized problems has yielded important scientific breakthroughs in the free boundary theory. Historically, regularizing methods in free boundary problems date back to Berestycki–Caffarelli–Nirenberg's pioneering work [5], where the linear elliptic scenario was addressed (cf. [64] for the analysis of elliptic PDEs of the flame propagation type) via a variational treatment for the following class of operators:

$$\mathcal{L}[u] := \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{N} b_i(x) D_i u(x) + c(x) u(x) = \beta_\varepsilon(u)$$

for C^1 coefficients. Before presenting the recent progresses in the fully nonlinear scenario, we must quote some fundamental contributions of several authors regarding homogeneous/inhomogeneous singular perturbation problems (one and two-phases and their parabolic counterpart), as well as variational problems with uniformly elliptic and degenerate structure, see [16, 17, 20, 44, 48, 49, 52, 54–56] for an extensive but incomplete list of such investigations:

$$\mathcal{L}u^\varepsilon(x) := \begin{cases} \operatorname{div}(A(x)\nabla u^\varepsilon) = \Gamma(x)\beta_\varepsilon(u^\varepsilon) & \text{uniformly elliptic operator,} \\ \operatorname{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon(x) & p\text{-Laplacian,} \\ \operatorname{div}\left(\frac{g(|\nabla u^\varepsilon|)}{|\nabla u^\varepsilon|}\nabla u^\varepsilon\right) = \beta_\varepsilon(u^\varepsilon) & g\text{-Laplacian in Orlicz–Sobolev spaces,} \\ \operatorname{div}(|\nabla u^\varepsilon|^{p_\varepsilon(x)-2}\nabla u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon(x) & p(x)\text{-Laplacian,} \\ u^\varepsilon \geq 0 & \text{in } \Omega, \\ \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon(x) & \text{two-phase problem for Laplacian,} \\ \Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon) + f_\varepsilon(x) & \text{two-phase problem for heat operator,} \\ \operatorname{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) & \text{two-phase problem for } p\text{-Laplacian.} \end{cases}$$

In the last two decades, nonlinear FBPs like (I-FBP-NH) have been widely studied in the literature via singular perturbation methods. In contrast with their variational counterpart (cf. [1, 20, 52, 54, 55, 64]), the analysis of non-variational singularly perturbed PDEs imposes significant challenging obstacles, mainly due to the lack of monotonicity formulae (cf. [50, 51]), energy estimates (cf. [20, 44]) and a stable notion of "weak formulation" of solutions (see [64]), just to cite a few.

In this scenario, Teixeira in [62] started the journey of investigation into fully nonlinear elliptic singular PDEs as follows:

$$F(x, D^2 u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \Omega \text{ with } u^\varepsilon \geq 0, \tag{1.12}$$

where $\beta_\varepsilon(u^\varepsilon) \to \delta_0$ (the Dirac delta measure). The author proves optimal Lipschitz regularity of solutions of (1.12), as well as H^1 compactness for Bellman's singular PDEs. Thereafter, in [59] the authors finish the analysis introduced in [62]. In effect, they prove, among other analytic and geometric properties, that the free boundary condition is driven by a new operator, namely F^*, *the recession profile*, which arises via a blow-up argument on the family of elliptic equations generated by the original operator F (we recommend [60] for the parabolic counterpart of such studies). Theorem 1.3 in [57] yields global Lipschitz regularity estimates to

$$\begin{cases} F(x, \nabla u^\varepsilon, D^2u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) & \text{in } \Omega, \\ u^\varepsilon(x) \geq 0 & \text{in } \Omega, \\ u^\varepsilon(x) = g(x) & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

and [56] studied a FBP like (1.13) with an inhomogeneous forcing term like (1.6). Finally, in [3], the authors prove similar existence, optimal regularity and geometric results for the class of fully nonlinear, anisotropic degenerate elliptic FBPs (with (A0)–(A1), (1.4) and (1.5) in force) as follows:

$$|\nabla u^\varepsilon|^p F(D^2u^\varepsilon) = \zeta_\varepsilon(x, u^\varepsilon) \quad \text{in } \Omega, \text{ with } p \geq 0 \text{ and } u^\varepsilon \geq 0,$$

thereby summarizing the current researches on singular perturbation methods in non-variational models.

1.3.2. Recent progresses on degenerate equations in non-divergence form. Now, we turn our attention towards regularity features of our model operator in (1.1). Regularity properties for fully nonlinear models with single degeneracy structure as follows:

$$\mathscr{G}_p(x, \nabla u, D^2u) := |\nabla u|^p F(x, D^2u) \quad \text{with } 0 < p < \infty, \tag{1.14}$$

have been an increasing focus of studies over the last decades due to their intrinsic connection to several qualitative/quantitative issues in pure mathematics (see [2, 6, 7, 9–12, 41]), as well as a number of geometric and FBPs (see [3, 21, 27, 28]). Additionally, we also refer the interested reader to [13, 14, 19, 23, 35, 37, 45, 46, 55, 57, 59, 61, 62, 65, 66] for an incomplete list of corresponding results in the uniformly elliptic scenario.

In contrast with (1.14), one of the main characteristics of the model case

$$u \mapsto [|\nabla u|^p + \mathfrak{a}(x)|\nabla u|^q]\, \Delta u \quad \text{(with (1.3) in force)} \tag{1.15}$$

is its transition between two distinct degeneracy rates, which depends on the values of the modulating function $\mathfrak{a}(\cdot)$. For this reason, the diffusion process presents a non-uniformly elliptic and doubly degenerate signature, which mixes up two different p-Laplacian type operators in non-divergence form (cf. [2, 9–12, 41]). Such a prototype in (1.15) can be understood as a non-variational extension of certain variational integrals of the calculus of variations with (p, q)-growth conditions as follows:

(DPF) $(W_0^{1,p}(\Omega) + g, L^m(\Omega)) \ni (w, f) \mapsto \min \displaystyle\int_\Omega \left(\frac{1}{p}|\nabla w|^p + \frac{\mathfrak{a}(x)}{q}|\nabla w|^q - fw\right) dx,$

where $\mathfrak{a} \in C^{0,\alpha}(\Omega, [0,\infty))$, for some $0 < \alpha \le 1 < p \le q < \infty$ and $m \in (N, \infty]$. Finally, notice that minimizers to (**DPF**) exhibits non-uniform and doubly degenerate ellipticity in a model with a kind of (p,q)-structure:

$$\operatorname{div}(\mathcal{A}(x,\nabla u)\nabla u) = f(x) \quad \text{in } \Omega, \quad \text{where } \mathcal{A}(x,\xi) := |\xi|^{p-2} + \mathfrak{a}(x)|\xi|^{q-2}$$

We recommend [4, 31–34, 53] as interesting related works.

Now, let us come back to non-variational models like (1.15). Regarding regularity estimates of fully nonlinear models with non-homogeneous degeneracy, the starting point was De Filippis' paper [30], where $C^{1,\alpha}_{\text{loc}}$-regularity for viscosity solutions of

$$[|\nabla u|^p + \mathfrak{a}(x)|\nabla u|^q]\, F(D^2u) = f \in L^\infty(\Omega), \quad \text{(with (A0)–(A1) and (1.3) in force)}$$

was addressed, for some $\alpha \in (0,1)$ depending on universal parameters. Later on, [24] established sharp gradient estimates to general models driven by (1.7), as well as a number of applications of such estimates in geometric free boundary and related nonlinear elliptic PDEs (cf. [21, 27]).

At this point, a natural question arises: which are the regularity and geometric features of solutions and level surfaces in problems of the type (1.1)? In particular, we are interested in geometric properties that are independent of the regularization parameter and that therefore can be carried over (in a uniform fashion) in the limit process.

To the best of the authors' acknowledgment, few advances are known concerning the regularity theory for inhomogeneous FBPs like (I-FBP-NH). As a matter of fact, many of these are available in the context of linear operators (see [1, 50, 51]) and in the uniformly elliptic scenario (cf. [59]). We must quote a recent work [35], where the authors deal with an inhomogeneous two-phase FBP driven by fully nonlinear elliptic operators. Getting further results on the limiting FBPs (I-FBP-NH), which include in particular the regularity of the free boundary, are challenging and open issues in such a line of investigation.

A program for developing the theory of (I-FBP-NH) is summarized as follows:

- ✓ Existence, uniform/geometric regularity estimates for certain regularizing solutions of (1.1).
- ✓ Existence and optimal regularity estimates of certain solutions of (I-FBP-NH), e.g., viscosity solutions obtained as a limit of singular perturbation problems.
- ✓ Measure theoretic properties of the free boundary, such as finite perimeter and density features for the positivity region.
- ✓ Strong regularity properties of the interfaces, e.g., Lipschitz or "flat" interfaces becoming regular enough (in an appropriate sense). See Chapters 4 and 5 in [18]; see also [22].

2. Background results

In the sequel, we will state an essential tool we will make use of, namely the fundamental estimate from Theorem 1.1 in [24] and Theorem 1 in [30]. Let us enunciate the statement of this regularity estimate.

Theorem 2.1 ($C^{1,\alpha}_{\text{loc}}$-estimates). *Let F be an operator satisfying* (A0)–(A2). *Suppose further the assumptions* (1.2) *and* (1.3) *are in force. Let u be a bounded viscosity solution to*

$$\mathcal{G}(x, \nabla u, D^2 u) = f(x,u) \in L^{\infty}(\Omega \times \mathbb{R}).$$

Then, $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$. Moreover, the following estimate holds true:

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C \cdot \left(\|u\|_{L^{\infty}(\Omega)} + 1 + \|f\|^{1/(p+1)}_{L^{\infty}(\Omega\times\mathbb{R})}\right)$$

for universal constants $\alpha \in (0,1)$ and $C > 0$.

Remark 2.2. We now recall some basic estimates that we will use through this manuscript. Assume that u is a non-negative viscosity solution to

$$\mathcal{G}(x, \nabla u, D^2 u) = f \in C^0(\Omega) \tag{2.1}$$

and that the assumptions (A0)–(A2), (1.2) and (1.3) hold. Then, we have:

(1) **Harnack's inequality**: if $f \in L^m(B_1) \cap C^0(B_1)$ with $m > N$, then

$$\sup_{B_{1/2}} u(x) \leq C(N,\lambda,\Lambda,p,q,L_1)\cdot\left\{\inf_{B_{1/2}} u(x) + \max\left\{\left\|\frac{f}{1+\mathfrak{a}}\right\|^{1/(p+1)}_{L^{\infty}(B_1)}, \left\|\frac{f}{1+\mathfrak{a}}\right\|^{1/(q+1)}_{L^{\infty}(B_1)}\right\}\right\}.$$

(2) **Gradient estimates**: if $f \in L^{\infty}(B_1)$, then, $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ and

$$|\nabla u(0)| \leq C(N,\lambda,\Lambda,p,\alpha,L_1,L_2,\|F\|_{C^{\omega}},\|\mathfrak{a}\|_{L^{\infty}})\cdot\left(\|u\|_{L^{\infty}(B_1)} + 1 + \|f\|^{1/(p+1)}_{L^{\infty}(B_1)}\right).$$

We present now a kind of "cutting lemma", which strongly relies on Lemma 6 in [41], and is concerned with the homogeneous, doubly degenerate problem. The following result can be inferred from Lemma 4.1 in [30] (see also Lemma 6 in [41]) by a careful inspection of the proof.

Lemma 2.3 (Cutting lemma). *Let F be an operator satisfying* (A0)–(A2), (1.2) *and* (1.3), *and let u be a viscosity solution of*

$$\mathcal{H}(x, Du) F(x, D^2 u) = 0 \quad \text{in } B_1(0).$$

Then u is a viscosity solution of

$$F(x, D^2 u) = 0 \quad \text{in } B_1(0).$$

Now, let us present a useful comparison tool. For that purpose, we shall assume the following: there exists a continuous function $\hat{\omega}\colon [0,\infty) \to [0,\infty)$ with $\hat{\omega}(0) = 0$, such that if $\mathrm{X}, \mathrm{Y} \in \mathrm{Sym}(N)$ and $\varsigma \in (0,\infty)$ satisfy

$$-\varsigma\begin{pmatrix}\mathrm{Id_N} & 0\\ 0 & \mathrm{Id_N}\end{pmatrix} \leq \begin{pmatrix}\mathrm{X} & 0\\ 0 & \mathrm{Y}\end{pmatrix} \leq 4\varsigma\begin{pmatrix}\mathrm{Id_N} & -\mathrm{Id_N}\\ -\mathrm{Id_N} & \mathrm{Id_N}\end{pmatrix}, \tag{2.2}$$

then

$$\mathcal{G}(x, \varsigma(x-y), \mathrm{X}) - \mathcal{G}(y, \varsigma(x-y), -\mathrm{Y}) \leq \hat{\omega}(\varsigma|x-y|^2) \quad \forall\, x, y \in \mathbb{R}^N, x \neq y. \tag{2.3}$$

We stress that such a condition is not necessary when $\mathcal{G}$ does not depend on x-variable. In this context, conditions (A0) and (A1) are sufficient to our purpose (cf. [19]).

Finally, we will assume the following: there exist a universal constant $C_\mathfrak{a} > 0$ and a modulus of continuity $\hat{\omega}_\mathfrak{a}\colon [0,\infty) \to [0,\infty)$ such that

$$|\mathcal{H}(x,\xi) - \mathcal{H}(y,\xi)| \le C_\mathfrak{a}\, \hat{\omega}_\mathfrak{a}(|x-y|)\, |\xi|^q \quad \forall\, (x,y,\xi) \in \Omega \times \Omega \times \mathbb{R}^N. \tag{2.4}$$

The proof of the following comparison principle is similar to those of Theorem 1.1 in [6], Theorem 1 in [7] and Theorem A.5 in [22]. For this reason, we will omit the proof.

Lemma 2.4 (Comparison principle). *Assume that* (A0)–(A1), (1.2), (1.3), (2.2) *and* (2.3) *hold. Let $f \in C^0(\bar{\Omega})$ and let h be a continuous increasing function satisfying $h(0) = 0$. Suppose u_1 and u_2 are, respectively, a viscosity supersolution and subsolution of*

$$\mathcal{G}(x, \nabla w, D^2 w) = h(w) + f(x) \quad \text{in } \Omega.$$

If $u_1 \ge u_2$ on $\partial\Omega$, then $u_1 \ge u_2$ in Ω.

Furthermore, if h is nondecreasing (*in particular, if $h \equiv 0$*)*, the result holds if u_1 is a strict supersolution or vice versa, if u_2 is a strict subsolution.*

Finally, we present a qualitative property known as the ABP estimate (see [29, 43]).

Theorem 2.5 (Alexandroff–Bakelman–Pucci estimate). *Assume that* (A0)–(A2) *hold. Then there exists $C = C(N, \lambda, p, q, \operatorname{diam}(\Omega)) > 0$ such that, for any viscosity sub-solution* (*respectively, super-solution*) *$u \in C^0(\overline{\Omega})$ of* (2.1) *in $\{x \in \Omega : u(x) > 0\}$* (*respectively, in $\{x \in \Omega : u(x) < 0\}$*)*, there holds*

$$\sup_{\Omega} u(x) \le \sup_{\partial\Omega} u^+(x) + C \cdot \operatorname{diam}(\Omega) \max\left\{ \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^+))}^{1/(p+1)}, \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^+))}^{1/(q+1)} \right\}$$

$$\left(\textit{resp.},\ \sup_{\Omega} u^-(x) \le \sup_{\partial\Omega} u^-(x) + C \cdot \operatorname{diam}(\Omega) \max\left\{ \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^-))}^{1/(p+1)}, \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^-))}^{1/(q+1)} \right\} \right),$$

where $\Gamma^+(u) := \{x \in \Omega : \exists\, \xi \in \mathbb{R}^N$ such that $u(y) \le u(x) + \langle \xi, y - x\rangle\ \forall\, y \in \Omega\}$.

Let us finish this section by commenting on how to construct viscosity solutions. The idea is to obtain a solution of Perron type, the least super-solution. Our approach holds by adapting the so-called method of sub-solutions and super-solutions to the viscosity theory to produce a solution.

Given a regular boundary datum g, a pair of sub- and super-solutions can be obtained by solving

$$\mathcal{G}(x, \nabla \underline{u}^\varepsilon, D^2 \underline{u}^\varepsilon) = \sup_{\Omega \times [0,+\infty)} \zeta_\varepsilon(x,t) \quad \text{and} \quad \mathcal{G}(x, \nabla \overline{u}^\varepsilon, D^2 \overline{u}^\varepsilon) = \inf_{\Omega \times [0,+\infty)} \zeta_\varepsilon(x,t) \tag{2.5}$$

satisfying $\underline{u}^\varepsilon = \overline{u}^\varepsilon = g$ on $\partial\Omega$. The existence of solutions of (2.5) follows, for instance, from ideas in Propositions 2 and 3 of [8].

Finally, fixed a pair of sub-solution and super-solution solutions of the equation (1.1), the following general procedure yields the existence of a Perron solution:

Theorem 2.6. *Let $\mathcal{G}$ be an elliptic fully nonlinear operator satisfying* (A0)–(A2), (1.2) *and* (1.3)*, and let $h \in C^{0,1}(\Omega \times [0,\infty))$ be a bounded, Lipschitz function in $\mathbb{R}^N$. Suppose that the equation*

$$\mathcal{G}(x, \nabla u, D^2 u) = h(x,u)$$

admits $u_\star, u^\star \in C^0(\overline{\Omega})$, sub- and super-solution, respectively, such that $u_\star \leq u^\star$ in Ω and $u_\star = u^\star = g \in C^0(\partial\Omega)$. Define the set of functions

$$\mathcal{S} := \{w \in C^0(\overline{\Omega}); u_\star \leq w \leq u^\star \text{ and } w \text{ is a super-solution of } \mathcal{G}(x, \nabla u, D^2 u) = h(x,u)\}.$$

Then

$$v(x) := \inf_{w \in \mathcal{S}} w(x)$$

is a continuous viscosity solution of

$$\begin{cases} \mathcal{G}(x, \nabla u, D^2 u) = h(x,u) & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

By using [19] and Theorem 8.5, the proof follows the same lines as those of Theorem 2.1 in [3]. For this reason, we will omit it.

3. Lipschitz regularity estimates

In this section, we derive uniform gradient estimates, which in particular provide compactness in the local uniform convergence topology. In view of the results to be proven in Section 4, such an estimate is indeed optimal.

Before starting the proof of the local Lipschitz estimate, we need to ensure the uniform bound for non-negative solutions to (1.1). Such a statement is a direct consequence of the Alexandroff–Bakelman–Pucci estimate (see Theorem 8.6).

Lemma 3.1. *Let u^ε be a non-negative viscosity solution to* (1.1)*. Then, there exists a constant* C(*universal*) > 0 *such that*

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)} + \mathrm{C} \cdot \mathrm{diam}(\Omega) \max\left\{ \left\| \frac{\mathcal{B}_0}{1+\mathfrak{a}} \right\|_{L^N(\Omega)}^{1/(p+1)}, \left\| \frac{\mathcal{B}_0}{1+\mathfrak{a}} \right\|_{L^N(\Omega)}^{1/(q+1)} \right\}.$$

Proof. Define $v^\varepsilon(x) := u^\varepsilon(x) - \|g\|_{L^\infty(\partial\Omega)}$. Now, notice that

$$\mathcal{G}(x, \nabla v^\varepsilon, D^2 v^\varepsilon) \geq \mathcal{B}_0 \quad \text{in } \Omega$$

in the viscosity sense. Moreover $v^\varepsilon \leq 0$ on $\partial\Omega$. Therefore, the Alexandrov–Bakelman–Pucci estimate (Theorem 8.6) provides the desired estimate. ∎

Proof of Theorem 1.4. First, we analyze the transition region $\{0 \leq u^\varepsilon \leq \varepsilon\} \cap \Omega'$. For $\varepsilon \leq \min\left\{1, \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)\right\}$, fix $x_0 \in \{0 \leq u^\varepsilon \leq \varepsilon\} \cap \Omega'$ and define the scaled function

$$v(x) := \frac{1}{\varepsilon} u^\varepsilon(x_0 + \varepsilon x) \quad \text{in } B_1.$$

It is straightforward to show that v fulfils, in the viscosity sense,

$$\mathscr{G}_{x_0,\varepsilon}(x, \nabla v(x), D^2 v(x)) = \varepsilon \zeta(x_0 + \varepsilon x, u^\varepsilon(x_0 + \varepsilon x)) \quad \text{in } B_1,$$

where (see equation (1.7))

$$\begin{cases} F_{x_0,\varepsilon}(x, \mathrm{X}) &:= \varepsilon F(x_0 + \varepsilon x, \frac{1}{\varepsilon}\mathrm{X}), \\ \mathcal{H}_{x_0,\varepsilon}(x, \xi) &:= \mathcal{H}(x_0 + \varepsilon x, \xi), \\ \mathfrak{a}_{x_0,\varepsilon}(x) &:= \mathfrak{a}(x_0 + \varepsilon x), \\ f_{x_0,\varepsilon}(x) &:= \varepsilon \zeta(x_0 + \varepsilon x, u^\varepsilon(x_0 + \varepsilon x)). \end{cases}$$

Hence, it follows from the structural assumption (1.4) that

$$0 \leq f_{x_0,\varepsilon}(x) \leq \mathcal{A} + \mathcal{B} := \mathrm{C}_\star.$$

Moreover, it is easy to check that the assumptions (A0)–(A2) and (1.2) and (1.3) are satisfied by $F_{x_0,\varepsilon}$, $\mathcal{H}_{x_0,\varepsilon}$ and $\mathfrak{a}_{x_0,\varepsilon}$ (with the same universal constants). Therefore, from the $C^{1,\alpha}_{\text{loc}}$ regularity estimate (see Theorem 2.1 and Remark 2.2, item (2)), we have

(3.1) $|\nabla v(0)| \leq \mathrm{C} \cdot \left\{\|v\|_{L^\infty(B_{1/2})} + 1 + \mathrm{C}_\star{}^{1/(p+1)}\right\}$ (for a constant $\mathrm{C}(\text{universal}) > 0$),

Additionally, since $v(0) = \frac{1}{\varepsilon}u^\varepsilon(x_0) \leq 1$, by Harnack's inequality (Theorem 8.3), we obtain

(3.2) $\|v\|_{L^\infty(B_{1/2})} \leq \mathrm{C}_0(N, \lambda, \Lambda, L_1, \mathfrak{a}, p, q, \mathrm{C}_\star)$ (for some $\mathrm{C}_0 > 0$ independent of ε).

Finally, by combining (3.1) and (3.2) we get

(3.3) $$|\nabla u^\varepsilon(x_0)| = |\nabla v(0)| \leq \mathrm{C}_1 \quad \text{(for some } \mathrm{C}_1 > 0 \text{ independent of } \varepsilon).$$

Now, we analyse the region $\{u^\varepsilon > \varepsilon\} \cap \Omega'$. Define

$$\Gamma_\varepsilon := \{x \in \Omega' \text{ such that } u^\varepsilon(x) = \varepsilon\},$$

and fix a point $\hat{x}_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$. Let us call $r_0 := \text{dist}(\hat{x}_0, \Gamma_\varepsilon)$ the distance from $\hat{x}_0$ to Γ_ε. Then, we define the re-normalized function $v_{\hat{x}_0,r_0} \colon B_1 \to \mathbb{R}$ as

$$v_{\hat{x}_0,r_0}(x) := \frac{u^\varepsilon(\hat{x}_0 + r_0 x) - \varepsilon}{r_0}.$$

It is easy to check that $v_{\hat{x}_0,r_0}$ satisfies, in the viscosity sense,

(3.4) $$\mathscr{G}_{\hat{x}_0,r_0}(x, \nabla v_{\hat{x}_0,r_0}(x), D^2 v_{\hat{x}_0,r_0}(x)) = r_0\, \zeta_\varepsilon(\hat{x}_0 + r_0 x, u^\varepsilon(\hat{x}_0 + r_0 x)),$$

where, as before,

$$\begin{cases} F_{\hat{x}_0,r_0}(x, \mathrm{X}) &:= r_0\, F(\hat{x}_0 + r_0 x, \frac{1}{r_0}\mathrm{X}), \\ \mathcal{H}_{\hat{x}_0,r_0}(x, \xi) &:= \mathcal{H}(\hat{x}_0 + r_0 x, \xi), \\ \mathfrak{a}_{\hat{x}_0,r_0}(x) &:= \mathfrak{a}(\hat{x}_0 + r_0 x), \\ f_{\hat{x}_0,r_0}(x) &:= r_0\, \zeta(\hat{x}_0 + r_0 x, u^\varepsilon(\hat{x}_0 + r_0 x)). \end{cases}$$

By construction, $u^{\varepsilon}(\hat{x}_0 + r_0 x) > \varepsilon$ for all $x \in B_1$. In particular,

$$v_{\hat{x}_0, r_0}(x) \geq 0 \quad \text{for every } x \in B_1. \tag{3.5}$$

Hence, it follows from the assumption (1.4) that

$$\|f_{\hat{x}_0,r_0}\|_{L^\infty(B_1)} \leq \mathrm{C}_2(\mathcal{B}, \mathrm{diam}(\Omega')).$$

By making use of the $C^{1,\alpha}_{\mathrm{loc}}$ regularity estimate (see Theorem 2.1 and Remark 2.2, item (2)), we conclude that

$$|\nabla u^{\varepsilon}(\hat{x}_0)| = |\nabla v_{\hat{x}_0,r_0}(0)| \leq \mathrm{C} \cdot \left(\frac{1}{r_0}\|u^{\varepsilon} - \varepsilon\|_{L^\infty(B_{r_0/2}(\hat{x}_0))} + 1 + \mathrm{C}_2^{1/(p+1)}\right). \tag{3.6}$$

It remains to show a uniform control for the term $\frac{1}{r_0}\|u^{\varepsilon} - \varepsilon\|_{L^\infty(B_{r_0/2}(\hat{x}_0))}$. For that purpose, let $z_0 \in \Gamma_\varepsilon$ be a point that achieves distance, i.e., $r_0 = |\hat{x}_0 - z_0|$. Now, from the Lipschitz estimate proven for points within $\{0 \leq u^{\varepsilon} \leq \varepsilon\} \cap \Omega'$, namely (3.3), we have

$$|\nabla u^{\varepsilon}(z_0)| \leq \mathrm{C}_0.$$

Hence,

$$\frac{\partial v_{\hat{x}_0,r_0}}{\partial \nu}(y_0) \leq |\nabla u^{\varepsilon}(z_0)| \leq \mathrm{C}_0 \quad \text{with} \quad v_{\hat{x}_0,r_0}(y_0) = 0 \quad \text{and} \quad y_0 := \frac{z_0 - \hat{x}_0}{r_0}. \tag{3.7}$$

Thus, from (3.4), (3.5) and (3.7) we are able to apply Lemma 8.7 and conclude that there exists a constant c(universal) > 0 such that

$$v_{\hat{x}_0,r_0}(0) \leq \mathrm{c}. \tag{3.8}$$

Moreover, from Harnack's inequality (see Theorem 8.3) we obtain (using (3.8))

$$\frac{1}{r_0}\|u^{\varepsilon} - \varepsilon\|_{L^\infty(B_{r_0/2}(\hat{x}_0))} = \sup_{B_{1/2}(0)} v_{\hat{x}_0,r_0}(x) \leq \mathrm{C}_0(N, \lambda, \Lambda, p, q, L_1, \mathfrak{a}, \mathrm{c}, \mathcal{B}, \mathrm{diam}(\Omega'))$$

which finishes the proof of the theorem. ∎

Remark 3.2. It is important to stress that for each $\varepsilon > 0$ fixed, viscosity solutions u^{ε} are in effect $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. In particular, when F is concave/convex, it follows from Corollary 1.1 in [24] that $u^{\varepsilon} \in C^{1,1/(p+1)}_{\mathrm{loc}}(\Omega)$. At this point, on one hand, for any $0 < \varsigma \ll 1$ small, near ε-layers, one obtains that

$$\lim_{\varepsilon \to 0^+} \|\nabla u^{\varepsilon}\|_{C^{0,\varsigma}(\Omega')} = +\infty.$$

On the other hand, Theorem 1.4 ensures that the Lipschitz norm of u^{ε} remains uniformly controlled (independently of ε). From such a point of view, our estimates are optimal.

4. Geometric non-degeneracy

4.1. Building barriers

As explained in the introduction, one of the main intricacies in dealing with singularly perturbed models with non-homogeneous degeneracy is to avoid that solutions degenerate along their transition surfaces. For this reason, a decisive devise for overcoming such an obstacle will be implementing a geometric non-degeneracy estimate.

In this section, we show that solutions grow in a linear fashion away from ε-level surfaces, inside $\{u^{\varepsilon} > \varepsilon\}$. In particular, this implies that *in measure* the two free boundaries do not intersect. The proof shall be based on building an appropriate barrier function. To this end, we shall look at elliptic models with non-homogeneous degeneracy as follows:

$$\mathcal{H}(x, \nabla w)\, \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 w) = \zeta(x, w) \quad \text{in } \mathbb{R}^N \quad (\text{with (1.2) and (1.3) in force}), \tag{4.1}$$

where the reaction term fulfils the non-degeneracy assumption (cf. (1.5))

$$\mathcal{J}^{*} := \inf_{\mathbb{R}^N \times [t_0, \mathrm{T}_0]} \zeta(x, t) > 0, \tag{4.2}$$

Proposition 4.1 (Barrier). *Let $0 < t_0 < \mathrm{T}_0 < 1$ be fixed. For a constant A_0(universal) > 0, there exists a radially symmetric profile $\Theta_{\mathrm{L}} \colon \mathbb{R}^N \to \mathbb{R}$ fulfilling*

(1) $\Theta_{\mathrm{L}} \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$;

(2) $t_0 \le \Theta_{\mathrm{L}}(x) \le \mathrm{T}_0$;

(3) Θ_{L} *is a (point-wise) super-solution to* (4.1);

(4) *for some κ_0(universal) > 0,*

$$\Theta_{\mathrm{L}}(x) \ge \kappa_0 \cdot 4\mathrm{L} \quad \textit{for } |x| \ge 4\mathrm{L}, \quad \textit{where } \mathrm{L} \ge \mathrm{L}_0 := \sqrt{\tfrac{\mathrm{T}_0 - t_0}{\mathrm{A}_0}}. \tag{4.3}$$

Proof. For α(universal) > 0 and A_0(universal) > 0 to be chosen a posteriori, let us define

$$\Theta_{\mathrm{L}}(x) := \begin{cases} t_0 & \text{for } 0 \le |x| < \mathrm{L}, \\ \mathrm{A}_0\,(|x| - \mathrm{L})^2 + t_0 & \text{for } \mathrm{L} \le |x| < \mathrm{L} + \sqrt{\frac{\mathrm{T}_0 - t_0}{\mathrm{A}_0}}, \\ \psi(\mathrm{L}) - \phi(\mathrm{L})/|x|^{\alpha} & \text{for } |x| \ge \mathrm{L} + \sqrt{\frac{\mathrm{T}_0 - t_0}{\mathrm{A}_0}}, \end{cases} \tag{4.4}$$

where

$$\begin{cases} \phi(\mathrm{L}) := \frac{2}{\alpha}\sqrt{(\mathrm{T}_0 - t_0)\mathrm{A}_0}\left(\mathrm{L} + \sqrt{\frac{\mathrm{T}_0 - t_0}{\mathrm{A}_0}}\right)^{1+\alpha}, \\ \psi(\mathrm{L}) := \mathrm{T}_0 + \phi(\mathrm{L})\left(\mathrm{L} + \sqrt{\frac{\mathrm{T}_0 - t_0}{\mathrm{A}_0}}\right)^{-\alpha}, \end{cases} \tag{4.5}$$

It is easy to check that $\Theta_{\mathrm{L}} \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. For this reason, we may compute the second order derivatives of Θ_{L} a.e. Moreover, from definition, $t_0 \le \Theta_{\mathrm{L}}(x) \le \mathrm{T}_0$ is easily verified.

In the sequel, we are going to show that Θ_{L} satisfies (point-wise) (4.1), as long as we perform appropriate choices on the parameters $\alpha, \mathrm{A}_0 > 0$.

In effect, for $0 \le |x| < \mathrm{L}$, the inequality in (4.1) (i.e., a viscosity supersolution) is clearly satisfied (due to (4.2)).

In the annular region $\mathrm{L} \le |x| < \mathrm{L} + \sqrt{(\mathrm{T}_0 - t_0)/\mathrm{A}_0}$, we obtain

$$|\nabla\Theta_{\mathrm{L}}(x)| = 2\mathrm{A}_0\,(|x| - \mathrm{L}) \le 2\sqrt{\mathrm{A}_0\,(\mathrm{T}_0 - t_0)}$$

and

$$D^2\Theta_{\mathrm{L}}(x) = 2\mathrm{A}_0\Big[\Big(\frac{1}{|x|^2} - \frac{(|x| - \mathrm{L})}{|x|^3}\Big)x \otimes x + \frac{(|x| - \mathrm{L})}{|x|}\,\mathrm{Id}_{\mathrm{N}}\Big] \le 4\mathrm{A}_0 \cdot \mathrm{Id}_{\mathrm{N}}.$$

Therefore, by using (1.2) and (1.3) we obtain

$$\begin{aligned}&\mathcal{H}(x, \nabla\Theta_{\mathrm{L}}(x))\,\mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_{\mathrm{L}}(x))\\ &\qquad \le 4\mathrm{A}_0\,N\Lambda L_2\Big[\big(2\sqrt{\mathrm{A}_0(\mathrm{T}_0 - t_0)}\,\big)^p + \|\mathfrak{a}\|_{L^\infty(\Omega)}\big(2\sqrt{\mathrm{A}_0(\mathrm{T}_0 - t_0)}\,\big)^q\Big].\end{aligned}$$

Now, thanks to the assumption (4.2) we are able to choose a positive constant $\mathrm{A}_0 = \mathrm{A}_0(N, \Lambda, L_2, p, q, \|\mathfrak{a}\|_{L^\infty(\Omega)}, \mathrm{T}_0 - t_0, \mathcal{I}^*)$ such that

$$\mathcal{H}(x, \nabla\Theta_{\mathrm{L}}(x))\,\mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_{\mathrm{L}}(x)) \le \mathcal{I}^*.$$

Thus, by using item (2) we conclude that

$$\mathcal{H}(x, \nabla\Theta_{\mathrm{L}}(x))\,\mathcal{M}^+(D^2\Theta_{\mathrm{L}}(x)) \le \mathcal{I}^* \le \zeta(x, \Theta_{\mathrm{L}}(x)).$$

Finally, let us analyse the region $|x| \ge \mathrm{L} + \sqrt{(\mathrm{T}_0 - t_0)/\mathrm{A}_0}$. Straightforward calculations give

$$D^2\Theta_{\mathrm{L}}(x) = \alpha\,\phi(\mathrm{L})\,|x|^{-(\alpha+2)}\Big(-\frac{(\alpha + 2)}{|x|^2}x \otimes x + \mathrm{Id}_{\mathrm{N}}\Big).$$

Thus,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_{\mathrm{L}}(x)) \le \alpha\,\phi(\mathrm{L})\,|x|^{-(\alpha+2)}\,\left[-(\alpha + 1)\lambda + (N - 1)\Lambda\right].$$

Therefore, selecting $\alpha \in [(N - 1)\Lambda/\lambda - 1,\ \infty)$, we get (using (4.2) and $\mathcal{H}(x, \xi) \ge 0$)

$$\mathcal{H}(x, \nabla\Theta_{\mathrm{L}}(x))\,\mathcal{M}^+(D^2\Theta_{\mathrm{L}}(x)) \le 0 \le \zeta(x, \Theta_{\mathrm{L}}(x)),$$

which ensures that Θ_{L} fulfils (4.1), as desired.

We now show that the super-solution Θ_L fulfils (4.3). From (4.5) (second line) we have

$$|x| \ge 4\mathrm{L} \ge 2(\mathrm{L} + \mathrm{L}_0) = 2\left(\frac{\phi(\mathrm{L})}{\psi(\mathrm{L}) - \mathrm{T}_0}\right)^{1/\alpha}.$$

Hence, for $\alpha > 0$,

$$\Theta_{\mathrm{L}}(x) = \psi(\mathrm{L}) - \frac{\phi(\mathrm{L})}{|x|^\alpha} \ge \psi(\mathrm{L}) - \frac{1}{2^\alpha}(\psi(\mathrm{L}) - \mathrm{T}_0) > \frac{1}{2^\alpha}(\psi(\mathrm{L}) - \mathrm{T}_0).$$

Therefore, by using (4.5),

$$\Theta_{\mathrm{L}}(x) \ge \kappa_0 \cdot 4\mathrm{L} \quad \text{for } \kappa_0 := \frac{\alpha^{-1}}{2^{\alpha+1}}\sqrt{\mathrm{A}_0(\mathrm{T}_0 - t_0)}. \qquad \blacksquare$$

4.2. Linear growth

To establish lower bounds on the growth of solutions to (1.1) inward the set $\{u^\varepsilon > \varepsilon\}$, the strategy will be to consider appropriate scaling versions of the universal barrier Θ_{L}.

Proof of Theorem 1.5. Let us assume, without loss of generality, $0 \in \{u^\varepsilon > \varepsilon\}$. Now, we set $\eta := d_\varepsilon(0)/2$ and consider the reaction term

$$\zeta(z,t) := \begin{cases} \varepsilon\zeta_\varepsilon(\varepsilon x, \varepsilon t), & \text{if } \varepsilon x \in \Omega, \\ \mathfrak{I}^* & \text{otherwise.} \end{cases}$$

Given the barrier Θ_{L} built-up previously, we define

$$\Theta_\varepsilon(x) := \varepsilon \cdot \Theta_{\frac{\eta}{4\varepsilon}}\left(\frac{x}{\varepsilon}\right).$$

One can verify that the scaled barrier Θ_ε satisfies

$$\mathscr{G}(x, \nabla\Theta_\varepsilon(x), D^2\Theta_\varepsilon(x)) \leq \zeta_\varepsilon(x, \Theta_\varepsilon(x)),$$

Moreover, by (4.3) and (4.4), we have that for $4\mathrm{L}_0\varepsilon \ll \eta$,

$$\Theta_\varepsilon(0) = t_0 \cdot \varepsilon \quad \text{and} \quad \Theta_\varepsilon(x) \geq \kappa_0 \cdot \eta \quad \text{on } \partial B_\eta. \tag{4.6}$$

We claim that there exists a $z_0 \in \partial B_\eta$ such that

$$\Theta_\varepsilon(z_0) \leq u^\varepsilon(z_0). \tag{4.7}$$

Indeed, if we assume $\Theta_\varepsilon > u^\varepsilon$ everywhere in ∂B_η, then

$$v^\varepsilon(x) := \min\{\Theta_\varepsilon(x), u^\varepsilon(x)\}$$

would be a super-solution to (1.1). However, v^ε is strictly below of u^ε, which contradicts the minimality of u^ε. Therefore, by (4.6) and (4.7), we conclude

$$\sup_{\overline{B_\eta}} u^\varepsilon(x) \geq u^\varepsilon(z_0) > \Theta_\varepsilon(z_0) \geq \kappa_0 \cdot \eta. \tag{4.8}$$

Furthermore, u^ε solves, in the viscosity sense,

$$\mathscr{B}_0 \leq \mathscr{G}(x, \nabla u^\varepsilon, D^2 u^\varepsilon) \leq \mathscr{B} \quad \text{in } B_{2\eta}.$$

Therefore, by Harnack's inequality (see Theorem 8.3 and Remark 8.4), we obtain

$$\sup_{B_\eta} u^\varepsilon \leq \mathrm{C}(N, \lambda, \Lambda, q, L_1) \cdot \left(u^\varepsilon(0) + \max\left\{(2\eta)^{\frac{p+2}{p+1}}\mathscr{B}^{\frac{1}{p+1}}, (2\eta)^{\frac{q+2}{q+1}}\mathscr{B}^{\frac{1}{q+1}}\right\}\right).$$

Thus, by (4.8),

$$u^\varepsilon(0) \geq \left(\mathrm{C}^{-1}\kappa_0 - 2^{\frac{q+2}{q+1}}\max\left\{\mathscr{B}^{\frac{1}{p+1}}\eta^{\frac{1}{p+1}}, \mathscr{B}^{\frac{1}{q+1}}\eta^{\frac{1}{q+1}}\right\}\right)\eta.$$

Finally, by taking

$$0 < \eta < \min\left\{\mathscr{B} \cdot \left(\mathrm{C}^{-1}\kappa_0 2^{-\frac{q+2}{q+1}}\right)^{p+1}, \mathscr{B} \cdot \left(\mathrm{C}^{-1}\kappa_0 2^{-\frac{q+2}{q+1}}\right)^{q+1}, \frac{\mathrm{diam}(\Omega)}{4}\right\},$$

we have

$$u^\varepsilon(0) \geq \mathrm{c} \cdot \eta.$$

for some constant $0 < \mathrm{c}(\text{universal}) < \mathrm{C}^{-1}\kappa_0$. ∎

5. Some important implications from Theorems 1.4 and 1.5

In this section, we discuss some implications of the sharp control of solutions, established in Sections 3 and 4.2. As a consequence of Lipschitz regularity, i.e., Theorem 1.4, and linear growth, i.e., Theorem 1.5, we obtain the complete control of u^ε in terms of $d_\varepsilon(x_0)$.

Corollary 5.1. *Given $\Omega' \Subset \Omega$, there exists a constant $\mathrm{C}(\Omega', \text{universal parameters}) > 0$ such that for $x_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$ and $0 < \varepsilon \leq \frac{1}{2} d_\varepsilon(x_0)$, there holds*

$$\mathrm{C}^{-1} d_\varepsilon(x_0) \leq u^\varepsilon(x_0) \leq \mathrm{C}\, d_\varepsilon(x_0).$$

Proof. Take $z_0 \in \partial\{u^\varepsilon > \varepsilon\}$, such that $|z_0 - x_0| = d_\varepsilon(x_0)$. Thus, it follows from Theorem 1.4 that

$$u^\varepsilon(x_0) \leq \mathrm{C}_0\, d_\varepsilon(x_0) + u^\varepsilon(z_0) \leq (\mathrm{C}_0 + 1)\; d_\varepsilon(x_0),$$

The first inequality is precisely the statement of Theorem 1.5. ∎

Next we will prove that the Perron type solutions are strongly non-degenerate near ε-layers. This means that the $\sup_{B_r(x_0)} u^\varepsilon$ (for $x_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$) is comparable to r. This is an important piece of information about the growth rate of u^ε away from ε-surfaces. The proof is, to a certain extent, a consequence of the proof of Theorem 1.5.

Proof of Theorem 1.6. Firstly, the estimate from above follows directly from Lipschitz regularity (Theorem 1.4). Now, as in the Theorem 1.5, we take $\Theta_\varepsilon(x) = \varepsilon\Theta_{\frac{\rho}{4\varepsilon}}(x)$. Thus,

$$u^\varepsilon(z_0) > \Theta_\varepsilon(z_0),$$

for some point $z_0 \in \partial B_\rho(x_0)$. Finally, we note that

$$\sup_{\overline{B_\rho(x_0)}} u^\varepsilon(x) \geq u^\varepsilon(z) > \Theta_\varepsilon(z_0) \geq \kappa_0 \cdot \rho.$$ ∎

Corollary 5.2. *Given $x_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$, $\varepsilon \ll \rho$ and $\rho \ll 1$ small enough (in a universal way), there exists a constant $0 < \mathrm{c}_0(\text{universal}) < 1$ such that*

$$\mathscr{L}^N(B_\rho(x_0) \cap \{u^\varepsilon > \varepsilon\}) \geq \mathrm{c}_0 \cdot \mathscr{L}^N(B_\rho(x_0))$$

where $\mathscr{L}^N(\mathrm{S})$ is the Lebesgue measure of the set $\mathrm{S} \subset \mathbb{R}^N$.

Proof. From strong non-degeneracy (Theorem 1.6), there exists $y_0 \in B_\rho(x_0)$ such that

$$u^\varepsilon(y_0) \geq \mathrm{c}_0\, \rho.$$

From Lipschitz regularity (Theorem 1.4), for $z_0 \in B_{\kappa\rho}(y_0)$, we have

$$u^\varepsilon(z_0) - \mathrm{C}\kappa\rho \geq u^\varepsilon(y_0).$$

Thus, by previous estimates, it is possible to choose $0 < \kappa \ll 1$ small (in a universal way) such that

$$z \in B_{\kappa\rho}(y_0) \cap B_\rho(x_0) \quad \text{and} \quad u^\varepsilon(z) > \varepsilon.$$

Finally, there exists a portion of $B_\rho(x_0)$ with volume of order $\sim \rho^N$ within $\{u^\varepsilon > \varepsilon\}$. Thus,

$$\mathscr{L}^N(B_\rho(x_0) \cap \{u^\varepsilon > \varepsilon\}) \geq \mathscr{L}^N(B_\rho(x_0) \cap B_{\kappa\rho}(y_0)) = \mathrm{c}_0\, \mathscr{L}^N(B_\rho(x_0)),$$

for some constant $0 < \mathrm{c}_0(\text{universal}) \ll 1$. ∎

Corollary 5.3. *Given $x_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$, $\varepsilon \ll \rho$ and $\rho \ll 1$ small enough (in a universal way), then*

$$\frac{1}{\rho} \fint_{B_\rho(x_0)} u^\varepsilon(x)\,dx \geq \mathrm{c}$$

for a constant c(*universal*) > 0 *also depending on* ε.

Proof. As in Corollary 5.2, there exists a constant $0 < \kappa$(universal) $\ll 1$ such that

$$\fint_{B_\rho(x_0)} u^\varepsilon(x)dx \geq C_N \fint_{B_\rho(x_0)\cap B_{\kappa\rho}(y_0)} u^\varepsilon(x)\,dx \geq \mathrm{c}\,\rho$$

for a constant $0 < \mathrm{c}$(universal) $\ll 1$ and some $y_0 \in \{u^\varepsilon > \varepsilon\} \cap \Omega'$. ■

5.1. A Harnack type inequality

For solutions of (1.1), a classical Harnack inequality is valid for balls that touch the free boundary along the ε-surfaces, i.e., $\partial\{u^\varepsilon > \varepsilon\} \cap \Omega'$.

Theorem 5.4. *Let u^ε be a solution of (1.1). Let also $x_0 \in \{u^\varepsilon > \varepsilon\}$ and $\varepsilon \leq d := d_\varepsilon(x_0)$. Then*

$$\sup_{B_{d/2}(x_0)} u^\varepsilon(x) \leq \mathrm{C} \cdot \inf_{B_{d/2}(x_0)} u^\varepsilon(x)$$

for a constant C(*universal*) > 0 *independent of* ε.

Proof. Let $z_1, z_2 \in \overline{B_{d/2}(x_0)}$ be points such that

$$\inf_{B_{d/2}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_1) \quad \text{and} \quad \sup_{B_{d/2}(x_0)} u^\varepsilon(x) = u^\varepsilon(z_2).$$

Since $d_\varepsilon(z_1) \geq d/2$, by Corollary 5.1,

$$u^\varepsilon(z_1) \geq \mathrm{C}_1 \cdot d. \tag{5.1}$$

Moreover, by strong non-degeneracy (Theorem 1.6),

$$u^\varepsilon(z_2) \leq \mathrm{C}_2 \cdot \left(\frac{d}{2} + u^\varepsilon(x_0)\right). \tag{5.2}$$

Next, by taking $y_0 \in \partial\{u^\varepsilon > \varepsilon\}$ such that $d = |x_0 - y_0|$ and $z \in \overline{B_d(y_0)} \cap \{u^\varepsilon > \varepsilon\}$, we obtain from Corollary 5.1 and Theorem 1.6 that

$$u^\varepsilon(x_0) \leq \sup_{B_d(z)} u^\varepsilon \leq \mathrm{C}_2 \cdot (d + u^\varepsilon(z)) \leq \mathrm{C}_3 \cdot d. \tag{5.3}$$

In conclusion, by combining (5.1), (5.2) and (5.3), we obtain

$$\sup_{B_{d/2}(x_0)} u^\varepsilon(x) \leq \mathrm{C} \cdot \inf_{B_{d/2}(x_0)} u^\varepsilon(x).$$ ■

5.2. Porosity of the level surfaces

As a consequence of the growth rate and the non-degeneracy (Theorems 1.4 and 1.5), we obtain the porosity of level sets.

Definition 5.5 (Porous set). A set $\mathrm{S} \subset \mathbb{R}^N$ is called porous with porosity $\delta > 0$ if $\exists\, \mathrm{R} > 0$ such that

$$\forall x \in \mathrm{S}, \quad \forall r \in (0, \mathrm{R}), \quad \exists y \in \mathbb{R}^N \ \text{ such that } \ B_{\delta r}(y) \subset B_r(x) \setminus \mathrm{S}.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $N - \mathrm{c}\delta^N$, where $\mathrm{c} = \mathrm{c}(N) > 0$ is a dimensional constant. In particular, a porous set has Lebesgue measure zero (see [68]).

Theorem 5.6 (Porosity). *Let u^ε be a solution of* (1.1). *Then the level sets $\partial\{u^\varepsilon > \varepsilon\} \cap \Omega'$ are porous with porosity constant independent of ε.*

Proof. Let $\mathrm{R} > 0$ and $x_0 \in \Omega' \Subset \Omega$ be such that $\overline{B_{4\mathrm{R}}(x_0)} \subset \Omega$.

Claim: The set $\partial\{u^\varepsilon > \varepsilon\} \cap B_{\mathrm{R}}(x_0)$ is porous.

Let $x \in \partial\{u^\varepsilon > \varepsilon\} \cap B_{\mathrm{R}}(x_0)$ be fixed. For each $r \in (0, \mathrm{R})$, we have $\overline{B_r(x)} \subset B_{2\mathrm{R}}(x_0) \subset \Omega$. Now, let $y \in \partial B_r(x)$ be such that $u^\varepsilon(y) = \sup_{\partial B_r(x)} u^\varepsilon(x)$. From strong non-degeneracy (Theorem 1.6),

$$u^\varepsilon(y) \geq \mathrm{c} \cdot r. \tag{5.4}$$

On the other hand, we know (see Theorem 1.4) that near the free boundary,

$$u^\varepsilon(y) \leq \mathrm{C} \cdot d_\varepsilon(y), \tag{5.5}$$

where $d_\varepsilon(y)$ is the distance from y to the set $\overline{B_{2\mathrm{R}}(x_0)} \cap \Gamma_\varepsilon$. Next, from (5.4) and (5.5) we get

$$d_\varepsilon(y) \geq \delta r \tag{5.6}$$

for a positive constant $\delta \in (0, 1)$.

Let now $y^* \in [x, y]$ (straight line segment connecting the points x and y) be such that $|y - y^*| = \delta r/2$. Hence, we have that

$$B_{\delta r/2}(y^*) \subset B_{\delta r}(y) \cap B_r(x). \tag{5.7}$$

In effect, for each $z \in B_{\delta r/2}(y^*)$,

$$|z - y| \leq |z - y^*| + |y - y^*| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r,$$

and

$$|z - x| \leq |z - y^*| + \big(|x - y| - |y^* - y|\big) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

so (5.7) follows.

Now, from (5.6), $B_{\delta r}(y) \subset B_{d_\varepsilon(y)}(y) \subset \{u^\varepsilon > \varepsilon\}$, and thus

$$B_{\delta r}(y) \cap B_r(x) \subset \{u^\varepsilon > \varepsilon\},$$

which provides together with (5.7) the following:

$$B_{\delta r/2}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{u^\varepsilon > \varepsilon\} \subset B_r(x) \setminus (\partial\{u^\varepsilon > \varepsilon\} \cap B_{\mathrm{R}}(x_0)),$$

thereby finishing the proof. ∎

It is important to stress that for FBPs modeled by a merely uniformly elliptic operator, one should not expect an improved Hausdorff estimate for the interface. In turn, when diffusion is driven by the Laplacian operator, then the Alt–Caffarelli theory in [1] establishes that $c\delta^N = 1$. At this point, a natural issue is: what should be the minimum structural assumption on F in order to obtain perimeter estimates for interfaces $\partial\{u^\varepsilon > \varepsilon\}$? We will address an answer to such a question in the next section.

6. Hausdorff measure estimates

In this section, we establish Hausdorff measure estimates of the approximating level surfaces. A necessary condition for the study of such an estimate is to impose non-degeneracy of the reaction term propagates up to the transition layer. Hereafter, in condition (1.5), we shall take $t_0 = 0$, i.e.,

$$\mathfrak{I} := \inf_{\Omega \times [0, \mathrm{T}_0]} \varepsilon \zeta_\varepsilon(x, \varepsilon t) > 0$$

for some $\mathrm{T}_0 > 0$ to be enforced. A condition at infinity on the governing operator F, which will be discussed soon, is also required in our analysis concerning Hausdorff estimates.

Next result states that, in measure, the Hessian of an approximating solution blows-up near the transition layer as $\varepsilon \to 0^+$. The proof follows the same lines as Proposition 6.1 in [3] and Proposition 5.1 in [58]. For this reason, we will omit it here.

Proposition 6.1. *Fix $\Omega' \Subset \Omega$, $\mathrm{C} \gg 1$ and $\rho < \mathrm{dist}(\Omega', \partial\Omega)$. There exists $\varepsilon_0 > 0$ such that, for $\varepsilon \le \varepsilon_0$, there holds*

$$\int_{B_\rho(x_\varepsilon)} (\zeta_\varepsilon(x, u^\varepsilon) - \mathrm{C})\, dx \ge 0 \quad \textit{for any } x_\varepsilon \in \partial\{u^\varepsilon > \varepsilon\} \cap \Omega'. \tag{6.1}$$

Proposition 6.1 implies that near the transition layer, the governing operator F gets evaluated at very large matrices. Such an insight motivates the following structural asymptotic condition on the nonlinearity.

Definition 6.2 (Asymptotic concavity). We say that a uniformly elliptic operator $F\colon \Omega \times \mathrm{Sym}(N) \to \mathbb{R}$ satisfies the C_F*-asymptotic concavity property* (respectively, the asymptotic convexity property) if there exist $\mathfrak{A} \in \mathcal{A}_{\lambda,\Lambda}$ and a non-negative continuous function $\mathrm{C}_F^*\colon \Omega \to \mathbb{R}$ such that

$$F(x, X) \le \mathrm{tr}(\mathfrak{A}(x) \cdot X) + \mathrm{C}_F^*(x) \quad \text{(respectively, } F(x, X) \ge \cdots\text{)}, \tag{\textbf{ACP}}$$

for all $(x, X) \in \Omega \times \mathrm{Sym}(N)$, where

$$\mathcal{A}_{\lambda,\Lambda} := \{\mathfrak{A} \in \mathrm{Sym}(N) \text{ such that } \lambda\, \mathrm{Id}_N \le \mathfrak{A} \le \Lambda\, \mathrm{Id}_N\}.$$

It is noteworthy that the condition (**ACP**) is weaker than the concavity (respectively, convexity) condition, which for instance is required in Evans–Krylov–Trudinger's $C^{2,\alpha}_{\text{loc}}$ regularity theory (see [37], [45], [46], [65] and [66]). Indeed, it means a sort of concavity (respectively, convexity) condition at the infinity of $\text{Sym}(N)$ for F. Furthermore, the concavity (respectively, convexity) assumption is precisely when $\mathrm{C}_F = 0$. From a geometric point of view, such a condition means that for each $x \in \Omega$ fixed, there exists a hyperplane which decomposes $\mathbb{R} \times \text{Sym}(N)$ in two semi-spaces such that the graph of $F(x, \cdot)$ is always below such a hyperplane (see [3], [58], [59] and references therein for some motivations and other details).

In turn, if F fulfils (**ACP**), then its recession operator is a concave (respectively, convex) operator; in other words,

$$\begin{aligned} F^*(x,X) = \lim_{\tau \to 0+} \tau F\Big(x, \frac{1}{\tau}X\Big) &\le \lim_{\tau \to 0+} \big[\text{tr}(\mathfrak{A}(x)\cdot X) + \tau \mathrm{C}^*_F(x)\big] \\ &= \text{tr}(\mathfrak{A}(x)\cdot X) \quad (\text{respectively, } \ge \text{tr}(\mathfrak{A}(x)\cdot X)). \end{aligned}$$

Example 6.3. Let us consider a C^1 uniformly elliptic operator $F\colon \text{Sym}(N) \to \mathbb{R}$. Then, its *recession profile* F^* should be understood as the "limiting operator" for the natural scaling on F. By way of illustration, for a number of operators, it is possible to verify the existence of the following limit:

$$\mathfrak{A}_{ij} := \lim_{\|X\| \to \infty} \frac{\partial F}{\partial X_{ij}}(X),$$

In such a context, we obtain $F^*(X) = \text{tr}(\mathfrak{A}_{ij} X)$. An interesting example is the class of Hessian type operators:

$$F_m(e_1(D^2u), \dots, e_N(D^2u)) := \sum_{j=1}^{N} \sqrt[m]{1 + e_j(D^2u)^m} - N,$$

where $m \in \mathbb{N}$ is an odd number. In this setting,

$$F^*(X) = \sum_{j=1}^{N} e_j(X) \quad (\text{the Laplacian operator}).$$

We recommend the reader Example 2.4 in [23], Section 6.2 in [24], Example 3.6 in [27] and Example 5 in [28] for a number of other illustrative examples.

Recently, [61] improved regularity estimates for viscosity solutions of *asymptotically concave* equations were proven (see also [23] for global regularity results). Such operators play an essential role in establishing finiteness of the $(N-1)$-Hausdorff measure in several fully nonlinear singularly perturbed FBPs, whose Hessian of solutions blows-up through the phase transition. For this reason, the limiting free boundary condition is ruled by F^* rather than by F (see [59] for an illustrating example).

Hereafter in this section, we assume the governing operator F has the asymptotically concave property (**ACP**).

Remark 6.4. Notice that if u^ε is a Perron's solution to (1.1), then we have that, in the viscosity sense,

$$F(x, D^2u^\varepsilon) = \zeta_\varepsilon(x, u^\varepsilon)\,\mathcal{H}(x, \nabla u^\varepsilon)^{-1} \quad \text{in } \{u^\varepsilon > \varepsilon\} \cap \Omega',$$

for any $\Omega' \Subset \Omega$. Hence, by Lipschitz regularity (Theorem 1.4) and (**N-HDeg**), one has

$$F(x, D^2u^\varepsilon) = \zeta_\varepsilon(x, u^\varepsilon)\,\mathcal{H}(x, \nabla u^\varepsilon)^{-1} \geq \zeta_\varepsilon(x, u^\varepsilon)\, L_2^{-1}(C_0^p + \|\mathfrak{a}\|_{L^\infty(\Omega)} C_0^q)^{-1}.$$

Therefore, by the (**ACP**) condition,

$$\begin{aligned}\int_{B_\rho(x_\varepsilon)} \mathfrak{A}_{ij}(x)\, D_{ij}u^\varepsilon(x)\, dx &\geq \int_{B_\rho(x_\varepsilon)} \left[\zeta_\varepsilon(x, u^\varepsilon)\,\mathcal{H}(x, \nabla u^\varepsilon)^{-1} - \mathrm{C}_F^*(x)\right] dx\\ &\geq \mathrm{C}_{p,q}^{-1} \int_{B_\rho(x_\varepsilon)} \left[\zeta_\varepsilon(x, u^\varepsilon) - \|\mathrm{C}_F^*\|_{L^\infty(\Omega)}\, \mathrm{C}_{p,q}\right] dx > 0,\end{aligned}$$

where $\mathrm{C}_{p,q} := L_2(C_0^p + \|\mathfrak{a}\|_{L^\infty(\Omega)} C_0^q)$ and we have used Proposition 6.1.

In the next result, Remark 6.4 will allow us to adapt some arguments available for elliptic linear problems (cf. [1]). The proof can be obtained following the same ideas as those of Lemma 6.3 in [3] and Lemma 4.1 in [59].

Lemma 6.5. *There exists a constant* $\mathrm{C}(\Omega', \textit{universal parameters}) > 0$ *such that, for each* $x_\varepsilon \in \partial\{u^\varepsilon > \varepsilon\} \cap \Omega'$ *and* $\rho \ll 1$, *there holds*

$$\int_{B_\rho(x_\varepsilon) \cap \{\varepsilon \leq u^\varepsilon < \mu\}} |\nabla u^\varepsilon|^2\, dx \leq \mathrm{C}\mu\, \rho^{N-1}.$$

Next, we recall some definitions and auxiliary results.

Definition 6.6 (δ-density). Given an open subset $\mathcal{O} \subset \mathbb{R}^N$, we say that $\mathcal{O}$ has the δ-density property in Ω, for $0 < \delta < 1$, if there exists $\tau > 0$ such that

$$\mathcal{L}^N(B_\delta(x) \cap \mathcal{O}) \geq \tau \cdot \mathcal{L}^N(B_\delta(x)).$$

Definition 6.7 (δ-neighborhood of a set). Given a measurable set $\mathrm{S} \subset \mathbb{R}^N$ and a positive constant $\delta > 0$, we denote

$$\mathcal{N}_\delta(\mathrm{S}) := \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, \mathrm{S}) < \delta\},$$

the δ-neighborhood of S in $\mathbb{R}^d$.

Next we will introduce the notion of $\mathcal{H}^j$-Hausdorff measure.

Definition 6.8 ($\mathcal{H}^j$-Hausdorff measure). Let $r_0 > 0$ be given, let $0 < \delta < r_0$ be fixed, and let $\mathrm{S} \subset \mathbb{R}^N$ be a given Borel set. For an arbitrary $j \in \mathbb{N} \setminus \{0\}$, we define the (δ, j)-Hausdorff content of S as follows:

$$\mathcal{H}_\delta^j(\mathrm{S}) := \inf\left\{\sum_i r_i^j \;:\; \mathrm{S} \subset \bigcup_i B_{r_i}(x_i) \;\text{ such that }\; r_i < \delta\right\},$$

where the infimum is taken over all covers $\{B_{r_i}(x_i)\}_i$ of S. The $\mathcal{H}^j$-Hausdorff measure of S is defined as

$$\mathcal{H}^j(\mathrm{S}) := \lim_{\delta \to 0^+} \mathcal{H}_\delta^j(\mathrm{S}).$$

Before establishing uniform bounds of the $\mathcal{H}^{N-1}$-Hausdorff measure of the level-surfaces $\partial\{u^\varepsilon > \varepsilon\}$, let us recall a classical result from measure theory.

Lemma 6.9 (Density property). *Let $\mathcal{O} \Subset \Omega$ be an open set.*

(1) *If there exists δ such that $\mathcal{O}$ has the δ-density property, then there exists a constant $\mathrm{C} = \mathrm{C}(\tau, N)$ such that*

$$|\mathcal{N}_\delta(\partial\mathcal{O}) \cap B_\rho(x)| \leq \frac{1}{2^N\tau}|\mathcal{N}_\delta(\partial\mathcal{O}) \cap B_\rho(x) \cap \mathcal{O}| + \mathrm{C}\delta\rho^{N-1},$$

with $x \in \partial\mathcal{O} \cap \Omega$ and $\delta \ll \rho$.

(2) *If $\mathcal{O}$ has uniform density in Ω along $\mathcal{O}$, then $|\partial A \cap \Omega| = 0$.*

Proof. Property (1) holds by using a covering argument, and (2) is a consequence of the Lebesgue differentiation theorem (see [38]). ■

Next, we obtain an N-dimensional measure estimate on ε-level layers, that is uniform with respect to the parameter ε. The proof holds the same lines as those of Lemma 6.5 in [3]. We omit it here.

Lemma 6.10. *Fixed $\Omega' \Subset \Omega$, there exists a constant $\mathrm{C}^*(\Omega', \text{universal parameters}) > 0$, such that if $\mathrm{C}^*\mu \leq 2\rho \ll \operatorname{dist}(\Omega', \partial\Omega)$ then, for $\mu, \varepsilon > 0$ small enough, and with $3\mathrm{C}_1\varepsilon < \mu \ll \rho$, we have*

$$\mathcal{L}^N\left(\{\mathrm{C}_1\varepsilon < u^\varepsilon < \mu\} \cap B_\rho(x_\varepsilon)\right) \leq \mathrm{C}^*\mu\,\rho^{N-1},$$

where $x_\varepsilon \in \partial\{u^\varepsilon > \varepsilon\} \cap \Omega'$, with $d_\varepsilon(x_\varepsilon) \ll \operatorname{dist}(\Omega', \partial\Omega)$ and $\mathrm{C}_1 > 1$.

Finally, we are ready to establish the $(N-1)$-Hausdorff estimate of approximating level sets (uniform with respect to the parameter ε).

Theorem 6.11. *Fixed $\Omega' \Subset \Omega$, there exists a constant $\mathrm{C}^*(\Omega', \text{universal parameters}) > 0$ such that*

$$\mathcal{L}^N\left(\mathcal{N}_\mu(\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon)\right) \leq \mathrm{C}\mu\,\rho^{N-1},$$

for $C_1 > 1$, $x_\varepsilon \in \partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\} \cap \Omega'$, $d_\varepsilon(x_\varepsilon) \ll \operatorname{dist}(\Omega', \partial\Omega)$ and $\mathrm{C}_1\varepsilon \ll \rho$. In particular,

$$\mathcal{H}^{N-1}(\{u^\varepsilon = \mathrm{C}_1\varepsilon\} \cap B_\rho(x_0)) \leq \mathrm{C}\cdot\rho^{N-1},$$

for constants $\mathrm{C}, \mathrm{C}_1 > 0$ independent of ε.

Proof. From Lipschitz regularity (Theorem 1.4), for $z \in \partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}$ and $y \in \mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon) \cap \{\mathrm{C}_1\varepsilon < u^\varepsilon\}$, we obtain

$$u^\varepsilon(y) \leq u^\varepsilon(z) + C|z-y| \leq \mu + \mathrm{C}\delta \leq \kappa\mu,$$

for $\mu = \mathrm{C}\delta$ and $\kappa(\text{universal}) > 0$. Therefore, the inclusion

$$\mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon) \cap \{C_1\varepsilon < u^\varepsilon\} \subset \{\mathrm{C}_1\varepsilon < u^\varepsilon < \kappa\mu\} \cap B_\rho(x_\varepsilon) \tag{6.2}$$

holds. On the other hand, by Corollary 5.2 and by taking δ as above, we verify that

$$\mathcal{L}^N(B_\delta(x) \cap \{u^\varepsilon > \mathrm{C}_1\varepsilon\}) \geq \mathrm{c}\cdot\mathcal{L}^N(B_\delta(x)) \quad \text{for } x \in \partial\{u^\varepsilon > \varepsilon\}.$$

Hence, we conclude that $\partial\{u^\varepsilon > \mathrm{C}_1\varepsilon\}$ has the δ-density property. Thus, Lemma 6.9 ensures the existence of a constant M(universal) > 0 such that

$$\begin{aligned}&\mathcal{L}^N(\mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon))\\&\qquad\le \mathrm{C}_2\,\mathcal{L}^N(\mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon) \cap \{\mathrm{C}_1\varepsilon < u^\varepsilon\}) + \mathrm{M}\delta\rho^{N-1}.\end{aligned}$$

Hence, by applying (6.2), we obtain

$$\mathcal{L}^N(\mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon)) \le \mathrm{C}_2\,\mathcal{L}^N(\{\mathrm{C}_1\varepsilon < u^\varepsilon < \kappa\mu\} \cap B_\rho(x_\varepsilon)) + \mathrm{M}\delta\rho^{N-1},$$

for some constant C_2(universal) > 0. Finally, for $\mu \ll \rho$, Lemma 6.10 yields

$$\mathcal{L}^N\left(\mathcal{N}_\delta(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_\rho(x_\varepsilon)\right) \le \mathrm{C}\delta\rho^{N-1} \quad \text{for some } \mathrm{C} > 0.$$

In order to conclude, we take a covering of $\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\} \cap B_\rho(x_\varepsilon)$ by balls $\{B_{r_j}\}_j$ centered at points along $\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\} \cap B_\rho(x_\varepsilon)$ with radius $\mu \ll 1$. Thus, we may write

$$\bigcup_j B_{r_j} \subset \mathcal{N}_\mu(\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_{\rho+\mu}(x_\varepsilon).$$

Therefore, there exist universal constants $\mathrm{C}_3, \mathrm{C}_4 > 0$ such that

$$\begin{aligned}&\mathcal{H}^{N-1}_\mu(\partial\{\mathrm{C}_1\varepsilon < u^\varepsilon\} \cap B_\rho(x_\varepsilon)) \le \mathrm{C}_3\sum_j \mathcal{L}^{N-1}(\partial B_{r_j}) = \frac{\mathrm{C}_3}{\mu}\sum_j \mathcal{L}(B_{r_j})\\&\quad\le \frac{\mathrm{C}_4}{\mu}\mathcal{L}^N(\mathcal{N}_\mu(\{\mathrm{C}_1\varepsilon < u^\varepsilon\}) \cap B_{\rho+\mu}(x_\varepsilon)) \le \mathrm{C}_4\mathrm{C}(\rho+\mu)^{N-1} = \mathrm{C}_4\mathrm{C}\rho^{N-1} + \mathrm{o}(1).\end{aligned}$$

We finish the proof of the theorem by letting $\mu \to 0^+$. ∎

7. Limiting scenario as $\varepsilon \to 0^+$

We will now establish geometric and measure theoretic properties for a limiting profile $\lim_{j\to\infty} u^{\varepsilon_{k_j}}(x)$, for a subsequence $\varepsilon_{k_j} \to 0$. In effect, from Lipschitz regularity, the family $\{u^{\varepsilon_k}\}$ is pre-compact in the $C^{0,1}_{\mathrm{loc}}(\Omega)$-topology. Thus, up to a subsequence, there exists a function u_0 obtained as the uniform limit of $u^{\varepsilon_{k_j}}$, as $\varepsilon_{k_j} \to 0$.

From now on, we will use the following definition when referring to u_0:

$$u_0(x) := \lim_{j\to\infty} u^{\varepsilon_{k_j}}(x).$$

Furthermore, we see that such a limiting function satisfies

(1) $u_0 \in [0, K_0]$ in $\overline{\Omega}$ for some constant K_0(universal) > 0 (independent of ε);
(2) $u_0 \in C^{0,1}_{\mathrm{loc}}(\Omega)$;
(3) $\mathcal{G}(x, \nabla u_0, D^2u_0) = f_0(x)$ in $\{u_0 > 0\}$, with $0 \le f_0 \in L^\infty(\Omega) \cap C^0(\Omega)$.

Notice that by combining item (3) with the regularity estimate established in Theorem 1.1 of [24], it follows that $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\{u_0 > 0\})$. However, such an estimate degenerates as we approach $\mathfrak{F}(u_0, \Omega')$. Nevertheless, from item (2), the gradient remains under control, even when $\mathrm{dist}(x_0, \mathfrak{F}(u_0, \Omega')) \to 0$.

In the next proof, we show that at each point $z_0 \in \mathfrak{F}(u_0, \Omega')$, there exists a cone with vertex z_0 that confines the graph of the limiting profile.

Proof of Theorem 1.8. Firstly, the upper estimate follows from the local Lipschitz continuity of u_0. Next, from Corollary 5.1, there exists $y_\varepsilon \in \{0 \le u^\varepsilon \le \varepsilon\} \cap \Omega^\Omega$ with $d_\varepsilon(x_0) = |x_0 - y_\varepsilon|$ such that

$$u^\varepsilon(x_0) \ge \mathrm{c} \cdot d_\varepsilon(x) = \mathrm{c}\,|x_0 - y_\varepsilon|,$$

for a constant c(universal) > 0. Thus, up to a subsequence, $y_\varepsilon \to y_0 \in \{u = 0\}$, and hence

$$u_0(x_0) \ge \mathrm{c}\,|x_0 - y_0| \ge \mathrm{c} \cdot \mathrm{dist}(x_0, \mathfrak{F}(u_0, \Omega')). \qquad \blacksquare$$

Theorem 7.1. *Given $\Omega' \Subset \Omega$, there exist universal positive constants C_0 and r_0 such that*

$$\mathrm{C}_0^{-1} r \le \sup_{B_r(x_0)} u_0(x) \le \mathrm{C}_0 (r + u_0(x_0))$$

for any $x_0 \in \overline{\{u_0 > 0\}} \cap \Omega'$ with $\mathrm{dist}(x_0, \partial\{u_0 > 0\}) \le \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega')$ and $r \le r_0$.

Proof. This is a consequence of passing to the limit as $\varepsilon \to 0$ in Theorem 1.6. $\blacksquare$

Our next result states that the set $\{u_0 > 0\}$ is the limit, in the Hausdorff distance, of $\{u^\varepsilon > \varepsilon\}$ as $\varepsilon \to 0$.

Theorem 7.2. *Given $\mathrm{C}_1 > 1$, the inclusions*

$$\{u_0 > 0\} \cap \Omega' \subset \mathcal{N}_\delta(\{u^{\varepsilon_k} > \mathrm{C}_1\varepsilon_k\}) \cap \Omega',$$
$$\{u^{\varepsilon_k} > \mathrm{C}_1\varepsilon_k\} \cap \Omega' \subset \mathcal{N}_\delta(\{u_0 > 0\}) \cap \Omega'$$

hold for $\delta \ll 1$ and $\varepsilon_k \ll \delta$.

Proof. We will just prove the first inclusion, since the second one is obtained similarly. Let us suppose, for the sake of contradiction, that there exist a subsequence $\varepsilon_k \to 0$ and points $x_k \in \{u_0 > 0\} \cap \Omega'$ such that

$$\mathrm{dist}(x_k, \{u^{\varepsilon_k} > \mathrm{C}_1\varepsilon_k\}) > \delta. \tag{7.1}$$

From Theorem 7.1, and taking $k \gg 1$, we obtain

$$u^{\varepsilon_k}(y_k) = \sup_{B_{\delta/2}(x_k)} u^{\varepsilon_k}(x) \ge \frac{1}{2} \cdot \sup_{B_{\delta/2}(x_k)} u_0(x_k) \ge \mathrm{c}\,\frac{\delta}{2} \ge \mathrm{C}_1\varepsilon_k$$

for some $y_k \in \overline{B_{\delta/2}(x_k)} \cap \{u^{\varepsilon_k} > \mathrm{C}_1\varepsilon_k\}$, which contradicts (7.1). This finishes the proof. $\blacksquare$

Theorem 7.3. *Given a sub-domain $\Omega' \Subset \Omega$, there exist constants* C(*universal*) > 0 *and* $\rho_0(\Omega', \textit{universal parameters}) > 0$ *such that, for any* $x_0 \in \mathfrak{F}(u_0, \Omega')$ *and* $\rho \leq \rho_0$, *there holds*

$$\mathrm{C}^{-1} \leq \frac{1}{\rho} \fint_{\partial B_\rho(x_0)} u_0(x)\, d\mathcal{H}^{N-1} \leq \mathrm{C}. \tag{7.2}$$

Proof. From Lipschitz regularity, the upper estimate is easily satisfied. To prove the other inequality, we consider $z_\varepsilon \in \partial\{u^\varepsilon > 0\} \cap \Omega'$ satisfying

$$|z_\varepsilon - x_0| = \mathrm{dist}(x_0, \partial\{u^\varepsilon > 0\}).$$

From Theorem 7.2, we have $z_\varepsilon \to x_0$. Thus, we may pass to the limit as $\varepsilon_k \to 0$ in the thesis of Corollary 5.3, thereby finishing the proof of the theorem. ∎

Remark 7.4. We will say that u_0 is *locally uniformly non-degenerate* in $\mathfrak{F}(u_0, \Omega')$ provided condition (7.2) is satisfied. Such a condition is another way of saying that u_0 enjoys Lipschitz regularity and non-degeneracy property (in a integral sense).

Next, we see that the set $\{u_0 > 0\}$ has uniform density along $\mathfrak{F}(u_0, \Omega')$.

Theorem 7.5. *Given $\Omega' \Subset \Omega$, there exists a constant* c_0(*universal*) > 0 *such that, for* $x_0 \in \mathfrak{F}(u_0, \Omega')$, *there holds*

$$\mathcal{L}^N(B_\rho(x_0) \cap \{u_0 > 0\}) \geq \mathrm{c}_0 \cdot \mathcal{L}^N(B_\rho(x_0)), \tag{7.3}$$

for $\rho \ll 1$. *In particular,* $\mathcal{L}^N(\mathfrak{F}(u_0, \Omega')) = 0$.

Proof. The estimate (7.3) follows as in the proof of Corollary 5.2. We conclude the proof by making use of the Lebesgue differentiation theorem and a covering argument (a Besicovitch–Vitali type result, see [38] for details). ∎

Finally, we are in a position to establish the Hausdorff measure estimate of the limiting free boundary.

Proof of Theorem 1.9. From Theorem 7.2, for $k \gg 1$ large enough, one has

$$\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0) \subset \mathcal{N}_{4\delta}(\partial\{u^{\varepsilon_k} > \mathrm{C}_1\varepsilon_k\}) \cap B_{2\rho}(x_0).$$

Now, by assuming, $\varepsilon_k \ll \delta \ll \rho \ll \mathrm{dist}(\Omega', \partial\Omega)$, the hypothesis of Theorem 6.11 are verified, thereby implying the following estimate for the δ-neighborhood:

$$\mathcal{L}^N(\mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_\rho(x_0)) \leq \mathrm{C} \cdot \delta\rho^{N-1}.$$

Next, let $\{B_{r_j}\}_{j \in N}$ be a covering of $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$ by balls with radii $\delta > 0$ and centered at free boundary points on $\mathfrak{F}(u_0, \Omega') \cap B_\rho(x_0)$. Hence,

$$\bigcup_{j \in \mathbb{N}} B_{r_j} \subset \mathcal{N}_\delta(\mathfrak{F}(u_0, \Omega')) \cap B_{\rho+\delta}(x_0).$$

Therefore, there exists a constant $\overline{\mathrm{C}}$(universal) > 0 such that

$$\mathcal{H}^{N-1}_{\delta}(\mathfrak{F}(u_0,\Omega')\cap B_\rho(x_0)) \le \overline{\mathrm{C}}\sum_j \mathcal{L}^{N-1}(\partial B_{r_j}) = N\,\frac{\overline{\mathrm{C}}}{\delta}\sum_j \mathcal{L}^N(B_{r_j})$$
$$\le N\,\frac{\overline{\mathrm{C}}}{\delta}\,\mathcal{L}^N(\mathcal{N}_\delta(\mathfrak{F}(u_0,\Omega'))\cap B_{\rho+\delta}(x_0)) \le \mathrm{C}(N)(\rho+\delta)^{N-1} = \mathrm{C}(N)\rho^{N-1} + o(\delta).$$

Finally, by letting $\delta \to 0^+$ we conclude the proof. ■

As a consequence of the previous statement, we conclude that $\mathfrak{F}(u_0,\Omega')$ has locally finite perimeter (see [38] for a precise definition). Furthermore, the reduced free boundary, i.e., $\mathfrak{F}_{\mathrm{red}}(u_0,\Omega') := \partial_{\mathrm{red}}\{u_0 > 0\} \cap \Omega'$ has a total $\mathcal{H}^{N-1}$ measure in the sense that

$$\mathcal{H}^{N-1}(\mathfrak{F}(u_0,\Omega') \setminus \mathfrak{F}_{\mathrm{red}}(u_0,\Omega')) = 0,$$

See Theorem 6.7 in [3] for a detailed proof. In particular, the limiting free boundary has an outward vector for $\mathcal{H}^{N-1}$ a. e. in $\mathfrak{F}_{\mathrm{red}}(u_0,\Omega')$ (see [38]).

7.1. Final comments: An ansatz on the free boundary condition

In the particular case coming from the homogeneous flame propagation theory,

$$\zeta_\varepsilon(t) = \frac{1}{\varepsilon}\,\zeta\Big(\frac{t}{\varepsilon}\Big),$$

where ζ is a continuous function supported in $[0,1]$, then the limiting function satisfies

$$F(x, D^2u) = 0 \quad \text{in } \{u > 0\},$$

in view of the cutting lemma in [41], Lemma 6. In this case, even though the gradient degeneracy is no longer present in the limiting equation, it does leave its *signature* on the expected linear behavior along the limiting transition boundary.

Let us analyze one-dimensional profiles, i.e., the limiting configuration of the equation

$$(|u^\varepsilon_x|^p + \kappa|u^\varepsilon_x|^q)\cdot u^\varepsilon_{xx} = \zeta_\varepsilon(u^\varepsilon) \quad \text{for } \kappa > 0.$$

By multiplying the above equation by $u^\varepsilon_x dx$, we find the differential equality

$$(|u^\varepsilon_x|^p u^\varepsilon_x + \kappa|u^\varepsilon_x|^q u^\varepsilon_x)\cdot(u^\varepsilon_{xx}dx) = \zeta_\varepsilon(u^\varepsilon)\cdot u^\varepsilon_x dx. \tag{7.4}$$

However,

$$\zeta_\varepsilon(u^\varepsilon)\cdot u^\varepsilon_x\,dx = \frac{d}{dx}\mathfrak{Z}_\varepsilon(u^\varepsilon),$$

where

$$\mathfrak{Z}_\varepsilon(x) := \int_0^{x/\varepsilon}\zeta(s)\,ds \to \int_0^1 \zeta(s)\,ds \quad \text{as } \varepsilon \to 0^+.$$

Performing a change of variables,

$$u^\varepsilon_x(x) = w(x) \quad \Longrightarrow \quad u^\varepsilon_{xx}dx = dw,$$

we can write

$$\int (|u_x^\varepsilon|^p u_x^\varepsilon + \kappa |u_x^\varepsilon|^q u_x^\varepsilon) \cdot u_{xx}^\varepsilon dx = \int (|w|^p + \kappa |w|^q)\, w\, dw.$$

Thus, computing anti-derivatives in (7.4) and letting $\varepsilon \to 0$, we obtain for the limiting function u that

$$\frac{1}{p+2}|u'(x_0)|^{p+2} + \frac{\kappa}{q+2}|u'(x_0)|^{q+2} = \int_0^1 \zeta(s)\, ds.$$

Therefore,

$$|u'(x_0)| \le \min\left\{ \sqrt[p+2]{(p+2)\int_0^1 \zeta(s) ds},\ \sqrt[q+2]{\left(\frac{q+2}{\kappa}\right)\int_0^1 \zeta(s) ds} \right\}.$$

In particular, by taking $p = 0 = \kappa$, we recover the classical free boundary condition in the isotropic flame propagation theory, see [5].

8. Appendix

8.1. Harnack's inequality

For the reader's convenience, in this Appendix we gather the statements of two fundamental results in elliptic regularity, namely the weak Harnack inequality and the local maximum principle. Such pivotal tools will provide a Harnack inequality (respectively, local Hölder regularity) to viscosity solutions.

Theorem 8.1 (Weak Harnack's inequality, [40], Theorem 2). *Let u be a non-negative continuous function such that*

$$F_0(x, \nabla u, D^2 u) \le 0 \quad \textit{in } B_1$$

in the viscosity sense. Assume that F_0 is uniformly elliptic in the X variable (see condition (A1)) and that $F_0 \in C^0(B_1 \times (\mathbb{R}^N \setminus B_{\mathrm{M_F}}) \times \mathrm{Sym}(N))$ for some $\mathrm{M_F} \ge 0$. Further assume that

$$(8.1) \quad |\xi| \ge \mathrm{M_F} \quad \textit{and} \quad F_0(x, \xi, X) \le 0 \quad \Longrightarrow \quad \mathcal{M}^-_{\lambda,\Lambda}(X) - \sigma(x)|\xi| - f_0(x) \le 0$$

for continuous functions f_0 and σ in B_1. Then, for any $q_1 > N$,

$$\|u\|_{L^{p_0}(B_{1/4})} \le C\left\{ \inf_{B_{1/2}} u + \max\left\{\mathrm{M_F}, \|f_0\|_{L^N(B_1)}\right\}\right\}$$

where $p_0 > 0$ is universal and the constant $C > 0$ depends on N, q_1, λ, Λ and $\|\sigma\|_{L^{q_1}(B_1)}$.

Theorem 8.2 (Local maximum principle, [40], Theorem 3). *Let u be a continuous function satisfying*

$$F_0(x, \nabla u, D^2 u) \ge 0 \quad \textit{in } B_1$$

in the viscosity sense. Assume that F_0 *is uniformly elliptic in the* X *variable and that* $F_0 \in C^0(B_1 \times (\mathbb{R}^N \setminus B_{\mathrm{M_F}}) \times \mathrm{Sym}(N))$ *for some* $\mathrm{M_F} \geq 0$. *Further assume that*

$$|\xi| \geq \mathrm{M_F} \quad \textit{and} \quad F_0(x, \xi, X) \geq 0 \quad \Longrightarrow \quad \mathcal{M}^+_{\lambda,\Lambda}(X) + \sigma(x)|\xi| + f_0(x) \geq 0 \tag{8.2}$$

for continuous functions f_0 *and* σ *in* B_1. *Then, for any* $p_1 > 0$ *and* $q_1 > N$,

$$\sup_{B_{1/4}} u \leq C \left\{\|u^+\|_{L^{p_1}(B_{1/2})} + \max\{\mathrm{M_F}, \|f_0\|_{L^N(B_1)}\}\right\},$$

where $C > 0$ *is a constant depending on* N, q_1, λ, Λ, $\|\sigma\|_{L^{q_1}(B_1)}$ *and* p_1.

Let us recall that such results were proved by Imbert [40] by following the strategy of the uniformly elliptic case, see [14], Section 4.2. Such a strategy is based on the so-called L^ε-lemma, which establishes a polynomial decay for the measure of the super-level sets of a non-negative super-solution for the Pucci extremal operator $\mathcal{M}^+_{\lambda,\Lambda}$:

$$\mathcal{L}^N(\{x \in B_1 : u(x) > t\} \cap B_1) \leq \frac{C}{t^\varepsilon}\cdot \tag{8.3}$$

Unfortunately, Imbert's paper has a gap in the proof of (8.3). Such an error was recently made up in a joint work with Silvestre, see [42], where an appropriate L^ε-estimate was addressed. In fact, their proof holds for "Pucci extremal operators for large gradients" defined, for a fixed τ, by

$$\widetilde{\mathcal{M}}^+_{\lambda,\Lambda}(D^2u, \nabla u) := \begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \Lambda|\nabla u| & \text{if } |\nabla u| \geq \tau, \\ +\infty & \text{otherwise ;} \end{cases}$$

$$\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}(D^2u, \nabla u) := \begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \Lambda|\nabla u| & \text{if } |\nabla u| \geq \tau, \\ -\infty & \text{otherwise.} \end{cases}$$

The L^ε-estimate is proved to hold whenever $\tau \leq \varepsilon_0$ is universal (see Theorem 5.1 in [42]). Moreover, notice that the ellipticity condition $\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}$ is consistent with (8.1) if we take $\sigma(x) \equiv \Lambda$. More precisely, if (8.1) holds and u is a super-solution for $F_0(x, \nabla w, D^2w) = 0$, then it is also a super-solution for $\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}$ with right-hand side f_0. An analogous reasoning is valid for $\widetilde{\mathcal{M}}^+_{\lambda,\Lambda}$ and (8.2).

Once the L^ε-estimate is derived, the proof of Theorem 8.1 is exactly as the one in [40] which is, in turn, a modification of the uniformly elliptic case in Theorem 4.8a of [14]. As for Theorem 8.2, it also follows from (8.3) by assuming (in a fist moment) that the L^ε norm of u^+ is small and by obtaining the general result by interpolation. Indeed, the smallness of the L^ε norm readily implies (8.3), which in turn gives that u is bounded (see Lemma 4.4 in [14], which is adapted in Section 7.2 of [40]).

Notice that our class of operators fits in this scenario by setting

$$F_0(x, \nabla v, D^2v) := \mathcal{H}(x, \nabla v)F(x, D^2v) - f(x)$$

and

$$f_0(x) := \frac{L_1^{-1} f^+(x)}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^q} \quad \text{for suitable } \varepsilon_0 > 0.$$

In effect, we have that whenever

$$\mathcal{H}(x, \nabla v)F(x, D^2 v) \le f(x) \quad \text{in } B_1$$

in the viscosity sense, then the ellipticity condition (A1) of F ensures that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 v) \le F(x, D^2 v) \le \frac{f(x)}{\mathcal{H}(x, \nabla v)} \le \frac{f^+(x)}{\mathcal{H}(x, \nabla v)}$$

whenever $|\nabla v| \ge \mathrm{M_F} = \varepsilon_0 > 0$, so that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 v) - \Lambda|\nabla v| - f_0(x) \le \left(\frac{1}{\mathcal{H}(x, \nabla v)} - \frac{L_1^{-1}}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^q}\right) f^+(x) \le 0.$$

Recall that the constants obtained in [42] are monotone with respect to τ and bounded away from zero and infinity, so we get a uniform estimate as (8.3) for supersolutions of $\mathcal{G}[v] := \mathcal{H}(x, \nabla v)F(x, D^2 v)$.

Therefore, in such a situation we have (recall $\sigma(x) \equiv \Lambda$), from Theorem 8.1,

$$\|v\|_{L^{p_0}(B_{1/4})} \le C \left\{ \inf_{B_{1/2}} v + \varepsilon_0 + \|f_0\|_{L^N(B_1)} \right\} \le \Xi_0, \tag{8.4}$$

where, if $\varepsilon_0 \in (0, 1]$,

$$\Xi_0 = C \left\{ \inf_{B_{1/2}} v + \min\left\{1, \left[(q+1)\sqrt[N]{|B_1|}L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{1/(q+1)}\right\}\right\}$$

and if $\varepsilon_0 \in (1, \infty)$,

$$\Xi_0 = C \left\{ \inf_{B_{1/2}} v + \min\left\{1, \left[(p+1)\sqrt[N]{|B_1|}L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{1/(p+1)}\right\}\right\}.$$

Notice that we have used in above inequalities that the function

$$(0, \infty) \ni t \mapsto \mathfrak{h}(t) = t + \frac{1}{t^s}\left(\sqrt[N]{|B_1|}\, L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right)$$

is optimized (i.e., it reaches its lowest upper bound) when

$$t^* = \left(s\sqrt[N]{|B_1|}\, L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right)^{1/(s+1)} \quad \text{for } s \in (0, \infty).$$

In conclusion, in any case, we obtain (since $0 < p \le q < \infty$)

$$\|v\|_{L^{p_0}(B_{1/4})} \le C \left\{ \inf_{B_{1/2}} v + (q+1)^{1/(q+1)}\, \Pi_{p,q}^{f^+,\mathfrak{a}} \right\},$$

where

$$\Pi_{p,q}^{f^+,\mathfrak{a}} := \max\left\{\left[\sqrt[N]{|B_1|}L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{\frac{1}{p+1}}, \left[\sqrt[N]{|B_1|}L_1^{-1}\left\|\frac{f^+}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{\frac{1}{q+1}}\right\}.$$

Similarly, from Theorem 8.2, if

$$\mathcal{H}(x, \nabla v)F(x, D^2 v) \geq f(x) \quad \text{in } B_1$$

in the viscosity sense, we again have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) \geq F(x, D^2 v) \geq \frac{f(x)}{\mathcal{H}(x, \nabla v)} \geq -\frac{f^-(x)}{\mathcal{H}(x, \nabla v)} \quad \text{whenever } \varepsilon_0 = \mathrm{M_F} \leq |\nabla u|,$$

and we can set, similarly as above,

$$f_0(x) := \frac{L_1^{-1} f^-(x)}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^p}$$

to get

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + \Lambda|\nabla v| + f_0(x) \geq \left(\frac{L_1^{-1}}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^p} - \frac{1}{\mathcal{H}(x, \nabla v)}\right) f^-(x) \geq 0.$$

Therefore, in such a setting, we have from Theorem 8.2,

$$\sup_{B_{1/2}} v \leq \|u^+\|_{L^{p_1}(B_1)} + \varepsilon_0 + \|f_0\|_{L^N(B_1)} \leq \Xi_1, \tag{8.5}$$

where, as before, if $\varepsilon_0 \in (0, 1]$,

$$\Xi_1 = C\left\{\|v^+\|_{L^{p_1}(B_1)} + \min\left\{1, \left[(q+1)\sqrt[N]{|B_1|}\, L_1^{-1}\left\|\frac{f^-}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{1/(q+1)}\right\}\right\},$$

and if $\varepsilon_0 \in (1, \infty)$,

$$\Xi_1 = C\left\{\|v^+\|_{L^{p_1}(B_1)} + \min\left\{1, \left[(p+1)\sqrt[N]{|B_1|}\, L_1^{-1}\left\|\frac{f^-}{1+\mathfrak{a}}\right\|_{L^\infty(B_1)}\right]^{1/(p+1)}\right\}\right\}.$$

Therefore, in any case (since $0 < p \leq q < \infty$),

$$\sup_{B_{1/2}} v \leq C\left\{\|v^+\|_{L^{p_1}(B_1)} + (q+1)^{1/(q+1)}\, \Pi^{f^-,\mathfrak{a}}_{p,q}\right\},$$

thereby finishing this analysis.

Finally, by combining (8.4) and (8.5), we obtain the following Harnack inequality for viscosity solutions.

Theorem 8.3 (Harnack's inequality). *Let u be a non-negative viscosity solution to*

$$F_0(x, \nabla v, D^2 v) = 0 \quad \textit{in } B_1.$$

Then,

$$\sup_{B_{1/2}} u(x) \leq \mathrm{C}\cdot\left\{\inf_{B_{1/2}} u(x) + (q+1)^{1/(q+1)}\, \Pi^{f,\mathfrak{a}}_{p,q}\right\},$$

where $\mathrm{C} > 0$ *depends only on* N, λ *and* Λ.

Remark 8.4 (Harnack's inequality, scaled version). For our purposes, it will be useful to obtain a scaled version of the Harnack inequality. Indeed, let v be a non-negative viscosity solution to

$$\mathscr{G}(x, \nabla v, D^2 v) = f(x) \quad \text{in } B_r \text{ for a fixed } r \in (0, \infty),$$

where (A0)–(A2), (1.2) and (1.3) are in force. Then,

$$\sup_{B_{r/2}} v(x) \leq \mathrm{C} \cdot \left\{ \inf_{B_{r/2}} v(x) + (q+1)^{\frac{1}{q+1}} \max\left\{ r^{\frac{p+2}{p+1}}, r^{\frac{q+2}{p+1}} \right\} \Pi_{p,q}^{f,\mathfrak{a}} \right\},$$

where $\mathrm{C}(N, \lambda, \Lambda) > 0$.

Finally, from Harnack's inequality (Theorem 8.3) and making use of arguments in Proposition 4.10 of [14], we can obtain, in a standard way, the following interior Hölder regularity result (cf. [30], Theorem 2).

Theorem 8.5 (Local Hölder estimate). *Let u be a viscosity solution to*

$$F_0(x, \nabla v, D^2 v) = 0 \quad \text{in } B_1,$$

where f is a continuous and bounded function. Then, $u \in C^{0,\alpha}_{\text{loc}}(B_1)$ for some universal $\alpha \in (0,1)$. Moreover,

$$\|u\|_{C^{0,\alpha}(\overline{B_{1/2}})} \leq \mathrm{C} \cdot \left\{ \|u\|_{L^\infty(B_1)} + (q+1)^{1/(q+1)} \, \Pi_{p,q}^{f} \right\},$$

where $\mathrm{C} > 0$ depends only on N, λ and Λ.

8.2. An Alexandroff–Bakelman–Pucci type estimate

In the sequel, we will deliver an ABP estimate adapted to our context of fully nonlinear models with non-homogeneous degeneracy (cf. Theorem 1 in [29] and Theorem 1.1 in [43]). Such an estimate is pivotal in order to obtain universal bounds for viscosity solutions in terms of data of the problem.

Theorem 8.6 (Alexandroff–Bakelman–Pucci estimate). *Assume* (A0)–(A2) *hold. Then there exists $C = C(N, \lambda, p, q, \text{diam}(\Omega)) > 0$ such that for any viscosity sub-solution (respectively, super-solution) $u \in C^0(\overline{\Omega})$ of* (2.1) *in $\{x \in \Omega : u(x) > 0\}$ (respectively, in $\{x \in \Omega : u(x) < 0\}$) satisfies*

$$\sup_{\Omega} u^+(x) \leq \sup_{\partial\Omega} u^+(x) + C \cdot \text{diam}(\Omega) \max\left\{ \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^+))}^{1/(p+1)}, \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^+))}^{1/(q+1)} \right\}$$

(respectively,

$$\sup_{\Omega} u^-(x) \leq \sup_{\partial\Omega} u^-(x) + C \cdot \text{diam}(\Omega) \max\left\{ \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^-))}^{1/(p+1)}, \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^N(\Gamma^+(u^-))}^{1/(q+1)} \right\})$$

where

$$\Gamma^+(u) := \left\{ x \in \Omega : \exists\, \xi \in \mathbb{R}^N \text{ such that } u(y) \leq u(x) + \langle \xi, y - x \rangle \ \forall y \in \Omega \right\}.$$

In particular,

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial\Omega)} + C \cdot \text{diam}(\Omega) \max\left\{ \left\| \frac{f}{1+\mathfrak{a}} \right\|_{L^N(\Omega)}^{1/(p+1)}, \left\| \frac{f}{1+\mathfrak{a}} \right\|_{L^N(\Omega)}^{1/(q+1)} \right\}.$$

Proof. We just prove the first estimate; the second one follows by similar reasoning. In the sequel, we will show that our class of operators fits into the framework of Theorem 1 in [40]. For that purpose, as before, let us consider

$$F_0(x,\xi,\mathrm{X}) := \mathcal{H}(x,\xi)F(x,\mathrm{X}) - f(x),$$
$$f_0(x) := \frac{L_1^{-1} f^+(x)}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^q} \quad \text{for fixed } \varepsilon_0 \in (0,\infty).$$

Now, if we have (in the viscosity sense)

$$\mathcal{H}(x,\nabla u)F(x,D^2u) \le f(x) \quad \text{in } B_1,$$

then condition (A1) and by supposing $|\nabla v| \ge \mathrm{M_F} = \varepsilon_0 > 0$ ensure that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le F(x,D^2u) \le \frac{f^+(x)}{\mathcal{H}(x,\nabla u)}.$$

Consequently,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \Lambda|\nabla u| - f_0(x) \le \left(\frac{1}{\mathcal{H}(x,\nabla u)} - \frac{L_1^{-1}}{\varepsilon_0^p + \mathfrak{a}(x)\varepsilon_0^q}\right) f^+(x) \le 0.$$

Therefore, u is a viscosity super-solution of a uniformly elliptic problem with "large" gradient. From ABP estimate in Theorem 1 of [40] we obtain that

$$\sup_{\Omega} u^-(x) \le \sup_{\partial\Omega} u^-(x) + C\cdot \mathrm{diam}(\Omega)\big(\varepsilon_0 + \|f_0\|_{L^N(\Gamma^+(u^-))}\big).$$

We split the analysis into two cases. First, if $\varepsilon_0 \in (0,1]$, then

$$\sup_{\Omega} u^-(x) \le \sup_{\partial\Omega} u^-(x) + C\cdot\mathrm{diam}(\Omega)\left(\varepsilon_0 + L_1^{-1}\frac{1}{\varepsilon_0^q}\left\|\frac{f^-}{1+\mathfrak{a}}\right\|_{L^N(\Gamma^+(u^-))}\right)$$

(8.6) $$\le \sup_{\partial\Omega} u^-(x) + C\cdot\mathrm{diam}(\Omega)\min\left\{1, \left((q+1)L_1^{-1}\left\|\frac{f^-}{1+\mathfrak{a}}\right\|_{L^N(\Gamma^+(u^-))}\right)^{1/(q+1)}\right\}$$

On the other hand, if $\varepsilon_0 \in (1,\infty)$, then

$$\sup_{\Omega} u^-(x) \le \sup_{\partial\Omega} u^-(x)$$

(8.7) $$+ C\cdot\mathrm{diam}(\Omega)\min\left\{1, \left((p+1)L_1^{-1}\left\|\frac{f^-}{1+\mathfrak{a}}\right\|_{L^N(\Gamma^+(u^-))}\right)^{1/(p+1)}\right\}.$$

Therefore, by combining inequalities (8.6) and (8.7) we conclude that

$$\sup_{\Omega} u^-(x) \le \sup_{\partial\Omega} u^-(x) + C(\mathrm{diam}(\Omega), p, q, L_1, \Lambda)$$
$$\cdot \max\left\{\left\|\frac{f^-}{1+\mathfrak{a}}\right\|^{1/(p+1)}_{L^N(\Gamma^+(u^-))}, \left\|\frac{f^-}{1+\mathfrak{a}}\right\|^{1/(q+1)}_{L^N(\Gamma^+(u^-))}\right\}.$$

■

8.3. An inhomogeneous Hopf type result

In this final part, we will present a pivotal tool in proving uniform Lipschitz estimates of solutions, namely a quantitative version of the Hopf lemma, in the inhomogeneous setting for fully nonlinear problems with non-homogeneous degeneracy (cf. Lemma 2.10 in [57] for the uniformly elliptic and homogeneous case).

Lemma 8.7 (Inhomogeneous Hopf type lemma). *Assume that* (A0)–(A1) *and* (1.2) *hold. Let u be a positive viscosity solution to*

$$\mathcal{G}(x, \nabla u, D^2 u) = f(x) \quad \text{in } B_R(z_0),$$

where $f \in L^\infty(B_R(z_0))$. Assume further that for some $x_0 \in \partial B_R(z_0)$,

$$u(x_0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x_0) \le \Im,$$

where ν is the inward normal direction at x_0. Then, for any $r \in (0,1)$, there exists a constant C_0(universal) > 0 such that

$$\sup_{B_{rR/2}(z_0)} u(x) \le \mathrm{C}_0 R \cdot \left\{ r\,\Im + \max\left\{ (r^{p+2}R)^{1/(p+1)}, (r^{p+2}R)^{1/(q+1)} \right\} \Pi_{p,q}^{f, r^{p-q}\mathfrak{a}(z_0+rRx)} \right\}.$$

Proof. First, it is sufficient to consider the scaled function $v_{z_0,R}\colon B_1 \to \mathbb{R}$ given by

$$v_{z_0,R}(x) := \frac{u(z_0 + rRx)}{R},$$

for $r \in (0,1)$ to be determined a posteriori.

In effect, $v_{z_0,R}$ is a non-negative viscosity solution of

$$\mathcal{H}_{z_0,R}(y, \nabla v_{z_0,R}) F_{z_0,R}(x, D^2 v_{z_0,R}) = f_{z_0,R}(x) \quad \text{in } B_1,$$

where

$$\begin{cases} F_{z_0,R}(x, \mathrm{X}) := r^2 R\, F\big(z_0 + rRx, \frac{1}{r^2 R}\mathrm{X}\big), \\ \mathcal{H}_{z_0,R}(x, \xi) := r^p\, \mathcal{H}\big(z_0 + rRx, \frac{1}{r}\xi\big), \\ \mathfrak{a}_{z_0,R}(x) := r^{p-q}\, \mathfrak{a}(z_0 + rRx), \\ f_{z_0,R}(x) := r^{p+2}\, R\, f(z + rRx). \end{cases}$$

Moreover, $F_{z_0,R}, \mathcal{H}_{z_0,R}$ and $\mathfrak{a}_{z_0,R}$ satisfy the structural assumptions (A0)–(A2), (1.2) and (1.3).

Now, let

$$\mathcal{A}_{1/2,1} := B_1 \setminus B_{1/2}$$

and define a barrier function $\Phi\colon \overline{\mathcal{A}_{1/2,1}} \to \mathbb{R}_+$ given by

$$\Phi(x) = \mu_0 \cdot (e^{-\delta|x|^2} - e^{-\delta}),$$

where $\mu_0, \delta > 0$ will be chosen later on. The gradient and the Hessian of Φ in $\mathcal{A}_{1/2,1}$ are

$$\nabla\Phi(x) = -2\mu_0\delta x e^{-\delta|x|^2} \quad \text{and} \quad D^2\Phi(x) = 2\mu_0\delta e^{-\delta|x|^2}(2\delta x \otimes x - \mathrm{Id_N}).$$

Next, we will show that such a barrier is a viscosity solution to

$$\mathcal{H}_{z_0,R}(x, \nabla\Phi) F_{z_0,R}(x, D^2\Phi) > f_{z_0,R}(x) \quad \text{in } \mathcal{A}_{1/2,1} \tag{8.8}$$

adjusting appropriately the values of $\mu_0, \delta > 0$ and $r > 0$.

Notice that for $\delta > \Lambda(N-1)/(2\lambda)$, the barrier Φ is a convex and decreasing function in the annular region $\mathcal{A}_{1/2,1}$. This and the uniform ellipticity of $F_{z_0,R}$ (see (A1)) give

$$\begin{aligned} F_{z_0,R}(x, D^2\Phi) &\geq \mathcal{M}^{-}_{\lambda,\Lambda}(D^2\Phi(x)) = 2\mu_0\,\delta\, e^{-\delta|x|^2}\,[2\delta\lambda - \Lambda(N-1)] \\ &\geq 2\mu_0\,\delta\, e^{-\delta}\,[2\delta\lambda - \Lambda(N-1)] \quad \text{in } \mathcal{A}_{1/2,1}. \end{aligned}$$

Assumption (1.2) further yields

$$\begin{aligned} \mathcal{H}_{z_0,R}(x, \nabla\Phi) &= r^p\,\mathcal{H}\Big(z_0 + rRx, \frac{1}{r}\nabla\Phi\Big) \\ &\geq r^p\Big(\frac{1}{r^p}|\nabla\Phi|^p + \mathfrak{a}(z_0 + rRx)\frac{1}{r^q}|\nabla\Phi|^q\Big) \geq (2\delta\,\mu_0\, e^{-\delta})^p \quad \text{in } \mathcal{A}_{1/2,1} \end{aligned}$$

(recall $q \geq p$). These two inequalities together yield

$$\begin{aligned} \mathcal{H}_{z_0,R}(x, \nabla\Phi) F_{z_0,R}(x, D^2\Phi) &\geq (2\delta\,\mu_0\, e^{-\delta})^{p+1}\,[2\delta\lambda - \Lambda(N-1)] \\ &> r^{p+2}\, R\, \|f\|_{L^\infty(\mathcal{A}_{rR/2,rR})}, \end{aligned}$$

which holds true provided we choose $r \ll 1$ small (depending on μ_0 and δ).

Therefore, Φ is a strict subsolution.

Now, by choosing

$$\mu_0 := (e^{-\delta/4} - e^{-\delta})^{-1} \cdot \inf_{\partial B_{1/2}} v_{z_0,R}(x) > 0,$$

it follows that

$$\Phi(x) \leq v_{z_0,R}(x) \quad \text{on } \partial\mathcal{A}_{1/2,1}.$$

Thus, by the comparison principle (see Lemma 2.4),

$$\Phi(x) \leq v_{z_0,R}(x) \quad \text{in } \mathcal{A}_{1/2,1}. \tag{8.9}$$

Therefore, if we write $y_0 := (x_0 - z_0)/(rR)$, and taking into account (8.9) and the hypotheses $u(x_0) = 0$, we obtain concerning the normal derivatives in the direction ν at x_0 the following:

$$\mu_0\,\delta\, e^{-\delta} \leq \frac{\partial\Phi(y_0)}{\partial\nu} \leq \frac{\partial v_{z_0,R}(y_0)}{\partial\nu} \leq r\,\mathfrak{I}.$$

Thus,

$$\inf_{\partial B_{1/2}} v_{z_0,R}(x) \leq r\,\mathfrak{I}\,\delta^{-1}\cdot(e^{-\frac{3}{4}\delta} - 1).$$

On the other hand, by the Harnack inequality (see Theorem 8.3) we have that

$$\begin{aligned} \sup_{B_{1/2}} v_{z_0,R}(x) &\leq \mathrm{C}\cdot\Big\{\inf_{B_{1/2}} v_{z_0,R} + (q+1)^{\frac{1}{q+1}} \max\big\{(r^{p+2}R)^{\frac{1}{p+1}}, (r^{p+2}R)^{\frac{1}{q+1}}\big\}\,\Pi_{p,q}^{f,\mathfrak{a}_{z_0,R}}\Big\} \\ &\leq \mathrm{C}\cdot\Big\{\inf_{\partial B_{1/2}} v_{z_0,R} + (q+1)^{\frac{1}{q+1}} \max\big\{(r^{p+2}R)^{\frac{1}{p+1}}, (r^{p+2}R)^{\frac{1}{q+1}}\big\}\,\Pi_{p,q}^{f,\mathfrak{a}_{z_0,R}}\Big\} \\ &\leq \mathrm{C}_0\cdot\Big\{r\,\mathfrak{I} + \max\big\{(r^{p+2}R)^{\frac{1}{p+1}}, (r^{p+2}R)^{\frac{1}{q+1}}\big\}\,\Pi_{p,q}^{f,\mathfrak{a}_{z_0,R}}\Big\}, \end{aligned}$$

and by using the definition of $v_{z_0,R}$ we conclude that

$$\sup_{B_{rR/2}(z_0)} u(x) \le \mathrm{C}_0 R \cdot \left\{ r\Im + \max\left\{ (r^{p+2}R)^{1/(p+1)}, (r^{p+2}R)^{1/(q+1)} \right\} \Pi_{p,q}^{f,\mathfrak{a}_{z_0,R}} \right\}. \quad \blacksquare$$

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References

[1] Alt, H. W. and Caffarelli, L. A.: Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144.

[2] Araújo, D. J., Ricarte, G. C. and Teixeira, E. V.: Geometric gradient estimates for solutions to degenerate elliptic equations. *Calc. Var. Partial Differential Equations* **53** (2015), no. 3-4, 605–625.

[3] Araújo, D. J., Ricarte, G. C. and Teixeira, E. V.: Singularly perturbed equations of degenerate type. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), no. 3, 655–678.

[4] Baroni, P., Colombo, M. and Mingione, G.: Regularity for general functionals with double phase. *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 62, 48 pp.

[5] Berestycki, H., Caffarelli, L. A. and Nirenberg, L.: Uniform estimates for regularization of free boundary problems. In *Analysis and partial differential equations*, pp. 567–619. Lecture Notes in Pure and Appl. Math. 122, Dekker, New York, 1990.

[6] Birindelli, I. and Demengel, F.: Comparison principle and Liouville type results for singular fully nonlinear operators. *Ann. Fac. Sci. Toulouse Math. (6)* **13** (2004), no. 2, 261–287.

[7] Birindelli, I. and Demengel, F.: Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. *Commun. Pure Appl. Anal.* **6** (2007), no. 2, 335–366.

[8] Birindelli, I. and Demengel, F.: The Dirichlet problem for singular fully nonlinear operators. In *Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference*, pp. 110–121. Discrete Contin. Dyn. Syst. 2007, suppl.

[9] Birindelli, I. and Demengel, F.: Regularity for radial solutions of degenerate fully nonlinear equations. *Nonlinear Anal.* **75** (2012), no. 17, 6237–6249.

[10] Birindelli, I. and Demengel, F.: $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations. *ESAIM Control Optim. Calc. Var.* **20** (2014), no. 4, 1009–1024.

[11] Birindelli, I. and Demengel, F.: Hölder regularity of the gradient for solutions of fully nonlinear equations with sub linear first order term. In *Geometric methods in PDE's*, pp. 257–268. Springer INdAM Ser. 13, Springer, Cham, 2015.

[12] Birindelli, I., Demengel, F. and Leoni, F.: $C^{1,\gamma}$ regularity for singular or degenerate fully nonlinear equations and applications. *NoDEA Nonlinear Differential Equations Appl.* **26** (2019), no. 5, article no. 40, 13 pp.

[13] Caffarelli, L. A.: Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math. (2)* **130** (1989), no. 1, 189–213.

[14] Caffarelli, L. A. and Cabré, X.: *Fully nonlinear elliptic equations*. American Mathematical Society Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995.

[15] Caffarelli, L. A., Lee, K.-A. and Mellet, A.: Singular limit and homogenization for flame propagation in periodic excitable media. *Arch. Ration. Mech. Anal.* **172** (2004), no. 2, 153–190.

[16] Caffarelli, L. A., Lederman, C. and Wolanski, N.: Uniform estimates and limits for a two phase parabolic singular perturbation problem. *Indiana Univ. Math. J.* **46** (1997), no. 2, 453–489.

[17] Caffarelli, L. A., Lederman, C. and Wolanski, N.: Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem. *Indiana Univ. Math. J.* **46** (1997), no. 3, 719–740.

[18] Caffarelli, L. A. and Salsa, S.: *A geometric approach to free boundary problems*. Graduate Studies in Mathematics 68, American Mathematical Society, Providence, RI, 2005.

[19] Crandall, M., Ishii, H. and Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

[20] Danielli, D., Petrosyan, A. and Shahgholian, H.: A singular perturbation problem for the p-Laplace operator. *Indiana Univ. Math. J.* **52** (2003), no. 2, 457–476.

[21] da Silva, J. V., Leitão, R. A. and Ricarte, G. C.: Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries. *Math. Nachr.* **294** (2021), no. 1, 38–55.

[22] da Silva, J. V., Rampasso, G. C., Ricarte, G. C. and Vivas, H.: Free boundary regularity for a class of one-phase problems with non-homogeneous degeneracy. To appear in *Israel J. Math.*, DOI: 10.1007/s11856-022-2392-5.

[23] da Silva, J. V. and Ricarte, G. C.: An asymptotic treatment for non-convex fully nonlinear elliptic equations: global Sobolev and BMO type estimates. *Commun. Contemp. Math.* **21** (2019), no. 7, 1850053, 28 pp.

[24] da Silva, J. V. and Ricarte, G. C.: Geometric gradient estimates for fully nonlinear models with non-homogeneous degeneracy and applications. *Calc. Var. Partial Differential Equations* **59** (2020), no. 5, article no. 161, 33 pp.

[25] da Silva, J. V., Rossi, J. D. and Salort, A.: Regularity properties for p-dead core problems and their asymptotic limit as $p \to \infty$. *J. Lond. Math. Soc. (2)* **99** (2019), no. 1, 69–96.

[26] da Silva, J. V. and Salort, A.: Sharp regularity estimates for quasi-linear elliptic dead core problems and applications. *Calc. Var. Partial Differential Equations* **57** (2018), no. 3, article no. 83, 24 pp.

[27] da Silva, J. V. and Vivas, H.: The obstacle problem for a class of degenerate fully nonlinear operators. *Rev. Mat. Iberoam.* **37** (2021), no. 5, 1991–2020.

[28] da Silva, J. V. and Vivas, H.: Sharp regularity for degenerate obstacle type problems: a geometric approach. *Discrete Contin. Dyn. Syst.* **41** (2021), no. 3, 1359–1385.

[29] Dávila, G., Felmer, P. and Quaas, A.: Alexandroff–Bakelman–Pucci estimate for singular or degenerate fully nonlinear elliptic equations. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 19-20, 1165–1168.

[30] De Filippis, C.: Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), no. 1, 110–132.

[31] De Filippis, C. and Mingione, G.: On the regularity of minima of non-autonomous functionals. *J. Geom. Anal.* **30** (2020), no. 2, 1584–1626.

[32] De Filippis, C. and Mingione, G.: Manifold constrained non-uniformly elliptic problems. *J. Geom. Anal.* **30** (2020), no. 2, 1661–1723.

[33] De Filippis, C. and Mingione, G.: Lipschitz bounds and nonautonomous integrals. *Arch. Ration. Mech. Anal.* **242** (2021), no. 2, 973–1057.

[34] De Filippis, C. and Oh, J.: Regularity for multi-phase variational problems. *J. Differential Equations* **267** (2019), no. 3, 1631–1670.

[35] De Silva, D., Ferrari, F. and Salsa, S.: Free boundary regularity for fully nonlinear nonhomogeneous two-phase problems. *J. Math. Pures Appl. (9)* **103** (2015), no. 3, 658–694.

[36] dos Prazeres, D. and Teixeira, E. V.: Cavity problems in discontinuous media. *Calc. Var. Partial Differential Equations* **55** (2016), no. 1, Art. 10, 15 pp.

[37] Evans, L. C.: Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.* **35** (1982), no. 3, 333–363.

[38] Evans, L. C. and Gariepy, L. C.: *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[39] Holmes, M. H.: *Introduction to perturbation methods*. Second edition. Texts in Applied Mathematics 20, Springer, New York, 2013.

[40] Imbert, C.: Alexandroff–Bakelman–Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations. *J. Differential Equations* **250** (2011), no. 3, 1553–1574.

[41] Imbert, C. and Silvestre, L.: $C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations. *Adv. Math.* **233** (2013), 196–206.

[42] Imbert, C. and Silvestre, L.: Estimates on elliptic equations that hold only where the gradient is large. *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 6, 1321–1338.

[43] Junges Miotto, T.: The Aleksandrov–Bakelman–Pucci estimates for singular fully nonlinear operators. *Commun. Contemp. Math.* **12** (2010), no. 4, 607–627.

[44] Karakhanyan, A.: Regularity for the two phase singular perturbation problems. *Proc. Lond. Math. Soc. (3)* **123** (2021), no. 5, 433–459.

[45] Krylov, N. V.: Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), no. 3, 487–523, 670.

[46] Krylov, N. V.: Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.* **47** (1983), no. 1, 75–108.

[47] Lederman, C., Vázquez, J. L. and Wolanski, N.: Uniqueness of solution to a free boundary problem from combustion. *Trans. Amer. Math. Soc.* **353** (2001), no. 2, 655–692

[48] Lederman, C. and Wolanski, N.: Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **27** (1998), no. 2, 253–288 (1999).

[49] Lederman, C. and Wolanski, N.: A two phase elliptic singular perturbation problem with a forcing term. *J. Math. Pures Appl. (9)* **86** (2006), no. 6, 552–589.

[50] Lederman, C. and Wolanski, N.: A local monotonicity formula for an inhomogeneous singular perturbation problem and applications. *Ann. Mat. Pura Appl. (4)* **187** (2008), no. 2, 197–220.

[51] Lederman, C. and Wolanski, N.: A local monotonicity formula for an inhomogeneous singular perturbation problem and applications. II. *Ann. Mat. Pura Appl. (4)* **189** (2010), no. 1, 25–46.

[52] Lederman, C. and Wolanski, N.: An inhomogeneous singular perturbation problem for the $p(x)$-Laplacian. *Nonlinear Anal.* **138** (2016), 300–325.

[53] Liu, W. and Dai, G.: Existence and multiplicity results for double phase problem. *J. Differential Equations* **265** (2018), no. 9, 4311–4334.

[54] Martínez, S. and Wolanski, N.: A singular perturbation problem for a quasi-linear operator satisfying the natural growth condition of Lieberman. *SIAM J. Math. Anal.* **41** (2009), no. 1, 318–359.

[55] Moreira, D. and Teixeira, E. V.: A singular perturbation free boundary problem for elliptic equations in divergence form. *Calc. Var. Partial Differential Equations* **29** (2007), no. 2, 161–190.

[56] Moreira, D. and Wang, L.: Singular perturbation method for inhomogeneous nonlinear free boundary problems. *Calc. Var. Partial Differential Equations* **49** (2014), no. 3-4, 1237–1261.

[57] Ricarte, G. C. and Silva, J. V.: Regularity up to the boundary for singularly perturbed fully nonlinear elliptic equations. *Interfaces Free Bound.* **17** (2015), no. 3, 317–332.

[58] Ricarte, G. C., Silva, J. V. and Teymurazyan, R.: Cavity type problems ruled by infinity Laplacian operator. *J. Differential Equations* **262** (2017), no. 3, 2135–2157.

[59] Ricarte, G. C. and Teixeira, E. V.: Fully nonlinear singularly perturbed equations and asymptotic free boundaries. *J. Funct. Anal.* **261** (2011), no. 6, 1624–1673.

[60] Ricarte, G. C., Teymurazyan, R. and Urbano, J. M.: Singularly perturbed fully nonlinear parabolic problems and their asymptotic free boundaries. *Rev. Mat. Iberoam.* **35** (2019), no. 5, 1535–1558.

[61] Silvestre, L. and Teixeira, E. V.: Regularity estimates for fully non linear elliptic equations which are asymptotically convex. In *Contributions to nonlinear elliptic equations and systems*, pp. 425–438. Progr. Nonlinear Differential Equations Appl. 86, Birkhäuser/Springer, Cham, 2015.

[62] Teixeira, E. V.: Optimal regularity of viscosity solutions of fully nonlinear singular equations and their limiting free boundary problems. XIV School on Differential Geometry (Portuguese). *Mat. Contemp.* **30** (2006), 217–237.

[63] Teixeira, E. V.: *Elliptic regularity and free boundary problems: an introduction*. Publicações Matemáticas do IMPA, 26o Colóquio Brasileiro de Matemática, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2007.

[64] Teixeira, E. V.: A variational treatment for elliptic equations of the flame propagation type: regularity of the free boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), no. 4, 633–658.

[65] Trudinger, N. S.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. *Trans. Amer. Math. Soc.* **278** (1983), no. 2, 751–769.

[66] Trudinger, N. S.: Regularity of solutions of fully nonlinear elliptic equations. *Boll. Un. Mat. Ital. A (6)* **3** (1984), no. 3, 421–430.

[67] Weiss, G. S.: A singular limit arising in combustion theory: fine properties of the free boundary. *Calc. Var. Partial Differential Equations* **17** (2003), no. 3, 311–340.

[68] Zajicek, L.: Porosity and σ-porosity. *Real Anal. Exchange* **13** (1987/88), 314–350.

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