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Holomorphic semigroups and Sarason's characterization of vanishing mean oscillation

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Abstract. It is a classical theorem of Sarason that an analytic function of bounded mean oscillation (BMOA) is of vanishing mean oscillation if and only if its rotations converge in norm to the original function as the angle of the rotation tends to zero. In a series of two papers, Blasco et al. have raised the problem of characterizing all semigroups of holomorphic functions (φ_t) that can replace the semigroup of rotations in Sarason's theorem. We give a complete answer to this question, in terms of a logarithmic vanishing oscillation condition on the infinitesimal generator of the semigroup (φ_t). In addition, we confirm the conjecture of Blasco et al. that all such semigroups are elliptic. We also investigate the analogous question for the Bloch and the little Bloch spaces, and surprisingly enough, we find that the semigroups for which the Bloch version of Sarason's theorem holds are exactly the same as in the BMOA case.

1. Introduction and main results

A *semigroup of analytic self maps* of the unit disc $\mathbb{D}$ is the flow of a (unique) holomorphic vector field G on $\mathbb{D}$ which is defined for all positive times $t \geq 0$. In other words it is the solution $(\varphi_t : \mathbb{D} \mapsto \mathbb{D})_{t\geq 0}$ of the Cauchy problem

$$\begin{cases} G(\varphi_t(z)) = \dfrac{\partial \varphi_t(z)}{\partial t}, \\ \varphi_0(z) \equiv z, \end{cases} \tag{1.1}$$

when this exists. In particular, G is called the *infinitesimal generator* of the semigroup. From the dynamical viewpoint, the continuous version of Denjoy–Wolff's theorem (see Theorem 8.3.1 in [11]) guarantees the existence of a unique point $\tau \in \overline{\mathbb{D}}$, which is called the *Denjoy–Wolff point* of (φ_t), and φ_t converges to τ uniformly on compact sets as $t \to +\infty$, except when (φ_t) consists of elliptic automorphisms of $\mathbb{D}$. This allows for a first classification of semigroups: the *elliptic* ones, when $\tau \in \mathbb{D}$, and the *non-elliptic*, when $|\tau| = 1$.

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There has been an increasing amount of literature on semigroups of analytic functions studying both the dynamical features of semigroups, such as slopes of orbits [7, 11], rates of convergence to the Denjoy–Wolff point [8] and boundary fixed points [13] to name a few, but also studying holomorphic semigroups in the context of (holomorphic) function spaces. This approach was pioneered by Berkson and Porta [6]. In this paper we will focus on some questions regarding this latter aspect.

Let $\mathcal{H}(\mathbb{D})$ be the Frechét space of holomorphic functions in the unit disc. In [6], Berkson and Porta considered the *semigroup of composition operators*

$$C_t(f) := f \circ \varphi_t, \quad f \in \mathcal{H}(\mathbb{D}),$$

associated to a semigroup (φ_t). They proved that each (φ_t) induces a *strongly continuous semigroup of operators* [17], Section 34, on the classical Hardy space of analytic functions H^p, $p > 0$, i.e., that each operator C_t is a bounded linear operator on H^p, the semigroup identity is satisfied, $C_t \circ C_s = C_{t+s}$, and furthermore C_t converges to the identity operator in the strong operator topology as $t \searrow 0$. Their work has been quite influential, and has naturally led to analogous considerations in a variety of spaces of analytic functions in the unit disc, among them the classical weighted Bergman spaces [22] and the Dirichlet space [23]. It turns out that the original results of Berkson and Porta continue to hold in these different settings virtually invariable. That is, the composition operators C_t associated to any given semigroup of analytic functions (φ_t) form a strongly continuous semigroup of composition operators in all these spaces.

A different phenomenon arises when one considers some of the most well known *non-separable* spaces of analytic functions. The first one to notice this, although not in the language of semigroup theory, was Sarason [21], in the setting of BMOA; the space of analytic functions of bounded mean oscillation. For more background on these spaces, the reader is referred to Section 2.

Theorem A (Sarason's theorem). *Let $\rho_t(z) = e^{it}z$, $t \geq 0$, be the family of rotations in the unit disc. Then for a function $f \in$ BMOA,*

$$\lim_{t \searrow 0} \|f \circ \rho_t - f\|_{\mathrm{BMOA}} = 0$$

if and only if f is of vanishing mean oscillation (VMOA).

As a matter of fact, the rotations $(\rho_t)_{t \geq 0}$ form a semigroup of analytic functions, and Sarason's theorem shows that the composition semigroup induced by (ρ_t) is *not* strongly continuous, rather there exists a *maximal closed subspace* of BMOA on which rotations induce a strongly continuous composition semigroup. We should mention here that Sarason formulates his theorem in the space of functions of bounded mean oscillation in the real line, but as he notices [21], p. 1, the result reported here is an equivalent reformulation of his.

Motivated by this observation, in a series of two papers [9, 10], Blasco, Contreras, Díaz-Madrigal, Martínez, Papadimitrakis and Siskakis studied composition semigroups in BMOA and in the Bloch space $\mathcal{B}$ as long as in their "little-oh" versions, VMOA and $\mathcal{B}_0$. It turns out that strong continuity depends on the characteristics of each specific semigroup (φ_t), which led the authors to introduce the *maximal subspace of strong continuity*:

$$[\varphi_t, X] := \{f \in X : \lim_{t \searrow 0} \|C_t(f) - f\|_X = 0\},$$

i.e., the maximal linear subspace on which (φ_t) induces a strongly continuous composition semigroup. It can be proven that when $X = \mathrm{BMOA}$ or $\mathcal{B}$, for all semigroups this is a closed subspace of X, see Proposition 1 in [10]. Furthermore, each (φ_t) generates a strongly continuous semigroup (C_t) in VMOA and $\mathcal{B}_0$, hence in general we have

$$X_0 \subseteq [\varphi_t, X] \subseteq X, \tag{1.2}$$

where $X_0 = \mathrm{VMOA}$ or $\mathcal{B}_0$.[1]

In view of Sarason's theorem, it is quite natural to ask about those (φ_t) for which the maximal subspace of strong continuity is *minimal*, in the sense of equation (1.2). In other words we want to find a characterization of semigroups for which $[\varphi_t, X] = X_0$.

Blasco et al. [10] prove that the following "logarithmic vanishing Bloch" condition on the infinitesimal generator:

$$\lim_{|z|\nearrow 1} \frac{1-|z|^2}{G(z)} \log \frac{1}{1-|z|^2} = 0, \tag{LVB}$$

is sufficient so that $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ holds. On the opposite direction, they prove that $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ implies (LVB) under the *a priori* hypothesis

$$\limsup_{|z|\nearrow 1} \frac{1-|z|^2}{|G(z)|} \log \frac{1}{1-|z|^2} < +\infty. \tag{1.3}$$

It should be noted that non-elliptic semigroups always fail to satisfy (1.3), therefore the theorem provides no information on the non-elliptic case. Furthermore, there exist non trivial elliptic semigroups that also fail to satisfy (1.3).

In the same work, the authors also investigate the case $[\varphi_t, \mathrm{BMOA}] = \mathrm{VMOA}$. The sufficient condition they obtain is a variation of the logarithmic vanishing Bloch condition adapted to the nature of BMOA. For obvious reasons, we shall call it a "logarithmic vanishing mean oscillation" condition:

$$\lim_{|a|\nearrow 1} \left(\log \frac{e}{1-|a|^2}\right)^2 \int_{\mathbb{D}} \frac{1-|\phi_a(z)|^2}{|G(z)|^2}\, dm(z) = 0, \tag{LVMO}$$

where $\phi_a(z) := (a-z)/(1-\bar{a}z)$ and dm is the normalized Lebesgue measure on $\mathbb{D}$. Similarly to the Bloch case, the necessity of this condition is proved under the assumption that

$$\limsup_{|a|\nearrow 1} \left(\log \frac{e}{1-|a|^2}\right)^2 \int_{\mathbb{D}} \frac{1-|\phi_a(z)|^2}{|G(z)|^2}\, dm(z) < +\infty. \tag{1.4}$$

It is known, and quite straightforward to verify, that (LVMO) $\Rightarrow$ (LVB), hence the sufficient condition in the BMOA case is apparently stronger than the one for the Bloch space, and analogously (1.4) $\Rightarrow$ (1.3). Hence, a fortiori, all non-elliptic semigroups and some elliptic ones fail to satisfy (1.4).

[1] In the rest of the paper we shall use the shorthand X and X_0 to mean that X is the Bloch space or BMOA and X_0 is either the little Bloch space or VMOA, respectively.

In view of the above results, it is unclear whether there exist non-elliptic semigroups such that the maximal subspace is minimal (either in the Bloch space or in BMOA). This problem has been already posed as a question in [9], Question 2. In this direction, the authors in [9] provide some necessary conditions for the minimality of the maximal subspace. Suppose that (φ_t) is a non-elliptic semigroup with Denjoy–Wolff point τ. Then Berkson–Porta's formula [11], Theorem 10.1.10, gives the following representation of G:

$$G(z) = (z - \tau)(\bar{\tau} z - 1)\, p(z),$$

where p is a holomorphic function of non-negative real part. Therefore p has a Herglotz representation by some non-negative Borel measure μ, supported on $\partial\mathbb{D}$. In Corollary 5 of [9], the authors prove that if $[\varphi_t, X] = X_0$, then μ has no atoms, i.e., $\mu\{\zeta\} = 0, \forall \zeta \in \partial\mathbb{D}$. Furthermore, they prove ([9], Corollary 6) that if (φ_t) is non-elliptic and $[\varphi_t, X] = X_0$, then the Koenigs function h satisfies

$$h \in \Big(\bigcap_{p<\infty} H^p \Big) \setminus \mathrm{BMOA}, \tag{1.5}$$

where H^p is the classical Hardy space.

We have been able to answer these questions, providing a complete characterization of the semigroups for which $[\varphi_t, X] = X_0$.

Theorem 1.1. *Let (φ_t) be a semigroup of analytic functions with infinitesimal generator G. The following are equivalent.*

(a) $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$,

(b) $[\varphi_t, \mathrm{BMOA}] = \mathrm{VMOA}$,

(c) (φ_t) *is an elliptic semigroup and G satisfies the logarithmic vanishing Bloch condition* (LVB),

(d) (φ_t) *is an elliptic semigroup and G satisfies the logarithmic vanishing mean oscillation condition* (LVMO).

The surprising aspect of this theorem is that not only the sufficient conditions of Blasco et al. are also necessary for the minimality of the maximal subspace under no further assumptions, but quite unexpectedly, the two conditions are equivalent. Hence the class of holomorphic semigroups which can replace the rotations in Sarason's theorem is exactly the same for the Bloch space and for BMOA. In particular, the implications (a) ⇒ (c) and (b) ⇒ (d) answer in the affirmative the question of Blasco et al. [9], i.e., that no non-elliptic semigroup has maximal subspace which coincides with either $\mathcal{B}_0$ or VMOA.

Plan of the paper

In Section 2 we give a quick overview of the concepts that will go into the proof of the main theorem. In particular, we discuss in more detail semigroups of analytic functions and the spaces BMOA and $\mathcal{B}$, as well as some weighted versions of them. In Section 3 we prove the main theorem. In fact, the central part of the proof is a construction presented in Proposition 3.5.

2. Background

In this section we shall discuss some of the background material and introduce some notation that we are going to use later. In the unit disc $\mathbb{D}$, we denote by δ the hyperbolic distance,

$$\delta(a,z) := \frac{1}{2}\log\frac{1+|\phi_a(z)|}{1-|\phi_a(z)|}, \quad \text{where} \quad \phi_a(z) := \frac{a-z}{1-\bar{a}z}, \quad a,z\in\mathbb{D}.$$

This is the distance corresponding to the hyperbolic Riemannian metric $ds/(1-s^2)$. The metric space $(\mathbb{D},\delta)$ is a model of the hyperbolic plane usually called the *Poincaré disc*. The functions ϕ_a are isometric automorphisms of the Poincaré disc and are also involutions ($\phi_a^{-1} = \phi_a$).

For a holomorphic function f defined on $\mathbb{D}$, we define its *hyperbolic translation* f_a with respect to $a \in \mathbb{D}$ as

$$f_a(z) := f(\phi_a(z)) - f(a).$$

Another fact that is going to be used repeatedly is the following approximation for the hyperbolic distance of a point $z \in \mathbb{D}$ to the origin:

$$1+\delta(0,z) \approx \log\frac{e}{1-|z|^2}.$$

Let us now take a closer look to holomorphic semigroups. An equivalent way to define a holomorphic semigroup (φ_t) is as a family $\{\varphi_t : t \geq 0\}$ of analytic self maps of the unit disc $\varphi_t\colon \mathbb{D}\to\mathbb{D}$ such that

(1) $\varphi_0(z) \equiv z$,

(2) $\varphi_t \circ \varphi_s = \varphi_{t+s}, \quad t,s \geq 0$,

(3) $\varphi_t(z) \to z$ uniformly on compact subsets of $\mathbb{D}$, as $t \searrow 0$.

It turns out that if (φ_t) is a semigroup then each φ_t is univalent, see Theorem 8.1.17 in [11]. In addition, for all members of a semigroup (φ_t) (other than the hyperbolic rotations), there exists a common "fixed point" $\tau \in \overline{\mathbb{D}}$ for which

$$\lim_{t\to\infty}\varphi_t(z) = \tau, \quad z\in\mathbb{D},$$

usually called the Denjoy–Wolff point of (φ_t). The concept of Denjoy–Wolff point of a semigroup plays a key role in the semigroup theory, and we can classify semigroups with respect to their Denjoy–Wolff point, τ, as follows (see Theorem 8.3.1 in [11]):

(1) If $\tau \in \mathbb{D}$, then there exists $\lambda \in \mathbb{C}\setminus\{0\}$ with $\Re(\lambda) \geq 0$ such that

$$\varphi_t'(\tau) = e^{-\lambda t}, \quad t \geq 0.$$

In addition, we either have $|\varphi_t'(\tau)| = 1$ for every $t > 0$, or $|\varphi_t'(\tau)| < 1$ for every $t > 0$.

(2) If $\tau \in \partial\mathbb{D}$, then there exists $\lambda \geq 0$ such that

$$\angle\lim_{z\to\tau}\varphi_t'(\tau) = e^{-\lambda t}, \quad t \geq 0.$$

The number λ is called the spectral value of the semigroup. We say that (φ_t) is *elliptic* if $\tau \in \mathbb{D}$, *parabolic* if $\tau \in \partial\mathbb{D}$ with spectral value $\lambda = 0$, and *hyperbolic* if $\tau \in \partial\mathbb{D}$ with spectral value $\lambda > 0$. The semigroup is called *non-elliptic* if it is either parabolic or hyperbolic.

If (φ_t) is a semigroup, then the limit

$$G(z) = \lim_{t\searrow 0} \frac{\varphi_t(z) - z}{t}$$

exists uniformly on compact subsets of $\mathbb{D}$. The function $G \in \mathcal{H}(\mathbb{D})$ is the infinitesimal generator of (φ_t) and characterizes the semigroup in a unique way. In addition, G satisfies the following relations:

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\,\frac{\partial \varphi_t(z)}{\partial z}, \quad z \in \mathbb{D},\ t \geq 0. \tag{2.1}$$

Due to the Berkson–Porta formula [6], we can represent the infinitesimal generator G in terms of the Denjoy–Wolff point τ of the semigroup as

$$G(z) = (\bar{\tau} z - 1)(z - \tau)p(z), \quad z \in \mathbb{D}, \tag{2.2}$$

where $\tau \in \overline{\mathbb{D}}$ and $p \in \mathcal{H}(\mathbb{D})$ with $\Re(p(z)) \geq 0$ for all $z \in \mathbb{D}$. Conversely, every function of this form is the infinitesimal generator of a holomorphic semigroup.

A geometric description of all holomorphic semigroups is provided by the so called *Koenigs function*, a conformal map which conjugates a given semigroup (φ_t) to a model semigroup.

When (φ_t) is an elliptic semigroup, with Denjoy–Wolff point $\tau \in \mathbb{D}$, the function h is the unique conformal map such that $h(\tau) = 0$, $h'(\tau) = 1$ and

$$h(\phi_t(z)) = e^{-\lambda t} h(z), \quad z \in \mathbb{D},\ t \geq 0.$$

In addition, we have that $h'(z)/h(z) = -\lambda/G(z)$. In the non-elliptic case, h is the unique conformal map such that $h(0) = 0$ and

$$h(\phi_t(z)) = h(z) + it, \quad z \in \mathbb{D},\ t \geq 0.$$

In this case, we have that $h'(z) = i/G(z)$.

For a semigroup (φ_t) with infinitesimal generator G and Denjoy–Wolff point τ, following the notation used in [9], Definition 4, we consider the function $\gamma\colon \mathbb{D} \to \mathbb{C}$, which we will call the *associated* γ*-symbol* of (φ_t). This function is defined as follows: if $\tau \in \mathbb{D}$, then

$$\gamma(z) := \int_{\tau}^{z} \frac{\zeta - \tau}{G(\zeta)}\, d\zeta,$$

while if $\tau \in \partial\mathbb{D}$, then

$$\gamma(z) := \int_{0}^{z} \frac{i}{G(\zeta)}\, d\zeta.$$

In the case where $\tau \in \partial\mathbb{D}$, then γ coincides with h, while if $\tau \in \mathbb{D}$, then $\gamma'(z) = -\frac{z-\tau}{\lambda}\frac{h'(z)}{h(z)}$.

The maximal subspace of strong continuity, for a semigroup of composition operators (C_t), can also be described in terms of the infinitesimal generator G, see Theorem 1 in [9]. If (C_t) acts on a Banach space $\mathcal{X}$ of analytic functions in the unit disc which contains the constant functions, and in addition we have that $\sup_{t\leq 1} \|C_t\|_{\mathcal{X}} < \infty$, then

$$[\varphi_t, \mathcal{X}] = \overline{\{f \in \mathcal{X} : Gf' \in \mathcal{X}\}}. \tag{2.3}$$

This description already indicates a connection between the maximal subspace and the so called *generalized Volterra operator*, defined for an analytic symbol g as

$$T_g(f)(z) := \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \quad f \in \mathcal{H}(\mathbb{D}).$$

This operator was first introduced by Pommerenke [19], who studied its boundedness properties on the Hardy space H^2 in connection to the analytic John–Nirenberg inequality. Since then, several authors studied these operators focusing on conditions on the symbol g under which T_g is bounded or compact. The survey papers [1, 24] contain much more information on the generalized Volterra operator.

It turns out that if $\mathcal{X}$ is Banach space of analytic functions, and g is the associated γ-symbol of (φ_t), under mild additional assumptions on $\mathcal{X}$, we have the following characterization for the maximal subspace of strong continuity (see Proposition 2 in [9]):

$$[\varphi_t, \mathcal{X}] = \overline{\mathcal{X} \cap (T_\gamma(\mathcal{X}) \oplus \mathcal{C})}, \tag{2.4}$$

where $\mathcal{C}$ is the set of all constant functions.

Finally, we introduce some definitions and we recall some theorems regarding the Banach spaces of analytic functions we are interested in. The space BMOA is the space of all analytic functions in the Hardy space H^2 which have bounded mean oscillation. Having to choose between many equivalent descriptions, we will use a description in terms of Carleson measures. We say that $f \in$ BMOA if and only if

$$\|f\|_*^2 := \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1-|z|^2)\, dm(z) < \infty,$$

where I is any arc on $\partial\mathbb{D}$ and $|I|$ is its length. Also, $S(I)$ is the so called *Carleson box*, which for us will be the closed hyperbolic halfplane in the Poincaré disc which has I as its boundary. There exist more "square" versions of Carleson boxes, but the invariant nature of this definition will simplify some of our computations.

The space BMOA is a Banach space, equipped with the norm

$$\|f\|_{\mathrm{BMOA}} := |f(0)| + \|f\|_*.$$

The closure of all polynomials in BMOA is the space VMOA, which has an equivalent description in terms of the following vanishing Carleson condition:

$$\lim_{|I|\searrow 0} \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1-|z|^2)\, dm(z) = 0.$$

The space BMOA is a subspace of the well-known Bloch space, denoted by $\mathcal{B}$. We say that a function $f \in \mathcal{H}(\mathbb{D})$ belongs to $\mathcal{B}$ if and only if

$$\sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|^2) < \infty.$$

The closure of polynomials in the Bloch norm is called the little Bloch space, denoted by $\mathcal{B}_0$. Equivalently, $f \in \mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1} |f'(z)|\,(1-|z|^2) = 0.$$

The spaces $\mathcal{B}$ and $\mathcal{B}_0$ are Banach spaces equipped with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|^2).$$

For more information on these spaces, see [26].

In Definition 3 of [9], the authors consider some weighted versions of BMOA and $\mathcal{B}$ which are closely related to the conditions (LVMO) and (LVB). Let $f \in \mathcal{H}(\mathbb{D})$. Then we say that f belongs to $\mathrm{BMOA}_{\log}$ if and only if

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\left(\log \frac{e}{|I|}\right)^2}{|I|} \int_{S(I)} |f'(z)|^2(1-|z|^2)\, dm(z) < \infty, \tag{2.5}$$

and $f \in \mathrm{VMOA}_{\log}$ if and only if

$$\lim_{|I|\to 0} \frac{\left(\log \frac{e}{|I|}\right)^2}{|I|} \int_{S(I)} |f'(z)|^2(1-|z|^2)\, dm(z) = 0. \tag{2.6}$$

Respectively, we say that f belongs to $\mathcal{B}_{\log}$ if and only if

$$\sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|^2)\left(\log \frac{e}{1-|z|^2}\right) < \infty, \tag{2.7}$$

and $f \in \mathcal{B}_{\log,0}$ if and only if

$$\lim_{|z|\to 1} |f'(z)| \left(\log \frac{e}{1-|z|^2}\right) (1-|z|^2) = 0. \tag{2.8}$$

These spaces naturally appeared in the study of multipliers for BMOA and the Bloch space. A function g is a pointwise multiplier of $\mathcal{B}$, i.e., $gf \in \mathcal{B}$ for all $f \in \mathcal{B}$, if and only if $g \in H^\infty \cap \mathcal{B}_{\log}$, [12], where H^∞ is the space of bounded analytic functions in the unit disc. An analogous result holds for BMOA: g is a multiplier for BMOA if and only if $g \in H^\infty \cap \mathrm{BMOA}_{\log}$, [18]. For our purposes, these spaces are interesting because they characterize the boundedness and compactness of T_g on BMOA and $\mathcal{B}$, see Theorems 5 and 6 in [9], and [16, 25].

Theorem B. *The operator* $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ *is bounded if and only if* $g \in \mathrm{BMOA}_{\log}$. *Furthermore, the following are equivalent*:

(i) $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ *is compact,*

(ii) $g \in \mathrm{VMOA}_{\log}$,

(iii) $T_g : \mathrm{BMOA} \to \mathrm{BMOA}$ *is weakly compact.*

An analogous result holds for the Bloch space. The operator $T_g : \mathcal{B} \to \mathcal{B}$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$. *Furthermore, the following are equivalent*:

(I) $T_g : \mathcal{B} \to \mathcal{B}$ *is compact,*

(II) $g \in \mathcal{B}_{\log,0}$,

(III) $T_g : \mathcal{B} \to \mathcal{B}$ *is weakly compact.*

In addition, Gantmacher's theorem [3], Theorem 5.23, implies that $T_g : X \to X$ is weakly compact if and only if $T_g(X) \subseteq X_0$.

Notation

For two quantities A and B depending on a number of parameters, we shall write $A \lesssim B$ if there exists some positive constant $C > 0$ not depending on the parameters such that $A \leq CB$. The set of parameters in question should be clear from the context. Similarly, we shall write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

3. Proof of main results

We can now turn towards the proof of Theorem 1.1, which can be divided on a macroscopic scale in two main parts. The first one regards the equivalence of parts (c) and (d) of Theorem 1.1. This equivalence is actually another manifestation of the rigidity properties of univalent functions.

The idea that we employ already appeared in [20], and was further refined in [5]. Here we shall adapt it in the weighted setting relevant to our problem. Although we could do all calculations for the logarithmic weight, we prefer to work with a more general class of weights since we think that this renders more clearly the idea of the proof. Let ω be a strictly positive weight of the class $C^1(\mathbb{D})$. We assume the following regularity condition on ω:

$$(1 - |z|^2)|\nabla\omega(z)| \leq C_\omega \, \omega(z), \quad \forall z \in \mathbb{D}, \tag{3.1}$$

for some $C_\omega > 0$.

Lemma 3.1. *Suppose that* ω *is a weight which satisfies* (3.1) *with some constant* $C_\omega < 1$. *Let also* $f \in \mathcal{H}(\mathbb{D})$ *be such that*

$$|f'(z)|(1 - |z|^2)\omega(z) \leq K, \quad \forall z \in \mathbb{D},$$

where $K > 0$. *Then,*

$$\int_0^1 \sup_{a \in \mathbb{D}, |z| \leq r} (\omega(a)|f_a(z)|)^2 \, dr < +\infty.$$

Proof. Let us start with a local oscillation estimate on ω. Let $z = re^{i\theta}$. We have

$$\log \frac{\omega(z)}{\omega(0)} \leq \int_0^r \frac{|\nabla\omega(se^{i\theta})|}{\omega(se^{i\theta})} \, ds \leq C_\omega \, \delta(0, z).$$

Since condition (3.1) is invariant under composition with Möbius transformations, see Proposition 3.1 in [2], we have that

$$\omega(z) \le e^{C_\omega \,\delta(z,w)}\, \omega(w), \quad z, w \in \mathbb{D}.$$

We proceed now with an estimate of the quantity that appears in the lemma. For $z \in \mathbb{D}$, we have

$$|f_a'(z)|\,(1-|z|^2) = |f'(\phi_a(z))|\,(1-|\phi_a(z)|^2) \;\le K\,\omega(\phi_a(z))^{-1}.$$

Hence for $z = re^{i\theta}$ we have

$$\begin{aligned}\omega(a)\,|f_a(z)| &\le K \int_0^r \frac{\omega(a)}{\omega(\phi_a(te^{i\theta}))}\,\frac{dt}{1-t^2} \le K \sup_{\{w\,:\,\delta(a,w)\le\delta(0,r)\}} \frac{\omega(a)}{\omega(w)}\,\delta(0,r)\\ &\le K\, e^{C_\omega\,\delta(0,r)}\,\delta(0,r).\end{aligned}$$

The lemma follows from the fact that the function

$$e^{2C_\omega \delta(0,r)}\,\delta(0,r)^2 = \Big(\frac{1+r}{1-r}\Big)^{C_\omega}\Big(\frac{1}{2}\log\frac{1+r}{1-r}\Big)^2$$

is integrable in $(0,1)$ if $C_\omega < 1$. ∎

The next proposition is a weighted version of Pommerenke's result [20].

Proposition 3.2. *Let $f\colon \mathbb{D} \to \mathbb{C}$ be univalent and let ω be a weight as in Lemma 3.1. Suppose also that*

$$\lim_{|z|\nearrow 1} |f'(z)|\,(1-|z|^2)\,\omega(z) = 0. \tag{3.2}$$

Then,

$$\lim_{|a|\nearrow 1} \omega(a)^2 \int_{\mathbb{D}} |f'(z)|^2\,(1-|\phi_a'(z)|^2)\,dm(z) = 0. \tag{3.3}$$

Proof. Let f be such a function. Setting $\mathbb{D}_r = \{z \in \mathbb{D} : |z| \le r\},\ r \in [0,1)$, we have

$$\begin{aligned}\omega(a)^2 \int_{\mathbb{D}} |f'(w)|^2\,(1-|\phi_a(w)|^2)\,dm(w) &= \omega(a)^2 \int_0^1 \int_{\mathbb{D}_r} |f_a'(z)|^2\;dm(z)\,dr\\ &= \omega(a)^2 \int_0^R \int_{\mathbb{D}_r} |f_a'(z)|^2\;dm(z)\,dr + \omega(a)^2 \int_R^1 \int_{\mathbb{D}_r} |f_a'(z)|^2\;dm(z)\,dr\\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Since f_a is univalent, the inner integral is the normalized area of the image $f_a(\mathbb{D}_r)$, hence

$$\int_{\mathbb{D}_r} |f_a'(z)|^2\;dm(z) \le \sup_{z\in\mathbb{D}_r} |f_a(z)|^2.$$

Let now $\varepsilon > 0$. By Lemma 3.1, there exists some $R_0 < 1$ such that

$$\mathrm{II} \leq \omega(a)^2 \int_{R_0}^1 \sup_{z \in \mathbb{D}_r} |f_a(z)|^2 \, dr < \varepsilon, \quad \forall a \in \mathbb{D}.$$

In order to estimate the integral I, notice first that the oscillation estimate in the proof of Lemma 3.1, for $z \leq R_0$, gives

$$\frac{\omega(a)}{\omega(\phi_a(z))} \leq e^{C_\omega \delta(0,R_0)}, \quad a \in \mathbb{D}.$$

Hence we have

$$\begin{aligned}
\mathrm{I} &= \omega(a)^2 \int_0^{R_0} \int_{\mathbb{D}_r} \Big(\frac{|f'(\phi_a(z))|(1-|\phi_a(z)|^2)}{1-|z|^2} \Big)^2 \, dm(z)\, dr \\
&= \int_0^{R_0} \int_{\mathbb{D}_r} \Big(\frac{\omega(a)}{\omega(\phi_a(z))} \Big)^2 \Big(\frac{|f'(\phi_a(z))|(1-|\phi_a(z)|^2)\,\omega(\phi_a(z))}{1-|z|^2} \Big)^2 \, dm(z)\, dr \\
&\leq e^{2C_\omega \delta(0,R_0)} \int_0^{R_0} \frac{1}{(1-r^2)^2} \int_{\mathbb{D}_r} |f'(\phi_a(z))|^2 (1-|\phi_a(z)|^2)^2 \omega(\phi_a(z))^2 \, dm(z)\, dr.
\end{aligned}$$

Now, since f satisfies (3.2), we can find $R_1 < 1$ such that

$$\omega(w)\,(1-|w|^2)\,|f'(w)| \leq e^{-C_\omega \delta(0,R_0)} (1-R_0^2)\sqrt{\varepsilon}, \quad \forall w : R_1 < |w| < 1.$$

Finally, there exists some $\delta > 0$ such that $|\phi_a(z)| > R_1$, if $|z| \leq R_0$ and $|a| > 1-\delta$. This gives

$$\mathrm{I} \leq (1-R_0^2)^2 \int_0^{R_0} \frac{1}{(1-r^2)^2} \, dr\, \varepsilon \leq \varepsilon.$$

Therefore we have proved that for each $\varepsilon > 0$, we can find $\delta > 0$ such that for $|a| > 1-\delta$,

$$\omega(a)^2 \int_{\mathbb{D}} |f'(w)|^2 (1-|\phi_a(w)|^2)\, dm(w) \leq 2\varepsilon. \qquad \blacksquare$$

Corollary 3.3. *If f is univalent, then*

$$f \in \mathcal{B}_{\log,0} \text{ if and only if } f \in \mathrm{VMOA}_{\log}.$$

Proof. It is sufficient to prove the direct implication. Consider the weight $\omega_K(z) := \log \frac{K}{1-|z|^2}$. For some $K > 0$ large enough, ω_K satisfies the hypothesis of Lemma 3.1. Then $f \in \mathcal{B}_{\log,0}$ is equivalent to (3.2), hence it satisfies (3.3), which is equivalent to $f \in \mathrm{VMOA}_{\log}$. $\blacksquare$

We now turn to the second part of the proof. Roughly speaking, the characterization of the maximal subspace of strong continuity as $\overline{(T_\gamma(X) \oplus \mathcal{C})} \cap X$ by Blasco et al. [9] allows us to approach the problem of studying the maximal subspace of strong continuity purely by functional analytic methods. The central part of the proof will therefore follow from a construction of a function in the range of T_g under some technical assumptions on g. The construction turns out to be quite explicit by pasting together some holomorphic "building blocks". These so called building blocks behave much like the logarithmic function $\ell_w(z) := \log(\frac{e}{1-\bar{w}z})$ in the sense that at a prescribed point (in this case w) achieves the biggest possible growth while keeping the BMOA or Bloch norm below a fixed threshold. Our construction requires some improved decay properties away from the point w. The exact definition of these functions is presented in the next lemma.

Lemma 3.4. *For a point $w \in \mathbb{D}$, we denote by w^* the hyperbolic midpoint between 0 and w. Let also I_w be the closed arc in the unit circle such that w is the point in $S(I_w)$ closest to the origin (see Figure 1). Then the function*

$$\beta_w(z) := \log \frac{e}{1 - \phi_{w^*}(z)\,\overline{w}}$$

satisfies the following properties:

(i) $\|\beta_w\|_{\mathcal{B}} \lesssim \|\beta_w\|_* \lesssim 1$,

(ii) $\Re \beta_w \geq 0$,

(iii) $|\Im \beta_w| \leq \pi/2$,

(iv) $\Re \beta_w(z) \approx \log \frac{e}{1-|w|^2}$ *for* $z \in S(I_w)$,

(v) *if $z \notin S(I_{w^*})$, then $|\beta_w(z)| \leq c_0$, where c_0 is an absolute constant. In particular, for all $\delta > 0$ there exists $\delta' > 0$ such that if $1 - |w| \leq \delta'$ and $1 - |z| \geq \delta$, then $|\beta_w(z)| \leq c_0$.*

Proof. To prove part (i), notice that the Möbius invariant part of the norm does not change after composing the function $\log \frac{e}{1-\overline{w}z}$ with the Möbius transformation ϕ_{w^*}. Also,

$$\beta_w(0) = \log \frac{e}{1 + w^*\,\overline{w}} \leq 1.$$

Parts (ii) and (iii) follow by the definition of the logarithm. To see (iv), notice that ϕ_{w^*} preserves the diameter passing through w, it maps w^* at 0, and it leaves invariant the hyperbolic distances, therefore it should map $S(I_w)$ to $S(I_{w^*})$. Hence, if $z \in S(I_w)$ and we define $y := \phi_{w^*}(z) \in S(I_{w^*})$,

$$\Re \beta_w(z) = \log \frac{e}{|1 - y\overline{w}|} \gtrsim \log \frac{e}{1 - |w^*|^2} \approx \log \frac{e}{1 - |w|^2}.$$

A similar geometric reasoning as before shows that $\phi_{w^*}(\mathbb{D} \setminus S(I_{w^*}))$ is the half disc which contains $-w$ and defined by the diameter perpendicular to the radius passing from w. Let therefore $z \in \mathbb{D}$ in this half plane, or equivalently $\Re(z\overline{w}) \leq 0$. Hence,

$$\left|\log \frac{e}{1 - z\overline{w}}\right| \leq \log \frac{e}{|1 - z\overline{w}|} + \frac{\pi}{2} \leq 1 + \frac{\pi}{2} \leq 3.$$

To verify the second part of property (v), it remains only to notice that for every $\delta > 0$ there exists $\delta' > 0$ such that if $1 - |w| \leq \delta'$, then the disc $\{|z| \leq 1 - \delta\}$ is contained in $\mathbb{D} \setminus S(I_{w^*})$. ∎

The next proposition is the main technical tool in the proof of our main theorem.

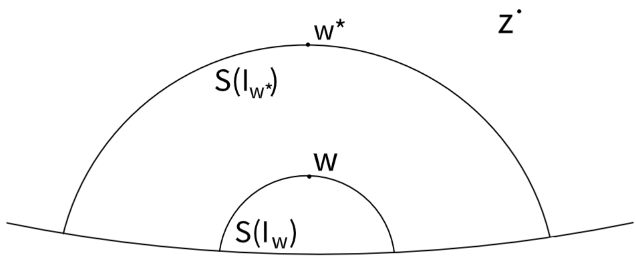

Figure 1. The construction in Lemma 3.4.

Proposition 3.5. *Let* $g \in \mathrm{BMOA} \setminus \mathrm{VMOA}_{\log}$. *Then there exists a function* $F \in \mathrm{BMOA}$ *such that* $T_g F \in \mathrm{BMOA} \setminus \mathrm{VMOA}$.

Proof. First note that if $g \in \mathrm{BMOA} \setminus \mathrm{VMOA}$, then the function $F \equiv 1$ satisfies the required properties. On the other hand, if $g \in \mathrm{BMOA}_{\log} \setminus \mathrm{VMOA}_{\log}$, then by Theorem B we have that $T_g(\mathrm{BMOA}) \subset \mathrm{BMOA}$ but $T_g(\mathrm{BMOA}) \not\subseteq \mathrm{VMOA}$; therefore, we can find a function F as claimed in the thesis of the theorem. We have therefore reduced the problem to the case $g \in \mathrm{VMOA} \setminus \mathrm{BMOA}_{\log}$.

In order to reduce the number of constants in the proof, we assume without loss of generality that

$$\int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2)\, dm(z) = 1 \le \|g\|_*.$$

Basic reduction of the problem. The basic step of the proof is the construction of a sequence of arcs $\{I_n\}$ and a sequence of functions F_n of the form

$$F_n(z) = \sum_{k=0}^{n} a_k\, \beta_{w_k}(z),$$

for some $w_k \in \mathbb{D}$ such that

(1) the coefficients $\{a_k\}$ satisfy $0 \le a_k \le 2^{-k}$,
(2) for all $n \in \mathbb{N}$ it holds

$$\frac{1}{|I_n|}\int_{S(I_n)} (\Re F_n(z))^2 |g'(z)|^2 (1-|z|^2)\, dm(z) \ge 1,$$

(3) for all $n \in \mathbb{N}$ we have $\|T_g F_n\|_* \le \max\{\|T_g F_{n-1}\|_* + 2^{-n} C(g), C(g)\}$, where $C(g)$ is a positive constant which depends only on g.

Suppose now that we can construct such a sequence of functions. Then we can finish the proof as follows: we have that

$$\sum_{k=0}^{\infty} a_k (\|\beta_{w_k}\|_* + |\beta_{w_k}(0)|) \lesssim \sum_{k=0}^{\infty} a_k < +\infty.$$

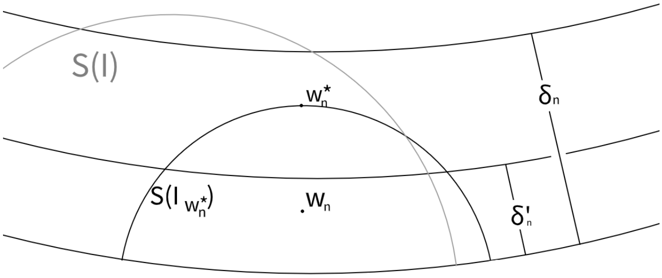

Figure 2. The n-th step in the construction.

Therefore

$$F := \sum_{k=0}^{\infty} a_k \beta_{w_k} \in \text{BMOA}.$$

Applying repeatedly property (3), we find that

$$\|T_g F_n\|_* \leq \max\left\{\|T_g F_0\|_* + \sum_{r=1}^{n} 2^{-r} C(g), C(g)\right\} \leq \|g\|_* + C(g).$$

There is a slight subtlety in the fact that since T_g is not continuous we cannot directly infer from (3) that $T_g F \in$ BMOA. But this problem is easily overcome. It suffices to prove that $|\Im F(z) g'(z)|^2 (1 - |z^2|)\, dm(z)$ and $(\Re F(z))^2 |g'(z)|^2 (1 - |z|^2)\, dm(z)$ are Carleson measures for the Hardy space. The first one is clearly a Carleson measure, since the imaginary part of F is bounded. Let $I \subseteq \partial\mathbb{D}$. By the monotone convergence theorem, we have

$$\begin{aligned}&\frac{1}{|I|} \int_{S(I)} (\Re F(z))^2 |g'(z)|^2 (1 - |z|^2)\, dm(z)\\ &\quad = \lim_{n\to\infty} \frac{1}{|I|} \int_{S(I)} (\Re F_n(z))^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) \leq \lim_{n\to\infty} \|T_g F_n\|_*^2 \leq C(g)^2.\end{aligned}$$

Finally, we should prove that $T_g F \notin$ VMOA. This is a simple consequence of (2):

$$\begin{aligned}&\frac{1}{|I_n|} \int_{S(I_n)} |F(z) g'(z)|^2 (1 - |z|^2)\, dm(z)\\ &\qquad \geq \frac{1}{|I_n|} \int_{S(I_n)} (\Re F_n(z))^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) \geq 1.\end{aligned}$$

Therefore it remains only to construct such a sequence of functions. This will be done in a recursive way.

Recursive definition Let $I_0 = \partial\mathbb{D}$, $w_0 = 0$ and $a_0 = 1$. Consequently $F_0 \equiv 1$ and conditions (1)–(3) are automatically satisfied. For the recursive step, suppose that $I_0, \ldots, I_{n-1}$,

$w_0, \dots w_{n-1}$ and $a_1, \dots a_{n-1}$ are defined and we want to proceed our construction. Since $g \in \mathrm{VMOA}$ and F_{n-1} is bounded, we can find some $\delta_n > 0$ such that

$$\sup_{|I| \le \delta_n} \frac{1}{|I|} \int_{S(I)} |F_{n-1}(z)\, g'(z)|^2 (1 - |z|^2)\, dm(z) \le 1. \tag{3.4}$$

By part (v) of Lemma 3.4, there also exists some $\delta_n' < \delta_n$ such that for any pair of complex numbers $z, w \in \mathbb{D}$ such that $1 - |w| \le \delta_n'$ and $1 - |z| \ge \delta_n$, then $|\beta_w(z)| \le 3$. We might also choose δ_n' such that $\sqrt{\delta_n'} \le 2^{-2n} \delta_n$.

Furthermore, by Lemma 3.4 (iv), since $g \notin \mathrm{BMOA}_{\log}$, there exists some $w_n \in \mathbb{D}$ which satisfies $1 - |w_n| \le \delta_n'$ such that

$$\frac{1}{|I_{w_n}|} \int_{I_{w_n}} \Re(\beta_{w_n}(z))^2\, |g'(z)|^2 (1 - |z|^2)\, dm(z) \ge 2^{2n}. \tag{3.5}$$

Notice furthermore that because g is of vanishing mean oscillation, the supremum

$$M_n^2 := \sup_{|I| \le \delta_n} \frac{1}{|I|} \int_{S(I)} \Re(\beta_{w_n}(z))^2\, |g'(z)|^2 (1 - |z|^2)\, dm(z) \ge 2^{2n} \tag{3.6}$$

is in fact a maximum which is attained for some interval $I_n \subseteq \partial\mathbb{D}$, $|I_n| \le \delta_n$. We claim that the function $F_n := F_{n-1} + M_n^{-1} \beta_{w_n}$ satisfies the required properties. We start by proving properties (1) and (2). By equation (3.6), it is clear that

$$a_n := M_n^{-1} \le 2^{-n}.$$

Also, since β_w has positive real part,

$$\begin{aligned}
&\frac{1}{|I_n|} \int_{S(I_n)} |F_n(z) g'(z)|^2 (1 - |z|^2)\, dm(z) \\
&\qquad \ge \frac{M_n^{-2}}{|I_n|} \int_{S(I_n)} |\Re(\beta_{w_n}(z))^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) = 1.
\end{aligned}$$

Finally, we need to proved the claimed estimate on $\|T_g F_n\|_*$. To obtain this, we consider two cases. First suppose that $|I| \ge \delta_n$. We start with a preliminary estimate:

$$\begin{aligned}
&\frac{1}{|I|} \int_{S(I)} |\beta_{w_n}(z) g'(z)|^2 (1 - |z|^2)\, dm(z) \\
&\qquad \le \frac{|I_{w_n^*}|}{|I|} \frac{1}{|I_{w_n^*}|} \int_{S(I_{w_n^*})} |\beta_{w_n}(z) g'(z)|^2 (1 - |z|^2)\, dm(z) \\
&\qquad\quad + \frac{1}{|I|} \int_{S(I) \setminus S(I_{w_n^*})} |\beta_{w_n}(z) g'(z)|^2 (1 - |z|^2)\, dm(z) \\
&\qquad \lesssim \frac{1 - |w_n^*|}{\delta_n} M_n^2 + \|g\|_*^2 \lesssim \frac{(1 - |w_n|)^{1/2}}{\delta_n} M_n^2 + \|g\|_*^2 \\
&\qquad \lesssim \frac{\delta_n'^{\,1/2}}{\delta_n} M_n^2 + \|g\|_*^2 \le 2^{-2n} M_n^2 + \|g\|_*^2 .
\end{aligned}$$

In this estimate we have used the fact that $|I_{w^*}| \approx 1 - |w_n^*| \approx (1 - |w_n|)^{1/2}$. Now the induction hypothesis together with the above calculation permit us to estimate as follows:

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_{S(I)} |F_n(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \\
&\qquad \le M_n^{-1}\Big(\frac{1}{|I|}\int_{S(I)} |\beta_{w_n}(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \\
&\qquad\quad + \Big(\frac{1}{|I|}\int_{S(I)} |F_{n-1}(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \\
&\qquad \lesssim 2^{-n} + M_n^{-1}\|g\|_* + \|T_g F_{n-1}\|_* \le 2^{-n} C(g) + \|T_g F_{n-1}\|_*,
\end{aligned}$$

where $C(g)$ is a positive constant depending only on g and not on n.

It remains to consider the case $|I| \le \delta_n$. In this case, equation (3.4) allows us to argue as follows:

$$\begin{aligned}
&\Big(\frac{1}{|I|}\int_{S(I)} |F_n(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \le \\
&\quad \le M_n^{-1}\Big(\frac{1}{|I|}\int_{S(I)} |\beta_{w_n}(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \\
&\qquad + \Big(\frac{1}{|I|}\int_{S(I)} |F_{n-1}(z)\, g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} \\
&\quad \lesssim M_n^{-1} \sup_{|I|\le\delta_n} \Big(\frac{1}{|I|}\int_{S(I)} \Re(\beta_{w_n}(z))^2 |g'(z)|^2 (1-|z|^2)\, dm(z)\Big)^{1/2} + \|g\|_* + 1 \\
&\quad \le C(g).
\end{aligned}$$

We have proved that

$$\|T_g F_n\|_* \le \max\{\|T_g F_{n-1}\|_* + 2^{-n} C(g), C(g)\},$$

which completes the induction step and the proof is complete. ∎

We also need the Bloch version of Proposition 3.5.

Proposition 3.6. *Let $g \in \mathcal{B} \setminus \mathcal{B}_{\log,0}$. Then there exists a function $F \in \mathcal{B}$ such that $T_g F \in \mathcal{B} \setminus \mathcal{B}_0$.*

Proof. The proof of this proposition is very similar to the proof of Proposition 3.5, and in fact a bit simpler, therefore we shall give only a rough sketch of it. A similar argument as the one used in the proof of Proposition 3.5 allows us to reduce the problem to the case $g \in \mathcal{B}_0 \setminus \mathcal{B}_{\log}$.

We shall construct inductively two sequences of points $\{z_n\}$ and $\{w_k\}$ in the unit disc such that the functions

$$F_n(z) = \sum_{k=0}^{n} a_k\, \beta_{w_k}(z)$$

satisfy:

(1) the coefficients $\{a_k\}$ satisfy $0 \le a_k \le 2^{-k}$,

(2) for all $n \in \mathbb{N}$, there holds

$$\Re(F_n(z_n))\,|g'(z_n)|\,(1-|z_n|^2) \ge 1,$$

(3) for all $n \in \mathbb{N}$, we have $\|T_g F_n\|_{\mathcal{B}} \le C(g)$, where $C(g)$ is a constant depending only on g.

Given this construction, we can complete the proof as in the case of functions of bounded mean oscillation.

Assume without loss of generality that $g'(0) = 1$. Then set $z_0 = w_0 = 0$ and $a_0 = 1$. For the inductive step, assume that the parameters are defined up to level $n-1$. We can find $\delta_n > 0$ such that

$$\sup_{|z|\ge 1-\delta_m} |F_{n-1}(z)\,g'(z)|\,(1-|z|^2) \le 1. \tag{3.7}$$

Choose $\delta'_n > 0$ as in Lemma 3.4 (v). Moreover, since $g \notin \mathcal{B}_{\log}$, there exists some $w_n \in \mathbb{D}$, $1-|w_n| \le \delta'_n$ such that

$$\Re\beta_{w_n}(w_n)\,|g'(w_n)|\,(1-|w_n|^2) \ge 2^n. \tag{3.8}$$

Finally, let z_n be a point, $1-|z_n| \le \delta_n$, where the supremum

$$M_n := \sup_{1-|z|\le\delta_n} \Re(\beta_{w_n}(z))\,|g'(z)|\,(1-|z|^2)$$

is attained. We finish the recursive step by setting $a_n = M_n^{-1}$. It remains to verify the properties (1)–(3). This is done in a similar way as in the proof of Proposition 3.5, and the details are left to the reader. ∎

We can now assemble all pieces in order to prove our main result. We shall first prove all equivalences for elliptic semigroups, and then we shall prove that no non-elliptic semigroup satisfies (a) or (b) of Theorem 1.1.

Proof of Theorem 1.1. We start by proving the equivalence of (c) and (d). Recall that if we assume that $\tau = 0$, then

$$\gamma'(z) = \frac{z}{G(z)} = -\frac{1}{p(z)},$$

but since $\Re(p) \ge 0$, from the Alexander–Noshiro–Warschawski criterion it follows that γ is a univalent function in $\mathcal{B}$, hence in BMOA. Notice that (c) is equivalent to $\gamma \in \mathcal{B}_{\log,0}$ and that (d) is equivalent to $\gamma \in \mathrm{VMOA}_{\log}$, hence the proof is a direct consequence of Corollary 3.3 and the fact that γ is univalent.

We proceed now to the proof of the equivalences (a) ⇔ (c) and (b) ⇔ (d). The strategy is quite similar for both, so we prove in detail that (b) ⇔ (d), and we sketch the proof for the other implication.

Let (φ_t) be an elliptic semigroup such that $[\varphi_t, \mathrm{BMOA}] = \mathrm{VMOA}$. We will show that γ must be in $\mathrm{VMOA}_{\log}$. To prove this, suppose that $\gamma \in \mathrm{BMOA} \setminus \mathrm{VMOA}_{\log}$. From

Proposition 3.5, we can find a function $F \in \mathrm{BMOA}$ such that $T_\gamma(F) \in \mathrm{BMOA} \setminus \mathrm{VMOA}$. From (2.4) we know that

$$\mathrm{BMOA} \cap \big(T_\gamma(\mathrm{BMOA}) \oplus \mathcal{C}\big) \subseteq [\varphi_t, \mathrm{BMOA}].$$

But this means that the function $T_\gamma(F) \in [\varphi_t, \mathrm{BMOA}]$, which is equal to VMOA by our assumption, and this is a contradiction.

Conversely, suppose that (d) holds, i.e., that the infinitesimal generator G satisfies (LVMO), which also implies (1.4). Then the result follows from Corollary 2 in [9].

For the equivalence of (a) and (c), one needs to follow the exact same reasoning as before. To be more specific, assuming that (a) holds, use Proposition 3.6 together with the fact that

$$\mathcal{B} \cap \big(T_\gamma(\mathcal{B}) \oplus \mathcal{C}\big) \subseteq [\varphi_t, \mathcal{B}]$$

to ensure that $\gamma \in \mathcal{B}_{\log,0}$ by contradiction, and for the converse implication apply Corollary 2 in [9] as before.

It remains to prove that non-elliptic semigroups cannot satisfy $[\varphi_t, \mathrm{BMOA}] = \mathrm{VMOA}$ or $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$. The argument is identical for both BMOA and $\mathcal{B}$, so we shall only deal with the Bloch space. Assume that (φ_t) is a non-elliptic semigroup such that $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$. Without loss of generality, we consider $\tau = 1$. Let also h be the associated Koenigs function and let $H(z) = h(z)/z$. Then the classical Koebe distortion theorem implies that $\log H \in \mathcal{B}$ (in fact, by Remark 2 on p. 87 of [9], we know that $\log H \in \mathcal{B}_0$, but we shall not need this extra information). Then we distinguish two cases. If $\log H \in \mathcal{B}_{\log,0}$, then we know from Theorem B that the operator

$$T_{\log H} : \mathcal{B} \to \mathcal{B}$$

is *compact*. Then if $\lambda \neq 0$ is a point in the spectrum of $T_{\log H}$, by the spectral theorem for compact operators [17], Section 21.2, it must be an eigenvalue. But this is impossible since $T_{\log H} f = \lambda f$ implies that $f \equiv 0$ (see Proposition 5.1 in [2]). Therefore $T_{\log H}$ has trivial spectrum. In particular, there exists $f \in \mathcal{B}$ such that

$$f(z) - T_{\log H} f(z) \equiv 1.$$

Solving this first order ODE we find that $f = H$, which implies that $h \in \mathcal{B}$, or equivalently by Pommerenke's theorem [20], that $h \in \mathrm{BMOA}$. This contradicts (1.5).

This leaves only the possibility that $\log H \in \mathcal{B} \setminus \mathcal{B}_{\log,0}$. Then we are again in a situation where we can apply Proposition 3.6. Therefore there exists $F \in \mathcal{B}$ such that $T_{\log H} F \in \mathcal{B} \setminus \mathcal{B}_0$. Notice that this is equivalent to the fact that the function

$$T_h\Big(\frac{F}{H}\Big)(z) = \int_0^z \frac{tF(t)h'(t)}{h(t)}\,dt$$

belongs to $\mathcal{B} \setminus \mathcal{B}_0$. Since in this case h is the γ-symbol of the semigroup, it remains to prove that the function F/H is a Bloch function in order to arrive at a contradiction.

We have

$$\begin{aligned}\Big|\Big(\frac{F}{H}\Big)'(z)\Big|(1-|z|^2) &\leq \frac{|F'(z)|(1-|z|^2)}{|H(z)|} + \frac{|F(z)H'(z)|(1-|z|^2)}{|H(z)|^2} \\ &\leq \|H^{-1}\|_{H^\infty}\big(\|F\|_{\mathcal{B}} + \|T_{\log H}F\|_{\mathcal{B}}\big) < \infty.\end{aligned}$$

In other words, we have shown that the function $T_h(F/H) \in \mathcal{B}$ and at the same time it belongs to the range $T_\gamma(\mathcal{B})$, hence by (2.4) in $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$, and this is a contradiction. ∎

Further remarks

We believe that the techniques that we have employed can be used to prove similar characterizations in other kind of spaces. In particular, recent studies investigated the maximal subspace of holomorphic semigroups in BMOA-type spaces [14], in the analytic Morrey spaces [15] and in the setting of mixed norm spaces [4]. It would be interesting to know whether a similar characterization of the minimality of the maximal subspace is possible in these settings, too.

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