

Slab theorem and halfspace theorem for constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We prove that a properly embedded annular end of a surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $0 < H \leq 1/2$ can not be contained in any horizontal slab. Moreover, we show that a properly embedded surface with constant mean curvature $0 < H \leq 1/2$ contained in $\mathbb{H}^2 \times [0, +\infty)$ and with finite topology is necessarily a graph over a simply connected domain of \mathbb{H}^2 . For the case $H = 1/2$, the graph is entire.

1. Introduction

The theory of constant mean curvature (CMC) $H > 0$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ drew a lot of attention after the work by Abresch and Rosenberg [1], where they described a Hopf-type holomorphic quadratic differential on any such surface, and characterized the CMC rotational spheres for $H > 1/2$ as the only immersed CMC spheres in this space [1, 12, 18, 19]. For $0 \leq H \leq 1/2$, there are no compact CMC examples. This is why $H = 1/2$ is called the critical value for the mean curvature in $\mathbb{H}^2 \times \mathbb{R}$. The CMC rotational simply connected examples for $0 < H \leq 1/2$ are entire graphs of paraboloid-type shape (see Section 2.3 for more details). The geometric behavior of CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for $H > 1/2$ is, in some sense, analogous to the one of surfaces of positive CMC in \mathbb{R}^3 . For instance, for these values of the mean curvature there exist spheres and a 1-parameter family of annuli invariant by a vertical translation similar to the Delaunay's examples (see [19] and references therein).

An important theorem by Hoffman and Meeks [11] in the classical theory of minimal surfaces in \mathbb{R}^3 is the halfspace theorem, saying that there are no properly immersed non-flat minimal surfaces in a halfspace. However, in $\mathbb{H}^2 \times \mathbb{R}$ this result does not hold: there are many entire minimal graphs and rotational annuli (called catenoids) contained in a slab constructed by Nelli and Rosenberg [17]. There are also other properly embedded minimal annuli of bounded height constructed in [7]. Notice that the existence of spheres for $H > 1/2$ and paraboloid-type graphs for $0 < H \leq 1/2$ implies that entire graphs of bounded height cannot exist in $\mathbb{H}^2 \times \mathbb{R}$ for $H > 0$ by the maximum principle.

In [4], Collin, Hauswirth and Rosenberg proved that a properly embedded simply connected minimal surface in a slab S of height less than π must be an entire graph. More generally, they proved that each end of a minimal surface properly embedded in S with finite topology is a graph outside a compact domain. For the case of CMC surfaces with $0 < H \leq 1/2$ the behavior is different. We prove that there are no examples in a slab.

Theorem (Slab theorem). *Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface (possibly with boundary) with constant mean curvature $0 < H \leq 1/2$ and at least one properly embedded annular end. Then M can not be contained in a horizontal slab $\mathbb{H}^2 \times [0, L]$, for any $L > 0$.*

In particular, there are no properly embedded CMC surfaces for $0 < H \leq 1/2$ with finite topology contained in a horizontal slab of $\mathbb{H}^2 \times \mathbb{R}$.

For any $0 < H \leq 1/2$, Manzano and Torralbo constructed in [14] properly immersed CMC surfaces contained in a slab. These examples are invariant by a group of symmetries induced by a tessellation of \mathbb{H}^2 by regular polygons. A fundamental domain of any of these examples is compact and its lift in $\mathbb{H}^2 \times \mathbb{R}$ has, by periodicity, only one end of infinite topology. If these examples were embedded, it would show that the hypothesis of having an annular end in the theorem above is sharp.

For CMC $H = 1/2$, there exists a halfspace theorem [9] for complete embedded CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$ lying on one side of a horocylinder (with some condition on the mean curvature vector); the only such examples are parallel horocylinders. Nelli and Sa Earp [20] also proved that the only CMC surfaces with $H = 1/2$ contained in the mean convex side of the rotationally invariant paraboloid-type entire graph are translated copies of the graph. If we think of CMC surfaces on one side of a horizontal slice, the only known examples are entire graphs. Here we prove that, for $0 < H \leq 1/2$, the only properly embedded CMC surfaces with finite topology contained in one side of a horizontal slice are graphs.

Theorem (Halfspace theorem). *Let $M \subset \mathbb{H}^2 \times [0, +\infty)$ be a properly embedded surface with constant mean curvature $0 < H \leq 1/2$ and finite topology. Then M is necessarily a graph over a simply connected domain of \mathbb{H}^2 . For $H = 1/2$, the graph is entire.*

2. Preliminaries

In this section we will set up some notations and introduce some classes of constant mean curvature (CMC) graphs in $\mathbb{H}^2 \times \mathbb{R}$ that we will use as barriers. Throughout this paper we consider the cylinder model for $\mathbb{H}^2 \times \mathbb{R}$. We consider $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ endowed with the hyperbolic metric $g_{-1} = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 denotes the Euclidean metric in \mathbb{R}^2 . We will then consider the unit solid cylinder with the product metric $g = g_{-1} + dt^2$ as model for $\mathbb{H}^2 \times \mathbb{R}$. In this model, there is a natural notion of asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ where $(\partial_\infty \mathbb{H}^2) \times \mathbb{R} = \mathbb{S}^1 \times \mathbb{R}$.

2.1. Constant mean curvature Scherk graphs

For any $H \in (0, 1/2]$, Hauswirth, Rosenberg and Spruck [10] described necessary and sufficient conditions over a compact admissible domain in order to guarantee the existence

of a graph with constant mean curvature H assuming infinite boundary values. A compact domain Ω in \mathbb{H}^2 is said to be admissible if it is simply connected and its boundary $\partial\Omega$ is a polygon with sides $\{A_i\}$ and $\{B_i\}$, with neither two consecutive A_i edges nor B_i edges, and all satisfying $\kappa(A_i) = 2H$ and $\kappa(B_i) = -2H$, where κ denotes the geodesic curvature with respect to the interior of Ω . The Jenkins–Serrin problem they considered consists of finding a solution to the equation for CMC graphs in Ω of mean curvature H (we will call it H -graph) which assumes boundary values $+\infty$ on each A_i and $-\infty$ on each B_i . In the case the domain is an ideal quadrilateral with two opposite edges A_1, A_2 and two opposite edges B_1, B_2 , Folha and Melo constructed complete examples – which are called complete Scherk H -graphs – for any $0 < H < \sqrt{2}/2$ (see Appendix in [8]). Since we will use as barriers these complete Scherk H -graphs for $0 < H < 1/2$, here we show their existence for this wider range of the mean curvature.

Proposition 2.1. *For any $0 < H < 1/2$, there exists a complete Scherk H -graph over an ideal domain, called Scherk domain, bounded by curves A_i and B_i (as above) with $\kappa(A_i) = 2H$ and $\kappa(B_i) = -2H$, for $i = 1, 2$.*

Proof. We use a Plateau conjugate method. Melo [16] constructed a complete minimal Scherk-type graph over an ideal quadrilateral in any $\widetilde{\text{PSL}}(2, \mathbb{R})$ space. Using Daniel’s correspondence [5] and the techniques by Castro-Infantes, Manzano and the third author in [3], these graphs correspond to complete Scherk H -graphs into $\mathbb{H}^2 \times \mathbb{R}$ as desired, with $0 < H < 1/2$. ■

2.2. Bigraph horizontal annuli with constant mean curvature $H = 1/2$

For $H = 1/2$, instead of using Scherk-type graphs as barriers we will use the one-parameter family of horizontal annuli $\{\mathcal{C}_a\}_{a>0}$ constructed by Daniel and the first author in Section 8 of [6], called horizontal catenoids, whose boundary at infinity consists of two vertical lines (see also [3] for an alternative construction). Up to an isometry, we can assume that any horizontal catenoid \mathcal{C}_a is symmetric with respect to the vertical plane $\{y = 0\}$ that separates both ends and with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$, and the lower half of \mathcal{C}_a is a graph which we denote by \mathcal{G}_a (see Figure 1). Each graph \mathcal{G}_a is isometric to a minimal surface Σ_a in the Heisenberg space Nil_3 , called its conjugate surface, and they share the same values of the angle function $\nu = \langle N, \partial_t \rangle$ (see Daniel’s correspondence in [5]). This minimal surface Σ_a is a helicoidal surface bounded by two vertical geodesics. These boundary geodesics correspond to the horizontal curves at height 0 of \mathcal{G}_a , where $\nu = 0$.

The parameter of the family $\{\mathcal{C}_a\}_{a>0}$ corresponds to the size of the neck of the annuli, where by *neck* we mean the (compact) intersection curve between the annulus and the vertical plane of symmetry $\{y = 0\}$.

When a goes to zero, the limit surface consists of the union of two tangent horocylinders (a pinching is produced in this case). The limit surface become vertical everywhere, even if we consider different translated copies of the annuli, so the limit domain of the domains where the graphs \mathcal{G}_a are defined is the union of two tangent horodisks and is foliated by divergence lines (sets of points where the gradient of the functions are unbounded), that are horocycles at the same two points at infinity.

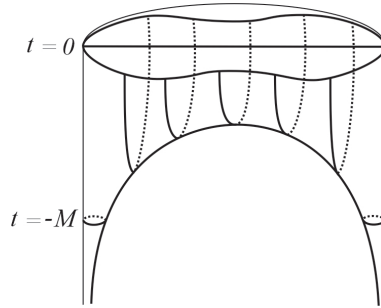


Figure 1. Half of an horizontal catenoid \mathcal{C}_a which is a graph \mathcal{G}_a over a domain in $\mathbb{H}^2 \times \{0\}$.

Now translate the catenoids so that any \mathcal{C}_a is tangent to $\mathbb{H}^2 \times \{0\}$ at the origin of \mathbb{H}^2 and this point is contained in the neck of the annulus. When a diverges, the necks of the annuli (all of them passing through the origin) become as large as we want. Hence, when a goes to $+\infty$, the graphs \mathcal{G}_a converge to the entire graph \mathcal{J} given explicitly by Sa Earp in equation (31) of [21] which is invariant by a one-parameter family of hyperbolic translations. Notice that in Remark 3.7 of [3] it is proved that the conjugated minimal surfaces of \mathcal{G}_a in Nil_3 converge to the minimal entire graph in Nil_3 invariant by the isometric translations along a horizontal geodesic, and Daniel proved in Example 5.6 of [5] that the conjugate surface of this entire minimal graph is the graph \mathcal{J} . In particular, for any $d > 0$, there exists a_0 such that \mathcal{R}_d is contained in the domain where the graph \mathcal{G}_a is defined for any $a \geq a_0$, where $\mathcal{R}_d \subset \mathbb{H}^2$ denotes the region bounded by the two equidistant curves at distance d to the horizontal geodesic $\{y = 0\} \subset \mathbb{H}^2$, see Figure 2.

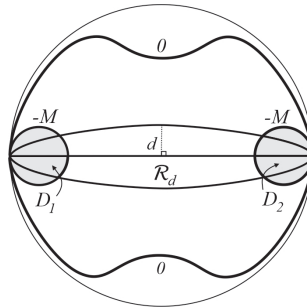


Figure 2. Projection of the graph \mathcal{G}_a and the domains D_1 and D_2 over which $\mathcal{G}_a \cap \{t < -M\}$ projects, for large M .

On the other hand, the conjugate surface $\Sigma_a \subset \text{Nil}_3$ is foliated by straight lines with a Gauss map which is horizontal at infinity. Hence, on each straight line the function v converges uniformly to zero at infinity. This shows that, outside a compact set, the unit normal vector at any point of the graph \mathcal{G}_a is arbitrarily close to horizontal. In particular, using different slide back sequences we can prove that, for $M > 0$ large enough, the translation of the surface $\mathcal{G}_a \cap \{t < -M\}$ converges to a horocylinder. This uniform convergence

proves that the horizontal curves $\mathcal{G}_a \cap \{t = -M\}$ are close to two horocycles and contain any half equidistant curves to a geodesic having the same points at infinity (corresponding to the ends of the horizontal catenoid). Up to an isometry we can assume that this geodesic is $\{y = 0\} \subset \mathbb{H}^2$. In particular, $\mathcal{G}_a \cap \{t < -M\}$ is a graph over two unbounded domains D_1 and D_2 so that $\mathcal{R}_d \setminus (D_1 \cup D_2)$ is compact for any $d > 0$.

2.3. Entire constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$

There is a well-known class of *entire* CMC graphs for any $H \in (0, 1/2]$ that are rotationally invariant complete vertical graphs with empty asymptotic boundary in $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$ (see, for instance, [18, 19]). We are going to call them *paraboloids*. One such surface is the graph of a convex function u that diverges to $+\infty$ when approaching $\partial_\infty \mathbb{H}^2$ and takes its global minimum at height 0 (up to a vertical translation). We will denote by \mathcal{P}^+ this surface throughout this paper. The symmetric surface with respect to the horizontal slice at height 0 will be denoted by \mathcal{P}^- .

When $H = 1/2$, \mathcal{P}^+ is the graph of the function (in polar coordinates)

$$u(r, \theta) = \frac{1}{\sqrt{1-r^2}}, \quad \text{where } 0 \leq r < 1.$$

The mean curvature vector of the paraboloid \mathcal{P}^+ points upwards, and we orient the surface by the unit vector field N^+ such that $\nu^+ = \langle N^+, \partial_t \rangle$ is positive. Since \mathcal{P}^+ is rotationally invariant, we can think of ν^+ as a function on $r \in [0, 1)$ in polar coordinates. Moreover, since \mathcal{P}^+ is the graph of a convex function whose tangent plane is becoming vertical at infinity, ν^+ is strictly decreasing and takes all values in $(0, 1]$. In particular, for any $\alpha \in (0, 1)$ there exist $k_\alpha, h_\alpha > 0$ such that the region of \mathcal{P}^+ where $\nu^+ \geq \alpha$ coincides with $\mathcal{P}^+ \cap (\mathbb{H}^2 \times [0, h_\alpha])$ and is bounded by a horizontal circle of radius k_α (see Figure 3). We observe that both h_α and k_α diverge as α goes to 0.

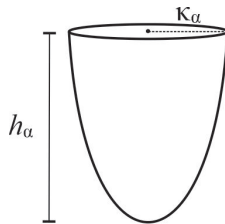


Figure 3. $\mathcal{P}^+ \cap (\mathbb{H}^2 \times [0, h_\alpha])$.

The paraboloid \mathcal{P}^+ (as well as \mathcal{P}^-) separates the ambient space into two connected components. We denote by \mathcal{P}_{int} the mean-convex component (the one where the mean curvature vector points to), and by \mathcal{P}_{ext} the other one. We will use this notation in the proof of the slab theorem.

There are many other examples of complete CMC graphs in $\mathbb{H}^2 \times \mathbb{R}$. In fact, for any $H \in (0, 1/2]$ there are families of CMC graphs invariant by a hyperbolic or a parabolic translation, not necessarily entire when $H < 1/2$ (see [21] or the appendix in [15]). The

intersection of the asymptotic boundary of all these examples with $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$ is contained in one or two vertical lines, and some of these examples are contained in one side of a horizontal slice.

Moreover, there exist many CMC $H = 1/2$ entire graphs which are not rotationally invariant, obtained as deformation of the paraboloid, with empty asymptotic boundary in $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$ (see [2] for more details).

3. Slab theorem

Collin, Hauswirth and Rosenberg [4] have proved that an annular end of a properly immersed minimal surface contained in a slab of height less than π is a multigraph outside a compact domain with a finite number of sheets. When $0 < H \leq 1/2$, the following theorem proves the non-existence of a properly embedded annular end with CMC H in a slab of any height L . We will denote by \mathcal{S}_L the horizontal slab $\mathbb{H}^2 \times [0, L]$, for any $L > 0$.

Theorem 3.1. *Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface (possibly with boundary) with constant mean curvature $0 < H \leq 1/2$ and at least one properly embedded annular end. Then M can not be contained in \mathcal{S}_L , for any $L > 0$.*

In particular, there are no properly embedded CMC surfaces for $0 < H \leq 1/2$ with finite topology contained in a horizontal slab of $\mathbb{H}^2 \times \mathbb{R}$.

Proof. Let us suppose, by contradiction, that there exists one such surface M contained in \mathcal{S}_L , and call E a properly embedded annular end of M . Then $E \subset \mathcal{S}_L$ is an annulus with compact boundary ∂E . Since E is properly embedded, there exists a compact disk D (not necessarily minimal) with $\partial D = \partial E$ such that $E \cup D$ is a surface that separates $\mathbb{H}^2 \times \mathbb{R}$ into two connected components. Along E , the mean curvature vector \vec{H} distinguishes these two components. We call interior component the one where \vec{H} points to, and exterior component the other one. We denote by N the unit normal vector to E such that $\vec{H} = HN$, and $\nu = \langle N, \partial_t \rangle$ denotes what we call the angle function of E .

Let us consider a paraboloid \mathcal{P}^+ with the same mean curvature $0 < H \leq 1/2$ as E (see Section 2.3). Its mean curvature vector is pointing upwards, and we orient the paraboloid by the unit vector field N^+ such that $\nu^+ = \langle N^+, \partial_t \rangle$ is positive. Since ν^+ takes all values in $(0, 1]$ and \mathcal{P}^+ is rotationally invariant, for any point $p \in E$ with $\nu(p) > 0$, we will be able to translate \mathcal{P}^+ in such a way that it passes through p with $N^+(p) = N(p)$. In the case $\nu(p) < 0$, we instead consider a translation of \mathcal{P}^- to find a paraboloid with the same mean curvature vector as E at p .

Claim 1. There exists $\alpha_0 \in (0, 1)$ such that $\{p \in E : |\nu(p)| \geq \alpha_0\}$ contains a sequence of diverging points $(p_n)_{n \in \mathbb{N}}$, i.e.,

$$(3.1) \quad d(p_n) := \text{dist}_{\mathbb{H}^2}(\pi(p_n), \pi(\partial E)) \rightarrow +\infty \quad \text{and} \quad |\nu(p_n)| \geq \alpha_0,$$

where $\pi: \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2$ denotes the (vertical) projection onto the first factor.

If not, for any sequence of points $p_n \in E$ such that $d(p_{n+1}) > d(p_n) \geq n$ (the sequence diverges horizontally), the sequence $\{\nu(p_n)\}_{n \in \mathbb{N}}$ converges uniformly to zero. We then

consider the isometry T_n which is the composition of a vertical and a horizontal translations mapping p_n to a fixed point $p_0 = (0, 0, L/2) \in \mathbb{H}^2 \times \mathbb{R}$, and call $E_n = T_n(E)$.

If there is a subsequence of $\{p_n\}_{n \in \mathbb{N}}$ such that the Gaussian curvature of E is uniformly bounded in a small neighborhood of any p_n , then there is a subsequence of the surfaces E_n which locally converges to a CMC surface E_∞ whose angle function ν_∞ vanishes identically in a neighborhood of p , and E_∞ is contained in a vertical cylinder over a complete curve of constant geodesic curvature $2H$. By the unique continuation theorem, there is a subsequence of $\{E_n\}_{n \in \mathbb{N}}$ converging to a complete vertical cylinder, a contradiction with the fact that all the terms in the sequence are contained in a fixed horizontal slab of height less than $3L$ ($\mathbb{H}^2 \times [-L, 2L]$).

Then we can assume that the curvature of E is not uniformly bounded in neighborhoods of p_n and, by passing to a subsequence if necessary, that $|A(p_n)| \geq n$ for any n , where A denotes the second fundamental form of E . We call B_n the connected component that contains p_n of the intersection of E with the extrinsic ball $\mathbb{B}(p_n, \delta)$ centered at p_n of uniform small radius $\delta > 0$, and we define

$$f_n(p) := \bar{d}(p, \partial B_n) |A(p)|,$$

for any $p \in B_n$, where \bar{d} denotes the extrinsic distance in $\mathbb{H}^2 \times \mathbb{R}$. The function f_n vanishes on ∂B_n and $f_n(p_n) = \delta |A(p_n)| \geq \delta n$. We then deduce that f_n attains its maximum at a point $q_n \in B_n$, and $f_n(q_n) \geq \delta n$. On the other hand, $\delta |A(q_n)| \geq f_n(q_n)$, from where we deduce that $|A(q_n)| \geq n$.

We now consider $r_n := \frac{1}{2} \bar{d}(q_n, \partial B_n)$ and $B'_n \subset B_n$ the connected component of $E \cap \mathbb{B}(q_n, r_n)$ that contains q_n . For any point $q \in B'_n$, it holds

$$2r_n = \bar{d}(q_n, \partial B_n) \leq \bar{d}(q_n, q) + \bar{d}(q, \partial B_n) \leq r_n + \bar{d}(q, \partial B_n).$$

Hence, $\bar{d}(q, \partial B_n) \geq r_n$. Since $2r_n |A(q_n)| = f_n(q_n) \geq f_n(q) \geq r_n |A(q)|$, we conclude that

$$|A(q)| \leq 2|A(q_n)| =: 2\lambda_n.$$

Since $\nu(p_n)$ converges uniformly to zero, we can assume, by passing to a subsequence if necessary, that $|\nu| < 1/n$ in B_n . Thus we have that $|\nu| < 1/n$ and $|A| \leq 2\lambda_n$ in B'_n .

Now we consider a blow up on the metric g of $\mathbb{H}^2 \times \mathbb{R}$ by a factor $\lambda_n \geq n$; more precisely, we define Σ_n as B'_n with the metric $g_n = \lambda_n g$. We can use the exponential map at the point q_n to lift the surface Σ_n to its tangent plane $T_{q_n}(\mathbb{H}^2 \times \mathbb{R}) \approx \mathbb{R}^3$, and we obtain a surface $\tilde{\Sigma}_n \subset \mathbb{R}^3$ which is a minimal surface with respect to the lifted metric \tilde{g}_n , where \tilde{g}_n is the metric such that the exponential map \exp_{q_n} is an isometry from $(\tilde{\Sigma}_n, \tilde{g}_n)$ to (Σ_n, g_n) . Therefore, $\tilde{\Sigma}_n \subset \mathbb{B}_{\mathbb{R}^3}(0, \lambda_n r_n)$ and, if \tilde{A} denotes the second fundamental form of $\tilde{\Sigma}_n$, we have $|\tilde{A}(0)| = 1$ and $|\tilde{A}(q)| \leq 2$ for all $q \in \tilde{\Sigma}_n$.

On one hand, we observe that $\lambda_n r_n$ diverges, as $2\lambda_n r_n = f_n(q_n) \geq f_n(p_n) \geq \delta n$. Then the balls $\mathbb{B}_{\mathbb{R}^3}(0, \lambda_n r_n)$ converge to \mathbb{R}^3 and the metrics g_n converge to the canonical metric g_0 of \mathbb{R}^3 . For a fixed n , the compact surfaces $\tilde{\Sigma}'_k := \tilde{\Sigma}_k \cap \mathbb{B}_{\mathbb{R}^3}(0, \lambda_n r_n)$ of \mathbb{R}^3 , with $k \geq n$, all pass through the origin 0 and have uniform bounded curvature. Then a subsequence of $\{\tilde{\Sigma}'_k\}_k$ converges to a minimal surface in (\mathbb{R}^3, g_0) passing through the origin 0 and $|\tilde{A}(0)| = 1$, where \tilde{A} also denotes the second fundamental form of this limit

surface. This argument holds for any n , so we can use a diagonal argument and obtain, as a limit of a subsequence of the surfaces $\tilde{\Sigma}_k$, a complete minimal surface $\tilde{\Sigma}$ in \mathbb{R}^3 with $0 \in \tilde{\Sigma}$ and $|\tilde{A}(0)| = 1$.

On the other hand, we knew that $|v| < 1/n$ in B'_n . Then we obtain that $|v| < 1/n$ in $\tilde{\Sigma}_n$, from where we deduce that the Gauss map of $\tilde{\Sigma}$ takes values in a neighborhood of the equator of \mathbb{S}^2 . Then this limit surface $\tilde{\Sigma}$ must be a vertical plane, which contradicts the fact that $|\tilde{A}(0)| = 1$. This proves Claim 1.

Take the sequence $\{p_n\}_{n \in \mathbb{N}}$ given by Claim 1. We can assume, by passing to a subsequence if necessary, that $v(p_n) > 0$ (or $v(p_n) < 0$) for any n . We consider, for each p_n , a translation of the paraboloid \mathcal{P}^+ (or \mathcal{P}^- , depending on the sign of $v(p_n)$) tangent to E at p_n with the same mean curvature vector at p_n . We denote such paraboloid by $\mathcal{P}(p_n)$.

Since the angle function v^+ of the paraboloid \mathcal{P}^+ is a decreasing function (by the convexity of u , see Section 2.3), there exists a unique $\beta \in (0, \alpha_0)$ such that $h_\beta = h_{\alpha_0} + L$. Hence, $\mathcal{P}(p_n) \cap (\mathbb{H}^2 \times [0, L])$ is the translation of a subdomain of $\mathcal{P}^\pm \cap (\mathbb{H}^2 \times [0, \pm h_\beta])$, where the sign \pm depends on the sign of the angle function v at p_n . For n large enough, $d(p_n) > 2k_\beta$ and the tangent paraboloid $\mathcal{P}(p_n)$ does not intersect ∂E .

The local intersection of E and $\mathcal{P}(p_n)$ near p_n consists of k curves, with $k \geq 2$, meeting at an equal angle at p_n . We denote by $\Gamma_n = E \cap \mathcal{P}(p_n)$ the intersection of the two surfaces. Since the intersection of $\mathcal{P}(p_n)$ with the slab containing E is compact, Γ_n is also compact.

The paraboloid $\mathcal{P}(p_n)$ divides $\mathbb{H}^2 \times \mathbb{R}$ into two components: a mean-convex one, $\mathcal{P}_{\text{int}}(p_n)$, and a non mean-convex one, $\mathcal{P}_{\text{ext}}(p_n)$. Since the intersection of $\mathcal{P}_{\text{int}}(p_n)$ with a horizontal slab is compact, any component of $E \setminus \mathcal{P}(p_n)$ contained in $\mathcal{P}_{\text{int}}(p_n)$ is necessarily compact.

Claim 2. There is no compact component Σ of $E \setminus \mathcal{P}(p_n)$ contained in $\mathcal{P}_{\text{ext}}(p_n)$ with boundary $\partial \Sigma \subset \mathcal{P}(p_n)$.

Suppose by contradiction this is not true. Then we can find a vertically translated copy $\tilde{\mathcal{P}}(p_n)$ of $\mathcal{P}(p_n)$ tangent to Σ at a point \tilde{p} such that Σ is contained in the mean-convex side of $\tilde{\mathcal{P}}(p_n)$. By the maximum principle, we get that the mean curvature vector of Σ at \tilde{p} points to the non mean-convex side of $\tilde{\mathcal{P}}(p_n)$. Let $\sigma(\mathcal{P})$ be the symmetric copy of $\mathcal{P}(p_n)$ with respect to $\mathbb{H}^2 \times \{0\}$. We can translate $\sigma(\mathcal{P})$ so that it is tangent to $\tilde{\mathcal{P}}(p_n)$ (and hence to Σ) at \tilde{p} . Then Σ and $\sigma(\mathcal{P})$ share the same mean curvature vector at \tilde{p} and Σ is contained in the non mean-convex side of $\sigma(\mathcal{P})$, a contradiction by the maximum principle, and Claim 2 is proved.

Locally at p_n , the set $E \setminus \Gamma_n$ has at least four components with at least two of them contained in the mean-convex component $\mathcal{P}_{\text{int}}(p_n)$. We call Σ_1 and Σ_2 two of these components.

Claim 3. Σ_1 and Σ_2 can be connected by an arc $\gamma \subset E \cap \mathcal{P}_{\text{int}}(p_n)$.

Suppose by contradiction this is not the case. Thus Γ_n bounds at least two distinct connected components R_1 and R_2 of $E \setminus \mathcal{P}(p_n)$ which are contained in $\mathcal{P}_{\text{int}}(p_n)$ whose boundaries meet at p_n . We consider vertical translations $T_s \mathcal{P}(p_n)$ of the paraboloid, where $T_s(p) = p + s \partial_t$ for $s \geq 0$, which foliate $\mathcal{P}_{\text{int}}(p_n)$. Since $E \cap \mathcal{P}_{\text{int}}(p_n)$ is com-

pact, there is a last leaf $T_{s_1}\mathcal{P}(p_n)$ of the foliation that meets R_1 . Then $T_{s_1}\mathcal{P}(p_n)$ and R_1 are tangent at a point q_1 and R_1 is below $T_{s_1}\mathcal{P}(p_n)$. By the maximum principle, the mean curvature vector of E and the mean curvature vector of the paraboloid $T_{s_1}\mathcal{P}(p_n)$ are opposite at q_1 . The component R_1 separates $\mathcal{P}_{\text{int}}(p_n)$ into two connected components and the mean-convex one R_1^+ is compact. Observe that R_1 and R_2 are local graphs near p_n . Since the mean curvature vector of E at p_n is pointing into $\mathcal{P}_{\text{int}}(p_n)$ and E is embedded, the component R_2 is completely contained in R_1^+ (see Figure 4) and the compact component of $\mathcal{P}_{\text{int}}(p_n) \setminus R_2$ is not mean-convex. We then reach a contradiction applying the maximum principle with the last leaf $T_{s_2}\mathcal{P}(p_n)$ of the foliation that meets R_2 . This proves Claim 3.

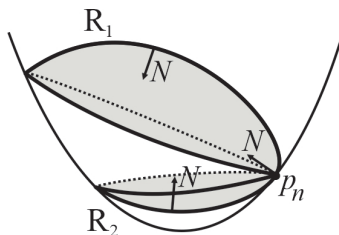


Figure 4. Connected components R_1 and R_2 of $E \setminus \mathcal{P}(p_n)$ in $\mathcal{P}_{\text{int}}(p_n)$ appearing in the proof of Claim 3.

Let γ_n be a compact arc in $E \cap \mathcal{P}_{\text{int}}(p_n)$ linking two points $q_1 \in \Sigma_1$ and $q_2 \in \Sigma_2$. We can complete γ_n by a compact segment $\gamma'_i \subset \Sigma_i$ with endpoints q_i and p_n such that $\alpha_n = \gamma_n \cup \gamma'_1 \cup \gamma'_2$ is a loop in E . If α_n is homologous to zero in E , then it is the boundary of a disk D which contains points in $\mathcal{P}_{\text{ext}}(p_n)$ close to p_n ; hence, the disk D has a subdomain in $\mathcal{P}_{\text{ext}}(p_n)$, a contradiction to Claim 2. This proves that α_n is in the homology class of ∂E and $\alpha_n \cup \partial E$ bounds a subannulus \mathcal{A}_n of E .

Since we can do this construction for a sequence of diverging points p_n , we can use a translation of the paraboloid \mathcal{P}^+ so that $\mathcal{P}^+ \cap \partial E = \emptyset$ and \mathcal{A}_n and \mathcal{P}^+ are tangent at a point. Then by the maximum principle we conclude that, along \mathcal{A}_n , the mean curvature vector of E points into the solid cylinder \mathcal{A}_n^+ bounded by the annulus \mathcal{A}_n and two disks with boundaries α_n and ∂E . We are then in the situation of a mean convex cylinder with boundaries contained in two compact subdomains $K(\alpha_n)$ and $K(\partial E)$ of $\mathbb{H}^2 \times \mathbb{R}$. Since all the curves α_n are contained in the intersection with the slab \mathcal{S}_L with translated copies of the mean-convex side of the same paraboloid, $\mathcal{P}^\pm \cap (\mathbb{H}^2 \times [0, \pm h_\beta])$, we can suppose that the compact sets $K(\alpha_n)$ are compact balls of uniform radius.

First assume $0 < H < 1/2$. For n large enough, we can suppose that there exists a Scherk domain Ω bounded by arcs A_1, A_2, B_1, B_2 with $\kappa(A_i) = 2H$ and $\kappa(B_i) = -2H$, where κ denotes the geodesic curvature with respect to the interior of Ω , such that B_i separates $K(\alpha_n)$ from $K(\partial E)$, for $i = 1, 2$ (see Figure 5). In particular, we can conclude that the vertical planes $A_1 \times \mathbb{R}$ and $A_2 \times \mathbb{R}$ do not intersect the annulus \mathcal{A}_n ; otherwise we could use hyperbolic translations of the CMC H -plane $A_i \times \mathbb{R}$ to get a contradiction with the maximum principle (notice that the mean curvature vector of \mathcal{A}_n points into the compact region bounded by it, so we have the correct orientation to apply

the maximum principle). Now moving the CMC H -planes $A_i \times \mathbb{R}$ towards the annulus \mathcal{A}_n using hyperbolic translations, we can guarantee the existence of two disjoint vertical CMC H -planes $\Gamma_1 \times \mathbb{R}$ and $\Gamma_2 \times \mathbb{R}$ such that the annulus \mathcal{A}_n is contained in the convex side of the two of them and Γ_i separates A_1 from A_2 , for $i = 1, 2$. We can then translate downwards the complete Scherk H -graph over Ω so that it is below the annulus \mathcal{A}_n . Now we get a contradiction using the maximum principle and vertical translations of the complete Scherk H -graph.

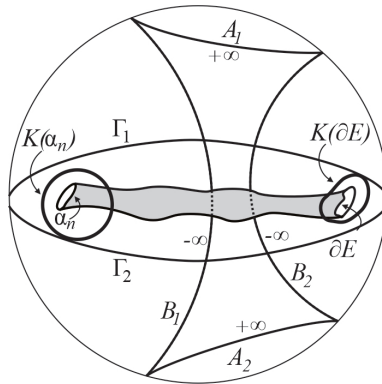


Figure 5. Projection of the annulus \mathcal{A}_n and of the complete Scherk H -graph for the case $0 < H < 1/2$.

In the case of $H = 1/2$, the argument is similar but a bit more subtle and using the graph given by the half of a horizontal catenoid described in Section 2.2 instead.

Let us first prove that, for any n , \mathcal{A}_n projects onto the region between two equidistant curves to a same geodesic. Up to an isometry, we can assume that the geodesic which minimizes the distance between $K(\alpha_n)$ and $K(\partial E)$ is contained in $\{y = 0\} \subset \mathbb{H}^2$, and let \mathcal{J} be the entire $1/2$ -graph invariant by translations along this geodesic described in Section 2.2.

We consider $d > 0$ so that the vertical projection of $K(\alpha_n) \cup K(\partial E)$ is contained in \mathcal{R}_d , where we recall that $\mathcal{R}_d \subset \mathbb{H}^2$ denotes the region bounded by the two equidistant curves at distance d to the horizontal geodesic $\{y = 0\} \subset \mathbb{H}^2$. We translate \mathcal{J} downwards so that it is much below \mathcal{A}_n and start translating upwards until it contains the equidistant curves $\partial \mathcal{R}_d \times \{L\}$ or intersects $K(\alpha_n) \cup K(\partial E)$ for the first time. By the maximum principle this translated copy of \mathcal{J} lies below \mathcal{A}_n , and the vertical projection of \mathcal{A}_n onto \mathbb{H}^2 is contained in \mathcal{R}_d . Since the compact balls $K(\alpha_n)$ have uniform radius, this distance d does not depend on n .

Let us now consider a horizontal catenoid \mathcal{C}_a such that \mathcal{R}_d is contained in the projection of \mathcal{C}_a in \mathbb{H}^2 . We can assume, up to a horizontal translation, that the vertical symmetry plane Π (which separates an end of \mathcal{C}_a from the other) intersects \mathcal{A}_n transversally and has $\Pi \cap K(\alpha_n) = \Pi \cap K(\partial E) = \emptyset$. For $M > L$ and n sufficiently large, we can suppose that $\mathcal{C}_a \cap \{-M \leq t \leq 0\}$ is a graph \mathcal{S} over a domain $\tilde{\Omega}$ such that $\tilde{\Omega} \cap K(\alpha_n) = \tilde{\Omega} \cap K(\partial E) = \emptyset$ as in Figure 6. Applying the maximum principle with vertical translations of the graph \mathcal{S} , we conclude that \mathcal{A}_n can not exist. ■

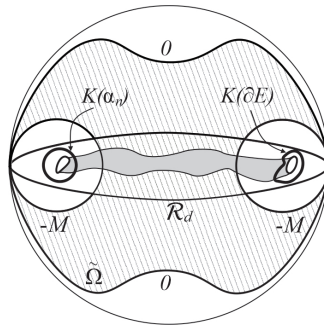


Figure 6. Projection of the annulus \mathcal{A}_n and of the graph \mathcal{G}_a for the case $H = 1/2$.

4. Halfspace theorem

Theorem 4.1. *Let M be a properly embedded CMC surface in $\mathbb{H}^2 \times [0, +\infty)$ with $0 < H \leq 1/2$ and finite topology. Then M is necessarily a graph over a simply connected domain of \mathbb{H}^2 . For $H = 1/2$, the graph is entire.*

Proof. Suppose by contradiction that there exist two distinct points $p_1, p_2 \in M$ such that $\pi(p_1) = \pi(p_2) = x$, where $\pi: \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2$ is the (vertical) projection onto the first component; that is, $p_1 = (x, t_1)$ and $p_2 = (x, t_2)$, with $t_1 \neq t_2$. Assume $t_1 < t_2$ and take $t_3 > 2t_2$ so that there exists a curve in M joining p_1, p_2 contained in $\{t < t_3\}$. We can assume that $M \cap \{t = t_3\}$ is transversal. Let us denote $\Sigma = M \cap \{t < t_3\}$.

Suppose that $\bar{\Sigma}$ is compact. Then there exists a minimum for the height function t , where the mean curvature vector \vec{H} of Σ coincides with ∂_t . Hence, we can start the Alexandrov reflection method for Σ with horizontal planes coming from below, and obtain that M is symmetric with respect to a horizontal plane below $\{t = t_2\}$, which implies that M is compact, a contradiction. Thus $\bar{\Sigma}$ cannot be compact.

For any $\rho > 0$, consider a translated paraboloid $T_t \mathcal{P}^-$ with $T_t(p) = p + t\partial_t$ so that the solid cylinder $D_{\mathbb{H}^2}(x, \rho) \times [0, t_3]$ is contained in $T_t \mathcal{P}_{\text{int}}^-$, where $D_{\mathbb{H}^2}(x, \rho)$ is the hyperbolic disk centered at x with radius ρ . Since M has finite topology, we can assume that, for $\rho > 0$ large enough, any connected component of $M_\rho = M \cap T_t \mathcal{P}_{\text{ext}}^-$ is either compact with boundary on $T_t \mathcal{P}^-$ or an annular end of M with boundary on $T_t \mathcal{P}^-$. Since M is properly embedded, the number of compact components of M_ρ is finite.

For $\rho' > \rho$ large enough, we consider the vertical cylinder $C_{\rho'} = D_{\mathbb{H}^2}(x, \rho') \times \mathbb{R}$ which contains entirely $T_t \mathcal{P}^- \cap \mathcal{S}_{t_3}$ and all compact components of M_ρ with boundary in the paraboloid, where we recall that \mathcal{S}_{t_3} denotes the slab $\mathbb{H}^2 \times [0, t_3]$. If a point of M is outside this cylinder, then it is part of an end of M which has its boundary on the paraboloid $T_t \mathcal{P}^-$. We will only consider the part of M_ρ contained in the slab \mathcal{S}_{t_3} and outside the cylinder $C_{\rho'}$. Let us consider $R_{\rho'} = \mathcal{S}_{t_3} \setminus C_{\rho'}$.

Claim 4. We can take $\rho' > \rho$ large enough so that $M_\rho \cap \partial C_{\rho'} \cap \mathcal{S}_{t_3}$ is transversal and any non-compact component of $M_\rho \cap R_{\rho'}$ satisfies $\nu > 0$.

Since M is properly embedded, there are a finite number of non-compact components of M_ρ . It then suffices to prove the claim for any of them. Let E be a non-compact compo-

ment of M_ρ whose boundary is in $T_t\mathcal{P}^-$ and suppose by contradiction that there exists a diverging sequence of points $\{p_n\}_{n \in \mathbb{N}}$ in $E \cap R_{\rho'}$ with $\nu(p_n) \leq 0$. If $\nu \equiv 0$ in a neighborhood of a point p_n , M would be (locally at p_n) a cylinder over a curve of constant geodesic curvature $2H$. By the analytic continuation theorem, M would be a complete CMC cylinder contradicting the fact that M is contained in a halfspace. Then in a neighborhood of any p_n , there must exist a point $q_n \in M$ where $\nu(q_n) < 0$. Hence, we can work with a sequence of points $q_n \in E \cap R_{\rho'}$ with $\nu(q_n) < 0$ and the sequence $\{q_n\}_{n \in \mathbb{N}}$ diverges to $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$.

Now we are going to argue as in the proof of Theorem 3.1. We observe that the third coordinate of the points q_n is bounded by t_3 . For any n , since $\nu(q_n) < 0$, we can consider a translation of the paraboloid \mathcal{P}^- , denoted by $\mathcal{P}(q_n)$, tangent to E at q_n with same mean curvature vector. A similar argument as in Claim 1 yields that there exists a subsequence of divergent points $\{q_n\}_{n \in \mathbb{N}}$ such that $\nu(q_n) \leq \alpha_0 < 0$, for some $\alpha_0 \in \mathbb{R}^-$, and the paraboloid $\mathcal{P}(q_n)$ does not intersect $\partial E \subset T_t\mathcal{P}^-$ for n large enough. Now arguing as in Claims 2 and 3, we can conclude that there is curve α in the homology class of ∂E contained in the adherence of the mean-convex component determined by the tangent paraboloid. Hence, there is a subannulus \mathcal{A} of E with boundary $\alpha \cup \partial E$ and we can use a complete Scherk H -graph if $0 < H < 1/2$ or half a horizontal catenoid \mathcal{C}_a if $H = 1/2$, to reach a contradiction. This proves that for $\rho' > 0$ large enough, any point p of M_ρ contained in $R_{\rho'}$ satisfies $\nu(p) > 0$. To conclude Claim 4, it suffices to take ρ' larger, if necessary, so that the intersection $M_\rho \cap \partial C_{\rho'} \cap S_{t_3}$ is transversal.

We denote by $\tilde{\Sigma} = M \cap C_{\rho'} \cap S_{t_3}$, with boundary $\partial \tilde{\Sigma} \subset \partial C_{\rho'} \cup \{t = t_3\}$. We observe that if $p = (y_0, t_0) \in \partial \tilde{\Sigma} \cap \partial C_{\rho'}$ then there is no point $(y_0, t) \in \partial \tilde{\Sigma} \cap \partial C_{\rho'}$ with $t_0 < t < t_3$, since $\nu(y_0, t_0) > 0$ and any other point would also satisfy $\nu(y_0, t) > 0$, contradicting the embeddedness property of the surface. Therefore, $\partial \tilde{\Sigma} \cap \partial C_{\rho'}$ consists of curves projecting graphically onto $\partial D_{\mathbb{H}^2}(x, \rho')$ and/or curves in $D_{\mathbb{H}^2}(x, \rho') \times \{t_3\}$.

Now we apply the Alexandrov reflection method to $\tilde{\Sigma}$. We observe that the minimum for the height function on $\tilde{\Sigma}$ can be attained at an interior point and/or a boundary point in $\partial C_{\rho'}$. On both cases (taking ρ' larger if necessary) $\tilde{\Sigma}$ is locally a graph near that points, and we can start the Alexandrov reflection for $\tilde{\Sigma}$ using the family of horizontal planes coming from below. We obtain that $\tilde{\Sigma}$ must be symmetric with respect to a horizontal plane below $\{t = t_2\}$, but this is not possible as either M would be compact (if $\partial \tilde{\Sigma} \cap \partial C_{\rho'}$ is empty) or $\partial \tilde{\Sigma} \cap \partial C_{\rho'}$ would contain a point with positive angle function ν , contradicting Claim 4.

This contradiction proves that no two distinct points in M have the same vertical projection. Standard arguments show that the mean curvature vector of M is then nowhere horizontal, so M is a complete multigraph. Manzano and the third author proved in [13] that any complete multigraph with $0 < H \leq 1/2$ (see [9] for the case $H = 1/2$) must be a graph over an unbounded domain $\Omega \subset \mathbb{H}^2$.

Now let us prove that Ω is simply connected. If this were not the case, there would be at least a Jordan curve γ in $\partial\Omega$. Since M is complete, the graph would necessarily diverge over γ . It was proved in [10] that this is only possible if γ has constant geodesic curvature $\pm 2H$, a contradiction since γ is closed.

Finally, observe that for $H = 1/2$ the graph is necessarily entire by the result in [9]. ■

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