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Stability estimates in inverse problems for the Schrödinger and wave equations with trapping

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Abstract. For a class of Riemannian manifolds with boundary that includes all negatively curved manifolds with strictly convex boundary, we establish Hölder type stability estimates in the geometric inverse problem of determining the electric potential or the conformal factor from the Dirichlet-to-Neumann map associated with the Schrödinger equation and the wave equation. The novelty in this result lies in the fact that we allow some geodesics to be trapped inside the manifold and have infinite length.

Dedicated to the memory of Slava Kurylev.

1. Introduction

In this article we study a geometric inverse problem associated with the anisotropic Schrödinger equation and the wave equation on a compact Riemannian manifold (M, g) with boundary ∂M.

Let Δ_g be the non-negative Laplace–Beltrami operator associated with the metric g, we consider two initial-value-problems. First, we consider the Schrödinger equation for finite time of propagation and with Dirichlet conditions:

$$\begin{cases} (i\partial_t - \Delta_g + q(x))u(t,x) = 0, & \text{in } (t,x) \in I \times M, \\ u(0,\cdot) = 0, & \text{in } x \in M, \\ u(t,x) = f(t,x), & \text{on } (t,x) \in I \times \partial M, \end{cases} \tag{1.1}$$

where $I = (0, T)$ for $T > 0$ fixed. Secondly, we consider the wave equation for infinite time of propagation and with Dirichlet conditions:

$$\begin{cases} (\partial_t^2 + \Delta_g + q(x))u(t,x) = 0, & \text{in } (t,x) \in I \times M, \\ u(0,\cdot) = 0, \ \ \partial_t u(0,\cdot) = 0, & \text{in } x \in M, \\ u(t,x) = f(t,x), & \text{on } (t,x) \in I \times \partial M, \end{cases} \tag{1.2}$$

where $I = (0, T)$ and T can be equal to $+\infty$.

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We aim at studying the problem of the stable recovery of the potential q, or alternatively conformal factor in a conformal class of a metric g, from the *Dirichlet-to-Neumann map* associated with (1.1) and (1.2). The Dirichlet-to-Neumann map (DN map in short) is the operator defined, for each $T < \infty$, by

$$\Lambda^S_{g,q} : H^1_0([0,T) \times \partial M) \to L^2((0,T) \times \partial M), \quad \Lambda^S_{g,q} f := -\partial_{\mathrm{n}} u^S|_{(0,T)\times\partial M},$$
$$\Lambda^W_{g,q} : H^1_0([0,T) \times \partial M) \to L^2((0,T) \times \partial M), \quad \Lambda^W_{g,q} f := -\partial_{\mathrm{n}} u^W|_{(0,T)\times\partial M},$$

where u^S solves (1.1), u^W solves (1.2), and ∂_{n} is the unit inward normal derivative at ∂M. Here $H^1_0([0,T) \times \partial M)$ denotes the closed subspace of functions in $H^1([0,T) \times \partial M)$ vanishing at $t = 0$. For the wave equation, we shall need to consider the case $T = \infty$, and we will show that there is $\nu_0 \geq 0$ depending only on $\|q\|_{L^\infty}$ so that for all $\nu \geq \nu_0$,

$$\Lambda^W_{g,q} : e^{\nu t} H^1_0(\mathbb{R}_+ \times \partial M) \to e^{\nu t} L^2(\mathbb{R}_+ \times \partial M)$$

is bounded.

By a stability estimate, we mean that there is a constant $C > 0$, possibly depending on some a priori bound on $\|q_j\|_{H^s(M)}$ for some $s \geq 0$, such that an estimate of the following form holds:

$$\|q_1 - q_2\|_{L^2(M)} \leq C F(\|\Lambda^{S/W}_{g,q_1} - \Lambda^{S/W}_{g,q_2}\|_{*,\nu}),$$

where the used norm for the Schrödinger/wave DN map are, respectively,

$$\|\cdot\|_* = \|\cdot\|_{H^1(I\times\partial M)\to L^2(I\times\partial M)} \quad \text{and} \quad \|\cdot\|_{*,\nu} = \|\cdot\|_{e^{\nu t}H^1_0(I\times\partial M)\to e^{\nu t}L^2(I\times\partial M)}$$

and F is a continuous function satisfying $F(0) = 0$; we shall write simply $\|\cdot\|_*$ for the wave case when $I = [0,T]$ with $T < \infty$, and $\nu = 0$. We say that the stability is of Hölder type if $F(x) = x^\beta$ for some $\beta > 0$, and it is said of log-type if $F(x) = \log(1/x)^{-\beta}$ for some $\beta > 0$. More generally, one can ask if there is a stability for the problem of recovering the metric, i.e.,

$$\|g_1 - \psi^* g_2\|_{L^2(M)} \leq C\, F(\|\Lambda^{S/W}_{g_1,0} - \Lambda^{S/W}_{g_2,0}\|_*)$$

for some diffeomorphism ψ (depending on g_1 and g_2). Here we have used the L^2 norm on M to measure $q_1 - q_2$, but one could also ask the same question for Sobolev or Hölder norms. Assuming a priori bounds on q in some large enough Sobolev spaces $H^{s_0}(M)$ allows to deduce (by interpolation) bounds on $\|q_1 - q_2\|_{H^s}$ for $s < s_0$ if one has bounds on $\|q_1 - q_2\|_{L^2}$ (and similarly for $g_1 - g_2$).

The problem of determination of the metric g or the potential q from $\Lambda^W_{g,q}$ was solved in general by Belishev–Kurylev [3] (see also [10]), but the stability estimates in the general setting appeared only recently in the work of Burago–Ivanov–Lassas–Lu [6] and are of $\log\log$ type (i.e., $F(x) = |\log|\log x||^{-\beta}$) for the case with no potential. When $g = g_{\mathrm{eucl}}$ is the Euclidean metric on a domain $M \subset \mathbb{R}^n$, a Hölder type stability was proved by Sun [19] and Alessandrini–Sun–Sylvester [1] for the determination of the potential q from $\Lambda^W_{g_{\mathrm{eucl}},q}$. In the case of non-Euclidean metrics, but close to the Euclidean metric on a ball in $\mathbb{R}^n$, Stefanov–Uhlmann [16] obtained Hölder estimates for the metric recovery (with no potential involved), and they extended this result in [17] to Riemannian metrics close to a *simple* metric g_0 with injective X-ray transform on symmetric 2-tensors. Such simple metrics are dense among simple metrics. We recall that simple metrics are Riemannian metrics with no conjugate points on a ball B in $\mathbb{R}^n$ with strictly convex boundary; in particular, all geodesics in B for such a metric have finite length with endpoints on the boundary ∂B. If g_0 is a fixed simple metric, Bellassoued and Dos Santos Ferreira [4, 5] proved Hölder stability of the inverse problem for both $\|\Lambda^S_{g_0,q_1} - \Lambda^S_{g_0,q_2}\|_*$ and $\|\Lambda^W_{g_0,q_1} - \Lambda^W_{g_0,q_2}\|_*$. When g is close to a fixed simple metric g_0 with injective X-ray transform on 2-tensors, Montalto [14] extended the previous result to the recovery of the pair (g, q) (and a magnetic potential term in addition) in a Hölder stable way. For non-simple metrics, we are aware of only two results showing strong stability: the first by Bao–Zhang [2], who prove for a non-trapping metric $g = c(x)^2 g_{\mathrm{eucl}}$, conformal to the Euclidean metric, and satisfying certain assumptions on their conjugate points, that if $\|\Lambda^W_{c^2 g_{\mathrm{eucl}}} - \Lambda^W_{\tilde{c}^2 g_{\mathrm{eucl}}}\|_*$ is small enough then the conformal factors agree $c = \tilde{c}$; the second, by Stefanov–Uhlmann–Vasy [18], is of the same kind but under the assumption that g_{eucl} is replaced by a metric g_0 so that the manifold (M, g_0) can be foliated by strictly convex hypersurfaces. In all these results, the time interval $I = (0, T)$ can be taken with $T > 0$ finite but large enough for the wave case (depending on the diameter of the domain), while for the Schrödinger case it can be taken finite and small using infinite speed of propagation.

All these mentioned results, where Hölder stability results hold, assume no trapped geodesic rays for the Riemannian manifold (M, g), i.e., geodesics staying inside the interior M° of M for infinite time. Existence of trapped geodesics means that some regions of the phase space are not accessible from the boundary by geodesic rays, and some waves can possibly stay (microlocally) trapped for a long time near these trapped rays, so that a part of the information can not be read off microlocally from the DN map at the boundary. It is thus an open question to understand how stable is the recovery of the coefficients of the wave equation or the Schrödinger equation when the metric is not simple. The difficulty to obtain such Hölder estimates lies in the fact that one usually reduces the inverse problem for the DN map to some X-ray tomography problem using wave packets or WKB solutions of the wave/Schrödinger equations that concentrate near single geodesics going from a point of the boundary to another point. It is likely that under general assumptions, no Hölder stability estimates hold but log stability estimates do; we mention the recent work of Koch–Rüland–Salo [11] about this question. Our purpose in this work is to address this stability question in a family of cases where the trapped set is sufficiently filamentary, the typical example being that of a non-simply connected Riemannian metric with negative curvature and strictly convex boundary.

Our main geometric assumptions are the hyperbolicity of the trapped set for the geodesic flow and the absence of conjugate points. We notice that these two assumptions are satisfied if (M, g) is negatively curved. Let us recall the precise definition of hyperbolic trapped set. Let $\varphi_t\colon SM \to SM$ be the geodesic flow for $t \in \mathbb{R}$, where $SM = \{(x, v) \in TM : |v|_{g(x)} = 1\}$ is the unit tangent bundle. We call, for every $z = (x, v) \in SM$, the escape time of SM in positive (+) and negative (−) times,

$$\tau_+(z) := \sup\{t \geq 0 \mid \forall s < t, \varphi_s(z) \in SM^\circ\} \in [0, +\infty],$$
$$\tau_-(z) := \inf\{t \leq 0 \mid \forall s > t, \varphi_s(z) \in SM^\circ\} \in [-\infty, 0].$$

In other words, $\pm\tau_\pm(z)$ is the time needed for the geodesic $(\varphi_{\pm t}(z))|_{t\geq 0}$ to reach $\partial_\pm SM \cup \partial_0 SM$. The incoming ($-$) and outgoing ($+$) tails in SM are defined by

$$\Gamma_\mp = \{z \in SM \mid \tau_\pm(z) = \pm\infty\},$$

and the trapped set for the flow on SM is the set $K := \Gamma_+ \cap \Gamma_-$. If ∂M is strictly convex for (M, g) (i.e., the second fundamental form of ∂M is positive), the trapped set K is a compact flow-invariant subset of the interior SM° of SM. We say that the trapped set $K \subset SM$ is a *hyperbolic set* if there exist $C > 0$ and $\nu > 0$ so that there is a continuous flow-invariant splitting over K,

$$T_K(SM) = \mathbb{R}X \oplus E_u \oplus E_s, \tag{1.3}$$

where X is the geodesic vector field on SM, and E_s, E_u are vector subspaces satisfying for all $z \in K$,

$$\|\mathrm{d}\varphi_t(z)\,w\| \leq Ce^{-\nu t}\|w\|, \qquad \forall t > 0,\ \forall w \in E_s(z), \tag{1.4}$$

$$\|\mathrm{d}\varphi_t(z)\,w\| \leq Ce^{-\nu |t|}\|w\|, \qquad \forall t < 0,\ \forall w \in E_u(z), \tag{1.5}$$

with respect to any fixed metric on SM. The notion of conjugate points can be defined as follows. If $\pi_0 \colon SM \to M$ is the projection and $\mathcal{V} := \ker d\pi_0 \subset T(SM)$ is the vertical bundle of the fibration, we say that there is no conjugate point if $\mathrm{d}\varphi_t(\mathcal{V}) \cap \mathcal{V} = \{0\}$ for all $t \neq 0$, where $\{0\}$ denotes the 0-section of $T(SM)$.

1.1. The case of the Schrödinger equation

The DN map associated with (1.1) is bounded (see Theorem 1 in [4]) as an operator from $H^1((0,T) \times \partial M)$ to $L^2((0,T) \times \partial M)$. Our first goal is to a obtain a Hölder stability estimate of the form

$$\|q_1 - q_2\|_{L^2(M)} \leq C\|\Lambda_{g,q_1} - \Lambda_{g,q_2}\|_*^\beta, \tag{1.6}$$

for some $\beta > 0$ for the Schrödinger equation on a bounded time interval $(0, T)$. Here we assume that q_1 and q_2 belong to the family of admissible electrical potentials,

$$\mathcal{Q}(N_0) := \{q \in W^{1,\infty}(M) \mid \|q\|_{W^{1,\infty}(M)} \leq N_0\}, \tag{1.7}$$

with $N_0 > 0$ fixed, and that q_1 and q_2 coincide on the boundary ∂M. It is known that the estimate (1.6) holds on simple manifolds [4] with $\beta = 1/8$. Our aim is to extend this result to the case of hyperbolic trapped set of the geodesic flow and no conjugate points.

Our first result gives the stable determination of the potential q from the DN map.

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension $d \geq 2$ with strictly convex boundary. Let $T, N_0 > 0$ fixed. Assume that the trapped set K is hyperbolic and there are no conjugate points. Then, there exists a constant $C = C(M, g, T, N_0) > 0$ such that, for any $q_1, q_2 \in \mathcal{Q}(N_0)$ with $q_1 = q_2$ on ∂M,*

$$\|q_1 - q_2\|_{L^2(M)} \leq C\|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*^\beta, \tag{1.8}$$

for some $\beta > 0$ depending only on (M, g).

We notice from our proof that the constant β can be expressed in terms of the volume entropy and dynamical quantities on the geodesic flow of (M, g), more precisely, the pressure of the unstable Jacobian of the geodesic flow on the trapped set and the maximal expansion rate of the flow.

In order to obtain a stability estimate for the conformal factor of the metric, we consider the family of admissible conformal factors given by

$$\mathscr{C}(N_0, k, \varepsilon) := \{c \in \mathcal{C}^\infty(M) \,|\, c > 0 \text{ in } \overline{M},\ \|1-c\|_{\mathcal{C}^0(M)} \le \varepsilon,\ \|c\|_{\mathcal{C}^k(M)} \le N_0\}. \tag{1.9}$$

Our second result gives the stable determination of the conformal factor.

Theorem 2. *Let (M, g) be a compact Riemannian manifold of dimension $d \ge 2$ with strictly convex boundary, hyperbolic trapped set K and no conjugate points. Let $T, N_0 > 0$ be fixed. Then, there exist $k \ge 1$ depending only on* $\dim(M)$*, $\varepsilon > 0$ depending on (M, g, N_0) and a constant $C = C(M, g, T, N_0) > 0$ such that, for any $c \in \mathscr{C}(N_0, k, \varepsilon)$ with $c = 1$ near ∂M,*

$$\|1-c\|_{L^2(M)} \le C\|\Lambda^S_{g,0} - \Lambda^S_{cg,0}\|_*^\beta, \tag{1.10}$$

for some $\beta > 0$ depending only on (M, g).

As far as we know, these two results are the first Hölder stability results for the Schrödinger equation when the principal symbol of the operator has trapped bicharacteristic rays.

1.2. The case of the wave equation

The DN map associated with (1.2) with $I = (0, T)$ is bounded as an operator mapping $H_0^1((0,T) \times \partial M)$ to $L^2((0,T) \times \partial M)$ (see [12,13]). In the case $I = (0, \infty)$, it is necessary to introduce a exponential weight in the time as $T \to +\infty$ to obtain boundedness of the DN map. For our result, due to the fact that some geodesics have infinite length (those that are trapped), we need to consider the wave equation for all positive time.

For $k, \ell \in \mathbb{N}_0$, let $\nu > 0$. We define the weighted Sobolev space $e^{\nu t} H^k(I; H^\ell(M))$ as the space of functions $f \in H^k(I; H^\ell(M))$, with finite norm

$$\|f\|_{e^{\nu t} H^k(I;H^\ell(M))} := \sum_{j=0}^k \Big(\int_0^\infty e^{-2\nu t} \|\partial_t^j f(t,\cdot)\|^2_{H^\ell(M)} \,\mathrm{d}t \Big)^{1/2}.$$

In particular, we denote $e^{\nu t} H^k(I \times M) := e^{\nu t} H^k(I; H^k(M))$. Similarly, we define the weighted Sobolev spaces $e^{\nu t} H^k(I; H^\ell(\partial M))$ on the boundary ∂M, and we denote $e^{\nu t} H^k(I \times \partial M) := e^{\nu t} H^k(I; H^k(\partial M))$.

The DN map associated with (1.2) is continuous from $e^{\nu t} H_0^1(I \times \partial M)$ to $e^{\nu t} L^2(I \times \partial M)$ for every $\nu \ge \nu_0$: this follows from Theorems 6.10 and 7.1 in [7], and can be checked that $\nu_0 \ge 0$ depends only on $\|q\|_{L^\infty}$, as we show in Lemma 4.3 and the comment that follows. We denote

$$\|\Lambda^W_{g,q}\|_{*,\nu} := \|\Lambda^W_{g,q}\|_{\mathscr{L}(e^{\nu t} H_0^1(I \times \partial M); e^{\nu t} L^2(I \times \partial M))}. \tag{1.11}$$

We next state our main result on the stable determination of the electric potential from the DN map.

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$ with strictly convex boundary, hyperbolic trapped set and no conjugate points. Let $N_0 > 0$ be fixed. There is ν_0 depending only on N_0 such that for every $\nu > \nu_0$, there exists $C > 0$ such that, for any $q_1, q_2 \in \mathcal{Q}(N_0)$ with $q_1 = q_2$ on ∂M,*

$$\|q_1 - q_2\|_{L^2(M)} \leq C \|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|^{\beta}_{*,\nu}, \tag{1.12}$$

for some $\beta > 0$ depending only on (M, g) and ν.

We finally state our main result on the stable recovery of the conformal factor.

Theorem 4. *Let (M, g) be a compact Riemannian manifold of dimension $d \geq 2$ with strictly convex boundary, hyperbolic trapped set and no conjugate points. Let $N_0 > 0$ be fixed. Then, there exist $\nu_0 > 0$, $k \geq 1$ depending only on d, and $\varepsilon > 0$ depending on (M, g, N_0), such that for all $\nu > \nu_0$, there is C depending on (M, g, N_0, ν) so that, for any $c \in \mathscr{C}(N_0, k, \varepsilon)$ with $c = 1$ near ∂M,*

$$\|1 - c\|_{L^2(M)} \leq C \|\Lambda^W_{g,0} - \Lambda^W_{cg,0}\|^{\beta}_{*,\nu}, \tag{1.13}$$

for some $\beta > 0$ depending only on (M, g) and ν.

1.3. Method of proof

To obtain the stability results, we use the general method of [4,5,17] of reducing the problem to some estimate on X-ray transform of $q_1 - q_2$. We however need to perform several important modifications due to trapping. Ultimately, we rely on some results of the second author [9] on the injectivity and stability estimates of the X-ray transform for the class of manifold under study, but it is not a simple reduction to that problem, as we now explain. We first follow the well-known route of constructing WKB solutions u of the Schrödinger/wave equation concentrating on each geodesic γ of length less or equal to $T_0 > 0$ with endpoints on the boundary. We use the universal covering of M to construct u since M is not assumed simply connected. We can then bound the integral of $q_1 - q_2$ along these geodesics by a constant times $\|\Lambda^{S/W}_{g,q_1} - \Lambda^{S/W}_{g,q_2}\|_{*,\nu}$. The non-simple metric assumption complicates that step compared to the simple metric case, due to the fact that geodesics self intersect. In the Schrödinger equation, using the infinite speed of propagation, we can take T_0 as large as we want by taking WKB solutions with frequencies $\lambda \gg T_0/T$, while for the wave equation we need to know the DN map on time $[0, \infty)$ to be able to let T_0 be arbitrarily large. We then use some estimate on the volume of the set of geodesics staying in M° for time $\leq T_0$: this volume decays exponentially in T_0. We deduce that the transform $I_0^* I_0(q_1 - q_2)$ of $q := q_1 - q_2$ can be controled in L^2 by

$$C e^{C_0 T_0} \|\Lambda^{S/W}_{g,q_1} - \Lambda^{S/W}_{g,q_2}\|^{1/4}_{*,\nu} \|q\|^{1/2}_{W^{1,\infty}} + C e^{-\varepsilon T_0} \|q\|_{L^\infty} \tag{1.14}$$

for some $C_0 > 0, C > 0, \varepsilon > 0$ independent of T_0. Here $I_0 \colon L^\infty(M) \to L^2_{\mathrm{loc}}(\partial SM \setminus \Gamma_-)$ is the X-ray transform defined by

$$I_0 q(z) := \int_0^{\tau_+(z)} q(\pi_0(\varphi_t(z)))\, \mathrm{d}t,$$

that extends continuously to $L^\infty(M) \to L^2(\partial SM)$ by [9]; here, $\pi_0 : SM \to M$ is the projection on the base. For simple metrics, it is well known (see [15]) that the normal operator $\Pi_0 := I_0^* I_0$ is an elliptic pseudo-differential operator of order -1, thus satisfying $\|\Pi_0 f\|_{H^s} \geq C_s \|f\|_{H^{s-1}}$ for all $s \geq 0$ and $C_s > 0$ depending on s. In [9], using anisotropic Sobolev spaces an Fredholm theory for vector fields generating Axiom A flows [8], it is shown that the same properties hold on Π_0 for metrics with no conjugate points and hyperbolic trapping. We can then bound the norm of $\|q_1 - q_2\|$ by a constant times some norm $\|\Pi_0(q_1 - q_2)\|$, which in turn is bounded by (1.14). Taking T_0 large enough (depending on $\|\Lambda^{S/W}_{g,q_1} - \Lambda^{S/W}_{g,q_2}\|_{*,\nu}$) and using interpolation estimates, we can then show that the second term of (1.14) can be absorbed into the first term, and we obtain the desired stability bound. The case of the recovery of the conformal factor is addressed using a similar type of arguments.

We make a final comment about the assumption $q_1 = q_2$ on ∂M (respectively, $c = 1$ near ∂M): this assumption could be removed by standard arguments provided the potentials q_i (respectively, for c) have uniform bounds in $\mathcal{C}^k(M)$ for k large enough. Since this amounts to construct geometrical optics solutions concentrated on very short geodesics almost tangent to ∂M, the proof is basically the same as in Section 3 of [17] and Theorem 2 in [14] in the case of the wave equation, and a slight variation in the case of the Schrödinger equation. We write this short argument in an Appendix, where $q_j \in \mathcal{C}^4(M)$ (respectively, $q_j \in \mathcal{C}^8(M)$) is sufficient for the wave (respectively, Schrödinger) equation.

Notations. In what follows, we shall use the notational convention of writing $C > 0$ for constants appearing in upper/lower bounds, where these constants may change from line to line, and we shall indicate its dependence on the parameters of our problem when this is important.

2. Geometric setting and dynamical properties of the geodesic flow

In this section we recall, for a Riemannian manifold (M, g) with strictly convex boundary, some notions about the geometry of the unit tangent bundle $SM := \{(x, v) \in TM \mid g_x(v, v) = 1\}$ and the dynamics of the geodesic flow on SM. Let

$$\pi_0 : SM \to M, \quad \pi_0(x, v) = x,$$

be the natural projection on the base. We will denote by X the geodesic vector field on SM defined by $Xf(x, v) = \partial_t f(\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))|_{t=0}$, where $\gamma_{(x,v)}(t)$ is the unit speed geodesic with initial condition $(\gamma_{(x,v)}(0), \dot\gamma_{(x,v)}(0)) = (x, v)$. We will denote by $\varphi_t(x, v) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))$ the geodesic flow, which in turn is the flow of the vector field X.

The incoming $(-)$ and outgoing $(+)$ boundaries of the unit tangent bundle of M are defined by

$$\partial_\pm SM := \{(x, v) \in SM \mid x \in \partial M,\ \mp g_x(v, \mathrm{n}) > 0\},$$

where n is the inward pointing unit normal vector field to ∂M. For any $(x, v) \in SM$, define the forward and backward escaping times by

$$\tau_+(x, v) = \sup\{t \geq 0 \mid \varphi_t(x, v) \in \partial SM \text{ or } \forall s \in (0, t), \varphi_s(x, v) \in SM^\circ\} \in [0, +\infty],$$
$$\tau_-(x, v) := -\tau_+(x, -v) \in [-\infty, 0],$$

where τ_+ satisfies $X\tau_+ = -1$ in SM with $\tau_+|_{\partial_+ SM} = 0$. For $(x,v) \in \partial_- SM$, the geodesic $\gamma_{(x,v)}$ with initial point x and tangent vector v either has infinite length (i.e., $\tau_+(x,v) = +\infty$) or it intersects ∂M at a boundary point $x' \in \partial M$ with tangent vector v' with $(x',v') \in \partial_+ SM$. The incoming $(-)$ and outgoing $(+)$ tails in SM are defined by

$$\Gamma_\mp = \{z \in SM \mid \tau_\pm(z) = \pm\infty\},$$

and the trapped set for the flow on SM is the set

$$K := \Gamma_+ \cap \Gamma_-.$$

This is a compact subset of SM° that is flow invariant ([9]). We define the subset $\mathcal{T}_+(t) \subset SM$ given by the points $(x,v) \in SM$ for which the orbit of the geodesic flow issued from (x,v) remains in SM after time t:

$$\mathcal{T}_+(t) := \{(x,v) \in SM \mid \tau_+(x,v) \geq t\}.$$

We define the *non-escaping mass function* $V(t)$ as

$$V(t) := \mathrm{Vol}(\mathcal{T}_+(t)),$$

where Vol is the volume with respect to the Liouville measure μ on SM. Let us also denote

$$\mathcal{T}_+^{\partial SM}(t) := \mathcal{T}_+(t) \cap \partial_- SM. \tag{2.1}$$

The *escape rate* $Q \leq 0$ measures the exponential rate of decay of $V(t)$. It is given by

$$Q := \limsup_{t \to +\infty} \frac{1}{t} \log V(t). \tag{2.2}$$

By Proposition 2.4 in [9], if the trapped set K is hyperbolic, then $Q = \mathrm{Pr}(-J_u)$ is the topological pressure of (minus) the unstable Jacobian $J_u := \partial_t \det(\mathrm{d}\varphi_t|_{E_u})|_{t=0}$ of the geodesic flow on the trapped set K, and it satisfies

$$Q = \mathrm{Pr}(-J_u) < 0.$$

Let $\mathrm{d}\mu_{\mathrm{n}}$ be the measure on ∂SM defined by

$$\mathrm{d}\mu_{\mathrm{n}}(x,v) := |g_x(v,\mathrm{n})|\, \iota^* |\mathrm{d}\mu(x,v)|,$$

where $|\mathrm{d}\mu|$ is the Liouville density, and $\iota \colon \partial SM \to SM$ is the inclusion map. When $\mathrm{Vol}(\Gamma_- \cup \Gamma_+) = 0$, then $\mathrm{Vol}_{\partial SM}(\Gamma_\pm \cap \partial_\pm SM) = 0$ and one can use Santaló's formula ([9], Section 2.5) to integrate functions in SM: for all $f \in L^1(SM)$,

$$\int_{SM} f \, \mathrm{d}\mu = \int_{\partial_- SM \setminus \Gamma_-} \int_0^{\tau_+(x,v)} f \circ \varphi_t(x,v) \, \mathrm{d}t \; \mathrm{d}\mu_{\mathrm{n}}(x,v). \tag{2.3}$$

It is convenient to view (M,g) as a strictly convex region of a larger smooth manifold (M_e, g_e) with strictly convex boundary so that each geodesic in $M_e \setminus M$ has finite length with endpoints on $\partial M_e \cup \partial M$. The existence of such extension is proved in [9], Section 2.1 and Lemma 2.3. Moreover, if (M,g) has hyperbolic trapped set and no conjugate points, one can choose (M_e, g_e) with the same properties as (M,g), as is shown in Lemma 2.3 of [9]. The vector field X and the flow φ_t are extended in SM_e and we define the function $\tau_\pm^e$ on SM_e just as we did for τ_+ on SM. The trapped set of the flow in SM_e is still $K \subset SM^\circ$, the incoming tail $\Gamma_\pm^e$ on SM_e is $\Gamma_\pm^e = \cup_{t \geq 0} \varphi_{\pm t}(\Gamma_\pm) \cap SM_e$ and $\Gamma_\pm^e \cap SM = \Gamma_\pm$.

3. The X-ray transform

In this section we recall from [9] the main properties of the X-ray transform acting on functions in our geometric setting. Let (M, g) be a smooth compact Riemannian manifold with strictly convex boundary and let M_e be a small extension with the same property.

The X-ray transform I is defined as the map

$$I : \mathcal{C}_c^\infty(SM \setminus (\Gamma_- \cup \Gamma_+)) \to \mathcal{C}_c^\infty(\partial_- SM \setminus \Gamma_-), \quad If(x,v) := \int_0^{\tau_+(x,v)} f \circ \varphi_t(x,v)\, \mathrm{d}t.$$

The X-ray transform can be extended to more general spaces. If $\mathrm{Vol}(K) = 0$, then Santaló's formula implies that the operator I extends as a bounded operator

$$I : L^1(SM) \to L^1(\partial_- SM; d\mu_{\mathrm{n}}).$$

When moreover the escape rate Q of (2.2) satisfies $Q < 0$ then, by Lemma 5.1 in [9], one has that

$$\forall p > 2, \quad I : L^p(SM) \to L^2(\partial_- SM, d\mu_{\mathrm{n}}). \tag{3.1}$$

For our purposes to extend the results of [4], it is more convenient to deal with the X-ray transform acting on functions in $\mathcal{C}^\infty(M)$. The projection $\pi_0 \colon SM_e \to M_e$ on the base induces a pullback map

$$\pi_0^* : \mathcal{C}_c^\infty(M_e^\circ) \to \mathcal{C}_c^\infty(SM_e^\circ), \quad \pi_0^* f := f \circ \pi_0,$$

and a pushforward map π_{0*} defined by duality:

$$\pi_{0*} : \mathcal{D}'(SM_e^\circ) \to \mathcal{D}'(M_e^\circ), \quad \langle \pi_{0*} u, f \rangle := \langle u, \pi_0^* f \rangle.$$

When acting on L^1 functions, the pushforward π_{0*} acts as

$$\pi_{0*} f(x) := \int_{S_x M} f(x,v)\, \mathrm{d}\omega_x(v).$$

where $\mathrm{d}\omega_x$ is the measure on $S_x M$ induced by g. The pullback by π_0 gives a bounded operator $\pi_0^* \colon L^p(M) \to L^p(SM)$ for all $p \in (1, \infty)$. We define the X-ray transform on functions by

$$I_0 = I \pi_0^*.$$

If $Q < 0$, then I_0 extends as a bounded operator

$$I_0 := I \pi_0^* : L^p(M) \to L^2(\partial_- SM, d\mu_{\mathrm{n}}), \tag{3.2}$$

for any $p > 2$. The adjoint $I_0^* \colon L^2(\partial_- SM, d\mu_{\mathrm{n}}) \to L^{p'}(M)$ is bounded for $1/p' + 1/p = 1$, and it is given precisely by $I_0^* = \pi_{0*} I^*$. The operator Π_0 is defined as the bounded self-adjoint operator

$$\Pi_0 : I_0^* I_0 = \pi_{0*} I^* I \pi_0^* : L^p(M) \to L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 2.$$

Similarly, we define the extended X-ray transform I_0^e associated with M_e, and

$$\Pi_0^e = I_0^{e*} I_0^e : L^p(M) \to L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 2. \tag{3.3}$$

Lemma 3.1 (Proposition 5.7 in [9]). *Assume that (M,g) has strictly convex boundary, no conjugate points and hyperbolic trapped set, and let (M_e,g_e) be an extension with the same properties, and with the same trapped set. The operator Π_0, respectively Π_0^e, is an injective elliptic pseudo-differential operator of order -1 in M°, respectively M_e°, with principal symbol $\sigma(\Pi_0^e)(x,\xi)=C_d|\xi|_g^{-1}$ for some constant $C_d>0$ depending only on $d=\dim M$. For each $k\in\mathbb{Z}$ and each compact subset $\Omega\subset M_e^\circ$ with smooth boundary, there exist $C_1,C_2>0$ such that for all $f\in C_c^\infty(\Omega)$,*

$$C_1\|f\|_{H^k(M_e)}\le\|\Pi_0^e f\|_{H^{k+1}(M_e)}\le C_2\|f\|_{H^k(M_e)}. \tag{3.4}$$

Moreover, a direct calculation yields, for $z\notin\Gamma_+^e\cup\Gamma_-^e$,

$$I^{e*}(I_0^e f)(z)=\int_{\tau_-^e(z)}^{\tau_+^e(z)}\pi_0^*f(\varphi_t(z))\,\mathrm{d}t,\quad I^*(I_0f)(z)=\int_{\tau_-(z)}^{\tau_+(z)}\pi_0^*f(\varphi_t(z))\,\mathrm{d}t,$$

and thus if $f\in L^p(M_e)$ satisfies $\operatorname{supp} f\subset M$, we have $I^{e*}(I_0^e f)=I^*(I_0f)$ on $SM\setminus(\Gamma_+\cup\Gamma_-)$. In particular, this implies that

$$(\Pi_0^e f)|_M=\Pi_0 f. \tag{3.5}$$

Since pseudo-differential operators of order -1 map $W^{s,p}_{\mathrm{comp}}(M_e^\circ)$ to $W^{s+1,p}_{\mathrm{loc}}(M_e^\circ)$ boundedly for all $(s,p)\in\mathbb{R}\times(1,\infty)$ (see Theorem 0.11.A in [20]), (3.5) implies that

$$\begin{aligned} f\in W_0^{s,p}(M)&\Longrightarrow \Pi_0 f\in W^{s+1,p}(M),\\ f\in W_0^{s,p}(M_e)&\Longrightarrow \Pi_0^e f\in W^{s+1,p}(M_e),\end{aligned} \tag{3.6}$$

where $W^{s,p}(M)$ denotes the Sobolev space (with s derivatives in L^p) on the manifold with boundary M, $W_0^{s,p}(M)$ is the closure of $\mathcal{C}_c^\infty(M^\circ)$ for the $W^{s,p}(M)$ topology, and similarly on M_e.

4. Geometrical optics solutions

We will assume along this section that (M,g) is a smooth compact Riemannian manifold with boundary such that

- the boundary ∂M is strictly convex,
- the metric g has no pairs of conjugate points,
- the trapped set K is hyperbolic.

We shall take an extension (M_e,g_e) of (M,g) with the same properties, and for notational simplicity, we will write g instead of g_e for the extended metric on M_e.

4.1. Geometrical optics for the Schrödinger equation

In this section we generalize the geometrical optics solutions given in Section 4 of [4] for simple manifolds to our geometric setting. Since the map $\exp_x^{-1}(M)\to M$, with $x\in M$, is no longer a diffeomorphism, but the exponential map behaves well on the universal cover of M, we then make the construction in the universal cover of M, periodize it with respect to the fundamental group $\pi_1(M)$, and then project it down to M.

We first recall the following.

Lemma 4.1 (Lemma 3.2 and equation (3.5) in [4]). *Let $T > 0$ and $q \in L^\infty(M)$. Assume that $F \in W^{1,1}([0,T]; L^2(M))$ is such that $F(0,\cdot) \equiv 0$. Then the unique solution v to*

$$\begin{cases} (i\partial_t - \Delta_g + q(x))v(t,x) = F(t,x) & \text{in } (0,T)\times M, \\ v(0,x) = 0 & \text{in } M, \\ v(t,x) = 0 & \text{on } (0,T)\times \partial M, \end{cases}$$

satisfies

$$v \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^2(M) \cap H_0^1(M)).$$

In addition, there is a $C > 0$ depending on (M,g), T and $\|q\|_{L^\infty}$ such that for any $\eta > 0$ small and $t \in [0,T]$,

$$\|v(t,\cdot)\|_{L^2(M)} \le C \int_0^T \|F(s,\cdot)\|_{L^2(M)}\,\mathrm{d}s, \tag{4.1}$$

$$\|v(t,\cdot)\|_{H_0^1(M)} \le C\left(\eta \|\partial_t F\|_{L^1([0,T];L^2(M))} + \eta^{-1}\|F\|_{L^1([0,T];L^2(M))}\right). \tag{4.2}$$

Let us consider extensions $M \Subset M_e \Subset M_{ee}$ of the manifold M and extend the metric g smoothly in a way that (M_e, g_e) has the same properties as (M,g). The potentials q_1 and q_2 may also be extended to M_{ee}, and their $W^{1,\infty}(M)$ norms may be bounded by N_0. Since q_1 and q_2 coincide on the boundary, their extension outside M can be taken so that $q_1 = q_2$ in $M_{ee} \setminus M$.

We first recall the construction, following [4], Section 4, of a geometric optics solution for simple manifolds and we will explain how to extend it to our setting. If (M,g) is a simple manifold, a geometric optics solution $u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^2(M))$ for the Schrödinger equation

$$\begin{aligned} (i\partial_t - \Delta_g + q(x))u = 0, &\quad \text{in } (0,T)\times M, \\ u(0,x) = 0, & \end{aligned} \tag{4.3}$$

can be constructed in terms of a function $\psi \in \mathcal{C}^2(M)$ satisfying the eikonal equation

$$|\nabla^g \psi(x)|_g = 1, \quad \forall x \in M,$$

and a function $a \in H^1(\mathbb{R}; H^2(M))$ solving the transport equation

$$\begin{aligned} &\frac{\partial a}{\partial t} + \mathrm{d}a(\nabla^g \psi) - \frac{1}{2}(\Delta_g \psi_y)a = 0, \quad \forall t \in \mathbb{R},\ x \in M, \\ &\text{with } a(t,x) = 0, \quad \forall x \in M, \quad \text{and } t \le 0, \text{ or } t \ge T_0, \end{aligned} \tag{4.4}$$

for some $T_0 > 0$ sufficiently large (which in the simple case is taken to satisfy $T_0 > 1 + \mathrm{Diam}(M_e)$, where $\mathrm{Diam}(M_e)$ is the g-diameter of M_e), and $\mathrm{d}a$ is the exterior derivative of a. More precisely, if ∂M_e is chosen close enough to ∂M so that (M_e, g) is a simple manifold, one can define, for any fixed $y \in \partial M_e$,

$$\psi(x) = \psi_y(x) := d_g(y,x), \quad x \in M_e.$$

Using geodesic polar coordinates we can write each $x \in M_e$ as

$$x = \exp_y(r(x)\,v(x)), \quad r(x) = d_g(y,x), \quad v(x) \in S_yM_e$$

where $\exp_y : TM_e \to M_e$ denotes the exponential map at y for the metric g. One defines a solution to the transport equation $a \in H^1(\mathbb{R}; H^2(M))$ in polar coordinates as

$$a(t,x) := \alpha^{-1/4}(r(x), v(x))\,\phi(t - r(x))\,b(y, v(x)),$$

where $\alpha = \alpha(r,v) = |\det(g_{ij}(r,v))|$ denotes the square of the volume element in geodesic polar coordinates, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies $\operatorname{supp}\phi \subset (0,\varepsilon_0)$ for $\varepsilon_0 > 0$ small, and b is a fixed initial data in $H^2(\partial_- SM_e)$.

A geometrical optics solution $u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^2(M))$ for (4.3) is then defined by $u(t,x) = G_\lambda(t,x) + v_\lambda(t,x)$, where

$$G_\lambda(t,x) := a(2\lambda t, x)\, e^{i\lambda(\psi_y(x) - \lambda t)},$$

and the remainder v_λ satisfies (see Lemma 4.1 in [4])

$$\begin{aligned} &v_\lambda(t,x) = 0, \quad \forall (t,x) \in (0,T) \times \partial M, \\ &v_\lambda(0,x) = 0. \end{aligned}$$

Moreover, there exists $C > 0$ such that, for all $\lambda \geq T_0/(2T)$,

$$\|v_\lambda(t,\cdot)\|_{H^k(M)} \leq C\lambda^{k-1}\|a\|_{H^1([0,T_0];H^2(M))}, \quad k = 0,1.$$

The constant C depends only on T and (M,g). One can also construct a geometrical optics solution $u(t,x)$ if the initial condition $u(0,x) = 0$ is replaced by the final condition $u(T,x) = 0$ provided $\lambda \geq T_0/2T$; in this case, v_λ satisfies $v_\lambda(T,x) = 0$.

In our setting, that is, assuming that the trapped set K is hyperbolic and that g has no conjugate points, the construction is a bit more subtle, since the exponential map $\exp_y : \exp_y^{-1}(M) \to M$ is no longer a diffeomorphism. We work on the universal cover $\widetilde{M}$ of M, which is a non-compact manifold with boundary (the boundary has infinitely many connected components), whose interior is diffeomorphic to a ball. One has

$$M = \widetilde{M}/\pi_1(M),$$

where the fundamental group $\pi_1(M)$ is identified with the group of deck transformations on $\widetilde{M}$, that is, the set of homeomorphisms $f : \widetilde{M} \to \widetilde{M}$ such that $\pi \circ f = \pi$, with the composition, where $\pi : \widetilde{M} \to M$ is the covering map. The metric g lifts to a smooth metric $\tilde{g}$ on $\widetilde{M}$ satisfying $\gamma^*\tilde{g} = \tilde{g}$ for all $\gamma \in \pi_1(M)$. More generally, we denote by $\tilde{\cdot}$ the lifted objects to the universal cover. If (M,g) is assumed to have no pair of conjugate points, $\tilde{g}$ does not have pairs of conjugate points. Thus, for each $y \in \widetilde{M}$ the exponential map

$$\widetilde{\exp}_y : U_y \subset T\widetilde{M} \to \widetilde{M}$$

is a diffeomorphism for some simply connected set U_y. Similarly, we define the universal cover $\widetilde{M}_e$ of M_e, and note that $\pi_1(M) = \pi_1(M_e)$ so that each deck transformation γ of $\widetilde{M}$

extends naturally as a deck transformation on $\widetilde{M}_e$. Let us fix $y \in \partial M_e$ and lift this point to $\tilde{y} \in \partial \widetilde{M}_e$. We can choose a fundamental domain $\mathcal{F} \subset \widetilde{M}$, so that $M = \mathcal{F}/\pi_1(M)$ via identification of the points of the boundary of $\mathcal{F}$ by the action of the elements of $\pi_1(M)$. Note that $\mathcal{F}$ has two types of boundary components, the boundary components $\partial_i \mathcal{F}$ in the interior $\widetilde{M}^\circ$ of $\widetilde{M}$ which are identified by elements of $\pi_1(M)$, and the boundary components $\mathcal{F} \cap \pi^{-1}(\partial M)$. Similarly, we choose a fundamental domain $\mathcal{F}_e$ for $\pi_1(M_e) \simeq \pi_1(M)$ in $\widetilde{M}_e$ extending $\mathcal{F}$, and denote by $\partial_i \mathcal{F}_e$ the interior boundary of $\mathcal{F}_e$. We can freely assume that $\tilde{y} \in \mathcal{F}_e$ does not belong to the closure of $\partial_i \mathcal{F}_e$. Recall the definition of the volume entropy of M:

$$h(M,g) := \limsup_{R\to\infty} \frac{1}{R} \log \operatorname{Vol}_g(B_g(\tilde{y};R)), \tag{4.5}$$

where $B_g(\tilde{y};R) \subset \widetilde{M}_g$ is the $\tilde{g}$-geodesic ball centered at $\tilde{y}$ of radius R. Since M is compact, one has $h(M,g) < \infty$ (for instance, by the Bishop–Gromov comparison theorem) and $h(M,g)$ is not depending on the choice of $\tilde{y}$.

We define on $\widetilde{M}$ the solution to the "lifted" eikonal equation $|\nabla_{\tilde{g}} \psi_{\tilde{y}}| = 1$ given by

$$\psi_{\tilde{y}}(\tilde{x}) := d_{\tilde{g}}(\tilde{y},\tilde{x}), \quad \tilde{x} \in \widetilde{M},$$

where $d_{\tilde{g}}$ denotes the distance associated to the lifted metric $\tilde{g}$ on $\widetilde{M}$. Notice that $\psi_{\tilde{y}}$ is well defined and smooth outside $\tilde{x} = \tilde{y}$ since $\tilde{g}$ has no conjugate points (for any $\tilde{x} \in \widetilde{M}$, there is a unique geodesic joining $\tilde{y}$ with $\tilde{x}$ and realizing $d_{\tilde{g}}(\tilde{y},\tilde{x})$). Let

$$\partial_- S_{\tilde{y}} \widetilde{M}_e := \{v \in S_{\tilde{y}} \widetilde{M}_e \mid \langle v, \nu(\tilde{y})\rangle > 0\}.$$

Using geodesic polar coordinates on $\widetilde{M}$, we can write each $x \in \widetilde{M}_e$ as

$$x = \widetilde{\exp}_{\tilde{y}}(r(x)v(x)), \quad r(x) = \psi_{\tilde{y}}(x), \quad v(x) \in S_{\tilde{y}} \widetilde{M}_e.$$

Fix $T_0 > 1 + \operatorname{Diam}(M_e)$. For any given $b \in H^2(\partial_- SM_e)$ with $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$ (recall definition (2.1)), we denote by $\tilde{b}$ its lift to $\partial_- S\mathcal{F}_e$. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \subset (0,\varepsilon_0)$, for $\varepsilon_0 > 0$ small. We define

$$a(t,x) := \alpha^{-1/4}(r(x),v(x))\,\phi(t-r(x))\,\tilde{b}(\tilde{y},v(x)), \tag{4.6}$$

where $\alpha(r,v) = |\det(g_{ij}(r,v))|$ denotes the square of the volume element of $\widetilde{M}_e$ in geodesic polar coordinates. We introduce the norm $\|a\|_*$ on functions on $[0,T_0] \times \widetilde{M}$ given by

$$\|a\|_* := \|a\|_{H^1([0,T_0];H^2(\widetilde{M}))}.$$

For any $\lambda > 0$, we set

$$G_\lambda(t,x) := \sum_{\gamma \in \pi_1(M)} a(2\lambda t, \gamma(x))\, e^{i\lambda(\psi_{\tilde{y}}(\gamma(x)) - \lambda t)}, \quad t \in (0,T),\ x \in \widetilde{M}_e, \tag{4.7}$$

where $\gamma(x)$ denotes the lift of the point $\pi(x) \in M_e$ to the fundamental domain $\gamma(\mathcal{F}_e)$. Notice that this definition does not depend on the choice of the lift $\tilde{y}$ but on the base point y, and since $\mathrm{supp}(a(2\lambda t,\cdot))$ is contained in a fixed compact set of $\widetilde{M}_e$ for $t \in [0,T]$, the sum in $\gamma \in \pi_1(M)$ is locally finite. Notice also that the condition $T_0 > 1 + \mathrm{Diam}(M_e)$ together with $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$ ensures that $G_\lambda(t,x)$ vanishes on $(0,T)\times\partial M$ provided that $2\lambda T \geq T_0$, since the solution to the transport equation crosses the whole manifold M_e in time T. Moreover, as $G_\lambda(t,\gamma x) = G_\lambda(t,x)$, the function G_λ descends to M_e (and will also be denoted G_λ downstair), and satisfies $G_\lambda \in H^1([0,T]; H^2(M_e))$.

Lemma 4.2. *Let $q \in L^\infty(M)$. For $T_0 > 0$ and $T > 0$, the Schrödinger equation*

$$\begin{aligned} (i\partial_t - \Delta_g + q(x))u &= 0, \quad \text{in } (0,T)\times M, \\ u(0,\cdot) &= 0, \quad \text{in } M, \end{aligned}$$

has a solution of the form

$$u(t,x) = G_\lambda(t,x) + v_\lambda(t,x),$$

with G_λ given by (4.7), such that

$$u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^2(M)),$$

where $v_\lambda(t,x)$ satisfies, for $\lambda \geq T_0/(2T)$,

$$\begin{aligned} v_\lambda(t,x) &= 0, \quad \forall (t,x) \in (0,T)\times\partial M, \\ v_\lambda(0,x) &= 0, \quad x \in M. \end{aligned}$$

Furthermore, for each $\varepsilon > 0$ there exists $C > 0$ depending only on $(\varepsilon, M, g, \|q\|_{L^\infty}, T)$, but not on T_0 or y, such that, for any $\lambda \geq T_0/(2T)$, the following estimate holds true:

$$\|v_\lambda(t,\cdot)\|_{H^k(M)} \leq C\, e^{(h+\varepsilon)T_0}\, \lambda^{k-1} \|a\|_*, \quad k = 0,1, \tag{4.8}$$

where $h = h(M,g)$ denotes the volume entropy of (M,g). The result remains valid after replacing the initial condition $u(0,\cdot) = 0$ by the final condition $u(T,\cdot) = 0$.

Proof. We prove the lemma with initial condition $u(0,\cdot) = 0$, the proof for $u(T,\cdot) = 0$ being analogous. As in the proof of Lemma 4.1 in [4], we consider for $x \in M_e$,

$$k(t,x) = -\sum_{\gamma\in\pi_1(M)} (i\partial_t - \Delta_{\tilde g} + \tilde q)\big(a(2\lambda t, \gamma(\tilde x))\, e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x)) - \lambda t)}\big),$$

where $\tilde x \in \widetilde{M}_e$ is a lift of x. Let v_λ be the solution, given by Lemma 4.1, to the homogeneous boundary value problem

$$\begin{cases} (i\partial_t - \Delta_g + q)v_\lambda(t,x) = k(t,x) & \text{in } (0,T)\times M, \\ v_\lambda(0,x) = 0, & \text{in } M, \\ v_\lambda(t,x) = 0 & \text{on } (0,T)\times\partial M. \end{cases}$$

We shall show that v_λ satisfies the estimate (4.8). A computation gives

$$
\begin{aligned}
-k(t,x) = &\sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x))-\lambda t)}(-\Delta_{\tilde g}+\tilde q(\gamma(\tilde x)))\,a(2\lambda t,\gamma(\tilde x))\\
&+2i\lambda \sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x))-\lambda t)}\Big(\partial_t\tilde a+\mathrm{d}a(\nabla^{\tilde g}\psi_{\tilde y})-\frac{a}{2}\Delta_{\tilde g}\psi_{\tilde y}\Big)(2\lambda t,\gamma(\tilde x))\\
&+\lambda^2\sum_{\gamma\in\pi_1(M)} a(2\lambda t,\gamma(\tilde x))\,e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x))-\lambda t)}\big(1-|\nabla^{\tilde g}\psi_{\tilde y}|^2_{\tilde g}\big).
\end{aligned}
$$

Using that a solves the transport equation, that $\psi_{\tilde y}$ solves the eikonal equation, and that $\Delta_{\tilde g}$ commutes with γ^* (since γ are isometries of $\tilde g$), we obtain

$$
\begin{aligned}
-k(t,x) &= \sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x))-\lambda t)}\,(-\Delta_{\tilde g}a+\tilde q a)\,(2\lambda t,\gamma(\tilde x))\\
&=: \sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(\tilde x))-\lambda t)}\,k_0(2\lambda t,\gamma(\tilde x)).
\end{aligned}
$$

Notice that $k_0\in H^1_0([0,T];L^2(M))$ for $\lambda\ge T_0/(2T)$ and that $k_0(s,\cdot)|_{\tilde M}=0$ for $s>T_0$. Then, using Lemma 4.1 we get

$$
v_\lambda\in\mathcal{C}^1([0,T];L^2(M))\cap\mathcal{C}([0,T];H^2(M)\cap H^1_0(M)).
$$

Moreover, using (4.1), there exists $C>0$ depending on (M,g), $T>0$, and $\|q\|_{L^\infty}$ such that

$$
\begin{aligned}
\|v_\lambda(t,\cdot)\|_{L^2(M)} &\le C\sum_{\gamma\in\pi_1(M)}\int_0^T \|k_0(2\lambda t,\cdot)\|_{L^2(\gamma(F))}\,\mathrm{d}t\\
&\le \frac{C}{\lambda}\sum_{\gamma\in\pi_1(M)}\int_0^{T_0}\|k_0(s,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\,\mathrm{d}s\le \frac{C(1+\|q\|_{L^\infty})\,T_0^{1/2}N(T_0)}{\lambda}\,\|a\|_*,
\end{aligned}
$$

where $N(T_0)$ denotes the number of fundamental domains which intersect the geodesic ball $B(\tilde y,T_0)$ of center $\tilde y$ and radius T_0 in $\tilde M$, that is,

$$
\begin{aligned}
N(T_0) &= \#\{\gamma\in\pi_1(M)\,|\,\exists x\in\mathcal{F},\ d_{\tilde g}(\gamma(x),\tilde y)\le T_0\}\\
&\le \#\big\{\gamma\in\pi_1(M)\,|\,\max_{x\in\mathcal{F}} d_{\tilde g}(\gamma(x),\tilde y)\le T_0+\mathrm{Diam}(M)\big\}.
\end{aligned}
$$

Clearly, $N(T_0)\,\mathrm{Vol}(M)\le\mathrm{Vol}(B(\tilde y,T_0+\mathrm{Diam}(M)))$, and therefore for each $\varepsilon>0$, there is $C>0$ (depending on $\mathrm{Diam}(M)$ and $\mathrm{Vol}(M)$) such that for all $T_0>0$ large enough,

$$
N(T_0)\le Ce^{(h+\varepsilon)T_0},
$$

where $h=h(M,g)$ is the volume entropy of (M,g) defined in (4.5). We notice that the constants $C>0$ above can be chosen independently of y and that h is in fact not depending on y.

Finally, by Lemma 4.1, there is $C > 0$ (depending on $(M, g, T, \|q\|_{L^\infty})$) such that for each $\eta > 0$,

$$\begin{aligned}\|\nabla^g v_\lambda(t,\cdot)\|_{L^2(M)} &\leq C\eta \sum_{\gamma\in\pi_1(M)} \int_0^T \big(\lambda^2 \|k_0(2\lambda t,\cdot)\|_{L^2(\gamma(\mathcal{F}))} + \lambda \|\partial_t k_0(2\lambda t,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\big)\, \mathrm{d}t \\ &\quad + C\eta^{-1} \sum_{\gamma\in\pi_1(M)} \int_0^T \|k_0(2\lambda t,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\, \mathrm{d}t.\end{aligned}$$

Choosing $\eta = \lambda^{-1}$, for each $\varepsilon > 0$ there is $C > 0$ such that for all $\lambda \geq T_0/(2T)$,

$$\begin{aligned}&\|\nabla^g v_\lambda(t,\cdot)\|_{L^2(M)} \\ &\leq C \sum_{\gamma\in\pi_1(M)} \Big(\int_0^{T_0} \|k_0(s,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\, \mathrm{d}s + \int_0^{T_0} \|\partial_t k_0(s,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\, \mathrm{d}s\Big) \leq Ce^{(h+\varepsilon)T_0}\|a\|_*.\end{aligned}$$

This concludes the proof. ∎

4.2. Geometrical optics for the wave equation

In this section we give the construction of geometric optics solutions for the wave equation with hyperbolic trapped set. First we use the following.

Lemma 4.3. *Let $q \in L^\infty(M)$. Then there exist a constant $C > 0$, depending only on $(M, g, \|q\|_{L^\infty})$, and a constant $\nu_0 \geq 0$, depending only on $\|q\|_{L^\infty}$, such that for all $0 < t \leq T$ and all $F \in L^2((0,T)\times M)$, there is a unique solution v to*

$$\left\{\begin{array}{ll} (\partial_t^2 + \Delta_g + q(x))v(t,x) = F(t,x) & \text{in } [0,T]\times M, \\ v(0,x) = 0, \quad \partial_t v(0,x) = 0 & \text{in } M, \\ v(t,x) = 0 & \text{on } [0,T]\times\partial M, \end{array}\right. \tag{4.9}$$

in $\mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}^0([0,T]; H_0^1(M))$ satisfying for each $\nu \geq \nu_0$,

(4.10)
$$\|v(t,\cdot)\|^2_{L^2(M)} + \|\partial_t v(t,\cdot)\|^2_{L^2(M)} + \|\nabla^g v(t,\cdot)\|^2_{L^2(M)} \leq C\int_0^t e^{\nu(t-s)}\|F(s,\cdot)\|^2_{L^2(M)}\mathrm{d}s,$$
$$\|e^{-\nu t/2}\partial_{\mathrm{n}} v\|_{L^2((0,T)\times\partial M)} \leq C\|e^{-\nu t/2}F\|_{L^2((0,T)\times M)},$$

where C depends only on $\|q\|_{L^\infty}$ and ν.

There is $C > 0$ as above such that for each $f \in H^1([0,T]\times\partial M)$ with $f(0,\cdot) = \partial_t f(0,\cdot) = 0$, there is a unique solution $u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}^0([0,T]; H_0^1(M))$ such that

$$\left\{\begin{array}{ll} (\partial_t^2 + \Delta_g + q(x))u(t,x) = 0 & \text{in } [0,T]\times M, \\ u(0,x) = 0, \quad \partial_t u(0,x) = 0 & \text{in } M, \\ u(t,x) = f(t,x) & \text{on } [0,T]\times\partial M, \end{array}\right. \tag{4.11}$$

and u satisfies

$$\|e^{-\nu t/2}\partial_{\mathrm{n}}u\|_{L^2([0,T]\times\partial M)} \leq C\|e^{-\nu t/2}f\|_{H^1([0,T]\times\partial M)}.$$

As a consequence, the operator $\Lambda_q^W : e^{\nu t/2}H_0^1(\mathbb{R}_+\times\partial M) \to e^{\nu t/2}L^2(\mathbb{R}_+\times\partial M)$ is bounded with norm depending only on (M,g), $\|q\|_{L^\infty}$ and ν.

Proof. The uniqueness and existence is done in [12], Chapter 3, Section 8 and 9, and is based on energy estimates. Here we want a uniform estimate in time involving the exponent ν_0, in particular for what concerns its dependence in q. Let v be a solution to (4.9). Then, for any $\nu \geq 0$, $v_\nu(t,x) = e^{-\nu t/2}v(t,x)$ satisfies the damped-wave equation

$$\begin{cases} (\partial_t^2 + \Delta_g + \nu^2/4 + q(x) + \nu\partial_t)v_\nu(t,x) = F_\nu(t,x) & \text{in } [0,T]\times M, \\ v_\nu(0,x) = 0, \quad \partial_t v_\nu(0,x) = 0 & \text{in } M, \\ v_\nu(t,x) = 0 & \text{on } [0,T]\times\partial M, \end{cases} \tag{4.12}$$

where $F_\nu(t,x) = e^{-\nu t/2}F(t,x)$. Similarly, let u be the solution to (4.11). Then $u_\nu(t,x) = e^{-\nu t/2}u(t,x)$ solves the boundary-value problem

$$\begin{cases} (\partial_t^2 + \Delta_g + \nu^2/4 + q(x) + \nu\partial_t)u_\nu(t,x) = 0 & \text{in } [0,T]\times M, \\ u_\nu(0,x) = 0, \quad \partial_t u_\nu(0,x) = 0 & \text{in } M, \\ u_\nu(t,x) = f_\nu(t,x) & \text{on } [0,T]\times\partial M, \end{cases} \tag{4.13}$$

where $f_\nu(t,x) = e^{-\nu/2t}f(t,x)$. First, multiply equation (4.12) by $\bar{v}_\nu$ and integrate in $[0,t]\times M$ to get

$$\begin{aligned} -\int_0^t\int_M |\partial_s v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s + \int_0^t\int_M |\nabla^g v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s + \int_0^t\int_M \Big(\frac{\nu^2}{4}+q\Big)|v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s \\ = -\int_M \partial_t v_\nu\,\bar{v}_\nu\,\mathrm{dv}_g - \nu\int_0^t\int_M \partial_s v_\nu\,\bar{v}_\nu\,\mathrm{dv}_g\,\mathrm{d}s + \int_0^t\int_M F_\nu\bar{v}_\nu\,\mathrm{dv}_g\,\mathrm{d}s. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\int_0^t\int_M |\nabla^g v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s + \frac{\nu^2}{4}\int_0^t\int_M |v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s \\ &\quad\leq \frac{1}{2}\int_M |\partial_t v_\nu(t)|^2\,\mathrm{dv}_g + \frac{1}{2}\int_M |v_\nu(t)|^2\,\mathrm{dv}_g + \frac{\nu}{2}\int_0^t\int_M (|\partial_s v_\nu|^2 + |v_\nu|^2)\,\mathrm{dv}_g\,\mathrm{d}s \\ &\quad\quad + \Big(\|q\|_{L^\infty} + \frac{1}{2}\Big)\int_0^t\int_M |v_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_M |F_\nu|^2\,\mathrm{dv}_g\,\mathrm{d}s \end{aligned}$$

and we get, for $C_\nu = \nu^2/4 - \nu/2 - \|q\|_{L^\infty} - 1/2$,

$$\begin{aligned} &\int_0^t \|\nabla^g v_\nu(s)\|_{L^2(M)}^2\,\mathrm{d}s + C_\nu\int_0^t \|v_\nu(s)\|^2\,\mathrm{d}s \\ &\leq \frac{1}{2}\|\partial_t v_\nu(t)\|_{L^2(M)}^2 + \frac{1}{2}\|v_\nu(t)\|_{L^2(M)}^2 + \frac{\nu}{2}\int_0^t \|\partial_s v_\nu\|_{L^2(M)}^2 + \frac{1}{2}\int_0^t \|F_\nu(s)\|_{L^2(M)}^2. \end{aligned} \tag{4.14}$$

We next multiply equation (4.12) by $\partial_t \bar{v}_\nu$, integrate in $[0,t]\times M$, take the real part and integrate by parts to obtain

$$\begin{aligned}
&\frac{\nu^2}{4}\|v_\nu(t)\|^2_{L^2(M)} + \|\partial_t v_\nu(t)\|^2_{L^2(M)} + \|\nabla^g v_\nu(t)\|^2_{L^2(M)}\\
&\qquad \le 2\int_0^t (\|F_\nu(s)\|_{L^2(M)}\|\partial_s v(s)\|_{L^2(M)})\,\mathrm{d}s + \|q\|_{L^\infty}\int_0^t \|v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s\\
&\qquad\quad + (\|q\|_{L^\infty} - \nu)\int_0^t \|\partial_s v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s\\
&\qquad \le \int_0^t \|F_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s + \|q\|_{L^\infty}\int_0^t \|v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s\\
&\qquad\quad + (1 + \|q\|_{L^\infty} - \nu)\int_0^t \|\partial_s v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s.
\end{aligned}$$

Defining $\Phi_{v_\nu}(t) := \|\partial_t v_\nu(t)\|^2_{L^2} + \|v_\nu(t)\|^2_{L^2} + \|\nabla^g v_\nu(t)\|^2_{L^2}$, using (4.14) and taking $\nu > 2$ large enough depending only on $\|q\|_{L^\infty}$, we obtain

$$\begin{aligned}
\frac{1}{2}\Phi_{v_\nu}(t) &\le \frac{3}{2}\int_0^t \|F_\nu(s,\cdot)\|^2_{L^2(M)}\,\mathrm{d}s + (\|q\|_{L^\infty} - C_\nu)\int_0^t \|v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s\\
&\quad + (1 + \|q\|_{L^\infty} - \nu)\int_0^t \|\partial_s v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s - \int_0^t \|\nabla^g v_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s.
\end{aligned}$$

Take ν large enough so that $C_\nu - \|q\|_{L^\infty} > 1$ and $\nu > 2 + \|q\|_{L^\infty}$; we then get

$$\Phi_{v_\nu}(t) + \int_0^t \Phi_{v_\nu}(s)\,\mathrm{d}s \le 3\int_0^t \|F_\nu(s)\|^2_{L^2(M)}\,\mathrm{d}s. \tag{4.15}$$

Since

$$\Phi_{v_\nu}(t) \ge \frac{2}{\nu^2+1}\,e^{-\nu t}\,\|\partial_t v\|^2_{L^2(M)} + \frac{1}{2}\,e^{-\nu t}\,\|v(t)\|^2_{L^2(M)},$$

this shows in particular the first estimate of (4.10). Next, let N be a smooth vector field equal to n the inward normal vector field on ∂M. Multiplying equation (4.12) by $\langle N, \nabla^g \bar{v}\rangle = \mathrm{d}\bar{v}_\nu(N)$, one has

$$\begin{aligned}
&\int_0^t\int_M F_\nu(s,\cdot)\,\mathrm{d}\bar{v}(N)\,\mathrm{dv}_g\,\mathrm{d}s\\
&\qquad = \int_0^t\int_M \partial_s^2 v_\nu\,\mathrm{d}\bar{v}_\nu(N)\,\mathrm{dv}_g\,\mathrm{d}s + \int_0^t\int_M \Delta_g v_\nu\,\mathrm{d}\bar{v}_\nu(N)\,\mathrm{dv}_g\,\mathrm{d}s\\
&\qquad\quad + \int_0^t\int_M \Big(\frac{\nu^2}{4} + q\Big) v_\nu\,\mathrm{d}\bar{v}_\nu(N)\,\mathrm{dv}_g\,\mathrm{d}s + \nu\int_0^t\int_M \partial_s v_\nu\,\mathrm{d}\bar{v}_\nu(N)\,\mathrm{dv}_g\,\mathrm{d}s\\
&\qquad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using integration by parts in s,

$$\begin{aligned}
2\mathrm{Re}(I_1 + I_4) &= 2\mathrm{Re}(\langle \partial_t v_\nu, \mathrm{d}\bar{v}_\nu(N)\rangle_{L^2(M)}) - \int_0^t \langle N, (\nabla^g|\partial_s v_\nu|_g^2)\rangle_{L^2(M)}\,\mathrm{d}s\\
&\quad + 2\nu\,\mathrm{Re}\Big(\int_0^t \langle \partial_s v_\nu, \mathrm{d}v_\nu(N)\rangle_{L^2(M)}\,\mathrm{d}s\Big),
\end{aligned}$$

which gives, after integrating by parts in $x \in M$ the second term, using $\partial_s v = 0$ on $(0,t)\times \partial M$ and the estimates (4.15) (for some $C_{\nu,\|q\|_{L^\infty}} > 0$ depending only on (M,g), ν, $\|q\|_{L^\infty}$),

$$|2\mathrm{Re}(I_1 + I_4)| \le C_{\nu,\|q\|_{L^\infty}} \|F_\nu\|^2_{L^2((0,t)\times M)}.$$

For the I_2 term, one gets by integration by parts,

$$2\mathrm{Re}(I_2) = 2\int_0^t \int_{\partial M} |\partial_{\mathrm{n}} v_\nu|^2 \, \mathrm{d}v_{\partial M}\, \mathrm{d}s + 2\mathrm{Re}\int_0^t \langle \nabla^g v_\nu, \nabla^g(\mathrm{d}v_\nu(N))\rangle_{L^2(M)}\, \mathrm{d}s,$$

and an easy computation shows

$$\begin{aligned}
&\langle \nabla^g v_\nu, \nabla^g(\mathrm{d}v_\nu(N))\rangle_{L^2(M)} \\
&\quad = \int_M \Big((\nabla^g N \nabla^g v_\nu, \nabla^g \bar{v}_\nu) + \frac{1}{2}\,\mathrm{div}(|\nabla^g v_\nu|^2 N) - \frac{1}{2}|\nabla^g v_\nu|^2\,\mathrm{div}(N)\Big)\,\mathrm{d}v_g \\
&\quad = \int_M \Big((\nabla^g N \nabla^g v_\nu, \nabla^g \bar{v}_\nu) - \frac{1}{2}|\nabla^g v|^2 \mathrm{div}(N)\Big)\,\mathrm{d}v_g - \frac{1}{2}\int_{\partial M} |\nabla^g v_\nu|^2\,\mathrm{d}v_{\partial M},
\end{aligned}$$

thus, using $|\nabla^g v_\nu|^2 = |\partial_{\mathrm{n}} v_\nu|^2$ (since $v_\nu = 0$ on ∂M), we obtain, by (4.15),

$$\left|2\mathrm{Re}(I_2) - \int_0^t \int_{\partial M} |\partial_{\mathrm{n}} v_\nu|^2\,\mathrm{d}v_{\partial M}\,\mathrm{d}s\right| \le C_{\nu,\|q\|_{L^\infty}} \|F_\nu\|^2_{L^2((0,t),L^2(M))}.$$

Finally, (4.15) directly gives $|I_3| \le C_{\nu,\|q\|_{L^\infty}} \|F_\nu\|^2_{L^2((0,t)\times M)}$, and we conclude that

$$\int_0^t \int_{\partial M} |\partial_{\mathrm{n}} v_\nu|^2\,\mathrm{d}v_{\partial M}\,\mathrm{d}s \le C_{\nu,\|q\|_{L^\infty}} \|F_\nu\|^2_{L^2((0,t),L^2(M))}. \tag{4.16}$$

The same results apply for solutions of the equation (4.12) on $[0,T]\times M$ with ν replaced by $-\nu$ and with boundary condition $v_{-\nu}(T) = \partial_t v_{-\nu}(T) = 0$.

Notice that if u_ν solves (4.13), then for any $F_{-\nu} \in L^2((0,T)\times M)$, one has, by duality with (4.12),

$$\ell(F_{-\nu}) := \int_0^T \int_M u_\nu(t,x) F_{-\nu}(t,x)\,\mathrm{d}v_g\,\mathrm{d}t = \int_0^T \int_{\partial M} f_\nu(t,x)\,\partial_{\mathrm{n}} v_{-\nu}(t,x)\,\mathrm{d}v_{\partial M}\,\mathrm{d}t,$$

where $v_{-\nu}$ is the solution to (4.12) with ν replaced by $-\nu$ and with boundary condition $v_{-\nu}(T) = \partial_t v_{-\nu}(T) = 0$ rather than $v_\nu(0) = \partial_t v_\nu(0) = 0$. By (4.16), there is $C > 0$ depending only on (M,g) so that

$$|\ell(F_{-\nu})| \le C\|f_\nu\|_{L^2((0,T)\times\partial M)} \|F_{-\nu}\|_{L^2((0,T)\times M)}.$$

Therefore,

$$\|u_\nu\|_{L^2((0,T)\times M))} \le C\|f_\nu\|_{L^2((0,T)\times \partial M)}.$$

Moreover, if $f_\nu \in H^1((0,T)\times M)$, then $w_\nu = \partial_t u_\nu$ solves the equation

$$\begin{cases} (\partial_t^2 + \Delta_g + \frac{\nu^2}{4}q + \nu\partial_t) w_\nu(t,x) = 0, & \text{in } [0,T]\times M \\ w_\nu(0,x) = \partial_t w_\nu(0,x) = 0, & \text{in } M \\ w_\nu(x,t) = \partial_t f_\nu(t,x), & \text{on } \partial M. \end{cases}$$

This implies that $w_\nu = \partial_t u_\nu \in L^2((0,T)\times M)$ by the uniqueness result (see Chapter 3, Section 8 and 9, in [12]), and with bound $\|w_\nu\|_{L^2((0,T)\times M))} \leq C\|\partial_t f_\nu\|_{L^2((0,T)\times\partial M)}$. Note that $\partial_t w_\nu = \partial_t^2 u_\nu = -(\Delta_g + \nu^2/4 + q + \nu\partial_t)u_\nu \in L^2((0,T); H^{-2}(M))$ and

$$\|\partial_t w_\nu\|_{L^2((0,T),H^{-2}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1([0,T]\times\partial M)}$$

for some constant $C_{\nu,\|q\|_{L^\infty}}$ depending only on $\|q\|_{L^\infty}$, ν and (M,g). On the other hand, $\partial_t^2 w_\nu = -(\Delta_g + \nu^2/4 + q + \nu\partial_t)w_\nu \in L^2((0,T); H^{-2}(M))$ with norm

$$\|\partial_t w_\nu\|_{L^2((0,T),H^{-2}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1([0,T]\times\partial M)}$$

for some constant as above, thus by interpolation with $w_\nu \in L^2((0,T); L^2(M))$, we also have $\partial_t w_\nu \in L^2((0,T); H^{-1}(M))$ ([12], Proposition 2.2) with bound

$$\|\partial_t w_\nu\|_{L^2((0,T);H^{-1}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1([0,T]\times\partial M)}.$$

In particular, we obtain

$$\|u_\nu\|_{L^2((0,T)\times M)} + \|\partial_t u_\nu\|_{L^2((0,T)\times M)} + \|\partial_t^2 u_\nu\|_{L^2((0,T);H^{-1}(M)} \\ \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T)\times\partial M)}.$$

We next observe that

$$\begin{cases} \Delta_g u_\nu = -\partial_t^2 u_\nu - (\nu_0^2/4 + q)u_\nu - \nu\partial_t u_\nu =: \psi_\nu & \text{in } M, \\ u_\nu = f_\nu & \text{on } \partial M. \end{cases}$$

Then, by elliptic regularity and since $\partial_t^2 u_\nu, \partial_t u_\nu \in L^2((0,T); H^{-1}(M))$, there is $C > 0$ depending only on (M,g) so that

$$\|u_\nu(t,\cdot)\|_{H^1(M)} \leq C\big(\|f_\nu(t,\cdot)\|_{H^1(\partial M)} + \|\psi_\nu(t,\cdot)\|_{H^{-1}(M)}\big).$$

Therefore, we obtain

$$\|u_\nu\|_{L^2((0,T)\times M)} + \|\partial_t u_\nu\|_{L^2((0,T)\times M)} + \|\nabla^g u_\nu\|_{L^2((0,T)\times M)} \\ \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T)\times\partial M)}. \tag{4.17}$$

Next, multiplying equation (4.13) by $\langle N, \bar{u}_\nu\rangle = d\bar{u}(N)$ and applying the same reasoning as we did above for v_ν and equation (4.12), it is direct to obtain the bound

$$\int_0^T \int_{\partial M} |\partial_{\mathrm{n}} u_\nu(t)|^2 \,\mathrm{dv}_{\partial M}\,\mathrm{d}t \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|^2_{H^1((0,T)\times\partial M)}. \tag{4.18}$$

This concludes the proof. ∎

Given $T > 0$, let $b \in H^2(\partial SM)$ with $b|_{\mathcal{T}_+^{\partial SM}(T)} = 0$. Let $a(t,\tilde{x})$ be a solution to the lifted transport equation defined by (4.6). We extend a to $\mathbb{R}_+ \times \widetilde{M}$ by zero and denote

$$\|a\|_* := \|a\|_{e^{\nu t}W^{1,1}(\mathbb{R}_+;H^2(\widetilde{M}))} + \|a\|_{e^{\nu t}W^{3,1}(\mathbb{R}_+;L^2(\widetilde{M}))}. \tag{4.19}$$

For $y \in \partial M_e$, we use the function $\psi_{\tilde{y}} = d_{\tilde{g}}(\tilde{y}, \cdot)$ on $\widetilde{M}_e$ defined above and define, for any $\lambda > 0$,

$$G_\lambda(t,x) := \sum_{\gamma \in \pi_1(M)} a(t,\gamma(x))\, e^{i\lambda(\psi_{\tilde{y}}(\gamma(x))-t)}. \tag{4.20}$$

The sum is locally finite and this function is $\pi_1(M)$ invariant, and thus descends to M_e.

Lemma 4.4. *Let $q \in L^\infty(M)$ and $\nu_0 \geq 0$ be as defined in Lemma 4.3. For any $\lambda > 0$ and $T > 0$, the equation*

$$\begin{cases} (\partial_t^2 + \Delta_g + q(x))u = 0, & \text{in } \mathbb{R}_+ \times M, \\ u(0,x) = \partial_t u(0,x) = 0, & x \in M, \end{cases}$$

has a solution of the form

$$u(t,x) = G_\lambda(t,x) + v_\lambda(t,x)$$

such that, for every $\nu > \nu_0$,

$$u \in e^{\nu t} H^1(\mathbb{R}_+; L^2(M)) \cap e^{\nu t} L^2(\mathbb{R}_+; H^1(M)),$$

and where $v_\lambda(t,x)$ satisfies

$$\begin{aligned} &v_\lambda(t,x) = 0, \quad \forall (t,x) \in \mathbb{R}_+ \times \partial M, \\ &v_\lambda(0,x) = 0, \quad \partial_t v_\lambda(0,x) = 0, \ x \in M, \end{aligned}$$

and for all $\varepsilon > 0$, there is $C > 0$ depending only on M, g, $\|q\|_{L^\infty}$, ν, ε so that

$$\lambda \|v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} + \|\partial_t v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} + \|\nabla^g v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} \leq C e^{(h+\varepsilon)T} \|a\|_*, \tag{4.21}$$

where $h = h(M,g)$ denotes the volume entropy. The result remains valid if one replaces the initial condition $u(0,x) = \partial_t u(0,x) = 0$ by the final condition $u(t,x) = \partial_t u(t,x) = 0$ for $t \geq T$. In this case, v_λ satisfies

$$v_\lambda(t,x) = 0, \quad \partial_t v_\lambda(t,x) = 0, \quad t \geq T, \ x \in M.$$

Proof. We prove the lemma with initial condition $u(0,\cdot) = \partial_t u(0,\cdot) = 0$. The proof for $u(T,\cdot) = \partial_t u(T,\cdot) = 0$ is analogous. As in the proof of Lemma 4.1 in [5], we consider

$$k(t,x) = -\sum_{\gamma \in \pi_1(M)} (\partial_t^2 + \Delta_{\tilde{g}} + \tilde{q}) \big(a(t,\gamma(x))\, e^{i\lambda(\psi_{\tilde{y}}(\gamma(x))-t)}\big).$$

Let v_λ be the solution, given by Lemma 4.3, to the boundary value problem

$$\begin{cases} (\partial_t^2 + \Delta_g + q)v_\lambda(t,x) = k(t,x) & \text{in } \mathbb{R}_+ \times M, \\ v_\lambda(0,x) = \partial_t v_\lambda(0,x) = 0, & \text{in } M, \\ v_\lambda(t,x) = 0, & \text{on } \mathbb{R}_+ \times \partial M. \end{cases} \tag{4.22}$$

To prove the claim it is sufficient to show that v_λ satisfies the estimate (4.21). A computation yields

$$\begin{aligned}-k(t,x) = &\sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}\,((\partial_t^2+\Delta_{\tilde g}+\tilde q)a)(t,\gamma(x))\\ &-2i\lambda\sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}\Big(\partial_t a+\mathrm{d}a(\nabla^{\tilde g}\psi_{\tilde y})-\frac{a}{2}\Delta_{\tilde g}\psi_{\tilde y}\Big)(t,\gamma(x))\\ &-\lambda^2\sum_{\gamma\in\pi_1(M)} a(t,\gamma(x))\,e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}(1-|\nabla^{\tilde g}\psi_{\tilde y}|^2).\end{aligned}$$

Using that a solves the transport equation and $\psi_{\tilde y}$ solves the eikonal equation, we obtain

$$\begin{aligned}-k(t,x) &= \sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}\,((\partial_t^2+\Delta_{\tilde g}+\tilde q)a)(t,\gamma(x))\\ &=: \sum_{\gamma\in\pi_1(M)} e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}\,k_0(t,\gamma(x)).\end{aligned}$$

Notice that $k_0\in H_0^1([0,T];L^2(\widetilde M))$. Extending k_0 to I by zero, we obtain

$$\|k_0\|_{e^{\nu t}L^1(I;L^2(\widetilde M))}+\|\partial_t k_0\|_{e^{\nu t}L^1(I;L^2(\widetilde M))}\le C\|a\|_*.$$

Moreover, the function

$$w_\lambda(t,x) := \int_0^t v_\lambda(s,x)\,\mathrm{d}s$$

solves the mixed hyperbolic problem (4.22) with right-hand side $k_1(t,x)=\int_0^t k(s,x)\,\mathrm{d}s$. For $t\ge T$, this is equal to

$$\begin{aligned}k_1(t,x) &= \frac{1}{i\lambda}\sum_{\gamma\in\pi_1(M)}\int_0^t k_0(s,\gamma(x))\,\partial_s(e^{i\lambda(\psi(\gamma(x))-s)})\,\mathrm{d}s\\ &= -\frac{1}{i\lambda}\sum_{\gamma\in\pi_1(M)}\int_0^t \partial_s k_0(s,\gamma(x))\,e^{i\lambda(\widetilde\psi(\gamma(x))-s)}\,\mathrm{d}s.\end{aligned}$$

Then, by Lemma 4.3, we obtain for ν_0 defined in Lemma 4.3,

$$\begin{aligned}\|v_\lambda(t,\cdot)\|_{L^2(M)} &\le C\int_0^t e^{\nu_0(t-s)}\,\|k_1(s,\cdot)\|_{L^2(M)}\,\mathrm{d}s\\ &\le \frac{C}{\lambda}\sum_{\gamma\in\pi_1(M)}\int_0^t e^{\nu_0(t-s)}\int_0^s \|\partial_r k_0(r,\cdot)\|_{L^2(\gamma(F))}\,\mathrm{d}r\,\mathrm{d}s\\ &\le \frac{C\,N(T)}{\lambda}\int_0^t e^{\nu_0(t-s)}\int_0^s \|\partial_r k_0(r,\cdot)\|_{L^2(\widetilde M)}\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

Therefore, using that $N(T) \leq Ce^{(h+\varepsilon)T}$ and the Minkowski integral inequality, we obtain for $\nu > \nu_0$,

$$\begin{aligned}
&\|v_\lambda\|_{e^{\nu t}L^2(I\times M)} \\
&\quad\leq \frac{Ce^{(h+\varepsilon)T}}{\lambda}\Big(\int_0^\infty\Big(\int_0^t\int_0^s e^{-(\nu-\nu_0)t}\,e^{-\nu_0 s}\,\|\partial_r k_0(r,\cdot)\|_{L^2(\widetilde{M})}\,\mathrm{d}r\,\mathrm{d}s\Big)^2\mathrm{d}t\Big)^{1/2} \\
&\quad\leq \frac{Ce^{(h+\varepsilon)T}}{\lambda}\int_0^\infty\int_r^\infty\Big(\int_s^\infty e^{-2(\nu-\nu_0)(t-s)}\,e^{-2\nu s}\|\partial_r k_0(r,\cdot)\|^2_{L^2(\widetilde{M})}\,\mathrm{d}t\Big)^{1/2}\mathrm{d}s\,\mathrm{d}r \\
&\quad\leq \frac{Ce^{(h+\varepsilon)T}}{\lambda}\int_0^\infty\int_r^\infty e^{-\nu(s-r)}\,e^{-\nu r}\|\partial_r k_0(r,\cdot)\|_{L^2(\widetilde{M})}\,\mathrm{d}s\,\mathrm{d}r \\
&\quad\leq \frac{Ce^{(h+\varepsilon)T}}{\lambda}\int_0^\infty e^{-\nu r}\|\partial_r k_0(r,\cdot)\|_{L^2(\widetilde{M})}\,\mathrm{d}r \leq \frac{Ce^{(h+\varepsilon)T}}{\lambda}\,\|a\|_*
\end{aligned}$$

for some constants C independent of (T,λ) (but depending on $\nu-\nu_0$). On the other hand, using again Lemma 4.3 for v_λ, we obtain

$$\begin{aligned}
\|\partial_t v_\lambda(t,\cdot)\|_{L^2(M)} + \|\nabla^g v_\lambda(t,\cdot)\|_{L^2(M)} &\leq C\sum_{\gamma\in\pi_1(M)}\int_0^t e^{\nu(t-s)}\,\|k_0(s,\cdot)\|_{L^2(\gamma(F))}\,\mathrm{d}s \\
&\leq Ce^{(h+\varepsilon)T}\int_0^t e^{\nu(t-s)}\,\|k_0(s,\cdot)\|_{L^2(\widetilde{M})}\,\mathrm{d}s.
\end{aligned}$$

Repeating the previous argument, we get:

$$\|\partial_t v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} + \|\nabla^g v_\lambda(t,\cdot)\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} \leq Ce^{(h+\varepsilon)T}\,\|a\|_*. \qquad\blacksquare$$

5. Stable determination of the electrical potential for the Schrödinger equation

In this section we shall prove Theorem 1, i.e., the stability estimate for the Schrödinger equation. The main idea relies in using geometric optics solutions with initial data $b \in H^2(\partial_- SM_e)$ with $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$, where $T_0 > 0$ is taken large. These solutions are concentrating on geodesics with endpoints on the boundary and with length at most T_0; when $T_0 \to \infty$, these geodesics will cover a set of full measure in M. Using these solutions, one can control the L^2 to norm of $\Pi_0^e q$ by the difference of the DN maps for q_1 and q_2, with a remainder term coming from L^2 estimates of the X-ray transform of q and $\Pi_0^e q$ on $\mathcal{T}_+^{\partial SM}(T_0)$; here $\Pi_0^e = I_0^{e*} I_0^e$ is the normal operator associated to the X-ray transform on M_e introduced in (3.3). As we shall see, using the estimate (3.4), this is enough to obtain our stability estimate since, due to the hyperbolicity of the trapped set, the volume of $\mathcal{T}_+(T_0)$ decays exponentially as $T_0 \to \infty$ and this remainder term can be absorbed in the stability estimate for T_0 sufficiently large.

5.1. Preliminary estimates

We first reformulate Lemmas 5.1 and 5.2 from [4] in our setting. Let $q = q_1 - q_2$ extended to M_e by $q = 0$ in $M_e \setminus M$. Recall that $\Lambda^S_{g,q}$ is the Dirichlet-to-Neumann map associated with the Schrödinger equation (1.1).

Lemma 5.1. *Let $q_1, q_2 \in W^{1,\infty}(M)$ with $q_1|_{\partial M} = q_2|_{\partial M}$ and set $q := q_1 - q_2$. There exist $C > 0$ depending only on $(M, g, \|q\|_{W^{1,\infty}})$ and $C_0 > 0$ depending only on (M, g) such that for any $T_0 > 0$, any $a_1, a_2 \in H^1(\mathbb{R}; H^2(\widetilde{M}))$ satisfying the transport equation (4.4) on $\widetilde{M}_e$ (for the same solution $\psi_{\tilde{y}}$ to the eikonal equation $|\nabla_{\tilde{g}} \psi_{\tilde{y}}| = 1$) with condition $a_j|_{\partial_- S\widetilde{M}} = \pi^* b_j$ for $b_1, b_2 \in H^2(\partial_- SM)$ satisfying $b_1|_{\mathcal{T}_+^{\partial SM}(T_0))} = b_2|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$, the following estimate holds true:*

$$\Big| \int_0^T \int_{\widetilde{M}} \tilde{q}(x)\, a_1(2\lambda t, x)\, \overline{a_2(2\lambda t, x)}\, \mathrm{dv}_{\tilde{g}}(x)\, \mathrm{d}t \Big| \\ \leq C e^{C_0 T_0} \big(\lambda^{-2} + \|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_* \big) \|a_1\|_* \, \|a_2\|_*,$$

for any $\lambda \geq T_0/(2T)$, where $\tilde{q} = \pi^* q \in W^{1,\infty}(\widetilde{M})$ is the lift of q.

Proof. By Lemma 4.2, one can construct a_2, $\psi_{\tilde{y}}$, such that for $\lambda \geq T_0/(2T)$ and $G_{2,\lambda}$ defined as in (4.7), the solution

$$u_2(t, x) = G_{2,\lambda}(t, x) + v_{2,\lambda}(t, x)$$

to the Schrödinger equation corresponding to the potential q_2,

$$\begin{cases} (i\partial_t - \Delta_g + q_2(x)) u(t, x) = 0, & \text{in } (0, T) \times M, \\ u(0, \cdot) = 0, & \text{in } M, \end{cases}$$

satisfies $v_{2,\lambda}(t, x) = 0$ for all $(t, x) \in (0, T) \times \partial M$, and

$$\lambda \|v_{2,\lambda}(t, \cdot)\|_{L^2(M)} + \|\nabla v_{2,\lambda}(t, \cdot)\|_{L^2(M)} \leq C e^{(h+\varepsilon)T_0} \|\tilde{a}_2\|_*. \tag{5.1}$$

Moreover, $u_2 \in \mathcal{C}^1([0, T]; L^2(M)) \cap \mathcal{C}([0, T]; H^2(M))$. We next denote by f_λ the restriction of $G_{2,\lambda}$ on $(0, T] \times \partial M$

$$f_\lambda(t, x) := G_{2,\lambda}(t, x) = \sum_{\gamma \in \pi_1(M)} a_2(2\lambda t, \gamma(x))\, e^{i\lambda(\psi(\gamma(x)) - \lambda t)}.$$

Let v be the solution to the non-homogeneous boundary value problem

$$\begin{cases} (i\partial_t - \Delta_g + q_1) v(t, x) = 0, & (t, x) \in (0, T) \times M, \\ v(0, x) = 0, & x \in M, \\ v(t, x) = u_2(t, x) = f_\lambda(t, x), & (t, x) \in (0, T) \times \partial M, \end{cases}$$

and denote $w = v - u_2$. Notice that w solves the following homogeneous boundary value problem for the Schrödinger equation:

$$\begin{cases} (i\partial_t - \Delta_g + q_1) w(t, x) = q(x) u_2(t, x), & (t, x) \in (0, T) \times M, \\ w(0, x) = 0, & x \in M, \\ w(t, x) = 0, & (t, x) \in (0, T) \times \partial M. \end{cases}$$

Since $q(x)u_2 \in W^{1,1}([0,T];L^2(M))$ with $u_2(0,\cdot) \equiv 0$, by Lemma 4.1 we obtain that

$$w \in \mathscr{C}^1([0,T];L^2(M)) \cap \mathscr{C}([0,T];H^2(M)\cap H_0^1(M)).$$

On the other hand, using Lemma 4.2, we construct a special solution

$$u_1 \in \mathscr{C}^1([0,T];L^2(M)) \cap \mathscr{C}([0,T];H^2(M))$$

to the backward Schrödinger equation

$$\begin{cases} (i\partial_t - \Delta_g + \overline{q}_1(x))u_1(t,x) = 0, & (t,x) \in (0,T)\times M, \\ u_1(T,x) = 0, & x \in M, \end{cases}$$

having the special form

$$u_1(t,x) = \sum_{\gamma\in\pi_1(M)} a_1(2\lambda t, \gamma(x))\, e^{i\lambda(\psi_{\tilde{y}}(\gamma(x))-\lambda t)} + v_{1,\lambda}(t,x),$$

which corresponds to the electric potential $\bar{q}_1$, where $v_{1,\lambda}$ vanishes on $(0,T)\times\partial M$, satisfies $v_{1,\lambda}(T,\cdot) = 0$, and

$$\forall t \in [0,T], \quad \lambda\|v_{1,\lambda}(t,\cdot)\|_{L^2(M)} + \|\nabla^g v_{1,\lambda}(t,\cdot)\|_{L^2(M)} \le Ce^{(h+\varepsilon)T_0}\|a_1\|_*. \tag{5.2}$$

By integration by parts and Green's formula, we obtain

$$\begin{aligned}\int_0^T\int_M (i\partial_t - \Delta_g + q_1)\, w\, \overline{u}_1 \,\mathrm{dv}_g\,\mathrm{d}t &= \int_0^T\int_M q\, u_2\, \overline{u}_1 \,\mathrm{dv}_g\,\mathrm{d}t \\ &= -\int_0^T\int_{\partial M} \partial_{\mathrm{n}} w\, \overline{u}_1 \,\mathrm{dv}_{g_{|\partial M}}\,\mathrm{d}t.\end{aligned}$$

We then obtain

$$\int_0^T\int_M q\, u_2\, \overline{u}_1 \,\mathrm{dv}_g\,\mathrm{d}t = -\int_0^T\int_{\partial M} (\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2})(f_\lambda)(t,x)\, \overline{g}_\lambda(t,x) \,\mathrm{dv}_{g_{|\partial M}}(x)\,\mathrm{d}t,$$

where the boundary data g_λ is given by

$$g_\lambda(t,x) := \sum_{\gamma\in\pi_1(M)} a_1(2\lambda t, \gamma(x))\, e^{i\lambda(\psi_{\tilde{y}}(\gamma(x))-\lambda t)}, \quad (t,x) \in (0,T)\times\partial M.$$

Using the definition of u_1 and u_2, we get

$$\begin{aligned}&\sum_{\gamma_1,\gamma_2\in\pi_1(M)} \int_0^T\int_{\mathscr{F}} \tilde{q}(x)\, a_2(2\lambda t, \gamma_1(x))\, \overline{a_1(2\lambda t, \gamma_2(x))}\, e^{i\lambda(\psi_{\tilde{y}}(\gamma_1(x))-\psi_{\tilde{y}}(\gamma_2(x)))} \,\mathrm{dv}_{\tilde{g}}(x)\,\mathrm{d}t \\ &\quad= -\int_0^T\int_{\partial M} \overline{g}_\lambda (\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}) f_\lambda \,\mathrm{dv}_{g_{|\partial M}}\,\mathrm{d}t - \int_0^T\int_M q\, v_{2,\lambda}\, \overline{v}_{1,\lambda} \,\mathrm{dv}_g\,\mathrm{d}t \\ &\qquad - \int_0^T\int_M q\, f_\lambda\, \overline{v}_{1,\lambda} \,\mathrm{dv}_g\,\mathrm{d}t - \int_0^T\int_M q\, v_{2,\lambda}\, \overline{g_\lambda} \,\mathrm{dv}_g\,\mathrm{d}t.\end{aligned}$$

By (5.2), there is $C > 0$ depending on $\|q\|_{L^\infty}$, T and $\varepsilon > 0$ so that

$$\Big|\int_0^T \int_M q f_\lambda \overline{v}_{1,\lambda}\, \mathrm{dv}_g\, \mathrm{d}t\Big| \leq \|q\|_{L^\infty} \sum_{\gamma\in\pi_1(M)} \int_0^T \|a_2(2\lambda t,\cdot)\|_{L^2(\gamma(\mathcal{F}))} \|v_{1,\lambda}(t,\cdot)\|_{L^2(M)}\, \mathrm{d}t$$
$$\leq C e^{2(h+\varepsilon)T_0} \lambda^{-2} \|a_2\|_* \|a_1\|_*,$$

where $\|\cdot\|_* = \|\cdot\|_{H^1([0,T_0];H^2(\widetilde{M}))}$ as before, and the second λ^{-1} comes from the change of variables in t. With the same argument, using (5.1) and (5.2),

$$\Big|\int_0^T \int_M q\, v_{2,\lambda}\, \overline{g}_\lambda\, \mathrm{dv}_g\, \mathrm{d}t\Big| + \Big|\int_0^T \int_M q\, v_{2,\lambda}\, \overline{v}_{1,\lambda}\, \mathrm{dv}_g\, \mathrm{d}t\Big| \leq C e^{2(h+\varepsilon)T_0}\, \lambda^{-2} \|a_2\|_* \|a_1\|_*.$$

Finally, using the trace theorem, we get

$$\Big|\int_0^T \int_{\partial M} ((\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}) f_\lambda)\, \overline{g}_\lambda\, \mathrm{dv}_{g|_{\partial M}}\, \mathrm{d}t\Big|$$
$$\leq C \|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*\, \|f_\lambda\|_{H^1((0,T)\times\partial M)}\, \|g_\lambda\|_{L^2((0,T)\times\partial M)}$$
$$\leq C e^{2(h+\varepsilon)T_0}\, \|a_1\|_*\, \|a_2\|_*\, \|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*.$$

The diagonal term gives, using that γ_1 and γ_2 are isometries of $\tilde{g}$ and preserve $\mathrm{dv}_{\tilde{g}}$,

$$\sum_{\gamma\in\pi_1(M)} \int_0^T \int_{\mathcal{F}} \tilde{q}(x)\, a_2(2\lambda t, \gamma(x))\, \overline{a_1(2\lambda t, \gamma(x))}\, \mathrm{dv}_{\tilde{g}}(x)\, \mathrm{d}t$$
$$= \int_0^T \int_{\widetilde{M}} \tilde{q}(x)\, a_2(2\lambda t, x)\, \overline{a_1(2\lambda t, x)}\, \mathrm{dv}_{\tilde{g}}(x)\, \mathrm{d}t.$$

The last point consists in bounding the off-diagonal terms

$$\Big|\sum_{\gamma_1\neq\gamma_2} \int_0^T \int_{\mathcal{F}} \tilde{q}(x)\, a_2(2\lambda t, \gamma_1(x))\, \overline{a_1(2\lambda t, \gamma_2(x))}\, e^{i\lambda(\psi_{\tilde{y}}(\gamma_1(x)) - \psi_{\tilde{y}}(\gamma_2(x)))}\, \mathrm{dv}_{\tilde{g}}(x)\, \mathrm{d}t\Big|$$
$$= \Big|\sum_{\gamma\neq \mathrm{Id}} \int_0^T \int_{\widetilde{M}} \tilde{q}(x)\, a_2(2\lambda t, x)\, \overline{a_1(2\lambda t, \gamma(x))}\, e^{i\lambda(\psi_{\tilde{y}}(x) - \psi_{\tilde{y}}(\gamma(x)))}\, \mathrm{dv}_{\tilde{g}}(x)\, \mathrm{d}t\Big|, \tag{5.3}$$

We will apply non stationary phase: to that aim, we need to bound below the norm

$$|\nabla^{\tilde{g}}(\psi_{\tilde{y}}(\cdot) - \psi_{\tilde{y}}(\gamma(\cdot))| = |\nabla^{\tilde{g}} d_{\tilde{g}}(\cdot, \tilde{y}) - (\mathrm{d}\gamma)^{-1} \nabla^{\tilde{g}} d_{\tilde{g}}(\cdot, \tilde{y}) \circ \gamma|.$$

This corresponds to bounding below the distance between two vectors $v_1 = \mathrm{d}\pi \nabla^{\tilde{g}} \psi_{\tilde{y}}(x)$ and $v_2 = \mathrm{d}\pi (\mathrm{d}\gamma)^{-1} (\nabla^{\tilde{g}} \psi_{\tilde{y}})(\gamma(x)) \in T_x M_e$ tangent to two geodesics α_1 and α_2 of length $\leq T_0$ in M_e, starting at $y \in \partial M_e$ and with endpoints $x \in M_e$, and α_1, α_2 being in two different homotopy classes. On the other hand, if $\mathrm{injrad}(M_{ee}, g)$ is the injectivity radius of (M_{ee}, g), then for $0 < \delta < \mathrm{injrad}(M_{ee}, g)$ and any C^1 curves $\alpha_1 : [0,1] \to M_e$ and $\alpha_2 : [0,1] \to M_e$ so that

$$\alpha_1(0) = \alpha_2(0), \quad \alpha_1(1) = \alpha_2(1), \quad \alpha_2([0,1]) \subset \{z \in M_e \mid d_g(z, \alpha_1([0,1])) < \delta\},$$

there is a homotopy $h\colon [0,1]\times M_e \to M_e$ so that $h(0,\alpha_1(t)) = \alpha_1(t)$ and $h(1,\alpha_2(t)) = \alpha_2(t)$. However by a standard estimate on flows of smooth vector fields, there is $C_0 > 0$ independent of $y, x \in M_e$ (C_0 depends on the $\mathscr{C}^1$ norm of the geodesic vector field X_{g_0}) such that

$$d_g(\alpha_2([0,1]), \alpha_1([0,1])) \leq |v_1 - v_2|_g\, e^{C_0 T_0}.$$

We then deduce that $|v_1 - v_2|_{\tilde g} \geq \delta\, e^{-C_0 T_0}$, and therefore we can apply one integration by parts in the second line of (5.3) and the usual change of coordinates $t = s/\lambda$ to obtain

$$\Big|\sum_{\gamma\in\pi_1(M)\setminus \mathrm{Id}} \int_0^T \int_{\widetilde M} \tilde q(x)\, a_2(2\lambda t, x)\, \overline{a_1(2\lambda t, \gamma(x))}\, e^{i\lambda(\psi_{\tilde y}(x) - \psi_{\tilde y}(\gamma(x)))}\, \mathrm{dv}_{\tilde g}(x)\, \mathrm{d}t\Big|$$
$$\leq \lambda^{-2}\, C e^{(2h+\varepsilon+C_0)T_0}\, \|a_2\|_*\, \|a_1\|_*,$$

where $C > 0$ now depends on the $\|q\|_{W^{1,\infty}(M)}$ norm instead of $\|q\|_{L^\infty}$. ∎

Notice that $C_0 > 0$ above depends only the *maximal expansion rate of the flow* defined as the smallest constant $\theta > 0$ such that for each $\varepsilon > 0$ there is $C > 0$ such that for all t large enough,

$$\|\mathrm{d}\varphi_t\| \leq C e^{(\theta+\varepsilon)|t|}. \tag{5.4}$$

Next, we show the following lemma which is key to relate the X-ray transform of q to the DN map of the Schrödinger equation.

Lemma 5.2. *Let $q_1, q_2 \in W^{1,\infty}(M)$ with $q_1|_{\partial M} = q_2|_{\partial M}$ and set $q := q_1 - q_2$. There are $C_0 > 0$ depending only on (M,g) and $C > 0$ depending on $(M, g, \|q_1\|_{W^{1,\infty}}, \|q_2\|_{W^{1,\infty}})$ such that for any $T_0 > 0$ and any $b \in H^2(\partial_- SM_e)$ such that $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$, the estimate*

$$\Big|\int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(sv))\, b(y,v)\, \mu(y,v)\, \mathrm{d}s\, \mathrm{d}\omega_y(v)\Big|$$
$$\leq C e^{C_0 T_0} \|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*^{1/2}\, \|b(y,\cdot)\|_{H^2(\partial_- S_y M_e)}$$

holds uniformly for any $y \in \partial M_e$, where $\mu(y,v) = g(\mathrm{n}_y, v)$, with n_y the inward unit normal of ∂M_e at y.

Proof. We take two solutions a_1, a_2 to the transport equation on the universal cover $\widetilde M_e$ defined as before by

$$a_1(t,x) = \alpha^{-1/4}\, \phi(t - r(x))\, \tilde b(\tilde y, v(x)),$$
$$a_2(t,x) = \alpha^{-1/4}\, \phi(t - r(x))\, \tilde \mu(\tilde y, v(x))\, \chi_{T_0}(\tau_+^e(y, v(x))),$$

where $r(x) = d_{\tilde g}(x, \tilde y)$, $\widetilde{\exp}_{\tilde y}(r(x)v(x)) = x$, $\chi_{T_0} \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ is supported in $[0, T_0+1]$ and is equal to 1 in $[0, T_0]$, and $\tilde\mu, \tilde b$ are lifts of μ, b to $S\widetilde M_e$ as before. Here we have used the natural identification $S_{\tilde y}\widetilde M_e \simeq S_y M_e$ to define $\tau_+^e(y, v(x))$. We write using geodesic polar coordinates $x = \widetilde{\exp}_{\tilde{y}}(rv)$ with $v \in \partial_- S_{\tilde{y}} \widetilde{M}_e$,

$$\begin{aligned}
&\int_0^T \int_{\widetilde{M}} \tilde{q}(x)\, a_1(2\lambda t, x)\, a_2(2\lambda t, x)\, \mathrm{d}v_g(x)\, \mathrm{d}t \\
&\quad = \int_0^T \int_{\partial_- S_{\tilde{y}} \widetilde{M}_e} \int_0^{\tau_+^e(y,v)} \tilde{q}(r, v)\, a_1(2\lambda t, r, v)\, a_2(2\lambda t, r, v)\, \alpha^{1/2}\, \mathrm{d}r\, \mathrm{d}\omega_{\tilde{y}}(v)\, \mathrm{d}t \\
&\quad = \int_0^T \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(rv))\, \phi^2(2\lambda t - r)\, b(y, v)\, \mu(y, v)\, \mathrm{d}r\, \mathrm{d}\omega_y(v)\, \mathrm{d}t \\
&\quad = \frac{1}{2\lambda} \int_0^{2\lambda T} \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(rv))\, \phi^2(t - r)\, b(y, v)\, \mu(y, \theta)\, \mathrm{d}r\, \mathrm{d}\omega_y(v)\, \mathrm{d}t.
\end{aligned}$$

By Lemma 5.1, we obtain the bound (using that $r \leq T_0 \leq 2\lambda T$ by our assumption)

$$\begin{aligned}
&\Big| \int_0^\infty \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(rv))\, \phi^2(t - r)\, b(y, v)\, \mu(y, v)\, \mathrm{d}r\, \mathrm{d}\omega_y(v)\, \mathrm{d}t \Big| \\
&\qquad \leq C\, e^{C_0 T_0} \big(\lambda^{-1} + \lambda \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_* \big)\, \|\phi\|_{H^3(\mathbb{R})}^2\, \|b(y, \cdot)\|_{H^2(S_y^- M_e)}.
\end{aligned}$$

Moreover, using the properties of the function ϕ, we also have

$$\begin{aligned}
&\int_0^\infty \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(rv))\, \phi^2(t - r)\, b(y, v)\, \mu(y, v)\, \mathrm{d}r\, \mathrm{d}\omega_y(v)\, \mathrm{d}t \\
&\qquad = \Big(\int_{-\infty}^\infty \phi^2(t)\, \mathrm{d}t \Big) \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(rv))\, b(y, v)\, \mu(y, v)\, \mathrm{d}r\, \mathrm{d}\omega_y(v).
\end{aligned}$$

Finally, to prove the lemma it suffices to take

$$\lambda = \frac{T_0}{2T} \cdot \Big(\frac{2\delta(N_0)}{\| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_*} \Big)^{1/2},$$

where $\delta(N_0) := \sup_{q \in \mathcal{Q}(N_0)} \|\Lambda_{g,q}^S\|_*$ is finite by Theorem 1 in [4]. ■

5.2. Proof of the stability estimate

By Lemma 5.2 we have, for any $y \in \partial M_e$ and $b \in H^2(\partial_- S M_e)$ such that $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$,

$$\begin{aligned}
&\Big| \int_{\partial_- S_y M_e} \int_0^{\tau_+^e(y,v)} q(\exp_y(sv))\, \mathrm{d}s\ b(y, v)\, \mu(y, v)\, \mathrm{d}\omega_y(v) \Big| \\
&\qquad \leq C e^{C_0 T_0}\, \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_*^{1/2}\, \|b(y, \cdot)\|_{H^2(\partial_- S_y M_e)},
\end{aligned}$$

where C, C_0 are uniform in y, T_0. Now we take a bump function $\chi_{T_0} \in \mathcal{C}_c^\infty(\mathbb{R})$ supported in the interval $[0, T_0)$ and equal to 1 on $[0, T_0 - 1]$, for $T_0 \gg 1$, and set

$$b(y, v) := \chi_{T_0}(\tau_+^e(y, v))\, I_0^e(\Pi_0^e q)(y, v). \tag{5.5}$$

Since Π_0^e is a pseudo-differential operator of order -1 on M_e ([9], Proposition 5.7) and $q = 0$ in $M_e \setminus M$, $\Pi_0^e q \in W^{2,p}(M_e)$ for all $p < \infty$. Integrating with respect to $y \in \partial M_e$, we obtain

$$
\begin{aligned}
(5.6) \qquad \Big| \int_{\partial_- SM_e} I_0^e(q)(y,v)\, I_0^e(\Pi_0^e q)(y,v)\, \mathrm{d}\mu_{\mathrm{n}}(y,v) \Big| & \\
\leq C\, e^{C_0 T_0}\, \|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*^{1/2}\, \|b\|_{H^2(\partial_- SM_e)} & \\
+ \Big| \int_{\mathcal{T}_+^{\partial SM}(T_0)} I_0^e q(y,v)\, I_0^e(\Pi_0^e q)(y,v)\, \mathrm{d}\mu_{\mathrm{n}}(y,v) \Big|. &
\end{aligned}
$$

Moreover, we can write

$$
(5.7) \qquad I_0^e(\Pi_0^e q)(y,v) = \int_0^{\tau_+^e(y,v)} \pi_0^*(\Pi_0^e q) \circ \varphi_t(y,v)\, \mathrm{d}t, \quad (y,v) \in \partial_- SM_e \setminus \Gamma_-.
$$

By the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\Big| \int_{\mathcal{T}_+^{\partial SM}(T_0)} I_0^e q(y,\theta)\, I_0^e(\Pi_0^e q)(y,\theta)\, \mathrm{d}\mu_{\mathrm{n}}(y,\theta) \Big| \\
&\leq \Big(\int_{\mathcal{T}_+^{\partial SM}(T_0)} |I_0^e q(y,\theta)|^2\, \mathrm{d}\mu_{\mathrm{n}}(y,\theta) \Big)^{1/2} \Big(\int_{\mathcal{T}_+^{\partial SM}(T_0)} |I_0^e(\Pi_0^e q)(y,\theta)|^2\, \mathrm{d}\mu_{\mathrm{n}}(y,\theta) \Big)^{1/2}.
\end{aligned}
$$

Since $q \in W^{1,\infty}(M) \subset H^1(M)$, then by Proposition 5.7 in [9,], $\Pi_0^e q \in H^2(M)$. By the Sobolev embedding theorem, we also have that

$$
\pi_0^* q,\ \pi_0^* \Pi_0^e q \in L^p(SM),
$$

for some $p > 2$. Let us now give an estimate on the L^2-norm of the X-ray transform of an L^p-function in $\mathcal{T}_+^{\partial SM}(T_0)$.

Lemma 5.3. *Let $Q < 0$ be the escape rate defined by (2.2) and let $p \in (2,\infty]$. Then there exists $C = C(Q, p, \dim M) > 0$ such that for all $f \in L^p(M)$ and $T_0 \gg 1$ large,*

$$
\int_{\mathcal{T}_+^{\partial SM}(T_0)} |I_0^e f(y,v)|^2\, \mathrm{d}\mu_{\mathrm{n}}(y,v) \leq C e^{QT_0/2}\, \|f\|^2_{L^p(M)}.
$$

Proof. We follow the argument in Lemma 5.1 of [9]. Using Hölder's inequality, with $1/p + 1/p' = 1$ and $r/p' = (p-1)/(p-2) > 1$, and Santaló's formula, we have

$$
\begin{aligned}
\|I_0^e f\|^2_{L^2(\mathcal{T}_+^{\partial SM}(T_0))} &= \int_{\mathcal{T}_+^{\partial SM}(T_0)} \Big| \int_0^{\tau_+^e(y,v)} \pi_0^* f(\varphi_t(y,v))\, \mathrm{d}t \Big|^2 d\mu_{\mathrm{n}} \\
&\leq \int_{\mathcal{T}_+^{\partial SM}(T_0)} \Big(\int_0^{\tau_+^e(y,v)} |\pi_0^* f(\varphi_t(y,v))|^p\, \mathrm{d}t \Big)^{2/p} \tau_+^e(y,v)^{2/p'}\, \mathrm{d}\mu_{\mathrm{n}} \\
&\leq \Big(\int_{\mathcal{T}_+^{\partial SM}(T_0)} \int_0^{\tau_+^e(y,v)} |\pi_0^* f(\varphi_t(y,v))|^p\, \mathrm{d}t\, \mathrm{d}\mu_{\mathrm{n}} \Big)^{2/p} \|\tau_+^e\|^{2/p'}_{L^{2r/p'}(\mathcal{T}_+^{\partial SM}(T_0))} \\
&\leq C \|f\|^2_{L^p(M)} \Big(\int_{T_0-L-1}^{\infty} (t+L+1)^{2r/p'-1} V(t)\, \mathrm{d}t \Big)^{1/r} \leq C\, \|f\|^2_{L^p(M)} e^{QT_0/2},
\end{aligned}
$$

with C depending only on (Q, p, M) and L some fixed constant satisfying $\tau_+ + L \geq \tau_+^e$ (thus depending only on M). In the third inequality, we have used equation (4.13) in [9], which states that for $L > 0$ as above and for all $T \gg L$,

$$\int_{\partial_- SM} \mathbf{1}_{[T,\infty)}(\tau_+^e)\, \mathrm{d}\mu_\nu \leq 2V(T - L - 1)$$

and then the Cavalieri principle gives (by definition of $V(t)$)

$$\int_{\mathcal{T}_+^{\partial SM}(T_0)} (\tau_+^e(x, v))^{2r/p'} \mathrm{d}\mu_\nu(x, v) \leq C \int_{T_0 - L - 1}^{\infty} (t + L + 1)^{2r/p' - 1} V(t)\, \mathrm{d}t. \qquad \blacksquare$$

Notice that as $p \to +\infty$ we have $r \to 1$, and that $C(Q, p, \dim M)$ can be taken uniform in p.

Lemma 5.4. *There are $C > 0$ and $\theta > 0$ such that for all $b \in H^2(\partial_- SM_e)$ given by (5.5),*

$$\|b\|_{H^2(\partial_- SM)} \leq Ce^{\theta T_0} \|q\|_{W^{1,\infty}(M)}.$$

Proof. First, by the implicit function theorem, $\tau_+^e : \partial_- SM_e \setminus \mathcal{T}_+^{\partial SM}(T_0) \to \mathbb{R}_+$ is a smooth function. We shall compute its $\mathcal{C}^2$-norm. Let ρ be a boundary defining function of M_e so that $|d\rho|_g = 1$ near ∂M_e and $d(\pi_0^* \rho)(X) = -g(v, \mathrm{n})$ at $\partial_+ SM_e$. The function τ_+^e is defined by the implicit equation

$$\pi_0^* \rho(\varphi_{\tau_+^e(x,v)}(x, v)) = 0.$$

Therefore, denoting $S(x, v) := \varphi_{\tau_+^e(x,v)}(x, v)$ one has, on $\partial_- SM_e \setminus \Gamma_-$,

$$\mathrm{d}\tau_+^e(x, v) = \frac{\mathrm{d}(\pi_0^* \rho)_{S(x,v)} . \mathrm{d}\varphi_{\tau_+^e(x,v)}}{g(S(x, v), \mathrm{n})}. \tag{5.8}$$

From standard estimates on flows of autonomous C^2-vector fields, there are $C, \theta > 0$ depending on $\|X\|_{C^2}$ such that for all $t \in \mathbb{R}$ for which the flow is defined,

$$\|\varphi_t\|_{C^2} \leq Ce^{\theta |t|}.$$

Using this, and the fact that $g(S(x, v), \mathrm{n}) > c_0 > 0$ for some c_0 if $\tau_+(x, v) > 1$, we see from the expression (5.8) and its derivative that there are $C > 0$, $\theta > 0$ independent of T_0 such that

$$\sup_{(x,v) \in \partial \mathcal{T}_+(T_0),\, \tau_+^e(x,v) > 1} \|\nabla \tau_+^e(x, v)\| + \|\nabla^2 \tau_+^e(x, v)\| \leq Ce^{2\theta T_0}, \tag{5.9}$$

where ∇ is any fixed Riemannian connection on ∂SM_e (for example that given by the Sasaki metric). First, by (3.2) and Lemma 3.1, for each $p > 2$ we have (using Sobolev embedding)

$$\|b\|_{L^2(\partial_- SM_e)} \leq C \|\Pi_0^e q\|_{L^p(M_e)} \leq C \|q\|_{L^2(M)}. \tag{5.10}$$

Next we compute $\mathrm{d}b$. Let $f := \Pi_0^e q$. Since $\mathrm{supp}(q) \subset M$, one can use (3.6) to deduce that $f \in W^{2,p}(M_e)$ for all $p < \infty$ and that $f \in \mathcal{C}^\infty(M_e \setminus M)$. For $z \in \partial_- SM_e$,

$$\mathrm{d}b(z) = \mathrm{d}\tau_+^e(z)\, \chi'_{T_0}(\tau_+^e(z))(I_0^e f)(z) + \chi_{T_0}(\tau_+^e(z)) \Big(\mathrm{d}\tau_+^e(z)\, \pi_0^* f(S(z)) + \int_0^{\tau_+^e(z)} \mathrm{d}(\pi_0^* f).\mathrm{d}\varphi_t(z)\, \mathrm{d}t\Big)$$

and therefore by (5.9) and (3.2), there are $C > 0, \theta > 0$ independent of T_0, q such that

$$\|\mathrm{d}b\|_{L^2} \leq C T_0\, e^{\theta T_0}\, \|q\|_{W^{1,\infty}(M)}. \tag{5.11}$$

Finally, we compute another derivative of b, and write

$$\begin{aligned}\nabla^2 b(z) = {}& \nabla^2 \tau_+^e(z)\, \chi'_{T_0}(\tau_+^e(z))(I_0^e f)(z) + (\mathrm{d}\tau_+^e \otimes \mathrm{d}\tau_+^e)(z)\, \chi''_{T_0}(\tau_+^e(z))(I_0^e f)(z) \\ & + 2\,\chi'_{T_0}(\tau_+^e(z))\, \mathrm{d}\tau_+^e(z) \otimes \Big(\mathrm{d}\tau_+^e(z)\, \pi_0^* f(S(z)) + \int_0^{\tau_+^e(z)} \mathrm{d}(\pi_0^* f).\mathrm{d}\varphi_t(z)\, \mathrm{d}t\Big) \\ & + \chi_{T_0}\, (\tau_+^e(z))(\nabla^2 \tau_+^e(z)\, \pi_0^* f(S(z)) + 2\mathrm{d}\tau_+^e(z) \otimes (\mathrm{d}(\pi_0^* f).\mathrm{d}S(z))) \\ & + \chi_{T_0}(\tau_+^e(z)) \int_0^{\tau_+^e(z)} \nabla(\varphi_t^* d(\pi_0^* f))\, \mathrm{d}t.\end{aligned}$$

Since $\pi_0^* f$ is smooth near $\partial_- SM_e$ and since the bounds (5.9) hold and $I_0^e f \in L^2$, we obtain that there are $C > 0, \theta > 0$ independent of T_0, q such that

$$\|\nabla^2 b\|_{L^2} \leq C T_0\, e^{2\theta T_0}\, \|q\|_{W^{1,\infty}(M)}. \tag{5.12}$$

Combining (5.12), (5.11) and (5.10), we get the desired result. ∎

Proof of Theorem 1. By (5.6), Lemma 5.3, Lemma 5.4, and using that $\|\Pi_0^e q\|_{L^p(M_e)} \leq C\|q\|_{L^\infty(M)}$ for each $p \in (2, \infty)$, we have that there are $C_0 > 0$ depending only on (M, g) and C depending on $(M, g, \|q\|_{W^{1,\infty}})$ such that

$$\int_{M_e} |\Pi_0^e(q)|^2\, \mathrm{dv}_g \leq C e^{C_0 T_0}\, \|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_*^{1/2} + C e^{Q T_0/2}. \tag{5.13}$$

Then, defining $\alpha := e^{-QT_0/2}$ and $m := -2C_0/Q > 0$, we deduce from (5.13) that

$$\|\Pi_0^e q\|_{L^2(M_e)}^2 \leq C\big(\alpha^m \|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_*^{1/2} + \alpha^{-1}\big).$$

We next take T_0 sufficiently large so that $\alpha = \|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_*^{-1/(2(m+1))}$. With this choice, we obtain

$$\|\Pi_0^e q\|_{L^2(M_e)}^2 \leq C\|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_*^{1/(2(m+1))}. \tag{5.14}$$

This holds in the regime

$$T_0 = \frac{2}{Q} \log\Big(\|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_*^{1/(2(m+1))}\Big).$$

Finally, by (3.4), there is $C > 0$ depending only on (M, g) such that

$$\|q\|_{L^2(M)} \le C\|\Pi_0^e q\|_{H^1(M_e)}, \quad \|\Pi_0^e q\|_{H^2(M_e)} \le C\|q\|_{H^1(M)}. \tag{5.15}$$

Using this, an interpolation estimate, and (5.14), we obtain

$$\|\Pi_0^e q\|^2_{H^1(M_e)} \le C\|\Pi_0^e q\|_{L^2(M_e)}\|\Pi_0^e q\|_{H^2(M_e)} \le C\|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*^{1/(4(m+1))}\|q\|_{H^1(M)}.$$

Finally, by the first bound of (5.15), we conclude that for $q_1, q_2 \in \mathcal{Q}(N_0)$,

$$\|q\|^2_{L^2(M)} \le C\|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_*^{1/(4(m+1))} N_0.$$

This concludes the proof. ■

6. Stable determination of the electrical potential for the wave equation

In this section, we shall prove Theorem 3.

6.1. Preliminary estimates

We start with a lemma very similar to Lemma 5.1, but now in the context of the wave equation. Its proof follows the lines of that of Lemma 5.1.

Lemma 6.1. *Let $q_1, q_2 \in W^{1,\infty}(M)$ with $q_1|_{\partial M} = q_2|_{\partial M}$ and set $q := q_1 - q_2$ There exist $C > 0$ depending only on $(M, g, \|q_i\|_{W^{1,\infty}(M)})$, $\nu > 0$ depending only on $\|q_i\|_{L^\infty}$ and $C_0 \ge 0$ depending only on (M, g, ν) such that for any $T > 0$, $\lambda > 1$, and for $a_1, a_2 \in H_0^1([0, T], H^2(\widetilde{M}))$ the functions constructed in (4.6) with function b given respectively by $b_1, b_2 \in H^2(\partial_- SM)$ satisfying $b_1|_{\mathcal{T}_+^{\partial SM}(T)} = b_2|_{\mathcal{T}_+^{\partial SM}(T)} = 0$, the following estimate holds true*:

$$\left|\int_0^T \int_{\widetilde{M}} \tilde{q}(x)\, a_1(t, x)\, \overline{a_2(t, x)}\, \mathrm{d}v_{\tilde{g}}(x)\, \mathrm{d}t\right| \le C\, e^{C_0 T}\big(\lambda^{-1} + \lambda\|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|_{*,\nu}\big)\, \|a_1\|_*\, \|a_2\|_*$$

where $\tilde{q}$ is the lift of q to $\widetilde{M}$.

Proof. We shall proceed as for the Schrödinger equation. Let $\lambda > 0$. By Lemma 4.4, there exist a_2, $\psi_{\tilde{y}}$ as in (4.6), such that for the $G_{2,\lambda}(t, x)$ given by (4.20) with $a = a_2$, the solution

$$u_2(t, x) = G_{2,\lambda}(t, x) + v_{2,\lambda}(t, x)$$

to the wave equation corresponding to the potential q_2

$$\begin{cases} (\partial_t^2 + \Delta_g + q_2(x))u(t, x) = 0, & \text{in } I \times M, \\ u(0, \cdot) = 0, \quad \partial_t u(0, \cdot) = 0, & \text{in } M, \end{cases}$$

satisfies $v_{2,\lambda}(t,x)=0$ for all $(t,x)\in(0,T)\times\partial M$, and (for $\varepsilon>0$ small)

$$\lambda\|v_{2,\lambda}\|_{e^{\nu t}L^2(I\times M)}+\|\nabla^g v_{2,\lambda}\|_{e^{\nu t}L^2(I\times M)}\le Ce^{(h+\varepsilon)T}\|a_2\|_*. \tag{6.1}$$

We next denote by f_λ the restriction of $G_{2,\lambda}$ to $[0,T]\times\partial M$,

$$f_\lambda(t,x):=G_{2,\lambda}(t,x)=\sum_{\gamma\in\pi_1(M)} a_2(t,\gamma(x))\,e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}.$$

Let v be the solution to the boundary value problem

$$\begin{cases}(\partial_t^2+\Delta_g+q_1)v(t,x)=0, & (t,x)\in I\times M,\\ v(0,x)=0,\quad \partial_t v(0,x)=0, & x\in M,\\ v(t,x)=u_2(t,x):=f_\lambda(t,x), & (t,x)\in I\times\partial M,\end{cases}$$

and denote $w=v-u_2$. Notice that w solves the following homogeneous boundary value problem for the wave equation:

$$\begin{cases}(\partial_t^2+\Delta_g+q_1)w(t,x)=q(x)u_2(t,x), & (t,x)\in I\times M,\\ w(0,x)=0,\quad \partial_t w(0,x)=0, & x\in M,\\ w(t,x)=0, & (t,x)\in I\times\partial M.\end{cases}$$

Since $q(x)u_2\in\mathcal{C}([0,T];L^2(M))$ with $u_2(0,\cdot)\equiv 0$, by Lemma 4.3, we obtain that

$$w\in\mathcal{C}^1(I;L^2(M))\cap\mathcal{C}(I;H_0^1(M)).$$

On the other hand, we construct a special solution

$$u_1\in e^{\nu t}H^1(I;L^2(M))\cap e^{\nu t}L^2(I;H^1(M))$$

to the backward wave equation

$$\begin{cases}(\partial_t^2+\Delta_g+\bar q_1(x))u_1(t,x)=0, & (t,x)\in(0,T)\times M,\\ u_1(T,x)=0,\quad \partial_t u_1(T,x)=0, & x\in M,\end{cases}$$

having the special form

$$u_1(t,x)=\sum_{\gamma\in\pi_1(M)} a_1(t,\gamma(x))\,e^{i\lambda(\psi_{\tilde y}(\gamma(x))-t)}+v_{1,\lambda}(t,x),$$

which corresponds to the electric potential q_1, where $v_{1,\lambda}$ satisfies $v_{1,\lambda}(T,\cdot)=\partial_t v_{1,\lambda}(T,\cdot)=0$, and for each $\varepsilon>0$ there is C independent of T,λ such that

$$\lambda\|v_{1,\lambda}\|_{e^{\nu t}L^2(I\times M)}+\|\nabla^g v_{1,\lambda}\|_{e^{\nu t}L^2(I\times M)}\le Ce^{(h+\varepsilon)T}\|a_1\|_*. \tag{6.2}$$

By integration by parts and Green's formula, we obtain

$$\int_0^T\int_M(\partial_t^2+\Delta_g+q_1)w\overline{u}_1\,\mathrm{d}v_g\,\mathrm{d}t=\int_0^\infty\int_M qu_2\overline{u}_1\,\mathrm{d}v_g\,\mathrm{d}t=-\int_0^T\int_{\partial M}\partial_{\mathrm{n}}w\overline{u}_1\,\mathrm{d}v_{g|_{\partial M}}\,\mathrm{d}t.$$

We therefore obtain

$$\int_0^T q\,u_2\,\overline{u}_1\,\mathrm{dv}_g\,\mathrm{d}t = \int_0^T \int_{\partial M} (\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2})(f_\lambda)\,\overline{g}_\lambda\,\mathrm{dv}_{g|_{\partial M}}\,\mathrm{d}t,$$

where the boundary data g_λ is given by

$$g_\lambda(t,x) := \sum_{\gamma\in\pi_1(M)} a_1(t,\gamma(x))\,e^{i\lambda(\psi_{\tilde{y}}(\gamma(x))-t)}, \quad (t,x)\in(0,T)\times\partial M.$$

Using the definition of u_1 and u_2, we get

$$\begin{aligned}
\sum_{\gamma_1,\gamma_2\in\pi_1(M)} \int_0^T \int_{\mathcal{F}} \tilde{q}(x)\,a_2(t,\gamma_1(x))\,\overline{a_1(t,\gamma_2(x))}\,\mathrm{dv}_{\tilde{g}}(x)\,\mathrm{d}t \\
= \int_0^T \int_{\partial M} \overline{g}_\lambda(\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}) f_\lambda\,\mathrm{dv}_{g|_{\partial M}}\,\mathrm{d}t - \int_0^T \int_M q\,v_{2,\lambda}\,\overline{v}_{1,\lambda}\,\mathrm{dv}_g\,\mathrm{d}t \\
- \sum_{\gamma\in\pi_1(M)} \int_0^T \int_{\mathcal{F}} q\,e^{i\lambda(\psi_{\tilde{y}}(\gamma(\cdot))-t)}\,a_2(t,\gamma(\cdot))\,\overline{v}_{1,\lambda}\,\mathrm{dv}_{\tilde{g}}\,\mathrm{d}t \\
- \sum_{\gamma\in\pi_1(M)} \int_0^T \int_{\mathcal{F}} q\,v_{2,\lambda}\,e^{-i\lambda(\psi_{\tilde{y}}(\gamma(\cdot))-t)}\,\overline{a_1(t,\gamma(\cdot))}\,\mathrm{dv}_{\tilde{g}}\,\mathrm{d}t.
\end{aligned}$$

By (6.1) and (6.2), for each $\varepsilon > 0$ small, there is $C > 0$ depending on g, $\|q\|_{L^\infty}$, ε such that for all T, λ,

$$\begin{aligned}
\sum_{\gamma\in\pi_1(M)} \Big(\Big| \int_0^T \int_{\mathcal{F}} q\,e^{i\lambda(\psi\circ\gamma-t)}\,a_2(t,\gamma(\cdot))\,\overline{v}_{1,\lambda}\,\mathrm{dv}_{\tilde{g}}\,\mathrm{d}t \Big| \\
+ \Big| \int_0^T \int_{\mathcal{F}} q\,e^{i\lambda(\psi\circ\gamma-t)}\,a_1(t,\gamma(\cdot))\,\overline{v}_{2,\lambda}\,\mathrm{dv}_{\tilde{g}}\,\mathrm{d}t \Big| \Big) \\
\le C \sum_{\gamma\in\pi_1(M)} \int_0^T \|a_2(t,\cdot)\|_{L^2(\gamma(\mathcal{F}))}\,\|v_{1,\lambda}(t,\cdot)\|_{L^2(M)}\,\mathrm{d}t \\
\le C\,e^{2(h+\nu+\varepsilon)T}\,\lambda^{-1}\|a_2\|_*\,\|a_1\|_*
\end{aligned}$$

and

$$\Big| \int_0^T \int_M q v_{2,\lambda}\overline{v}_{1,\lambda}\,\mathrm{dv}_g\,\mathrm{d}t \Big| \le Ce^{2(h+\nu+\varepsilon)T}\lambda^{-2}\,\|a_1\|_*\,\|a_2\|_*.$$

We also have, using that for all $\varepsilon > 0$,

$$\|f_\lambda\|_{H^1([0,T]\times\partial M} \le C\lambda e^{(h+\varepsilon)T}\|a_2\|_* \quad \text{and} \quad \|g_\lambda\|_{L^2([0,T]\times\partial M} \le Ce^{(h+\varepsilon)T}\|a_1\|_*$$

for some $C > 0$ depending only on the metric g and ε, that

$$\begin{aligned}
\Big| \int_0^T \int_{\partial M} (\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2})(f_\lambda)\,\overline{g}_\lambda\,\mathrm{dv}_{g|_{\partial M}}\,\mathrm{d}t \Big| \\
\le Ce^{2(h+\nu+\varepsilon)T}\lambda\,\|a_1\|_*\,\|a_2\|_*\,\|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|_{*,\nu}.
\end{aligned}$$

Finally, using that

$$\sum_{\gamma\in\pi_1(M)}\int_0^T\int_{\mathcal{F}}\tilde{q}(x)\,a_2(t,\gamma(x))\,\overline{a_1(t,\gamma(x))}\,\mathrm{dv}_{\tilde{g}}(x)\,\mathrm{d}t$$
$$=\int_0^T\int_{\widetilde{M}}\tilde{q}(x)\,a_2(t,x)\,\overline{a_1(t,x)}\,\mathrm{dv}_{\tilde{g}}(x)\,\mathrm{d}t,$$

and bounding the off-diagonal terms as in (5.3), the lemma holds. ■

We then obtain the following Lemma comparable to Lemma 5.2:

Lemma 6.2. *There exist $C>0$ depending only on $(M,g,\|q_1\|_{W^{1,\infty}},\|q_2\|_{W^{1,\infty}})$, ν depending on $(\|q_1\|_{L^\infty},\|q_2\|_{L^\infty})$ and $C_0>0$ depending on (M,g,ν) such that, for any $T>1$, $b\in H^2(\partial_-SM_e)$ such that $b|_{\mathcal{T}_+^{\partial SM}(T)}=0$, one has that the inequality*

$$\Big|\int_{\partial_-S_yM_e}\int_0^{\tau_+^e(y,v)}q(\exp_y(sv))\,b(y,v)\,\mu(y,v)\,\mathrm{d}s\,\mathrm{d}\omega_y(v)\Big|$$
$$\le Ce^{C_0T_0}\,\|\Lambda^W_{g,q_1}-\Lambda^W_{g,q_2}\|_{*,\nu}^{1/2}\,\|b(y,\cdot)\|_{H^2(\partial_-S_yM_e)}$$

holds uniformly for any $y\in\partial M_e$, where $\mu(y,v)=g(\mathrm{n}_y,v)$.

Proof. Take a_1,a_2 solutions to the transport equation on the universal cover $\widetilde{M}$ as before. Then, as in the proof of Lemma 5.2, we obtain

$$\int_0^T\int_{\widetilde{M}}\tilde{q}\,a_1\,a_2\,\mathrm{dv}_{\tilde{g}}\,\mathrm{d}t$$
$$=\int_0^T\int_{\partial_-S_y(M_e)}\int_0^{\tau_+^e(y,v)}q(\exp_y(rv))\,\phi^2(t-r)\,b(y,v)\,\mu(y,v)\,\mathrm{d}r\,\mathrm{d}\omega_y(v)\,\mathrm{d}t$$
$$=\|\phi\|^2_{L^2(\mathbb{R})}\int_{\partial_-S_y(M_e)}\int_0^{\tau_+^e(y,v)}q(\exp_y(rv))\,b(y,v)\,\mu(y,v)\,\mathrm{d}r\,\mathrm{d}\omega_y(v).$$

Combining this with Lemma 6.1 and choosing $\lambda=\|\Lambda^W_{g,q_1}-\Lambda^W_{g,q_2}\|_{*,\nu}^{-1/2}$ yields the desired result. ■

6.2. Proof of the stability estimate

Using Lemma 6.2, there is $C>0$ such that for any $y\in\partial M$ and $b\in H^2(\partial_-SM_e)$ such that $b|_{\mathcal{T}_+^{\partial SM}(T))}=0$,

$$\Big|\int_{\partial_-S_y(M_e)}\int_0^{\tau_+(y,v)}I_0^e(q)(y,v)\,b(y,v)\,\mathrm{d}\mu_\nu(y,v)\Big|$$
$$\le Ce^{C_0T}\,\|\Lambda^W_{g,q_1}-\Lambda^W_{g,q_2}\|_*^{1/2}\,\|b(y,\cdot)\|_{H^2(\partial_-S_y(M_e))}. \tag{6.3}$$

Now we take a bump function $\chi_T\in\mathscr{C}_c^\infty(\mathbb{R})$ as in Section 5.2 and choose b by (5.5) with T instead of T_0

Proof of Theorem 3. The proof is then almost the same as the proof of Theorem 1; we then just briefly describe it. From (6.3), the same as (5.6) holds with $\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}$ replaced by $\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}$. Then using Lemmas 5.3 and 5.4 with $\|\Pi^e_0 q\|_{L^p(M_e)} \leq \|q\|_{L^\infty(M)}$, we obtain

$$\int_{M_e} |\Pi^e_0(q)|^2 \,\mathrm{dv}_g \leq Ce^{C_0 T_0} \|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|^{1/2}_{*,\nu} \|q\|_{W^{1,\infty}(M)} + Ce^{QT/2} \|q\|^2_{L^\infty(M)}.$$

The last part of the proof is exactly the same as for Theorem 1 by choosing T so that $e^{-QT/2} = \|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|^{-1/(2(m+1))}_{*,\nu}$ with $m = -2C_0/Q$. ■

7. Stable determination of the conformal factor for the Schrödinger equation

In this section we prove Theorem 2. The proof follows the argument of Theorem 3 in [4], and we indicate the main modifications. Let $c \in \mathscr{C}(N_0, k, \varepsilon)$ be such that $c = 1$ near the boundary ∂M. We denote

$$\rho_0(x) := 1 - c(x), \quad \rho_1(x) := c^{d/2}(x) - 1, \quad \rho_2(x) := c^{d/2-1}(x) - 1,$$
$$\rho(x) := \rho_2(x) - \rho_1(x) = c^{d/2-1}(x)(1 - c(x)), \tag{7.1}$$

where recall that $d = \dim(M)$. We modify the construction of geometric optics solutions (see Section 6.1 in [4]) in our geometrical setting similarly to what was discussed in previous sections, using the universal cover $\widetilde{M}$. We consider two solutions $\widetilde{\psi}_1, \widetilde{\psi}_2$, respectively to the lifted eikonal equations $|\nabla_{\tilde{g}} \widetilde{\psi}_1| = 1$ and $|\nabla_{\tilde{c}\tilde{g}} \widetilde{\psi}_2| = 1$, a solution a_2 to the lifted transport equation

$$\partial_t a_2 + \mathrm{d}a_2(\nabla^{\tilde{g}} \widetilde{\psi}_1) - \frac{a_2}{2} \Delta_{\tilde{g}} \widetilde{\psi}_1 = 0, \tag{7.2}$$

given in geodesical polar coordinates with respect to g, as in (4.6), by

$$a_2(t,x) = \alpha^{-1/4} \phi(t - r(x)) \tilde{b}(\tilde{y}, v(x)), \quad \tilde{y} \in \partial SM_e,$$

for some $b \in H^2(\partial_- SM)$ satisfying that $b|_{\mathcal{T}^{\partial SM}_+(T_0)} = 0$, and $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\operatorname{supp} \phi \subset (0, \varepsilon_0)$, $\varepsilon_0 > 0$ small, and a solution a_3 to the lifted transport equation

$$\partial_t a_3 + da_3(\nabla^{\tilde{c}\tilde{g}} \widetilde{\psi}_2) - \frac{a_3}{2} \Delta_{\tilde{c}\tilde{g}} \widetilde{\psi}_2 = -\frac{1}{2i} a_2(t,x) \Big(1 - \frac{1}{\tilde{c}}\Big) e^{i\lambda(\widetilde{\psi}_1 - \widetilde{\psi}_2)}$$

which satisfies (analogously to (6.10) in [4]) the bound

$$\|a_3\|_* \leq C\lambda^2 \|1 - c\|_{\mathcal{C}^2(M)} \|a_2\|_*, \tag{7.3}$$

for some $C > 0$ depending on (M, g) (it does not depend on T_0, notice that the derivatives of the difference between $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ can be bounded in terms of derivatives of c), where $\|\cdot\|_* = \|\cdot\|_{H^1([0,T_0];H^2(\widetilde{M}))}$. Then, Lemma 6.2 in [4] becomes in our setting:

Lemma 7.1. *The equation*

$$\begin{cases} (i\partial_t - \Delta_{cg})u = 0, & \text{in } (0,T)\times M,\\ u(0,x) = 0, & \text{in } M,\end{cases}$$

has a solution of the form

$$u_2(t,x) = \sum_{\gamma\in\pi_1(M)} \Big(\frac{1}{\lambda} a_2(2\lambda t, \gamma(x)) e^{i\lambda(\widetilde{\psi}_1(\gamma(x))-\lambda t)} + a_3(2\lambda t, \gamma(x);\lambda) e^{i\lambda(\widetilde{\psi}_2(\gamma(x))-\lambda t)}\Big) + v_{2,\lambda}(t,x),$$

which satisfies, for $\lambda \geq T_0/(2T)$,

$$\lambda\|v_{2,\lambda}(t,\cdot)\|_{L^2(M)} + \|\nabla^g v_{2,\lambda}(t,\cdot)\|_{L^2(M)} + \lambda^{-1}\|\partial_t v_{2,\lambda}(t,\cdot)\|_{L^2(M)} \leq Ce^{C_0T_0}\big(\lambda^2\|\rho_0\|_{\mathcal{C}^2(M)} + \lambda^{-1}\big)\|a_2\|_*$$

for $C > 0$ depending on (M,g,N_0) and $C_0 > 0$ depending on (M,g).

The constant C_0 above can be written in terms of the max of the volume entropies of the metrics cg for $|c-1|_{\mathcal{C}^1(M)} \leq 1/2$ (this can be bounded by a uniform constant times the volume entropy of (M,g)).

Moreover, reasoning as in previous sections, Lemma 6.3 in [4] becomes in our setting:

Lemma 7.2. *There exist constants $C > 0$ depending on (M,g,N_0) and $C_0 > 0$ depending on (M,g) such that, for any $a_1, a_2 \in H^1([0,T_0]; H^2(\widetilde{M}))$ solving (7.2) associated to $b_1, b_2 \in H^2(\partial_- SM_e)$ with $b_1|_{\mathcal{T}_+^{\partial SM}(T_0)} = b_2|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$, the estimate*

$$\Big|\sum_{\gamma\in\pi_1(M)}\int_0^T\int_M \rho(x)\, a_1(2\lambda t,\gamma(x))\, a_2(2\lambda t,\gamma(x))\,\mathrm{d}v_g(x)\,\mathrm{d}t\Big| \leq C\lambda^{-1}e^{C_0T_0}\left(\|\rho_0\|_{\mathcal{C}^1(M)}(\lambda^{-1}+\lambda^3\|\rho_0\|_{\mathcal{C}^2(M)}) + \lambda\|\Lambda_g^S - \Lambda_{cg}^S\|_*\right)\|a_1\|_*\|a_2\|_*$$

holds for any $\lambda \geq T_0/(2T)$.

Finally, Lemma 6.4 in [4] becomes in our setting:

Lemma 7.3. *There exist $C > 0$ and $C_0 > 0$ as in the previous lemma such that, for any $b \in H^2(\partial_- SM_e)$ with $b|_{\mathcal{T}_+^{\partial SM}(T_0)} = 0$ the estimate*

$$\Big|\int_{\partial_- SM_e} I_0^e(\rho)(y,v)\, b(y,v)\,\mathrm{d}\mu_{\mathrm{n}}(y,v)\Big| \leq Ce^{C_0T_0}\big((\lambda^{-1}+\lambda^3\|\rho_0\|_{\mathcal{C}^2(M)})\|\rho_0\|_{\mathcal{C}^1(M)} + \lambda\|\Lambda_g^S - \Lambda_{cg}^S\|_*\big)\|b\|_{H^2(\partial_- SM)}$$

holds for any $\lambda \geq T_0/(2T)$. Here, I_0^e is the X-ray transform for g on functions on M_e.

Proof of Theorem 2. We take b as in (5.5) with q replaced by ρ. By Lemmas 7.3, 5.3 and 5.4, there are $C > 0, C_0 > 0$ depending on (M,g,N_0,ε) such that

$$\|\Pi_0^e\rho\|^2_{L^2(M_e)} \leq Ce^{C_0T_0}\big((\lambda^{-1}+\lambda^3\|\rho_0\|_{\mathcal{C}^2(M)})\|\rho_0\|_{\mathcal{C}^1(M)} + \lambda\|\Lambda_g^S - \Lambda_{cg}^S\|_*\big)\|\rho\|_{W^{1,p}(M)} + Ce^{QT_0/2}\|\rho\|^2_{L^p(M)}$$

for some $p > 2$. By interpolation,

$$\begin{aligned}\|\Pi_0^e\rho\|_{H^1(M_e)}^2 &\leq C\,\|\Pi_0^e\rho\|_{L^2(M_e)}\,\|\Pi_0^e\rho\|_{H^2(M_e)}\\ &\leq Ce^{C_0T_0/2}\big((\lambda^{-1}+\lambda^3\|\rho_0\|_{\mathcal{C}^2(M)})\|\rho_0\|_{\mathcal{C}^1(M)}+\lambda\|\Lambda_g^S-\Lambda_{cg}^S\|_*\big)^{1/2}\\ &\quad\cdot\|\rho\|_{W^{1,p}(M)}^{1/2}\,\|\rho\|_{H^1(M)}+Ce^{QT_0/4}\,\|\rho\|_{L^p(M)}\|\rho\|_{H^1(M)}.\end{aligned}$$

We use (3.4) to deduce the bound

$$\begin{aligned}\|\rho\|_{L^2(M)}^2 \leq{}& Ce^{C_0T_0/2}\,(\lambda^{-1}+\lambda^3\|\rho_0\|_{\mathcal{C}^2(M)})^{1/2}\,\|\rho_0\|_{\mathcal{C}^1(M)}^{1/2}\,\|\rho\|_{W^{1,p}(M)}^{1/2}\|\rho\|_{H^1(M)}\\ &+Ce^{C_0T_0/2}\lambda^{1/2}\|\rho_0\|_{\mathcal{C}^1(M)}^{3/2}\|\Lambda_g^S-\Lambda_{cg}^S\|_*^{1/2}+Ce^{QT_0/4}\|\rho\|_{L^p(M)}\|\rho\|_{H^1(M)}.\end{aligned}$$

Taking

$$\lambda=\frac{T_0}{2T}\cdot\Big(\frac{\|\rho_0\|_{\mathcal{C}^2(M)}}{2N_0}\Big)^{-1/4},$$

we obtain for $T_0 > 0$ large,

$$\begin{aligned}&\|\rho\|_{L^2(M)}^2\\ &\leq Ce^{C_0T_0}\big(\|\rho_0\|_{\mathcal{C}^2(M)}^{17/8}+\|\rho_0\|_{\mathcal{C}^2(M)}^{11/8}\|\Lambda_g^S-\Lambda_{cg}^S\|_*^{1/2}\big)+Ce^{QT_0/4}\|\rho\|_{L^p(M)}\|\rho\|_{H^1(M)}\\ &\leq Ce^{C_0T_0}\big(\varepsilon^\ell\|\rho_0\|_{\mathcal{C}^2(M)}^{33/16}+\|\Lambda_g^S-\Lambda_{cg}^S\|_*^{1/2}\big)+Ce^{QT_0/4}\,\|\rho\|_{L^p(M)}\,\|\rho\|_{H^1(M)},\end{aligned}$$

where C depends on N_0, $\ell=\frac{1}{16}(1-2/k)$, and we have used the interpolation estimate

$$\|\rho_0\|_{\mathcal{C}^2(M)}\leq C\,\|\rho_0\|_{\mathcal{C}(M)}^{1-2/k}\leq C\varepsilon^{1-2/k}.$$

By interpolation, choosing $k > \max(s,s')$ large enough, we have for some $\delta,\delta' > 0$ small,

$$\|\rho_0\|_{\mathcal{C}^2(M)}\leq C\|\rho_0\|_{H^{d/2+2+\delta}(M)}\leq C\|\rho_0\|_{L^2(M)}^{32/33+\delta}\|\rho_0\|_{H^s(M)}^{1/33-\delta}\leq C\|\rho_0\|_{L^2(M)}^{32/33+\delta}, \tag{7.4}$$

$$\|\rho\|_{H^1(M)}\leq\|\rho\|_{L^2(M)}^{1-\delta'}\|\rho\|_{H^{s'}(M)}^{\delta'}\leq C\|\rho\|_{L^2(M)}^{1-\delta'}, \tag{7.5}$$

$$\|\rho\|_{L^p(M)}\leq\|\rho\|_{L^2(M)}^{1-\delta'}\|\rho\|_{L^{p'}(M)}^{\delta'}\leq C\|\rho\|_{L^2(M)}^{1-\delta'}, \tag{7.6}$$

where $C > 0$ depends on N_0. Thus, using that

$$C^{-1}\|\rho_0\|_{L^2}\leq\|\rho\|_{L^2}\leq C\|\rho_0\|_{L^2},$$

we see that there are $C > 0$ depending on (M,g,N_0,k) and $C_0 > 0$ depending on (M,g) such that

$$\|\rho_0\|_{L^2(M)}^2\leq Ce^{C_0T_0}\,\varepsilon^\ell\,\|\rho_0\|_{L^2(M)}^{2+\delta}+Ce^{C_0T_0}\,\|\Lambda_g^S-\Lambda_{cg}^S\|_*^{1/2}+Ce^{QT_0/4}\,\|\rho_0\|_{L^2(M)}^{2-2\delta'}.$$

We finally take T_0 sufficiently large so that

$$Ce^{QT_0/4}<\frac{1}{2}\,\|\rho_0\|_{L^2}^{2\delta'}.$$

This allows us to absorb the third term of the right-hand side into the left-hand side:

$$\|\rho_0\|^2_{L^2(M)} \le C\varepsilon^{\ell}\,\|\rho_0\|^{-2m\delta'}_{L^2(M)}\,\|\rho_0\|^{2+\delta}_{L^2(M)} + C\,\|\rho_0\|^{-2m\delta'}_{L^2(M)}\,\|\Lambda^S_g - \Lambda^S_{cg}\|^{1/2}_*,$$

for $m = 4C_0/|Q| > 0$. Choosing $\delta > 2m\delta'$ and taking ε sufficiently small, we can absorb the first term of the right-hand side into the left-hand side. Therefore, there exist $\beta > 0$ depending on (M,g) and $C > 0$ depending on (M,g,N_0,k,ε) such that

$$\|\rho_0\|_{L^2(M)} \le C\,\|\Lambda^S_g - \Lambda^S_{cg}\|^{\beta}_*,$$

and the proof is complete. ■

8. Stable determination of the conformal factor for the wave equation

In this section we sketch the proof of Theorem 4. We just indicate the modifications with respect to the proof of Theorem 2 in [5]. We will assume that the conformal factor satisfies $c \in \mathscr{C}(N_0,k,\varepsilon)$ and is such that $c = 1$ near the boundary ∂M, and we use the notation (7.1). First, Lemma 6.2 in [5] becomes in our setting:

Lemma 8.1. *The equation*

$$\begin{cases} (\partial_t^2 + \Delta_{cg})u = 0, & \text{in } (0,T)\times M,\\ u(0,x) = 0, & \text{in } M, \end{cases}$$

has a solution of the form

$$u_2(t,x) = \sum_{\gamma\in\pi_1(M)} \Big(\frac{1}{\lambda}\,a_2(t,\gamma(x))\,e^{i\lambda(\psi_1(\gamma(x))-t)} + a_3(t,\gamma(x);\lambda)\,e^{i\lambda(\psi_2(\gamma(x))-t)}\Big) + v_{2,\lambda}(t,x)$$

such that there is $\nu_0 > 0$ (given by Lemma 4.3) so that for all $\nu > \nu_0$, $\exists C > 0$ depending only on (M,g,N_0,ν), and $C_0 > 0$ depending only on (M,g,ν) so that $\forall\lambda > 1$,

$$\lambda\|v_{2,\lambda}\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} + \|\nabla^g v_{2,\lambda}\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} + \lambda^{-1}\|\partial_t v_{2,\lambda}\|_{e^{\nu t}L^2(\mathbb{R}_+\times M)} \le Ce^{C_0T}\big(\|\rho_0\|_{\mathscr{C}^2(M)}\,\lambda^2 + \lambda^{-1}\big)\|a_2\|_*.$$

Moreover, Lemma 6.3 in [5] is replaced by:

Lemma 8.2. *There exist constants $C_0 > 0$ depending on (M,g,ν) and $C > 0$ depending on (M,g,N_0,ν) such that, for any $a_1,a_2 \in H^1([0,T];H^2(\widetilde{M}))$ solving (7.2) associated to $b_1,b_2 \in H^2(\partial_-SM_e)$ with $b_1|_{\mathcal{T}_+^{\partial SM}(T)} = b_2|_{\mathcal{T}_+^{\partial SM}(T)} = 0$, the estimate*

$$\Big|\sum_{\gamma\in\pi_1(M)} \int_0^T\!\!\int_M \rho(x)\,a_1(t,\gamma(x))\,\overline{a_2(t,\gamma(x))}\,\mathrm{dv}_g(x)\,\mathrm{d}t\Big| \le Ce^{C_0T}\,\|\rho_0\|_{\mathscr{C}^1(M)}\,\big(\lambda^{-1} + \lambda^3\,\|\rho_0\|_{\mathscr{C}^2(M)}\big)\,\|a_1\|_*\,\|a_2\|_* + Ce^{C_0T}\,\lambda\,\|\Lambda^W_g - \Lambda^W_{cg}\|_{*,\nu}\,\|a_1\|_*\,\|a_2\|_*,$$

holds for all $\lambda > 1$.

Finally, Lemma 6.4 in [5] is replaced by:

Lemma 8.3. *There exist $C > 0$ depending on (M, g, N_0, ν) and $C_0 > 0$ depending on (M, g, ν) such that, for any $b \in H^2(\partial_- SM_e)$ with $b|_{\mathcal{T}_+^{\partial SM}(T)} = 0$, the estimate*

$$\Big| \int_{\partial_- SM_e} I_0^e(\rho)(y, v)\, b(y, v)\, \mathrm{d}\mu_\nu(y, v) \Big| \\ \leq C e^{C_0 T} \left((\lambda^{-1} + \lambda^3 \|\rho_0\|_{\mathcal{C}^2(M)}) \|\rho_0\|_{\mathcal{C}^1(M)} + \lambda \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu} \right) \|b\|_{H^2(\partial_- SM)}$$

holds for all $\lambda > 1$.

Proof of Theorem 4. We take b as in (5.5) with q replaced by ρ. By Lemmas 7.3, 5.3 and 5.4, there are $C > 0$ depending on (M, g, N_0, ν) and $C_0 > 0$ depending on (M, g) such that

$$\|\Pi_0^e \rho\|_{L^2(M_e)}^2 \leq C e^{C_0 T} \left(\left(\frac{1}{\lambda} + \lambda^3 \|\rho_0\|_{\mathcal{C}^2(M)} \right) \|\rho_0\|_{\mathcal{C}^1(M)} + \lambda \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu} \right) \\ \cdot \|\rho\|_{W^{1,p}(M)} + C e^{QT/2} \|\rho\|_{L^p(M)}^2$$

for some $p > 2$. By interpolation,

$$\|\Pi_0^e \rho\|_{H^1(M_e)}^2 \leq C \|\Pi_0^e \rho\|_{L^2(M_e)} \|\Pi_0^e \rho\|_{H^2(M_e)} \\ \leq C e^{C_0 T/2} \left(\left(\frac{1}{\lambda} + \lambda^3 \|\rho_0\|_{\mathcal{C}^2(M)} \right) \|\rho_0\|_{\mathcal{C}^1(M)} + \lambda \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu} \right)^{1/2} \|\rho\|_{\mathcal{C}^1(M)}^{3/2} \\ + C e^{QT/4} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)}.$$

We use (3.4) to deduce the bound

$$\|\rho\|_{L^2(M)}^2 \leq C e^{C_0 T/2} (\lambda^{-1} + \lambda^3 \|\rho_0\|_{\mathcal{C}^2(M)})^{1/2} \|\rho_0\|_{\mathcal{C}^1(M)}^{1/2} \|\rho\|_{\mathcal{C}^1(M)}^{3/2} \\ + C e^{C_0 T/2} \lambda^{1/2} \|\rho_0\|_{\mathcal{C}^1(M)}^{3/2} \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{1/2} + C e^{QT/4} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)}.$$

Taking $\lambda = \|\rho_0\|_{\mathcal{C}^2(M)}^{-1/4}$, we obtain, for $T > 0$ large,

$$\|\rho\|_{L^2(M)}^2 \leq C e^{C_0 T} \left(\|\rho_0\|_{\mathcal{C}^2(M)}^{17/8} + \|\rho_0\|_{\mathcal{C}^2(M)}^{11/8} \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{1/2} \right) \\ + C e^{QT/4} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)} \\ \leq C e^{C_0 T} \left(\varepsilon^\ell \|\rho_0\|_{\mathcal{C}^2(M)}^{33/16} + \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{1/2} \right) + C e^{QT/4} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)},$$

where $\ell = \frac{1}{16}(1 - 2/k)$. Choosing $k > \max(s, s')$ large enough as in the previous section, we have for some $\delta, \delta' > 0$ small the interpolation estimates (7.4), (7.5), and (7.6). Thus we get that there are $C > 0$ depending on (M, g, N_0, k) and $C_0 > 0$ depending on (M, g) such that

$$\|\rho_0\|_{L^2(M)}^2 \leq C e^{C_0 T} \varepsilon^\ell \|\rho_0\|_{L^2(M)}^{2+\delta} + C e^{C_0 T} \|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{1/2} + C e^{QT/4} \|\rho_0\|_{L^2(M)}^{2-2\delta'}.$$

We finally take T sufficiently large so that

$$C e^{QT/4} < \frac{1}{2} \|\rho\|_{L^2}^{2\delta'}.$$

This allows us to absorb the third term of the right-hand side into the left-hand side:

$$\|\rho_0\|_{L^2(M)}^2 \leq C\,\varepsilon^{\ell}\,\|\rho_0\|_{L^2}^{-2m\delta'}\,\|\rho_0\|_{L^2}^{2+\delta} + C\,\|\rho_0\|_{L^2}^{-2m\delta'}\,\|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{1/2},$$

for $m = 4C_0/|Q| > 0$. Choosing $\delta > 2m\delta'$ and taking ε sufficiently small (depending only on $\|1-c\|_{L^2}$), we can absorb the first term of the right-hand side into the left-hand side. Therefore, there exist $\beta > 0$ depending on (M, g, ν) and $C > 0$ depending on $(M, g, N_0, k, \varepsilon)$ such that

$$\|\rho_0\|_{L^2(M)} \leq C\,\|\Lambda_g^W - \Lambda_{cg}^W\|_{*,\nu}^{\beta},$$

and the proof is complete. ∎

9. Appendix

In this appendix, we discuss how to remove the boundary condition $q_1 = q_2$ on ∂M in Theorems 1 and 3, at the cost of assuming more regularity on q_j.

9.1. Boundary stability: the wave case

We follow the argument of [14], Section 3. Assuming that $q_1, q_2 \in \mathscr{C}^4(M)$, one can construct two geometrical optics solutions $(\partial_t^2 + \Delta_g + q_j)u_j = 0$ concentrated on a small geodesic close to a boundary point $x_0 \in \partial M$:

$$u_j = e^{i\lambda(t-\psi)}(a_0 + \lambda^{-1}a_1^j + \lambda^{-2}a_2^j) + v_j =: u_j^1 + v_j,$$

where ψ is a suitable solution to the eikonal equation $|\nabla_g\psi| = 1$ with $\psi|_{U\cap\partial M} = x'.\omega$ for some $\omega \in T_{x_0}\partial M \simeq \mathbb{R}^{n-1}$ satisfying $|\omega'| < 1$ (but close to 1), a_0 solves a transport equation involving only the metric and some derivatives of ψ, $a_0|_{\partial M} = \chi \in \mathscr{C}_c^\infty((0,T')\times\partial M;\mathbb{R}^+)$ is some cutoff function equal to 1 near (t_0, x_0), and a_1^j, a_2^j solve the transport equation

$$i\big(2\partial_t + 2\nabla^g\psi - \Delta_g\psi\big)a_k^j = -(\partial_t^2 + \Delta_g + q_j)a_{k-1}^j, \quad a_k^j|_{\partial M} = 0. \tag{9.1}$$

By Lemma 4.3, one can construct the remainder term v_j so that $\|\partial_{\mathrm{n}} v_j\|_{\mathscr{C}^0([0,T'];L^2(\partial M))} = \mathcal{O}(\lambda^{-2})$ for some small $T' > 0$, with uniform dependence with respect to $\|q_j\|_{\mathscr{C}^4(M)}$ and $v_j|_{(0,T')\times\partial M} = 0$. Using that

$$\Lambda_{g,q_j}^W(u_j|_{\partial M}) = e^{it\lambda(t-\psi)}\big(i\lambda(\partial_{x_n}\psi)a_0 - \partial_{x_n}a_0 - \lambda^{-1}\partial_{x_n}a_1\big) + \mathcal{O}(\lambda^{-2}),$$

that $\|u_j^1\|_{H^1([0,T']\times\partial M)} \leq C\lambda$, and choosing $\lambda \sim \|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|_*^{-1/3}$, one gets the bound

$$\big\|\partial_{x_n}a_1^1 - \partial_{x_n}a_1^2\big\|_{L^2([0,T']\times\partial M)} \leq C\,\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|_*^{1/3}.$$

This implies $2|\partial_{x_n}a_1^1 - \partial_{x_n}a_1^2| = |(q_1-q_2)\chi|$ by restricting (9.1) to $(0,T')\times\partial M$. Therefore, there is an open neighborhood $V \subset U \cap \partial M$ of x_0 of uniform size and $C > 0$ uniform in $\|q_j\|_{\mathscr{C}^4}$ such that

$$\|q_1 - q_2\|_{L^2(V)} \leq C\,\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|_*^{1/3}.$$

By interpolation and Sobolev embeddings, we get

$$\|q_1 - q_2\|_{\mathcal{C}^1(\partial M)} \leq C\|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|_*^{\mu}$$

for some $\mu > 0$. We finally replace q_2 by

$$\tilde{q}_2 = q_2 + \theta(\|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|_*^{-\mu/2} d_g(\cdot, \partial M))(q_1 - q_2)$$

in Theorem 3 for some $\theta \in \mathcal{C}_c^\infty([0,1))$ equal to 1 near 0 so that $\tilde{q}_2 = q_1$ on ∂M.

9.2. Boundary stability: the Schrödinger case

We use a similar argument for the Schrödinger equation. Assume that $q_j \in \mathcal{C}^8(M)$. We construct a geometric optic solution of the form

$$u_j(t,x) = e^{i\lambda(\psi(t,x)-\lambda t)}\Big(\sum_{k=0}^{4} a_k^j(2\lambda t, x)\,\lambda^{-k}\Big) + v_j(t,x) =: u_j^1(t,x) + v_j(t,x),$$

where $a_0^j(s,x)$ solves the same transport equation as for the wave equation (thus not depending on q_j) with $a_0^j|_{\partial M} = \chi$, and a_k^j for $k \geq 1$ solves

$$i\big(2\partial_t + 2\nabla^g\psi - \Delta_g\psi\big)a_k^j(t,x) = (\Delta_g - q_j)a_{k-1}^j(t,x), \quad a_k^j|_{\partial M} = 0, \tag{9.2}$$

and the remainder can be estimated by $\|\partial_{\rm n} v_j\|_{L^2((0,T)\times\partial M)} = \mathcal{O}(\lambda^{-3})$ using Lemma 3.2 in [4] and the fact that

$$\|\lambda^{-4}e^{i\lambda(\psi-\lambda t)}((\Delta_g - q_j)a_4^j)(2\lambda\cdot,\cdot))\|_{H^1([0,T];L^2(M))} = \mathcal{O}(\lambda^{-3})$$

(we loose λ^2 from the ∂_t derivative but gain one λ^{-1} from the change of variable $t \mapsto t/\lambda$ in the dt integral as a_k^j are supported in time interval of size $\mathcal{O}(\lambda^{-1})$). We then obtain

$$\Lambda^S_{g,q_j}(u_j|_{\partial M}) = e^{i\lambda(\psi-\lambda t)}\big(i\lambda(\partial_{x_n}\psi)a_0 - \partial_{x_n}a_0 - \lambda^{-1}\partial_{x_n}a_1\big)(2\lambda t, x) + \mathcal{O}_{L^2}(\lambda^{-3})$$

Proceeding as for the wave equation, we deduce that

$$\|\partial_{x_n}a_1^1 - \partial_{x_n}a_1^2\|_{L^2([0,T]\times\partial M)} \leq C\|\Lambda^W_{g,q_1} - \Lambda^W_{g,q_2}\|_*^{1/3},$$

and the end of the proof is the same as for the wave equation.

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