Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1320

Published by EMS Press

Cauchy's surface area formula in the Heisenberg groups

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Abstract. We show an analogue of Cauchy's surface area formula for the Heisenberg groups $\mathbb{H}_n$ for $n \geq 1$, which states that the p-area of any compact hypersurface Σ in $\mathbb{H}_n$ with its p-normal vector defined almost everywhere on Σ is the average of its projected p-areas onto the orthogonal complements of all p-normal vectors of the Pansu spheres (up to a constant). The formula provides a geometric interpretation of the p-areas defined by Cheng–Hwang–Malchiodi–Yang in $\mathbb{H}_1$ and Cheng–Hwang–Yang in $\mathbb{H}_n$ for $n \geq 2$. We also characterize the projected areas for rotationally symmetric domains in $\mathbb{H}_n$; namely, for any rotationally symmetric domain with boundary in $\mathbb{H}_n$, its projected p-area onto the orthogonal complement of any normal vector of the Pansu spheres is a constant, independent of the choice of the projected directions.

1. Introduction

One of the most important problems in integral geometry is how to obtain geometric information (for instance, lengths, surface areas, and volumes) of objects by lower dimensional geometric quantities. Cauchy's surface area formula in $\mathbb{R}^n$ provides one of the solutions to this problem. It says that the surface area of any convex body in the n-dimensional Euclidean space is equal to the average of the areas of its orthogonal projections onto all subspaces. By a convex body we mean a compact convex set in $\mathbb{R}^n$ with non-empty interior. The formula was originally proved by Augustin Cauchy in 1841 [4] for $n = 2$, and in 1850 [5] for $n = 3$. After that, the formula was generalized by Kubota [19], Minkowski [23] and Bonnesen [3]. The literature on the formula and its applications to other scientific fields is quite large; for instance, we refer the interested readers to [13, 18, 29, 30] for the viewpoints of integral geometry and convex geometry, and to [32] and [22], p. 290, for the applications to the measurement of elementary particles in chemistry. Recently, a simple proof of the formula, that used as key observation an algebraic property for the Minkowski sum of sets in $\mathbb{R}^n$, was presented in [31].

Let K be any compact convex subset in $\mathbb{R}^n$. If V is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, we denote by $K|V$ the orthogonal projection of K onto V. Let ω_{n-1} be the

Mathematics Subject Classification (2020): Primary 53C17; Secondary 53C65, 53C23.
Keywords: Sub-Riemannian manifolds, pseudo-hermitian geometry, Cauchy surface area formula, Heisenberg groups.

$(n-1)$-dimensional volume of the unit ball in $\mathbb{R}^{n-1}$. We first recall the classical result derived by Cauchy.

Theorem 1.1 (Cauchy's surface area formula, Theorem 5.5.2 in [18]). *For any n-dimensional compact convex subset K in $\mathbb{R}^n$, if $S(K)$ denotes the surface area of K, we have*

$$S(K) = \frac{1}{2\omega_{n-1}} \int_{u \in \mathbb{S}^{n-1}} \mu_{n-1}(K|u^{\perp})\, dS_u,$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional standard unit sphere, dS_u is the surface area element at $u \in \mathbb{S}^{n-1}$, and $\mu_{n-1}(K|u^{\perp})$ is the $(n-1)$-dimensional volume of the orthogonal projection of K onto the subspace $u^{\perp} = \{v \in \mathbb{R}^n; v \text{ is perpendicular to } u\}$.

The term $\mu_{n-1}(K|u^{\perp})$ is the projected area of K onto the subspace in $\mathbb{R}^n$ perpendicular to the vector u, and it can be explicitly represented by the integral formula

$$\mu_{n-1}(K|u^{\perp}) = \int_{v \in \mathbb{S}^{n-1}} |u \star v|\, dS_v, \tag{1.1}$$

where v is the outward unit normal vector to the surface of K, dS_v is the surface area element at $v \in \mathbb{S}^{n-1}$, and $u \star v$ is the standard inner product of vectors u and v in $\mathbb{R}^n$. The identity (1.1) can be proved by applying the discrete form of the analogous identity on an n-polygon and taking the limit as the value n goes to infinity. See [18], p. 56, for details. In particular, when $K = \mathbb{S}^{n-1}$, the following lemma shows that the surface area of the orthogonal projection of the standard unit sphere is independent of the choice of the projected direction. This natural property for $\mathbb{S}^n$ plays a key role in the proof of Theorem 1.1. Basically, the proof of the main result (Theorem 1.5) is inspired by that of Lemma 1.2.

Lemma 1.2 ([18], Lemma 5.5.1). *For any $u \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$,*

$$\mu_{n-1}(\mathbb{S}^{n-1}|u^{\perp}) = \int_{v \in \mathbb{S}^{n-1}} |u \star v|\, dS_v = 2\omega_{n-1}. \tag{1.2}$$

Notice that since the projection of the sphere onto the $(n-1)$-dimensional subspace $u^{\perp}$ is counted twice (from the "front" and the "back" of the subspace $u^{\perp}$), the coefficient $1/2$ comes in as shown in (1.2). Roughly speaking, the identities (1.1) and (1.2) precise the following geometric intuition: since the integrand $|u \star v|$ on the right-hand side of those identities is the absolute value of the cosine of the angle between the vectors u and v, the quantity $|u \star v|\, dS_v$ is the projected area of the infinitesimal area at $v \in \mathbb{S}_{n-1}$ onto the plane $u^{\perp}$.

We also mention that the reversed statement of Lemma 1.2 is not true in general. To be precise, a convex body in $\mathbb{R}^n$ with equal projected areas along any direction is *not* necessarily a sphere. For instance, a Reuleaux triangle in $\mathbb{R}^2$ is not a circle but a convex body with constant width and equal projected area along any direction.

Here is the key observation for the proofs of Theorem 1.1 and Lemma 1.2. Both proofs are based on the invariant translations in the homogeneous space $\mathbb{R}^n$. Suppose $u(0)$ is the direction that K (or $\mathbb{S}^{n-1}$ for Lemma 1.2) is projected along. With the inner product $u \star v$ in (1.2) we mean that the value $u(p) \star v(p)$ is taken at the boundary point $p \in \partial K$

of K in the tangent space $T\mathbb{R}^n_p$ of $\mathbb{R}^n$ at p by considering the parallel transport of the vector $u(0)$ at the origin to the outward unit normal $u(p)$ on the boundary ∂K. Hence, when considering the Euclidean space $\mathbb{R}^n$ as a Lie group with the natural left translation $L_p q := p + q$ for any points p and q in $\mathbb{R}^n$, we can write

$$|u(p) \star v(p)| = |L_{p*}u(0) \star v(p)|, \tag{1.3}$$

where L_{p*} is the pushforward of the left translation L_p. In the present work, we will show the analogues of Theorem 1.1 and Lemma 1.2 for the Heisenberg groups, which are regarded as flat models of pseudohermitian manifolds. The notion for parallel transport in $\mathbb{R}^n$ shall be replaced by left invariant translations in the Heisenberg groups $\mathbb{H}_n$ for $n \geq 1$. Instead of using the standard unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ to derive Cauchy's surface area formula (namely, Lemma 1.2), we shall use the Pansu spheres (defined in next section, equation (2.4)), that seem to be a more natural model in $\mathbb{H}_n$. See Remark 3.1 for a geometric interpretation of this approach. We also mention that the approach of group translations weaken the assumption of convexity of domains in the Euclidean spaces. Here we only need that the boundary is of class C^2 and such that most of the p-normal vectors can be defined on it. See the paragraph before Definition 1.3 below.

Now we give some background (see next section for more details). The Heisenberg group $\mathbb{H}_n$ for $n \geq 1$ is defined as the Euclidean space $\mathbb{R}^{2n+1}$ with contact structure ξ (also called *distribution*). At each point in $\mathbb{H}_n$, there is a contact plane ξ of dimension $2n$. Suppose Σ is a hypersurface in $\mathbb{H}_n$. The singular set of Σ is the set of points of Σ at which the tangent plane of Σ coincides with the contact plane; otherwise, points on Σ are called nonsingular points. At any nonsingular point, the intersection of the tangent plane $T\Sigma$ and the contact plane ξ is of dimension $2n-1$ and so there is a unique unit vector N (called the *p-normal vector*) contained in ξ and perpendicular to the intersection $T\Sigma \cap \xi$ with respect to the Levi metric. The volume and area elements in $\mathbb{H}_n$ are the usual Euclidean volumes and the *p-areas*, respectively. The p-area, introduced by Cheng–Hwang–Yang [7] for $n = 1$ and Cheng–Hwang–Malchiodi–Yang [9] for $n \geq 2$, comes from a variation of the surface Σ in the normal direction fN for some suitable function f with compact support on the regular points of Σ. Such invariant area measure in $\mathbb{H}_n$ coincides with the three dimensional Hausdorff measure of Σ, considered in [12], [1], and [2]. Notice that although the p-normal vectors are not defined at the singular points of Σ, the p-area element is globally defined on Σ and vanishes at the singular points ([9], p. 135). Moreover, the p-area element can be represented explicitly in a variety of forms, including differential forms (see Appendix in [9] for $\mathbb{H}_n$), local coordinates for graphs ([7], p. 261), and recently for parametrized surfaces [16] by the author. These notions, especially in the framework of geometric measure theory, have been used to study existence or regularity properties of minimizers for the relative perimeter or extremizers of isoperimetric inequalities (see, e.g., [11, 14, 17, 20, 21, 25, 26, 28]).

Recall [27] that the Pansu spheres $\mathcal{P}^n_\lambda$ in $\mathbb{H}_n$ can be defined by rotating a (helix) geodesic joining the points $(0, \ldots, 0, \pm\frac{\pi}{4\lambda^2}) \in \mathbb{R}^{2n+1}$ (the "North" and "South" poles, respectively) about the x_{2n+1}-axis such that its "equator" is a standard sphere $\mathbb{S}^{2n-1}_{1/\lambda}$ in $\mathbb{R}^{2n}$ centered at the origin with radius $1/\lambda$. It is rotationally symmetric and topologically equivalent to the standard unit sphere $\mathbb{S}^{2n}$, and its parametrization can be exactly expressed as in (2.4). In [24], Monti analyzed the symmetrization of $\mathcal{P}^n_\lambda$, and Cheng et al. [8] used a notion of umbilicity to characterize the Pansu spheres in $\mathbb{H}_n$ for $n \geq 2$. There are only two singular points, at the North and South poles of $\mathcal{P}_\lambda^n$, and so the p-normal vectors are well-defined globally on $\mathcal{P}_\lambda^n$ except for the poles. In Theorem 1.5 and Proposition 1.6, we will show that, in $\mathbb{H}_n$, the Pansu spheres are the natural analogues of the standard unit spheres in the Euclidean space in Theorem 1.1 and Lemma 1.2.

We make a remark for the requirements of the hypersurfaces we are concerned with. In Theorem B in [9], the authors assume that the surfaces Σ in $\mathbb{H}_1$ have C^2-regularity with bounded p-mean curvatures, and obtain that the set of all singular points in Σ consists of only isolated points and smooth curves. In both cases, the measure of the singular set is zero in the sense of p-area elements, and so this set does not influence the results of the integrals for the p-normal vectors even though the p-normals are not defined on the singular sets (for example, the right-hand sides of (1.4) and (1.5)). Therefore, for our purposes, whenever in the article we say "*the normal vectors are defined on the hypersurface* Σ *a.e.*", it will mean that the normal vectors are globally defined on Σ except possibly for a set of measure zero in the sense of p-area elements. Besides, the inner product of vectors in $\mathbb{H}_n$ is always adopted with respect to the Levi metric, and we denote by $\tilde{N}(p)$ (respectively, $N(p)$) the unit p-normal vector at p on the Pansu sphere $\mathcal{P}_\lambda^n$ (respectively, any hypersurface Σ) in $\mathbb{H}_n$.

The following definition was motivated by Lemma 1.2 and the identity (1.3) for compact hypersurfaces in the Heisenberg groups $\mathbb{H}_n$.

Definition 1.3. Let Σ be any compact hypersurface in the Heisenberg group $\mathbb{H}_n$, $n \geq 1$, with the p-normal vectors defined a.e. on Σ. Given a unit p-normal vector $\tilde{N}(p)$ at p on the Pansu sphere $\mathcal{P}_\lambda^n$, the projected p-area of Σ onto the orthogonal complement $\tilde{N}(p)^\perp := \{u \in T_p\mathbb{H}_n\,;\, u \cdot \tilde{N}(p) = 0\}$ is defined by

$$\mathcal{A}(\Sigma|\tilde{N}^\perp(p)) = \int_{q\in\Sigma} |L_{qp^{-1}*}\tilde{N}(p) \cdot N(q)|\, d\Sigma_q, \tag{1.4}$$

where $L_{qp^{-1}*}$ is the pushforward of the left translation $L_{qp^{-1}} := L_q \circ L_{p^{-1}}$ in $\mathbb{H}_n$ and $N(q)$ is the unit p-normal vector of Σ at $q \in \Sigma$.

In Definition 1.3 we only consider the projected p-areas of the surface Σ onto the p-normal vectors of the Pansu spheres $\mathcal{P}_\lambda^n$. There are two reasons: first, by applying the pushforward $L_{p^{-1}{}_*}$ of the left translation $L_{p^{-1}}$ on the unit p-normal vector $\tilde{N}(p)$ of $\mathcal{P}_\lambda^n$, it is clear that the set $\{L_{p^{-1}{}_*}\tilde{N}(p)|\text{ all } p \in \mathcal{P}_\lambda^n\} = \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{H}_n$, and hence the directions that the projection is along with are not full of all possible positions in $\mathbb{H}_n$ (namely, $\mathbb{S}^{2n}$). This raises the question of considering the projected p-areas along *any* direction, that is, along any vector $u \in \mathbb{S}^{2n}$. However, Proposition 1.4 below shows that for $n = 1$, when Σ is a rotationally symmetric surface and we consider the projections of Σ along arbitrary directions u in $\mathbb{H}_1$, the projected p-areas of Σ depend on the choice of the projected directions. As a result, the projected p-areas of both the Pansu spheres and the standard unit sphere $\mathbb{S}^2$ will *not* be constant if the projected directions are in $\mathbb{S}^{2n}$. This observation suggests that using the p-normal vectors of the Pansu spheres as projected vectors would be better than the usual Euclidean normal vectors. Secondly, the projected p-area of the Pansu spheres along its p-normal vectors is a constant, independent of the choice of the projected directions, and the value can be exactly obtained (Proposition 1.6), but that of the spheres $\mathbb{S}^{2n}$ involve an integral which does not seem to have the closed form (3.2). Therefore, considering the p-normal vectors of the Pansu spheres is a natural generalization of projected p-areas in the Heisenberg groups.

Next result shows that the projected p-areas of any rotationally symmetric compact surface depend on the projected directions if we consider *arbitrary* directions in $\mathbb{S}^2 \subset \mathbb{H}_1$.

Proposition 1.4. *Let Σ be any rotationally symmetric compact hypersurface in the Heisenberg group $\mathbb{H}_1$ with the p-normal vectors defined a.e. on Σ. Suppose $\Sigma = \Sigma^+ \cup \Sigma^-$ can be represented by*

$$\Sigma^{\pm} : (r, \theta) \to (r \cos\theta, r \sin\theta, h^{\pm}(r)),$$

for some functions $h^+ \geq 0$, $h^- \leq 0$, and for $0 \leq \theta \leq 2\pi$, $0 \leq r \leq R$, where R is some positive number. If h^+ and h^- satisfy any of the following conditions:

(1) $\frac{dh^+}{dr}$ *or* $\frac{dh^-}{dr}$ *is continuous on the interval* $[0, R]$,

(2) $|\frac{dh^+}{dr}|$ *or* $|\frac{dh^-}{dr}|$ *is bounded by* $r/\sqrt{R^2 - r^2}$ *on the interval* $[0, R)$,

then for any vector $u(0) := (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ in $\mathbb{S}^2$, $0 \leq \alpha \leq \pi, 0 \leq \beta \leq 2\pi$, the projected p-area of Σ onto the subspace $u(0)^{\perp}$ perpendicular to $u(0)$ is given by

$$\mathcal{A}(\Sigma|u(0)^{\perp}) = \int_{q\in\Sigma} |L_{q*}u(0) \cdot N(q)|\, d\Sigma_q = |\sin\alpha| C, \tag{1.5}$$

where $N(q)$ is the unit p-normal vector at $q \in \Sigma$ and C is a constant independent of the choices of the vector $u(0)$. In particular, when $\alpha = \pi/2$, the identity (1.5) *becomes* (1.8) *for $n = 1$.*

Notice that in Proposition 1.4, the sharp case, namely $h_r = r/\sqrt{R^2 - r^2}$, occurs if and only if the compact hypersurface Σ is the standard sphere $\mathbb{S}^2(R)$ with radius R in $\mathbb{H}_1$. Similarly, the sharp case in Theorem 1.7 occurs for $\Sigma = \mathbb{S}^{2n}(R)$, the standard sphere with radius R in $\mathbb{H}_n$.

Our main result is an analogue of Theorem 1.1 for the Heisenberg groups, and shows that the p-area $\mathcal{A}(\Sigma)$ of any compact convex surface $\Sigma \subset \mathbb{H}_n$ is the average of the projected p-areas onto the orthogonal complements of all unit p-normal vectors in $\mathcal{P}^n_{\lambda}$ over the volume of the $(2n-1)$-dimensional Euclidean sphere.

Theorem 1.5. *Let Σ be any compact hypersurface in the Heisenberg group $\mathbb{H}_n$, $n \geq 1$, with the p-normal vectors defined a.e. on Σ. Let $\tilde{N}(p)$ be a unit p-normal vector at p on the Pansu sphere $\mathcal{P}^n_{\lambda}$, and let $\mathcal{A}(\Sigma|\tilde{N}(p)^{\perp})$ be the projected p-area of Σ onto the orthogonal complement $\tilde{N}(p)^{\perp}$. Then the p-area of Σ is given by*

$$\mathcal{A}(\Sigma) = \frac{1}{2C_n\omega_{2n-1}} \int_{p\in\mathcal{P}^n_{\lambda}} \mathcal{A}(\Sigma|\tilde{N}(p)^{\perp})\, d\Sigma_p, \tag{1.6}$$

where $C_n = \frac{\sqrt{\pi}}{\lambda^{2n+1}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}$ is a dimensional constant, $\Gamma(x)$ is the Gamma function, and ω_{2n-1} is the volume of the $(2n-1)$-dimensional Euclidean sphere. Moreover, the p-area of the surface Σ can be represented by the average projected p-areas of Σ along all the *p-normal vectors of the Pansu sphere, namely,*

$$\mathcal{A}(\Sigma) = \frac{S_{2n-1}}{2\omega_{2n-1}\mathcal{A}(\mathcal{P}^n_\lambda)} \int_{p\in\mathcal{P}^n_\lambda} \mathcal{A}(\Sigma|\tilde{N}(p)^\perp)\, d\Sigma_p, \tag{1.7}$$

where S_{2n-1} *is the* (*Euclidean*) *surface area of the* $(2n-1)$*-sphere in* $\mathbb{R}^{2n}$.

The proof of Theorem 1.5 is based on the observation that the projected p-area of the Pansu sphere $\mathcal{P}^n_\lambda$ in $\mathbb{H}_n$ along its p-normal vectors is a constant; it is a property analogous to that of Lemma 1.2 for spheres in $\mathbb{R}^n$.

Proposition 1.6. *Given the Pansu sphere* $\mathcal{P}^n_\lambda$ *in* $\mathbb{H}_n$, $n \geq 1$, *defined by* (2.4) *and any p-normal vector* $\tilde{N}(p)$ *for some fixed* $p \in \mathcal{P}^n_\lambda$, *denote by* $\tilde{N}(q)$ *and* $L_{qp^{-1}*}\tilde{N}(p)$, *respectively, the unit p-normal vector at* $q \in \mathcal{P}^n_\lambda$ *and the pushforward of the left translation* $L_{qp^{-1}}$ *operated on* $\tilde{N}(p)$. *Then the projected p-area of the Pansu sphere along* $\tilde{N}(p)$ *is a dimensional constant, namely,*

$$\mathcal{A}(\mathcal{P}^n_\lambda|\tilde{N}(p)^\perp) = \int_{q\in\mathcal{P}^n_\lambda} |L_{qp^{-1}*}\tilde{N}(p)\cdot\tilde{N}(q)|\, d\Sigma_q = 2C_n\,\omega_{2n-1},$$

where C_n *is the constant defined in Theorem* 1.5.

In fact, Proposition 1.6 can be generalized to any rotationally symmetric surfaces about the x_{2n+1}-axis in $\mathbb{H}_n$. This also shows that the reversed statement of Proposition 1.6 is *not* true in general, which is a result similar to that of $\mathbb{R}^n$, as we have mentioned in the paragraph after Lemma 1.2 that there exist subsets which are not spheres in $\mathbb{R}^n$ with constant width (e.g., the Reuleaux triangles).

Theorem 1.7. *Let* $\Sigma = \Sigma^+ \cup \Sigma^-$ *be a rotationally symmetric compact hypersurface in* $\mathbb{H}_n$ *obtained by rotating a hypersurface in* $\mathbb{R}^{2n}$ *about the* x_{2n+1}*-axis, where* $\Sigma^+ = \{x_{2n+1} = h^+ \geq 0\}$ *and* $\Sigma^- = \{x_{2n+1} = h^- \geq 0\}$ *for some functions* $h^\pm = h^\pm(r)$ *defined on* $[0, R]$ *for some* $R > 0$. *If* h^+ *and* h^- *satisfy any of the following conditions*:

(1) $\frac{dh^+}{dr}$ *or* $\frac{dh^-}{dr}$ *is continuous on the interval* $[0, R]$,

(2) $|\frac{dh^+}{dr}|$ *or* $|\frac{dh^-}{dr}|$ *is bounded by* $\frac{r}{\sqrt{R^2-r^2}}$ *on the interval* $[0, R)$,

and denoting by $\tilde{N}(p)$ *any unit p-normal vector at* p *in the Pansu sphere* $\mathcal{P}^n_\lambda$ *defined by* (2.4), *then the projected p-area of* Σ *onto* $\tilde{N}(p)^\perp$ *is given by*

$$\mathcal{A}(\Sigma|\tilde{N}(p)^\perp) = \int_{q\in\Sigma} |L_{qp^{-1}*}\tilde{N}(p)\cdot N(q)|\, d\Sigma_q = C, \tag{1.8}$$

where $N(q)$ *is the unit p-normal vector at* $q \in \Sigma$ *and* C *is a constant independent of the choice of* $\tilde{N}(p)$. *In particular, if* $\Sigma = \mathcal{P}^n_\lambda$, *then the projected p-area is exactly the same as that of Proposition* 1.6.

Since the p-areas for Pansu sphere $\mathcal{P}^n_\lambda$ can be obtained by equation (2.8) below, we get, as an immediate application of Theorem 1.5, that the expected value of the function $F_{\tilde{N}(p)}(\Sigma) := \mathcal{A}(\Sigma|\tilde{N}(p)^\perp)$ can be written as

$$\mathrm{Exp}(F_{\tilde{N}(p)}) := \frac{1}{\mathcal{A}(\mathcal{P}^n_\lambda)} \int_{p\in\mathcal{P}^n_\lambda} \mathcal{A}(\Sigma|\tilde{N}(p)^\perp)\, d\Sigma_p = \mathcal{A}(\Sigma)\,\omega_{2n-1}S_{2n-1}.$$

Roughly speaking, the number $\mathrm{Exp}(F_{\tilde{N}(p)}(\Sigma))$ is the average of the projected p-areas of the surface Σ onto a randomly-chosen plane $\tilde{N}(p)^{\perp}$.

This paper is organized as follows. In Section 2 we recall some fundamental background about the Heisenberg groups regarded as pseudohermitian manifolds. The precise expressions for the Pansu spheres, the p-normal vectors for rotationally symmetric surfaces, and the p-areas will also be derived. Section 3 contains the proofs of our results and the explanation of the geometric meanings of Proposition 1.6 in Remark 3.1.

2. Preliminaries

Let (M, J, Θ) be a $(2n+1)$-dimensional pseudohermitian manifold with an integrable CR structure J and a global contact form Θ such that the bilinear form $G := \frac{1}{2} d\Theta(\cdot, J\cdot)$ is positive definite on the contact bundle $\xi := \ker \Theta$. The metric G is usually called the Levi metric. Consider a hypersurface $\Sigma \subset M$. A point $p \in \Sigma$ is called singular if ξ coincides with $T\Sigma$ at p. Otherwise, p is called nonsingular and $V := \xi \cap T\Sigma$ is $(2n-1)$-dimensional in this case. There is a unique (up to a sign) unit vector $N \in \xi$ that is perpendicular to V with respect to the Levi metric G. We call N the *Legendrian normal* or the *p-normal vector* ("p" stands for "pseudohermitian"). Suppose that Σ bounds a domain D in M. In [7], the authors defined the p-area $2n$-form $d\Sigma$ by computing the first variation, away from the singular set, of the standard volume $\Theta \wedge (d\Theta)^n$ along the p-normal vector N:

$$\delta_{fN} \int_D \Theta \wedge (d\Theta)^n = c(n) \int_\Sigma f \, d\Sigma,$$

where f is a C^∞-smooth function on Σ with compact support away from the singular points, and $c(n) = 2^n n!$ is a normalization constant. The sign of N is determined by requiring that $d\Sigma$ is positive with respect to the induced orientation on Σ. Notice that the p-area $2n$-form can be continuously extended to the set of singular points and vanishes on the set, so that we can talk about the p-area of Σ by integrating $d\Sigma$ over Σ ([9], p. 135).

One of the most interesting examples of pseudohermitian manifolds is the n-dimensional Heisenberg group $\mathbb{H}_n$, which can be regarded as a flat pseudohermitian manifold $(\mathbb{R}^{2n+1}, \Theta_0, J_0)$. Two survey articles [6,33] gave some recent developments on geometric analysis in $\mathbb{H}_n$. Here $\Theta_0 := dz + \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$ at a point $(\vec{X}, z) := (x, y, z) := (x_1, y_1, \ldots, x_n, y_n, z) \in \mathbb{R}^{2n+1}$ and $J_0(\mathring{e}_{x_j}) = \mathring{e}_{y_j}$, $J_0(\mathring{e}_{y_j}) = -\mathring{e}_{x_j}$, where

$$\mathring{e}_{x_j} = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z}, \quad \mathring{e}_{y_j} = \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial z}$$

for $j = 1, \ldots, n$, span the contact plane $\xi_0 := \ker \Theta_0$. Notice that the $\mathring{e}_{x_j}$ and the $\mathring{e}_{y_j}$ form an orthonormal basis with respect to the Levi metric $G_0 := (\sum_{j=1}^{n} dx_j \wedge dy_j)(\cdot, J_0 \cdot)$. The group $\mathbb{H}_n$ is also a Lie group with the natural left translation defined by

$$\begin{aligned} L_{(\vec{X},z)}(\vec{X}', z') &:= L_{(x_1,y_1,\ldots,x_n,y_n,z)}(x_1', y_1', \ldots, x_n', y_n', z') \\ &= \Big(x_1 + x_1', y_1 + y_1', \ldots, x_n + x_n', y_n + y_n', z + z' + \sum_{i=1}^{n} (y_i x_i' - x_i y_i')\Big). \end{aligned}$$

Recall that the p-area form $d\Sigma_p$ at the point $p = (x_1, y_1, \ldots, x_n, y_n, z) \in \mathbb{H}_n$ for the graph $z = f(x, y)$ (see equation (2.7) in [7]) is given by

$$d\Sigma_p = D(p)\, dx_1\, dy_1 \cdots dx_n\, dy_n, \tag{2.1}$$

where

$$D(p) = \Big[\sum_{j=1}^{n} (f_{x_j} - y_j)^2 + (f_{y_j} + x_j)^2\Big]^{1/2}. \tag{2.2}$$

In general (see [7], p. 260), the p-normal vector $N(p)$ for any graph $z = f(x, y)$ at the point $p = (x_1, y_1, \ldots, x_n, y_n, z)$ is defined by

$$N(p) = \frac{-1}{D(p)} \sum_{j=1}^{n} \big[(f_{x_j} - y_j)\mathring{e}_{x_j}(p) + (f_{y_j} + x_j)\mathring{e}_{y_j}(p)\big]. \tag{2.3}$$

We also recall [27] that the Pansu sphere $\mathcal{P}^n_\lambda$ with radius $1/\lambda$ in $\mathbb{H}_n$ is the union of the graphs of the functions f and $-f$, where

$$f(x, y) = \frac{1}{2\lambda^2}\left(\lambda\sqrt{x^2 + y^2}\sqrt{1 - \lambda^2(x^2 + y^2)} + \cos^{-1}\lambda\sqrt{x^2 + y^2}\right), \tag{2.4}$$

where $\sqrt{x^2 + y^2} \leq 1/\lambda$, $x := (x_1, \ldots, x_n)$, $y := (y_1, \ldots, y_n)$, $x^2 := \sum_{j=1}^n x_j^2$, and $y^2 := \sum_{j=1}^n y_j^2$. Note that the intersection of the Pansu sphere $\mathcal{P}^n_\lambda$ and the plane $\{z = 0\}$ is the standard sphere $\mathbb{S}^{2n-1}_{1/\lambda}$ centered at the origin with radius $1/\lambda$. Denote $r = \sqrt{x^2 + y^2} = \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}$. Clearly, if we take the partial derivatives with respect to x_j and y_j for $j = 1, \ldots, n$, we have, respectively,

$$f_{x_j} := \frac{\partial f}{\partial x_j}(x, y) = \frac{-\lambda x_j r}{\sqrt{1 - \lambda^2 r^2}} \quad \text{and} \quad f_{y_j} := \frac{\partial f}{\partial y_j}(x, y) = \frac{-\lambda y_j r}{\sqrt{1 - \lambda^2 r^2}}. \tag{2.5}$$

When the graph is a Pansu sphere $\mathcal{P}^n_\lambda$, by substituting (2.5) into (2.2), we have

$$D(p) = \frac{r}{\sqrt{1 - \lambda^2 r^2}}, \tag{2.6}$$

and so by (2.6), (2.3) and (2.5), the p-normal vector at the point p in $\mathcal{P}^n_\lambda$ can be written in terms of r, namely,

$$\tilde{N}(p) = \sum_{j=1}^{n} \Big[\Big(\lambda x_j + \frac{\sqrt{1 - \lambda^2 r^2}}{r} y_j\Big)\mathring{e}_{x_j}(p) + \Big(\lambda y_j - \frac{\sqrt{1 - \lambda^2 r^2}}{r} x_j\Big)\mathring{e}_{y_j}(p)\Big]. \tag{2.7}$$

By (2.1), (2.6) and (2.4), and using spherical coordinates on $\mathbb{R}^{2n}$, a straight computation shows that the p-area of the Pansu sphere in $\mathbb{H}_n$ is given by

$$\begin{aligned} \mathcal{A}(P^n_\lambda) &= \int_{p \in \mathcal{P}^n_\lambda} D(p)\, dx_1\, dy_1 \cdots dx_n\, dy_n \\ &= 2\int_0^{1/\lambda}\int_{\mathbb{S}^{2n-1}} \frac{r^{2n}}{\sqrt{1 - \lambda^2 r^2}}\, dS^{2n-1}\, dr = S_{2n-1}\frac{\sqrt{\pi}\,\Gamma(n + 1/2)}{\lambda^{2n+1}\Gamma(n + 1)}, \end{aligned} \tag{2.8}$$

where S_{2n-1} is the (Euclidean) surface area of the $(2n-1)$-sphere in $\mathbb{R}^{2n}$, and $\Gamma(x)$ is the Gamma function. In particular, when $n = 1$, $\mathcal{A}(\mathcal{P}^1_\lambda) = \pi^2/\lambda^3$. The last integral with respect to r can be obtained by setting $\lambda r = \sin\theta$. Then we have

$$\int_0^{1/\lambda} \frac{r^{2n}}{\sqrt{1-\lambda^2 r^2}}\, dr = \frac{1}{\lambda^{2n+1}} \int_0^{\pi/2} \sin^{2n}\theta\, d\theta.$$

To deal with the integral of power $2n$ of sine function, we set $I_n = \int_0^{\pi/2} \sin^{2n}\theta d\theta$ and use integration by parts to have the reduction formula $I_n = \frac{2n-1}{2n} I_{n-1}$. By induction, one gets

$$I_n = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!},$$

where the double factorials are defined by

$$\begin{aligned}(2n-1)!! &= (2n-1)(2n-3)(2n-5)\cdot\dots\cdot 3\cdot 1,\\ (2n)!! &= (2n)(2n-2)(2n-4)(2n-6)\cdots 4\cdot 2.\end{aligned}$$

Then

$$I_n = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!} = \frac{\pi}{2}\frac{(2n-1)!!}{2^n n!} = \frac{\sqrt{\pi}}{2}\Big(\frac{\frac{(2n-1)!!}{2^n}\sqrt{\pi}}{n!}\Big) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}.$$

3. The proofs

Proof of Proposition 1.4. We point out that although the proof below is only for $n = 1$; the same argument can be applied to higher dimensional Heisenberg groups.

Let $q = (r\cos\theta, r\sin\theta, h(r))$ be a point on Σ. Denote by h^+ and h^- the graphs of $\{h \geq 0\}$ and $\{h < 0\}$ respectively, and let $h_r^\pm := \frac{dh^\pm}{dr}$. Using (2.1), (2.2) and (2.3) for $n = 1$ and setting $\eta := \cos\alpha - r\sin\theta\sin\alpha\cos\beta + r\cos\theta\sin\alpha\sin\beta$, we have

$$\begin{aligned}
&|L_{q*}u(0)\cdot N(q)|\, d\Sigma_q\\
&= \Big|\big(\sin\alpha\cos\beta\,\mathring{e}_1(q) + \sin\alpha\sin\beta\,\mathring{e}_2(q) + \eta T(q)\big)\\
&\quad\cdot \frac{-1}{D(q)}\big((h_r^+\cos\theta - r\sin\theta)\,\mathring{e}_1(q) + (h_r^+\sin\theta + r\cos\theta)\,\mathring{e}_2(q)\big)\Big| D(q)\, r\, dr\, d\theta\\
&= r|\sin\alpha|\,|h_r^+\cos(\theta-\beta) - r\sin(\theta-\beta)|\, dr\, d\theta\\
&= r|\sin\alpha|\sqrt{(h_r^+)^2 + r^2}\,\Big|\Big(\frac{h_r^+}{\sqrt{(h_r^+)^2+r^2}}\cos(\theta-\beta) - \frac{r}{\sqrt{(h_r^+)^2+r^2}}\sin(\theta-\beta)\Big)\Big|\, dr\, d\theta\\
&= r|\sin\alpha|\sqrt{(h_r^+)^2 + r^2}\,\big|\cos(\theta-\beta+\phi)\big|\, dr\, d\theta,
\end{aligned}$$

where $\cos\phi = h_r^+/\sqrt{h_r^2 - r^2}$. We have a similar result for the function h^-.

Since the angles ϕ and β are both independent of the angle θ, by the continuity of h_r^+ and h_r^- the projected p-area of Σ is given by

$$\begin{aligned}\mathcal{A}(\Sigma|u(0)^\perp) &= \iint_{q\in\Sigma} |L_{q_*}u(0)\cdot N(q)|\,d\Sigma_q \\ &= |\sin\alpha| \int_0^R r\sqrt{(h_r^+)^2+r^2}\int_0^{2\pi} |\cos(\theta-\beta+\phi)|\,d\theta\,dr \\ &\quad + |\sin\alpha| \int_0^R r\sqrt{(h_r^-)^2+r^2}\int_0^{2\pi} |\cos(\theta-\beta+\phi)|\,d\theta\,dr \\ &= 4|\sin\alpha|\Big(\int_0^R r\sqrt{(h_r^+)^2+r^2}\,dr + \int_0^R r\sqrt{(h_r^-)^2+r^2}\,dr\Big). \end{aligned} \tag{3.1}$$

If the functions h_r^+ and h_r^- are continuous on $[0, R]$, then both integrals on the right-hand side of (3.1) are finite, and so $\mathcal{A}(\Sigma|u(0)^\perp) = C|\sin\alpha|$ for some constant C, depending only on $h^+(r)$ and $h^-(r)$. Otherwise, if any of $|h^+(r)|$ and $|h^-(r)|$ is bounded by $r/\sqrt{R^2-r^2}$ on the interval $[0, R)$, say $h^+(r)$, then we may choose a positive number $M > \sqrt{1+R^2}$ such that the following estimate holds:

$$\begin{aligned}\int_0^R r\sqrt{(h_r^+)^2+r^2}\,dr &\le \int_0^R r\sqrt{\frac{r^2}{R^2-r^2}+r^2}\,dr = \int_0^R r^2\sqrt{\frac{1+R^2-r^2}{R^2-r^2}}\,dr \\ &< R^2M\int_0^R \sqrt{\frac{1}{R^2-r^2}}\,dr = R^2M\lim_{a\to R^-}\tan^{-1}\Big(\frac{r}{\sqrt{R^2-r^2}}\Big)\Big|_0^a \\ &= \frac{R^2M\pi}{2} < \infty. \end{aligned} \tag{3.2}$$

Therefore, by the assumptions for $h^+(r)$, $h^-(r)$, and (3.1), we conclude that the projected p-area satisfies $\mathcal{A}(\Sigma|u(0)^\perp) < C|\sin\alpha|$, for some constant C only depending on the function $h^\pm(r)$.

When $\alpha = \pi/2$, the vector $u(0)$ is a unit vector on the xy-plane, which is the pushforward of a p-normal vector $\tilde{N}(p)$ for some point p on the Pansu sphere $\mathcal{P}_\lambda^1$. Thus, we have $u(0) = L_{p^{-1}*}\tilde{N}(p)$, and the result follows immediately. ∎

In the following calculations, p will always stand for a fixed point on the Pansu sphere $\mathcal{P}_\lambda^n$ and q will be an arbitrary point on $\mathcal{P}_\lambda^n$. Notice that since the p-normal vector $\tilde{N}(p)$ of $\mathcal{P}_\lambda^n$ is on the contact plane ξ_p at p, $\tilde{N}(p)$ can be carried parallel to the vector $L_{qp^{-1}*}\tilde{N}(p) \in \xi_q$ by the pushforward $L_{qp^{-1}*}$ of the left translation $L_{qp^{-1}} := L_q \circ L_{p^{-1}}$.

Proof of Proposition 1.6. First, we calculate the inner product

$$L_{qp^{-1}*}\tilde{N}(p)\cdot\tilde{N}(q)$$

at a point $q \in \mathcal{P}_\lambda^n$ with respect to the Levi metric. Write $p = (x_j, y_j, f(x_j, y_j))$ and $q = (\bar{x}_j, \bar{y}_j, f(\bar{x}_j, \bar{y}_j))$, where f is the function defined by (2.4). Without loss of generality, we may assume that q is on the upper half $\{f \ge 0\}$ of $\mathcal{P}_\lambda^n$. Let r (respectively, $\bar{r}$) denote the radius of the spherical coordinates of the points $(x_1, y_1, \dots, x_n, y_n)$ (respectively,

$(\bar{x}_1, \bar{y}_1, \dots, \bar{x}_n, \bar{y}_n))$ of $\mathbb{R}^{2n}$. Since $\{\mathring{e}_{x_j}(p), \mathring{e}_{y_j}(p)\}$ is an orthonormal basis in the contact plane ξ_p for any $p \in \mathbb{H}_n$, by (2.7) and (2.6), we have

$$
\begin{aligned}
(3.3)\quad & |L_{qp^{-1}*}\tilde{N}(p) \cdot \tilde{N}(q)|D(q) \\
&= \Big| \sum_{j=1}^{n} \Big[\Big(\lambda x_j + \frac{\sqrt{1-\lambda^2 r^2}}{r} y_j \Big) \mathring{e}_{x_j}(q) + \Big(\lambda y_j - \frac{\sqrt{1-\lambda^2 r^2}}{r} x_j \Big) \mathring{e}_{y_j}(q) \Big] \\
&\quad \cdot \sum_{j=1}^{n} \Big[\Big(\lambda \bar{x}_j + \frac{\sqrt{1-\lambda^2 \bar{r}^2}}{\bar{r}} \bar{y}_j \Big) \mathring{e}_{\bar{x}_j}(q) + \Big(\lambda \bar{y}_j - \frac{\sqrt{1-\lambda^2 \bar{r}^2}}{\bar{r}} \bar{x}_j \Big) \mathring{e}_{\bar{y}_j}(q) \Big] \Big| \cdot \frac{\bar{r}}{\sqrt{1-\lambda^2 \bar{r}^2}} \\
&= \Big| \sum_{j=1}^{n} \Big[\Big(\lambda x_j + \frac{\sqrt{1-\lambda^2 r^2}}{r} y_j \Big) \Big(\lambda \bar{x}_j + \frac{\sqrt{1-\lambda^2 \bar{r}^2}}{\bar{r}} \bar{y}_j \Big) \\
&\quad + \Big(\lambda y_j - \frac{\sqrt{1-\lambda^2 r^2}}{r} x_j \Big) \Big(\lambda \bar{y}_j - \frac{\sqrt{1-\lambda^2 \bar{r}^2}}{\bar{r}} \bar{x}_j \Big) \Big] \Big| \cdot \frac{\bar{r}}{\sqrt{1-\lambda^2 \bar{r}^2}} \\
&= \Big| \sum_{j=1}^{n} \Big[\Big(\frac{\lambda^2 \bar{r}}{\sqrt{1-\lambda^2 \bar{r}^2}} + \frac{\sqrt{1-\lambda^2 r^2}}{r} \Big) (x_j \bar{x}_j + y_j \bar{y}_j) \\
&\quad + \Big(-\frac{\lambda \bar{r}}{r} \frac{\sqrt{1-\lambda^2 r^2}}{\sqrt{1-\lambda^2 \bar{r}^2}} + \lambda \Big) (x_j \bar{y}_j - y_j \bar{x}_j) \Big] \Big|.
\end{aligned}
$$

Observe that in spherical coordinates, each component x_j can be represented as

$$x_j = (\text{radial distance}) \cdot (\text{product of sine and cosine functions})$$

for $j = 1, \dots, n$. Thus, one can write the components $x_j = r a_j$ for some product a_j of trigonometric functions. Similarly, one has $y_j = r b_j$, $\bar{x}_j = \bar{r} \bar{a}_j$, and $\bar{y}_j = \bar{r} \bar{b}_j$. Notice that the points $(a, b) = (a_1, b_1, \dots, a_n, b_n)$ and $(\bar{a}, \bar{b}) = (\bar{a}_1, \bar{b}_1, \dots, \bar{a}_n, \bar{b}_n)$ are both on the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$. Next, substituting x_j, y_j, $\bar{x}_j$, $\bar{y}_j$ by r, $\bar{r}$, a_j, b_j, $\bar{a}_j$, $\bar{b}_j$ in the last equation of (3.3), we get

$$
(3.4)\qquad |L_{qp^{-1}*}\tilde{N}(p) \cdot \tilde{N}(q)|D(q) = \Big| \sum_{j=1}^{n} \big[A(a_j \bar{a}_j + b_j \bar{b}_j) + B(a_j \bar{b}_j - b_j \bar{a}_j) \big] \Big|,
$$

where

$$
A = \frac{\lambda^2 \bar{r}^2 r}{\sqrt{1-\lambda^2 \bar{r}^2}} + \bar{r}\sqrt{1-\lambda^2 r^2} \quad \text{and} \quad B = -\frac{\lambda \bar{r}^2 \sqrt{1-\lambda^2 r^2}}{\sqrt{1-\lambda^2 \bar{r}^2}} + \lambda r \bar{r}.
$$

We emphasize the key observation that

$$
(3.5)\qquad A^2 + B^2 = \frac{\bar{r}^2}{1-\lambda^2 \bar{r}^2},
$$

which is independent of the choice of the point p.

Now consider the unit vectors $u = (a_1, b_1, \dots, a_n, b_n)$ and $\bar{v} = (\bar{a}_1, \bar{b}_2, \dots, \bar{a}_n, \bar{b}_n)$ in $\mathbb{R}^{2n}$ with respect to the usual Euclidean norm. Let $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the canonical almost complex structure defined by $J(u) = (-b_1, a_1, \dots, -b_n, a_n)$. If θ is the angle between two unit vectors u and $\bar{v}$ (in the usual Euclidean norm), and we denote by $u \star \bar{v}$ the Euclidean inner product of u and $\bar{v}$, then we have $u \star \bar{v} = \cos\theta$ and $J(u) \star \bar{v} = \pm \sin\theta$.

The sign depends on the chosen orientation of $\mathbb{R}^{2n}$. Therefore, the right-hand side of (3.4) can be represented as

$$\text{(3.6)}\quad \Big|\sum_{j=1}^{n}\big[A(a_j\bar{a}_j+b_j\bar{b}_j)+B(a_j\bar{b}_j-b_j\bar{a}_j)\big]\Big| = |A(u\star\bar{v})+B(J(u)\star\bar{v})|$$

$$= \sqrt{A^2+B^2}\,\Big|\frac{A}{\sqrt{A^2+B^2}}\cos\theta_q \pm \frac{B}{\sqrt{A^2+B^2}}\sin\theta_q\Big| \overset{(3.5)}{=} \frac{\bar{r}}{\sqrt{1-\lambda^2\bar{r}^2}}|\cos(\theta_q\mp\alpha)|$$

for some value $\alpha=\alpha(r,\bar{r},\lambda)$ such that $\cos(\alpha)=A/\sqrt{A^2+B^2}$. We add the subscript θ_q to indicate that the angle θ is a function of the point q.

Recall that the p-area form $d\Sigma_q$ at the point q in $\mathbb{H}_n$ for any graph $z=f(x,y)$ is $d\Sigma_q=D(q)dx_1dy_1\cdots dx_ndy_n$ (see equation (2.7) in [7]). Finally, combine (3.4), (3.6) and use the integral for spherical coordinates, one has

$$\begin{aligned}
&\int_{q\in\mathcal{P}_\lambda^n}|L_{qp^{-1}}{}_*\tilde{N}(p)\cdot\tilde{N}(q)|\,d\Sigma_q\\
&\quad=\int_{q\in\mathcal{P}_\lambda^n}|L_{qp^{-1}}{}_*\tilde{N}(p)\cdot\tilde{N}(q)|D(q)\,dx_1\,dy_1\cdots dx_n\,dy_n\\
&\quad=\int_{p\in\mathcal{P}_\lambda^n}\Big|\sum_{j=1}^{n}\big[A(a_j\bar{a}_j+b_j\bar{b}_j)+B(a_j\bar{b}_j-b_j\bar{a}_j)\big]\Big|\,dx_1\,dy_1\cdots dx_n\,dy_n\\
&\quad=2\int_0^{1/\lambda}\int_0^{2\pi}\frac{\bar{r}\cdot\bar{r}^{2n-1}}{\sqrt{1-\lambda^2\bar{r}^2}}|\cos(\theta_q\mp\alpha)|\,dS_q\,d\bar{r}\\
&\quad=2\int_0^{1/\lambda}\frac{\bar{r}^{2n}}{\sqrt{1-\lambda^2\bar{r}^2}}\int_0^{2\pi}|\cos(\theta_q\mp\alpha)|\,dS_q\,d\bar{r}\\
&\quad\overset{(*)}{=}4\omega_{2n-1}\int_0^{1/\lambda}\frac{\bar{r}^{2n}}{\sqrt{1-\lambda^2\bar{r}^2}}\,d\bar{r}=2\,\frac{\sqrt{\pi}\,\omega_{2n-1}}{\lambda^{2n+1}}\,\frac{\Gamma(n+1/2)}{\Gamma(n+1)},
\end{aligned}$$

where dS_q is the $(2n-1)$-dimensional (Euclidean) surface area element at q and ω_{2n-1} is the volume of the unit $(2n-1)$-ball in $\mathbb{R}^{2n-1}$. Note that in $(*)$ above, for any fixed $\bar{r}\in(0,1/\lambda]$ (equivalently, α is a constant with respect to q), Lemma 1.2 implies that the integral $\int_0^{2\pi}|\cos(\theta_q\mp\alpha)|dS_q=2\omega_{2n-1}$, the projected area of the $(2n-1)$-dimensional sphere onto the $(2n-1)$-subspace in $\mathbb{R}^{2n}$. The last integral can be obtained from (2.8). ■

Remark 3.1. In the Euclidean space, the projected area of K onto the subspace $u^\perp$ can be represented as the integral of the inner product $u\star v$ as shown in (1.1). Suppose the unit vector u is based at the origin and ends at somewhere on $\mathbb{S}^{n-1}$. The proof of Lemma 1.2 uses the fact that, by parallel transport in $\mathbb{R}^n$, the base point of u is parallel moved to any point $p\in\mathbb{S}^{n-1}$ and makes the inner product with the unit outward normal vectors $v(p)$ at p. Similarly, in $\mathbb{H}_n$ the integral of $|L_{qp^{-1}*}\tilde{N}(p)\cdot\tilde{N}(q)|$ over $\mathcal{P}_\lambda^n$ in Proposition 1.6 can be regarded as the projected p-area of $\mathcal{P}_\lambda^n$ onto the subspace $\tilde{N}(p)^\perp:=\{u\in T_p\mathbb{H}_n|\ u\cdot N(p)=0$ with respect to the Levi metric$\}$. For instance, in $\mathbb{H}_1$, we have the orthonormal basis $\{e_1(p),N(p),T\}$ for $e_1(p)\in T_p\Sigma\cap\xi_p$, $N(p)=J(e_1(p))\in\xi_p$, and $T(p)=\frac{\partial}{\partial z}|_p$. Then $N(p)^\perp=\mathrm{span}_p\{e_1(p),T(p)\}$. Geometrically, $N(p)^\perp$ is a plane (Euclidean) perpendicular to the xy-plane.

The next lemma shows that the map $L_{p^{-1}}{}_*$ is surjective.

Lemma 3.2. *Let $\mathcal{P}_\lambda^n$ be the Pansu sphere in the Heisenberg group $\mathbb{H}_n$ defined by (2.4) and denote by ξ_0 the contact plane at the origin in $\mathbb{H}_n$. For any $p \in \mathcal{P}_\lambda^n$, the map*

$$p \mapsto N(0) := L_{p^{-1}}{}_*(\tilde{N}(p)) \subset \mathbb{S}^{2n-1} \subset \xi_0$$

which assigns to any unit p-normal vector $\tilde{N}(p)$ at $p \in \mathcal{P}_\lambda^n$ a unit p-normal vector $N(0)$ in $\mathbb{S}^{2n-1}$, is surjective.

Proof. For any unit vector $N(0)$ on $\mathbb{S}^{2n-1} \subset \xi_0$ in $\mathbb{H}_n$, one may assume that $N(0) = (x_1, y_1, \ldots, x_n, y_n, 0)$ with $\sum_{j=1}^n (x_j^2 + y_j^2) = 1$. Set $p = \frac{1}{\lambda} N(0)$. Then the point p is in the intersection $\mathcal{P}_\lambda^n \cap \{z = 0\}$. Construct the unit p-normal $\tilde{N}(p)$ by replacing the x_j and the y_j on right-hand side of (2.7) by $\frac{1}{\lambda} x_j$ and $\frac{1}{\lambda} y_j$, respectively. It is clear that the vector $\tilde{N}(p)$ is the p-normal vector at $p \in \mathcal{P}_\lambda^n$ satisfying $L_{p^{-1}}{}_* \tilde{N(p)} = N(0)$. ∎

The map in Lemma 3.2 is a generalization of the Gauss map defined by Chiu–Ho [10] in $\mathbb{H}_1$. In that paper, the authors studied the degree of the Gauss map for horizontally regular curves with some topological properties (see Theorems 1.6 and 2.1 in [10]).

Proof of Theorem 1.5. For any p-normal vector $N(q)$ at $q \in \Sigma$, one has that $L_{q^{-1}}{}_* N(q) \subset \mathbb{S}^{2n-1}$. Moreover, Lemma 3.2 implies that there exists a p-normal vector $\tilde{N}(q')$ at some $q' \in \mathcal{P}_\lambda^n$ such that $L_{q'q^{-1}}{}_* N(q) = \tilde{N}(q')$; equivalently,

$$L_{q^{-1}}{}_* N(q) = L_{q'^{-1}*} \tilde{N}(q').$$

Thus, for any fixed p-normal vector $\tilde{N}(p)$ at $p \in \mathcal{P}_\lambda^n$ we have

$$\begin{aligned} |L_{qp^{-1}*} \tilde{N}(p) \cdot N(q)| &= |L_{p^{-1}}{}_* \tilde{N}(p) \cdot L_{q^{-1}}{}_* N(q)| \\ &= |L_{p^{-1}}{}_* \tilde{N}(p) \cdot L_{q'^{-1}}{}_* \tilde{N}(q')| = |L_{q'p^{-1}}{}_* \tilde{N}(p) \cdot \tilde{N}(q')|. \end{aligned} \tag{3.7}$$

Finally, using (3.7) and Proposition 1.6, one gets

$$\begin{aligned} \int_{p \in \mathcal{P}_\lambda^n} \mathcal{A}(\Sigma | \tilde{N}(p)^\perp)\, d\Sigma_p &= \int_{p \in \mathcal{P}_\lambda^n} \int_{q \in \Sigma} |L_{qp^{-1}*} \tilde{N}(p) \cdot N(q)|\, d\Sigma_q d\Sigma_p \\ &= \int_{q \in \Sigma} \int_{p \in \mathcal{P}_\lambda^n} |L_{qp^{-1}*} \tilde{N}(p) \cdot N(q)|\, d\Sigma_p\, d\Sigma_q \\ &= \int_{q \in \Sigma} \int_{p \in \mathcal{P}_\lambda^n} |L_{q'p^{-1}}{}_* \tilde{N}(p) \cdot \tilde{N}(q')|\, d\Sigma_p\, d\Sigma_q \\ &= 2C_n \omega_{2n-1} \int_{q \in \Sigma} d\Sigma_q = 2C_n \omega_{2n-1} \mathcal{A}(\Sigma). \end{aligned}$$

This completes the proof of (1.6). The second result (1.7) can be immediately obtained using (2.8) and (1.6). ∎

We now prove Theorem 1.7. For simplicity, we shall just prove the case $\mathbb{H}_1$; the proof for higher dimensions is the same. When $n \geq 2$, the constant C in Theorem 1.7 depends only on the dimension n of $\mathbb{H}_n$.

Proof of Theorem 1.7 for $n = 1$. Let Σ be any rotationally symmetric compact surface in $\mathbb{H}_1$ defined by $(r, \theta) \to (r\cos\theta, r\sin\theta, h(r))$, for $0 \le r \le R$ for some real number R and $0 \le \theta \le 2\pi$. Let $\mathcal{P}^1_\lambda$ be the Pansu sphere in $\mathbb{H}_1$ defined by (2.4) and let p be a fixed point on $\mathcal{P}^1_\lambda$. Denote by h^+ and h^- the graphs of $\Sigma^+ := \{h \ge 0\}$ and $\Sigma^- := \{h < 0\}$, respectively. By using (2.1), (2.2), (2.3) and (2.7) for $n = 1$, and writing $p = (\tilde r\cos\theta, \tilde r\sin\theta, h^+(r))$, one has

$$
\begin{aligned}
&|L_{qp^{-1}*}\tilde N(p)\cdot N(q)|\,d\Sigma_q\\
&= \Big| L_{qp^{-1}*}\big((\lambda\tilde r\cos\theta + \sqrt{1-\lambda^2\tilde r^2}\sin\theta)\mathring{e}_1(p) + (\lambda\tilde r\sin\theta - \sqrt{1-\lambda^2\tilde r^2}\cos\theta)\mathring{e}_2(p)\big)\\
&\quad\cdot\frac{-1}{D(q)}\big((h_r^+\cos\phi - r\sin\phi)\mathring{e}_1(q) + (h_r^+\sin\phi + r\cos\phi)\mathring{e}_2(q)\big)\Big| D(q)\,r\,dr\,d\phi\\
&= \Big|\big((\lambda\tilde r\cos\theta + \sqrt{1-\lambda^2\tilde r^2}\sin\theta)\mathring{e}_1(q) + (\lambda\tilde r\sin\theta - \sqrt{1-\lambda^2\tilde r^2}\cos\theta)\mathring{e}_2(q)\big)\\
&\quad\cdot\big((h_r^+\cos\phi - r\sin\phi)\mathring{e}_1(q) + (h_r^+\sin\phi + r\cos\phi)\mathring{e}_2(q)\big)\Big|\,r\,dr\,d\phi\\
&= |(\lambda h_r^+\tilde r - r\sqrt{1-\lambda^2\tilde r^2})\cos(\theta-\phi) + (\lambda r\tilde r + h_r^+\sqrt{1-\lambda^2\tilde r^2})\sin(\theta-\phi)|\,r\,dr\,d\phi
\end{aligned}
$$

Set $A := \lambda h_r^+\tilde r - r\sqrt{1-\lambda^2\tilde r^2}$ and $B := \lambda r\tilde r + h_r^+\sqrt{1-\lambda^2\tilde r^2}$, so $A^2 + B^2 = (h_r^+)^2 + r^2$. We point out that the term $A^2 + B^2$ does not involve any terms with $\tilde r$ and θ, that is, it is independent of the choice of the p-normal $\tilde N(p)$ of the Pansu sphere. Now we set $\cos\psi = A/\sqrt{A^2+B^2}$ and $\sin\psi = B/\sqrt{A^2+B^2}$, and notice that the angle ψ does not depend on the angle ϕ, and so, when taking the integral with respect to the angle ϕ, the angle ϕ can be regarded as a constant. Then one has the following formula for the projected p-area:

$$
\begin{aligned}
\mathcal{A}(\Sigma^+|\tilde N(p)^\perp) &= \iint_{q\in\Sigma} |L_{qp^{-1}*}\tilde N(p)\cdot N(q)|\,d\Sigma_q\\
&= \iint_{q\in\Sigma} r\sqrt{A^2+B^2}\left|\frac{A}{\sqrt{A^2+B^2}}\cos(\theta-\phi) + \frac{B}{\sqrt{A^2+B^2}}\sin(\theta-\phi)\right| dr\,d\phi\\
&:= \iint_{q\in\Sigma} r\sqrt{(h_r^+)^2 + r^2}\,\left|\cos\psi\cos(\theta-\phi) + \sin\psi\sin(\theta-\phi)\right| dr\,d\phi\\
&= \int_0^R r\sqrt{(h_r^+)^2+r^2}\Big(\int_0^{2\pi} |\cos(\theta-\phi-\psi)|\,d\phi\Big)dr = C\int_0^R r\sqrt{(h_r^+)^2+r^2}\,dr
\end{aligned}
\tag{3.8}
$$

for some constant C. Here we have used the fact that the angles ψ and θ are independent of ϕ. By using the assumptions for the functions $h^\pm(r)$ and applying the same argument as in Proposition 1.4 (below equation (3.1)), we conclude that the projected p-area $\mathcal{A}(\Sigma|\tilde N(p)^\perp) = \mathcal{A}(\Sigma^+|\tilde N(p)^\perp) + \mathcal{A}(\Sigma^-|\tilde N(p)^\perp)$ is a constant, as desired. This completes the proof of Theorem 1.7. ∎

As we have seen, Cauchy's surface area formula in $\mathbb{R}^n$ gives the measure of codimension one for convex subsets, and it can be generalized to the measures of any higher codimensions. The related notion in integral geometry is called quermassintegral, see [29]. Recently, the author showed several results of integral geometry in $\mathbb{R}^n$ to $\mathbb{H}^n$ (see [15] for Crofton's formula and the containment problems, and [16]). However, it is still not clear if we can have a natural concept for quermassintegral in$\mathbb{H}_n$, $n \geq 1$, to deal with higher-codimension measure for subsets. It is also not clear if numerous concepts (e.g., the support functions and the intrinsic volumes for convex bodies) in convex geometry can possibly be developed in $\mathbb{H}^n$. Those might be the interesting topics worth to study for future research.

Acknowledgements. The author thanks Prof. Hung-Lin Chiu (National Tsing Hua University, Taiwan) for the suggestion of considering rotationally symmetric surfaces.

Funding. This work was supported by Ministry of Science and Technology, Taiwan, grant number: 108-2115-M-024-007-MY2.

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Received April 28, 2021; revised September 30, 2021. Published online December 4, 2021.

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