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Local well-posedness for the gKdV equation on the background of a bounded function

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Abstract. We prove the local well-posedness for the generalized Korteweg–de Vries equation in $H^s(\mathbb{R})$, $s > 1/2$, under general assumptions on the nonlinearity $f(x)$, on the background of an $L^\infty_{t,x}$-function $\Psi(t,x)$, with $\Psi(t,x)$ satisfying some suitable conditions. As a consequence of our estimates, we also obtain the unconditional uniqueness of the solution in $H^s(\mathbb{R})$. This result not only gives us a framework to solve the gKdV equation around a Kink, for example, but also around a periodic solution, that is, to consider localized non-periodic perturbations of a periodic solution. As a direct corollary, we obtain the unconditional uniqueness of the gKdV equation in $H^s(\mathbb{R})$ for $s > 1/2$. We also prove global existence in the energy space $H^1(\mathbb{R})$, in the case where the nonlinearity satisfies $|f''(x)| \lesssim 1$.

1. Introduction

1.1. The model

The initial value problem for the k-Korteweg–de Vries equation (k-KdV)

$$\begin{cases} \partial_t u + \partial_x^3 u \pm u^k \partial_x u = 0, & t \in \mathbb{R},\ x \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

has been extensively studied in the last five decades and is one of the most famous equations in the context of dispersive PDEs. This family of equations includes the celebrated Korteweg–de Vries (KdV) equation (case $k = 1$), which was derived as a model for the unidirectional propagation of nonlinear dispersive long waves [30], and subsequently found in the study of collision-free hydro-magnetic waves [14]. Nowadays, the KdV equation has shown applications to several physical situations, such as, for example, in plasma physics, for the study of ion-acoustic waves in cold plasma [4, 49], as well as some relationships with the Fermi–Pasta–Ulam problem [31, 50–52]. Moreover, some connections with algebraic geometry were given in [9] (see also [36] and the references therein). On the other hand, in [48] it has been shown that this equation also describes pressure waves

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in a liquid-gas bubble mixture, as well as waves in elastic rods (see [45]). We refer to [36] for a more extensive description of all of these (and more) physical applications.

In the case where $k = 2$, we find another fairly celebrated equation, the so-called modified KdV equation, which also models the propagation of weak nonlinear dispersive waves. In this regard, a large class of hyperbolic models has been reduced to the latter two equations. It is worth to notice that there is a deep relationship between these two models given by the Miura transformation [35].

These two cases ($k = 1, 2$) correspond to completely integrable systems, in terms of the existence of a Lax-pair, and both of them have been solved via inverse scattering. An interesting property of (1.1) is that these are the only two cases on which this equation corresponds to a completely integrable system (see [12, 13]).

One of the most important features of equation (1.1) is the existence of solitary wave solutions of both types, localized solitary waves and kink solutions. In the completely integrable cases, these solutions correspond to soliton solutions, that is, they preserve their shape and speed after collision with objects of the same type.

In this work we seek to study the initial value problem associated with the following generalization of the k-KdV equation (1.1):

$$\begin{cases} \partial_t v + \partial_x(\partial_x^2 v + f(v)) = 0, \\ v(0,x) = \Phi(x), \end{cases} \tag{1.2}$$

where $v = v(t,x)$ stands for a real-valued function, the nonlinearity f is also real-valued, and $t, x \in \mathbb{R}$. Motivated by the study of Kink solutions, here we do not intend to assume any decay of the initial data $\Phi(x)$ but, for the moment, only that $\Phi \in L^\infty(\mathbb{R})$. Instead, we decompose the solution $v(t,x)$ in the following fashion:

$$v(t,x) = u(t,x) + \Psi(t,x), \tag{1.3}$$

where we assume that $\Psi \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is a given function (see (1.7) below for the specific hypotheses on Ψ) and we seek for $u(t) \in H^s(\mathbb{R})$. Then it is natural to rewrite the above IVP in terms of the Cauchy problem associated with the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_t \Psi + \partial_x\big(\partial_x^2 u + \partial_x^2 \Psi + f(u+\Psi)\big) = 0, \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}). \end{cases} \tag{1.4}$$

We stress that equation (1.4) is nothing else than equation (1.2) once replacing the decomposition given in (1.3). In the case where $f(x) = x^2$ and $\Psi = \Psi(x)$ is a time-independent function belonging to the so-called Zhidkov class

$$\mathcal{Z}(\mathbb{R}) := \big\{\Psi \in \mathcal{D}'(\mathbb{R}) : \Psi \in L^\infty(\mathbb{R}),\ \Psi' \in H^\infty(\mathbb{R})\big\},$$

this equations has been used to model the evolution of bores on the surface of a channel, incorporating nonlinear and dispersive effects intrinsic to such propagation [5].

Important. In this work we only assume that $f:\mathbb{R} \to \mathbb{R}$ is a real-analytic function satisfying that its Taylor expansion around zero has infinite radius of convergence, that is, there exists a family $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that, for all $x \in \mathbb{R}$, the nonlinearity $f(x)$ can be represented as

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad \text{with } \limsup_{k\to+\infty} \sqrt[k]{|a_k|} = 0. \tag{1.5}$$

Notice that any polynomial $p(x)$ satisfies the previous hypothesis, as well as $\exp(x)$, $\sinh(x)$, $\cosh(x)$, $\sin(x)$, $\cos(x)$, $p(\sin(x))$, etc.

It is worth to notice that, since equation (1.4) can be regarded as a perturbation of the gKdV equation (1.2) with initial data $v(0,\cdot) \in H^s(\mathbb{R})$, one might think that, in order to prove local well-posedness for equation (1.4), it is reasonable to proceed by using the contraction principle as in [24]. However, it seems that this does not even hold in the case where $f(x) = x^2$, due to the occurrence of the term $\Psi\partial_x u$, since Ψ is not integrable, which makes this problem more involved even for the KdV case.

As mentioned before, one of our main motivations comes from studying Kink solutions. For instance, we can consider the defocusing modified Korteweg–de Vries (mKdV) equation, that is, $f(u) = -u^3$, as well as the Gardner equation, that is, $f(u) = u^2 - \beta u^3$. Both equations are well known to have Kink solutions given by (respectively, see [17])

$$\begin{aligned} \Psi_{\mathrm{mKdV},c}(t,x) &= \pm\sqrt{c}\tanh\Big(\sqrt{\tfrac{c}{2}}(x+ct)\Big), & c > 0, \\ \Psi_{\mathrm{Gardner},c}(t,x) &= \frac{1}{3\beta} \pm \frac{1}{\sqrt{\beta}}\Phi_{\mathrm{mKdV},c}\Big(t, x - \frac{t}{3\beta}\Big), & \beta > 0. \end{aligned}$$

Moreover, at the same time we also seek to give a framework to study localized non-periodic perturbations of periodic solutions for the generalized model (1.4), such as, for example, the famous cnoidal and dnoidal wave solutions of the KdV and the mKdV equations (respectively)

$$\Psi_{\mathrm{cn},c}(t,x) := \alpha + \beta\mathrm{cn}^2(\gamma(x-ct),\kappa), \quad \Psi_{\mathrm{dn},c}(t,x) := \beta\mathrm{dn}(\gamma(x-ct),\kappa), \tag{1.6}$$

with $c > 0$ and $(\alpha,\beta,\gamma,\kappa) \in \mathbb{R}^4$ satisfying some suitable conditions, where $\mathrm{cn}(\cdot,\cdot)$ and $\mathrm{dn}(\cdot,\cdot)$ stand for the Jacobi elliptic cnoidal and dnoidal functions, respectively (see [30]).

It is important to mention that the gKdV equation (1.2) enjoys (at least formally) several conservation laws, such as the conservations of the mean, the mass and the energy, which are given by (respectively)

$$\begin{aligned} I_1(v(t)) &:= \int_{\mathbb{R}} v(t,x)\,dx = I_1(v_0), \\ I_2(v(t)) &:= \int_{\mathbb{R}} v^2(t,x)\,dx = I_2(v_0), \\ I_3(v(t)) &:= \int_{\mathbb{R}} \big(v_x^2(t,x) - F(v(t,x))\big)\,dx = I_3(v_0), \end{aligned}$$

where $F(\cdot)$ stands for a primitive of $f(\cdot)$. However, due to the presence of $\Psi(t,x)$, none of these quantities are well-defined for solutions of equation (1.4). Although, a suitable modification of the energy functional I_3 shall play a key role in proving global well-posedness in $H^1(\mathbb{R})$ when $f(x)$ grows at most as x^2 (see Theorem 1.8 below).

Important. In the sequel we shall always assume that the given function $\Psi(t,x)$ satisfies the following hypotheses:

$$\begin{cases} \partial_t\Psi \in L^\infty(\mathbb{R}^2), \\ \Psi \in L^\infty(\mathbb{R}, W^{s+1^+,\infty}(\mathbb{R})), \\ (\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)) \in L^\infty(\mathbb{R}, H^{s^+}(\mathbb{R})). \end{cases} \tag{1.7}$$

Remark 1.1. Note that any function $\Psi = \Psi(x) \in L^\infty(\mathbb{R})$ such that $\Psi' \in H^\infty(\mathbb{R})$, for example, Ψ being a Kink, satisfies all the conditions in (1.7). Hence, equation (1.4) together with conditions (1.7) contain as particular cases all the frameworks considered in [5, 11, 18, 53]. However, we do not require that $\Psi(t,x)$ has well-defined limits at $\pm\infty$ as in those previous works. For instance, if $\Psi = \Psi(t,x)$ solves the gKdV equation (1.2), then the latter expression in (1.7) is identically zero, and hence the third hypothesis is immediately satisfied. In particular, we can consider $\Psi(t,x)$ being a periodic solution of the gKdV equation. Nevertheless, $\Psi(t,x)$ does not need to be a solution, neither to have a small H^{s+}-norm once replaced in the equation. For example, we can solve equation (1.4) with Ψ being

$$\Psi(t,x) = 1 + 4\tanh(x+t) + \cos(\log(1 + x^2 + t^2)).$$

Notice that this function does not have symmetries (neither odd nor even), nor well-defined limits at $\pm\infty$; also, none of its derivatives has exponential decay. Clearly, it does not solve the equation either, whereas it satisfies all the conditions in (1.7) for any $s > 1/2$ (for example).

1.2. Unconditional uniqueness

The generalized Korteweg–de Vries equation (1.2) has been proven to be locally well-posed (LWP) for regular localized initial data in [1, 6, 21, 46]. Since then, considerable effort has been devoted to understand the Cauchy problem (1.1) with rough data. In the seminal work of Kenig, Ponce and Vega, LWP for equation (1.1) has been established in H^s-spaces, for all $k \in \mathbb{Z}_+$, with s moving in a range that depends on k. In the case where $k \geq 4$, these results are sharp, in the sense that they reach the critical index given by the scaling invariance [24]. This proof relies on Strichartz estimates, along with local smoothing effect and maximal estimates. Then a normed functional space is constructed based on these estimates, which allows them to solve (1.1) via a fixed point argument. The solutions obtained in this way are obviously unique in such a resolution space. However, as explained in [43], the question of whether this uniqueness holds for solutions that do not belong to these resolution spaces turns out to be far from trivial at this level of regularity. This type of question was first raised by Kato in [22] in the context of Schrödinger equations. We refer to such uniqueness in $L^\infty((0,T), H^s(\mathbb{R}))$, without intersecting with any other auxiliary functional space as *unconditional uniqueness*. This type of uniqueness has been shown to be useful, for example, to pass to the limit in perturbative analysis when one of the coefficients of the equation tends to zero (see [38] for instance).

1.3. Main results

In the remainder of this work, we focus on studying the Cauchy problem associated with (1.4). Before going further, let us give a precise definition of what we mean by a solution.

Definition 1.2. Let $T>0$ and $s>1/2$, both being fixed. Consider $u\in L^\infty((0,T),H^s(\mathbb{R}))$. We say that $u(t,x)$ is a solution to (1.4) emanating from the initial data $u_0\in H^s(\mathbb{R})$ if $u(t,x)$ satisfies (1.4) in the distributional sense, that is, for any test function $\phi\in C_0^\infty((-T,T)\times\mathbb{R})$, we have

$$\begin{aligned}(1.8)\quad &\int_0^\infty\int_{\mathbb{R}}\big[(\phi_t+\phi_{xxx})u+\phi_x(f(u+\Psi)-f(\Psi))\big]\,dx\,dt+\int_{\mathbb{R}}\phi(0,x)u_0(x)\,dx\\ &\qquad=-\int_0^\infty\int_{\mathbb{R}}\big[(\phi_t+\phi_{xxx})\Psi+\phi_x f(\Psi)\big]\,dx\,dt-\int_{\mathbb{R}}\phi(0,x)\Psi(0,x)\,dx.\end{aligned}$$

Remark 1.3. Notice that, for $u\in L^\infty((0,T),H^s(\mathbb{R}))$ with $s>1/2$, we have that u^p is well defined for all $p\in\mathbb{N}$, which, along with (1.5) and (1.7), implies that

$$f(u+\Psi)-f(\Psi)\in L^\infty((0,T),H^s(\mathbb{R})).$$

Thus, relation (1.8) and the hypotheses in (1.7) forces that $u_t\in L^\infty((0,T),H^{s-3}(\mathbb{R}))$, and hence (1.4) is satisfied in $L^\infty((0,T),H^{s-3}(\mathbb{R}))$. In particular, we infer that

$$(1.9)\qquad u\in C([0,T],H^{s-3}(\mathbb{R})).$$

Moreover, from hypotheses (1.7), we infer that $\Psi\in C([0,T],L^\infty(\mathbb{R}))$, which in turn, together with (1.8) and (1.9), forces the initial condition $u(0)=u_0$. On the other hand, notice that we also have $u\in C_w([0,T],H^s(\mathbb{R}))$ and $u\in C([0,T],H^\theta(\mathbb{R}))$ for all $\theta<s$. Finally, we stress that the above relations also ensure that u satisfies the Duhamel formula associated with (1.4) in $C([0,T],H^{s-3}(\mathbb{R}))$.

Our first main result give us the (unconditional) local well-posedness for (1.4).

Theorem 1.4 (Local well-posedness). *Let $s>1/2$ be fixed. Consider $f:\mathbb{R}\to\mathbb{R}$ to be any real-analytic function such that its Taylor expansion around zero has infinite radius of convergence. Consider also $\Psi(t,x)$ satisfying the conditions in (1.7). Then, for any $u_0\in H^s(\mathbb{R})$, there exist $T=T(\|u_0\|_{H^s})>0$ and a solution u to the* IVP (1.4) *such that*

$$u\in C([0,T],H^s(\mathbb{R}))\cap L_T^2W_x^{s^-,\infty}\cap X_T^{s-1,1}.$$

Furthermore, the solution is unique in the class

$$u\in L^\infty((0,T),H^s(\mathbb{R})).$$

Also, the data-to-solution map $\Phi:u_0\mapsto u$ is continuous from $H^s(\mathbb{R})$ into $C([0,T],H^s(\mathbb{R}))$.

Remark 1.5. We refer to the next section for a definition of Bourgain spaces $X^{s,b}$ and their corresponding time-restricted versions $X_T^{s,b}$.

Remark 1.6. Notice that the previous theorem allows us both to take $f(x)$ being any polynomial but also to consider $f(x) = e^x$. In particular, if $f(x) = x^2$ or $f(x) = x^3$, the previous theorem allows us to take $\Psi(t, x)$ being, for instance, a periodic solution such as the cnoidal or dnoidal wave solutions (or any other traveling wave solution) given in (1.6), respectively.

As a direct corollary of the previous theorem, by considering $\Psi \equiv 0$, we infer the unconditional uniqueness for the gKdV equation (1.2), for initial data $v_0 \in H^s(\mathbb{R})$ with $s > 1/2$.

Theorem 1.7. *The Cauchy problem associated with* (1.2) *is unconditionally locally well-posed in* $H^s(\mathbb{R})$ *for* $s > 1/2$.

Finally, under some extra conditions on the growth of $f(x)$, we prove global well-posedness (GWP) for equation (1.4).

Theorem 1.8 (GWP in $H^1(\mathbb{R})$). *Assume further that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|f''(x)| \lesssim 1 \quad \textit{for all } x \in \mathbb{R}.$$

If the initial data $u_0 \in H^s(\mathbb{R})$, *with* $s \geq 1$, *then the local solution* $u(t)$ *provided by Theorem* 1.4 *can be extended for any* $T > 0$.

Remark 1.9. Note that the previous theorem gives the GWP, in particular, for $f(x) = x^2$ but also for $f(x) = \sin(x)$ or $f(x) = \cos(x)$ as nonlinearities.

Remark 1.10. We stress that Theorems 1.4 and 1.8 give us the local and global well-posedness for $H^s(\mathbb{R})$-perturbations, $s > 1/2$ and $s \geq 1$, respectively, of regular periodic solutions of the KdV equation, in particular, for $H^s(\mathbb{R})$-perturbations of periodic traveling waves solutions.

From the above results we are able to deduce local well-posedness for equation (1.2) on Zhidkov spaces. To this end, we introduce $\mathcal{Z}^s(\mathbb{R})$ as the function space given by

$$\mathcal{Z}^s(\mathbb{R}) := \big\{\Psi \in \mathcal{D}'(\mathbb{R}) : \Psi \in L^\infty(\mathbb{R}),\ \Psi' \in H^{s-1}(\mathbb{R})\big\},$$

endowed with the natural norm $\|\Psi\|_{\mathcal{Z}^s} := \|\Psi\|_{L^\infty} + \|\Psi'\|_{H^{s-1}}$.

Theorem 1.11. *Let* $s > 1/2$. *Consider* $f : \mathbb{R} \to \mathbb{R}$ *to be any real-analytic function such that its Taylor expansion around zero has infinite radius of convergence. Then, for any* $v_0 \in \mathcal{Z}^s(\mathbb{R})$, *there exist* $T = T(\|v_0\|_{\mathcal{Z}^s}) > 0$ *and a solution* v *to the* IVP (1.2) *such that*

$$v \in C([0,T], \mathcal{Z}^s(\mathbb{R})) \cap L^2_T W_x^{s^-,\infty} \quad \textit{and} \quad v - v_0 \in C([0,T], H^s(\mathbb{R})).$$

Furthermore, the solution is unique in the class

$$v(t) - v_0 \in L^\infty((0,T), H^s(\mathbb{R})).$$

Also, the data-to-solution map $\Phi : v_0 \mapsto v(t)$ *is continuous from* $\mathcal{Z}^s(\mathbb{R})$ *into* $C([0,T], \mathcal{Z}^s(\mathbb{R}))$. *Moreover, if* $s \geq 1$ *and* $f(x)$ *satisfies* $|f''(x)| \lesssim 1$ *for all* $x \in \mathbb{R}$, *then the solution* $v(t)$ *can be extended for all times* $T > 0$.

Our method of proof relies in the improvements of the energy method, recently developed in [39, 40, 43], along with symmetrization arguments previously used in [8], for example. However, due to the presence of $\Psi(t,x)$ (which breaks the symmetry) and the general nonlinearity, the present analysis shall be more involved than the previous cases.

At this point it is important to remark that local well-posedness in such a general framework has never been established for equation (1.4) before. However, the smooth case is by no means a new result, but rather a suitable rewriting of the previous proofs (see [1, 18] for example). For the sake of completeness, we prefer to state this theorem and give a brief proof of its most important parts (see Section 5). In fact, the key point to prove LWP for equation (1.2) in the smooth case are the commutator estimates which, in the case of equation (1.4), can be performed with almost no changes (with respect to [1, 18]) thanks to our hypothesis on Ψ.

1.4. Previous literature

Concerning the local well-posedness of equation (1.1), there exists a vast literature for each case of $k \in \mathbb{Z}_+$. In the case where $k = 1$, Kenig, Ponce and Vega [25] showed the LWP in $H^s(\mathbb{R})$ for $s > -3/4$ via the contraction principle. Later, Guo and Kishimoto independently proved GWP for $s = -3/4$ (see [16, 28]). This result is sharp [7], in the sense that the flow map fails to be uniformly continuous in $H^s(\mathbb{R})$ for $s < -3/4$. Then GWP was proved for $s > -3/4$ by using the I-method [8]. Without asking for the uniform continuity but just continuity of the data-to-solution map, by using the complete integrability of the equation, Killip and Visan [27] showed GWP in $H^{-1}(\mathbb{R})$, which is the lowest index one can obtain due to the result of Molinet [37], which ensures that this map cannot be continuous below $H^{-1}(\mathbb{R})$. On the other hand, Zhou [54] demonstrated the unconditional uniqueness in $L^2(\mathbb{R})$. In the periodic case, LWP was proved in $H^s(\mathbb{T})$ for $s \geq -1/2$ by Kenig, Ponce and Vega [25]. These local-in-time solutions are also shown to exist on an arbitrary time interval. Moreover, the unconditional uniqueness in $L^2(\mathbb{T})$ was established in [3]. In this case the best result is due to Kappeler and Topalov [20] in $H^{-1}(\mathbb{T})$.

Concerning the mKdV case, that is when $k = 2$, the result of Kenig, Ponce and Vega ensures the LWP for $s \geq 1/4$ on the line [24]. It has been proven that this result is sharp, in the sense that the flow map fails to be uniformly continuous in $H^s(\mathbb{R})$ as soon as $s < 1/4$, for both the focusing mKdV [26] and the defocusing one [7]. Then GWP was shown for $s > 1/4$ in [8] by using the I-method (see also [10]). Moreover, unconditional uniqueness in $H^s(\mathbb{R})$ for $s > 1/3$ was established by Molinet et al. in [40], and recently improved for $s > 1/4$ by Kwon et al. in [32]. On the other hand, in the periodic case, unconditional LWP for $s \geq 1/3$ was proved by Molinet et al. in [41], by using the improved energy method developed in [43] together with the construction of modified energies (see also [44]). Furthermore, global well-posedness has been shown in $H^s(\mathbb{T})$ for $s \geq 1/2$ in [8].

In the case where $k = 3$, we refer to [15] for the LWP in $H^s(\mathbb{R})$ for $s \geq -1/6$, and to [24] for the local well-posedness in the case where $k \geq 4$, up to the critical index given by the scaling invariance (inclusive). These results do not address unconditional uniqueness.

Regarding equation (1.4), as far as we know this equation has never been treated in such a general framework, and hence there is no abundant specific literature for it. However, in the case of the KdV equation, i.e., $f(x) = x^2$, with $\Psi = \Psi(x)$ being a time-independent function belonging to the Zhidkov class, we find the result of Iorio et al. [18]

for regular data (see also [5]). To the best of our knowledge, the best result to date (in the previously mentioned framework) is given by Gallo [11], where LWP was established for the KdV case for $s > 1$ under the same hypothesis on $\Psi(x)$ as in the work of Iorio et al [18]. Note that Theorem 1.11 extends both results [11, 18] to the whole range $s \in (1/2, 1]$ and also provides the GWP in the case $s \geq 1$. On the other hand, in the case of general nonlinearity $f(x)$, under some extra conditions concerning the values of $\Psi(x)$ at $\pm\infty$ and the value of the integral of $f(x)$ on the region $[\Psi(-\infty), \Psi(+\infty)]$, Zhidkov [53] established local well-posedness for data in $H^2(\mathbb{R})$. In the same work, he also proved the $H^1(\mathbb{R})$ orbital stability of Kinks of equation (1.4) for H^2 solutions. Then, by using this stability property, he showed the global existence of H^2 solutions for small H^1-perturbations of such Kinks. In order to prove these results, Zhidkov assumed, among other hypotheses, that $\Psi'(x) > 0$ for all $x \in \mathbb{R}$, and that $\Psi(x)$ converges exponentially fast to its limits at $\pm\infty$. As we already mentioned, Theorem 1.4 contains (and improves) the results in [5, 11, 18, 53]. In particular, notice that Theorem 1.11 allows us to extend the existence and the stability result of Zhidkov by only considering H^1-solutions which are H^1-close to those Kinks. Finally, we point out that, in the case $f(x) = x^2$, Theorem 1.4 is related to the existence problem for the KdV equation with variable coefficients, which has recently been treated by a similar approach in [42].

During the review process of this work the author found out that, simultaneously, in the specific case of the KdV equation, T. Laurens proved global well-posedness in $H^{-1}(\mathbb{R})$ on the background of a smooth step-like function (see [33, 34]).

1.5. Organization of this paper

This paper is organized as follows. In Section 2 we introduce all the notations that we shall use in the sequel, and then we state a series of preliminary results needed in our analysis. In Section 3 we prove the main a priori energy estimates for solutions and for the difference of solutions. In Section 4 we establish Theorem 1.4. Then in Section 5 we sketch the proof of the LWP in the smooth case (see Theorem 5.1 below). Finally, in Section 6 we prove the global well-posedness result, i.e., Theorem 1.8.

2. Preliminaries

2.1. Basic notations

For any pair of positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also denote by $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, for $\alpha \in \mathbb{R}$, we denote by α^+, respectively α^-, a number slightly greater, respectively lesser, than α. Furthermore, we shall occasionally use the notation $F(x)$ to denote a primitive of the nonlinearity $f(x)$, that is, $F(s) = \int_0^s f(s')ds'$.

Now, for $u(t,x) \in \mathcal{S}'(\mathbb{R}^2)$, $\mathcal{F}u = \hat{u}$ shall denote its space Fourier transform, whereas $\mathcal{F}_t u$, respectively $\mathcal{F}_{t,x}u$, shall denote its time Fourier transform, respectively space-time Fourier transform. Additionally, for $s \in \mathbb{R}$, we introduce the Bessel and Riesz potentials of order $-s$, namely, J_x^s and D_x^s, given by (respectively)

$$J_x^s u := \mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}u\big) \quad \text{and} \quad D_x^s u := \mathcal{F}^{-1}(|\xi|^s\mathcal{F}u).$$

We also denote by $U(t)$ the unitary group associated with the linear part of (1.1), that is, the Airy group

$$U(t)g := e^{-\partial_x^3} g = \mathcal{F}^{-1}(e^{it\xi^3}\mathcal{F} g).$$

On the other hand, throughout this work we consider a fixed smooth cutoff function η satisfying

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \le \eta \le 1, \quad \eta|_{[-1,1]} = 1 \quad \text{and} \quad \operatorname{supp}\eta \subset [-2,2]. \tag{2.1}$$

We define $\phi(\xi) := \eta(\xi) - \eta(2\xi)$ and, for $\ell \in \mathbb{Z}$, we denote by ϕ_{2^ℓ} the function given by

$$\phi_{2^\ell}(\xi) := \phi(2^{-\ell}\xi).$$

Additionally, we shall denote by ψ_{2^ℓ} the function given by

$$\psi_{2^\ell}(\tau,\xi) := \phi_{2^\ell}(\tau - \xi^3) \quad \text{for } \ell \in \mathbb{N}\setminus\{0\}, \qquad \psi_1(\tau,\xi) := \eta(\tau-\xi^3).$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic. Unless stated otherwise, we work with homogeneous dyadic decomposition for the space-frequency and time-frequency variables, and nonhomogeneous decompositions for modulation variables, i.e., these variables range over numbers of the form $\{2^\ell : \ell \in \mathbb{Z}\}$ and $\{2^\ell : \ell \in \mathbb{N}\}$, respectively. We denote these sets by $\mathbb{D}$ and $\mathbb{D}_{\mathrm{nh}}$, respectively. Then with the previous notations and definitions, we have that

$$\sum_{N\in\mathbb{D}} \phi_N(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}\setminus\{0\}, \qquad \operatorname{supp}\phi_N \subset \Big\{\frac{1}{2}N \le |\xi| \le 2N\Big\}.$$

In the same fashion,

$$\sum_{L\in\mathbb{D}_{\mathrm{nh}}} \psi_L(\tau,\xi) = 1 \quad \text{for all } (\tau,\xi) \in \mathbb{R}^2.$$

We define the Littlewood–Paley multipliers by the following identities:

$$P_N u := \mathcal{F}_x^{-1}(\phi_N \mathcal{F} u), \quad R_K u := \mathcal{F}_t^{-1}(\phi_K \mathcal{F}_t u) \quad Q_L u := \mathcal{F}_{t,x}^{-1}(\psi_L \mathcal{F}_{t,x} u). \tag{2.2}$$

With these definitions at hand, we introduce the operators

$$P_{\ge N} := \sum_{M\ge N} P_M, \quad P_{\le N} := \sum_{M\le N} P_M, \quad Q_{\ge L} := \sum_{\tilde{L}\ge L} Q_{\tilde{L}}, \quad Q_{\le \tilde{L}} := \sum_{\tilde{L}\le L} Q_{\tilde{L}}.$$

In addition, we borrow some notations from [47]. For a natural number $k \ge 2$ and $\xi \in \mathbb{R}$, we denote the $(k-1)$-dimensional affine-hyperplane of $\mathbb{R}^k$ by

$$\Gamma^k(\xi) := \big\{(\xi_1,\dots,\xi_k) \in \mathbb{R}^k : \xi_1 + \cdots + \xi_k = \xi\big\},$$

endowed with the natural measure

$$\int_{\Gamma^k(\xi)} F(\xi_1,\dots,\xi_k)\, d\Gamma^k(\xi) := \int_{\mathbb{R}^k} F(\xi_1,\dots,\xi_{k-1},\xi-\xi_1-\cdots-\xi_{k-1})\, d\xi_1\cdots d\xi_{k-1},$$

for any function $F\colon \Gamma^k(\xi)\to\mathbb{C}$. Moreover, when $\xi=0$ we shall simply denote by $\Gamma^k=\Gamma^k(0)$ with the obvious modifications.

To finish this first subsection, we introduce the notation for the pseudoproduct operator that we shall repeatedly use in the sequel. Let χ be a (possibly complex-valued) measurable bounded function on $\mathbb{R}^2$. We define the operator $\Pi=\Pi_\chi$ on $\mathcal{S}(\mathbb{R})^2$ by the expression

$$\mathcal{F}(\Pi(f,g))(\xi):=\int_{\mathbb{R}}\hat{f}(\xi_1)\hat{g}(\xi-\xi_1)\chi(\xi,\xi_1)\,d\xi_1. \tag{2.3}$$

This bilinear operator behaves as a product operator in the sense that it satisfies the following properties:

$$\Pi(f,g)=fg \ \text{ when } \chi\equiv 1 \quad\text{and}\quad \int\Pi_\chi(f,g)h=\int f\,\Pi_{\chi_1}(g,h)=\int\Pi_{\chi_2}(f,h)g,$$

where $\chi_1(\xi,\xi_1)=\overline{\chi}(\xi_1,\xi)$ and $\chi_2(\xi,\xi_1)=\overline{\chi}(\xi-\xi_1,\xi)$ for all trio of functions $f,g,h\in\mathcal{S}(\mathbb{R})$. Throughout this paper we shall also use pseudoproduct operators with k-entries, which are defined as in (2.3) with the obvious modifications.

2.2. Function spaces

For $s,b\in\mathbb{R}$, we define the Bourgain space $X^{s,b}$ associated with the linear part of (1.1) as the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|u\|_{X^{s,b}}^2:=\int_{\mathbb{R}^2}(1+|\tau-\xi^3|)^{2b}(1+|\xi|)^{2s}|\mathcal{F}_{t,x}[u](\tau,\xi)|^2\,d\xi\,d\tau.$$

We recall that these spaces satisfy

$$\|u\|_{X^{s,b}}\sim\|U(-t)u\|_{H^{s,b}_{t,x}},\quad \text{where } \|u\|_{H^{s,b}_{t,x}}:=\|J_x^sJ_t^bu\|_{L^2_{t,x}}.$$

Additionally, we define the frequency-enveloped spaces associated with $H^s(\mathbb{R})$ as follows: Let $s\in\mathbb{R}$ and $\kappa>1$ be fixed. Consider a sequence $\{\omega_N\}_{N\in\mathbb{D}}$ of positive real numbers satisfying $\omega_N\le\omega_{2N}\le 2^\varepsilon\omega_N$, for some $\varepsilon>0$ such that $\varepsilon<\min\{\delta_1,\delta_2\}$, where δ_1 and δ_2 are the small numbers associated with the choices we make in the hypotheses

$$\Psi\in L_t^\infty W_x^{s+1^+,\infty} \quad\text{and}\quad (\partial_t\Psi+\partial_x^3\Psi+\partial_xf(\Psi))\in L_t^\infty H_x^{s^+}. \tag{2.4}$$

In other words, if we assume (2.4) for some $\delta_1,\delta_2>0$ small, then we assume, in particular, that ω_N satisfies

$$\frac{\omega_N}{N^{\delta_*}}\xrightarrow{N\to+\infty}0,\quad \delta_*:=\min\{\delta_1,\delta_2\}.$$

Furthermore, we also assume that $\omega_N\to 1$ as $N\to 0$. Then we define the space $H^s_\omega(\mathbb{R})$ associated with $\{\omega_N\}$ as the completion of the Schwarz space $\mathcal{S}(\mathbb{R})$ under the norm

$$\|f\|_{H^s_\omega}^2:=\sum_N\omega_N^2\|P_Nf\|_{H^s}^2\sim\sum_N\omega_N^2N^{2s}\|P_Nf\|_{L^2}^2.$$

Of course, by definition, we have $H^s_\omega(\mathbb{R}) \subseteq H^s(\mathbb{R})$. Moreover, if $\omega_N = 1$ for all $N \in \mathbb{D}$, then $H^s_\omega = H^s$. The goal we seek by introducing these frequency-enveloped spaces is to be able to prove the continuity part in Theorem 1.4.

Finally, we define the restriction-in-time version of all of the above spaces. Let $T > 0$ be fixed, and consider F to be any normed space of space-time functions. We define its restriction in time version F_T as the space of functions $u\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfying

$$\|u\|_{F_T} := \inf\big\{\|\tilde{u}\|_F \mid \tilde{u}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \text{with } \tilde{u} = u \text{ on } [0,T]\times\mathbb{R}\big\} < +\infty.$$

2.3. Extension operator

In this subsection we introduce the extension operator that we shall use in the sequel. We borrow this definition from [40]. The key property of this operator is that it defines a bounded operator from $L^\infty_T H^s_\omega \cap X^{s-1,1}_T \cap L^2_T W^{r,\infty}_x$ into $L^\infty_t H^s_\omega \cap X^{s-1,1} \cap L^2_t W^{r,\infty}_x$ with $r < s$.

Definition 2.1. Given $T \in (0,2)$ and $u\colon [0,T]\times\mathbb{R}\to\mathbb{R}$, we define the extension operator ρ_T by the following identity:

$$\rho_T[u](t) := U(t)\eta(t)U(-\mu_T(t))u(\mu_T(t)), \tag{2.5}$$

where η corresponds to the function given in (2.1) and μ_T is the continuous function

$$\mu_T(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } t \in (0,T), \\ T & \text{if } t \geq T. \end{cases}$$

Remark 2.2. Notice that, directly from the definition, we have $\rho_T[u](t,x) = u(t,x)$ for all $(t,x) \in [0,T]\times\mathbb{R}$.

The next lemma give us the main properties of this operator (see [40]).

Lemma 2.3. *Let $T \in (0,2)$ be fixed. Consider $(s,r,\theta,b) \in \mathbb{R}^4$ with $r < s$ and $b \in (1/2,1]$. Then the following holds*:

$$\rho_T\colon L^\infty_T H^s_\omega \cap X^{\theta,b}_T \cap L^2_T W^{r,\infty}_x \mapsto L^\infty_t H^s_\omega \cap X^{\theta,b} \cap L^2_t W^{r,\infty}_x.$$

In other words, we have the following inequality:

$$\|\rho_T[u]\|_{L^\infty_t H^s_\omega} + \|\rho_T[u]\|_{X^{\theta,b}} + \|\rho_T[u]\|_{L^2_t W^{r,\infty}_x} \lesssim \|u\|_{L^\infty_T H^s_\omega} + \|u\|_{X^{\theta,b}_T} + \|u\|_{L^2_T W^{r,\infty}_x}.$$

Moreover, the implicit constant involved in the latter inequality can be chosen independent of (T,s,r,θ,b).

2.4. Resolution space

From now on, for any $s \in \mathbb{R}$ and any sequence $\{\omega_N\}_{N\in\mathbb{D}}$ satisfying the hypotheses in Section 2.2, we define the resolution space

$$\mathcal{M}^s_\omega := L^\infty_t H^s_\omega \cap X^{s-1,1},$$

endowed with the natural norm

$$\|u\|_{\mathcal{M}^s_\omega} := \|u\|_{L^\infty_t H^s_\omega} + \|u\|_{X^{s-1,1}}. \tag{2.6}$$

When $\omega_N \equiv 1$, we simply write $\mathcal{M}^s = \mathcal{M}^s_\omega$. Before going further we recall the following basic lemma concerning Sobolev spaces.

Lemma 2.4 (See [2] for example). *Let $a, b, c \in \mathbb{R}$ be a triplet of real numbers satisfying*

$$a \geq c, \quad b \geq c, \quad a + b \geq 0 \quad \textit{and} \quad a + b - c > \frac{n}{2}.$$

Then the map $(f, g) \mapsto f \cdot g$ is a continuous bilinear form from $H^a(\mathbb{R}^n) \times H^b(\mathbb{R}^n)$ into $H^c(\mathbb{R}^n)$.

The following lemma ensures us that $L^\infty_T H^s_\omega$-solutions also belong to $\mathcal{M}^s_{\omega,T}$, whereas the difference of two solutions in $L^\infty_T H^s_x$ take place in $\mathcal{M}^{s-1}_T$.

Lemma 2.5. *Let $s > 1/2$ and $T \in (0, 2)$ be given. Let $u \in L^\infty((0, T), H^s_\omega(\mathbb{R}))$ be a solution to equation (1.4) with initial data $u_0 \in H^s_\omega(\mathbb{R})$. Then $u \in \mathcal{M}^s_{\omega,T}$ and the following inequality holds*:

$$\begin{aligned} \|u\|_{\mathcal{M}^s_{\omega,T}} \lesssim \big(1 + T^{1/2}\mathcal{F}_1(\|u\|_{L^\infty_T H^s_x}, \|\Psi\|_{L^\infty_T W^{s^+,\infty}_x})\big)\|u\|_{L^\infty_T H^s_\omega} \\ + T^{1/2}\|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{L^\infty_T H^{s-1}_x}, \end{aligned} \tag{2.7}$$

where $\mathcal{F}_1 : \mathbb{R}^2 \to \mathbb{R}_+$ is a smooth function. Also, for any pair $u, v \in L^\infty((0, T), H^s(\mathbb{R}))$ of solutions to equation (1.4) associated with initial data $u_0, v_0 \in H^s(\mathbb{R})$, the following holds:

$$\|u-v\|_{\mathcal{M}^{s-1}_T} \lesssim \big(1+T^{1/2}\mathcal{F}_2(\|u\|_{L^\infty_T H^s_x}, \|v\|_{L^\infty_T H^s_x}, \|\Psi\|_{L^\infty_T W^{s^+,\infty}_x})\big)\|u-v\|_{L^\infty_T H^{s-1}_x}, \tag{2.8}$$

for some smooth function $\mathcal{F}_2 : \mathbb{R}^3 \to \mathbb{R}_+$.

Proof. First of all, we have to extend the functions $u(t)$ and $v(t)$ from $(0, T)$ to the whole line $\mathbb{R}$. Hence, we benefit from the extension operator ρ_T defined in (2.5), which we use to take extensions $\tilde{u} := \rho_T[u]$, $\tilde{v} := \rho_T[v]$ defined on $\mathbb{R}^2$, both supported in $(-2, 2)$. For the sake of notation, we drop the tilde in the sequel.

Once we have extended (in time) both solutions, comparing inequality (2.7) with the definition of the $\mathcal{M}^s_T$ norm in (2.6), it is clear that it is enough to estimate the $X^{s-1,1}$-norm. In fact, let us start out by writing the solution in its Duhamel form

$$u(t) = U(t)u_0 + c\int_0^t U(t-t')\big(\partial_t\Psi + \partial_x^3\Psi + \partial_x f(u+\Psi)\big)\,dt'.$$

Then, by using standard linear estimates in Bourgain spaces, we obtain

$$\begin{aligned} \|u\|_{X^{s-1,1}_T} &\lesssim \|u\|_{L^\infty_T H^{s-1}_x} + \|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{X^{s-1,0}_T} \\ &\quad + \|\partial_x(f(u+\Psi) - f(\Psi))\|_{X^{s-1,0}_T} \\ &\lesssim \|u\|_{L^\infty_T H^{s-1}_x} + T^{1/2}\|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{L^\infty_T H^{s-1}_x} \\ &\quad + T^{1/2}\|f(u+\Psi) - f(\Psi)\|_{L^\infty_T H^s_x}. \end{aligned} \tag{2.9}$$

Now, from the classical Sobolev estimates for products, it is not difficult to see that there exists a constant $c > 0$ such that, for $k, m \in \mathbb{N}$, with $k \geq m$,

$$\|g^{k-m}h^m\|_{H^s} \leq c^k \|g\|_{H^s}^{k-m}\|h\|_{W^{s+,\infty}}^m, \quad s > 1/2. \tag{2.10}$$

Then, to control the contribution of the latter term in (2.9), it is enough to use the Taylor expansion to get

$$\begin{aligned}\|f(u+\Psi)-f(\Psi)\|_{L_T^\infty H_x^s} &\lesssim \sum_{k=1}^\infty \sum_{m=0}^{k-1} c^k |a_k| \binom{k}{m} \|u\|_{L_T^\infty H_x^s}^{k-m}\|\Psi\|_{L_T^\infty W_x^{s+,\infty}}^m \\ &\lesssim \|u\|_{L_T^\infty H_x^s} \sum_{k=1}^\infty k c^k |a_k| \big(\|u\|_{L_T^\infty H_x^s} + \|\Psi\|_{L_T^\infty W_x^{s+,\infty}}\big)^{k-1}.\end{aligned}$$

This concludes inequality (2.7) thanks to the hypothesis on the coefficients a_k in (1.5). Now we turn to show (2.8). In order to do so, let us define $w := u - v$. Notice that $w(t,x)$ solves the following equation:

$$\partial_t w + \partial_x\big(\partial_x^2 w + f(u+\Psi) - f(v+\Psi)\big) = 0.$$

Thus, writing $w(t,x)$ in its Duhamel form, and then using standard linear estimates for Bourgain spaces, we get

$$\|w\|_{X_T^{s-2,1}} \lesssim \|w\|_{L_T^\infty H_x^{s-2}} + T^{1/2}\|f(u+\Psi)-f(v+\Psi)\|_{L_T^\infty H_x^{s-1}}.$$

Then, by using Taylor expansion, proceeding in a similar fashion as above, it is not difficult to see that

$$\begin{aligned}&\|f(u+\Psi)-f(v+\Psi)\|_{L_T^\infty H_x^{s-1}} \\ &\quad\lesssim \|w\|_{L_T^\infty H_x^{s-1}} \sum_{k=1}^\infty k c^k |a_k| \big(\|u\|_{L_T^\infty H_x^s} + \|v\|_{L_T^\infty H_x^s} + \|\Psi\|_{L_T^\infty W_x^{s+,\infty}}\big)^{k-1},\end{aligned}$$

where we have used again (2.10), as well as Lemma 2.4. The proof is complete. ∎

2.5. Preliminary lemmas

For $T > 0$ fixed, we consider $\mathbf{1}_T(t)$, the characteristic function on $[0,T]$. Then, for η given in (2.1) and $R > 0$, we decompose $\mathbf{1}_T(t)$ as

$$\mathbf{1}_T(t) = \mathbf{1}_{T,R}^{\text{low}}(t) + \mathbf{1}_{T,R}^{\text{high}}(t), \quad \text{where } \mathcal{F}_t(\mathbf{1}_{T,R}^{\text{low}})(\tau) = \eta\Big(\frac{\tau}{R}\Big)\mathcal{F}_t(\mathbf{1}_T)(\tau). \tag{2.11}$$

The following lemma gives us some basic estimates that shall be particularly convenient to take advantage of the above decomposition along with Bourgain spaces.

Lemma 2.6 (See [43]). *For any $R > 0$, $T > 0$ and $q \geq 1$, the following bounds hold*:

$$\|\mathbf{1}_{T,R}^{\text{high}}\|_{L^q} \lesssim \min\{T, R^{-1}\}^{1/q} \quad \textit{and} \quad \|\mathbf{1}_{T,R}^{\text{low}}\|_{L^\infty} \lesssim 1. \tag{2.12}$$

Moreover, if $L \gg R$, then for all $u \in L^2(\mathbb{R}^2)$, the following inequality holds:

$$\|Q_L(\mathbf{1}_{T,R}^{\mathrm{low}}u)\|_{L^2} \lesssim \|Q_{\sim L}u\|_{L^2}.$$

Furthermore, for any $s \in \mathbb{R}$ and any $p \in [1,\infty]$, the operator $Q_{\leq L}$ is bounded in $L_t^pH_x^s$ uniformly in L.

Proof. We point out that the only part which is not strictly contained in [43] is given by the first inequality in (2.12). However, this proof follows very similar lines, except for one straightforward step, and hence we shall be brief. In fact, a direct computation yields

$$\begin{aligned}\|\mathbf{1}_{T,R}^{\mathrm{high}}\|_{L^q} &= \Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\Big(\mathbf{1}_T(t)-\mathbf{1}_T\Big(t-\frac{s}{R}\Big)\Big)\mathcal{F}^{-1}\eta(s)\,ds\Big|^q\,dt\Big)^{1/q}\\ &\leq \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big|\mathbf{1}_T(t)-\mathbf{1}_T\Big(t-\frac{s}{R}\Big)\Big|^q|\mathcal{F}^{-1}\eta(s)|^q\,dt\Big)^{1/q}ds\\ &\lesssim \int_{\mathbb{R}}\min\Big\{T,\frac{|s|}{R}\Big\}^{1/q}|\mathcal{F}^{-1}\eta(s)|\,ds \lesssim \min\{T,R^{-1}\}^{1/q},\end{aligned}$$

where to obtain the first inequality, we have used Minkowski integral inequality. The proof is complete. ∎

Lemma 2.7 ([43]). *Let $k \geq 3$ be a fixed parameter, and consider $L_1,\dots,L_k \geq 1$ and $N_1,\dots,N_k > 0$ to be a list of $2k$ dyadic numbers. Consider also $\{u_i\}_{i=1}^k \subset \mathcal{S}'(\mathbb{R}^2)$, all of them being real-valued, and let $\chi \in L^\infty(\mathbb{R}^2;\mathbb{C})$. Additionally, assume that $N_1 \geq N_2 \geq N_3 \geq 2^9k\max\{N_4,\dots,N_k\}$.*[1] *Then the following identity holds*:

$$\int_{\mathbb{R}^2}\Pi_\chi(Q_{L_1}P_{N_1}u_1,Q_{L_2}P_{N_2}u)\prod_{i=3}^k Q_{L_i}P_{N_i}u = 0,$$

unless $L_{\max} \geq (2^9k)^{-1}N_1N_2N_3$, where $L_{\max} := \max\{L_1,\dots,L_k\}$.

Finally, we prove some basic lemmas concerning the application of Hölder's inequality with some particular instances of pseudo-product operators that shall appear in the next section.

Lemma 2.8. *Let $k \geq 2$ be a fixed natural number. Consider k functions $u_1,\dots,u_k \in L^2(\mathbb{R})$ such that each of them is supported in the annulus $\{|\xi_i| \sim N_i\}$. Additionally consider a continuous function $a \in C(\mathbb{R}^k)$. Then the following inequality holds*:

$$\Big|\int_{\Gamma^k}a(\xi_1,\dots,\xi_k)u_1(\xi_1)\dots u_k(\xi_k)\,d\Gamma^k\Big| \lesssim (N_3\dots N_k)^{1/2}\|a\|_{L^\infty(\Omega_k)}\prod_{i=1}^k\|u_i\|_{L^2},$$

where

$$\Omega_k := \big\{(\xi_1,\dots,\xi_k) \in \Gamma^k : \forall i = 1,\dots,k,\ \xi_i \in \operatorname{supp} u_i\big\}.$$

[1] If $k = 3$, we omit the last inequality.

Proof. Let us start by assuming that $k \geq 3$, since the case $k = 2$ is direct. In fact, first of all we get rid of $a(\xi_1, \ldots, \xi_k)$ simply by bounding as follows:

$$\Big| \int_{\Gamma^k} a(\xi_1, \ldots, \xi_k) u_1(\xi_1) \ldots u_k(\xi_k)\, d\Gamma^k \Big| \leq \|a\|_{L^\infty(\Omega_k)} \int_{\Gamma^k} |u_1(\xi_1) \ldots u_k(\xi_k)|\, d\Gamma^k.$$

Then it is enough notice that we can bound the latter integral in the above inequality by

$$\int_{\Gamma^k} |u_1(\xi_1) \cdots u_k(\xi_k)|\, d\Gamma^k \leq \sup_{\xi_3, \ldots, \xi_k} \int_{\mathbb{R}} |u_1(-\xi_2 - \cdots - \xi_k) u_2(\xi_2)|\, d\xi_2 \times \prod_{i=3}^{k} \int_{\mathbb{R}} |u_i(\xi)|\, d\xi.$$

Finally, by the Cauchy–Schwarz inequality and the supports hypotheses, we have

$$\int_{\mathbb{R}} |u_i(\xi)|\, d\xi \lesssim N_i^{1/2} \|u_i\|_{L^2} \quad \text{and} \quad \Big| \int_{\mathbb{R}} u_1(-\xi_2 - \cdots - \xi_k) u_2(\xi_2)\, d\xi_2 \Big| \lesssim \|u_1\|_{L^2} \|u_2\|_{L^2}.$$

The proof is complete. ∎

Lemma 2.9. *Let $k \geq 2$ and $m \in [2, k+1]$ be two fixed natural numbers. Additionally, consider a family of dyadic numbers $\{N_i\}_{i=1}^{k+2}$, all of them being fixed. Let $u_1, \ldots, u_{k+1} \in L^2(\mathbb{R})$ such that each of them has Fourier transform supported on the ball $\{|\xi_i| \leq N_i\}$, respectively. Furthermore, assume that $M = \max\{N_3, \ldots, N_{k+1}\}$. Then the following holds*:

$$\Big| \int_{\Gamma^{k+1}} a(\xi_1, \ldots, \xi_k) \hat{u}_1(\xi_1) \ldots \hat{u}_{k+1}(\xi_{k+1})\, d\Gamma^{k+1} \Big| \lesssim M \|u_1\|_{L^2} \|u_2\|_{L^2} \prod_{i=3}^{k+1} \|u_i\|_{L^\infty},$$

where the implicit constant depends polynomially on k and $a(\xi_1, \ldots, \xi_k)$ stands for the following quantity:

$$a(\xi_1, \ldots, \xi_k) := \sum_{i=1}^{m} \phi_{N_{k+2}}^2(\xi_i)\, \xi_i, \quad \textit{where } \xi_{k+1} = -\xi_1 - \cdots - \xi_k. \tag{2.13}$$

Proof. In fact, first of all notice that, except for the terms associated with $i = 1, 2$ in the definition of $a(\xi_1, \ldots, \xi_k)$, the proof follows directly from Plancherel's theorem and Hölder's inequality. Indeed, going back to physical-variables and then using Hölder's inequality as well as Bernstein inequalities, we obtain

$$\begin{aligned}
&\Big| \int_{\Gamma^{k+1}} \big(a(\xi_1, \ldots, \xi_k) - \phi_{N_{k+2}}^2(\xi_1)\xi_1 - \phi_{N_{k+2}}^2(\xi_2)\xi_2 \big) \hat{u}_1(\xi_1) \cdots \hat{u}_{k+1}(\xi_{k+1})\, d\Gamma^{k+1} \Big| \\
&\quad \lesssim \|u_1\|_{L^2} \|u_2\|_{L^2} \sum_{j=3}^{m} \|\partial_x P_{N_{k+2}}^2 u_j\|_{L^\infty} \prod_{i=3, i\neq j}^{k+1} \|u_i\|_{L^\infty} \\
&\quad \lesssim M \|u_1\|_{L^2} \|u_2\|_{L^2} \prod_{i=3}^{k+1} \|u_i\|_{L^\infty}.
\end{aligned}$$

Thus, we can restrict ourselves to study the above integral when replacing $a(\xi_1,\dots,\xi_k)$ with the symbol

$$\widetilde{a}(\xi_1,\dots,\xi_k):=\phi_{N_{k+2}}^2(\xi_1)\xi_1+\phi_{N_{k+2}}^2(\xi_2)\xi_2.$$

Next, we split this symbol into two parts as follows:

$$\begin{aligned}\widetilde{a}(\xi_1,\dots,\xi_k)&=\phi_{N_{k+2}}^2(\xi_1)(\xi_1+\xi_2)-(\phi_{N_{k+2}}^2(\xi_1)-\phi_{N_{k+2}}^2(\xi_2))\xi_2\\&=:\widetilde{a}_1(\xi_1,\dots,\xi_k)+\widetilde{a}_2(\xi_1,\dots,\xi_k).\end{aligned}$$

Notice now that due to the additional restriction imposed by Γ^{k+1} (that is, $\xi_1+\dots+\xi_{k+1}=0$), in this domain we can rewrite $\widetilde{a}_1(\xi_1,\dots,\xi_k)$ as

$$\widetilde{a}_1(\xi_1,\dots,\xi_k)=-\phi_{N_{k+2}}^2(\xi_1)(\xi_3+\dots+\xi_{k+1}).$$

Hence, this case also follows from the above analysis. Therefore, it only remains to consider the case of $\widetilde{a}_2$. For the sake of clarity, we shall assume now that $k=2$, the proof for the general case shall be clear from this one. In fact, by using Plancherel's theorem, integration by parts and then Hölder's inequality, we immediately obtain that

$$\begin{aligned}&\Big\vert\int_{\Gamma^3}\widetilde{a}_2(\xi_1,\xi_2)\hat{u}_1(\xi_1)\hat{u}_2(\xi_2)\hat{u}_3(\xi_3)\,d\Gamma^3\Big\vert\\&=\Big\vert\int_{\mathbb{R}}(u_{2,x}P_{N_{k+2}}^2u_1-u_1P_{N_{k+2}}^2u_{2,x})u_3\,dx\Big\vert\\&=\Big\vert\int_{\mathbb{R}}u_2\partial_x(u_3P_{N_{k+2}}^2u_1-P_{N_{k+2}}^2(u_1u_3))\,dx\Big\vert\\&\lesssim\Vert u_1\Vert_{L^2}\Vert u_2\Vert_{L^2}\Vert\partial_xu_3\Vert_{L^\infty}+\Vert u_2\Vert_{L^2}\Vert[P_{N_{k+2}}^2\partial_x,u_3]u_1\Vert_{L^2}.\end{aligned}$$

Then, since $\Vert\partial_xu_3\Vert_{L^\infty}\lesssim M\Vert u_3\Vert_{L^\infty}$, it only remains to control the latter factor of the above inequality. In order to do that, first notice that by direct computations, we can write

$$[P_{N_{k+2}}^2\partial_x,u_3]u_1(x)=\int_{\mathbb{R}}K(x,y)u_1(y)\,dy,$$

where the kernel $K(x,y)$ can be written as

$$K(x,y)=icN_{k+2}{}^2\int_{\mathbb{R}}e^{iN_{k+2}(x-y)\eta}\eta\phi^2(\eta)(u_3(y)-u_3(x))\,d\eta,$$

for some constant $c\in\mathbb{R}$. Thus, as an application of the mean value theorem, it is not difficult to see that there exists a function $g\in L^1(\mathbb{R})$ such that

$$|K(x,y)|\lesssim N_{k+2}\Vert\partial_xu_3\Vert_{L^\infty}g(N_{k+2}(x-y)).$$

Notice that the latter inequality implies, in particular, the following uniform bound:

$$\sup_{y\in\mathbb{R}}\int_{\mathbb{R}}|K(x,y)|\,dx+\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}|K(x,y)|\,dy\lesssim\Vert\partial_xu_3\Vert_{L^\infty},$$

where the implicit constant does not depends on N_{k+2}. Therefore, applying Schur's lemma, and then Bernstein's inequality in the resulting right-hand side, we obtain that

$$\Vert[P_{N_{k+2}}^2\partial_x,u_3]u_1\Vert_{L^2}\lesssim\Vert u_1\Vert_{L^2}\Vert\partial_xu_3\Vert_{L^\infty}\lesssim M\Vert u_1\Vert_{L^2}\Vert u_3\Vert_{L^\infty},$$

which concludes the proof of the lemma. ∎

2.6. Strichartz estimates

In this subsection we seek to prove a refined Strichartz estimate for solutions to the linear Airy equation with a general source term. The proof we present here is just a slight modification of the arguments already used in [29,39,40]. Before getting into the details, let us recall the classical smoothing effect derived in [23] that shall be useful in the sequel:

$$\|e^{-t\partial_x^3}D_x^{1/4}u_0\|_{L_t^4L_x^\infty}\lesssim\|u_0\|_{L_x^2}. \tag{2.14}$$

Now we are ready to state our refined Strichartz estimate.

Lemma 2.10. *Let $T\in(0,1]$ and consider $\delta\geq 0$ to be a fixed parameter. Let $u(t,x)$ be any solution defined on $[0,T]$ to the following linear equation*:

$$\partial_t u+\partial_x^3u=F. \tag{2.15}$$

Then there exists $\kappa_1,\kappa_2>0$ such that, for any $\theta>0$, the following inequality holds:

$$\|u\|_{L_T^2L_x^\infty}\lesssim T^{\kappa_1}\|J_x^{-\frac14(1-\delta)+\theta}u\|_{L_T^\infty L_x^2}+T^{\kappa_2}\|J_x^{-\frac14(1+3\delta)+\theta}F\|_{L_T^2L_x^2}. \tag{2.16}$$

Proof. Let $u(t,x)$ be a solution to equation (2.15) defined on $[0,T]$. We use a nonhomogeneous Littlewood–Paley decomposition for the solution, that is, we write $u=\sum_N u_N$, where $u_N=P_Nu$, and N is a nonhomogeneous dyadic number. In the sequel we shall also use the notation F_N for P_NF. At this point it is important to notice that, on the one hand, from Minkowski's inequality, we know that

$$\|u\|_{L_T^2L_x^\infty}\leq\sum_N\|u_N\|_{L_T^2L_x^\infty}\lesssim\sup_N N^\theta\|u_N\|_{L_T^2L_x^\infty},$$

for any $\theta>0$. While on the other hand, by using the low-frequency projector $P_{\leq 1}$, from the Hölder and Bernstein inequalities, we see that

$$\|P_{\leq 1}u\|_{L_T^2L_x^\infty}\lesssim T^{1/2}\|P_{\leq 1}u\|_{L_T^\infty L_x^2}.$$

Then, from the inequalities above, we infer that it is enough to show that for any $\delta>0$ and any $N>1$ dyadic number, the following holds:

$$\|u_N\|_{L_T^2L_x^\infty}\lesssim T^{\kappa_1}\big\|D_x^{-\frac14(1-\delta)}u_N\big\|_{L_T^\infty L_x^2}+T^{\kappa_2}\big\|D_x^{-\frac14(1+3\delta)}F_N\big\|_{L_T^2L_x^2}. \tag{2.17}$$

Now, in order to prove (2.17), we chop the time-interval $[0,T]$ into several pieces of length $T^\kappa N^{-\delta}$, where $\kappa\in[1,2)$ stands for a small number that shall be fixed later. In other words, we have

$$[0,T]=\bigcup_{j\in J}I_j,\quad\text{where } I_j:=[a_j,b_j],\ |I_j|\sim T^\kappa N^{-\delta}\text{ and }\#J\sim T^{1-\kappa}N^\delta.$$

On the other hand, notice that $u_N(t)$ solves the integral equation

$$u_N(t)=e^{-(t-a_j)\partial_x^3}u_N(a_j)+\int_{a_j}^t e^{-(t-t')\partial_x^3}F_N(t')\,dt'\quad\text{for all } t\in I_j.$$

Therefore, by using the classical Strichartz estimate (2.14), as well as the Hölder and Bernstein inequalities, we obtain

$$\begin{aligned}
\|u_N\|_{L^2_TL^\infty_x} &= \Big(\sum_j \|u_N\|^2_{L^2_{I_j}L^\infty_x}\Big)^{1/2} \\
&\leq (T^\kappa N^{-\delta})^{1/4}\Big(\sum_j \|u_N\|^2_{L^4_{I_j}L^\infty_x}\Big)^{1/2} \\
&\lesssim (T^\kappa N^{-\delta})^{1/4}\Big(\sum_j \|D_x^{-1/4}u_N(a_j)\|^2_{L^2_x}\Big)^{1/2} \\
&\quad + (T^\kappa N^{-\delta})^{1/4}\Big(\sum_j \Big\|\int_{a_j}^t e^{-(t-t')\partial_x^3}F_N(t')\,dt'\Big\|^2_{L^4_{I_j}L^\infty_x}\Big)^{1/2} \\
&\lesssim (T^\kappa N^{-\delta})^{1/4}(T^{1-\kappa}N^\delta)^{1/2}\|D_x^{-1/4}u_N\|_{L^\infty_TL^2_x} \\
&\quad + (T^\kappa N^{-\delta})^{1/4}\Big(\sum_j T^\kappa N^{-\delta}\int_{I_j}\|D_x^{-1/4}F_N\|^2_{L^2_x}\,dt\Big)^{1/2} \\
&\lesssim T^{1/2-\kappa/4}\|D_x^{-1/4+\delta/4}u_N\|_{L^\infty_TL^2_x} + T^{3\kappa/4}\|D_x^{-1/4-3\delta/4}F_N\|_{L^2_TL^2_x},
\end{aligned}$$

which concludes the proof of (2.16) by choosing, for example, $\kappa = 1$. ∎

3. Energy estimates

3.1. A priori estimates for solutions

The goal of this section is to prove the following proposition that give us the key improved energy estimate for smooth solutions of (1.4).

Proposition 3.1. *Let $s > 1/2$ and $T \in (0,2)$ be fixed. Consider $u \in L^\infty((0,T), H^s_\omega(\mathbb{R}))$ to be a solution to equation (1.4) associated with initial data $u_0 \in H^s_\omega(\mathbb{R})$. Then the following inequality holds:*

$$\begin{aligned}
(3.1)\quad \|u\|^2_{L^\infty_TH^s_\omega} &\lesssim \|u_0\|^2_{H^s_\omega} + T\|u\|_{L^\infty_TH^s_\omega}\|\partial_t\Psi+\partial_x^3\Psi+\partial_xf(\Psi)\|_{L^\infty_TH^{s+}_x} + T^{1/4}\|u\|^2_{L^\infty_TH^s_\omega} \\
&\quad \times \mathcal{Q}_*\big(\|u\|_{L^\infty_TH^{1/2+}_x}, \|\Psi\|_{L^\infty_TW^{s+1+,\infty}_x}, \|\partial_t\Psi\|_{L^\infty_{t,x}}, \|\partial_t\Psi+\partial_x^3\Psi+\partial_xf(\Psi)\|_{L^\infty_TH^{-1/2+}_x}\big),
\end{aligned}$$

where $\mathcal{Q}_*\colon \mathbb{R}^4 \to \mathbb{R}_+$ is a smooth function.

Proof. First of all, in order to take advantage of Bourgain spaces, we have to extend the function $u(t)$ from $(0,T)$ to the whole line $\mathbb{R}$. Hence, we benefit from the extension operator ρ_T defined in (2.5), which we use to take an extension $\tilde u := \rho_T[u]$, defined on $\mathbb{R}^2$, such that $\|\tilde u\|_{\mathcal{M}} \leq 2\|u\|_{\mathcal{M}_T}$. For the sake of notation, we drop the tilde in the sequel.

Now we seek to prove (3.1). We begin by applying the frequency projector P_N to equation (1.4), with $N > 0$ dyadic but arbitrary. Notice that, on account of Remark 1.3, we have

$$P_Nu \in C([0,T],H^\infty) \quad \text{and} \quad \partial_tP_Nu \in L^\infty((0,T),H^\infty).$$

Therefore, taking the L^2_x-scalar product of the resulting equation against $P_N u$, multiplying the result by $\omega_N^2\langle N\rangle^{2s}$ and then integrating on $(0,t)$ with $0<t<T$, we obtain

$$\omega_N^2\langle N\rangle^{2s}\|P_N u(t)\|_{L^2}^2=\omega_N^2\langle N\rangle^{2s}\|P_N u_0\|_{L^2}^2 \\ -\omega_N^2\langle N\rangle^{2s}\int_0^t\int_{\mathbb{R}}P_N\big(\partial_t\Psi+\partial_x^3\Psi+\partial_x f(u+\Psi)\big)P_N u.$$

Thus, by applying Bernstein's inequality, we are lead to

$$\|P_N u(t)\|_{H^s_\omega}^2\lesssim\|P_N u_0\|_{H^s_\omega}^2 \\ +\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}P_N\big(\partial_t\Psi+\partial_x^3\Psi+\partial_x f(u+\Psi)\big)P_N u\Big|.$$

Thus, from the previous computation, we infer that, in order to conclude the proof of the proposition, we need to control the sum over all $N>0$ dyadic of the second term in the latter inequality. We divide the analysis into several steps, each of which dedicated to bound one of the following integrals:

$$\text{(3.2)}\qquad \begin{aligned}\mathrm{I}&:=\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}\partial_x P_N(f(u+\Psi)-f(\Psi))P_N u\Big|,\\ \mathrm{II}&:=\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}P_N(\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi))P_N u\Big|.\end{aligned}$$

Before going further we recall that due to the analyticity hypothesis (1.5), we can write

$$f(u(t,x)+\Psi(t,x))=\sum_{k=0}^{\infty}a_k(u(t,x)+\Psi(t,x))^k\quad\text{and}\quad f(\Psi(t,x))=\sum_{k=0}^{\infty}a_k\Psi^k(t,x).$$

With this in mind, from now on, for any $n,m\in\mathbb{N}$, we shall denote by I_{u^n} and $\mathrm{I}_{u^n\Psi^m}$ the quantity I above once $f(u+\Psi)$ is replaced by u^k and $u^k\Psi^m$, respectively, that is,

$$\text{(3.3)}\qquad \mathrm{I}_{u^k}:=\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}\partial_x P_N(u^k)P_N u\Big|,$$

$$\text{(3.4)}\qquad \mathrm{I}_{u^k\Psi^m}:=\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}\partial_x P_N(u^k\Psi^m)P_N u\Big|.$$

We point out that, in the sequel, we shall systematically omit most of the factors depending on k by hiding them in a $\lesssim_k$-sign. This sign is defined exactly as "$\lesssim$" in Section 2 but allowing the constant c to depend on k. Notice that, in order to make sense of the sum (in k) of all the following bounds, we only need to be careful that the final implicit constant depends at most as c^k for some constant $c>0$.

Step 1. We begin by controlling II right away. In fact, by using hypothesis (1.7), we infer that it is enough to use the Cauchy–Schwarz and Bernstein inequalities to obtain

$$\mathrm{II}\lesssim\int_0^T\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\|P_N(\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi))\|_{L^2_x}\|P_N u(s,\cdot)\|_{L^2_x}\,ds \\ \lesssim T\|u\|_{L^\infty_T H^s_\omega}\|\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi)\|_{L^\infty_T H^{s+}_x},$$

where we have used the hypotheses made on ω_N in Section 2.2. This concludes the proof of the first case.

Step 2. Now we aim to control the general case for I_{u^k} in (3.3) for all $k\geq 1$, that is, we aim to control the following quantity:

$$\begin{aligned}\text{(3.5)}\qquad \mathrm{I}_{u^k}&=\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\\&\quad\times\sup_{t\in(0,T)}\Big\vert\int_0^t\int_{\Gamma^{k+1}}a_k(\xi_1,\dots,\xi_{k+1})\hat{u}(s,\xi_1)\dots\hat{u}(s,\xi_{k+1})\,d\Gamma^{k+1}\,ds\Big\vert,\end{aligned}$$

where the symbol $a_k(\xi_1,\dots,\xi_{k+1})$ is explicitly given by

$$a_k(\xi_1,\dots,\xi_{k+1}):=i\phi_N^2(\xi_{k+1})\xi_{k+1}.$$

We point out that in the previous identity (3.5), we have used both the fact that $u(t,\cdot)$ is real-valued as well as the fact that ϕ_N is even. Then, in order to deal with this case, we symmetrize the multiplier $a(\xi_1,\dots,\xi_{k+1})$, that is, from now on we consider

$$\widetilde{a}_k(\xi_1,\dots,\xi_{k+1}):=[a_k(\xi_1,\dots,\xi_{k+1})]_{\mathrm{sym}}=\frac{i}{k+1}\sum_{i=1}^{k+1}\phi_N^2(\xi_i)\xi_i.$$

Notice that, since $\phi_N^2(\xi_1)\xi_1+\phi_N^2(\xi_2)\xi_2\equiv 0$ on Γ^2, the case $k=1$ immediately vanishes, and hence, from now on we can assume that $k\geq 2$. Thus, by using frequency decomposition and the above symmetrization, the problem of bounding (3.5) is reduced to control the following quantity:

$$\begin{aligned}\text{(3.6)}\qquad &\sum_{N>0}\omega_N^2\langle N\rangle^{2s}\\&\times\sup_{t\in(0,T)}\Big\vert\int_0^t\sum_{N_1,\dots,N_{k+1}}\int_{\Gamma^{k+1}}\widetilde{a}_k(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^{k+1}\phi_{N_i}(\xi_i)\hat{u}(t',\xi_i)\,d\Gamma^{k+1}\,dt'\Big\vert.\end{aligned}$$

Moreover, by symmetry, without loss of generality we can always assume that[2]

$$N_1\geq N_2\geq N_3\geq N_4=\max\{N_4,\dots,N_{k+1}\}.$$

Before going further, note that the case $N\lesssim 1$ can be treated right away. In fact, from Lemma 2.9 and Bernstein's inequality, we see that[3]

$$\begin{aligned}&\sum_{N\lesssim 1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big\vert\int_0^t\sum_{N_1,\dots,N_{k+1}}\int_{\Gamma^{k+1}}\widetilde{a}_k(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^{k+1}\phi_{N_i}(\xi_i)\hat{u}(t',\xi_i)\,d\Gamma^{k+1}dt'\Big\vert\\&\lesssim_k\int_0^T\sum_{N\lesssim 1}\omega_N^2\langle N\rangle^{2s}\sum_{N_1,\dots,N_{k+1}}\Big\vert\int_{\Gamma^{k+1}}\widetilde{a}_k(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^{k+1}\phi_{N_i}(\xi_i)\hat{u}(t',\xi_i)\,d\Gamma^{k+1}\Big\vert dt'\end{aligned}$$

[2] If $k=2$, we only assume $N_1\geq N_2\geq N_3$. Notice also that this assumption shall introduce a factor k^4 into the following estimates.

[3] Notice that here, and in all of the bounds below, we obtain a constant c^{k-1} coming from using Sobolev's embedding $k-1$ times.

$$\lesssim_k \int_0^T \sum_{N_1,\ldots,N_{k+1}} \omega_N^2 N_3 \|P_{N_1}u(t',\cdot)\|_{L_x^2}\|P_{N_2}u(t',\cdot)\|_{L_x^2}\prod_{i=3}^{k+1} N_i^{1/2}\|P_{N_i}u(t',\cdot)\|_{L_x^2}\,dt'$$

$$\lesssim_k \int_0^T \|u(t',\cdot)\|_{H_\omega^s}^2\|u(t',\cdot)\|_{H_x^{1/2+}}^{k-1}\,dt' \lesssim_k T\|u\|_{L_T^\infty H_\omega^s}^2\|u\|_{L_T^\infty H_x^{1/2+}}^{k-1}.$$

Therefore, in the sequel we just need to consider the sum over frequencies $N \gg 1$. More precisely, from now on we assume that $N \geq 8^8 k$. On the other hand, from the explicit form of $\widetilde{a}_k$, it is not difficult to see that $\widetilde{a}_k \equiv 0$, unless $N_1 \geq \frac{1}{2}N$. Furthermore, due to the additional constraint[4] imposed by Γ^{k+1}, we must also have that $N_2 \geq \frac{1}{2k}N_1$. Therefore, roughly (up to a constant involving k), we have that $N_1 \sim N_2$ with $N_1 \geq \frac{1}{2}N$.[5] Then we split the analysis into three possible cases. First, we divide the space into two regions, namely, either

$$N_3 \geq 2^9 kN_4 \quad \text{or} \quad N_3 < 2^9 kN_4. \tag{3.7}$$

Then, only for the second case, we split the space again into two regions, namely,

$$N_1 < 8kN \quad \text{and} \quad N_1 \geq 8kN. \tag{3.8}$$

The only reason why we separate both cases in (3.8) is to be able to justify how we sum over the set $N \gg 1$; they can certainly be treated simultaneously though. We choose to separate them for the sake of clarity.

Before getting into the details, let us introduce some notation for each of the regions under study. From now on we set[6]

$$\begin{aligned}
\mathbf{N}^1 &:= \big\{(N_1,\ldots,N_{k+1}) \in \mathbb{D}^{k+1} : N_3 \geq 2^9 kN_4\big\},\\
\mathbf{N}^2 &:= \big\{(N_1,\ldots,N_{k+1}) \in \mathbb{D}^{k+1} : N_1 < 8kN \text{ and } N_3 < 2^9 kN_4\big\},\\
\mathbf{N}^3 &:= \big\{(N_1,\ldots,N_{k+1}) \in \mathbb{D}^{k+1} : N_1 \geq 8kN \text{ and } N_3 < 2^9 kN_4\big\},
\end{aligned}$$

and denote by $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$, the corresponding contribution of (3.6) associated with each of these regions, respectively.[7]

Notice that all of the above regions require that $k \geq 3$ to be well defined. However, we point out that the case $k = 2$ shall follow directly from the analysis that we shall carry out to deal with the first region above, that is, the region $\mathbf{N}^1$.[8]

Step 2.1. In this first sub-step we seek to deal with the first case in (3.7), that is, to control the contribution of the region $N_3 \geq 2^9 kN_4$. We aim to take advantage of classical

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[4]By this we mean the condition $\xi_1 + \cdots + \xi_{k+1} = 0$. In the sequel, each time we mention "the constraint imposed by Γ^k", we refer to the previous condition with k frequencies.

[5]Notice that, in the sequel, we shall repeatedly use these relations to absorb factors like $\langle N\rangle^s$ with $\|P_{N_2}u\|_{L^2}$, in the sense that we shall write $\langle N\rangle^s\|P_{N_2}u\|_{L^2} \lesssim_k \|P_{N_2}u\|_{H^s}$. Due to the above relations, in the worst case this type of bounds shall involve a factor k^s due to the use of $N_2 \sim N$.

[6]Recall that we are also assuming that $N_1 \geq N_2 \geq N_3 \geq N_4 = \max\{N_4,\ldots,N_{k+1}\}$.

[7]That is, the quantity obtained once restricting the inner sum in (3.6) to $\mathbf{N}^1$ and $\mathbf{N}^2$, respectively.

[8]In other words, roughly speaking, when $k = 2$, we could think of N_4 as being equal to 0, and hence the relation that defines $\mathbf{N}^1$ is always satisfied. Hence, if $k = 2$, we only have one case, which corresponds to $\mathbf{N}^1$.

Bourgain estimates. In order to do so, we begin using the decomposition given in (2.11), from which we infer that it is enough to control the following quantities:

$$\mathcal{G}_{1,R}^{\mathrm{high}} := \sum_{N\gg 1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{\widetilde{a}_k}\big(\mathbf{1}_{t,R}^{\mathrm{high}}P_{N_1}u,\mathbf{1}_t P_{N_2}u,\dots,P_{N_k}u\big)P_{N_{k+1}}u\Big|,$$

$$\mathcal{G}_{1,R}^{\mathrm{low,high}} := \sum_{N\gg 1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}$$
$$\times\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{\widetilde{a}_k}\big(\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\mathbf{1}_{t,R}^{\mathrm{high}}P_{N_2}u,P_{N_3}u,\dots,P_{N_k}u\big)P_{N_{k+1}}u\Big|,$$

$$\mathcal{G}_{1,R}^{\mathrm{low,low}} := \sum_{N\gg 1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}$$
$$\times\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{\widetilde{a}_k}\big(\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,P_{N_3}u,\dots,P_{N_k}u\big)P_{N_{k+1}}u\Big|,$$

where R stands for a large real number that shall be fixed later. For the sake of clarity, we split the analysis into two steps. Before getting into it, let us recall the definition of the resonant relation for $(k+1)$-terms, which is given by

$$\Omega_k(\xi_1,\dots,\xi_{k+1}) = \xi_1^3+\cdots+\xi_{k+1}^3.$$

We emphasize that, as an abuse of notation, sometimes we also write Ω_k with only k entries. However, in that case, Ω_k is given by $\xi_1^3+\cdots+\xi_k^3-(\xi_1+\cdots+\xi_k)^3$. Notice that both definitions are equivalent due to the constraint imposed by Γ^k.

Step 2.1.1. We begin by considering the case of $\mathcal{G}_{1,R}^{\mathrm{high}}$. The idea is to take advantage of the operator $\mathbf{1}_{t,R}^{\mathrm{high}}$ using Lemma 2.6. In fact, by choosing

$$R(N,N_1,\dots,N_{k+1}) := N_1N_3, \tag{3.9}$$

we can bound $\mathcal{G}_{1,R}^{\mathrm{high}}$ using the first inequality in Lemma 2.6, Lemma 2.9, as well as Sobolev's embedding, in the following fashion:

$$\begin{aligned}
\mathcal{G}_{1,R}^{\mathrm{high}} &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^1}T^{1/4}\omega_N^2\langle N\rangle^{2s}\|\mathbf{1}_{T,R}^{\mathrm{high}}\|_{L^{4/3}}\Big\|\int_{\mathbb{R}}\Pi_{\widetilde{a}_k}(P_{N_1}u,\dots,P_{N_k}u)P_{N_{k+1}}u\Big\|_{L^\infty}\\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^1}T^{1/4}\omega_N^2\langle N\rangle^{2s}N_3R^{-3/4}\|P_{N_1}u\|_{L_t^\infty L_x^2}\|P_{N_2}u\|_{L_t^\infty L_x^2}\prod_{i=3}^{k+1}\|P_{N_i}u\|_{L^\infty_{t,x}}\\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^1}T^{1/4}N_1^{-1/2}\|P_{N_1}u\|_{L_t^\infty H_\omega^s}\|P_{N_2}u\|_{L_t^\infty H_\omega^s}\\
&\qquad\times\prod_{i=3}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}\\
&\lesssim_k T^{1/4}\|u\|_{L_t^\infty H_\omega^s}^2\|u\|_{L_t^\infty H_x^{1/2+}}^{k-1},
\end{aligned}$$

where we have used the fact that $\omega_N/\omega_{N_i}\lesssim 1$, $i=1,2$, thanks to the hypothesis on ω_N in Section 2.2. To finish this first case, we point out that, thanks to the operator $\mathbf{1}_{t,R}^{\mathrm{high}}$ acting on the factor $P_{N_2}u$, the same estimates also hold for $\mathcal{G}_{1,R}^{\mathrm{low,high}}$.

Step 2.1.2. Now we consider the last term in the decomposition, that is, $\mathcal{G}_{1,R}^{\mathrm{low,low}}$. In fact, first of all, for the sake of notation, let us define the following functional

$$\mathcal{I}(u_1,\dots,u_{k+1}):=\sum_{N\gg1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{\widetilde{a}_k}(u_1,\dots,u_k)u_{k+1}\Big|.$$

Then we claim that, due to the relationship between the frequencies belonging to $\mathbf{N}^1$, the resonant relation satisfies

$$|\Omega_k(\xi_1,\dots,\xi_k)|\sim N_1N_2N_3.$$

In fact, let us start by recalling that, due to the additional constraint imposed by Γ^{k+1}, we have the relation $\xi_1+\cdots+\xi_{k+1}=0$. Then, by using the bound $N_3>2^9kN_4$, we deduce

$$\begin{aligned}|\Omega_k(\xi_1,\dots,\xi_k)|&=|\xi_1^3+\xi_2^3+\xi_3^3+\cdots+\xi_{k+1}^3|\\&=|\xi_2^3+\xi_3^3-(\xi_2+\xi_3+\cdots+\xi_{k+1})^3|+O(N_4^3)\\&=3|(\xi_2+\xi_3)\xi_2\xi_3|+O(N_1^2N_4)\\&=3|\xi_1\xi_2\xi_3|+O(N_1^2N_4)\sim N_1N_2N_3.\end{aligned}\tag{3.10}$$

Therefore, taking advantage of the above relation, we can now decompose $\mathcal{G}_{1,R}^{\mathrm{low,low}}$ with respect to modulation variables in the following fashion:

$$\begin{aligned}|\mathcal{G}_{1,R}^{\mathrm{low,low}}|&\le\mathcal{I}\big(Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,P_{N_3}u,\dots,P_{N_{k+1}}u\big)\\&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,P_{N_3}u,\dots,P_{N_{k+1}}u\big)\\&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,Q_{\gtrsim N^*}P_{N_3}u,\dots,P_{N_{k+1}}u\big)\\&\quad\ \vdots\\&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,Q_{\ll N^*}P_{N_3}u,\dots,Q_{\gtrsim N^*}P_{N_{k+1}}u\big)\\&=:\mathcal{I}_1+\cdots+\mathcal{I}_{k+1},\end{aligned}$$

where N^* stands for $N^*:=N_1N_2N_3$. At this point it is important to notice that, since in this case we have $N_2\ge\frac18N_1$, then we must also have that $N^*\gg N_1N_3=R$, which allows us to use the last inequality in Lemma 2.6. Thus, bounding in a similar fashion as before, by using the Hölder and Bernstein inequalities, as well as Lemma 2.6, Lemma 2.9 and the classical Bourgain estimates, we obtain

$$\begin{aligned}\mathcal{I}_1&\lesssim_k\sum_{N\gg1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}N_3\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{L_t^2L_x^2}\|\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}u\|_{L_t^2L_x^2}\prod_{i=3}^{k+1}\|P_{N_i}u\|_{L_t^\infty L_x^\infty}\\&\lesssim_k\sum_{N\gg1}\sum_{\mathbf{N}^1}N_2^{(-1)^+}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{X^{s-1,1}}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\\&\qquad\times\|P_{N_2}u\|_{L_t^\infty H_x^s}\prod_{i=3}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}\\&\lesssim_k T^{1/2}\|u\|_{X^{s-1,1}}\|u\|_{L_t^\infty H_x^s}\|u\|_{L_t^\infty H_x^{1/2+}}^{k-1}.\end{aligned}$$

Notice that, we have absorbed ω_N with $N_2^{-\varepsilon}$ thanks to the assumptions made in Section 2.2. Moreover, it is not difficult to see that, by following the same lines (up to trivial modifications), we can also bound $\mathcal{I}_2$, obtaining the same bound. On the other hand, to control $\mathcal{I}_3$, we use again both Lemma 2.6 and 2.9, as well as the Hölder and Bernstein inequalities to obtain

$$\begin{aligned}
\mathcal{I}_3 &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^1}\omega_N^2\langle N\rangle^{2s}N_3\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{L_t^2L_x^2} \\
&\quad\times\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}u\|_{L_t^\infty L_x^2}\|Q_{\gtrsim N^*}P_{N_3}u\|_{L_t^2L_x^\infty}\prod_{i=4}^{k+1}\|P_{N_i}u\|_{L_t^\infty L_x^\infty} \\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^1}N_2^{(-1)^+}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_1}u\|_{L_t^\infty H_x^s}\|P_{N_2}u\|_{L_t^\infty H_x^s}\min\{N_3^{1^-}N_1^{-1},N_3^{-(0^+)}\} \\
&\quad\times\|Q_{\gtrsim N^*}P_{N_3}u\|_{X^{(-1/2)+,1}}\prod_{i=4}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}} \\
&\lesssim_k T^{1/2}\|u\|_{L_t^\infty H_x^s}^2\|u\|_{X^{(-1/2)+,1}}\|u\|_{L_t^\infty H_x^{1/2+}}^{k-2}.
\end{aligned}$$

Notice that all the remaining cases $\mathcal{I}_i$, $i=4,\dots,k+1$, follow very similar lines to the latter case (up to trivial modifications), and they provide exactly the same bound. We omit the proof of these cases.

Step 2.2. Now we aim to deal with the region $N_1<8kN$. In fact, in this case, it is enough to notice that, combining both the hypotheses and the additional constraint imposed by Γ^{k+1}, we can write[9]

$$N_1\in[\tfrac{1}{2}N,4kN]\quad\text{and}\quad N_2\in[\tfrac{1}{2k}N_1,N_1].$$

Therefore, up to a factor k, we deduce that $N_1\sim N$ and $N_2\sim N$. Hence, by using the Hölder and Bernstein inequalities, as well as Lemma 2.9, we get that

$$\begin{aligned}
|\mathcal{G}_2| &\lesssim_k \int_0^T\sum_{N\gg 1}\sum_{\mathbf{N}^2}\omega_N^2\langle N\rangle^{2s}\left|\int_{\Gamma^{k+1}}\widetilde{a}_k(\xi_1,\dots,\xi_{k+1})\prod_{i=1}^{k+1}\phi_{N_i}(\xi_i)\hat{u}(s,\xi_i)\,d\Gamma^{k+1}\right|ds \\
&\lesssim_k \int_0^T\sum_{N\gg 1}\sum_{\mathbf{N}^2}\min\{N_3,N_3^{-(0^+)}\}\|P_{N_1}u(s,\cdot)\|_{H_\omega^s}\|P_{N_2}u(s,\cdot)\|_{H_\omega^s}\|J_x^{1/2^+}P_{N_3}u(s,\cdot)\|_{L_x^\infty} \\
&\quad\times\|J_x^{1/2^+}P_{N_4}u(s,\cdot)\|_{L_x^\infty}\prod_{i=5}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u(s,\cdot)\|_{H_x^{1/2+}}\,ds \\
&\lesssim_k \|u\|_{L_T^\infty H_\omega^s}^2\|J_x^{1/2^+}u\|_{L_T^2L_x^\infty}^2\|u\|_{L_T^\infty H_x^{1/2+}}^{k-3}.
\end{aligned}$$

We emphasize that, in this case, to sum over $N\gg 1$ we have used the fact that, for any function $f\in H_\omega^s(\mathbb{R})$, the sequence $\{\|P_{2^n}f\|_{H_\omega^s}\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$.

[9] Notice that this shall introduce a factor k into the following estimates.

Step 2.3. Finally, it only remains to consider the case where $N_1 \geq 8kN$. In fact, in this case, note that inequality $N_1 \geq 8kN$ implies, in particular, that $N_2 \geq 4N$, and hence we must also have $N_3 \geq \frac{1}{2}N$, otherwise $\widetilde{a}_k \equiv 0$. Then we can proceed similarly as in the latter step but using the factor N_3^- to sum over $N \gg 1$. Note that, in this case, we also have to use the fact that $\|P_{N_1}u(s,\cdot)\|_{H^s_\omega}$ and $\|P_{N_2}u(s,\cdot)\|_{H^s_\omega}$ are both square summable. The proof of Step 3 is finished.

Step 3: Finally, we consider the general case $\mathrm{I}_{u^k\Psi^m}$, where $k, m \geq 1$. As we have mentioned before, for small frequencies $N \lesssim 1$ we can directly bound the sum by simply using Hölder's inequality as follows:

$$\begin{aligned} &\sum_{N\lesssim 1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\mathbb{R}}\partial_x P_N(u^k\Psi^m)P_Nu\Big| \\ &\qquad \lesssim_k T\|u\|^2_{L^\infty_tH^s_\omega}\|u\|^{k-1}_{L^\infty_tH^{1/2+}_x}\|\Psi\|^m_{L^\infty}. \end{aligned} \tag{3.11}$$

Therefore, in the sequel we only consider the case where $N \gg k$.[10] On the other hand, notice that, by using Plancherel's theorem, we can rewrite the remaining quantity as

$$\sum_{N\gg 1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\int_{\Gamma^{k+2}}\mathbf{a}_k(\xi_1,\dots,\xi_{k+2})\hat{u}(\xi_1)\cdots\hat{u}(\xi_{k+1})\widehat{\Psi}^m(\xi_{k+2})\,d\Gamma^{k+2}\Big|, \tag{3.12}$$

where the symbol $\mathbf{a}_k(\xi_1,\dots,\xi_{k+2})$ is explicitly given by

$$\mathbf{a}_k(\xi_1,\dots,\xi_{k+2}) := i\phi_N^2(\xi_{k+1})\xi_{k+1}.$$

In the same spirit as for Steps 2, in order to deal with this case, we perform a symmetrization argument. Indeed, by symmetrizing the symbol we are led to consider

$$\tilde{\mathbf{a}}_k(\xi_1,\dots,\xi_{k+2}) = [\mathbf{a}_k(\xi_1,\dots,\xi_{k+2})]_{\mathrm{sym}} := \frac{i}{k+1}\sum_{i=1}^{k+1}\phi_N^2(\xi_i)\xi_i.$$

Then, by using frequency decomposition, the problem of bounding (3.12) is reduced to control the following quantity

$$\begin{aligned} &\sum_{N\gg 1}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_0^t\sum_{N_1,\dots,N_{k+2}}\int_{\Gamma^{k+2}}\tilde{\mathbf{a}}_k(\xi_1,\dots,\xi_{k+2})\phi_{N_{k+2}}(\xi_{k+2}) \\ &\qquad\times\widehat{\Psi}^m(\xi_{k+2})\prod_{i=1}^{k+1}\phi_{N_i}(\xi_i)\hat{u}(\xi_i)\Big|. \end{aligned} \tag{3.13}$$

Hence, by symmetry, without loss of generality, from now on we assume that $N_1 \geq N_2 \geq N_3 \geq N_4 = \max\{N_4,\dots,N_{k+1}\}$.[11] We point out that, in this case, we consider $N_{k+2} \in \mathbb{D}_{\mathrm{nh}}$.[12] Before going further, notice that there is an important case that can be

[10] Notice that this introduces another factor k into inequality (3.11) coming from the use of $N \lesssim 1$ when controlling the operator ∂_x.

[11] For the cases $k = 1, 2$ we only assume that $N_1 \geq N_2$ and $N_1 \geq N_2 \geq N_3$, respectively. Once again, notice that these assumptions introduces a factor k^4 into the following estimates.

[12] Since $N_{k+2} \in \mathbb{D}_{\mathrm{nh}}$, when $N_{k+2} = 1$ we consider $\eta(\xi_{k+2})$ instead of $\phi_{N_{k+2}}(\xi_{k+2})$ in (3.13), where $\eta(\cdot)$ is defined in (2.1).

treated without any further decomposition. In fact, let us consider the region $8^9 k N_{k+2} \geq N$. We begin by restricting ourselves to the case $N_2 \geq 1$. Let us denote the set of indexes associated with all the above constraints by $\mathbf{N}_{k+2}$. Then, by using Plancherel's theorem to go back to physical variables, taking advantage of the fact that $\Psi \in W_x^{(s+1)^+,\infty}$, we can control $\mathrm{I}_{u^k\Psi^m}$, in this region, by[13]

$$\begin{aligned}
&\sum_{N\gg 1} \omega_N^2 \langle N\rangle^{2s} \sup_{t\in(0,T)} \bigg| \int_0^t \sum_{\mathbf{N}_{k+2}} \int_{\Gamma^{k+2}} \tilde{\mathbf{a}}_k(\xi_1,\dots,\xi_{k+2}) \phi_{N_{k+2}}(\xi_{k+2}) \\
&\qquad \times \widehat{\Psi}^m(\xi_{k+2}) \prod_{i=1}^{k+1} \phi_{N_i}(\xi_i)\hat{u}(\xi_i) \bigg| \\
&\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}_{k+2}} \omega_N^2 \langle N\rangle^{2s} N N_{k+2}^{-(s+1)^+} \| P_{N_1} u(t',\cdot)\|_{L^2_x} \\
&\qquad \times \| P_{N_2} u(t',\cdot)\|_{L^2_x} \| P_{N_{k+2}}(\Psi^m(t',\cdot))\|_{W_x^{(s+1)+,\infty}} \prod_{i=3}^{k+1} \| P_{N_i} u(t',\cdot)\|_{L^\infty_x} \, dt' \\
&\lesssim_k T \|u\|_{L^\infty_T H^s_\omega}^2 \|u\|_{L^\infty_T H_x^{1/2+}}^{k-1} \|\Psi\|^m_{L^\infty_T W_x^{(s+1)+,\infty}}.
\end{aligned}$$

Here, we have absorbed one of the factors ω_N with $N_{k+1}^{-\varepsilon}$, thanks to the hypotheses made in Section 2.2. Notice that, to deal with the case $N_2 \leq 1$, it is enough to sum over N_2 inside the absolute value (before using Hölder's inequality), so that we obtain a factor $\|P_{\lesssim 1} u\|_{L^2}$ in the right-hand side (without any series in N_2). Therefore, in the sequel, we can assume that $8^9 k N_{k+2} < N$. Also, in this region, we have $\phi_N(\xi_{k+2}) \equiv 0$, and hence we can write

$$\tilde{\mathbf{a}}_k(\xi_1,\dots,\xi_{k+2}) = \frac{i}{k+1} \sum_{i=1}^{k+2} \phi_N^2(\xi_i)\xi_i.$$

This is somehow important since it shall allow us to use Lemma 2.9 with no problems.

Now, in the same spirit as in Step 2, we split the analysis into several cases, namely,

(1) $N_3 < 8^8 k N_{k+2}$,

(2) $N_3 \geq 8^8 k N_{k+2}$ and $N_3 \geq 2^9 k N_4$,

(3) $N_3 \geq 8^8 k N_{k+2}$ and $N_3 < 2^9 k N_4$.

Before getting into the details, let us introduce the notation for each of these regions. From now on, we denote by $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ the set of indexes associated with each of these regions,[14] and by $\mathrm{G}_1, \mathrm{G}_2$ and G_3, the corresponding contribution of (3.13) associated with each one of them, respectively.

In the same spirit as in Step 2, notice that all of the above regions require that $k \geq 3$ to be well defined. However, the case $k = 1$ shall follow directly from the analysis we shall carry out to deal with the first region above, that is, the region $\mathcal{N}_1$. On the other hand, the case $k = 2$ shall follow from the analysis associated with cases (1) and (2) above.[15]

[13] Notice that here we obtain another factor k^{s+1+} coming from the relation $8^9 k N_{k+2} \geq N$.

[14] Recall we are also assuming that $N_1 \geq N_2 \geq N_3 \geq N_4 = \max\{N_4,\dots,N_{k+1}\}$ and $9^9 k N_{k+2} < N$.

[15] In other words, roughly speaking, when $k = 1$, we could think of N_3 as being equal to 0, and hence the inequality of the first case is always satisfied, while when $k = 2$, we could think of N_4 being zero, and hence we still have two cases, namely, (1) and (2).

Step 3.1. We begin by studying the contribution of $I_{u^k\Psi^m}$ in the region $\mathcal{N}_1$. In fact, notice that, in this case, due to the hypotheses $\Psi \in W^{(s+1)^+,\infty}$ as well as the fact that $N_{k+2} \gtrsim N_3 \geq \max\{N_4, \dots, N_{k+1}\}$, we can control the whole sum G_1 directly from Lemma 2.9 and then use Bernstein inequalities to get the bound

$$
\begin{aligned}
|\mathrm{G}_1| &\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathcal{N}_1} \omega_N^2 \langle N\rangle^{2s} N_{k+2}(N_3 \dots . N_{k+1})^{1/2} \|P_{N_1}u(t',\cdot)\|_{L^2_x} \\
&\qquad \times \|P_{N_2}u(t',\cdot)\|_{L^2_x} \|P_{N_{k+2}}(\Psi^m(t',\cdot))\|_{L^\infty_x} \prod_{i=3}^{k+1} \|P_{N_i}u(t',\cdot)\|_{L^2_x}\, dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathcal{N}_1} N_{k+2}^{-(0^+)} \|P_{N_1}u(t',\cdot)\|_{H^s_\omega} \|P_{N_2}u(t',\cdot)\|_{H^s_\omega} \\
&\qquad \times \|P_{N_{k+2}}(\Psi^m(t',\cdot))\|_{W^{1+,\infty}_x} \prod_{i=3}^{k+1} \min\{N_i^{1/2}, N_i^{-(0^+)}\} \|P_{N_i}u(t',\cdot)\|_{H^{1/2+}_x}\, dt' \\
&\lesssim_k T\|u\|^2_{L^\infty_T H^s_\omega} \|u\|^{k-1}_{L^\infty_T H^{1/2+}_x} \|\Psi^m\|_{L^\infty_t W^{1+,\infty}_x}.
\end{aligned}
$$

We point out that, in the estimates above, to sum over the indexes N, N_1 and N_2, we have used the fact that $N_1 \in [\frac{1}{2}N, 4kN]$ and $N_2 \in [\frac{1}{2k}N_1, N_1]$.[16]

Step 3.2. Now we seek to control the contribution of (3.13) in the region $\mathcal{N}_2$. We aim to take advantage of classical Bourgain estimates. Similarly, as in the previous steps, we begin using the decomposition given in (2.11), from which we infer that it is enough to control the following quantities:

$$
\begin{aligned}
\mathrm{G}^{\mathrm{high}}_{2,R} &:= \sum_{N\gg 1}\sum_{\mathcal{N}^2} \omega_N^2\langle N\rangle^{2s} \\
&\qquad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{\tilde{\mathbf{a}}_k}\big(\mathbf{1}^{\mathrm{high}}_{t,R} P_{N_1}u, \mathbf{1}_t P_{N_2}u, P_{N_3}u, \dots, P_{N_{k+1}}u\big) P_{N_{k+2}}\Psi \Big|, \\
\mathrm{G}^{\mathrm{low,high}}_{2,R} &:= \sum_{N\gg 1}\sum_{\mathcal{N}^2} \omega_N^2\langle N\rangle^{2s} \\
&\qquad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{\tilde{\mathbf{a}}_k}\big(\mathbf{1}^{\mathrm{low}}_{t,R} P_{N_1}u, \mathbf{1}^{\mathrm{high}}_{t,R} P_{N_2}u, P_{N_3}u, \dots, P_{N_{k+1}}u\big) P_{N_{k+2}}\Psi \Big|, \\
\mathrm{G}^{\mathrm{low,low}}_{2,R} &:= \sum_{N\gg 1}\sum_{\mathcal{N}^2} \omega_N^2\langle N\rangle^{2s} \\
&\qquad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{\tilde{\mathbf{a}}_k}\big(\mathbf{1}^{\mathrm{low}}_{t,R} P_{N_1}u, \mathbf{1}^{\mathrm{low}}_{t,R} P_{N_2}u, P_{N_3}u, \dots, P_{N_{k+1}}u\big) P_{N_{k+2}}\Psi \Big|,
\end{aligned}
$$

where R stands for a large real number to be fixed. We split the analysis into two steps.

Step 3.2.1. We start by bounding $\mathrm{G}^{\mathrm{high}}_{2,R}$. We shall proceed in a similar fashion as in Step 2.1.1. In fact, we define again $R(N, N_1, \dots, N_{k+2}) := N_1N_3$. Then, by using the

[16] Once again, this introduces a factor k into the previous estimates. We stress that to avoid repeating arguments. In the sequel we shall no longer point out these dependencies.

first inequality in Lemma 2.6, Lemma 2.9, as well as Sobolev's embedding, we obtain

$$\begin{aligned}
\mathrm{G}_{2,R}^{\mathrm{high}} &\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2} T^{1/4}\omega_N^2\langle N\rangle^{2s}\|\mathbf{1}_{T,R}^{\mathrm{high}}\|_{L^{4/3}}\Big\|\int_{\mathbb{R}}\Pi_{\tilde{\mathbf{a}}_k}(P_{N_1}u,\dots,P_{N_{k+1}}u)P_{N_{k+2}}\Psi\Big\|_{L^\infty}\\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2} T^{1/4}\omega_N^2\langle N\rangle^{2s}N_2^{-1/2}\|P_{N_1}u\|_{L^\infty_tL^2_x}\|P_{N_2}u\|_{L^\infty_tL^2_x}\\
&\quad\times\|P_{N_{k+2}}\Psi\|_{L^\infty}\prod_{i=3}^{k+1}\|P_{N_i}u\|_{L^\infty_{t,x}}\\
&\lesssim_k T^{1/4}\|u\|_{L^\infty_tH^s_\omega}^2\|u\|_{L^\infty_tH^{1/2+}_x}^{k-1}\|\Psi\|_{L^\infty_TL^\infty_x}.
\end{aligned}$$

To finish this first case, we point out that, thanks to the operator $\mathbf{1}_{t,R}^{\mathrm{high}}$ acting on the factor $P_{N_2}u$, the same estimates also hold for $\mathrm{G}_{2,R}^{\mathrm{low,high}}$.

Step 3.2.2. To conclude the proof of Step 3.2 it only remains to consider $\mathcal{G}_{1,R}^{\mathrm{low,low}}$. As before, we begin by introducing some useful notation. We denote by $\mathcal{I}$ the functional given by

$$\mathcal{I}(u_1,\dots,u_{k+2}) := \sum_{N\gg 1}\sum_{\mathcal{N}^2}\omega_N^2\langle N\rangle^{2s}\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{\tilde{\mathbf{a}}_k}(u_1,\dots,u_{k+1})u_{k+2}\Big|.$$

Now notice that, proceeding in the exact same fashion as in (3.10), together with the fact that, in this case, $N_3\geq\max\{8^8kN_{k+2},2^9kN_4\}$, provides the relation

$$|\Omega_{k+1}(\xi_1,\dots,\xi_{k+2})|\sim N_1N_2N_3.$$

Thus, in order to take advantage of the above relation, we decompose $\mathrm{G}_{2,R}^{\mathrm{low,low}}$ with respect to modulation variables in the following fashion:

$$\begin{aligned}
|\mathrm{G}_{2,R}^{\mathrm{low,low}}| &\leq \mathcal{I}\big(Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\,\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,\,P_{N_3}u,\dots,P_{N_{k+1}}u,\,P_{N_{k+2}}\Psi\big)\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\,Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,\,P_{N_3}u,\dots,P_{N_{k+1}}u,\,P_{N_{k+2}}\Psi\big)\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,\,Q_{\gtrsim N^*}P_{N_3}u,\dots,P_{N_{k+1}}u,\,P_{N_{k+2}}\Psi\big)\\
&\quad\vdots\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,\,Q_{\ll N^*}P_{N_3}u,\dots,Q_{\gtrsim N^*}P_{N_{k+1}}u,\,P_{N_{k+2}}\Psi\big)\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}u,\,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}u,\,Q_{\ll N^*}P_{N_3}u,\dots,Q_{\ll N^*}P_{N_{k+1}}u,\,P_{N_{k+2}}\Psi\big)\\
&\qquad=:\mathcal{I}_1+\dots+\mathcal{I}_{k+2},
\end{aligned}$$

where once again $N^*:=N_1N_2N_3$. At this point it is important to notice that, since in this case we have $N_2\geq\frac{1}{8}N\gg 1$, then we must also have $N^*\gg N_1N_3=R$, which allows us to use the last inequality in Lemma 2.6. Thus, bounding in a similar fashion as before, by using the Hölder and Bernstein inequalities, as well as Lemma 2.6, Lemma 2.9 and the classical Bourgain estimates, we obtain

$$\begin{aligned}
\mathcal{I}_1 &\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2}\omega_N^2\langle N\rangle^{2s}N_3\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{L^2_tL^2_x}\|\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}u\|_{L^2_tL^2_x}\\
&\quad\times\|P_{N_{k+2}}(\Psi^m)\|_{L^\infty_{t,x}}\prod_{i=3}^{k+1}\|P_{N_i}u\|_{L^\infty_{t,x}}
\end{aligned}$$

$$\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2} N_2^{(-1)^+}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{X^{s-1,1}}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_2}u\|_{L_t^\infty H_x^s}$$
$$\times\|P_{N_{k+2}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{i=3}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}$$
$$\lesssim_k T^{1/2}\|u\|_{X^{s-1,1}}\|u\|_{L_t^\infty H_x^s}\|u\|^{k-1}_{L_t^\infty H_x^{1/2+}}\|\Psi\|^m_{L_{t,x}^\infty}.$$

It is not difficult to see that, by following the same lines (up to trivial modifications), we can also bound $\mathcal{J}_2$, obtaining the same bound. On the other hand, to control $\mathcal{J}_3$, we use again both Lemma 2.6 and 2.9, as well as the Hölder and Bernstein inequalities to obtain

$$\mathcal{J}_3\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2}\omega_N^2\langle N\rangle^{2s}N_3\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}u\|_{L_t^2L_x^2}\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}u\|_{L_t^\infty L_x^2}$$
$$\times\|Q_{\gtrsim N^*}P_{N_3}u\|_{L_t^2L_x^\infty}\|P_{N_{k+2}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{i=4}^{k+1}\|P_{N_i}u\|_{L_t^\infty L_x^\infty}$$
$$\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2}N_2^{(-1)^+}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_1}u\|_{L_t^\infty H_x^s}\|P_{N_2}u\|_{L_t^\infty H_x^s}$$
$$\times\min\{N_3^{1^-}N_1^{-1},N_3^{-(0^+)}\}\|Q_{\gtrsim N^*}P_{N_3}u\|_{X^{(-1/2)+,1}}\|P_{N_{k+2}}(\Psi^m)\|_{L_{t,x}^\infty}$$
$$\times\prod_{i=4}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}$$
$$\lesssim_k T^{1/2}\|u\|^2_{L_t^\infty H_x^s}\|u\|_{X^{(-1/2)+,1}}\|u\|^{k-2}_{L_t^\infty H_x^{1/2+}}\|\Psi\|^m_{L_{t,x}^\infty}.$$

Notice that all the remaining cases $\mathcal{J}_i$, $i=4,\dots,k+1$, follow very similar lines to the latter case (up to trivial modifications), and hence we omit them. Finally, to control $\mathcal{J}_{k+2}$, notice that, since all factors $P_{N_i}u$ have an operator $Q_{\ll N^*}$ in front of them, the factor $P_{N_{k+2}}(\Psi^m)$ is forced to be resonant, and hence in this case we can write $Q_{\gtrsim N^*}P_{N_{k+2}}(\Psi^m)=P_{N_{k+2}}(\Psi^m)$, otherwise $\mathcal{J}_{k+2}=0$ thanks to Lemma 2.7. Moreover, notice also that in the region $\mathcal{N}^2$, we have, in particular, $N_1N_2N_3\gg N_{k+2}^3$, and hence we infer that $|\tau_{k+2}-\xi_{k+2}^3|\sim|\tau_{k+2}|$, thus we actually have $P_{N_{k+2}}(\Psi^m)=R_{\gtrsim N^*}P_{N_{k+2}}(\Psi^m)$. Therefore, by using Lemmas 2.6, 2.9 as well as Bernstein's inequality and the above properties, we obtain

$$\mathcal{J}_{k+2}\lesssim_k \sum_{N\gg 1}\sum_{\mathcal{N}^2}(N_1N_2)^{(-1)^+}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|^2_{L^2}\|P_{N_1}u\|_{L_t^\infty H_x^s}\|P_{N_2}u\|_{L_t^\infty H_x^s}$$
$$\times\|\partial_tR_{\gtrsim N^*}P_{N_{k+2}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{i=3}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}$$
$$\lesssim_k T\|u\|^2_{L_t^\infty H_x^s}\|u\|^{k-1}_{L_t^\infty H_x^{1/2+}}\|\partial_t\Psi\|_{L_{t,x}^\infty}\|\Psi\|^{m-1}_{L_{t,x}^\infty}.$$

Step 3.3. To finish this step it only remains to consider G_3. In this case it is enough to proceed in the same fashion as in Steps 2.2 and 2.3. In fact, note that

$$N_3\geq 8^8(k+1)N_{k+2}\implies N_2\in[\tfrac{1}{2k}N_1,N_1].$$

Therefore, by using the Hölder and Bernstein inequalities, as well as Lemma 2.9, we get

$$
\begin{aligned}
|\mathrm{G}_3| \lesssim_k & \int_0^T \sum_{N\gg 1}\sum_{\mathcal{N}^3}\langle N\rangle^{2s}(N_1N_2)^{-s}\min\{N_3,N_3^{-(0^+)}\}\|P_{N_1}u(t',\cdot)\|_{H^s_\omega} \\
& \times \|P_{N_2}u(t',\cdot)\|_{H^s_\omega}\|J_x^{1/2^+}P_{N_3}u(t',\cdot)\|_{L^\infty_x}\|J_x^{1/2^+}P_{N_4}u(t',\cdot)\|_{L^\infty_x} \\
& \times \|P_{N_{k+2}}(\Psi^m(t',\cdot))\|_{L^\infty_x}\prod_{i=5}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u(t',\cdot)\|_{H_x^{1/2+}}\,dt' \\
\lesssim_k & \|u\|^2_{L^\infty_T H^s_\omega}\|J_x^{1/2^+}u\|^2_{L^2_TL^\infty_x}\|u\|^{k-3}_{L^\infty_TH_x^{1/2+}}\|\Psi\|^m_{L^\infty_{t,x}}.
\end{aligned}
$$

We emphasize that to sum over the indexes N, N_1, N_2 and N_3 in the case $N_3 \ll N_2$, we have used the fact that $\|P_{N_1}u(s,\cdot)\|_{H^s}$ and $\|P_{N_2}u(s,\cdot)\|_{H^s}$ are both square summable, as well as the factor N_3^-, as in the proof of Steps 2.2 and 2.3. The proof of Step 3 is complete.

Now we explain how we control the contribution of the $L^2_TL^\infty_x$ terms. To this end, we use the Strichartz estimate (2.16) with $\delta = 1$, from which we obtain

$$
\begin{aligned}
\|J_x^{1/2^+}u\|_{L^2_TL^\infty_x} \lesssim{}& T^{1/4}\|u\|_{L^\infty_TH_x^{1/2+}} + T^{3/4}\|\partial_t\Psi+\partial_x^3\Psi+\partial_xf(\Psi)\|_{L^\infty_TH_x^{-1/2+}} \\
& + T^{3/4}\|u\|_{L^\infty_TH_x^{1/2+}}\sum_{k=1}^{\infty}kc^k|a_k|\big(\|u\|_{L^\infty_TH_x^{1/2+}}+\|\Psi\|_{L^\infty_TW_x^{1/2+,\infty}}\big)^{k-1}.
\end{aligned}
$$

Gathering all the above estimates, and then using Lemma 2.5, we conclude the proof of the proposition. ■

3.2. A priori estimates for the difference of two solutions

In this subsection we seek to establish the key a priori estimate at the regularity level $s-1$ for the difference of two solutions. In the sequel, we explicitly consider $\omega_N = 1$ for all $N \in \mathbb{D}$, and hence $H^s_\omega(\mathbb{R}) = H^s(\mathbb{R})$.

Proposition 3.2. *Let $s > 1/2$ and $T \in (0,2)$ be fixed. Let $u, v \in L^\infty((0,T),H^s(\mathbb{R}))$ be two solutions to equation (1.4) associated with initial data $u_0, v_0 \in H^s(\mathbb{R})$. Then the following inequality holds*:

$$
\begin{aligned}
\|u-v\|^2_{L^\infty_TH^{s-1}_x} \lesssim{}& \|u_0-v_0\|^2_{H^{s-1}} + T^{1/4}\|u-v\|^2_{L^\infty_TH^{s-1}_x} \\
& \times \mathcal{Q}^*\big(\|u\|_{L^\infty_TH^s_x},\|v\|_{L^\infty_TH^s_x},\|\Psi\|_{L^\infty_tW_x^{s+1,\infty}},\|\partial_t\Psi\|_{L^\infty_{t,x}}\big),
\end{aligned}
$$

where $\mathcal{Q}^*\colon\mathbb{R}^4 \to \mathbb{R}_+$ is a smooth function.

Proof. As before, in order to take advantage of Bourgain spaces, we have to extend the functions u and v from $(0,T)$ to the whole line $\mathbb{R}$. Hence, by using the extension operator, we take extensions $\tilde{u} := \rho_T[u]$ and $\tilde{v} := \rho_T[v]$, supported in $(-2,2)$. For the sake of notation, we drop the tilde in the sequel. On the other hand, we point out that in the sequel we assume that $s \in (1/2,1]$. The case $s > 1$ is simpler and follows very similar arguments.

Now, let $w := u - v$. Then $w(t,x)$ satisfies the equation

$$\partial_t w + \partial_x\big(\partial_x^2 w + f(u+\Psi) - f(v+\Psi)\big) = 0. \tag{3.14}$$

We proceed as in the previous proposition, taking the frequency projector P_N to (3.14) with $N > 0$ dyadic, then taking the L_x^2-scalar product of the resulting equation against $P_N w$ and multiplying the result by $\langle N\rangle^{2s-2}$. Finally, integrating in time on $(0,t)$ for $0 < t < T$, and then applying Bernstein's inequality, we are lead to

$$\|P_N w(t)\|_{H_x^{s-1}}^2 \lesssim \|P_N w_0\|_{H^{s-1}}^2 + \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \int_{\mathbb{R}} P_N\big(f(u+\Psi) - f(v+\Psi)\big)\partial_x P_N w \right|.$$

As before, we split the analysis in several steps, each of which is devoted to different ranges of (k,i). Notice that the philosophy behind the estimates below is the same one from the proof of the last proposition. However, since in this case we have more (different) functions, we must have several more cases as well, since we cannot order all the frequencies appearing in $P_N(u^{k-i}v^{i-1}\Psi w)$, as we did in the previous proposition when there was only u^k.

As we shall see, the estimates above do not depend of how many u^{k-i} or v^{i-1} we have. Thus, to simplify the notation we shall write

$$\sum_{N>0} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \int_{\mathbb{R}} P_N(z_3 \ldots z_k \Psi^m w)\partial_x P_N w \right|,$$

for some $k \geq 3$ and $m \geq 0$, where each z_i denotes either u or v (not necessarily all being the same).

Step 1. Let us start by considering the case where we only have products of z_i, that is, where no power of Ψ is involved. In other words, we seek to bound the following quantity:

$$\sum_{N>0} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \int_{\mathbb{R}} P_N(z_3 \ldots z_k w)\partial_x P_N w \right|, \tag{3.15}$$

where $k \geq 3$. Before going further, we emphasize once again that in the sequel we assume $s \in (1/2, 1]$. Then, in the same fashion as in the previous proposition, we begin by symmetrizing the underlying symbol in (3.15), which allows us to reduce the problem to studying the symbol

$$d_k(\xi_1,\ldots,\xi_k) := \frac{i}{2}\phi_N^2(\xi_1)\xi_1 + \frac{i}{2}\phi_N^2(\xi_2)\xi_2, \tag{3.16}$$

where ξ_1 and ξ_2 denote the frequencies of each of the occurrences of w in (3.15), respectively. Hence, by frequency decomposition, it is enough to control the following quantity:

$$\sum_{N>0} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \sum_{N_1,\ldots,N_k} \int_{\Gamma^k} d_k(\xi_1,\ldots,\xi_k) \times \prod_{i=1}^2 \phi_{N_i}(\xi_i)\hat{w}(\xi_i) \prod_{j=3}^k \phi_{N_j}(\xi_j)\hat{z}_j(\xi_j) \right|. \tag{3.17}$$

Note that, by symmetry, we can always assume $N_1 \geq N_2$, $N_3 \geq N_4 = \max\{N_4, \dots, N_{k+1}\}$. Now, for the sake of simplicity, let us denote by I the following functional:

$$\mathrm{I}(N_1, \dots, N_k, u_1, \dots, u_k) := \int_{\Gamma^k} d_k(\xi_1, \dots, \xi_k)\phi_{N_1}(\xi_1)u_1(\xi_1)\dots\phi_{N_k}(\xi_k)u_k(\xi_k)\, d\Gamma^k.$$

Then, with this notation at hand, we define the set of admissible indexes

$$\mathbf{N}_k := \mathbb{D}^k \setminus \big\{(N_1, \dots, N_k) \in \mathbb{D}^k : \forall (u_1, \dots, u_k) \in H^1(\mathbb{R})^k, \\ \mathrm{I}(N_1, \dots, N_k, u_1, \dots, u_k) = 0\big\}. \tag{3.18}$$

Before going further, let us rule out right away the case $N_2 \lesssim 1$. In fact, if we set $\mathbf{N}_k^2 := \mathbf{N}_k \cap \{N_2 < 8^8 k\}$, then, from Lemma 2.9, Plancherel's theorem and the Hölder and Bernstein inequalities, we obtain

$$\begin{aligned}
&\sum_{N>0} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \Big| \int_0^t \sum_{\mathbf{N}_k^2} \int_{\Gamma^k} d_k(\xi_1, \dots, \xi_k) \prod_{i=1}^2 \phi_{N_i}(\xi_i)\hat{w}(t', \xi_i) \prod_{j=3}^k \phi_{N_j}(\xi_j)\hat{z}_j(t', \xi_j)\Big| \\
&\lesssim_k \int_0^T \sum_{N>0} \sum_{\mathbf{N}_k^2} \langle N\rangle^{2s-2} \Big| \int_{\Gamma^k} d_k(\xi_1, \dots, \xi_k) \prod_{i=1}^2 \phi_{N_i}(\xi_i)\hat{w}(t', \xi_i) \prod_{j=3}^k \phi_{N_j}(\xi_j)\hat{z}_j(t', \xi_j)\Big| \\
&\lesssim_k \int_0^T \sum_{N>0} \sum_{\mathbf{N}_k^2} \langle N\rangle^{2s-2} \min\{N, N_3\} \| P_{N_1} w_1(t', \cdot)\|_{L^2_x} \\
&\qquad \times \| P_{N_2} w_2(t', \cdot)\|_{L^\infty_x} \| P_{N_3} z_3(t', \cdot)\|_{L^2_x} \prod_{j=4}^k \| P_{N_j} z_j(t', \cdot)\|_{L^\infty_x}\, dt' \\
&\lesssim_k \int_0^T \sum_{N>0} \sum_{\mathbf{N}_k^2} \langle N\rangle^{2s-2} \langle N_1\rangle^{1-s} \langle N_3\rangle^{-s} N_2^{1/2} \min\{N, N_3\} \| P_{N_1} w_1(t', \cdot)\|_{H^{s-1}_x} \\
&\qquad \times \| P_{N_2} w_2(t', \cdot)\|_{H^{s-1}_x} \| P_{N_3} z_3(t', \cdot)\|_{H^s_x} \prod_{j=4}^k \min\{N_j^{1/2}, N_j^{-}\} \| P_{N_j} z_j(t', \cdot)\|_{H^{1/2+}_x}\, dt' \\
&\lesssim_k T \| w\|^2_{L^\infty_T H^{s-1}_x} \| z_3\|_{L^\infty_T H^s_x} \prod_{i=4}^k \| z_i\|_{L^\infty_T H^{1/2+}_x}.
\end{aligned}$$

Here, we have used the fact that either $N \lesssim 1$, and then there is nothing to proof, or $N \gg 1$, and then, roughly speaking, we have $\{N_1 \sim N$ and $N_3 \gtrsim N_1\}$. In fact, if $N \geq 8^9 k$, then, thanks to both facts, the explicit form of the symbol d_k and the definition of Γ^k, we must have that $N_1 \in [\frac{1}{2} N, 2N]$ and $N_3 \geq \frac{1}{2k} N_1$. Besides, in the case $N_4 \ll N_3$, we have used the fact that $\| P_{N_1} w(t, \cdot)\|_{H^{s-1}_x}$ and $\| P_{N_3} z_3(t, \cdot)\|_{H^s_x}$ are both square summable. Notice lastly that, in particular, the previous computations allows us to rule out the case $N \leq 8^7 k$.

Now, in order to deal with the remaining region, we split the analysis into three cases, namely,

$$\begin{aligned}
&\mathbf{N}^1 := \big\{(N_1, \dots, N_k) \in \mathbb{D}^k : N_2 \geq 8^8 k,\ N_3 < 2^9 k N_4\big\} \cap \mathbf{N}_k, \\
&\mathbf{N}^2 := \big\{(N_1, \dots, N_k) \in \mathbb{D}^k : N_2 \geq 8^8 k,\ N_3 \geq 2^9 k N_4,\ N_2 < 2^9 k N_4\big\} \cap \mathbf{N}_k, \\
&\mathbf{N}^3 := \big\{(N_1, \dots, N_k) \in \mathbb{D}^k : N_2 \geq 8^8 k,\ N_3 \geq 2^9 k N_4,\ N_2 \geq 2^9 k N_4\big\} \cap \mathbf{N}_k.
\end{aligned} \tag{3.19}$$

We denote the contribution of (3.15) associated with each of these regions by $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3$, respectively. Notice that the case $k = 3$ shall follow directly from the bound exposed for $\mathbf{N}^3$, while $\mathbf{N}^1$ and $\mathbf{N}^2$ only concern the cases $k \geq 4$.

Step 1.1. Let us begin by considering the contribution of (3.17) associated with $\mathbf{N}^1$. We recall once again that $N > 8^7 k$. In fact, in this case we can proceed directly from Lemma 2.9, Plancherel's theorem as well as the Hölder and Bernstein inequalities to obtain

$$\begin{aligned}
\mathrm{D}_1 &\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^1} \langle N\rangle^{2s-2} \Big| \int_{\Gamma^k} d_k(\xi_1,\dots,\xi_k) \prod_{i=1}^2 \phi_{N_i}(\xi_i)\hat{w}(t',\xi_i) \prod_{j=3}^k \phi_{N_j}(\xi_j)\hat{z}_j(t',\xi_j) \Big| dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^1} \langle N\rangle^{2s-2} \min\{N,N_3\} \|P_{N_1} w(t',\cdot)\|_{L^2_x} \|P_{N_2} w(t',\cdot)\|_{L^2_x} \\
&\quad \times \|P_{N_3} z_3(t',\cdot)\|_{L^\infty_x} \|P_{N_4} z_4(t',\cdot)\|_{L^\infty_x} \prod_{j=5}^k \|P_{N_j} z_j(t',\cdot)\|_{L^\infty_x} \, dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^1} \langle N\rangle^{s-1} \langle N_1\rangle^{1-s} \min\{N,N_3\} \|P_{N_1} w(t',\cdot)\|_{H^{s-1}_x} \|P_{N_2} w(t',\cdot)\|_{H^{s-1}_x} \\
&\quad \times \langle N_3\rangle^{-(1^+)} \|J_x^{1/2^+} P_{N_3} z_3(t',\cdot)\|_{L^\infty_x} \|J_x^{1/2^+} P_{N_4} z_4(t',\cdot)\|_{L^\infty_x} \prod_{j=5}^k \|P_{N_j} z_j(t',\cdot)\|_{L^\infty_x} \, dt' \\
&\lesssim_k \|w\|^2_{L^\infty_T H^{s-1}_x} \|J_x^{1/2^+} z_3\|_{L^2_T L^\infty_x} \|J_x^{1/2^+} z_4\|_{L^2_T L^\infty_x} \prod_{j=5}^k \|z_j\|_{L^\infty_T H^{1/2+}_x}.
\end{aligned}$$

Here, we have used the fact that $s \in (1/2, 1]$ so that $\langle N\rangle^{s-1}\langle N_2\rangle^{1-s} \lesssim 1$, and that, on $\mathbf{N}^1$, the following inequalities hold: $\langle N\rangle^{s}\langle N_1\rangle^{-s} \lesssim 1$ and $\langle N\rangle^{-1}\langle N_1\rangle \min\{N,N_3\} \lesssim_k N_3$.

Step 1.2. Now we consider the case of $\mathbf{N}^2$. Indeed, in a similar fashion as above, recalling that $s \in (1/2, 1]$ and that $N > 8^7 k$, then, by using Lemma 2.9, Plancherel's theorem as well as the Hölder and Bernstein inequalities, we infer that

$$\begin{aligned}
\mathrm{D}_2 &\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^2} \langle N\rangle^{2s-2} \Big| \int_{\Gamma^k} d_k(\xi_1,\dots,\xi_k) \prod_{i=1}^2 \phi_{N_i}(\xi_i)\hat{w}(t',\xi_i) \prod_{j=3}^k \phi_{N_j}(\xi_j)\hat{z}_j(t',\xi_j) \Big| dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^2} \langle N\rangle^{2s-2} \min\{N,N_3\} \|P_{N_1} w(t',\cdot)\|_{L^2_x} \|P_{N_2} w(t',\cdot)\|_{L^\infty_x} \\
&\quad \times \|P_{N_3} z_3(t',\cdot)\|_{L^2_x} \|P_{N_4} z_4(t',\cdot)\|_{L^\infty_x} \prod_{j=5}^k \|P_{N_j} z_j(t',\cdot)\|_{L^\infty_x} \, dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1} \sum_{\mathbf{N}^2} \langle N\rangle^{2s-2} \langle N_1\rangle^{1-s} \langle N_3\rangle^{-s} N_2^- N_4^- \min\{N,N_3\} \|P_{N_1} w\|_{H^{s-1}_x} \\
&\quad \times \|J_x^{(-1/2)^+} P_{N_2} w\|_{L^\infty_x} \|P_{N_3} z_3\|_{H^s_x} \|J_x^{1/2^+} P_{N_4} z_4\|_{L^\infty_x} \prod_{j=5}^k \|P_{N_j} z_j\|_{L^\infty_x} \, dt' \\
&\lesssim_k \|w\|_{L^\infty_T H^{s-1}_x} \|J_x^{(-1/2)^+} w\|_{L^2_T L^\infty_x} \|z_3\|_{L^\infty_T H^s_x} \|J_x^{1/2^+} z_4\|_{L^2_T L^\infty_x} \prod_{j=5}^k \|z_j\|_{L^\infty_T H^{1/2+}_x}.
\end{aligned}$$

Here we have used that, due to our current hypotheses, we always have that $N_3 \gtrsim N$. In fact, if $N_2 \leq \frac{1}{16}N$, and since $N_4 \ll N_3$, due to the explicit form of d_k and the definition of Γ^k, we infer that

$$N_1 \in [\tfrac{1}{2}N, 2N] \quad \text{and} \quad N_3 \in [\tfrac{1}{4}N_1, 4N_1].$$

On the other hand, if $N_2 \geq \frac{1}{8}N$, then, since $N_3 \geq N_4 \gtrsim N_2$, we obtain the desired relation.

Step 1.3. Finally, we are ready to treat the remaining case in (3.19), that is, we now deal with the region $\mathbf{N}^3$. In order to do so, we begin using the decomposition given in (2.11), from which we infer that it is enough to control the following quantities:

$$\mathcal{D}_{3,R}^{\text{high}} := \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\left|\int_{\mathbb{R}^2}\Pi_{d_k}(\mathbf{1}_{t,R}^{\text{high}}P_{N_1}w, \mathbf{1}_tP_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_k\right|,$$

$$\mathcal{D}_{3,R}^{\text{low,high}} := \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\left|\int_{\mathbb{R}^2}\Pi_{d_k}(\mathbf{1}_{t,R}^{\text{low}}P_{N_1}w, \mathbf{1}_{t,R}^{\text{high}}P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_k\right|,$$

$$\mathcal{D}_{3,R}^{\text{low,low}} := \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\left|\int_{\mathbb{R}^2}\Pi_{d_k}(\mathbf{1}_{t,R}^{\text{low}}P_{N_1}w, \mathbf{1}_{t,R}^{\text{low}}P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_k\right|,$$

where R stands for a large real number that shall be fixed later. For the sake of clarity we split the analysis into two steps.

Step 1.3.1. We begin by considering the case of $\mathcal{D}_{3,R}^{\text{high}}$. Once again, the idea is to take advantage of the operator $\mathbf{1}_{t,R}^{\text{high}}$ by using Lemma 2.6. In fact, thanks to our current hypothesis, we see that we can choose once again R being equal to $R := N_1N_3$. Moreover, due to the definition of Γ^k again, since $\min\{N_2, N_3\} \geq 2^9kN_4$, we infer that $N_3 \leq 8N_1$, and hence either we have

$$\{N_1 \geq 16N \text{ and } N_3 \in [\tfrac{1}{2}N_1, 2N_1]\} \quad \text{or} \quad \{N_1 \in [\tfrac{1}{2}N, 8N] \text{ and } N_3 \leq 8N_1\}. \tag{3.20}$$

Therefore, we can then bound $\mathcal{G}_{3,R}^{\text{high}}$ by using the first inequality in Lemma 2.6, Lemmas 2.8 and 2.9, as well as Sobolev's embedding in the following fashion:

$$\begin{aligned}
\mathcal{D}_{3,R}^{\text{high}} &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^3}T^{1/4}\langle N\rangle^{2s-2}\|\mathbf{1}_{T,R}^{\text{high}}\|_{L^{4/3}} \\
&\qquad \times \left\|\int_{\mathbb{R}}\Pi_{d_k}(\mathbf{1}_{t,R}^{\text{high}}P_{N_1}w, \mathbf{1}_tP_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_k\right\|_{L_t^\infty} \\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^3}T^{1/4}\langle N\rangle^{2s-2}\min\{N, N_3\}R^{-3/4} \\
&\qquad \times \|P_{N_1}w\|_{L_t^\infty L_x^2}\|P_{N_2}w\|_{L_t^\infty L_x^2}\prod_{j=3}^k N_j^{1/2}\|P_{N_j}z_j\|_{L_t^\infty L_x^2} \\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^3}T^{1/4}\langle N\rangle^{s-1}\langle N_1\rangle^{1-s}(N_1N_3)^{-3/4}\min\{N, N_3\} \\
&\qquad \times \|P_{N_1}u\|_{L_t^\infty H_x^{s-1}}\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\prod_{j=3}^k \min\{N_j^{1/2}, N_j^{-(0^+)}\}\|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}} \\
&\lesssim_k T^{1/4}\|w\|_{L_t^\infty H_x^{s-1}}^2\prod_{j=3}^k\|z_j\|_{L_t^\infty H_x^{1/2+}}.
\end{aligned}$$

As before, notice that we have used the fact that $s \in (1/2, 1]$ so that we have the following inequality $\langle N\rangle^{s-1}\langle N_2\rangle^{1-s} \lesssim 1$. To finish this first case, we point out that, thanks to the operator $\mathbf{1}_{t,R}^{\mathrm{high}}$ acting on the factor $P_{N_2}w$, the same estimates also hold for $\mathcal{D}_{3,R}^{\mathrm{low,high}}$.

Step 1.3.2. Now we consider the last term in the decomposition, that is, $\mathcal{D}_{3,R}^{\mathrm{low,low}}$. In fact, first of all, let us recall the notation introduced in the proof of the previous proposition (adapted to the current symbol):

$$\mathcal{I}_k(u_1,\dots,u_k) := \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{d_k}(u_1,u_2)u_3\cdots u_k\Big|.$$

Then, by using the hypothesis $\min\{N_2,N_3\}\geq 2^9kN_4$, following the same computations as in (3.10), we infer that the resonant relation satisfies

$$|\Omega_k(\xi_1,\dots,\xi_k)| \sim N_1N_2N_3.$$

Thus, we are in a proper setting to take advantage of Bourgain spaces. In order to do so, we decompose $\mathcal{D}_{3,R}^{\mathrm{low,low}}$ with respect to modulation variables in the following fashion:

$$\begin{aligned}
\mathcal{D}_{3,R}^{\mathrm{low,low}} &\leq \mathcal{I}\big(Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w, \mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w, P_{N_3}z_3,\dots,P_{N_k}z_k\big)\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w, Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w, P_{N_3}z_3,\dots,P_{N_k}z_k\big)\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w, Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w, Q_{\gtrsim N^*}P_{N_3}z_3,\dots,P_{N_k}z_k\big)\\
&\quad\ \vdots\\
&\quad+\mathcal{I}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w, Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w, Q_{\ll N^*}P_{N_3}z_3,\dots,Q_{\gtrsim N^*}P_{N_k}z_k\big)\\
&=: \mathcal{I}_1+\dots+\mathcal{I}_k,
\end{aligned}$$

where N^* stands for $N^* := N_1N_2N_3$. At this point it is important to recall that, since $N_2\geq 8^9k$, we also have that $N^*\gg N_1N_3 = R$, which allows us to use the last inequality in Lemma 2.6. Thus, bounding in a similar fashion as before, by using the Hölder and Bernstein inequalities, as well as Lemmas 2.6, 2.8 and 2.9, and the classical Bourgain estimates, we obtain

$$\begin{aligned}
\mathcal{I}_k &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}\min\{N,N_3\}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{L_t^2L_x^2}\\
&\quad\times\|\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}w\|_{L_t^2L_x^2}\prod_{j=3}^{k}N_j^{1/2}\|P_{N_j}z_j\|_{L_t^\infty L_x^2}\\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{s-1}\langle N_1\rangle^{1-s}N_2^{-1}N_3^{-1}\min\{N,N_3\}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{X^{s-2,1}}\\
&\quad\times\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\prod_{j=3}^{k+1}\min\{N_j^{1/2},N_j^{-(0^+)}\}\|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}}\\
&\lesssim_k T^{1/2}\|w\|_{X^{s-2,1}}\|w\|_{L_t^\infty H_x^{s-1}}\prod_{j=3}^{k}\|z_j\|_{L_t^\infty H_x^{1/2+}},
\end{aligned}$$

where we have used again (3.20), the fact that $N > 8^7k$ and that $s \in (1/2, 1]$, so that we have $\langle N\rangle^{s-1}\langle N_2\rangle^{1-s} \lesssim 1$.[17] Moreover, it is not difficult to see that, by following the same lines (up to trivial modifications), we can also bound $\mathcal{J}_2$, obtaining exactly the same bound. On the other hand, to control $\mathcal{J}_3$, we use again Lemmas 2.6, 2.8 and 2.9, as well as the Hölder and Bernstein inequalities, to obtain

$$\begin{aligned}\mathcal{J}_3 \lesssim_k &\sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{2s-2}N_3^{1/2}\min\{N,N_3\}\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{L_t^2L_x^2}\\ &\times\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}w\|_{L_t^\infty L_x^2}\|Q_{\gtrsim N^*}P_{N_3}z_3\|_{L_t^2L_x^2}\prod_{j=4}^{k}N_j^{1/2}\|P_{N_j}z_j\|_{L_t^\infty L_x^2}\\ \lesssim_k&\sum_{N\gg 1}\sum_{\mathbf{N}^3}\langle N\rangle^{s-1}\langle N_1\rangle^{-s}\langle N_3\rangle^{1/2^-}N_2^{-1}N_3^{-1/2}\min\{N,N_3\}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_1}w\|_{L_t^\infty H_x^{s-1}}\\ &\times\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\|Q_{\gtrsim N^*}P_{N_3}z_3\|_{X^{(-1/2)+,1}}\prod_{j=4}^{k+1}\min\{N_j^{1/2},N_j^{-(0^+)}\}\|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}}\\ \lesssim_k&\ T^{1/2}\|w\|_{L_t^\infty H_x^{s-1}}^2\|z_3\|_{X^{(-1/2)+,1}}\prod_{j=4}^{k}\|z_j\|_{L_t^\infty H_x^{1/2+}}.\end{aligned}$$

Notice that all the remaining cases $\mathcal{J}_i$, $i = 4,\dots,k$, follow very similar lines to the latter case above (up to trivial modifications), and they provide exactly the same bound. Hence, in order to avoid over-repeated computations, we omit the proof of these cases.

Step 2. Now we seek to bound the case where we only have powers of Ψ. In particular, no z_i is involved. More concretely, in this step we seek to study the following quantity:

$$\sum_{N>0}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_0^t\sum_{N_1,N_2,N_3}\int_{\mathbb{R}}\Pi_{d_3}(P_{N_1}w,P_{N_2}w)P_{N_3}(\Psi^m)\Big|, \tag{3.21}$$

where $d_3(\xi_1,\xi_2,\xi_3)$ stands for the symbol given in (3.16), where ξ_1 and ξ_2 denote the frequencies of each of the occurrences of w in (3.21), respectively.

Now notice that, by symmetry, we can always assume $N_1 \geq N_2$. Moreover, it is not difficult to see that, due to the additional constraint[18] given by Γ^3, in this case, we have that $\max\{N_2,N_3\} \geq \frac14 N_1$ and $N_3 \leq 8N_1$, and hence, either we have $N_2 \sim N_1$ or $N_3 \sim N_1$, or both. By similar reasons, if $N_3 \geq 16N$, then we must have $N_1 \in [\frac14 N_3, 4N_3]$, otherwise the inner integral in (3.21) vanishes.

On the other hand, following the same lines of the previous step, we can directly bound the case $\mathbf{N}_3^2 := \mathbf{N}_3 \cap \{N_2 \lesssim 1\}$, where $\mathbf{N}_3$ is the set defined in (3.18). In fact, from Lemma 2.9, Plancherel's theorem and Hölder's inequality, we obtain

$$\begin{aligned}\sum_{N>0}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_0^t\sum_{\mathbf{N}_3^2}\int_{\Gamma^3}d(\xi_1,\xi_2,\xi_3)\phi_{N_1}(\xi_1)\hat w(\xi_1)\phi_{N_2}(\xi_2)\hat w(\xi_2)\phi_{N_3}(\xi_3)\widehat{\Psi}^m(\xi_3)\Big|\\ \lesssim_k T\|w\|_{L_T^\infty H_x^{s-1}}^2\|\Psi^m\|_{L_T^\infty W_x^{1+,\infty}},\end{aligned}$$

[17] We point out that, in order to avoid over-repeated sentences, in what follows we shall no longer emphasize that $s \in (1/2, 1]$ and that $N > 8^7k$.

[18] We recall that, with this phrase we are referring to the condition $\xi_1 + \xi_2 + \xi_3 = 0$.

and hence in the sequel we can assume that $N_2 \geq 8^8 k$. Now, let us consider the region $\mathbf{N}_{\gg} := \mathbf{N}_3 \cap \{N_3 \geq 16N\}$. Then, by using Plancherel's theorem and then the Hölder and Bernstein inequalities, we get

$$\sum_{N\gg 1} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \sum_{\mathbf{N}_{\gg}} \int_{\Gamma^3} d(\xi_1,\xi_2,\xi_3)\phi_{N_1}(\xi_1)\hat{w}(\xi_1)\phi_{N_2}(\xi_2)\hat{w}(\xi_2)\phi_{N_3}(\xi_3)\widehat{\Psi}^m(\xi_3)\right|$$

$$\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_{\gg}} \langle N_3\rangle^{-(0^+)} \|P_{N_1} w(t',\cdot)\|_{H^{s-1}_x} \|P_{N_2} w(t',\cdot)\|_{H^{s-1}_x} \|P_{N_3}(\Psi^m(t',\cdot))\|_{W^{1+,\infty}_x} dt'$$

$$\lesssim_k T\|w\|^2_{L^\infty_T H^{s-1}_x} \|\Psi^m\|_{L^\infty_T W^{1+,\infty}_x}.$$

Hence, from now on we assume that $N_3 \leq 8N$, which in turn forces $N_1 \in [\frac{1}{4}N, 32N]$ thanks to the additional constraint given by Γ^3. Then, denoting this remaining region by $\mathbf{N}_{\leq}$, recalling that $\max\{N_2, N_3\} \geq \frac{1}{4}N_1$, we can bound the remaining portion of (3.21) from Lemma 2.9 and Bernstein's inequality as follows:

$$\sum_{N\gg 1} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \sum_{\mathbf{N}_{\leq}} \int_{\Gamma^3} d(\xi_1,\xi_2,\xi_3)\phi_{N_1}(\xi_1)\hat{w}(\xi_1)\phi_{N_2}(\xi_2)\hat{w}(\xi_2)\phi_{N_3}(\xi_3)\widehat{\Psi}^m(\xi_3)\right|$$

$$\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_{\leq}} \min\{N_3, N_3^{-(0^+)}\} \|P_{N_1} w\|_{H^{s-1}_x} \|P_{N_2} w\|_{H^{s-1}_x} \|P_{N_3}(\Psi^m)\|_{W^{1+,\infty}_x}$$

$$\lesssim_k T\|w\|^2_{L^\infty_T H^{s-1}_x} \|\Psi^m\|_{L^\infty_T W^{1+,\infty}_x},$$

where, in this case, we have used the fact that $\|P_{N_1} w(t,\cdot)\|_{H^{s-1}_x}$ and $\|P_{N_2} w(t,\cdot)\|_{H^{s-1}_x}$ are both square summable to sum over the region $N_3 \ll N$.

Step 3. Finally, it only remains to bound the "crossed terms". More specifically, in this step we aim to estimate the contribution of the following quantity:

(3.22)
$$\sum_{N>0} \langle N\rangle^{2s-2} \sup_{t\in(0,T)} \left| \int_0^t \sum_{N_1,\dots,N_{k+1}} \int_{\mathbb{R}} \Pi_{d_{k+1}}(P_{N_1} w, P_{N_2} w) P_{N_3} z_3 \cdots P_{N_k} z_k P_{N_{k+1}}(\Psi^m)\right|,$$

where we assume $k \geq 3$ and d_{k+1} is the symbol given in (3.16). We emphasize that, as in the previous proposition, we consider $N_{k+1} \in \mathbb{D}_{\mathrm{nh}}$.[19] Now notice that, by symmetry, we can always assume that $N_1 \geq N_2$ and $N_3 \geq N_4 = \max\{N_4, \dots, N_k\}$. Moreover, in contrast with Step 1, in this case, by using either Lemma 2.9 or Plancherel's theorem together with Hölder's inequality, the factor coming from the symbol d_{k+1} shall be of order

$$\min\{N, N_{\max}\} \quad \text{instead of} \quad \min\{N, N_3\},$$

as in the previous case, where we have adopted the notation $N_{\max} := \max\{N_3, N_{k+1}\}$. On the other hand, due to the definition of Γ^{k+1}, we infer that

$$\max\{N_2, N_3, N_{k+1}\} \geq \frac{1}{2k} N_1.$$

[19] Hence, when $N_{k+1} = 1$, we consider $\eta(\xi_{k+1})$ instead of $\phi_{N_{k+1}}(\xi_{k+1})$ in (3.22).

In fact, more generally, we have

$$\max\{N_1,N_2,N_3,N_4,N_{k+1}\}\setminus\{\max\{N_1,N_3,N_{k+1}\}\}\geq \frac{1}{2k}\max\{N_1,N_3,N_{k+1}\}.$$

Therefore, roughly speaking, the two largest frequencies are always equivalent (up to a factor depending on k). Now, let us start by ruling out the case $N_2<9^9k$. Indeed, letting $\mathbf{N}^k_{\lesssim}:=\mathbf{N}_{k+1}\cap\{N_2\lesssim 1\}$, from Lemma 2.9, Plancherel's theorem and Hölder's inequality, we get

$$\begin{aligned}\sum_{N>0}\langle N\rangle^{2s-2}&\sup_{t\in(0,T)}\Big|\int_0^t\sum_{\mathbf{N}^k_{\lesssim}}\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w,P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|\\
&\lesssim_k\int_0^T\sum_{N>0}\sum_{\mathbf{N}^k_{\lesssim}}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_3\rangle^{-s}N_2^{1/2}N_{k+1}^{-(1^+)}\min\{N,N_{\max}\}\|P_{N_1}w\|_{H^{s-1}_x}\\
&\quad\times\|P_{N_2}w\|_{H^{s-1}_x}\|P_{N_3}z_3\|_{H^s_x}\|P_{N_{k+1}}(\Psi^m)\|_{W^{1+,\infty}_x}\prod_{j=4}^k\min\{N_j^{1/2},N_j^-\}\|P_{N_j}z_j\|_{H^{1/2+}_x}\,dt'\\
&\lesssim_k T\|w\|^2_{L^\infty_TH^{s-1}_x}\|z_3\|_{L^\infty_TH^s_x}\|\Psi^m\|_{L^\infty_TW^{1+,\infty}_x}\prod_{j=4}^k\|z_j\|_{L^\infty_TH^{1/2+}_x},\end{aligned}$$

where we have used the fact that, if $N\gg 1$, then $N_1\in[\frac{1}{2}N,2N]$, as well as the fact that $\|P_{N_1}w(t,\cdot)\|_{H^{s-1}_x}$ and $\|P_{N_3}z_3(t,\cdot)\|_{H^s_x}$ are both square summable. Once again, notice that the previous bound allows us to assume in the sequel that $N\geq 9^8k$. Now, it is not difficult to see that, in the remaining region, we can further assume that $N_3\geq 8^8k$. In fact, let us assume that $N_3<8^8k$. Then, in this case, either we have $N_{k+1}\sim N_1$ or $\{N_{k+1}\ll N_1$ and $N_1\sim N_2\sim N\}$. Hence, denoting this region by $\mathbf{N}^{k,3}_{\leq}$, then, by using the Hölder and Bernstein inequalities, we have

$$\begin{aligned}\sum_{N>0}\langle N\rangle^{2s-2}&\sup_{t\in(0,T)}\Big|\int_0^t\sum_{\mathbf{N}^{k,3}_{\leq}}\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w,P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|\\
&\lesssim_k\int_0^T\sum_{N>0}\sum_{\mathbf{N}^{k,3}_{\leq}}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}N_{k+1}^{-(1^+)}\min\{N,N_{\max}\}\|P_{N_1}w\|_{H^{s-1}_x}\\
&\quad\times\|P_{N_2}w\|_{H^{s-1}_x}\|P_{N_{k+1}}(\Psi^m)\|_{W^{1+,\infty}_x}\prod_{j=3}^k\min\{N_j^{1/2},N_j^-\}\|P_{N_j}z_j\|_{H^{1/2+}_x}\,dt'\\
&\lesssim_k T\|w\|^2_{L^\infty_TH^{s-1}_x}\|\Psi^m\|_{L^\infty_TW^{1+,\infty}_x}\prod_{j=3}^k\|z_j\|_{L^\infty_TH^{1/2+}_x},\end{aligned}$$

where, to sum over the region $\{N_{k+1}\ll N$ and $N_1\sim N_2\sim N\}$, we have used the fact that $\|P_{N_1}w(s,\cdot)\|_{H^{s-1}_x}$ and $\|P_{N_2}w(s,\cdot)\|_{H^{s-1}_x}$ are both square summable. Moreover, there is another important case that can be directly treated. Let us define the set

$$\mathbf{N}^{k,1}_{\gg}:=\mathbf{N}_{k+1}\cap\{N_2\geq 9^9k\}\cap\{N_3\geq 8^8k\}\cap\big\{2^9kN_{k+1}>\min\{N_1,N_3\}\big\}.$$

The latter constraint implies, up to a factor k, that $\min\{N_1, N_3\} \lesssim_k N_{k+1}$. Then, proceeding in a similar fashion as above, noticing that $\min\{N, N_{\max}\} \lesssim_k \min\{N, N_{k+1}\}$, from the Hölder and Bernstein inequalities, we obtain

$$\sum_{N\gg 1}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_0^t\sum_{\mathbf{N}_{\gg}^{k,1}}\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w, P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|$$

$$\lesssim_k\int_0^T\sum_{N\gg 1}\sum_{\mathbf{N}_{\gg}^{k,1}}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}N_3^{-(0^+)}N_{k+1}^{-(1^+)}\min\{N, N_{\max}\}\|P_{N_1}w\|_{H_x^{s-1}}$$

$$\times\|P_{N_2}w\|_{H_x^{s-1}}\|P_{N_3}z_3\|_{H_x^{1/2+}}\|P_{N_{k+1}}(\Psi^m)\|_{W_x^{1+,\infty}}\prod_{j=4}^k\min\{N_j^{1/2}, N_j^{-}\}\|P_{N_j}z_j\|_{H_x^{1/2+}}dt'$$

$$\lesssim_k T\|w\|_{L_T^\infty H_x^{s-1}}^2\|\Psi^m\|_{L_T^\infty W_x^{1+,\infty}}\prod_{j=3}^k\|z_j\|_{L_T^\infty H_x^{1/2+}}.$$

Having dealt with the above cases, we can now easily deal with the region $2^9kN_{k+1} > N_2$. Indeed, denoting this region by $\mathbf{N}_{\gg}^{k,2}$, from the Hölder and Bernstein inequalities, we see that

$$\sum_{N\gg 1}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_0^t\sum_{\mathbf{N}_{\gg}^{k,2}}\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w, P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|$$

$$\lesssim_k\int_0^T\sum_{N\gg 1}\sum_{\mathbf{N}_{\gg}^{k,2}}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}N_2^{1/2}N_3^{-s}N_{k+1}^{-(1^+)}\min\{N, N_3\}\|P_{N_1}w\|_{H_x^{s-1}}$$

$$\times\|P_{N_2}w\|_{H_x^{s-1}}\|P_{N_3}z_3\|_{H_x^s}\|P_{N_{k+1}}(\Psi^m)\|_{W_x^{1+,\infty}}\prod_{j=4}^k\min\{N_j^{1/2}, N_j^{-}\}\|P_{N_j}z_j\|_{H_x^{1/2+}}dt'$$

$$\lesssim_k T\|w\|_{L_T^\infty H_x^{s-1}}^2\|z_3\|_{L_T^\infty H_x^s}\|\Psi^m\|_{L_T^\infty W_x^{1+,\infty}}\prod_{j=4}^k\|z_j\|_{L_T^\infty H_x^{1/2+}}.$$

Here, we have used the fact that $\|P_{N_1}w(s,\cdot)\|_{H_x^{s-1}}$ and $\|P_{N_3}z_3(s,\cdot)\|_{H_x^s}$ are both square summable, so that we are able to re-sum in the region $N_1\sim N_3\gg\max\{N_2, N_4, N_{k+1}\}$. Therefore, in the sequel we can assume that $\min\{N_1, N_2, N_3\}\geq 2^9kN_{k+1}$. This concludes all the straightforward cases. Now, in order to deal with the remaining region, we split the analysis into three cases, namely,

$$\begin{aligned}\mathbf{N}_k^1 &:= \big\{(N_1,\ldots,N_{k+1})\in\mathbb{D}^k\times\mathbb{D}_{\mathrm{nh}} : N_2\geq 9^9k,\ N_3\geq 8^8k,\\ &\qquad\min\{N_1, N_2, N_3\}\geq 2^9kN_{k+1},\ N_3 < 2^9kN_4\big\}\cap\mathbf{N}_{k+1},\\ \mathbf{N}_k^2 &:= \big\{(N_1,\ldots,N_{k+1})\in\mathbb{D}^k\times\mathbb{D}_{\mathrm{nh}} : N_2\geq 9^9k,\ N_3\geq 8^8k,\\ &\qquad\min\{N_1, N_2, N_3\}\geq 2^9kN_{k+1},\ N_3\geq 2^9kN_4,\ N_2 < 2^9kN_4\big\}\cap\mathbf{N}_{k+1},\\ \mathbf{N}_k^3 &:= \big\{(N_1,\ldots,N_{k+1})\in\mathbb{D}^k\times\mathbb{D}_{\mathrm{nh}} : N_2\geq 9^9k,\ N_3\geq 8^8k,\\ &\qquad\min\{N_1, N_2, N_3\}\geq 2^9kN_{k+1},\ N_3\geq 2^9kN_4,\ N_2\geq 2^9kN_4\big\}\cap\mathbf{N}_{k+1}.\end{aligned}$$

We denote by $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3$ the contribution of (3.22) associated with each of these regions, respectively. Notice that the case $k=3$ shall follow directly from the bound exposed for $\mathbf{N}_k^3$, while $\mathbf{N}_k^1$ and $\mathbf{N}_k^2$ only concern the cases $k\geq 4$.

Step 3.1. Let us begin by considering the case of $\mathbf{N}_k^1$. In fact, in this case, by using Lemma 2.9, Plancherel's theorem as well as the Hölder and Bernstein inequalities, we get

$$\begin{aligned}
\mathrm{D}_1 &\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_k^1}\langle N\rangle^{2s-2}\Big|\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w,P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|\,dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_k^1}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}N_3^{-(1^+)}N_{k+1}^{-(1^+)}\min\{N,N_3\}\|P_{N_1}w\|_{H^{s-1}_x} \\
&\quad\times\|P_{N_2}w\|_{H^{s-1}_x}\|J_x^{1/2^+}P_{N_3}z_3\|_{L^\infty_x}\|J_x^{1/2^+}P_{N_4}z_4\|_{L^\infty_x}\|P_{N_{k+1}}(\Psi^m)\|_{W^{1^+,\infty}_x}\prod_{j=5}^k\|P_{N_j}z_j\|_{L^\infty_x} \\
&\lesssim_k \|w\|^2_{L^\infty_TH^{s-1}_x}\|J_x^{1/2^+}z_3\|_{L^2_TL^\infty_x}\|J_x^{1/2^+}z_4\|_{L^2_TL^\infty_x}\|\Psi^m\|_{W^{1^+,\infty}_x}\prod_{j=5}^k\|z_j\|_{L^\infty_TH^{1/2^+}_x},
\end{aligned}$$

where, to sum over the region $\{N_3\ll N \text{ and } N_1\sim N_2\sim N\}$, we have used the fact that $\|P_{N_1}w(s,\cdot)\|_{H^{s-1}_x}$ and $\|P_{N_2}w(s,\cdot)\|_{H^{s-1}_x}$ are both square summable, while the case $N_3\gtrsim N$ follows directly thanks to the factor N_3^{-}.

Step 3.2. Now we consider the contribution of (3.22) associated with $\mathbf{N}_k^2$. Indeed, proceeding similarly as above, by using Lemma 2.9, Plancherel's theorem as well as the Hölder and Bernstein inequalities, we infer that

$$\begin{aligned}
\mathrm{D}_2 &\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_k^2}\langle N\rangle^{2s-2}\Big|\int_{\mathbb{R}}\Pi_{d_{k+1}}(P_{N_1}w,P_{N_2}w)P_{N_3}z_3\cdots P_{N_k}z_kP_{N_{k+1}}(\Psi^m)\Big|\,dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_k^2}\langle N\rangle^{2s-2}\min\{N,N_3\}\|P_{N_1}w(t',\cdot)\|_{L^2_x}\|P_{N_2}w(t',\cdot)\|_{L^\infty_x} \\
&\qquad\times\|P_{N_3}z_3(t',\cdot)\|_{L^2_x}\|P_{N_4}z_4(t',\cdot)\|_{L^\infty_x}\|P_{N_{k+1}}(\Psi^m)\|_{L^\infty_{t,x}}\prod_{j=5}^k\|P_{N_j}z_j(t',\cdot)\|_{L^\infty_x}\,dt' \\
&\lesssim_k \int_0^T \sum_{N\gg 1}\sum_{\mathbf{N}_k^2}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_3\rangle^{-s}N_2^{-}N_4^{-}N_{k+1}^{-1}\min\{N,N_3\}\|P_{N_1}w\|_{H^{s-1}_x} \\
&\qquad\times\|J_x^{(-1/2)^+}P_{N_2}w\|_{L^\infty_x}\|P_{N_3}z_3\|_{H^s_x}\|J_x^{1/2^+}P_{N_4}z_4\|_{L^\infty_x} \\
&\qquad\times\|P_{N_{k+1}}(\Psi^m)\|_{W^{1,\infty}_x}\prod_{j=5}^k\|P_{N_j}z_j\|_{L^\infty_x}\,dt' \\
&\lesssim_k \|w\|_{L^\infty_TH^{s-1}_x}\|J_x^{(-1/2)^+}w\|_{L^2_TL^\infty_x}\|z_3\|_{L^\infty_TH^s_x}\|J_x^{1/2^+}z_4\|_{L^2_TL^\infty_x} \\
&\qquad\times\|\Psi\|^m_{L^\infty_tW^{1,\infty}_x}\prod_{j=5}^k\|z_j\|_{L^\infty_TH^{1/2^+}_x}.
\end{aligned}$$

Here we have used the fact that, due to our current hypotheses, we always have that $N_3 \gtrsim N$. In fact, if $N_2 \leq \frac{1}{16}N$, since $N_4 \ll N_3$ and $N_{k+1} \ll \min\{N_1, N_2, N_3\}$, by using the explicit form of d_{k+1}, we infer that

$$N_1 \in [\tfrac{1}{2}N, 2N] \quad \text{and} \quad N_3 \in [\tfrac{1}{4}N_1, 4N_1].$$

On the other hand, if $N_2 \geq \frac{1}{8}N$, then, since $N_3 \geq N_4 \gtrsim N_2$, we obtain the desired relation.

Step 3.3. Finally, we are ready to treat the remaining case, that is, we now deal with the region $\mathbf{N}_k^3$. In order to do so, we begin using the decomposition given in (2.11), from which we infer that it is enough to control the following quantities:

$$\begin{aligned}
\mathcal{D}_{3,R}^{\text{high}} &:= \sum_{N\gg 1}\sum_{\mathbf{N}_k^3} \langle N\rangle^{2s-2} \\
&\quad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{d_{k+1}}(\mathbf{1}_{t,R}^{\text{high}} P_{N_1}w, \mathbf{1}_t P_{N_2}w) P_{N_{k+1}}(\Psi^m) \prod_{j=3}^k P_{N_j} z_j \Big|, \\
\mathcal{D}_{3,R}^{\text{low,high}} &:= \sum_{N\gg 1}\sum_{\mathbf{N}_k^3} \langle N\rangle^{2s-2} \\
&\quad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{d_{k+1}}(\mathbf{1}_{t,R}^{\text{low}} P_{N_1}w, \mathbf{1}_{t,R}^{\text{high}} P_{N_2}w) P_{N_{k+1}}(\Psi^m) \prod_{j=3}^k P_{N_j} z_j \Big|, \\
\mathcal{D}_{3,R}^{\text{low,low}} &:= \sum_{N\gg 1}\sum_{\mathbf{N}_k^3} \langle N\rangle^{2s-2} \\
&\quad \times \sup_{t\in(0,T)} \Big| \int_{\mathbb{R}^2} \Pi_{d_k}(\mathbf{1}_{t,R}^{\text{low}} P_{N_1}w, \mathbf{1}_{t,R}^{\text{low}} P_{N_2}w) P_{N_{k+1}}(\Psi^m) \prod_{j=3}^k P_{N_j} z_j \Big|,
\end{aligned}$$

where R stands for $R := N_1N_3$, as in the previous cases. For the sake of clarity we split the analysis into two steps.

Step 3.3.1. We begin by considering the case of $\mathcal{D}_{3,R}^{\text{high}}$. Once again, the idea is to take advantage of the operator $\mathbf{1}_{t,R}^{\text{high}}$ by using Lemma 2.6. Notice that, since in this case we have $N_3 \leq 8N_1$, then, roughly speaking, either we have

$$\{N_1 \gg N,\ N_2 \sim N \text{ and } N_1 \sim N_3\} \quad \text{or} \quad \{N_1 \sim N \text{ and } \max\{N_2, N_3\} \sim N_1\}. \tag{3.23}$$

We can then bound $\mathcal{G}_{3,R}^{\text{high}}$ by using the first inequality in Lemma 2.6, Lemma 2.9, as well as Sobolev's embedding, in the following fashion:

$$\begin{aligned}
\mathcal{D}_{3,R}^{\text{high}} &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3} T^{1/4}\langle N\rangle^{2s-2} \|\mathbf{1}_{T,R}^{\text{high}}\|_{L^{4/3}} \\
&\quad \times \Big\| \int_{\mathbb{R}} \Pi_{d_{k+1}}(\mathbf{1}_{t,R}^{\text{high}} P_{N_1}w, \mathbf{1}_t P_{N_2}w) P_{N_{k+1}}(\Psi^m) \prod_{j=3}^k P_{N_j} z_j \Big\|_{L_t^\infty} \\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3} T^{1/4}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}(N_1N_3)^{-3/4}\min\{N, N_3\}\|P_{N_1}w\|_{L_t^\infty H_x^{s-1}} \\
&\quad \times \|P_{N_2}w\|_{L_t^\infty H_x^{s-1}} \|P_{N_{k+1}}(\Psi^m)\|_{L_x^\infty} \prod_{j=3}^k \min\big\{N_j^{1/2}, N_j^{-(0^+)}\big\} \|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}} \\
&\lesssim_k T^{1/4}\|w\|_{L_t^\infty H_x^{s-1}}^2 \|\Psi\|_{L_{t,x}^\infty}^m \prod_{j=3}^k \|z_j\|_{L_t^\infty H_x^{1/2+}}.
\end{aligned}$$

Once again, notice that thanks to the operator $\mathbf{1}_{t,R}^{\mathrm{high}}$ acting on the factor $P_{N_2}w$, the same estimates also hold for $\mathcal{D}_{3,R}^{\mathrm{low,high}}$.

Step 3.3.2. Now we consider the last term in the decomposition, that is, $\mathcal{D}_{3,R}^{\mathrm{low,low}}$. First of all, let us recall the notation introduced in the proof of the previous proposition (adapted to the current symbol)

$$\mathcal{J}_k(u_1,\dots,u_{k+1}) := \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}\langle N\rangle^{2s-2}\sup_{t\in(0,T)}\Big|\int_{\mathbb{R}^2}\Pi_{d_{k+1}}(u_1,u_2)u_3\cdots u_{k+1}\Big|.$$

Then, by using the hypothesis $\min\{N_2,N_3\}\geq 2^9kN_4$ and following the same computations as in (3.10), we infer that the resonant relation satisfies

$$|\Omega_k(\xi_1,\dots,\xi_k)|\sim N_1N_2N_3.$$

Therefore, taking advantage of the above relation, we can now decompose $\mathcal{D}_{3,R}^{\mathrm{low,low}}$ with respect to modulation variables in the following fashion:

$$\begin{aligned}
\mathcal{D}_{3,R}^{\mathrm{low,low}} &\leq \mathcal{J}\big(Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w,\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w,P_{N_3}z_3,\dots,P_{N_k}z_k,P_{N_{k+1}}(\Psi^m)\big)\\
&+\mathcal{J}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w,Q_{\gtrsim N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w,P_{N_3}z_3,\dots,P_{N_k}z_k,P_{N_{k+1}}(\Psi^m)\big)\\
&+\mathcal{J}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w,Q_{\gtrsim N^*}P_{N_3}z_3,\dots,P_{N_k}z_k,P_{N_{k+1}}(\Psi^m)\big)\\
&\ \ \vdots\\
&+\mathcal{J}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w,Q_{\ll N^*}P_{N_3}z_3,\dots,Q_{\gtrsim N^*}P_{N_k}z_k,P_{N_{k+1}}(\Psi^m)\big)\\
&+\mathcal{J}\big(Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_1}w,Q_{\ll N^*}\mathbf{1}_{t,R}^{\mathrm{low}}P_{N_2}w,Q_{\ll N^*}P_{N_3}z_3,\dots,Q_{\ll N^*}P_{N_k}z_k,P_{N_{k+1}}(\Psi^m)\big)\\
&=: \mathcal{J}_1+\dots+\mathcal{J}_{k+1},
\end{aligned}$$

where N^* stands for $N^*:=N_1N_2N_3$. At this point it is important to recall that, since $N_2\geq 9^9k$, we also have that $N^*\gg N_1N_3=R$, which allows us to use the last inequality in Lemma 2.6. Thus, bounding in a similar fashion as before, by using the Hölder and Bernstein inequalities, as well as Lemmas 2.6 and 2.9, and the classical Bourgain estimates, we obtain

$$\begin{aligned}
\mathcal{J}_1 &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}\langle N\rangle^{2s-2}\min\{N,N_3\}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{L_t^2L_x^2}\\
&\quad\times\|\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}w\|_{L_t^2L_x^2}\|P_{N_{k+1}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{j=3}^{k}N_j^{1/2}\|P_{N_j}z_j\|_{L_t^\infty L_x^2}\\
&\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}\langle N\rangle^{2s-2}\langle N_1\rangle^{1-s}\langle N_2\rangle^{1-s}N_2^{-1}N_3^{-1}\min\{N,N_3\}\|Q_{\gtrsim N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{X^{s-2,1}}\\
&\quad\times\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\|P_{N_{k+1}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{j=3}^{k+1}\min\{N_j^{1/2},N_j^-\}\|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}}\\
&\lesssim_k T^{1/2}\|w\|_{X^{s-2,1}}\|w\|_{L_t^\infty H_x^{s-1}}\|\Psi\|_{L_{t,x}^\infty}^m\prod_{j=3}^{k}\|z_j\|_{L_t^\infty H_x^{1/2+}},
\end{aligned}$$

where we have used again (3.23). Moreover, it is not difficult to see that, by following the same lines (up to trivial modifications), we can also bound $\mathcal{J}_2$, obtaining the same bound. On the other hand, to control $\mathcal{J}_3$, we use again Lemmas 2.6 and 2.9, as well as the Hölder and Bernstein inequalities to get that

$$\begin{aligned}\mathcal{J}_3 &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}\langle N\rangle^{2s-2}N_3^{1/2}\min\{N,N_3\}\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_1}w\|_{L_t^2L_x^2}\|Q_{\ll N^*}\mathbf{1}_{T,R}^{\mathrm{low}}P_{N_2}w\|_{L_t^\infty L_x^2}\\ &\quad\times\|Q_{\gtrsim N^*}P_{N_3}z_3\|_{L_t^2L_x^2}\|P_{N_{k+1}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{j=4}^{k}N_j^{1/2}\|P_{N_j}z_j\|_{L_t^\infty L_x^2}\\ &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}T^{1/2}\langle N\rangle^{2s-2}\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}\langle N_3\rangle^{+}\min\{N,N_3\}\|P_{N_1}w\|_{L_t^\infty H_x^{s-1}}\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\\ &\quad\times\|Q_{\gtrsim N^*}P_{N_3}z_3\|_{X^{(-1/2)+,1}}\|P_{N_{k+1}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{j=4}^{k+1}\min\{N_j^{1/2},N_j^{-(0^+)}\}\|P_{N_j}z_j\|_{L_t^\infty H_x^{1/2+}}\\ &\lesssim_k T^{1/2}\|w\|_{L_t^\infty H_x^{s-1}}^2\|z_3\|_{X^{(-1/2)+,1}}\|\Psi\|_{L_{t,x}^\infty}^m\prod_{j=4}^{k}\|z_j\|_{L_t^\infty H_x^{1/2+}}.\end{aligned}$$

Notice that all the remaining cases $\mathcal{J}_i$, $i=4,\dots,k$, follow very similar lines to the latter case (up to trivial modifications), and they provide exactly the same bound as above, and hence we omit their proof. Finally, to control $\mathcal{J}_{k+1}$ notice that, since all factors $P_{N_i}u$ have an operator $Q_{\ll N^*}$ in front of them, then the factor $P_{N_{k+1}}(\Psi^m)$ is forced to be resonant, and hence in this case we can write $Q_{\gtrsim N^*}P_{N_{k+1}}(\Psi^m)=P_{N_{k+1}}(\Psi^m)$, otherwise $\mathcal{J}_{k+1}=0$ thanks to Lemma 2.7. Moreover, notice also that, in the region $\mathbf{N}_k^3$, we have in particular that $N_1N_2N_3\gg N_{k+1}^3$, and hence we infer that $|\tau_{k+1}-\xi_{k+1}^3|\sim|\tau_{k+1}|$, thus we can write $P_{N_{k+2}}(\Psi^m)=R_{\gtrsim N^*}P_{N_{k+2}}(\Psi^m)$. Therefore, by using Lemmas 2.6 and 2.9, as well as Bernstein's inequality and the above properties, we infer that

$$\begin{aligned}\mathcal{J}_{k+1} &\lesssim_k \sum_{N\gg 1}\sum_{\mathbf{N}_k^3}\langle N\rangle^{2s-2}\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}N_3^{-1}\min\{N,N_3\}\|\mathbf{1}_{T,R}^{\mathrm{low}}\|_{L^2}^2\|P_{N_1}w\|_{L_t^\infty H_x^{s-1}}\\ &\quad\times\|P_{N_2}w\|_{L_t^\infty H_x^{s-1}}\|\partial_tR_{\gtrsim N^*}P_{N_{k+2}}(\Psi^m)\|_{L_{t,x}^\infty}\prod_{i=3}^{k+1}\min\{N_i^{1/2},N_i^{-(0^+)}\}\|P_{N_i}u\|_{L_t^\infty H_x^{1/2+}}\\ &\lesssim_k T\|w\|_{L_t^\infty H_x^{s-1}}^2\|\partial_t\Psi\|_{L_{t,x}^\infty}\|\Psi\|_{L_{t,x}^\infty}^{m-1}\prod_{j=3}^{k}\|z_j\|_{L_t^\infty H_x^{1/2+}}.\end{aligned}$$

Now we explain how we control the contribution of the $L_T^2L_x^\infty$ terms. To this end, recalling the equation solved by $w(t,x)$, we use the Strichartz estimate (2.16) with $\delta=1$, from which we obtain

$$\begin{aligned}\|J_x^{(-1/2)^+}w\|_{L_T^2L_x^\infty} &\lesssim T^{1/4}\|w\|_{L_T^\infty H_x^{(-1/2)+}}+T^{3/4}\|w\|_{L_T^\infty H_x^{(-1/2)+}}\\ &\quad\times\sum_{k=1}^{\infty}kc^k|a_k|\big(\|u\|_{L_T^\infty H_x^{1/2+}}+\|v\|_{L_T^\infty H_x^{1/2+}}+\|\Psi\|_{L_T^\infty W_x^{1/2+,\infty}}\big)^{k-1}.\end{aligned}$$

Thus, gathering all the above estimate, and then using Lemma 2.5, we conclude the proof of the proposition. The proof is complete. ■

4. Unconditional well-posedness in H^s for $s > 1/2$

4.1. Existence and unconditional uniqueness

In this section we shall assume Theorem 5.1 hold, that is, we assume equation (1.4) is locally well-posed in $H^{3/2^+}(\mathbb{R})$. In the next section we shall sketch the main ideas of its proof (see [1] for further details, for example).

Before going further, for the sake of simplicity and by abusing the notation, recalling that Ψ is a given function, from now on we let

$$\begin{aligned}\|\!|\!|\Psi|\!|\!|_r &:= \|\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi)\|_{L^\infty_t H^r_x},\\ \mathcal{Q}_*(\|u\|_{L^\infty_T H^{1/2+}_x}) &:= \mathcal{Q}_*\big(\|u\|_{L^\infty_T H^{1/2+}_x},\|\Psi\|_{L^\infty_t W^{s+1+,\infty}_x},\|\partial_t\Psi\|_{L^\infty_{t,x}},|\!|\!|\Psi|\!|\!|_{s+}\big).\end{aligned}$$

Now, consider a sequence $\{\Psi_n\}_{n\in\mathbb{N}}\subset L^\infty(\mathbb{R}^2)$ satisfying the hypotheses in (1.7) with $s'=3/2^+$ for all $n\in\mathbb{N}$, such that

$$\|\partial_t\Psi_n-\partial_t\Psi\|_{L^\infty_{t,x}}+\|\Psi_n-\Psi\|_{L^\infty_t W^{s+1+,\infty}_x}+|\!|\!|\Psi_n-\Psi|\!|\!|_{s+}\xrightarrow{n\to+\infty}0.$$

Let $u\in C([0,T_0],H^\infty(\mathbb{R}))$ be a smooth solution to equation (1.4) associated with Ψ_n, with minimal existence time

$$T^\star=T^\star(\|u_0\|_{H^{3/2+}},|\!|\!|\Psi_n|\!|\!|_{3/2+},\|\Psi_n\|_{L^\infty_t W^{5/2,\infty}_x})>0,$$

emanating from initial data $u_0\in H^\infty(\mathbb{R})$. Then, according to Proposition 3.1, there exist a constant $c>0$ such that, after an application of the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|u\|^2_{L^\infty_T H^s_\omega}\leq\|u_0\|^2_{H^s_\omega}+cT\|u\|^2_{L^\infty_T H^s_\omega}+cT|\!|\!|\Psi_n|\!|\!|^2_{s+}\\ +cT^{1/4}\|u\|^2_{L^\infty_T H^s_\omega}\mathcal{Q}_*(\|u\|_{L^\infty_T H^{1/2+}_x}),\end{aligned}\tag{4.1}$$

for all $0<T\leq\min\{1,T_0\}$. We stress that $\mathcal{Q}_*$ only involves norms of Ψ_n associated with s and not with $s'=3/2^+$. Notice also that they do not depend on T either. Thus, we can consider the function

$$F(T):=cT+cT^{1/4}\mathcal{Q}_*\big(\|u\|_{L^\infty_T H^{1/2+}_x}\big),\quad T\in[0,T_0].$$

At this point it is important to recall that $\|u\|_{L^\infty_T H^s_\omega}\to\|u_0\|_{H^s_\omega}$ as $T\to0$. Moreover, notice that $F(0)=0$, and hence, thanks to the continuity of $T\mapsto F(T)$, we infer the existence of $T_*=T_*(\|u_0\|_{H^s_\omega})>0$ small enough such that

$$F(T')<1/2\quad\text{for all } T'<T_*.$$

In particular, the above inequality along with (4.1) implies that

$$\|u\|_{L^\infty_{T'}H^s_\omega}\lesssim\|u_0\|_{H^s_\omega}+|\!|\!|\Psi|\!|\!|_{s+}\quad\text{for all } T'<T_*.\tag{4.2}$$

Note that, from (4.2), we infer that the minimal existence time[20] can be chosen only depending on $\|u_0\|_{H^s_\omega}$ and $|||\Psi|||_{s^+}$. On the other hand, by using Proposition 3.2, we have

$$\|u-v\|^2_{L^\infty_T H^{s-1}_x} \lesssim \|u_0-v_0\|^2_{H^{s-1}} + T^{1/4}\|u-v\|^2_{L^\infty_T H^{s-1}_x}\mathcal{Q}^*(\|u\|_{L^\infty_T H^s_x}, \|v\|_{L^\infty_T H^s_x}),$$

where, as above, by an abuse of notation, we are letting

$$\mathcal{Q}^*(\|u\|_{L^\infty_T H^s_x}, \|v\|_{L^\infty_T H^s_x}) := \mathcal{Q}^*(\|u\|_{L^\infty_T H^s_x}, \|v\|_{L^\infty_T H^s_x}, \|\Psi\|_{L^\infty_t W^{s+1,\infty}_x}, \|\partial_t\Psi\|_{L^\infty_{t,x}}).$$

Therefore, a similar continuity argument as before yield us to the existence of a positive time $\tilde{T}_* = \tilde{T}_*(\|u_0\|_{H^s}, \|v_0\|_{H^s}) > 0$ such that

$$\|u-v\|_{L^\infty_{\tilde{T}'} H^{s-1}} \lesssim \|u_0-v_0\|_{H^{s-1}} \quad \text{for all } \tilde{T}' < \tilde{T}_*. \tag{4.3}$$

Now, let us consider an initial data $u_0 \in H^s(\mathbb{R})$ with $s > 1/2$. Consider a smooth sequence of functions $\{u_{0,n}\}_{n\in\mathbb{N}}$ strongly converging to u_0 in $H^s(\mathbb{R})$. Let $u_n(t)$ be the solution to equation (1.4) associated with Ψ_n, with initial data $u_{0,n}$. Note that the above analysis assures us that we can define the whole family of solutions $\{u_n\}$ in a common existence time interval $[0, T^*]$, for some $T^* > 0$ only depending on $\|u_0\|_{H^s}$ and $|||\Psi|||_{s^+}$. Then, thanks to estimate (4.2) with $\omega_N \equiv 1$, $\{u_n\}_{n\in\mathbb{N}}$ defines a bounded sequence in $C([0, T^*], H^s(\mathbb{R}))$, and hence we can extract a subsequence (still denoted by u_n) converging in the weak-$\star$ topology of $L^\infty_{T^*} H^s_x$ to some limit u. Moreover, from this latter convergence we also infer that $\partial_x f(u_n + \Psi_n)$ converges in a distributional sense to $\partial_x f(u + \Psi)$. Therefore, the limit object u solves equation (1.4) with Ψ, in a distributional sense. Furthermore, from (4.3), we get that $\{u_n\}$ defines a Cauchy sequence in $C([0, T^*], H^{s-1}(\mathbb{R}))$, and hence $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $L^\infty((0, T^*), H^{s-1}(\mathbb{R}))$. By the same reasons, from estimate (4.3), we conclude that this solution is the only one in the class $L^\infty((0, T), H^s(\mathbb{R}))$. On the other hand, the above results ensure that the map

$$[0, T^*] \ni t \mapsto u(t) \in H^s(\mathbb{R})$$

is weakly continuous. In fact, let $\varphi \in H^s$ arbitrary, and consider $\tilde{\varphi} \in H^{s+1}$ to be any function satisfying $\|\varphi - \tilde{\varphi}\|_{H^s} \leq \varepsilon/\|u\|_{L^\infty_{T^*} H^s_x}$. Then, for all $t, t' \in (0, T^*)$, we have

$$\begin{aligned}|\langle u(t)-u(t'), \varphi\rangle|_{H^s} &\leq \varepsilon + |\langle J^{s-1}_x(u(t)-u(t')), J^{s+1}\tilde{\varphi}\rangle_{H^s}| \\ &\leq \varepsilon + 2\|u_n - u\|_{L^\infty_T H^{s-1}_x}\|\tilde{\varphi}\|_{H^{s+1}} + \|u_n(t)-u_n(t')\|_{H^{s-1}_x}\|\tilde{\varphi}\|_{H^{s+1}}.\end{aligned}$$

Then, choosing n sufficiently large and using the strong convergence result in H^{s-1}, we deduce that we can control the right-hand side of the above inequality by 3ε, and hence $u(t)$ is weakly continuous from $[0, T^*]$ into $H^s(\mathbb{R})$. Moreover, recalling that, due to (4.3), $\{u_n\}$ defines a Cauchy sequence in $C([0, T^*], H^{s-1}(\mathbb{R}))$, we infer in particular that $u \in C([0, T^*], H^{s-1}(\mathbb{R}))$.

[20] By this we mean the quantity T $< T_*$, that we can choose, with which we know that the solution must exist at least on the interval [0, T]. That is, the solution is guaranteed to exist at least up to time T.

4.2. Continuity of the flow map

We are finally ready to prove both, the continuity of the flow map and the continuity of $u(t)$ with values in $H^s(\mathbb{R})$. Before getting into the details, let us recall the following standard lemma.

Lemma 4.1 ([29]). *Let $\{f_n\}_{n\in\mathbb{N}} \subset H^s(\mathbb{R})$ satisfying $f_n \to f$ in $H^s(\mathbb{R})$. Then there exists an increasing sequence $\{\omega_N\}_{N\in\mathbb{D}} \subset \mathbb{R}$ of positive numbers satisfying $\omega_N \leq \omega_{2N} \leq 2^+\omega_N$, with*

$$\{\omega_N \nearrow +\infty \ \text{as} \ N \to +\infty\} \quad \text{and} \quad \{\omega_N \to 1 \ \text{as} \ N \to 0\},$$

such that

$$\sup_{n\in\mathbb{N}} \sum_{N>0} \omega_N^2 \langle N\rangle^{2s} \| P_N f_n\|_{L^2}^2 < \infty.$$

With this in mind, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of solutions in $L^\infty([0,T], H^s(\mathbb{R}))$ associated with initial datum $u_n(0)$ satisfying $u_n(0) \to u(0)$ in $H^s(\mathbb{R})$. Now, we use the previous lemma with $f_n = u_n(0)$ and $f = u(0)$. Consider $\{\omega_N\}_{N\in\mathbb{D}}$ given by the previous lemma. Then it follows from Proposition 3.1 and estimate (4.2) that

$$\sup_{n\in\mathbb{N}} \sup_{t\in(0,T)} \big(\|u_n(t)\|_{H^s_\omega} + \|u(t)\|_{H^s_\omega}\big) < +\infty. \tag{4.4}$$

Note that the strong continuity in $C([0,T], H^{s-1}(\mathbb{R}))$, together with the boundedness in H^s_ω implies, in particular, the strong continuity of the map $[0,T] \ni t \mapsto u(t)$ in $H^s(\mathbb{R})$. In fact, first notice that it is enough to prove the continuity at $t = 0$. Thus, interpolating[21] the $H^s(\mathbb{R})$-norm, we obtain that

$$\|u(t) - u(0)\|_{H^s(\mathbb{R})} \lesssim \|u(t) - u(0)\|_{H^{s-1}(\mathbb{R})}^{\theta} \|u(t) - u(0)\|_{H^s_\omega(\mathbb{R})}^{1-\theta},$$

for some $\theta \in (0,1)$. Then, noticing that the first term on the right-hand side of the latter inequality goes to zero as t goes to zero, and since the second term is bounded on $[0,T]$, we conclude the strong continuity of the map $[0,T] \ni t \mapsto u(t)$ in $H^s(\mathbb{R})$.

Finally, to complete the proof of Theorem 1.4, it only remains to show the continuity of the flow map. Consider $\{u_n\}$ and u as above. We intend to control $\|u_n - u\|_{L^\infty_T H^s_x}$. First of all, by using the triangular inequality, we have

$$\|u_n - u\|_{L^\infty_T H^s_x} \leq \|u_n - P_{\leq N} u_n\|_{L^\infty_T H^s_x} + \|P_{\leq N} u_n - P_{\leq N} u\|_{L^\infty_T H^s_x} + \|P_{\leq N} u - u\|_{L^\infty_T H^s_x},$$

for any $N \in \mathbb{D}$. Then, take $\varepsilon \in (0,1)$ arbitrary but fixed. We claim that, as a particular consequence of (4.4), there exists $N_* \gg 1$ dyadic, such that for all $t \in [0,T]$, we have

$$\sup_{n\in\mathbb{N}} \|u_n(t) - P_{\leq N_*} u_n(t)\|_{H^s_x} + \|u(t) - P_{\leq N_*} u(t)\|_{H^s_x} < \frac{1}{2}\varepsilon.$$

In fact, it is enough to notice that

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \sup_{t\in(0,T)} \|u_n(t) - P_{\leq N_*} u_n(t)\|_{H^s_x} + \|u(t) - P_{\leq N_*} u(t)\|_{H^s_x} \\ &\quad \lesssim \sup_{n\in\mathbb{N}} \sup_{t\in(0,T)} \frac{1}{\omega_{N_*}} \Big(\sum_{N>N_*} \omega_N^2 \langle N\rangle^{2s} \big(\|u_n(t)\|_{L^2_x}^2 + \|u(t)\|_{L^2_x}^2\big)\Big)^{1/2}. \end{aligned}$$

[21] Recall that, due to Lemma 4.1, we know that $\{\omega_N\}_N$ is a non-trivial weight, in the sense that $\omega_N \nearrow +\infty$.

Therefore, since $\omega_N \nearrow +\infty$ as $N \to +\infty$, we conclude the proof of the claim. On the other hand, from the strong convergence in $C([0,T], H^{s-1}(\mathbb{R}))$ deduced in the previous subsection, we infer the existence of n_* such that, for all $n \geq n_*$ and all $t \in (0,T)$, we have

$$\|P_{\leq N_*}u_n(t) - P_{\leq N_*}u(t)\|_{H^s_x} \leq 2N_*\|P_{\leq N_*}u_n(t) - P_{\leq N_*}u(t)\|_{H^{s-1}_x} < \frac{1}{2}\,\varepsilon.$$

Gathering the last two estimates, we conclude the proof of the continuity of the flow map, and hence, the proof of Theorem 1.4.

4.3. Proof of Theorem 1.11

The proof of Theorem 1.11 is a direct consequence of Theorem 1.4 along with the following lemma, proved in [11, 18].

Lemma 4.2. *Let $\Phi \in \mathcal{Z}^s(\mathbb{R})$ for $s > 1/2$. Then there exists $\Psi \in C_b^\infty$ and $v \in H^s$ such that*

$$\Phi = u + \Psi, \quad \textit{with } \Psi' \in H^\infty(\mathbb{R}).$$

Moreover, the maps $\Phi \mapsto \Psi$ and $\Phi \mapsto u$ can be defined as linear maps such that for every $\tilde{s} > 1/2$, the following holds: The map $\Phi \mapsto \Psi$ is continuous from $\mathcal{Z}^s$ into $\mathcal{Z}^{\tilde{s}}$, whereas the map $\Phi \mapsto v$ is continuous from $\mathcal{Z}^s$ into H^s.

Proof. As we already mentioned, the proof follows almost the same lines as the corresponding versions in [11, 18]. However, due to the hypothesis $s > 1/2$, we need a slight modification of the argument. In fact, we shall actually explicitly define the function Ψ. Indeed, let us consider

$$\Psi(x) := (k * \Phi)(x), \quad \text{where } k(x) := \frac{1}{(4\pi)^{1/2}}\, e^{-x^2/4}.$$

Then it immediately follows that $\Psi \in C_b^\infty(\mathbb{R})$ and that $\Psi' \in H^\infty(\mathbb{R})$. Therefore, $\Phi - \Psi \in L^\infty \subset \mathcal{S}'$. Now, by direct computations, we obtain

$$\mathcal{F}(\Phi-\Psi) = (1-e^{-\xi^2})\widehat{\Phi}(\xi) = (1+|\xi|^2)^{1/4}\Big(\frac{1-e^{-\xi^2}}{\xi}\Big) \times \xi(1+|\xi|^2)^{-1/4}\,\widehat{\Phi}(\xi) =: \mathrm{I} \times \mathrm{II}.$$

Then it is enough to notice that $\mathrm{I} \in L^\infty$ and that, due to hypothesis $\Phi \in \mathcal{Z}^s(\mathbb{R})$, we get $\mathrm{II} \in L^2$. It is not difficult to see that from the above computation, we have $u := \Phi - \Psi \in H^s$. Finally, to obtain the continuity part of the statement, it is enough to notice that

$$\|\Psi'\|^2_{H^{\tilde{s}-1}} \leq \|\Phi'\|^2_{H^{s-1}} \sup_{\xi\in\mathbb{R}}\big((1+\xi^2)^{\tilde{s}-s}e^{-2\xi^2}\big) \lesssim \|\Phi'\|^2_{H^{s-1}}.$$

On the other hand, by straightforward computations from the definition of Ψ, we also obtain that $\|\Psi\|_{L^\infty} \leq \|\Phi\|_{L^\infty}$, which give us the continuity of the map $\Phi \mapsto \Psi$ from $\mathcal{Z}^s$ into $\mathcal{Z}^{\tilde{s}}$. Furthermore, proceeding similarly as above, we also get that

$$\|u\|^2_{H^s} \leq \|\Phi'\|^2_{H^{s-1}} \sup_{\xi\in\mathbb{R}}\Big(\frac{(1+\xi^2)(1-e^{-\xi^2})^2}{\xi^2}\Big) \lesssim \|\Phi'\|^2_{H^{s-1}},$$

which give us the continuity of $\Phi \mapsto u$ from $\mathcal{Z}^s(\mathbb{R})$ into $H^s(\mathbb{R})$. The proof is complete. ∎

Therefore, by using the above lemma, we can decompose the initial data $v(0,\cdot)$ associated with the IVP (1.2) into two functions $u_0 \in H^s(\mathbb{R})$ and $\Psi \in \mathcal{Z}^\infty(\mathbb{R})$. Hence, it is enough to write (1.2) in terms of the Cauchy problem (1.4), with $\Psi = \Psi(x)$ being a time-independent function belonging to $\Psi \in \mathcal{Z}^\infty(\mathbb{R})$. Notice that Ψ satisfies all the hypotheses in (1.7). Thus, Theorem 1.11 follows by using Theorem 1.4 with the above decomposition.

5. Local well-posedness in $H^{3/2^+}(\mathbb{R})$

This section is devoted to show the following result, that gives us the LWP for smooth initial data.

Theorem 5.1 (LWP for smooth data). *The Cauchy problem associated with (1.4) is locally well-posed in $H^s(\mathbb{R})$ for $s > 3/2$, with minimal existence time*

$$T = T\big(\|u_0\|_{H^s_x}, \|\Psi\|_{L^\infty_t W^{s+1^+,\infty}_x}, \|\!|\Psi|\!\|_s\big) > 0.$$

To establish the existence and uniqueness of smooth solutions to the IVP (1.4), we use the parabolic regularization method, that is, we consider solutions to the following equation:

$$\partial_t u + \partial_x^3 u - \mu\partial_x^2 u = -(\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)) - \partial_x(f(u+\Psi) - f(\Psi)), \tag{5.1}$$

for $\mu > 0$. Roughly, the idea is to start by showing LWP of the above equation, and then take the limit $\mu \to 0$. Since these ideas are (nowadays) fairly standard and have been used multiple times in many different contexts, we shall be brief and only sketch their main estimates. We refer to [1] and [18] for further details.

Before going further, let us recall some preliminary lemmas needed to prove Theorem 1.8. The following lemma give us the main estimate to prove the LWP of (5.1) (see [19]).

Lemma 5.2. *Let $\mu > 0$ be fixed. Let $W_\mu(t)$ to be the free group associated with the linear part of (5.1), that is,*

$$W_\mu(t) := \exp((\mu\partial_x^2 - \partial_x^3)t).$$

Then, for all $s \in \mathbb{R}$, $r \geq 0$ and all $f \in H^s(\mathbb{R})$, the following holds:

$$\|W_\mu(t)f\|_{H^{s+r}} \lesssim_r \Big(1 + \frac{1}{(2\mu t)^r}\Big)^{1/2}\|f\|_{H^s}.$$

As a direct consequence of the previous property, we have the following result.

Lemma 5.3. *Let $\mu > 0$ be fixed. Consider $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then there exists $T = T(\|u_0\|_{H^s}, \mu) > 0$ and a unique solution $u_\mu(t)$ to equation (5.1) satisfying*

$$u_\mu \in C([0,T], H^s(\mathbb{R})) \cap C((0,T], H^\infty(\mathbb{R})).$$

The previous lemma can be proven by writing $u_\mu(t)$ in its equivalent Duhamel form, and then proceeding by standard fixed point arguments, using Lemma 5.2. We omit its proof.

In the sequel we shall need the following lemma that combines commutator estimates with Sobolev inequalities.

Lemma 5.4 ([21]). *Let $s > 3/2$ and $r > 1$. Then, for all $f, g \in \mathcal{S}(\mathbb{R})$, the following holds:*

$$|\langle fg_x, g\rangle_{H^s}| \lesssim \|f_x\|_{H^{r-1}}\|g\|_{H^s}^2 + \|f_x\|_{H^{s-1}}\|g\|_{H^s}\|g\|_{H^r},$$

where the implicit constant only depends on s and r.

The next step is to show that the previously found solution $u_\mu(t)$ can be extended to an interval of existence independent of $\mu > 0$.

Lemma 5.5. *Let $\mu > 0$ be fixed. Let $u_\mu \in C([0,T], H^s(\mathbb{R}))$ be the solution to equation (5.1) given by the previous lemma, with initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then $u_\mu(t)$ can be extended to an interval $T' = T'(\|u_0\|_{H^s}) > 0$ independent of μ. Moreover, there exists a continuous function $\rho\colon [0,T'] \to \mathbb{R}$ such that*

$$\|u_\mu(t)\|_{H_x^s}^2 \le \rho(t), \quad \text{with } \rho(0) = \|u_0\|_{H^s}^2.$$

Proof. In fact, directly taking the derivative of the H^s-norm, using (5.1), after suitable integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\|u_\mu(t)\|_{H_x^s}^2 \le {} & -2\langle u_\mu, \partial_x(f(u+\Psi) - f(\Psi))\rangle_{H_x^s} \\ & - 2\langle u_\mu, \partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\rangle_{H_x^s}\end{aligned} \tag{5.2}$$

For the latter term above, from Cauchy–Schwarz, we can see that

$$|\langle u_\mu, \partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\rangle_{H_x^s}| \le \|u_\mu(t)\|_{H_x^s}^2 + \|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{L_t^\infty H_x^s}^2.$$

On the other hand, to estimate the first term in the right-hand side of (5.2), we write

$$\begin{aligned}\partial_x(f(u_\mu+\Psi) - f(\Psi)) = {} & u_{\mu,x}\sum_{k=1}^\infty\sum_{m=0}^{k-1} a_k(k-m)\binom{k}{m}u_\mu^{k-m-1}\Psi^m \\ & + \Psi_x\sum_{k=1}^\infty\sum_{m=1}^{k-1} a_k m\binom{k}{m}u_\mu^{k-m}\Psi^{m-1} =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Then, by the classical Sobolev estimates for products as well as Lemma 5.4, we infer that there exists a constant $c_1 > 0$ such that

$$|\langle u_\mu, \mathrm{I}\rangle|_{H_x^s} \lesssim \|u_\mu\|_{H_x^s}^2\sum_{k=1}^\infty k|a_k|c_1^k(\|u_\mu\|_{H_x^s} + \|\Psi\|_{L_t^\infty W_x^{s+,\infty}})^{k-1}.$$

In a similar fashion, there exists another constant $c_2 > 0$ such that

$$|\langle u_\mu, \mathrm{II}\rangle_{H_x^s}| \lesssim \|u_\mu\|_{H_x^s}^2\sum_{k=1}^\infty k^2|a_k|c_2^k(\|u_\mu\|_{H_x^s} + \|\Psi\|_{L_t^\infty W_x^{s+1+,\infty}})^{k-1}.$$

Therefore, gathering the above estimates, recalling that Ψ is given, we infer that there exists a smooth function $\mathcal{F}_*\colon \mathbb{R} \to \mathbb{R}_+$ such that

$$\frac{d}{dt}\|u_\mu(t)\|_{H_x^s}^2 \lesssim \|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{L_t^\infty H_x^s}^2 + \|u_\mu(t)\|_{H_x^s}^2\mathcal{F}_*(\|u_\mu(t)\|_{H_x^s}).$$

Then, denoting by C_Ψ the first term in the right-hand side above, it is enough to consider $\rho(t)$ to be the solution of the equation

$$\dot\rho(t) = C_\Psi + \rho(t)\mathcal{F}_*(\rho^{1/2}(t)), \quad \rho(0) = \|u_0\|_{H^s}^2.$$

Notice that the solution exists thanks to the Cauchy–Lipschitz theorem. Taking $T_* > 0$ to be the maximal existence time of $\rho(t)$, we conclude $\|u_\mu(t)\|_{H_x^s}^2 \leq \rho(t)$ for all $t \leq T_*$. ■

5.1. Proof of Theorem 5.1

By using the latter lemma we can now take a sequence of initial data $u_{0,\mu} \in H^s(\mathbb{R})$ strongly converging to some u_0 in $H^s(\mathbb{R})$. Then, by the uniform (in μ) bound we infer that, up to a subsequence, we can pass to the limit in the sequence of solutions $u_\mu(t)$, which converge in the weak-⋆ topology of $L^\infty((0,T), H^s(\mathbb{R}))$ to some limit object $u(t)$. It is not difficult to see, reasoning similarly as in the previous section, that $u(t)$ solves the equation in the distributional sense and the map $[0,T] \ni t \mapsto u(t) \in H^s(\mathbb{R})$ is weakly continuous. Let us now consider the uniqueness of the solution. To this end, let us consider $w := u - v$, with u and v solutions of the equation. Recall that then w solves

$$\partial_t w + \partial_x\big(\partial_x^2 w + f(u+\Psi) - f(v+\Psi)\big) = 0.$$

Then, taking the L^2-scalar product of the above equation with against w, we obtain

$$\begin{aligned}\frac{d}{dt}\|w\|_{L_x^2}^2 &= -\langle w, \partial_x(f(u+\Psi) - f(v+\Psi))\rangle_{L_x^2} \\ &\lesssim \|w\|_{L_x^2}^2 \sum_{k=1}^{\infty} c^k |a_k| \big(\|u\|_{H_x^1} + \|v\|_{H_x^1} + \|\Psi\|_{L_t^\infty W_x^{1+,\infty}}\big)^{k-1}.\end{aligned}$$

Thus, a direct application of Grönwall's inequality, recalling that $\|u(t)\|_{H_x^s}^2 + \|v(t)\|_{H_x^s}^2 \leq 2\rho(t)$, implies the uniqueness.

The strong continuity of the solution with values in $H^s(\mathbb{R})$, as well as the continuity of the flow-map can be proven by classical Bona–Smith arguments. We omit this proof.

6. Proof of Theorem 1.8

In this section we seek to prove the global well-posedness, Theorem 1.8. We recall that in this case we assume that

$$|f''(x)| \lesssim 1 \quad \text{for all } x \in \mathbb{R}, \tag{6.1}$$

which shall allow us to use Grönwall's inequality. We emphasize once again that, due to the presence of $\Psi(t,x)$, equation (1.4) has no evident conservation laws. Our first lemma states that the L^2-norm of the solution grows at most exponentially fast in time.

Lemma 6.1. *Let $u(t) \in C([0,T], H^1(\mathbb{R}))$ be a solution to equation (1.4) emanating from initial data $u_0 \in H^1(\mathbb{R})$. Then, for all $t \in [0,T]$, we have*

$$\|u(t)\|_{L_x^2}^2 \leq C_{u_0,\Psi} \exp(C_\Psi t), \tag{6.2}$$

where $C_\Psi > 0$ is a positive constant that only depends on Ψ, while $C_{u_0,\Psi} > 0$ depends on Ψ and u_0.

Proof. In fact, multiplying equation (1.4) by $u(t)$ and then integrating in space, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)\,dx &= -\int u\partial_x(f(u+\Psi)-f(\Psi)) - \int u(\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi))\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

Notice that, thanks to our hypotheses on Ψ, we can immediately bound II by using Young's inequality for products:

$$|\mathrm{II}| \leq \|u(t)\|_{L^2_x}^2 + \|\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi)\|_{L^\infty_t L^2_x}^2.$$

Now, the estimate for I is more delicate, since we require to integrate by parts. Also, we must be careful while splitting the integral into several integrals, since there might be terms that do not integrate (due to Ψ). Hence, in this case we can proceed as follows:

$$\begin{aligned}|\mathrm{I}| &= \lim_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1} u_x(f(u+\Psi)-f(\Psi))\Big|\\ &= \lim_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1} (u_x+\Psi_x)f(u+\Psi)-\Psi_x f(u+\Psi)-\Psi_x f(\Psi)\\ &\quad + \Psi_x f(\Psi) - u_x f(\Psi) - \Psi_x u f'(\Psi) + \Psi_x u f'(\Psi)\Big|\\ &\leq \limsup_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1}(u_x+\Psi_x)f(u+\Psi)-\Psi_x f(\Psi)\Big|\\ &\quad + \limsup_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1} u_x f(\Psi)+\Psi_x u f'(\Psi)\Big|\\ &\quad + \limsup_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1}\Psi_x f(u+\Psi)-\Psi_x f(\Psi)-\Psi_x u f'(\Psi)\Big|\\ &=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

Now, for I_1, notice that we can write the integrand as a full derivative, and hence we have

$$\begin{aligned}\mathrm{I}_1 &= \limsup_{R_1,R_2\to+\infty}\Big|\int_{-R_2}^{R_1}\partial_x(F(u+\Psi)-F(\Psi))\Big|\\ &\leq \limsup_{R_1\to+\infty}|F(u+\Psi)-F(\Psi)|(R_1)+\limsup_{R_2\to+\infty}|F(u+\Psi)-F(\Psi)|(-R_2)=0,\end{aligned}$$

where in the last equality we have used the fact that F is smooth and that $u(t)\in H^1(\mathbb{R})$, so, in particular, $u(t)\to 0$ as $x\to\pm\infty$ for all $t\in[0,T]$. On the other hand, for I_2, we integrate by parts to obtain

$$\mathrm{I}_2 \leq \limsup_{R_1\to+\infty}|uf(\Psi)|(R_1)+\limsup_{R_2\to+\infty}|uf(\Psi)|(-R_2)=0,$$

since $u(t)\in H^1(\mathbb{R})$, $\Psi\in L^\infty(\mathbb{R}^2)$ and f is smooth. Then, gathering all the above estimates, and then using Hölder's inequality along with hypothesis (6.1), we deduce that

$$|\mathrm{I}| \lesssim \Big|\int_{\mathbb{R}}\Psi_x\big(f(u+\Psi)-f(\Psi)-uf'(\Psi)\big)\Big| \lesssim \|\Psi_x\|_{L^\infty_{t,x}}\|u(t)\|_{L^2_x}^2.$$

Therefore, Grönwall's inequality provides (6.2). The proof is complete. ∎

Now, in order to control the H^1-norm, we consider the following modified energy functional

$$\begin{aligned}\mathcal{E}(u(t)) &:= \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx \\ &\quad - \int_{\mathbb{R}} \big(F(u(t,x)+\Psi(t,x)) - F(\Psi(t,x)) - u(t,x) f(\Psi(t,x))\big)\,dx.\end{aligned}$$

It is worth to notice that the previous functional is well defined for all times $t \in [0,T]$. The following lemma give us the desired control on the growth of the H^1-norm of the solution $u(t)$, and hence it finishes the proof of Theorem 1.8.

Lemma 6.2. *Let $u(t) \in C([0,T], H^1(\mathbb{R}))$ be a solution to equation (1.4) emanating from initial data $u_0 \in H^1(\mathbb{R})$. Then, for all $t \in [0,T]$, we have*

$$\|u(t)\|_{H^1_x} \lesssim C^*_{u_0,\Psi} \exp(C^*_\Psi t).$$

where $C^*_\Psi > 0$ is a positive constants that only depends on Ψ, while $C^*_{u_0,\Psi} > 0$ depends on Ψ and u_0.

Proof. First of all, by using the continuity of the flow with respect to the initial data, given by Theorem 1.4, we can assume $u(t)$ is sufficiently smooth so that all the following computations hold. Now, let us begin by explicitly computing the time derivative of the energy functional. In fact, by using equation (1.4), after suitable integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &= -\int u_{xx}u_t - \int u_t(f(u+\Psi) - f(\Psi)) - \int \Psi_t\big(f(u+\Psi) - f(\Psi) - uf'(\Psi)\big) \\ &= \int u_{xx}\partial_x(f(u+\Psi) - f(\Psi)) + \int u\partial_x^2(\Psi_t + \partial_x^3\Psi + \partial_x f(\Psi)) \\ &\quad + \int u_{xxx}(f(u+\Psi) - f(\Psi)) + \int (f(u+\Psi) - f(\Psi))\partial_x(f(u+\Psi) - f(\Psi)) \\ &\quad + \int (f(u+\Psi) - f(\Psi))(\Psi_t + \partial_x^3\Psi + \partial_x f(\Psi)) \\ &\quad - \int \Psi_t\big(f(u+\Psi) - f(\Psi) - uf'(\Psi)\big) \qquad (6.3) \\ &= \int u\partial_x^2(\Psi_t + \partial_x^3\Psi + \partial_x f(\Psi)) + \int (f(u+\Psi) - f(\Psi))(\Psi_t + \partial_x^3\Psi + \partial_x f(\Psi)) \\ &\quad - \int \Psi_t\big(f(u+\Psi) - f(\Psi) - uf'(\Psi)\big) \\ &\lesssim \big(1 + \|\Psi_t\|_{L^\infty_{t,x}} + \|\Psi\|^2_{L^\infty_{t,x}} + \|\partial_t\Psi + \partial_x^3\Psi + \partial_x f(\Psi)\|_{L^\infty_{t,x}}\big)\|u(t)\|^2_{L^2_x} \\ &\quad + \|\Psi_t + \partial_x^3\Psi + \partial_x f(\Psi)\|^2_{L^\infty_t H^2_x}.\end{aligned}$$

On the other hand, by using the Gagliardo–Nirenberg interpolation inequality, and then applying Young's inequality for products, we have

$$\Big|\int_{\mathbb{R}} u^3(t,x)\,dx\Big| \leq C\|u_x\|^{1/2}_{L^2_x}\|u(t)\|^{5/2}_{L^2_x} \leq \frac{\varepsilon^4}{4}\|u_x(t)\|^2_{L^2_x} + \frac{3}{4\varepsilon^{4/3}}\|u(t)\|^{10/3}_{L^2_x}.$$

Thus, by using the above inequality together with our current hypothesis on $f(x)$, we deduce

$$\Big|\int \big(F(u+\Psi)-F(\Psi)-uf(\Psi)\big)\Big| \lesssim \|\Psi\|_{L^\infty_{t,x}}\|u(t)\|^2_{L^2_x}+\|u(t)\|^3_{L^3_x}$$
$$\lesssim \|\Psi\|_{L^\infty_{t,x}}\|u(t)\|^2_{L^2_x}+\frac{\varepsilon^4}{4}\|u_x(t)\|^2_{L^2_x}+\frac{3}{4\varepsilon^{4/3}}+\|u(t)\|^{10/3}_{L^2_x}.$$

Therefore, integrating (6.3) on $[0,T]$, and then plugging the latter inequality in the resulting right-hand side, together with the conclusion of Lemma 6.1, letting $C_\varepsilon := 1-\frac{1}{4}\varepsilon^4$, we infer that

$$C_\varepsilon\int_{\mathbb{R}} u_x^2(t,x)\,dx \lesssim \int u_{0,x}^2-\int \big(F(u_0+\Psi_0)-F(\Psi_0)-u_0f(\Psi_0)\big)$$
$$+\,C_{u_0,\Psi}\big(1+\|\Psi_t\|_{L^\infty_{t,x}}+\|\Psi\|^2_{L^\infty_{t,x}}+\|\partial_t\Psi+\partial_x^3\Psi+\partial_x f(\Psi)\|_{L^\infty_{t,x}}\big)e^{10C_\Psi t/3},$$

where $C_{u_0,\Psi}$ and C_Ψ are the constants founded in the previous lemma. Then, choosing $\varepsilon>0$ small, we conclude the proof of the lemma. ∎

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