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# Tridiagonal kernels and left-invertible operators with applications to Aluthge transforms

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**Abstract.** Given scalars  $a_n (\neq 0)$  and  $b_n, n \geq 0$ , the tridiagonal kernel or band kernel with bandwidth 1 is the positive definite kernel  $k$  on the open unit disc  $\mathbb{D}$  defined by

$$k(z, w) = \sum_{n=0}^{\infty} ((a_n + b_n z)z^n)((\bar{a}_n + \bar{b}_n \bar{w})\bar{w}^n) \quad (z, w \in \mathbb{D}).$$

This defines a reproducing kernel Hilbert space  $\mathcal{H}_k$  (known as tridiagonal space) of analytic functions on  $\mathbb{D}$  with  $\{(a_n + b_n z)z^n\}_{n=0}^{\infty}$  as an orthonormal basis. We consider shift operators  $M_z$  on  $\mathcal{H}_k$  and prove that  $M_z$  is left-invertible if and only if  $\{|a_n/a_{n+1}|\}_{n \geq 0}$  is bounded away from zero. We find that, unlike the case of weighted shifts, Shimorin models for left-invertible operators fail to bring to the foreground the tridiagonal structure of shifts. In fact, the tridiagonal structure of a kernel  $k$ , as above, is preserved under Shimorin models if and only if  $b_0 = 0$  or that  $M_z$  is a weighted shift. We prove concrete classification results concerning invariance of tridiagonality of kernels, Shimorin models, and positive operators. We also develop a computational approach to Aluthge transforms of shifts. Curiously, in contrast to direct kernel space techniques, often Shimorin models fail to yield tridiagonal Aluthge transforms of shifts defined on tridiagonal spaces.

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## 1. Introduction

The theory of left-invertible weighted shifts or multiplication operators  $M_z$  on “diagonal” reproducing kernel Hilbert spaces is one of the most useful in operator theory, function theory, and operator algebras (see the classic by Shields [15]). Given a bounded sequence of positive real numbers  $\alpha = \{\alpha_n\}_{n \geq 0}$ , and an orthonormal basis  $\{e_n\}_{n \geq 0}$  of an infinite-dimensional Hilbert space  $\mathcal{H}$  (complex separable), the operator  $S_\alpha$ , defined by

$$(1.1) \quad S_\alpha e_n = \alpha_n e_{n+1} \quad (n \geq 0),$$

is called a *weighted shift* with weights  $\{\alpha_n\}_{n \geq 0}$ . In this case,  $S_\alpha$  is bounded ( $S_\alpha \in \mathcal{B}(\mathcal{H})$  in short) and  $\|S_\alpha\| = \sup_n \alpha_n$ . If the sequence  $\{\alpha_n\}_{n \geq 0}$  is bounded away from zero, then  $S_\alpha$  is a left-invertible but non-invertible operator. Note that the multiplication operator  $M_z$  on (most of the) diagonal reproducing kernel Hilbert spaces is the function theoretic counterpart of left-invertible weighted shifts which includes the Dirichlet shift, the Hardy shift, and the weighted and unweighted Bergman shifts, etc.

The main focus of this article is to study shifts on the “next best” concrete analytic kernels, namely, tridiagonal kernels. This notion was introduced by Adams and McGuire [2] in 2001 (see also the motivating paper by Adams, McGuire and Paulsen [3]). However, in spite of its natural appearance and potential applications, far less attention has been paid to the use of tridiagonal kernels in the aforementioned subjects. On the other hand, Shimorin [17] developed the idea of analytic models of left-invertible operators at about the same time as Adams and McGuire, which has been put forth as a key model for left-invertible operators by a number of researchers [6, 7, 10, 13].

In the present paper we consider the next level of shifts on tridiagonal spaces, namely, left-invertible shifts on tridiagonal spaces. We also discuss the pending and inevitable comparisons between Shimorin’s analytic models of left-invertible operators and Adams and McGuire’s theory of left-invertible shifts on tridiagonal spaces. In particular (and curiously enough), we find that, unlike the case of weighted shifts, Shimorin models fail to bring to the foreground the tridiagonal structure of shifts. We resolve this dilemma by presenting a complete classification of tridiagonal kernels that are preserved under Shimorin models.

We also prove a number of results concerning left-invertible properties of shifts on tridiagonal spaces, new tridiagonal spaces from the old, classifications of quasinormal operators, rank-one perturbations of left inverses, a computational approach to Aluthge transforms of shifts, etc. Again, curiously enough, some of our definite computations in the setting of tridiagonal kernels verify that the direct reproducing kernel Hilbert space technique is somewhat more powerful than Shimorin models. We also provide a family of instructive examples and supporting counterexamples.

To demonstrate the main contribution of this paper, it is now necessary to disambiguate central concepts. Needless to say, the theory of reproducing kernel Hilbert spaces will play a central role in this paper. Briefly stated, the essential idea of reproducing kernel Hilbert space [5] is to single out the role of positive definiteness of inner products, multipliers and bounded point evaluations of function Hilbert spaces. We denote by  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  the open unit disc in  $\mathbb{C}$ . Let  $\mathcal{E}$  be a Hilbert space. A function  $k: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$  is called

an *analytic kernel* if  $k$  is positive definite, that is,

$$\sum_{i,j=1}^n \langle k(z_i, z_j) \eta_j, \eta_i \rangle_{\mathcal{E}} \geq 0,$$

for all  $\{z_i\}_{i=1}^n \subseteq \mathbb{D}$ ,  $\{\eta_i\}_{i=1}^n \subseteq \mathcal{E}$  and  $n \in \mathbb{N}$ , and  $k$  analytic in the first variable. In this case, there exists a Hilbert space  $\mathcal{H}_k$ , which we call *analytic reproducing kernel Hilbert space* (*analytic Hilbert space*, in short), of  $\mathcal{E}$ -valued analytic functions on  $\mathbb{D}$  such that  $\{k(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\}$  is a total set in  $\mathcal{H}_k$  with the *reproducing property*  $\langle f, k(\cdot, w)\eta \rangle_{\mathcal{H}_k} = \langle f(w), \eta \rangle_{\mathcal{E}}$  for all  $f \in \mathcal{H}_k$ ,  $w \in \mathbb{D}$  and  $\eta \in \mathcal{E}$ . The *shift operator* on  $\mathcal{H}_k$  is the multiplication operator  $M_z$  (which will be assumed to be bounded) defined by

$$(M_z f)(w) = wf(w) \quad (f \in \mathcal{H}_k, w \in \mathbb{D}).$$

Note that there exist  $C_{mn} \in \mathcal{B}(\mathcal{E})$  such that  $k(z, w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n$ ,  $z, w \in \mathbb{D}$ . We say that  $\mathcal{H}_k$  is a *diagonal reproducing kernel Hilbert space* (and  $k$  is a *diagonal kernel*) if  $C_{mn} = 0$  for all  $|m - n| \geq 1$ . We say that  $k$  is a *tridiagonal kernel* (or *band kernel with bandwidth 1*) if

$$(1.2) \quad C_{mn} = 0 \quad (|m - n| \geq 2).$$

In this case, we say that  $\mathcal{H}_k$  is a *tridiagonal space*. Now let  $\{a_n\}_{n \geq 0}$  and  $\{b_n\}_{n \geq 0}$  be a sequences of scalars. In this paper, we will always assume that  $a_n \neq 0$  for all  $n \geq 0$ . Set

$$f_n(z) = (a_n + b_n z) z^n \quad (n \geq 0).$$

Assume that  $\{f_n\}_{n \geq 0}$  is an orthonormal basis of an analytic Hilbert space  $\mathcal{H}_k$ . Then  $\mathcal{H}_k$  is a tridiagonal space, as the well known fact from the reproducing kernel theory implies that

$$(1.3) \quad k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)} \quad (z, w \in \mathbb{D}).$$

We now turn to Shimorin's analytic model of left-invertible operators [17], which says that if  $T \in \mathcal{B}(\mathcal{H})$  is left-invertible and analytic (that is,  $\bigcap_{n=0}^{\infty} T^n \mathcal{H} = \{0\}$ ), then there exists an analytic Hilbert space  $\mathcal{H}_k (\subseteq \mathcal{O}(\mathbb{D}, \mathcal{W}))$  such that  $T$  and  $M_z$  on  $\mathcal{H}_k$  are unitarily equivalent, where  $\mathcal{W} = \ker T^* = \mathcal{H} \ominus T\mathcal{H}$  is the *wandering subspace* of  $T$ , and  $\mathcal{O}(\mathbb{D}, \mathcal{W})$  is the set of  $\mathcal{W}$ -valued analytic functions on  $\mathbb{D}$ . The *Shimorin kernel*  $k$  is explicit (see (2.11)) and involves the *Shimorin left inverse* of  $T$ :

$$(1.4) \quad L_T = (T^* T)^{-1} T^*,$$

The representation of the Shimorin kernel is useful in studying wandering subspaces of invariant subspaces of weighted shifts [16, 17]. See Chapter 6 of [9] and [14] in the context of the wandering subspace problem, and [13] and the extensive list of references therein for recent developments and implementations of Shimorin models.

We prove the following set of results. In Section 2, we present basic properties and constructions of tridiagonal spaces and Shimorin models. We introduce the core concept

of this paper: An *analytic tridiagonal kernel* is a scalar kernel  $k$  as in (1.3) such that  $\mathbb{C}[z] \subseteq \mathcal{H}_k$ ,

$$\sup_{n \geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1$$

(which ensures that  $M_z$  on  $\mathcal{H}_k$  is bounded), and  $\{\frac{a_n}{a_{n+1}}\}_{n \geq 0}$  is bounded away from zero. An analytic Hilbert space is called *analytic tridiagonal space* if the kernel function is an analytic tridiagonal kernel. In Proposition 2.7, we prove (a well-known fact) that weighted shifts behave well under Shimorin's analytic models.

In Section 3, we prove that  $\{\frac{a_n}{a_{n+1}}\}_{n \geq 0}$  is bounded away is equivalent to the fact that  $M_z$  on  $\mathcal{H}_k$  is left-invertible (see Theorems 3.2 and 3.5). We compute representations of Shimorin left inverses of shifts on analytic tridiagonal spaces (see Proposition 3.1 and Theorem 3.4).

Section 4 starts with Example 4.1, which shows that Shimorin kernels do not necessarily preserve the tridiagonal structure of kernels. We are nevertheless able to prove in Theorem 4.2 that it does for a kernel  $k$  of the form (1.3) if and only if  $M_z$  on  $\mathcal{H}_k$  is a weighted shift or  $b_0 = 0$ .

The main result of Section 5 classifies positive operators  $P$  on a tridiagonal space  $\mathcal{H}_k$  such that  $K(z, w) := \langle Pk(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k}$  defines a tridiagonal kernel on  $\mathbb{D}$ . More specifically, if

$$P = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \ddots \\ \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \bar{c}_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

denote the matrix representation of  $P$  with respect to the basis  $\{(a_n + b_n z)z^n\}_{n \geq 0}$  of  $\mathcal{H}_k$ , then the kernel  $K$  is tridiagonal if and only if (see Theorem 5.2)

$$c_{0n} = (-1)^{n-1} \frac{\bar{b}_1 \cdots \bar{b}_{n-1}}{\bar{a}_2 \cdots \bar{a}_n} c_{01}, \quad n \geq 2,$$

and

$$c_{mn} = (-1)^{n-m-1} \frac{\bar{b}_{m+1} \cdots \bar{b}_{n-1}}{\bar{a}_{m+2} \cdots \bar{a}_n} c_{m,m+1} \quad \text{for all } 1 \leq m \leq n-2.$$

Section 6 deals with quasinormal shifts. Suppose  $M_z$  is non-normal on an analytic tridiagonal space  $\mathcal{H}_k$ . Denote by  $P_{\mathbb{C}f_0}$  the orthogonal projection of  $\mathcal{H}_k$  onto  $\mathbb{C}f_0$ . In Theorem 6.2, we prove that  $M_z$  is quasinormal if and only if there exists  $r > 0$  such that

$$M_z^* M_z - M_z M_z^* = r P_{\mathbb{C}f_0}.$$

In Section 7, we compute Aluthge transforms of shifts. The notion of Aluthge transforms was introduced by Aluthge [4] in his study of  $p$ -hyponormal operators. Let  $\mathcal{H}$  be a Hilbert space,  $T \in \mathcal{B}(\mathcal{H})$ , and let  $T = U|T|$  be the polar decomposition of  $T$ . Here, and throughout this note,  $|T| = (T^*T)^{1/2}$  and  $U$  is the unique partial isometry such that  $\ker U = \ker T$ . The *Aluthge transform* of  $T$  is the bounded linear operator

$$\tilde{T} = |T|^{1/2} U |T|^{1/2}.$$

The Aluthge transform of  $\tilde{T}$  turns  $T$  into a more “normal” operator while keeping intact the basic spectral properties of  $T$ , see [11]. Evidently, the main difficulty associated with  $\tilde{T}$  is to compute or represent the positive part  $|T|$ . This is certainly not true for weighted shifts: Since  $|S_\alpha| = \text{diag}(\alpha_0, \alpha_1, \alpha_2, \dots)$  (cf. Proposition 2.7), it follows that  $\tilde{S}_\alpha = S_{\sqrt{\alpha}}$ , where

$$\sqrt{\alpha} := \{\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \dots\}.$$

Therefore,  $\tilde{S}_\alpha$  is also a weighted shift, namely,  $S_{\sqrt{\alpha}}$ . Here we consider the next natural step, which is the computation of  $\tilde{M}_z$ , where  $M_z$  is a left-invertible shift on some analytic Hilbert space  $\mathcal{H}_k$ . We prove that  $\tilde{M}_z$  is also a left-invertible shift on some analytic Hilbert space  $\mathcal{H}_{\tilde{k}}$ . The kernel  $\tilde{k}$  can be obtained either via Shimorin’s model (see Theorem 7.3), which we call the *Shimorin–Aluthge kernel of  $M_z$* , or by a direct approach (see Theorem 7.7), which we call the *standard Aluthge kernel of  $M_z$* . In Theorem 7.5, we prove that if  $\mathbb{C}[z] \subseteq \mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$ , then  $L_{M_z}$  and  $L_{\tilde{M}_z}$  are similar up to the perturbation of an operator of rank at most one. Moreover, in this setting Shimorin–Aluthge kernels are somewhat more explicit (see Theorem 7.6).

In Section 8 we consider truncated spaces (subclass of analytic tridiagonal spaces) in order to pinpoint more definite results, instructive examples, and counterexamples. A truncated space of order  $r (\geq 2)$  is an analytic tridiagonal space  $\mathcal{H}_k$  with  $k$  as in (1.3) such that

$$b_n = 0 \quad (n \neq 2, 3, \dots, r).$$

The computational advantage of a truncated space is that it annihilates a rank one operator (see (7.3)) associated with  $L_{M_z}$  of the shift  $M_z$ . As a result, in this case we are able to prove a complete classification of tridiagonal Shimorin–Aluthge kernels of shifts. This is the content of Theorem 8.3. Curiously, the classification criterion of Theorem 8.3 is also the classification criterion of tridiagonality of standard Aluthge kernels (see Corollary 8.4).

In Section 9, we comment on the assumptions in the definition of truncated kernels. We point out, at the other extreme, if one considers a (non-truncated) tridiagonal kernel  $k$  with

$$b_0 = b_1 = 1 \quad \text{or} \quad b_0 = 1,$$

and all other  $b_i$ ’s are equal to 0, then the standard Aluthge kernel of  $M_z$  is a tridiagonal but the Shimorin–Aluthge kernel of  $M_z$  is not. This is the main content of Example 9.1. We conclude the paper by two observations concerning tridiagonal structures of standard Aluthge kernels and kernels of the form  $(z, w) \mapsto \langle |M_z|^{-2}k(\cdot, w), k(\cdot, z) \rangle$ .

We remark that some of the observations outlined in Sections 7 and 8 are based on several more general results that have an independent interest in broader operator theory and function theoretic contexts.

## 2. Preparatory results and examples

In this section, we set up some definitions, collect some known facts about tridiagonal reproducing kernel Hilbert spaces and Shimorin analytic models, and observe some auxiliary results which are needed throughout the paper. We also explain the idea of Shimorin with the example of diagonal kernels (or, equivalently, weighted shifts).

We start with tridiagonal spaces. Here we avoid finer technicalities [2] and introduce only the necessary features of tridiagonal spaces. Let  $\mathcal{E}$  be a Hilbert space, let  $k$  be a  $\mathcal{B}(\mathcal{E})$ -valued analytic kernel on  $\mathbb{D}$ , and let  $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathcal{E})$  be the corresponding reproducing kernel Hilbert space. Then there exists a sequence  $\{C_{mn}\}_{m,n \geq 0} \subseteq \mathcal{B}(\mathcal{E})$  such that

$$k(z, w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n \quad (z, w \in \mathbb{D}).$$

Recall that (see (1.2))  $k$  is a *tridiagonal kernel* if  $C_{mn} = 0$ ,  $|m - n| \geq 2$ . We say that  $\mathcal{H}_k$  is a *tridiagonal space* if  $k$  is tridiagonal. We now single out two natural tridiagonal spaces.

**Definition 2.1.** A tridiagonal space  $\mathcal{H}_k$  is called *semi-analytic tridiagonal space* if  $\mathbb{C}[z] \subseteq \mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$ , and there exist scalars  $\{a_n\}_{n \geq 0}$  and  $\{b_n\}_{n \geq 0}$ , with  $a_n \neq 0$  for all  $n \geq 0$ , such that

$$(2.1) \quad \sup_{n \geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1,$$

and  $\{f_n\}_{n \geq 0}$  is an orthonormal basis of  $\mathcal{H}_k$ , where

$$(2.2) \quad f_n(z) = (a_n + b_n z) z^n \quad (n \geq 0).$$

Note that the conditions in (2.1) ensure that the shift  $M_z$  is a bounded linear operator on  $\mathcal{H}_k$ , see Theorem 5 of [2]. We refer the reader to Theorem 2 of [2] on the containment of polynomials.

**Definition 2.2.** A semi-analytic tridiagonal space  $\mathcal{H}_k$  is said to be *analytic tridiagonal space* if the sequence  $\left\{ \left| \frac{a_n}{a_{n+1}} \right| \right\}_{n \geq 0}$  is bounded away from zero, that is, there exists  $\epsilon > 0$  such that

$$(2.3) \quad \left| \frac{a_n}{a_{n+1}} \right| > \epsilon \quad (n \geq 0).$$

A scalar kernel  $k$  is called *semi-analytic (analytic) tridiagonal kernel* if the corresponding reproducing kernel Hilbert space  $\mathcal{H}_k$  is a semi-analytic (an analytic) tridiagonal space.

It is important to note that (2.3) is essential for left invertibility of  $M_z$ . As we will see in Theorem 3.5, if  $\mathcal{H}_k (\supseteq \mathbb{C}[z])$  is a tridiagonal space corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$  as in (2.2), and if  $\{a_n\}_{n \geq 0}$  and  $\{b_n\}_{n \geq 0}$  satisfy the conditions in (2.1), then condition (2.3) is equivalent to the left invertibility of  $M_z$  on  $\mathcal{H}_k$ . Also recall that the weighted shift  $S_\alpha$  with weights  $\{\alpha_n\}_{n \geq 0}$  (see (1.1)) is bounded if and only if  $\sup_{n \geq 0} \alpha_n < \infty$ . In this case,  $S_\alpha$  is left-invertible if and only if  $\{\alpha_n\}_{n \geq 0}$  is bounded away from zero (cf. Proposition 2.7). By translating this into the setting of analytic Hilbert spaces, see Proposition 7 of [15], it is clear that the conditions in Definition 2.2 are natural. For instance, if  $b_n = 0$ ,  $n \geq 0$ , then (2.3) is a necessary and sufficient condition for left invertibility of shifts on diagonal kernels.

Suppose  $k$  is a semi-analytic tridiagonal kernel. Note that  $k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)}$  (see (1.3)). Now fix  $n \geq 0$ , and write  $z^n = \sum_{m=0}^{\infty} \alpha_m f_m$  for some  $\alpha_m \in \mathbb{C}$ ,  $m \geq 0$ . Then

$$z^n = \alpha_0 a_0 + \sum_{m=1}^{\infty} (\alpha_{m-1} b_{m-1} + \alpha_m a_m) z^m.$$

Thus, comparing coefficients, we have  $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$ , and  $\alpha_n = \frac{1}{a_n}$ , as the  $a_i$ 's are non-zero scalars. Since

$$\alpha_{n+j-1} b_{n+j-1} + \alpha_{n+j} a_{n+j} = 0,$$

it follows that  $\alpha_{n+j} = -\frac{\alpha_{n+j-1} b_{n+j-1}}{a_{n+j}}$ , and thus

$$\alpha_{n+j} = \frac{(-1)^j}{a_n} \frac{b_n b_{n+1} \cdots b_{n+j-1}}{a_{n+1} \cdots a_{n+j}} \quad (j \geq 1).$$

This implies

$$(2.4) \quad z^n = \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_{n+j}}{\prod_{j=0}^{m-1} a_{n+j+1}} \right) f_{n+m} \quad (n \geq 0),$$

where  $\prod_{j=0}^{-1} x_{n+j} := 1$ . With this, we now proceed to compute  $M_z$ , see Section 3 of [2]. Let  $n \geq 0$ . Then  $M_z f_n = a_n z^{n+1} + b_n z^{n+2}$  implies that

$$\begin{aligned} M_z f_n &= \frac{a_n}{a_{n+1}} f_{n+1} + \left( b_n - \frac{a_n b_{n+1}}{a_{n+1}} \right) z^{n+2} \\ &= \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2} \left( \frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}} \right) z^{n+2}, \end{aligned}$$

that is,

$$(2.5) \quad M_z f_n = \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2} c_n z^{n+2},$$

where

$$(2.6) \quad c_n = \frac{a_n}{a_{n+2}} \left( \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \right) \quad (n \geq 0).$$

Then (2.4) implies that

$$(2.7) \quad M_z f_n = \left( \frac{a_n}{a_{n+1}} \right) f_{n+1} + c_n \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_{n+2+j}}{\prod_{j=0}^{m-1} a_{n+3+j}} \right) f_{n+2+m} \quad (n \geq 0),$$

and hence, with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , we have (see also [2], p. 729)

$$(2.8) \quad [M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ -\frac{c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & -\frac{c_1 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ddots \\ -\frac{c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & -\frac{c_2 b_4}{a_5} & c_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

The matrix representation of the conjugate of  $M_z$  is going to be useful in what follows:

$$(2.9) \quad [M_z^*] = \begin{bmatrix} 0 & \frac{\bar{a}_0}{\bar{a}_1} & \bar{c}_0 & -\frac{\bar{c}_0 \bar{b}_2}{\bar{a}_3} & \frac{\bar{c}_0 \bar{b}_2 \bar{b}_3}{\bar{a}_3 \bar{a}_4} & \cdots \\ 0 & 0 & \frac{\bar{a}_1}{\bar{a}_2} & \bar{c}_1 & -\frac{\bar{c}_1 \bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & 0 & 0 & \frac{\bar{a}_2}{\bar{a}_3} & \bar{c}_2 & \ddots \\ 0 & 0 & 0 & 0 & \frac{\bar{a}_3}{\bar{a}_4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

In particular,  $M_z$  is a weighted shift if and only if  $c_n = 0$  for all  $n \geq 0$ . Also, by (2.6), we have  $c_n = 0$  if and only if  $\frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n}$ ,  $n \geq 0$ . Therefore, we have the following observation.

**Lemma 2.3.** *The shift  $M_z$  on a semi-analytic tridiagonal space  $\mathcal{H}_k$  is a weighted shift corresponding to the basis  $\{f_n\}_{n \geq 0}$  if and only if  $c_n = 0$  for all  $n \geq 0$ , or, equivalently,  $\{\frac{b_n}{a_n}\}_{n \geq 0}$  is a constant sequence.*

The proof of the following lemma uses the assumption that  $\mathbb{C}[z] \subseteq \mathcal{H}_k$ .

**Lemma 2.4.** *If  $\mathcal{H}_k$  is a semi-analytic tridiagonal space, then  $\ker M_z^* = \mathbb{C}f_0$ .*

*Proof.* Clearly, (2.9) implies  $f_0 \in \ker M_z^*$ . On the other hand, from  $\mathbb{C}[z] \subseteq \mathcal{H}_k$ , we deduce that  $f_n = M_z(a_n z^{n-1} + b_n z^n) \in \text{ran } M_z$  for all  $n \geq 1$ , and thus  $\text{span}\{f_n : n \geq 1\} \subseteq \text{ran } M_z$ . The result now follows from the fact that  $\mathbb{C}f_0 = (\text{span}\{f_n : n \geq 1\})^\perp \supseteq \ker M_z^*$ . ■

Now we briefly describe the construction of Shimorin's analytic models of left-invertible operators. Let  $\mathcal{H}$  be a Hilbert space, and let  $T \in \mathcal{B}(\mathcal{H})$ . We say that  $T$  is *left-invertible* if there exists  $X \in \mathcal{B}(\mathcal{H})$  such that  $XT = I_{\mathcal{H}}$ . It is easy to check that this equivalently means that  $T$  is bounded below, which is also equivalent to the invertibility of  $T^*T$ . Following Shimorin, a bounded linear operator  $X \in \mathcal{B}(\mathcal{H})$  is *analytic* if

$$(2.10) \quad \bigcap_{n=0}^{\infty} X^n \mathcal{H} = \{0\}.$$

Note that from the viewpoint of analytic Hilbert spaces, shifts are always analytic. Indeed, let  $\mathcal{H}_k \subseteq \mathcal{O}(\Omega, \mathcal{E})$ , where  $\Omega \subseteq \mathbb{C}$  is a domain, and suppose the shift  $M_z$  is bounded on  $\mathcal{H}_k$ . If  $f \in \bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k$ , then for each  $n \geq 0$ , there exists  $g_n \in \mathcal{H}_k$  such that  $f = z^n g_n$ . Since  $\Omega$  is a domain and  $f$  is analytic on  $\Omega$ , we see that  $f \equiv 0$ , that is,  $\bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k = \{0\}$ .

Now let  $T \in \mathcal{B}(\mathcal{H})$  be a bounded below operator. We call  $L_T := (T^*T)^{-1}T^*$  the *Shimorin left inverse*, to distinguish it from other left inverses of  $T$  (see (1.4)). Set

$$\mathcal{W} = \ker T^* = \mathcal{H} \ominus T\mathcal{H},$$

and  $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{r(L_T)}\}$ , where  $r(L_T)$  is the spectral radius of  $L_T$ . Then

$$(2.11) \quad k_T(z, w) = P_{\mathcal{W}}(I - zL_T)^{-1}(I - \bar{w}L_T^*)^{-1}|_{\mathcal{W}} \quad (z, w \in \Omega)$$

defines a  $\mathcal{B}(\mathcal{W})$ -valued analytic kernel  $k_T : \Omega \times \Omega \rightarrow \mathcal{B}(\mathcal{W})$ , which we call the *Shimorin kernel* of  $T$  (see Corollary 2.14 of [17]). We lose no generality by assuming, as we shall



do, that  $\Omega = \mathbb{D}$ . If, in addition,  $T$  is analytic, then the unitary  $U: \mathcal{H} \rightarrow \mathcal{H}_k$ , defined by

$$(2.12) \quad (Uf)(z) = \sum_{n=0}^{\infty} (P_{\mathcal{W}} L_T^n f) z^n \quad (f \in \mathcal{H}, z \in \mathbb{D}),$$

satisfies  $UT = M_z U$ , see [17]. More precisely, we have the following result.

**Theorem 2.5** ([17]). *Let  $T \in \mathcal{B}(\mathcal{H})$  be an analytic left-invertible operator. Then  $T$  on  $\mathcal{H}$  and  $M_z$  on  $\mathcal{H}_{k_T}$  are unitarily equivalent.*

Denote by  $P_{\mathcal{W}}$  the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{W} = \ker T^*$ . It follows that

$$(2.13) \quad P_{\mathcal{W}} = I_{\mathcal{H}} - T L_T.$$

This plays an important role (in the sense of Wold decomposition of left-invertible operators) in the proof of the above theorem. The following equality will be very useful in what follows.

**Lemma 2.6.** *If  $T$  is a left-invertible operator on  $\mathcal{H}$ , then  $L_T L_T^* = |T|^{-2}$ .*

*Proof.* This follows from the fact that  $L_T L_T^* = (T^* T)^{-1} T^* T (T^* T)^{-1} = (T^* T)^{-1}$ . ■

In the case of left-invertible weighted shifts  $S_{\alpha}$  (see (1.1)), it is known that the shift  $M_z$  on  $\mathcal{H}_{k_{S_{\alpha}}}$  corresponding to the Shimorin kernel  $k_{S_{\alpha}}$  is also a weighted shift (for instance, see Example 5.2 of [13] in the context of bilateral weighted shifts). Nonetheless, we sketch the proof here for the sake of completeness.

**Proposition 2.7.** *Let  $S_{\alpha}$  be the weighted shift with weights  $\{\alpha_n\}_{n \geq 0}$ . If  $\{\alpha_n\}_{n \geq 0}$  is bounded away from zero, then  $S_{\alpha}$  is left-invertible, and the Shimorin kernel  $k_{S_{\alpha}}$  is diagonal.*

*Proof.* Let  $\{e_n\}_{n \geq 0}$  be an orthonormal basis of a Hilbert space  $\mathcal{H}$ , and let  $S_{\alpha} e_n = \alpha_n e_{n+1}$  for all  $n \geq 0$ . Observe that  $S_{\alpha}^* e_n = \alpha_{n-1} e_{n-1}$ ,  $n \geq 1$ , and  $S_{\alpha}^* e_0 = 0$ . Then  $\mathcal{W} = \ker S_{\alpha}^* = \mathbb{C} e_0$ , and

$$S_{\alpha}^* S_{\alpha} e_n = \alpha_n^2 e_n \quad (n \geq 0).$$

Since  $S_{\alpha}^* S_{\alpha}$  is a diagonal operator and  $\{\alpha_n\}_{n \geq 0}$  is bounded away from zero, it follows that  $S_{\alpha}^* S_{\alpha}$  is invertible, and hence  $S_{\alpha}$  is left-invertible. Then the Shimorin left inverse  $L_{S_{\alpha}} := (S_{\alpha}^* S_{\alpha})^{-1} S_{\alpha}^*$  is given by

$$(2.14) \quad L_{S_{\alpha}} e_n = \begin{cases} 0 & \text{if } n = 0, \\ \frac{1}{\alpha_{n-1}} e_{n-1} & \text{if } n \geq 1. \end{cases}$$

Therefore,  $L_{S_{\alpha}}$  is the backward shift, and for all  $m \geq 1$ ,

$$(2.15) \quad L_{S_{\alpha}}^m e_n = \begin{cases} 0 & \text{if } m > n, \\ \frac{1}{\alpha_0 \cdots \alpha_{n-1}} e_0 & \text{if } m = n, \\ \frac{1}{\alpha_{n-1} \cdots \alpha_{n-m}} e_{n-m} & \text{if } m < n. \end{cases}$$

Moreover,

$$L_{S_{\alpha}}^{*m} e_n = \frac{1}{\alpha_n \alpha_{n+1} \cdots \alpha_{n+m-1}} e_{n+m},$$

for all  $n \geq 0$  and  $m \geq 1$ . In particular,

$$L_{S_\alpha}^{*m} e_0 = \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{m-1}} e_m \quad (m \geq 1),$$

and thus, for each  $(m, n) \neq (0, 0)$ , we have clearly

$$P_{\mathcal{W}} L_{S_\alpha}^m L_{S_\alpha}^{*n} e_0 = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{(\alpha_0 \cdots \alpha_{n-1})^2} e_0 & \text{if } m = n. \end{cases}$$

This immediately gives

$$k_{S_\alpha}(z, w) = \sum_{n=0}^{\infty} (P_{\mathcal{W}} L_{S_\alpha}^n L_{S_\alpha}^{*n} |_{\mathcal{W}})(z\bar{w})^n \quad (z, w \in \mathbb{D}),$$

where  $\mathcal{W} = \mathbb{C}e_0$ . In particular, the Shimorin kernel  $k_{S_\alpha}$  is a diagonal kernel. Finally, identifying  $\mathcal{W}$  with  $\mathbb{C}$  and setting  $\beta_n = \frac{1}{\alpha_0 \cdots \alpha_{n-1}}$ ,  $n \geq 1$ , we get

$$k_{S_\alpha}(z, w) = 1 + \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} (z\bar{w})^n \quad (z, w \in \mathbb{D}). \quad \blacksquare$$

Notice in the above that the Shimorin left inverse  $L_{S_\alpha}$  is the backward shift corresponding to the weight sequence  $\{\frac{1}{\alpha_n}\}_{n \geq 0}$ , that is,

$$L_{S_\alpha} = \begin{bmatrix} 0 & \frac{1}{\alpha_0} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\alpha_1} & 0 & \ddots \\ 0 & 0 & 0 & \frac{1}{\alpha_2} & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

In the setting of Proposition 2.7, we now turn to the unitary map  $U: \mathcal{H} \rightarrow \mathcal{H}_{k_{S_\alpha}}$ , where  $\mathcal{H}_{k_{S_\alpha}} \subseteq \mathcal{O}(\mathbb{D}, \mathcal{W})$ , and

$$(Uf)(z) = \sum_{n=0}^{\infty} (P_{\mathcal{W}} L_{S_\alpha}^n f) z^n,$$

for all  $f \in \mathcal{H}$  and  $z \in \mathbb{D}$  (see (2.12)). Set  $f_n = Ue_n$ ,  $n \geq 0$ . Since  $\mathcal{W} = \mathbb{C}e_0$ , (2.14) yields  $f_0 = Ue_0 = P_{\mathcal{W}} e_0 = e_0$ . On the other hand, if  $n \geq 1$ , then (2.15) implies that

$$P_{\mathcal{W}} L_{S_\alpha}^m e_n = \begin{cases} \frac{1}{\beta_n} e_0 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

and hence  $f_n = \frac{1}{\beta_n} z^n e_0$ . Therefore,  $\{e_0\} \cup \{\frac{1}{\beta_n} z^n e_0\}_{n \geq 1}$  is the orthonormal basis of  $\mathcal{H}_{k_{S_\alpha}}$  corresponding to  $U$ . Moreover, for each  $n \geq 1$ , we have

$$M_z \left( \frac{1}{\beta_n} z^n e_0 \right) = \frac{1}{\beta_n} z^{n+1} e_0 = \alpha_n \frac{1}{\beta_{n+1}} z^{n+1} e_0 = \alpha_n \left( \frac{1}{\beta_{n+1}} z^{n+1} e_0 \right),$$

and hence  $M_z$  on  $\mathcal{H}_{k_{S_\alpha}}$  is also a weighted shift with the same weights  $\{\alpha_n\}_{n \geq 0}$ .

### 3. Tridiagonal spaces and left-invertibility

The main contribution of this section is the left invertibility and representations of Shimorin left inverses of shifts on tridiagonal reproducing kernel Hilbert spaces. Recall that the conditions in (2.1) ensures that the shift  $M_z$  is bounded on the semi-analytic tridiagonal space  $\mathcal{H}_k$ . Here we use the remaining condition (2.3) to prove that  $M_z$  is left-invertible.

Before we state and prove the result, we need to construct a specific bounded linear operator. The choice of this operator is not accidental, as we will see in Theorem 3.4 that it is nothing but the Shimorin left inverse of  $M_z$ . For each  $n \geq 1$ , set

$$(3.1) \quad d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}.$$

**Proposition 3.1.** *Let  $k$  be an analytic tridiagonal kernel corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n(z) = (a_n + b_n z)z^n$ ,  $n \geq 0$ . Then the linear operator  $L$ , represented by*

$$[L] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \dots \\ 0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & -\frac{d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & \frac{d_1 b_1 b_2}{a_2 a_3} & -\frac{d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ 0 & -\frac{d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & -\frac{d_3 b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , defines a bounded linear operator on  $\mathcal{H}_k$ .

*Proof.* For each  $n \geq 1$ , we have clearly

$$d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} = \frac{a_{n+1}}{a_n} \frac{b_n}{a_{n+1}} - \frac{a_n}{a_{n-1}} \frac{b_{n-1}}{a_n},$$

and hence

$$|d_n| \leq \left| \frac{a_{n+1}}{a_n} \right| \left| \frac{b_n}{a_{n+1}} \right| + \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{b_{n-1}}{a_n} \right|.$$

Since  $\{\left|\frac{a_n}{a_{n+1}}\right|\}_{n \geq 0}$  is bounded away from zero (see (2.3)), we have that  $\sup_{n \geq 0} \left|\frac{a_{n+1}}{a_n}\right| < \infty$ . This and the second assumption then imply that  $\{d_n\}$  is a bounded sequence.

Let  $S$  denote the matrix obtained from  $[L]$  by deleting all but the superdiagonal elements of  $[L]$ . Similarly, let  $L_0$  denote the matrix obtained from  $[L]$  by deleting all but the diagonal elements of  $[L]$ , and in general, assume that  $L_i$  denotes the matrix obtained from  $[L]$  by deleting all but the  $i$ -th subdiagonal of  $[L]$ ,  $i = 0, 1, 2, \dots$ . Since

$$L = S + \sum_{i \geq 0} L_i,$$

it clearly suffices to prove that  $S$  and  $\{L_i\}_{i \geq 0}$  are bounded, and  $S + \sum_{i \geq 0} L_i$  is absolutely

convergent. Note that

$$\|S\| = \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| < \infty.$$

Moreover, our assumption

$$\limsup_{n \geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1$$

implies that there exist  $r < 1$  and  $n_0 \in \mathbb{N}$  such that

$$\left| \frac{b_n}{a_{n+1}} \right| < r \quad (n \geq n_0).$$

Set

$$M = \sup_{n \geq 1} \left\{ \left| \frac{b_n}{a_{n+1}} \right|, |d_n| \right\}.$$

Then  $\|L_i\| \leq M^{i+1}$  for all  $i = 0, \dots, n_0$ , and

$$\|L_i\| \leq M^{n_0+1} r^{i-n_0} \quad (i > n_0),$$

from which it follows that

$$\begin{aligned} \|S\| + \sum_{i \geq 0} \|L_i\| &= \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + \sum_{i \geq n_0+1} \|L_i\| \\ &\leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + M^{n_0+1} \left( \sum_{i \geq n_0+1} r^{i-n_0} \right) \\ &\leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_n} \right| + \sum_{0 \leq i \leq n_0} \|L_i\| + M^{n_0+1} \frac{r}{1-r}, \end{aligned}$$

and this completes the proof of the theorem. ■

We are now ready to prove that  $M_z$  is left-invertible.

**Theorem 3.2.** *In the setting of Proposition 3.1, we have  $LM_z = I_{\mathcal{X}_k}$ .*

*Proof.* We consider the matrix representations of  $M_z$  and  $L$  as in (2.8) and Proposition 3.1, respectively. Let  $[L][M_z] = (\alpha_{mn})_{m,n \geq 0}$ . Clearly, it suffices to prove that  $\alpha_{mn} = \delta_{mn}$ . It is easy to see that  $\alpha_{m,m+k} = 0$  for all  $k \geq 1$ . Now, by (2.6) and (3.1), we have

$$(3.2) \quad c_n = -\frac{a_n}{a_{n+2}} d_{n+1} \quad (n \geq 0).$$

Note that the  $n$ -th column,  $n \geq 0$ , of  $[M_z]$  is the transpose of

$$\left( \underbrace{0, \dots, 0}_{n+1}, \frac{a_n}{a_{n+1}}, c_n, -\frac{c_n b_{n+2}}{a_{n+3}}, \dots, (-1)^{m-n-2} \frac{c_n b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m}, (-1)^{m-n-1} \frac{c_n b_{n+2} \cdots b_m}{a_{n+3} \cdots a_{m+1}}, \dots \right),$$

and the  $m$ -th row,  $m \geq 0$ , of  $[L]$  is given by

$$\left(0, (-1)^{m-1} \frac{d_1 b_1 \cdots b_{m-1}}{a_2 \cdots a_m}, (-1)^{m-2} \frac{d_2 b_2 \cdots b_{m-1}}{a_3 \cdots a_m}, (-1)^{m-3} \frac{d_3 b_3 \cdots b_{m-1}}{a_4 \cdots a_m}, \dots, -\frac{d_{m-1} b_{m-1}}{a_m}, d_m, \frac{a_{m+1}}{a_m}, 0, 0, \dots\right).$$

Now, if  $n \leq (m-2)$ , then the  $\alpha_{mn}$  (the  $(m, n)$ -th entry of  $[L][M_z]$ ) is given by

$$\begin{aligned} \alpha_{mn} &= (-1)^{m-n-1} \frac{d_{n+1} b_{n+1} \cdots b_{m-1}}{a_{n+2} \cdots a_m} \frac{a_n}{a_{n+1}} \\ &+ (-1)^{m-n-2} \frac{d_{n+2} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} c_n \\ &+ (-1)^{m-n-3} \frac{d_{n+3} b_{n+3} \cdots b_{m-1}}{a_{n+4} \cdots a_m} \left(-c_n \frac{b_{n+2}}{a_{n+3}}\right) \\ &+ \cdots + \left(-\frac{d_{m-1} b_{m-1}}{a_m}\right) (-1)^{m-n-3} c_n \frac{b_{n+2} \cdots b_{m-2}}{a_{n+3} \cdots a_{m-1}} \\ &+ d_m (-1)^{m-n-2} c_n \frac{b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} + \frac{a_{m+1}}{a_m} (-1)^{m-n-1} c_n \frac{b_{n+2} \cdots b_m}{a_{n+3} \cdots a_m a_{m+1}}, \end{aligned}$$

and hence, using (3.2), we obtain

$$\begin{aligned} \alpha_{mn} &= (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+1} \cdots b_{m-1}}{a_{n+1} a_{n+2} \cdots a_m} \\ &+ (-1)^{m-n-2} \left(-\frac{a_n}{a_{n+2}} d_{n+1}\right) \frac{d_{n+2} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} \\ &+ (-1)^{m-n-2} \left(-\frac{a_n}{a_{n+2}} d_{n+1}\right) \left(\frac{b_{n+2}}{a_{n+3}}\right) \left(\frac{d_{n+3} b_{n+3} \cdots b_{m-1}}{a_{n+4} \cdots a_m}\right) \\ &+ \cdots + (-1)^{m-n-2} \left(-\frac{a_n}{a_{n+2}} d_{n+1}\right) \frac{d_{m-1} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} \\ &+ (-1)^{m-n-2} \left(-\frac{a_n}{a_{n+2}} d_{n+1}\right) \frac{d_m b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} \\ &+ (-1)^{m-n-1} \left(-\frac{a_n}{a_{n+2}} d_{n+1}\right) \left(\frac{b_{n+2} \cdots b_m}{a_{n+3} \cdots a_m^2}\right) \\ &= (-1)^{m-n-1} d_{n+1} \left(\frac{a_n b_{n+1} \cdots b_{m-1}}{a_{n+1} a_{n+2} \cdots a_m} + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{n+2}\right. \\ &\quad + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{n+3} + \cdots + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{m-1} \\ &\quad \left. + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_m - \frac{a_n b_{n+2} \cdots b_m}{a_{n+2} a_{n+3} \cdots a_m^2}\right) \\ &= (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} \\ &\quad \times \left(\frac{b_{n+1}}{a_{n+1}} + (d_{n+2} + d_{n+3} + \cdots + d_{m-1} + d_m) - \frac{b_m}{a_m}\right). \end{aligned}$$

Recall from (3.1) that  $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$ ,  $n \geq 1$ . Then

$$\alpha_{mn} = (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} \left( \left( \frac{b_{n+1}}{a_{n+1}} - \frac{b_m}{a_m} \right) + \left( \frac{b_m}{a_m} - \frac{b_{n+1}}{a_{n+1}} \right) \right) = 0.$$

For the case  $n = m - 1$ , we have

$$\alpha_{m,m-1} = d_m \left( \frac{a_{m-1}}{a_m} \right) + \frac{a_{m+1}}{a_m} (c_{m-1}) = \left( \frac{a_{m-1}}{a_m} \right) d_m + \frac{a_{m+1}}{a_m} \left( -\frac{a_{m-1}}{a_{m+1}} d_m \right) = 0,$$

and finally,  $\alpha_{mm} = \left( \frac{a_{m+1}}{a_m} \right) \left( \frac{a_m}{a_{m+1}} \right) = 1$  completes the proof.  $\blacksquare$

In view of Theorem 3.2, let us point out, in particular (see the discussion following (2.10)), that shifts on analytic tridiagonal spaces are always analytic:

**Proposition 3.3.** *If  $k$  is an analytic tridiagonal kernel, then  $M_z$  is an analytic left-invertible operator on  $\mathcal{H}_k$ .*

Now let  $\mathcal{H}_k$  be an analytic tridiagonal space. Our aim is to compute the Shimorin left inverse  $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$  of  $M_z$  on  $\mathcal{H}_k$ . What we prove in fact is that  $L$  in Proposition 3.1 is the Shimorin left inverse of  $M_z$ . First note that

$$(3.3) \quad L_{M_z} z^n = z^{n-1} \quad (n \geq 1).$$

Indeed,

$$L_{M_z} z^n = (M_z^* M_z)^{-1} M_z^* M_z z^{n-1} = (M_z^* M_z)^{-1} (M_z^* M_z) z^{n-1}.$$

Therefore,  $L_{M_z}$  is the backward shift on  $\mathcal{H}_k$  (a well known fact about Shimorin left inverses). On the other hand, by Lemma 2.4, we have

$$L_{M_z} f_0 = (M_z^* M_z)^{-1} M_z^* f_0 = 0,$$

and hence  $L_{M_z} f_0 = 0$ , which in particular yields

$$(3.4) \quad L_{M_z} 1 = -\frac{b_0}{a_0}.$$

Let  $n \geq 1$ . Using (3.1), we have  $L_{M_z} f_n = L_{M_z} (a_n z^n + b_n z^{n+1}) = a_n z^{n-1} + b_n z^n$ , which implies

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}} (a_{n-1} z^{n-1} + b_{n-1} z^n) + \left( b_n - \frac{a_n b_{n-1}}{a_{n-1}} \right) z^n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n a_n z^n,$$

and hence  $L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n (a_n z^n + b_n z^{n+1}) - d_n b_n z^{n+1}$ . By (2.4), we have

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \left( \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^m b_{n+j}}{\prod_{j=0}^m a_{n+1+j}} f_{n+1+m} \right).$$

This is precisely the left inverse  $L$  of  $M_z$  in Proposition 3.1. Whence the next statement.

**Theorem 3.4.** *Let  $\mathcal{H}_k$  be an analytic tridiagonal space. If  $L$  is as in Proposition 3.1, then the Shimorin left inverse  $L_{M_z}$  of  $M_z$  is given by  $L_{M_z} = L$ . In particular,  $L_{M_z} f_0 = 0$ , and*

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \left( \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^m b_{n+j}}{\prod_{j=0}^m a_{n+1+j}} f_{n+1+m} \right) \quad (n \geq 1),$$

where  $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$  for all  $n \geq 1$ . Moreover, the matrix representation of  $L_{M_z}$  with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$  is given by

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \dots \\ 0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & -\frac{d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & \frac{d_1 b_1 b_2}{a_2 a_3} & -\frac{d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ 0 & -\frac{d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & -\frac{d_3 b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Next we verify that the bounded away assumption of  $\{\frac{a_n}{a_{n+1}}\}_{n \geq 0}$  in (2.3) is also a necessary condition for left-invertible shifts.

**Theorem 3.5.** *Let  $\mathcal{H}_k$  be a semi-analytic tridiagonal space corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n(z) = (a_n + b_n z) z^n$ ,  $n \geq 0$ . Then  $M_z$  is left-invertible if and only if  $\{\frac{a_n}{a_{n+1}}\}_{n \geq 0}$  is bounded away from zero or, equivalently,  $\mathcal{H}_k$  is an analytic tridiagonal space.*

*Proof.* In view of Theorem 3.2 we only need to prove the necessary part. Consider the Shimorin left inverse  $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$ . Using the fact that  $\mathbb{C}[z] \subseteq \mathcal{H}_k$ , one can show, along the similar line of computation preceding Theorem 3.4 (note that, by assumption,  $L_{M_z}$  is bounded), that the matrix representation of  $L_{M_z}$  with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$  is precisely given by the one in Theorem 3.4. Then for each  $n \geq 0$ , we have

$$\|(M_z^* M_z)^{-1} M_z^* \|_{\mathcal{B}(\mathcal{H}_k)} \geq \|(M_z^* M_z)^{-1} M_z^* f_n \|_{\mathcal{H}_k} \geq \left| \frac{a_{n+1}}{a_n} \right|,$$

which implies that

$$\left| \frac{a_n}{a_{n+1}} \right| \geq \frac{1}{\|(M_z^* M_z)^{-1} M_z^* \|_{\mathcal{B}(\mathcal{H}_k)}},$$

and hence the sequence is bounded away from zero. ■

## 4. Tridiagonal Shimorin models

As emphasized already in Proposition 2.7, if  $k$  is a diagonal kernel, then  $k_{M_z}$  is also a diagonal kernel. However, as we will see in the example below, Shimorin kernels are not compatible with tridiagonal kernels. This consequently motivates one to ask: How

to determine whether or not the Shimorin kernel  $k_{M_z}$  of a tridiagonal kernel  $k$  is also tridiagonal? We have a complete answer to this question:  $k_{M_z}$  is tridiagonal if and only if  $b_0 = 0$  or  $M_z$  is a weighted shift on  $\mathcal{H}_k$ . This is the main content of this section.

**Example 4.1.** Let  $a_n = 1$  for all  $n \geq 0$ , let  $b_0 = \frac{1}{2}$ , and let  $b_n = 0$  for all  $n \geq 1$ . Let  $\mathcal{H}_k$  denote the analytic tridiagonal space corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n = (a_n + b_n z)z^n$  for all  $n \geq 0$ . Since  $f_0 = 1 + \frac{1}{2}z$  and  $f_n = z^n$  for all  $n \geq 1$ , by (2.8), we have

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \ddots \\ \frac{1}{2} & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

By Theorem 3.4, the Shimorin left inverse  $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$  is given by

$$L_{M_z} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Recall, in this case, that  $\mathcal{W} = \mathbb{C} f_0$ . It is easy to see that  $L_{M_z} f_1 = f_0 - \frac{1}{2} f_1$ ,  $L_{M_z}^* f_0 = f_1$ ,  $L_{M_z}^* f_1 = -\frac{1}{2} f_1 + f_2$ , and  $L_{M_z}^* f_2 = f_3$ . Then

$$L_{M_z}^{*3} f_0 = -\frac{1}{2} L_{M_z}^* f_1 + L_{M_z}^* f_2 = \frac{1}{4} f_1 - \frac{1}{2} f_2 + f_3,$$

and hence  $P_{\mathcal{W}} L_{M_z} L_{M_z}^{*3} f_0 = \frac{1}{4} P_{\mathcal{W}} (L_{M_z} f_1)$ , as  $P_{\mathcal{W}} L_{M_z} f_j = 0$  for all  $j \neq 1$ . Thus,

$$P_{\mathcal{W}} L_{M_z} L_{M_z}^{*3} f_0 = \frac{1}{4} f_0 \neq 0,$$

which implies that the Shimorin kernel  $k_{M_z}$ , as defined in (2.11), is not a tridiagonal kernel.

Throughout this section,  $\mathcal{H}_k$  will be an analytic tridiagonal space corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n(z) = (a_n + b_n z)z^n$ ,  $n \geq 0$ . Recall that the Shimorin kernel  $k_{M_z}: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{W})$  is given by (see (2.11) and also Theorem 2.5)

$$k_{M_z}(z, w) = P_{\mathcal{W}} (I - z L_{M_z})^{-1} (I - \bar{w} L_{M_z}^*)^{-1} |_{\mathcal{W}} \quad (z, w \in \mathbb{D}).$$

Here, of course,  $\mathcal{W} = \mathbb{C} f_0$  is the one-dimensional space generated by the vector  $f_0$ . So one may regard  $k_{M_z}$  as a scalar kernel. We are now ready for the main result of this section.



**Theorem 4.2.** *The Shimorin kernel  $k_{M_z}$  of  $M_z$  is tridiagonal if and only if  $M_z$  on  $\mathcal{H}_k$  is a weighted shift or  $b_0 = 0$ .*

*Proof.* We split the proof into several steps.

**Step 1.** We first denote  $L_{M_z} = L$  and

$$X_{mn} = P_{\mathcal{W}} L^m L^{*n} |_{\mathcal{W}} \quad (m, n \geq 0),$$

for simplicity. First observe that Theorem 3.4 implies that  $L^m f_0 = 0$ ,  $m \geq 1$ , and hence  $X_{m0} = 0 = X_{m0}^* = X_{0m}$  for all  $m \geq 1$ . Then the formal matrix representation of the Shimorin kernel  $k_{M_z}$  is given by

$$(4.1) \quad [k_{M_z}] = \begin{bmatrix} I_{\mathcal{W}} & 0 & 0 & 0 & \dots \\ 0 & X_{11} & X_{12} & X_{13} & \dots \\ 0 & X_{12}^* & X_{22} & X_{23} & \dots \\ 0 & X_{13}^* & X_{23}^* & X_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Clearly, in view of the above,  $k_{M_z}$  is tridiagonal if and only if  $X_{mn} f_0 = 0$  for all  $m, n \neq 0$  and  $|m - n| \geq 2$ .

**Step 2.** In this step we aim to compute matrix representations of  $L^p$  and  $L^{*p}$ ,  $p \geq 1$ , with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$ . The matrix representation of  $[L]$  in Theorem 3.4 is instructive. It also follows that

$$(4.2) \quad [L^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{\bar{a}_1}{\bar{a}_0} & \bar{d}_1 & -\frac{\bar{d}_1 \bar{b}_1}{\bar{a}_2} & \frac{\bar{d}_1 \bar{b}_1 \bar{b}_2}{\bar{a}_2 \bar{a}_3} & -\frac{\bar{d}_1 \bar{b}_1 \bar{b}_2 \bar{b}_3}{\bar{a}_2 \bar{a}_3 \bar{a}_4} & \dots \\ 0 & \frac{\bar{a}_2}{\bar{a}_1} & \bar{d}_2 & -\frac{\bar{d}_2 \bar{b}_2}{\bar{a}_3} & \frac{\bar{d}_2 \bar{b}_2 \bar{b}_3}{\bar{a}_3 \bar{a}_4} & \ddots \\ 0 & 0 & \frac{\bar{a}_3}{\bar{a}_2} & \bar{d}_3 & -\frac{\bar{d}_3 \bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & 0 & 0 & \frac{\bar{a}_4}{\bar{a}_3} & \bar{d}_4 & \ddots \\ 0 & 0 & 0 & 0 & \frac{\bar{a}_5}{\bar{a}_4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Here we redo the construction taking into account the general  $p \geq 1$ , and proceed as in the proof of Theorem 3.4. However, the proof is by no means the same, and the general case is quite involved. Assume that  $n \geq 1$ . We need to consider two cases:  $n \geq p$  and  $n \leq p - 1$ . Suppose  $n \geq p$ . By (3.3) and (3.4), we have

$$L^p f_n = a_n L^p z^n + b_n L^p z^{n+1} = a_n z^{n-p} + b_n z^{n-p+1},$$

which implies

$$\begin{aligned} L^p f_n &= \frac{a_n}{a_{n-p}} (a_{n-p} z^{n-p} + b_{n-p} z^{n-p+1}) + \left( b_n - \frac{a_n}{a_{n-p}} b_{n-p} \right) z^{n-p+1} \\ &= \frac{a_n}{a_{n-p}} f_{n-p} + d_n^{(p)} z^{n-p+1}, \end{aligned}$$

where

$$(4.3) \quad d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}} b_{n-p} \quad (n \geq p).$$

Hence, by (2.4),

$$L^p f_n = \frac{a_n}{a_{n-p}} f_{n-p} + \frac{d_n^{(p)}}{a_{n-p+1}} \left( f_{n-p+1} - \frac{b_{n-p+1}}{a_{n-p+2}} f_{n-p+2} + \frac{b_{n-p+1} b_{n-p+2}}{a_{n-p+2} a_{n-p+3}} f_{n-p+3} - \dots \right),$$

that is,

$$L^p f_n = \frac{a_n}{a_{n-p}} f_{n-p} + \frac{d_n^{(p)}}{a_{n-p+1}} \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_{n-p+j+1}}{\prod_{j=0}^{m-1} a_{n-p+j+2}} \right) f_{n-p+m+1},$$

for all  $n \geq p$ . Here and in what follows, we define  $\prod_{j=0}^{-1} x_j := 1$ .

We now let  $p = 1$  and  $n = 1$ . Then, by Theorem 3.4, we have

$$(4.4) \quad Lf_1 = \frac{a_1}{a_0} f_0 + d_1 f_1 + \left( -\frac{d_1 b_1}{a_2} \right) f_2 + \left( \frac{d_1 b_1 b_2}{a_2 a_3} \right) f_3 + \dots$$

Finally, let  $1 \leq n \leq p - 1$ . Then  $p > 1$ , and again by (3.3) and (3.4), we have

$$\begin{aligned} L^p f_n &= L^p (a_n z^n + b_n z^{n+1}) = a_n L^{p-n} 1 + b_n L^{p-n-1} 1 \\ &= a_n \left( \frac{-b_0}{a_0} \right)^{p-n} + b_n \left( \frac{-b_0}{a_0} \right)^{p-n-1}, \end{aligned}$$

and hence  $L^p f_n = a_n \left( \frac{-b_0}{a_0} \right)^{p-n-1} \left[ \frac{b_n}{a_n} - \frac{b_0}{a_0} \right]$ . We set

$$(4.5) \quad \beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0} \quad (n \geq 1)$$

and

$$(4.6) \quad \beta_n^{(p)} = a_n \left( \frac{-b_0}{a_0} \right)^{p-n-1} \beta_n \quad (1 \leq n \leq p - 1).$$

Then  $L^p f_n = \beta_n^{(p)}$  and (2.4) implies that

$$L^p (f_n) = \frac{\beta_n^{(p)}}{a_0} \sum_{m=0}^{\infty} (-1)^m \left( \frac{\prod_{j=0}^{m-1} b_j}{\prod_{j=0}^{m-1} a_{j+1}} \right) f_m,$$

for all  $1 \leq n \leq p - 1$ . Then

$$(4.7) \quad [L^2] = \begin{bmatrix} 0 & \frac{\beta_1^{(2)}}{a_0} & \frac{a_2}{a_0} & 0 & 0 & \dots \\ 0 & -\frac{\beta_1^{(2)} b_0}{a_0 a_1} & \frac{d_2^{(2)}}{a_1} & \frac{a_3}{a_1} & 0 & \ddots \\ 0 & \frac{\beta_1^{(2)} b_0 b_1}{a_0 a_1 a_2} & -\frac{d_2^{(2)} b_1}{a_1 a_2} & \frac{d_3^{(2)}}{a_2} & \frac{a_4}{a_2} & \ddots \\ 0 & -\frac{\beta_1^{(2)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \frac{d_2^{(2)} b_1 b_2}{a_1 a_2 a_3} & -\frac{d_3^{(2)} b_2}{a_2 a_3} & \frac{d_4^{(2)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

and in general, for each  $p \geq 2$ , we have

$$(4.8) \quad [L^p] = \begin{bmatrix} 0 & \frac{\beta_1^{(p)}}{a_0} & \frac{\beta_2^{(p)}}{a_0} & \dots & \frac{\beta_{p-1}^{(p)}}{a_0} & \frac{a_p}{a_0} & 0 & 0 & \dots \\ 0 & -\frac{\beta_1^{(p)} b_0}{a_0 a_1} & -\frac{\beta_2^{(p)} b_0}{a_0 a_1} & \dots & -\frac{\beta_{p-1}^{(p)} b_0}{a_0 a_1} & \frac{d_p^{(p)}}{a_1} & \frac{a_{p+1}}{a_1} & 0 & \ddots \\ 0 & \frac{\beta_1^{(p)} b_0 b_1}{a_0 a_1 a_2} & \frac{\beta_2^{(p)} b_0 b_1}{a_0 a_1 a_2} & \dots & \frac{\beta_{p-1}^{(p)} b_0 b_1}{a_0 a_1 a_2} & -\frac{d_p^{(p)} b_1}{a_1 a_2} & \frac{d_{p+1}^{(p)}}{a_2} & \frac{a_{p+2}}{a_2} & \ddots \\ 0 & -\frac{\beta_1^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & -\frac{\beta_2^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \dots & -\frac{\beta_{p-1}^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \frac{d_p^{(p)} b_1 b_2}{a_1 a_2 a_3} & -\frac{d_{p+1}^{(p)} b_2}{a_2 a_3} & \frac{d_{p+2}^{(p)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Hence, for each  $p \geq 2$ , we have

$$(4.9) \quad [L^{*p}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \frac{\bar{\beta}_1^{(p)}}{a_0} & -\frac{\bar{\beta}_1^{(p)} \bar{b}_0}{a_0 a_1} & \frac{\bar{\beta}_1^{(p)} \bar{b}_0 \bar{b}_1}{a_0 a_1 a_2} & -\frac{\bar{\beta}_1^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{a_0 a_1 a_2 a_3} & \ddots \\ \frac{\bar{\beta}_2^{(p)}}{a_0} & -\frac{\bar{\beta}_2^{(p)} \bar{b}_0}{a_0 a_1} & \frac{\bar{\beta}_2^{(p)} \bar{b}_0 \bar{b}_1}{a_0 a_1 a_2} & -\frac{\bar{\beta}_2^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{a_0 a_1 a_2 a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{\bar{\beta}_{p-1}^{(p)}}{a_0} & -\frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0}{a_0 a_1} & \frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0 \bar{b}_1}{a_0 a_1 a_2} & -\frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{a_0 a_1 a_2 a_3} & \ddots \\ \frac{\bar{a}_p}{a_0} & \frac{\bar{d}_p^{(p)}}{a_1} & -\frac{\bar{d}_p^{(p)} \bar{b}_1}{a_1 a_2} & \frac{\bar{d}_p^{(p)} \bar{b}_1 \bar{b}_2}{a_1 a_2 a_3} & \ddots \\ 0 & \frac{\bar{a}_{p+1}}{a_1} & \frac{\bar{d}_{p+1}^{(p)}}{a_2} & -\frac{\bar{d}_{p+1}^{(p)} \bar{b}_2}{a_2 a_3} & \ddots \\ 0 & 0 & \frac{\bar{a}_{p+2}}{a_2} & \frac{\bar{d}_{p+2}^{(p)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Step 3.** We now identify condition on the sequence  $\{\beta_n^{(n+2)}\}_{n \geq 1}$  implied by the requirement that  $X_{m,m+2} = 0$ ,  $m \geq 1$ . Before proceeding further, we record here the following crucial observation: Suppose  $\beta_n^{(p)} = 0$  for some  $p$  and  $n$  such that  $1 \leq n \leq p-1$ . Then, by (4.6), we have

$$(4.10) \quad \beta_n^{(q)} = 0 \quad (q \geq p).$$

Now assume  $m \geq 1$ . The matrix representation in (4.9) implies

$$(4.11) \quad L^{*m+2} f_0 = \frac{1}{\bar{a}_0} (\bar{\beta}_1^{(m+2)} f_1 + \bar{\beta}_2^{(m+2)} f_2 + \dots + \bar{\beta}_{m+1}^{(m+2)} f_{m+1} + \bar{a}_{m+2} f_{m+2}).$$

Observe that, by Theorem 3.4, we have

$$P_{\mathcal{W}} L(f_i) = \begin{cases} \frac{a_1}{a_0} f_0 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Let us now assume that  $m \geq 2$ . Then (4.8) implies

$$(4.12) \quad P_{\mathcal{W}} L^m(f_i) = \begin{cases} \frac{\beta_i^{(m)}}{a_0} f_0 & \text{if } 1 \leq i \leq m-1, \\ \frac{a_m}{a_0} f_0 & \text{if } i = m, \\ 0 & \text{if } i \geq m+1. \end{cases}$$

Since  $X_{m,m+2} = P_{\mathcal{W}} L^m L^{*m+2} |_{\mathcal{W}}$ , this yields

$$(4.13) \quad X_{m,m+2} f_0 = \frac{1}{|a_0|^2} (\bar{\beta}_1^{(m+2)} \beta_1^{(m)} + \bar{\beta}_2^{(m+2)} \beta_2^{(m)} + \cdots + \bar{\beta}_{m-1}^{(m+2)} \beta_{m-1}^{(m)} + \bar{\beta}_m^{(m+2)} a_m) f_0.$$

In particular, if  $m = 1$ , then we have

$$X_{13} f_0 = \frac{1}{\bar{a}_0} \left( \bar{\beta}_1^{(3)} \frac{a_1}{a_0} \right) f_0,$$

and hence  $X_{13} = 0$  if and only if  $\beta_1^{(3)} = 0$ . By (4.13), applied with  $m = 2$ , we have

$$X_{24} f_0 = \frac{1}{|a_0|^2} (\bar{\beta}_1^{(4)} \beta_1^{(2)} + \bar{\beta}_2^{(4)} a_2) f_0.$$

Assume that  $\beta_1^{(3)} = 0$ . By (4.10), we have  $\beta_1^{(4)} = 0$ , and, consequently,

$$X_{24} f_0 = \bar{\beta}_2^{(4)} \frac{a_2}{|a_0|^2} f_0.$$

Hence we obtain  $X_{24} = 0$  if and only if  $\beta_2^{(4)} = 0$ . Therefore, if  $X_{m,m+2} = 0$  for all  $m \geq 1$ , then by induction, it follows that  $\beta_m^{(m+2)} = 0$  for all  $m \geq 1$ . The converse also follows from the above computation.

Thus, we have proved that

$$X_{m,m+2} = 0 \text{ for all } m \geq 1 \text{ if and only if } \beta_m^{(m+2)} = 0 \text{ for all } m \geq 1.$$

**Step 4.** Our aim is to prove the following claim:

*Suppose  $X_{i,i+2} = 0$  for all  $i = 1, \dots, m$ , and  $m \geq 1$ .  
Then  $X_{mn} = 0$  for all  $n = m + 3, m + 4, \dots$ , and  $m \geq 1$ .*

To this end, let  $n = m + j$  and  $j \geq 3$ . Then the matrix representation in (4.9) (or the equality (4.11)) implies

$$L^{*n} f_0 = \frac{1}{\bar{a}_0} (\bar{\beta}_1^{(n)} f_1 + \bar{\beta}_2^{(n)} f_2 + \cdots + \bar{\beta}_{n-1}^{(n)} f_{n-1} + \bar{a}_n f_n),$$

and then

$$P_{\mathcal{W}} L^m L^{*n} f_0 = \frac{1}{\bar{a}_0} \sum_{i=1}^{n-1} \bar{\beta}_i^{(n)} P_{\mathcal{W}} L^m (f_i) + \frac{\bar{a}_n}{\bar{a}_0} P_{\mathcal{W}} L^m f_n = \frac{1}{\bar{a}_0} \sum_{i=1}^m \bar{\beta}_i^{(n)} P_{\mathcal{W}} L^m (f_i),$$

since  $P_{\mathcal{W}} L^m f_i = 0$ ,  $i > m$ , which follows from the matrix representation of  $L^m$  in (4.8). Hence, by (4.12) (or directly from (4.8)), we have

$$P_{\mathcal{W}} L^m L^{*n} f_0 = \frac{1}{|a_0|^2} (\bar{\beta}_1^{(n)} \beta_1^{(m)} + \bar{\beta}_2^{(n)} \beta_2^{(m)} + \cdots + \bar{\beta}_{m-1}^{(n)} \beta_{m-1}^{(m)} + a_m \bar{\beta}_m^{(n)}) f_0.$$

Now note that  $X_{i,i+2} = 0$ , that is,  $\beta_i^{(i+2)} = 0$ ,  $i = 1, \dots, m$ , by assumption. Since  $i + 2 \leq m + j$  for all  $i = 1, \dots, m$ , by (4.10), we have

$$\beta_i^{(n)} = \beta_i^{(m+j)} = 0 \quad (i = 1, \dots, m).$$

Hence,  $P_{\mathcal{W}} L^m L^{*n} f_0 = 0$ , that is,  $X_{m,m+i} = 0$ ,  $i = 3, 4, \dots$ , which proves the claim.

**Step 5.** So far all we have proved is that  $X_{mn} = 0$  for all  $|m - n| \geq 2$  if and only if  $\beta_m^{(m+2)} = 0$  for all  $m \geq 1$ . Now, by (4.6) and (4.5), we have

$$\beta_n^{(n+2)} = a_n \left( -\frac{b_0}{a_0} \right) \beta_n,$$

where  $\beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0}$  for all  $n \geq 1$ . Thus,  $\beta_n^{(n+2)} = 0$  for all  $n \geq 1$  if and only if  $b_0 = 0$  or  $\beta_n = 0$  for all  $n \geq 1$ . On the other hand, Lemma 2.3 implies that  $\beta_n = 0$  for all  $n \geq 1$  if and only if  $M_z$  is a weighted shift.

Finally, by Proposition 2.7, we know that if  $M_z$  is a left-invertible weighted shift, then the Shimorin kernel is also a diagonal kernel. This completes the proof of Theorem 4.2. ■

In fact, we have proved the following general statement (see the matrix representation (4.1)): The Shimorin kernel  $k_{M_z}$  of  $M_z$  on an analytic tridiagonal space is  $(2j + 1)$ -diagonal for any finite  $j \geq 1$  if and only if  $M_z$  is a weighted shift or  $b_0 = 0$ . As a result, the Shimorin kernel of  $M_z$  on an analytic tridiagonal space is either tridiagonal or cannot be  $(2j + 1)$ -diagonal for any finite  $j > 1$ . We are thankful to Professor Indranil Biswas for bringing this to our attention.

## 5. Positive operators and tridiagonal kernels

Our aim is to classify positive operators  $P$  on a tridiagonal space  $\mathcal{H}_k$  such that

$$\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto \langle Pk(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k}$$

is also a tridiagonal kernel. While this problem is of independent interest, the motivation for our interest in this question also comes from Theorem 7.7 (also see the paragraph preceding Corollary 9.2). We start with a simple example.

**Example 5.1.** We consider the same example as in Example 4.1. Note that  $M_z$  is left-invertible and not a weighted shift with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$  of  $\mathcal{H}_k$ . Then, by Lemma 2.6, we have

$$|M_z|^{-2} = L_{M_z} L_{M_z}^* = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots \\ -\frac{1}{2} & \frac{5}{4} & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let

$$|M_z|^{-1} = \begin{bmatrix} \alpha & \beta & 0 & 0 & \dots \\ \beta & \gamma & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where  $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$  is the positive square root of  $\begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}$ . A straightforward calculation shows that  $\frac{\alpha}{2} + \beta \neq 0$ . Define  $K: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  by

$$K(z, w) = \langle |M_z|^{-1} k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}).$$

A simple computation then shows that

$$K(z, w) = \alpha + \left(\frac{\alpha}{2} + \beta\right)\bar{w} + \left(\frac{\alpha}{2} + \beta\right)z + \left(\frac{\alpha}{4} + \beta + \gamma\right)z\bar{w} + \sum_{n \geq 2} z^n \bar{w}^n,$$

that is,  $K$  is also a tridiagonal kernel.

The following is a complete classification of positive operators  $P$  for which  $(z, w) \mapsto \langle Pk(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k}$  defines a tridiagonal kernel.

**Theorem 5.2.** *Let  $\mathcal{H}_k$  be a tridiagonal space corresponding to the orthonormal basis  $f_n(z) = (a_n + b_n z)z^n$ ,  $n \geq 0$ . Let  $P$  be a positive operator on  $\mathcal{H}_k$  with matrix representation*

$$P = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \dots \\ \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \ddots \\ \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \bar{c}_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

with respect to the basis  $\{f_n\}_{n \geq 0}$ . Then the positive definite scalar kernel  $K$ , defined by

$$K(z, w) = \langle Pk(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}),$$

is tridiagonal if and only if

$$c_{0n} = (-1)^{n-1} \frac{\bar{b}_1 \cdots \bar{b}_{n-1}}{\bar{a}_2 \cdots \bar{a}_n} c_{01} \quad (n \geq 2)$$

and

$$c_{mn} = (-1)^{n-m-1} \frac{\bar{b}_{m+1} \cdots \bar{b}_{n-1}}{\bar{a}_{m+2} \cdots \bar{a}_n} c_{m,m+1} \quad (1 \leq m \leq n-2).$$

Equivalently,  $K$  is tridiagonal if and only if

$$P = \begin{bmatrix} c_{00} & c_{01} & -\frac{\bar{b}_1}{\bar{a}_2} c_{01} & \frac{\bar{b}_1 \bar{b}_2}{\bar{a}_2 \bar{a}_3} c_{01} & \dots \\ \bar{c}_{01} & c_{11} & c_{12} & -\frac{\bar{b}_2}{\bar{a}_3} c_{12} & \ddots \\ -\frac{\bar{b}_1}{\bar{a}_2} \bar{c}_{01} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \frac{\bar{b}_1 \bar{b}_2}{\bar{a}_2 \bar{a}_3} \bar{c}_{01} & -\frac{\bar{b}_2}{\bar{a}_3} \bar{c}_{12} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

*Proof.* Note that for each  $w \in \mathbb{D}$ , by (1.3), we have  $k(\cdot, w) = \sum_{m=0}^{\infty} \overline{f_m(w)} f_m$ , and thus

$$Pk(\cdot, w) = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m-1} \bar{c}_{nm} \overline{f_n(w)} + \sum_{n=m}^{\infty} c_{mn} \overline{f_n(w)} \right) f_m.$$

where  $\sum_{n=0}^{-1} x_n := 0$ . Then

$$\begin{aligned} \langle Pk(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} &= \sum_{m=0}^{\infty} f_m(z) \left( \sum_{n=0}^{m-1} \bar{c}_{nm} \overline{f_n(w)} + \sum_{n=m}^{\infty} c_{mn} \overline{f_n(w)} \right) \\ &= \sum_{m=0}^{\infty} (a_m z^m + b_m z^{m+1}) \left( \sum_{n=0}^{m-1} \bar{c}_{nm} (\bar{a}_n \bar{w}^n + \bar{b}_n \bar{w}^{n+1}) + \sum_{n=m}^{\infty} c_{mn} (\bar{a}_n \bar{w}^n + \bar{b}_n \bar{w}^{n+1}) \right) \\ &= \sum_{m, n \geq 0} \alpha_{mn} z^m \bar{w}^n, \end{aligned}$$

where  $\alpha_{mn}$  denotes the coefficient of  $z^m \bar{w}^n$ ,  $m, n \geq 0$ . Our interest here is to compute  $\alpha_{mn}$ ,  $|m - n| \geq 2$ . Clearly,  $\alpha_{mn} = \bar{\alpha}_{nm}$  for all  $m, n \geq 0$ , and

$$(5.1) \quad \alpha_{0n} = a_0(\bar{a}_n c_{0n} + \bar{b}_{n-1} c_{0, n-1}) \quad (n \geq 2)$$

and

$$(5.2) \quad \begin{aligned} \alpha_{mn} &= a_m(\bar{a}_n c_{mn} + \bar{b}_{n-1} c_{m, n-1}) \\ &\quad + b_{m-1}(\bar{a}_n c_{m-1, n} + \bar{b}_{n-1} c_{m-1, n-1}) \quad (1 \leq m < n). \end{aligned}$$

Suppose  $n \geq 2$ . By (5.1),  $\alpha_{0n} = 0$  if and only if  $c_{0n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{0, n-1}$ . In particular, if  $n = 2$ , then  $c_{02} = -\frac{\bar{b}_1}{\bar{a}_2} c_{01}$ , and hence, by (5.1) again, we have

$$c_{0n} = (-1)^{n-1} \frac{\prod_{i=1}^{n-1} \bar{b}_i}{\prod_{i=2}^n \bar{a}_i} c_{01} \quad (n \geq 2).$$

Therefore,  $\alpha_{0n} = 0$  for all  $n \geq 2$  if and only if the above identity holds for all  $n \geq 2$ .

Next we want to consider the case  $m, n \neq 0$  and  $|m - n| \geq 2$ . Assume that  $n \geq 3$ . Then (5.2), along with (5.1), implies

$$\begin{aligned} \alpha_{1n} &= a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1, n-1}) + b_0(\bar{a}_n c_{0n} + \bar{b}_{n-1} c_{0, n-1}) \\ &= a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1, n-1}) + \frac{b_0}{a_0} \alpha_{0n}. \end{aligned}$$

Therefore, if  $\alpha_{0n} = 0$  for all  $n \geq 3$ , then  $\alpha_{1n} = a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1, n-1})$ . Hence,  $\alpha_{1n} = 0$  if and only if  $\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1, n-1} = 0$ , which is equivalent to

$$c_{1n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{1, n-1}.$$

Therefore, under the assumption that  $\alpha_{1n} = 0$  and  $n \geq 4$ , (5.2) along with (5.1) implies

$$\begin{aligned} \alpha_{2n} &= a_2(\bar{a}_n c_{2n} + \bar{b}_{n-1} c_{2, n-1}) + b_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1, n-1}) \\ &= a_2(\bar{a}_n c_{2n} + \bar{b}_{n-1} c_{2, n-1}). \end{aligned}$$

Then  $\alpha_{2n} = 0$ ,  $n \geq 4$ , if and only if  $c_{2n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{2, n-1}$ . Consequently, by induction, for all  $m, n \neq 0$  and  $|m - n| \geq 2$ , we have that  $\alpha_{mn} = 0$  if and only if  $\bar{a}_n c_{mn} + \bar{b}_{n-1} c_{m, n-1} = 0$

or, equivalently,

$$c_{mn} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{m,n-1}.$$

Finally, observe that

$$c_{mn} = (-1)^{n-m-1} \frac{\bar{b}_{n-1} \cdots \bar{b}_{m+1}}{\bar{a}_n \cdots \bar{a}_{m+2}} c_{m,m+1}$$

for all  $1 \leq m \leq n-2$ . This completes the proof of the theorem.  $\blacksquare$

We will return to this in Theorem 8.3 and Corollary 8.4.

## 6. Quasinormal operators

A bounded linear operator  $T \in \mathcal{B}(\mathcal{H})$  is said to be *quasinormal* if  $T^*T$  and  $T$  commutes, that is,

$$[T^*, T]T = 0,$$

where  $[T^*, T] = T^*T - TT^*$  is the commutator of  $T$ . In this section, we present a complete classification of quasinormality of  $M_z$  on analytic tridiagonal spaces. Here, however, we do not need to assume that  $M_z$  is left-invertible.

To motivate our result on quasinormality, we first consider the known case of weighted shifts. Recall that the weighted shift  $S_\alpha$  corresponding to the weight sequence (of positive real numbers)  $\{\alpha_n\}_{n \geq 0}$  is given by  $S_\alpha e_n = \alpha_n e_{n+1}$  for all  $n \geq 0$ . Then (see the proof of Proposition 2.7)

$$S_\alpha S_\alpha^* e_{n+1} = \alpha_n^2 e_{n+1},$$

and hence  $(S_\alpha^* S_\alpha - S_\alpha S_\alpha^*) S_\alpha = 0$  if and only if  $(S_\alpha^* S_\alpha - S_\alpha S_\alpha^*) S_\alpha e_n = 0$  for all  $n \geq 0$ , which is equivalent to

$$\alpha_n(\alpha_{n+1}^2 - \alpha_n^2) = 0$$

for all  $n$ . Thus, we have proved Problem 139 of [8]:

**Lemma 6.1** ([8]). *The weighted shift  $S_\alpha$  is quasinormal if and only if the weight sequence  $\{\alpha_n\}_{n \geq 0}$  is a constant sequence.*

Now we turn to  $M_z$  on a semi-analytic tridiagonal space  $\mathcal{H}_k$ . Suppose  $[M_z^*, M_z] = rP_{f_0}$ , where  $r$  is a non-negative real number and  $P_{f_0}$  denotes the orthogonal projection of  $\mathcal{H}_k$  onto the one-dimensional space  $\mathbb{C}f_0$ . Then  $[M_z^*, M_z]M_z = rP_{f_0}M_z$  implies that

$$([M_z^*, M_z]M_z)f_n = rP_{f_0}(zf_n).$$

Now, by (2.7), we have

$$zf_n = \sum_{i=n+1}^{\infty} \beta_i f_i,$$

for some scalars  $\beta_i \in \mathbb{C}$ ,  $i \geq n+1$ . Notice that  $\beta_{n+1} = \frac{a_n}{a_{n+1}} \neq 0$ . This shows that  $P_{f_0}(zf_n) = 0$ , and hence

$$([M_z^*, M_z]M_z)f_n = 0 \quad (n \geq 0),$$



that is,  $M_z$  is quasinormal. Conversely, assume that  $M_z$  is a non-normal and quasinormal operator. Then  $[M_z^*, M_z]M_z = 0$  implies that  $\text{ran } M_z \subseteq \ker[M_z^*, M_z]$ , and therefore, by Lemma 2.4, we have

$$\mathbb{C} f_0 = \ker M_z^* \supseteq \overline{\text{ran}}[M_z^*, M_z].$$

Clearly this implies  $[M_z^*, M_z] = rP_{f_0}$  for some non-zero scalar  $r$ . Then

$$r\|f_0\|^2 = \langle rP_{f_0}f_0, f_0 \rangle_{\mathcal{H}_k} = \langle [M_z^*, M_z]f_0, f_0 \rangle_{\mathcal{H}_k} = \|M_z f_0\|^2 - \|M_z^* f_0\|^2 = \|M_z f_0\|^2,$$

as  $M_z^* f_0 = 0$ , which implies

$$r = \frac{\|M_z f_0\|^2}{\|f_0\|^2} > 0.$$

Thus, we have proved the following theorem.

**Theorem 6.2.** *Let  $\mathcal{H}_k$  be a semi-analytic tridiagonal space. Assume that  $M_z$  is a non-normal operator on  $\mathcal{H}_k$ . Then  $M_z$  is quasinormal if and only if there exists a positive real number  $r$  such that*

$$M_z^* M_z - M_z M_z^* = rP_{f_0},$$

where  $P_{f_0}$  denotes the orthogonal projection of  $\mathcal{H}_k$  onto the one-dimensional space  $\mathbb{C} f_0$ .

In more algebraic terms, this result can be formulated as follows: First we recall the matrix representation of  $M_z$  (see (2.8))

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ -\frac{c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & -\frac{c_1 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ddots \\ -\frac{c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & -\frac{c_2 b_4}{a_5} & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

For each  $n \geq 0$ , we denote by  $R_n$  and  $C_n$  the  $n$ -th row and  $n$ -th column, respectively, of  $[M_z]$ . We then identify each of these column and row vectors with elements in  $\mathcal{H}_k$ . Then  $R_n, C_n \in \mathcal{H}_k$ ,  $n \geq 0$ . Using the matrix representation  $[M_z^*]$  (see (2.9)) and  $[M_z]$ , we get

$$\langle R_0, R_n \rangle_{\mathcal{H}_k} = 0,$$

for all  $n \geq 0$ , and consequently

$$[[M_z^*, M_z]] = \begin{bmatrix} \langle C_0, C_0 \rangle_{\mathcal{H}_k} & \langle C_1, C_0 \rangle_{\mathcal{H}_k} & \langle C_2, C_0 \rangle_{\mathcal{H}_k} & \dots \\ \langle C_0, C_1 \rangle_{\mathcal{H}_k} & \langle C_1, C_1 \rangle_{\mathcal{H}_k} - \langle R_1, R_1 \rangle_{\mathcal{H}_k} & \langle C_2, C_1 \rangle_{\mathcal{H}_k} - \langle R_1, R_2 \rangle_{\mathcal{H}_k} & \dots \\ \langle C_0, C_2 \rangle_{\mathcal{H}_k} & \langle C_1, C_2 \rangle_{\mathcal{H}_k} - \langle R_2, R_1 \rangle_{\mathcal{H}_k} & \langle C_2, C_2 \rangle_{\mathcal{H}_k} - \langle R_2, R_2 \rangle_{\mathcal{H}_k} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, the following corollary holds.

**Corollary 6.3.** *Let  $\mathcal{H}_k$  be a semi-analytic tridiagonal space. Then  $M_z$  on  $\mathcal{H}_k$  is quasinormal if and only if*

$$\langle C_0, C_0 \rangle_{\mathcal{H}_k} = r, \quad \langle C_0, C_i \rangle_{\mathcal{H}_k} = 0 \quad (i \geq 1),$$

and

$$\langle C_n, C_m \rangle_{\mathcal{H}_k} - \langle R_m, R_n \rangle_{\mathcal{H}_k} = 0 \quad \text{for all } 1 \leq m \leq n.$$

It is easy to see that a quasinormal operator is always subnormal [8]. However, a complete classification of subnormality of  $M_z$  on tridiagonal spaces is rather more subtle and not quite as clear-cut as in the quasinormal situation. In fact, the general classification of subnormality of  $M_z$  on tridiagonal spaces is not known (however, see [1]).

## 7. Aluthge transforms of shifts

Recall that the *Aluthge transform* of an operator  $T \in \mathcal{B}(\mathcal{H})$  is the bounded linear operator

$$\tilde{T} = |T|^{1/2}U|T|^{1/2}.$$

In this section, we prove that the Aluthge transform of a left-invertible shift on an analytic Hilbert space is again an explicit shift on some analytic Hilbert space. We present two approaches to this problem, one based on Shimorin's analytic models of left-invertible operators and one is based on rather direct reproducing kernel Hilbert space techniques.

We begin with the following simple fact concerning Aluthge transforms of left-invertible operators.

**Lemma 7.1.** *If  $T$  is a left-invertible operator on  $\mathcal{H}$ , then*

$$\tilde{T} = |T|^{1/2}T|T|^{-1/2},$$

and  $\ker \tilde{T}^* = |T|^{-1/2} \ker T^*$ . In particular,  $\tilde{T}$  is similar to  $T$ .

*Proof.* Indeed,  $\tilde{T} = |T|^{1/2}U|T|^{1/2} = |T|^{1/2}(U|T|)|T|^{-1/2} = |T|^{1/2}T|T|^{-1/2}$ , as  $T^*T$  is invertible. The second equality follows from the first. ■

Suppose in addition that  $T$  is a shift on an analytic Hilbert space. In Theorem 7.3 (under an additional assumption that  $T$  is analytic), and then in Theorem 7.7 again, we prove that  $\tilde{T}$ , up to unitary equivalence, is also a shift on an explicit analytic Hilbert space. In connection with Lemma 2.6, we now prove the following proposition.

**Proposition 7.2.** *If  $T$  is a left-invertible operator on  $\mathcal{H}$ , then the Shimorin left inverse  $L_{\tilde{T}}$  of the Aluthge transform  $\tilde{T}$  is given by*

$$L_{\tilde{T}} = |T|^{1/2}((L_T|T|T)^{-1}L_T)|T|^{1/2} = |T|^{1/2}((T^*|T|T)^{-1}T^*)|T|^{1/2}.$$

*Proof.* By Lemma 7.1, we know that  $\tilde{T} = |T|^{1/2}T|T|^{-1/2}$ . Since  $|T|^{1/2}$  is invertible and  $L_T T = I$ , we have

$$|T|^{1/2}L_T|T|^{-1/2}\tilde{T} = I,$$

which implies that  $\tilde{T}$  is left-invertible, and hence  $(\tilde{T}^*\tilde{T})^{-1}$  exists. By Lemma 7.1 again, we have

$$\tilde{T}^*\tilde{T} = (|T|^{-1/2}T^*|T|^{1/2})(|T|^{1/2}T|T|^{-1/2}) = |T|^{-1/2}(T^*|T|T)|T|^{-1/2}.$$

Since  $(\tilde{T}^* \tilde{T})$  and  $|T|^{-1/2}$  both are invertible, we conclude that  $T^*|T|T$  is invertible. Moreover, the above equality implies

$$(\tilde{T}^* \tilde{T})^{-1} = |T|^{1/2}(T^*|T|T)^{-1}|T|^{1/2}.$$

Then

$$\begin{aligned} L_{\tilde{T}} &= (\tilde{T}^* \tilde{T})^{-1} \tilde{T}^* = (|T|^{1/2}(T^*|T|T)^{-1}|T|^{1/2})|T|^{-1/2} T^* |T|^{1/2} \\ &= |T|^{1/2} ((T^*|T|T)^{-1} T^*) |T|^{1/2}. \end{aligned}$$

On the other hand, since  $T^* = |T|^2 L_T$ , we have  $T^*|T|T = |T|^2 L_T |T|T$ , and hence

$$(T^*|T|T)^{-1} = (L_T |T|T)^{-1} |T|^{-2}.$$

Therefore,  $(\tilde{T}^* \tilde{T})^{-1} = |T|^{1/2} (L_T |T|T)^{-1} |T|^{-3/2}$ , which gives

$$\begin{aligned} L_{\tilde{T}} &= (\tilde{T}^* \tilde{T})^{-1} \tilde{T}^* = |T|^{1/2} (L_T |T|T)^{-1} |T|^{-2} (T^* |T|^{1/2}) \\ &= |T|^{1/2} (L_T |T|T)^{-1} L_T |T|^{1/2}, \end{aligned}$$

and this completes the proof.  $\blacksquare$

Then the above, along with Theorem 2.5 and Lemma 7.1 implies the following.

**Theorem 7.3.** *Let  $\mathcal{E}$  be a Hilbert space, and let  $k: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$  be an analytic kernel. Suppose  $M_z$  is left-invertible on  $\mathcal{H}_k$ . Then the Aluthge transform  $\tilde{M}_z$  is unitarily equivalent to the shift  $M_z$  on  $\mathcal{H}_{\tilde{k}} \subseteq \mathcal{O}(\mathbb{D}, \tilde{\mathcal{W}})$ , where*

$$\begin{aligned} \tilde{k}(z, w) &= P_{\tilde{\mathcal{W}}}(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_{\tilde{\mathcal{W}}} \quad (z, w \in \mathbb{D}), \\ \tilde{\mathcal{W}} &= \ker \tilde{M}_z^* = |M_z|^{-1/2} \ker M_z^*, \\ L &= |M_z|^{1/2} ((L_{M_z} |M_z| M_z)^{-1} L_{M_z}) |M_z|^{1/2}. \end{aligned}$$

**Definition 7.4.** The kernel  $\tilde{k}$  is called the *Shimorin–Aluthge kernel* of  $M_z$ .

Under some additional assumptions on scalar-valued analytic kernels, we now prove that, up to similarity and a perturbation of an operator of rank at most one,  $L_{\tilde{M}_z}$  and  $L_{M_z}$  are the same. As far as concrete examples are concerned, these assumptions are indispensable and natural (cf. Lemma 2.4).

**Theorem 7.5.** *Let  $k: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  be an analytic kernel,  $\mathbb{C}[z] \subseteq \mathcal{H}_k$ , and let  $\{f_n\} \subseteq \mathbb{C}[z]$  be an orthonormal basis of  $\mathcal{H}_k$ . Assume that  $M_z$  on  $\mathcal{H}_k$  is left-invertible,  $\ker M_z^* = \mathbb{C}f_0$  and*

$$f_n \in \text{span}\{z^m : m \geq 1\} \quad (n \geq 1).$$

Then  $L_{\tilde{M}_z}$  and  $L_{M_z}$  are similar up to the perturbation of an operator of rank at most one.

*Proof.* Since  $\ker M_z^* = \mathbb{C}f_0$ ,  $L_{M_z} f_0 = 0$  and  $L_{M_z} z^n = L_{M_z} M_z(z^{n-1}) = z^{n-1}$ , by the definition of  $L_{M_z}$ . This implies  $L_{M_z} z^n = z^{n-1}$ ,  $n \geq 1$  (see also (3.3)). In particular,  $L_{M_z} f_n \in \mathbb{C}[z]$  for all  $n \geq 0$ . Moreover, for each  $n \geq 1$ , we have

$$\begin{aligned} L_{\tilde{M}_z} (|M_z|^{1/2} z^n) &= |M_z|^{1/2} ((L_{M_z} |M_z| M_z)^{-1} L_{M_z}) |M_z| z^n \\ &= |M_z|^{1/2} (L_{M_z} |M_z| M_z)^{-1} (L_{M_z} |M_z| M_z) z^{n-1}, \end{aligned}$$

that is,  $L_{\tilde{M}_z}(|M_z|^{1/2}z^n) = |M_z|^{1/2}z^{n-1}$ . Therefore, we have

$$(|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2})z^n = L_{M_z}z^n = z^{n-1} \quad (n \geq 1).$$

Then  $(|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2} - L_{M_z})f_n = 0$  for all  $n \geq 1$ , which gives

$$(|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2} - L_{M_z})|_{\overline{\text{span}}\{f_n:n \geq 1\}} = 0.$$

Finally, we have clearly

$$(|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2} - L_{M_z})f_0 = (|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2})f_0,$$

and hence

$$(7.1) \quad F := |M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2} - L_{M_z}$$

is of rank at most one, and consequently  $L_{\tilde{M}_z}|M_z|^{1/2} = |M_z|^{1/2}(L_{M_z} + F)$ . This completes the proof of the theorem.  $\blacksquare$

The following analysis of  $F$ , defined as in (7.1), will be useful in what follows. Note that

$$(7.2) \quad L_{\tilde{M}_z}|M_z|^{1/2} = |M_z|^{1/2}(L_{M_z} + F).$$

Let  $g \in \mathcal{H}_k$ . Clearly, since  $L_{M_z}f_0 = 0$ , we have

$$Fg = \langle g, f_0 \rangle_{\mathcal{H}_k} (|M_z|^{-1/2}L_{\tilde{M}_z}|M_z|^{1/2}f_0).$$

Then Lemma 2.6 implies that

$$(7.3) \quad Fg = \langle g, f_0 \rangle_{\mathcal{H}_k} ((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0) \quad (g \in \mathcal{H}_k).$$

As we will see in Section 8, the appearance of the finite rank operator  $F$  causes severe computational difficulties for Shimorin–Aluthge kernels of shifts. On the other hand, combining Theorem 2.5, Proposition 7.2 and (7.2), we have the following theorem.

**Theorem 7.6.** *In the setting of Theorem 7.5, the Aluthge transform  $\tilde{M}_z$  of  $M_z$  on  $\mathcal{H}_k$  is unitarily equivalent to the shift  $M_z$  on  $\mathcal{H}_{\tilde{k}}$ , where*

$$\begin{aligned} \tilde{k}(z, w) &= P_{\mathcal{W}}(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_{\mathcal{W}}, \\ \mathcal{W} &= |M_z|^{-1/2} \ker M_z^* = \mathbb{C}(|M_z|^{-1/2}f_0), \\ L &= |M_z|^{1/2}(L_{M_z} + F)|M_z|^{-1/2}, \\ Fg &= \langle g, f_0 \rangle_{\mathcal{H}_k} ((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0) \quad (g \in \mathcal{H}_k). \end{aligned}$$

We now revisit Theorem 7.3 from a direct reproducing kernel Hilbert space standpoint. Indeed, there is a rather more concrete proof of Theorem 7.3 which avoids using the analytic model of left-invertible operators. In this case, also, the reproducing kernel of the corresponding Aluthge transform is explicit. Part of the proof follows the same line of argumentation as the proof of the reproducing kernel property of range spaces (cf. [3]). To the reader's benefit, we include all necessary details.

**Theorem 7.7.** *Let  $\mathcal{E}$  be a Hilbert space, and let  $k: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$  be an analytic kernel. Assume that the shift  $M_z$  is left-invertible on  $\mathcal{H}_k$ . Then*

$$\langle \tilde{k}(z, w)\eta, \zeta \rangle_{\mathcal{E}} = \langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}, \eta, \zeta \in \mathcal{E})$$

*defines a kernel  $\tilde{k}: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E})$ . Moreover, the shift  $M_z$  on  $\mathcal{H}_k$  defines a bounded linear operator, and there exists a unitary  $U: \mathcal{H}_k \rightarrow \mathcal{H}_{\tilde{k}}$  such that  $U\tilde{M}_z = M_zU$ .*

*Proof.* Define  $\tilde{\mathcal{H}} = |M_z|^{-1/2}\mathcal{H}_k$ . Then  $\tilde{\mathcal{H}} (= \mathcal{H}_k)$  is an  $\mathcal{E}$ -valued function Hilbert space endowed with the inner product  $\langle |M_z|^{-1/2}f, |M_z|^{-1/2}g \rangle_{\tilde{\mathcal{H}}} = \langle f, g \rangle_{\mathcal{H}_k}$  for all  $f, g \in \mathcal{H}_k$ . For each  $f \in \mathcal{H}_k$ ,  $w \in \mathbb{D}$  and  $\eta \in \mathcal{E}$ , we have

$$\begin{aligned} \langle |M_z|^{-1/2}f, |M_z|^{-1}(k(\cdot, w)\eta) \rangle_{\tilde{\mathcal{H}}} &= \langle f, |M_z|^{-1/2}(k(\cdot, w)\eta) \rangle_{\mathcal{H}_k} \\ &= \langle |M_z|^{-1/2}f, k(\cdot, w)\eta \rangle_{\mathcal{H}_k}, \end{aligned}$$

and hence, by the reproducing property of  $\mathcal{H}_k$ , it follows that

$$(7.4) \quad \langle |M_z|^{-1/2}f, |M_z|^{-1}(k(\cdot, w)\eta) \rangle_{\tilde{\mathcal{H}}} = \langle (|M_z|^{-1/2}f)(w), \eta \rangle_{\mathcal{E}}.$$

This says that  $\{|M_z|^{-1}(k(\cdot, w)\eta) : w \in \mathbb{D}, \eta \in \mathcal{E}\}$  reproduces the values of functions in  $\tilde{\mathcal{H}}$ , and furthermore, the evaluation operator  $ev_w: \tilde{\mathcal{H}} \rightarrow \mathcal{E}$  is continuous. Indeed,

$$\begin{aligned} |\langle ev_w(|M_z|^{-1/2}f), \eta \rangle_{\mathcal{E}}| &= |(\langle |M_z|^{-1/2}f \rangle)(w), \eta \rangle_{\mathcal{E}}| \\ &= |\langle |M_z|^{-1/2}f, |M_z|^{-1}(k(\cdot, w)\eta) \rangle_{\tilde{\mathcal{H}}}| \\ &\leq \| |M_z|^{-1/2}f \|_{\tilde{\mathcal{H}}} \| |M_z|^{-1}(k(\cdot, w)\eta) \|_{\tilde{\mathcal{H}}} \\ &= \| |M_z|^{-1/2}f \|_{\tilde{\mathcal{H}}} \| |M_z|^{-1/2}(k(\cdot, w)\eta) \|_{\mathcal{H}_k}. \end{aligned}$$

Since

$$\|k(\cdot, w)\eta\|_{\mathcal{H}_k}^2 = \langle k(\cdot, w)\eta, k(\cdot, w)\eta \rangle_{\mathcal{H}_k} = \langle k(w, w)\eta, \eta \rangle_{\mathcal{E}} = \|k(w, w)^{1/2}\eta\|_{\mathcal{E}}^2,$$

it follows that

$$\begin{aligned} \| |M_z|^{-1/2}(k(\cdot, w)\eta) \|_{\mathcal{H}_k} &\leq \| |M_z|^{-1/2} \|_{\mathcal{B}(\mathcal{H}_k)} \|k(\cdot, w)\eta\|_{\mathcal{H}_k} \\ &= \| |M_z|^{-1/2} \|_{\mathcal{B}(\mathcal{H}_k)} \|k(w, w)^{1/2}\eta\|_{\mathcal{E}} \\ &\leq \| |M_z|^{-1/2} \|_{\mathcal{B}(\mathcal{H}_k)} \|k(w, w)^{1/2}\|_{\mathcal{B}(\mathcal{E})} \|\eta\|_{\mathcal{E}}, \end{aligned}$$

which implies that

$$|\langle ev_w(|M_z|^{-1/2}f), \eta \rangle_{\mathcal{E}}| \leq (\| |M_z|^{-1/2} \|_{\mathcal{B}(\mathcal{H}_k)} \|k(w, w)^{1/2}\|_{\mathcal{B}(\mathcal{E})}) \| |M_z|^{-1/2}f \|_{\tilde{\mathcal{H}}} \|\eta\|_{\mathcal{E}}.$$

Therefore,  $\tilde{\mathcal{H}}$  is an  $\mathcal{E}$ -valued reproducing kernel Hilbert space corresponding to the kernel function

$$\tilde{k}(z, w) = ev_z \circ ev_w^* \quad (z, w \in \mathbb{D}).$$

Clearly, (7.4) implies that  $ev_w^*\eta = |M_z|^{-1}(k(\cdot, w)\eta)$  for all  $w \in \mathbb{D}$  and  $\eta \in \mathcal{E}$ . Since  $\langle \tilde{k}(z, w)\eta, \zeta \rangle_{\mathcal{E}} = \langle ev_w^*\eta, ev_z^*\zeta \rangle_{\mathcal{E}}$ , it follows that

$$\begin{aligned} \langle \tilde{k}(z, w)\eta, \zeta \rangle_{\mathcal{E}} &= \langle |M_z|^{-1}(k(\cdot, w)\eta), |M_z|^{-1}(k(\cdot, z)\zeta) \rangle_{\tilde{\mathcal{H}}} \\ &= \langle |M_z|^{-1/2}(k(\cdot, w)\eta), |M_z|^{-1/2}(k(\cdot, z)\zeta) \rangle_{\mathcal{H}_k}, \end{aligned}$$

that is,  $\langle \tilde{k}(z, w)\eta, \zeta \rangle_{\mathcal{E}} = \langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta \rangle_{\mathcal{H}_k}$ ,  $z, w \in \mathbb{D}$ ,  $\eta, \zeta \in \mathcal{E}$ . Therefore, as a reproducing kernel Hilbert space corresponding to the kernel  $\tilde{k}$ , we have  $\mathcal{H}_{\tilde{k}} = \tilde{\mathcal{H}}$ . Define the unitary map  $U: \mathcal{H}_k \rightarrow \mathcal{H}_{\tilde{k}}$  by

$$Uh = |M_z|^{-1/2}h \quad (h \in \mathcal{H}_k),$$

and recall from Lemma 7.1 that  $\tilde{M}_z^* = |M_z|^{-1/2}M_z^*|M_z|^{1/2}$ . Let  $f \in \mathcal{H}_k$ ,  $w \in \mathbb{D}$ , and let  $\eta \in \mathcal{E}$ . Then

$$\begin{aligned} \langle (U\tilde{M}_zU^*(|M_z|^{-1/2}f))(w), \eta \rangle_{\mathcal{E}} &= \langle U\tilde{M}_zU^*(|M_z|^{-1/2}f), |M_z|^{-1}(k(\cdot, w)\eta) \rangle_{\mathcal{H}_{\tilde{k}}} \\ &= \langle \tilde{M}_zU^*(|M_z|^{-1/2}f), |M_z|^{-1/2}(k(\cdot, w)\eta) \rangle_{\mathcal{H}_k} \\ &= \langle f, \tilde{M}_z^*|M_z|^{-1/2}(k(\cdot, w)\eta) \rangle_{\mathcal{H}_k} \\ &= \langle f, |M_z|^{-1/2}M_z^*(k(\cdot, w)\eta) \rangle_{\mathcal{H}_k}. \end{aligned}$$

But since  $M_z^*(k(\cdot, w)\eta) = \bar{w}k(\cdot, w)\eta$ , we have

$$\begin{aligned} \langle (U\tilde{M}_zU^*(|M_z|^{-1/2}f))(w), \eta \rangle_{\mathcal{E}} &= w \langle f, |M_z|^{-1/2}(k(\cdot, w)\eta) \rangle_{\mathcal{H}_k} \\ &= \langle w(|M_z|^{-1/2}f)(w), \eta \rangle_{\mathcal{E}}, \end{aligned}$$

which implies that

$$U\tilde{M}_zU^*(|M_z|^{-1/2}f) = z(|M_z|^{-1/2}f) \quad (f \in \mathcal{H}_k).$$

Thus, the shift  $M_z$  on  $\mathcal{H}_{\tilde{k}}$  is a bounded linear operator and  $U\tilde{M}_z = M_zU$ . ■

**Definition 7.8.** The kernel  $\tilde{k}$  is called the *standard Aluthge kernel* of  $M_z$ .

In particular, if  $k$  is a scalar-valued kernel, then  $\tilde{k}(\cdot, w) = U(|M_z|^{-1/2}k(\cdot, w))$  and

$$\tilde{k}(z, w) = \langle |M_z|^{-1}k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}).$$

Therefore, if the shift on a tridiagonal space  $\mathcal{H}_k$  is left-invertible, then there are two ways to compute the Aluthge kernel  $\tilde{k}$ : use Theorem 7.3, or use the one above. However, we note that from a general computational point of view, neither approach is completely satisfactory and definite. On the other hand, often the standard Aluthge kernel approach (and sometimes both standard Aluthge kernel and Shimorin–Aluthge kernel methods) lead to satisfactory results. We will discuss this in the following section.

## 8. Truncated tridiagonal kernels

In this section, we introduce a (perhaps both deliberate and accidental) class of analytic tridiagonal kernels from a computational point of view. Let  $\mathcal{H}_k$  be an analytic tridiagonal space corresponding to the kernel

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \quad (z, w \in \mathbb{D}),$$



Now, by Lemma 2.6,  $|M_z|^{-2} = L_{M_z} L_{M_z}^*$ , which implies

$$\left[ |M_z|^{-2} \right] = \begin{bmatrix} \left| \frac{a_1}{a_0} \right|^2 & 0 & 0 \\ 0 & A_{r+1}^2 & 0 \\ 0 & 0 & D^2 \end{bmatrix},$$

where

$$D^2 = \text{diag} \left( \left| \frac{a_{r+3}}{a_{r+2}} \right|^2, \left| \frac{a_{r+4}}{a_{r+3}} \right|^2, \dots \right),$$

and  $A_{r+1}^2$  is a positive definite matrix of order  $r + 1$ . Using this, one easily completes the proof.  $\blacksquare$

From the computational point of view, it is useful to observe that  $A_{r+1}^2 = L_{r+1} L_{r+1}^*$ , where

$$L_{r+1} = \begin{bmatrix} \frac{a_2}{a_1} & 0 & 0 & 0 & 0 \\ d_2 & \frac{a_3}{a_2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-2} \frac{d_2 b_2 \cdots b_{r-1}}{a_3 \cdots a_r} & (-1)^{r-3} \frac{d_3 b_3 \cdots b_{r-1}}{a_4 \cdots a_r} & \cdots & \frac{a_{r+1}}{a_r} & 0 \\ (-1)^{r-1} \frac{d_2 b_2 \cdots b_r}{a_3 \cdots a_r a_{r+1}} & (-1)^{r-2} \frac{d_3 b_3 \cdots b_r}{a_4 \cdots a_r a_{r+1}} & \cdots & d_{r+1} & \frac{a_{r+2}}{a_{r+1}} \end{bmatrix}.$$

In other words,  $A_{r+1}^2$  admits a lower-upper triangular factorization. This is closely related to the Cholesky factorizations/decompositions of positive-definite matrices in the setting of infinite-dimensional Hilbert spaces (see [3] and [12]).

We recall from Theorem 7.6 that the Shimorin–Aluthge kernel of  $M_z$  is given by

$$\tilde{k}(z, w) = P_{\tilde{W}} (I - z L_{\tilde{M}_z})^{-1} (I - \bar{w} L_{\tilde{M}_z}^*)^{-1} |_{\tilde{W}} \quad (z, w \in \mathbb{D}),$$

where  $\tilde{W} = |M_z|^{-1/2} \ker M_z^*$ ,

$$(8.1) \quad L_{\tilde{M}_z} = |M_z|^{1/2} (L_{M_z} + F) |M_z|^{-1/2},$$

and

$$Fg = \langle g, f_0 \rangle \mathcal{H}_k \left( (M_z^* |M_z| M_z)^{-1} M_z^* |M_z| f_0 \right) \quad (g \in \mathcal{H}_k).$$

We now come to the key point.

**Lemma 8.2.** *If  $k$  is a truncated kernel, then  $F = 0$  and  $L_{\tilde{M}_z} |M_z|^{1/2} = |M_z|^{1/2} L_{M_z}$ .*

*Proof.* The matrix representation of  $|M_z|^{-1}$  in Lemma 8.1 implies that  $|M_z| f_0 = \left| \frac{a_0}{a_1} \right| f_0$ , and hence

$$M_z^* |M_z| f_0 = \left| \frac{a_0}{a_1} \right| M_z^* f_0 = 0,$$

by Lemma 2.4. Therefore, the proof follows from the definition of  $F$  and (8.1).  $\blacksquare$

We are finally ready to state and prove the result we are aiming for.



**Theorem 8.3.** *Let  $\mathcal{H}_k$  be a truncated space of order  $r$ . Then the Shimorin–Aluthge kernel is tridiagonal if and only if*

$$c_{mn} = (-1)^{n-m-1} \frac{\bar{b}_{m+1} \cdots \bar{b}_{n-1}}{\bar{a}_{m+2} \cdots \bar{a}_n} c_{m,m+1},$$

for all  $1 \leq m \leq n-2$  and  $3 \leq n \leq r+1$ , where  $c_{mn}$  are the entries of the middle block submatrix of order  $r+1$  of  $[|M_z|^{-1}]$  in Lemma 8.1.

*Proof.* We split the proof into several steps.

**Step 1.** First observe that  $\tilde{k}(z, w) = \sum_{m,n=0}^{\infty} \tilde{X}_{mn} z^m \bar{w}^n$ , where  $\tilde{X}_{mn} = P_{\tilde{w}} L_{\tilde{M}_z}^m L_{\tilde{M}_z}^{*n} |_{\tilde{w}}$  for all  $m, n \geq 0$ . Now Lemma 8.2 implies that

$$L_{\tilde{M}_z}^m L_{\tilde{M}_z}^{*n} = |M_z|^{1/2} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*n} |M_z|^{1/2},$$

and  $P_{\tilde{w}} = I - \tilde{M}_z L_{\tilde{M}_z}$  by (2.13). Since

$$\tilde{M}_z = |M_z|^{1/2} M_z |M_z|^{-1/2} \quad \text{and} \quad L_{\tilde{M}_z} = |M_z|^{1/2} L_{M_z} |M_z|^{-1/2},$$

we have

$$P_{\tilde{w}} = |M_z|^{1/2} (I - M_z L_{M_z}) |M_z|^{-1/2} = |M_z|^{1/2} P_{\mathcal{W}} |M_z|^{-1/2},$$

that is,  $P_{\tilde{w}} |M_z|^{1/2} = |M_z|^{1/2} P_{\mathcal{W}}$ , which implies

$$(8.2) \quad \tilde{X}_{mn} = |M_z|^{1/2} P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*n} |_{\mathcal{W}} \quad (m, n \geq 0).$$

As a passing remark, we note that the above equality holds so long as the finite rank operator  $F = 0$  (this observation will be also used in Example 9.1).

**Step 2.** Now we compute the matrix representation of  $L_{M_z}^p$ ,  $p \geq 1$ . By Theorem 3.4, we have

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & 0 & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & 0 & -\frac{d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In particular, this yields

$$P_{\mathcal{W}} L_{M_z} f_j = \begin{cases} \frac{a_1}{a_0} f_0 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we let  $p \geq 2$ . Recall from (4.6) the definition  $\beta_n^{(p)} = a_n \left(\frac{-b_0}{a_0}\right)^{p-n-1} \beta_n$  for all  $n = 1, \dots, p-1$ , where  $\beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0}$ . Since  $b_0 = 0$ , we have  $\beta_n^{(p)} = 0$ ,  $1 \leq n < p-1$ , and

$$\beta_{p-1}^{(p)} = a_{p-1} \beta_{p-1} = a_{p-1} \left( \frac{b_{p-1}}{a_{p-1}} - \frac{b_0}{a_0} \right),$$

that is,  $\beta_{p-1}^{(p)} = b_{p-1}$  for all  $p \geq 2$ . In particular, since  $b_1 = 0$ , we have  $\beta_1^{(2)} = b_1 = 0$ . Also recall from (4.3) the definition  $d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}} b_{n-p}$ ,  $n \geq p$ . Therefore, by (4.7), the matrix representation of  $L_{M_z}^2$  is given by

$$[L_{M_z}^2] = \begin{bmatrix} 0 & 0 & \frac{a_2}{a_0} & 0 & 0 & \cdots \\ 0 & 0 & \frac{d_2^{(2)}}{a_1} & \frac{a_3}{a_1} & 0 & \ddots \\ 0 & 0 & 0 & \frac{d_3^{(2)}}{a_2} & \frac{a_4}{a_2} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

and in general, by (4.8), we have

$$(8.3) \quad [L_{M_z}^p] = \begin{bmatrix} 0 & \cdots & 0 & \frac{b_{p-1}}{a_0} & \frac{a_p}{a_0} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \frac{d_p^{(p)}}{a_1} & \frac{a_{p+1}}{a_1} & 0 & \ddots \\ 0 & \cdots & 0 & 0 & 0 & \frac{d_{p+1}^{(p)}}{a_2} & \frac{a_{p+2}}{a_2} & \ddots \\ 0 & \cdots & 0 & 0 & 0 & -\frac{d_{p+1}^{(p)} b_2}{a_2 a_3} & \frac{d_{p+2}^{(p)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad (p \geq 2).$$

Then

$$(8.4) \quad [L_{M_z}^{*p}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \frac{\bar{b}_{p-1}}{a_0} & 0 & 0 & 0 & \ddots \\ \frac{\bar{a}_p}{a_0} & \frac{\bar{d}_p^{(p)}}{a_1} & 0 & 0 & \ddots \\ 0 & \frac{\bar{a}_{p+1}}{a_1} & \frac{\bar{d}_{p+1}^{(p)}}{a_2} & -\frac{\bar{d}_{p+1}^{(p)} \bar{b}_2}{\bar{a}_2 \bar{a}_3} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad (p \geq 2).$$

**Step 3.** We prove that  $\tilde{X}_{0n} = |M_z|^{1/2} P_{\mathcal{W}} |M_z|^{-1} L_{M_z}^{*n} |_{\mathcal{W}} = 0$  for all  $n \geq 1$ . In what follows, the above matrix representations and the one of  $|M_z|^{-1}$  in Lemma 8.1 will be used repeatedly. By (4.2), we have  $L_{M_z}^* f_0 = \frac{\bar{a}_1}{a_0} f_1$ , and hence

$$\begin{aligned} \tilde{X}_{01} f_0 &= |M_z|^{1/2} P_{\mathcal{W}} |M_z|^{-1} L_{M_z}^* f_0 \\ &= |M_z|^{1/2} P_{\mathcal{W}} \left( \frac{\bar{a}_1}{a_0} [c_{11} f_1 + \bar{c}_{12} f_2 + \cdots] \right) = 0. \end{aligned}$$

On the other hand, if  $n \geq 2$ , then

$$L_{M_z}^{*n} f_0 = \frac{\bar{b}_{n-1}}{\bar{a}_0} f_{n-1} + \frac{\bar{a}_n}{\bar{a}_0} f_n,$$

and hence  $|M_z|^{-1} f_0 \perp L_{M_z}^{*n} f_0$ . This implies that  $\tilde{X}_{0n} = 0$  for all  $n \geq 2$ . Therefore, all entries in the first row (and hence, also in the first column) of the formal matrix representation of  $\tilde{k}(z, w)$  are zero except the  $(0, 0)$ -th entry (which is  $I_{\mathcal{W}}$ ). Hence (see also (4.1)),

$$[\tilde{k}(z, w)] = \begin{bmatrix} I_{\mathcal{W}} & 0 & 0 & 0 & \cdots \\ 0 & \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} & \ddots \\ 0 & \tilde{X}_{12}^* & \tilde{X}_{22} & \tilde{X}_{23} & \ddots \\ 0 & \tilde{X}_{13}^* & \tilde{X}_{23}^* & \tilde{X}_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

**Step 4.** Our only interest here is to analyze the finite rank (of rank at most one) operator  $\tilde{X}_{m,m+k}$ ,  $m \geq 1$ ,  $k \geq 2$ . The matrix representation in (8.4) implies

$$(8.5) \quad L_{M_z}^{*m+k} f_0 = \frac{1}{\bar{a}_0} (\bar{b}_{m+k-1} f_{m+k-1} + \bar{a}_{m+k} f_{m+k}),$$

and hence

$$(8.6) \quad |M_z|^{-1} L_{M_z}^{*m+k} f_0 = \frac{1}{\bar{a}_0} (\bar{b}_{m+k-1} |M_z|^{-1} f_{m+k-1} + \bar{a}_{m+k} |M_z|^{-1} f_{m+k}).$$

There are three cases to be considered.

*Case I* ( $m+k = r+2$ ). Note that  $b_{r+1} = 0$ . Then

$$|M_z|^{-1} L_{M_z}^{*r+2} f_0 = \frac{1}{\bar{a}_0} (\bar{a}_{r+2} |M_z|^{-1} f_{r+2}),$$

by (8.6), and thus

$$L_{M_z}^m |M_z|^{-1} L_{M_z}^{*r+2} f_0 = \frac{\bar{a}_{r+2}}{\bar{a}_0} L_{M_z}^m |M_z|^{-1} f_{r+2} = \frac{\bar{a}_{r+2}}{\bar{a}_0} \left| \frac{a_{r+3}}{a_{r+2}} \right| L_{M_z}^m f_{r+2}.$$

By (8.3), we have  $P_{\mathcal{W}} L_{M_z}^m f_{r+2} = P_{\mathcal{W}} L_{M_z}^m f_{m+k} = 0$  (note that  $k \geq 2$ ), and hence

$$P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*r+2} f_0 = 0,$$

that is,  $\tilde{X}_{m,m+k} = 0$ . It is easy to check that the equality also holds for  $m = 1$ .

*Case II* ( $m+k-1 \geq r+2$ ). In this case,  $b_{m+k-1} = 0$  and

$$|M_z|^{-1} f_{m+k} = \left| \frac{a_{m+k+1}}{a_{m+k}} \right| f_{m+k}.$$

Again, by (8.3), we have  $P_{\mathcal{W}} L_{M_z}^m f_{m+k} = 0$ ,  $k \geq 2$ , and thus in this case also  $\tilde{X}_{m,m+k} = 0$ . Again, it is easy to check that the equality holds for  $m = 1$ .

*Case III* ( $m+k < r+2$ ). We again stress that  $m \geq 1$  and  $k \geq 2$ . It is useful to observe, by virtue of (8.3) (also see (4.12)), that

$$P_{\mathcal{W}} L_{M_z}^m f_j = \begin{cases} \frac{b_{m-1}}{a_0} f_0 & \text{if } j = m-1, \\ \frac{a_m}{a_0} f_0 & \text{if } j = m, \\ 0 & \text{otherwise.} \end{cases}$$

Now set  $s = m + k - 1$ . The matrix representation of  $|M_z|^{-1}$  in Lemma 8.1 implies that

$$|M_z|^{-1} f_s = c_{1s} f_1 + c_{2s} f_2 + \cdots + c_{ss} f_s + \bar{c}_{s,s+1} f_{s+1} + \cdots + \bar{c}_{s,r+1} f_{r+1}.$$

By (8.3) and the above equality, we have

$$(8.7) \quad P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} f_s = \left( c_{m-1,s} \frac{b_{m-1}}{a_0} + c_{m,s} \frac{a_m}{a_0} \right) f_0.$$

Next, set  $t = m + k$ . Again, the matrix representation of  $|M_z|^{-1}$  in Lemma 8.1 implies that

$$|M_z|^{-1} f_t = c_{1t} f_1 + c_{2t} f_2 + \cdots + c_{tt} f_t + \bar{c}_{t,t+1} f_{t+1} + \cdots + \bar{c}_{t,r+1} f_{r+1},$$

and, again, by (8.3) and the above equality, we have

$$(8.8) \quad P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} f_t = \left( c_{m-1,t} \frac{b_{m-1}}{a_0} + c_{m,t} \frac{a_m}{a_0} \right) f_0.$$

It is easy to see that the equalities (8.7) and (8.8) also hold for  $m = 1$ . The equality in (8.5) becomes

$$|M_z|^{-1} L_{M_z}^{*m+k} f_0 = \frac{1}{\bar{a}_0} (\bar{b}_s |M_z|^{-1} f_s + \bar{a}_t |M_z|^{-1} f_t),$$

and hence, the one in (8.6) implies

$$P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*m+k} f_0 = \frac{1}{|a_0|^2} [\bar{b}_s (c_{m-1,s} b_{m-1} + c_{m,s} a_m) + \bar{a}_t (c_{m-1,t} b_{m-1} + c_{m,t} a_m)] f_0.$$

This shows that  $P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*m+k} f_0 = 0$  if and only if

$$\bar{b}_s (c_{m-1,s} b_{m-1} + c_{m,s} a_m) + \bar{a}_t (c_{m-1,t} b_{m-1} + c_{m,t} a_m) = 0.$$

**Step 5.** So far all we have proved is that  $\tilde{k}$  is tridiagonal if and only if

$$(8.9) \quad b_{m-1} (\bar{b}_{m+k-1} c_{m-1,m+k-1} + \bar{a}_{m+k} c_{m-1,m+k}) + a_m (\bar{b}_{m+k-1} c_{m,m+k-1} + \bar{a}_{m+k} c_{m,m+k}) = 0,$$

for all  $m \geq 1$ ,  $k \geq 2$  and  $m + k < r + 2$ .

If  $m = 1$ , then using the fact that  $b_0 = 0$ , we have  $c_{1,k+1} = -\frac{\bar{b}_k}{\bar{a}_{1+k}} c_{1,k}$ ,  $2 \leq k < r + 1$ , and hence

$$c_{1n} = (-1)^{n-2} \frac{\prod_{i=2}^{n-1} \bar{b}_i}{\prod_{i=3}^n \bar{a}_i} c_{12} \quad (3 \leq n \leq r + 1).$$

Similarly, if  $m = 2$ , then (8.9) together with the assumption that  $b_1 = 0$  implies that

$$(8.10) \quad c_{2n} = (-1)^{n-3} \frac{\prod_{i=3}^{n-1} \bar{b}_i}{\prod_{i=4}^n \bar{a}_i} c_{23} \quad (4 \leq n \leq r + 1).$$

Next, if  $m = 3$ , then (8.9) again implies

$$b_2(\bar{b}_{k+2} c_{2,k+2} + \bar{a}_{k+3} c_{2,k+3}) + a_3(\bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3}) = 0 \quad (k < r - 1).$$

On the other hand, by (8.10), we have  $c_{2,k+3} = -\frac{\bar{b}_{k+2}}{\bar{a}_{k+3}} c_{2,k+2}$ , and hence

$$\bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3} = 0,$$

which implies

$$c_{3,k+3} = -\frac{\bar{b}_{k+2}}{\bar{a}_{k+3}} c_{3,k+2} \quad (k < r - 1).$$

Now, evidently the recursive situation is exactly the same as that of the proof of Theorem 5.2 (more specifically, see (5.2)). This completes the proof of the theorem. ■

As is clear by now, by virtue of Theorem 5.2, the classification criterion of the above theorem is also a classification criterion of tridiagonality of standard Aluthge kernels. Therefore, we have the following corollary.

**Corollary 8.4.** *If  $\mathcal{H}_k$  is a truncated space, then the Shimorin–Aluthge kernel of  $M_z$  is tridiagonal if and only if the standard Aluthge kernel of  $M_z$  is tridiagonal.*

## 9. Final comments and results

First we comment on the assumptions in the definition of truncated kernels (see Section 8). The main advantage of the truncated space corresponding to a truncated kernel is that  $F = 0$ , where  $F$  is the finite rank operator as in (7.3). In this case, as already pointed out, we have  $L_{\tilde{M}_z} = |M_z|^{1/2} L_{M_z} |M_z|^{-1/2}$ . This brings a big cut down in computation. On the other hand, quite curiously, if

$$b_0 = b_1 = 1 \quad \text{or} \quad b_0 = 1,$$

and all other  $b_i$ 's are equal to 0, then the corresponding standard Aluthge kernel of  $M_z$  is tridiagonal kernel but the corresponding Shimorin–Aluthge kernel of  $M_z$  is not a tridiagonal kernel. Since computations are rather complicated in the presence of  $F$ , we only present the result for the following (convincing) case.

**Example 9.1.** Let  $a_n = b_0 = b_1 = 1$  and  $b_m = 0$  for all  $n \geq 0$  and  $m \geq 2$ . Let  $\mathcal{H}_k$  denote the tridiagonal space corresponding to the basis  $\{(a_n + b_n z) z^n\}_{n \geq 0}$ . By (2.8) and Theorem 3.4, we have

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad [L_{M_z}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ddots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

respectively. Hence, applying  $L_{M_z} L_{M_z}^* = |M_z|^{-2}$  (see Lemma 2.6) to this, we obtain

$$|M_z|^{-2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \ddots \\ 0 & -1 & 2 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Suppose  $\alpha = \frac{3+\sqrt{5}}{2}$  and  $\beta = \frac{3-\sqrt{5}}{2}$ . It is useful to observe that  $(1-\alpha)(1-\beta) + 1 = 0$ . Set

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{1/2},$$

where

$$a = \frac{1}{\sqrt{5}}[\sqrt{\alpha}(1-\beta) - \sqrt{\beta}(1-\alpha)], \quad b = \frac{1}{\sqrt{5}}[-\sqrt{\alpha} + \sqrt{\beta}],$$

$$c = \frac{1}{\sqrt{5}}[-\sqrt{\alpha}(1-\alpha) + \sqrt{\beta}(1-\beta)].$$

Clearly,

$$|M_z|^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

From this it follows that  $|M_z|f_0 = f_0$ , and hence the finite rank operator  $F$ , as in (7.3), is given by

$$Fg = \langle g, f_0 \rangle_{\mathcal{H}_k} ((M_z^* |M_z| M_z)^{-1} M_z^* |M_z| f_0) = 0 \quad (g \in \mathcal{H}_k).$$

Then  $F = 0$ , and hence (7.2) implies that  $L_{\tilde{M}_z} = |M_z|^{1/2} L_{M_z} |M_z|^{-1/2}$ . By (8.2) (see also Step 1 in the proof of Theorem 8.3), the coefficient of  $z^m \bar{w}^n$  of the Shimorin–Aluthge kernel  $\tilde{k}$  is given by  $\tilde{X}_{mn} = |M_z|^{1/2} P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^n |_{\mathcal{W}}$ ,  $m, n \geq 0$ . We compute the coefficient of  $z \bar{w}^3$  as

$$\begin{aligned} P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^{*3} f_0 &= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^{*2} f_1 \\ &= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^* f_2 \\ &= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} (-f_2 + f_3) \\ &= P_{\mathcal{W}} L_{M_z} (-bf_1 - cf_2 + f_3) \\ &= P_{\mathcal{W}} L_{M_z} (-bf_1) \\ &= -bf_0. \end{aligned}$$

But  $b = \frac{1}{\sqrt{5}}[-\sqrt{\alpha} + \sqrt{\beta}] \neq 0$ , and hence  $\tilde{X}_{13} \neq 0$ . This implies that the Shimorin–Aluthge kernel is not tridiagonal. On the other hand, the matrix representation of  $|M_z|^{-1}$  implies right away that the standard Aluthge kernel is tridiagonal (see Theorem 5.2).

Now we return to standard Aluthge kernels of shifts (see the definition following Theorem 7.7). Let  $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$  be a reproducing kernel Hilbert space. Suppose  $M_z$  on  $\mathcal{H}_k$  is left-invertible. Then Theorem 7.7 says that  $\tilde{M}_z$  and  $M_z$  on  $\mathcal{H}_{\tilde{k}}(\subseteq \mathcal{O}(\mathbb{D}))$  are unitarily equivalent, where

$$\tilde{k}(z, w) := \langle |M_z|^{-1}k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} = (|M_z|^{-1}k(\cdot, w))(z),$$

for all  $z, w \in \mathbb{D}$ . In the following, as a direct application of Theorem 5.2, we address the issue of tridiagonal representation of the shift  $M_z$  on  $\mathcal{H}_k$ .

**Corollary 9.2.** *In the setting of Theorem 7.7, assume in addition that  $\mathcal{E} = \mathbb{C}$  and  $\mathcal{H}_{\tilde{k}}$  is a tridiagonal space with respect to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n(z) = (a_n + b_n z)z^n$ ,  $n \geq 0$ . Then  $\mathcal{H}_k$  is a tridiagonal space if and only if*

$$U|M_z|U^* = \begin{bmatrix} c_{00} & c_{01} & -\frac{\bar{b}_1}{a_2}c_{01} & \frac{\bar{b}_1\bar{b}_2}{a_2a_3}c_{01} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & -\frac{\bar{b}_2}{a_3}c_{12} & \ddots \\ -\frac{b_1}{a_2}\bar{c}_{01} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \frac{b_1b_2}{a_2a_3}\bar{c}_{01} & -\frac{b_2}{a_3}\bar{c}_{12} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

with respect to the basis  $\{f_n\}_{n \geq 0}$ .

*Proof.* Recall from Theorem 7.7 that  $\mathcal{H}_{\tilde{k}} = |M_z|^{-1/2}\mathcal{H}_k$  and  $Uh = |M_z|^{-1/2}h$ ,  $h \in \mathcal{H}_k$ , defines the intertwining unitary. Set  $P := U|M_z|U^*$ . Then  $P \in \mathcal{B}(\mathcal{H}_{\tilde{k}})$  is a positive operator, and for any  $z, w \in \mathbb{D}$ , we have

$$\begin{aligned} \langle P\tilde{k}(\cdot, w), \tilde{k}(\cdot, z) \rangle_{\mathcal{H}_{\tilde{k}}} &= \langle |M_z|U^*\tilde{k}(\cdot, w), U^*\tilde{k}(\cdot, z) \rangle_{\mathcal{H}_k} \\ &= \langle |M_z||M_z|^{-1/2}k(\cdot, w), |M_z|^{-1/2}k(\cdot, z) \rangle_{\mathcal{H}_k} \\ &= \langle k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k}, \end{aligned}$$

as  $U(|M_z|^{-1/2}k(\cdot, w)) = \tilde{k}(\cdot, w)$ . Hence,  $k(z, w) = \langle P\tilde{k}(\cdot, w), \tilde{k}(\cdot, z) \rangle_{\mathcal{H}_{\tilde{k}}}$ ,  $z, w \in \mathbb{D}$ . The result now follows from Theorem 5.2.  $\blacksquare$

In particular, if  $\tilde{k}$  is a tridiagonal kernel, then for  $k$  to be a tridiagonal kernel, it is necessary (as well as sufficient) that  $U|M_z|U^*$  is of the form as in the above statement.

We conclude this paper with the following curious observation which stems from the matrix representations of Shimorin left inverses of shifts on analytic tridiagonal spaces (see Theorem 3.4). Let  $\mathcal{H}_k$  be an analytic tridiagonal space. Recall that  $L_{M_z}$  denotes the Shimorin left inverse of  $M_z$ . By Lemma 2.6, we have  $|M_z|^{-2} = L_{M_z}L_{M_z}^*$ . From the matrix representation of  $L_{M_z}$  in Theorem 3.4, one can check that the matrix representation of  $|M_z|^{-2}$  satisfies the conclusion of Theorem 5.2. Consequently, the positive definite scalar kernel

$$K(z, w) = \langle |M_z|^{-2}k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D}),$$

is a tridiagonal kernel. On the other hand, consider

$$a_n = \begin{cases} 2 & \text{if } n = 2, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad b_n = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then the shift  $M_z$  on the analytic tridiagonal space  $\mathcal{H}_k$  corresponding to the orthonormal basis  $\{f_n\}_{n \geq 0}$ , where  $f_n(z) = (a_n + b_n z)z^n$ ,  $n \geq 0$ , is left-invertible. However, a moderate computation reveals that the matrix representation of  $|M_z|^{-1}$  does not satisfy the conclusion of Theorem 5.2. In other words, the positive definite scalar kernel

$$K(z, w) = \langle |M_z|^{-1}k(\cdot, w), k(\cdot, z) \rangle_{\mathcal{H}_k} \quad (z, w \in \mathbb{D})$$

is not a tridiagonal kernel.

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