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The regularity problem for degenerate elliptic operators in weighted spaces

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Abstract. We study the solvability of the regularity problem for degenerate elliptic operators in the block case for data in weighted spaces. More precisely, let L_w be a degenerate elliptic operator with degeneracy given by a fixed weight $w \in A_2(dx)$ in $\mathbb{R}^n$, and consider the associated block second order degenerate elliptic problem in the upper-half space $\mathbb{R}^{n+1}_+$. We obtain non-tangential bounds for the full gradient of the solution of the block case operator given by the Poisson semigroup in terms of the gradient of the boundary data. All this is done in the spaces $L^p(v\,dw)$, where v is a Muckenhoupt weight with respect to the underlying natural weighted space $(\mathbb{R}^n, w\,dx)$. We recover earlier results in the non-degenerate case (when $w \equiv 1$, and with or without weight v). Our strategy is also different and more direct thanks in particular to recent observations on change of angles in weighted square function estimates and non-tangential maximal functions. Our method gives as a consequence the (unweighted) $L^2(dx)$-solvability of the regularity problem for the block operator

$$\mathbb{L}_\alpha u(x,t) = -|x|^\alpha \operatorname{div}_x \big(|x|^{-\alpha} A(x) \nabla_x u(x,t)\big) - \partial_t^2 u(x,t)$$

for any complex-valued uniformly elliptic matrix A and for all $-\varepsilon < \alpha < 2n/(n+2)$, where ε depends just on the dimension and the ellipticity constants of A.

1. Introduction

The study of divergence form degenerate elliptic equations was pioneered in the series of papers [22–24], where real symmetric elliptic matrices with some degeneracy expressed in terms of $A_2(dx)$-weights were considered (here and elsewhere, $A_2(dx) \equiv A_2(\mathbb{R}^n, dx)$). The goal of this paper is to obtain the solvability of the regularity problem for second order divergence form degenerate elliptic operators with complex coefficients and with boundary data in weighted Lebesgue spaces. To set the stage, let us introduce the class of operators that we consider here. Let A be an $n \times n$ matrix of complex L^∞-valued coeffi-

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cients defined on $\mathbb{R}^n$, $n \geq 2$. We assume that this matrix satisfies the following uniform ellipticity (or "accretivity") condition: there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda\,|\xi|^2 \leq \operatorname{Re} A(x)\,\xi \cdot \bar{\xi} \quad \text{and} \quad |A(x)\,\xi \cdot \bar{\zeta}| \leq \Lambda\,|\xi|\,|\zeta|, \tag{1.1}$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi \cdot \zeta = \xi_1\,\zeta_1 + \cdots + \xi_n\,\zeta_n$, and therefore $\xi \cdot \bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$. Associated with this matrix and a given weight $w \in A_2(dx)$ (which is fixed from now on, unless stated otherwise), we define the second order divergence form degenerate elliptic operator

$$L_w u = -w^{-1} \operatorname{div}(w\,A\,\nabla u), \tag{1.2}$$

which is understood as a maximal-accretive operator on $L^2(\mathbb{R}^n, w dx) \equiv L^2(w)$ with domain $\mathcal{D}(L_w)$ by means of a sesquilinear form. Note that writing $A_w = w\,A$, one has that A_w is a degenerate elliptic matrix in the sense that

$$\lambda\,|\xi|^2\,w(x) \leq \operatorname{Re} A_w(x)\,\xi \cdot \bar{\xi} \quad \text{and} \quad |A_w(x)\,\xi \cdot \bar{\zeta}| \leq \Lambda\,|\xi|\,|\zeta|\,w(x), \tag{1.3}$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. Conversely, if A_w is degenerate elliptic matrix satisfying the previous conditions one can trivially see that $A := w^{-1}\,A_w$ is uniformly elliptic.

The prominent case $w \equiv 1$ gives the class of uniformly elliptic operators. The celebrated resolution of the Kato problem in [5] established that if L is a uniformly divergence form elliptic operator (that is, $L = L_w$ with $w \equiv 1$), then $\sqrt{L} f$ is comparable to ∇f in $L^2(\mathbb{R}^n, dx) \equiv L^2(dx)$. This led to a new Calderón–Zygmund theory developed by the first named author in [1] to establish the boundedness in Lebesgue spaces of the associated functional calculus, vertical square function, Riesz transforms, reverse inequalities, etc. A key ingredient in that theory is the use of the so-called off-diagonal or Gaffney estimates satisfied by the associated heat semigroup and its gradient. This was later extended in [7–9], where the same operators were shown to satisfy weighted norm inequalities with Muckenhoupt weights. Conical square functions have been also considered in [6, 29]. Some of the previous results in conjunction with the theory of Hardy spaces for uniformly elliptic operators from [27, 28] led to [31], where the solvability of the regularity problem in the block case (see (1.4) and (1.5) below with $w \equiv 1$) for data in Lebesgue spaces was obtained. This amounted to control non-tangentially the full gradient of the solution given by the Poisson semigroup in terms of the gradient of the boundary datum. In turn, using the weighted Hardy space theory developed in [29, 30, 32], the solvability of the regularity problem in the block case for data in Lebesgue spaces with Muckenhoupt weights has been recently studied in [13].

Concerning the Kato problem in the general case, where L_w is a degenerate elliptic operator as above with a generic $w \in A_2(dx)$, [19] (see also [17, 18]) showed that $\sqrt{L_w} f$ is comparable to ∇f in $L^2(w)$. The boundedness of the associated operators (functional calculus, Riesz transform, reverse inequalities, vertical square functions, etc.), both in the natural Lebesgue spaces $L^p(w)$ and also in weighted spaces $L^p(v\,dw)$ with $v \in A_\infty(w)$ was considered in [16]. A particular case of interest was that on which, under further assumptions in w, the authors showed the equivalence of $\sqrt{L_w} f$ and ∇f in $L^2(dx)$ by simply taking $v = w^{-1}$. That is, the $L^2(dx)$-problem Kato problem was solved for a class of degenerate elliptic operators that goes beyond that of uniformly elliptic. For instance, [16] considered $L_\gamma = -|\cdot|^\gamma \operatorname{div}(|\cdot|^{-\gamma} A(\cdot)\nabla)$, where A is a uniformly elliptic matrix, $\gamma \in (-\varepsilon, 2n/(n+2))$, and ε depends on the dimension and the ellipticity constants of A. Some work has been also done concerning conical square functions with respect to the heat or Poisson semigroup generated by L_w and their gradients. For example, in [12] the last three authors of the present paper established the boundedness and the comparability of some conical square functions extending to the degenerate case the results from [29]. Moreover, in [33], the last named author has made a deeper study of the vertical and conical square functions and some non-tangential maximal functions arising from degenerate elliptic operators. On another direction, in [10] the authors considered L^2-boundary value problems for degenerate elliptic equations and systems. In particular, they initiated the study of Dirichlet and Neumann problems in the degenerate setting using the so-called first order method.

Our goal in this paper is to contribute to this theory by studying the solvability of the regularity problem for degenerate elliptic operators and also propose other methods, as it is explained after Theorem 1.1. More precisely, consider the degenerate elliptic operator $L_w = -w^{-1} \operatorname{div}(w\, A\, \nabla)$, where $w \in A_2(dx)$ and A is an $n \times n$ matrix of complex L^∞-valued coefficients defined on $\mathbb{R}^n$, $n \geq 2$, which is a uniformly elliptic matrix (see (1.1)) with ellipticity constants $0 < \lambda \leq \Lambda < \infty$. Introduce the $(n+1) \times (n+1)$ block matrix

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.4}$$

which is $(n+1) \times (n+1)$ uniformly elliptic with ellipticity constants $0 < \min\{\lambda, 1\} \leq \max\{\Lambda, 1\}$. This gives rise to the block degenerate elliptic operator in $\mathbb{R}^{n+1}$,

$$\mathbb{L}_w u = -w^{-1} \operatorname{div}_{x,t}(w\, \mathbb{A} \nabla_{x,t} u) = -w^{-1} \operatorname{div}_x(w\, A \nabla_x u) - \partial_t^2 u = (L_w)_x u - \partial_t^2 u. \tag{1.5}$$

Here and elsewhere, $\nabla_{x,t}$ denotes the full gradient, while the symbols ∇ and ∇_x refer just to the spatial derivatives. Note that in the previous equality we have used that w does not depend on the t variable, hence by applying Fubini's theorem it is not difficult to see that, with a slight abuse of notation, if we write $w(x,t) := w(x)$ for every $(x,t) \in \mathbb{R}^{n+1}$, then $w \in A_2(\mathbb{R}^{n+1}, dx)$ since $w \in A_2(dx)$.

The operator $-L_w$ generates a C^0-semigroup $\{e^{-tL_w}\}_{t>0}$ of contractions on $L^2(w)$ which is called the heat semigroup. This and the subordination formula (see (3.1) below) yield that $\{e^{-t\sqrt{L_w}}\}_{t>0}$ is a C^0-semigroup of contractions on $L^2(w)$. Hence, whenever $f \in C_c^\infty(\mathbb{R}^n)$, the function (called semigroup solution) given by the semigroup formula $u(x,t) := e^{-t\sqrt{L_w}} f(x)$, with $(x,t) \in \mathbb{R}^{n+1}_+$, is a strong solution in $C^2((0,\infty); L^2(w)) \cap C((0,\infty); \mathcal{D}(L_w))$ of the evolution equation $\partial_t^2 u(t) = L_w u(t)$ satisfied in $L^2(w)$ for all $t > 0$. But we can rather interpret $\mathbb{L}_w u = 0$ in $\mathbb{R}^{n+1}_+$ in a weak sense and u is also weak solution of it: by this we mean that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, dw\, dt)$ satisfies

$$\iint_{\mathbb{R}^{n+1}_+} \mathbb{A}(x) \nabla_{x,t} u(x,t) \cdot \nabla_{x,t} \psi(x,t)\, dw(x)\, dt = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^{n+1}_+). \tag{1.6}$$

Also, $u(\cdot, t) \to f$ in $L^2(w)$ as $t \to 0^+$ by the semigroup continuity (see, e.g., [20]). As usual, $dw(x) \equiv w(x)dx$.

Consider the L^2-non-tangential maximal function $\mathcal{N}_w$ defined in [10]:

$$\mathcal{N}_w h(x) := \sup_{t>0} \Big(\fint\!\!\fint_{W(x,t)} |h(y,s)|^2 \, dw(y)\, ds \Big)^{1/2}, \quad h \in L^2_{\rm loc}(\mathbb{R}^{n+1}_+, dw\, dt), \tag{1.7}$$

where $W(x,t) := (c_0^{-1}t, c_0 t) \times B(x, c_1 t)$ is a Whitney region and $c_0 > 1$, $c_1 > 0$ are fixed parameters throughout the paper.

Note that our assumption $w \in A_2(dx)$ implies that w is a doubling measure in $\mathbb{R}^n$, hence $(\mathbb{R}^n, w, |\cdot|)$ is a space of homogeneous type. Given $1 < p < \infty$ and $v \in A_\infty(w)$, we say that the weighted regularity problem $(R^{\mathbb{L}_w})_{L^p(v dw)}$ is solvable if for every $f \in C_c^\infty(\mathbb{R}^n)$ the weak solution of $\mathbb{L}_w u = 0$ in $\mathbb{R}^{n+1}_+$, given by $u(x,t) := e^{-t\sqrt{L_w}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, satisfies the following weighted non-tangential maximal function estimate:

$$\|\mathcal{N}_w(\nabla_{x,t} u)\|_{L^p(v dw)} \le C \|\nabla f\|_{L^p(v dw)}. \tag{1.8}$$

Once this estimate is under control, one can extend the semigroup to general data. However, the status of convergence to the boundary of the solution needs a specific treatment that is not addressed here.

As in [1,9,16], we denote by $(p_-(L_w), p_+(L_w))$ and by $(q_-(L_w), q_+(L_w))$ the maximal open intervals on which the heat semigroup $\{e^{-tL_w}\}_{t>0}$ and the gradient of the heat semigroup $\{\sqrt{t}\nabla e^{-tL_w}\}_{t>0}$ are respectively uniformly bounded on $L^p(w)$. That is,

$$p_-(L_w) := \inf\Big\{ p \in (1,\infty) : \sup_{t>0} \|e^{-tL_w}\|_{L^p(w)\to L^p(w)} < \infty \Big\},$$

$$p_+(L_w) := \sup\Big\{ p \in (1,\infty) : \sup_{t>0} \|e^{-tL_w}\|_{L^p(w)\to L^p(w)} < \infty \Big\},$$

$$q_-(L_w) := \inf\Big\{ p \in (1,\infty) : \sup_{t>0} \|\sqrt{t}\nabla e^{-tL_w}\|_{L^p(w)\to L^p(w)} < \infty \Big\},$$

$$q_+(L_w) := \sup\Big\{ p \in (1,\infty) : \sup_{t>0} \|\sqrt{t}\nabla e^{-tL_w}\|_{L^p(w)\to L^p(w)} < \infty \Big\}.$$

We need to introduce some extra notation (see Section 2). Set $r_w := \inf\big\{p : w \in A_p(dx)\big\}$, and note that $1 \le r_w < 2$ since $w \in A_2(dx)$. Given $0 \le p_0 < q_0 \le \infty$ and $v \in A_\infty(w) = A_\infty(\mathbb{R}^n, w\,dx)$, define

$$\mathcal{W}_v^w(p_0, q_0) := \big\{ p \in (p_0, q_0) : v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w) \big\}.$$

We are now ready to state our main result.

Theorem 1.1. *Let $w \in A_2(dx)$ and let $\mathbb{L}_w$ be a block degenerate elliptic operator in $\mathbb{R}^{n+1}_+$ as above. Let $v \in A_\infty(w)$ be such that*

$$\mathcal{W}_v^w\big(\max\{r_w, q_-(L_w)\}, q_+(L_w)\big) \neq \emptyset. \tag{1.9}$$

Then, for every $p \in \mathcal{W}_v^w\big(\max\big\{r_w, \frac{n r_w q_-(L_w)}{n r_w + q_-(L_w)}\big\}, q_+(L_w)\big)$ and every $f \in C_c^\infty(\mathbb{R}^n)$, if one sets $u(x,t) = e^{-t\sqrt{L_w}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, then

$$\|\mathcal{N}_w(\nabla_{x,t} u)\|_{L^p(v dw)} \le C \|\nabla f\|_{L^p(v dw)} \tag{1.10}$$

and $(R^{\mathbb{L}_w})_{L^p(v dw)}$ is solvable.

Let us compare this result with some previous work. When $w \equiv 1$ (that is, we are working with the class of uniformly – or non-degenerate – elliptic operators) and $v \equiv 1$, then, clearly, $r_w = 1$, $\mathcal{W}_v^w(\max\{r_w, q_-(L_w)\}, q_+(L_w)) = (q_-(L_w), q_+(L_w)) \neq \emptyset$, and our result gives (1.10) in the range $\big(\max\big\{1, \frac{n q_-(L_w)}{n+q_-(L_w)}\big\}, q_+(L_w)\big)$, hence we fully recover Theorem 4.1 in [31]. If we still assume that $w \equiv 1$ and we let $v \in A_\infty(w) = A_\infty(dx)$, then our assumption (1.9) agrees with that in Theorem 1.10 of [13] and the range of p's here is slightly worse than the one in that result (the lower end-point in Theorem 1.10 of [13] has been pushed down using an extra technical argument that we have chosen not to follow here).

Sharpness of the exponents is a delicate issue. It is not known even when $w \equiv v \equiv 1$. Let us discuss this situation for information and relate to other estimates. The upper bound $p < q_+(L)$ in (1.10) is by definition sharp for the L^p boundedness of $\sqrt{t}\nabla e^{-tL}$. It is shown in [4], Section 12, that this is also sharp for the L^p bound of $t\nabla e^{-t\sqrt{L}}$. But global L^p control of such non-tangential maximal functions with inside L^2 averages does not imply L^p boundedness when $p > 2$: namely, $\|\nabla_{x,t} e^{-t\sqrt{L}} f\|_p \lesssim \|\nabla f\|_p$ is not necessary and the maximal estimate could be obtained for other reasons as seen in [6] for some square functions. Consider the lower bound in p. First, it is the same as the one found in [1] for $\|\sqrt{L} f\|_p \lesssim \|\nabla f\|_p$, and for this one sharpness is not known (although it is believable that it should be). Secondly, when $p \le 2$, a control of this non-tangential maximal function does imply L^p boundedness. More precisely, one deduces $\|\nabla_{x,t} e^{-t\sqrt{L}} f\|_p \lesssim \|\nabla f\|_p$ uniformly in $t > 0$ (and one recovers $\|\sqrt{L} f\|_p \lesssim \|\nabla f\|_p$ when $t \to 0$). The same lower bound in p is shown using the theory of Hardy spaces adapted to L in [4], Section 9, but here too it is said that sharpness is not known.

Our methods to prove Theorem 1.1, in particular the estimate involving $\partial_t u$, are also novel. The above works used advanced technology of Hardy spaces adapted to operators: developing them in our context is probably a new challenge in itself. Instead, we rely on recent change of angle formulas for weighted conical square function estimates (see Section 2.4) and also the ones we prove for non-tangential weighted maximal functions (see Lemma 3.3), which allow us to implement more directly standard tools in the field.

An important consequence of our method is that we can obtain the solvability of the regularity problem corresponding to data in unweighted Lebesgue spaces. The main idea consists in taking $v = w^{-1}$ in Theorem 1.1. The following result focuses on the case of the L^2-solvability (more general results are presented in Section 4, see Corollaries 4.1 and 4.2).

Corollary 1.2. *Let $w \in A_2(dx)$ and let $\mathbb{L}_w$ be a block degenerate elliptic operator in $\mathbb{R}^{n+1}_+$ as above. Given $\Theta \ge 1$, there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda) \in (0, \frac{1}{2n}]$, such that for every $w \in A_{1+\varepsilon}(dx) \cap \mathrm{RH}_{\max\{\frac{2}{1-\varepsilon}, 1+(1+\varepsilon)\frac{n}{2}\}}(dx)$ with $0 \le \varepsilon < \varepsilon_0$ and $[w]_{A_2(dx)} \le \Theta$, then*

$$\|\mathcal{N}_w(\nabla_{x,t} u)\|_{L^2(dx)} \le C \|\nabla f\|_{L^2(dx)}, \quad \text{for every } f \in C_c^\infty(\mathbb{R}^n), \tag{1.11}$$

where $u(x,t) = e^{-t\sqrt{L_w}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$. Hence $(R^{\mathbb{L}_w})_{L^2(dx)}$ is solvable.

Furthermore, if we set

$$\mathbb{L}_\alpha u(x,t) = -|x|^\alpha \operatorname{div}_x \big(|x|^{-\alpha} A(x) \nabla_x u(x,t)\big) - \partial_t^2 u(x,t),$$

where A is an $n \times n$ matrix of complex L^∞-valued coefficients defined on $\mathbb{R}^n$, $n \geq 2$, satisfying the uniform ellipticity condition (1.1)*, then there exists $0 < \varepsilon < 1/2$ small enough (depending only on the dimension and the ratio Λ/λ) such that if $-\varepsilon < \alpha < 2n/(n+2)$, then* (1.11) *holds in this scenario and $(R^{\mathbb{L}_\alpha})_{L^2(dx)}$ is solvable.*

The plan of the paper is as follows. In Section 2 we introduce notations and definitions, and we recall some known results. We also obtain estimates for some inhomogeneous vertical and conical square functions which are interesting in their own right (see Propositions 2.14 and 2.18). To prove our main result, Theorem 1.1, we split the main estimate into two independent pieces, one regarding $\mathcal{N}_w(\nabla_x u)$ and the other one related to $\mathcal{N}_w(\partial_t u)$, see respectively Propositions 3.1 and 3.2 in Section 3. In Section 4 we study the solvability of the regularity problem in unweighted Lebesgue spaces and, in particular, we prove Corollary 1.2.

2. Preliminaries

We shall use the following notation: dx denotes the usual Lebesgue measure in $\mathbb{R}^n$, dw denotes the measure in $\mathbb{R}^n$ given by the weight w, and $v dw$ or $d(vw)$ denotes the one given by the product weight vw. Besides, throughout the paper n will denote the dimension of the underlying space $\mathbb{R}^n$ and we shall always assume $n \geq 2$.

Given a ball B, let r_B denote the radius of B. We write λB for the concentric ball with radius λr_B, $\lambda > 0$. Moreover, we set $C_1(B) = 4B$ and, for $j \geq 2$, $C_j(B) = 2^{j+1}B \setminus 2^j B$.

2.1. Weights

We need to introduce some classes of Muckenhoupt weights. Namely, $A_\infty(dx)$, on which the underlying measure space is $(\mathbb{R}^n, dx)$, and then fix $w \in A_\infty(dx)$ and consider the class $A_\infty(w)$ where the "weighted" underlying space is $(\mathbb{R}^n, dw)$.

2.1.1. $A_\infty(dx)$ weights. By a weight w we mean a non-negative, locally integrable function. For brevity, we will often write dw for $w\,dx$. In particular, we write $w(E) = \int_E dw$ and $L^p(w) = L^p(\mathbb{R}^n, dw)$. We will use the following notation for averages: given a set E such that $0 < w(E) < \infty$,

$$\fint_E f\,dw = \frac{1}{w(E)} \int_E f\,dw,$$

or, if $0 < |E| < \infty$,

$$\fint_E f\,dx = \frac{1}{|E|} \int_E f\,dx.$$

Abusing slightly the notation, for $j \geq 1$, we set

$$\fint_{C_j(B)} f\,dw = \frac{1}{w(2^{j+1}B)} \int_{C_j(B)} f\,dw.$$

We state some definitions and basic properties of Muckenhoupt weights. For further details, see [21,25,26]. Consider the Hardy–Littlewood maximal function

$$\mathcal{M} f(x) := \sup_{B \ni x} \fint_B |f(y)|\, dy.$$

It is well known that given a weight w, $\mathcal{M}$ is bounded on $L^p(w)$ if and only if $w \in A_p(dx)$, $1 < p < \infty$, where we say that $w \in A_p(dx)$, $1 < p < \infty$, if

$$[w]_{A_p(dx)} := \sup_B \Big(\fint_B w(x)\, dx \Big) \Big(\fint_B w(x)^{1-p'}\, dx \Big)^{p-1} < \infty.$$

Here and below, the sups run over the collection of balls $B \subset \mathbb{R}^n$. When $p = 1$, $\mathcal{M}$ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ if and only if $w \in A_1(dx)$, that is, if

$$[w]_{A_1(dx)} := \sup_B \Big(\fint_B w(x)\, dx \Big) \Big(\operatorname*{ess\,sup}_{x \in B} w(x)^{-1} \Big) < \infty.$$

We also introduce the reverse Hölder classes. We say that $w \in \mathrm{RH}_s(dx)$, $1 < s < \infty$, if

$$[w]_{\mathrm{RH}_s(dx)} := \sup_B \Big(\fint_B w(x)\, dx \Big)^{-1} \Big(\fint_B w(x)^s\, dx \Big)^{1/s} < \infty,$$

and

$$[w]_{\mathrm{RH}_\infty(dx)} := \sup_B \Big(\fint_B w(x)\, dx \Big)^{-1} \Big(\operatorname*{ess\,sup}_{x \in B} w(x) \Big) < \infty.$$

It is also well known that

$$A_\infty(dx) := \bigcup_{1 \le p < \infty} A_p(dx) = \bigcup_{1 < s \le \infty} \mathrm{RH}_s(dx).$$

Throughout the paper we shall use in several places the following properties. Namely, if $w \in \mathrm{RH}_s(dx)$, $1 < s \le \infty$,

$$\frac{w(E)}{w(B)} \le [w]_{\mathrm{RH}_s(dx)} \Big(\frac{|E|}{|B|} \Big)^{1/s'}, \quad \forall E \subset B, \tag{2.1}$$

where B is any ball in $\mathbb{R}^n$. Analogously, if $w \in A_p(dx)$, $1 \le p < \infty$, then

$$\Big(\frac{|E|}{|B|} \Big)^p \le [w]_{A_p(dx)} \frac{w(E)}{w(B)}, \quad \forall E \subset B. \tag{2.2}$$

This implies in particular that w is a doubling measure, that is,

$$w(\lambda B) \le [w]_{A_p(dx)}\, \lambda^{np}\, w(B), \quad \forall B,\ \forall \lambda > 1. \tag{2.3}$$

We continue by introducing some important notation. Weights in the classes $A_p(dx)$ and $\mathrm{RH}_s(dx)$ have a self-improving property: if $w \in A_p(dx)$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}(dx)$, and similarly if $w \in \mathrm{RH}_s(dx)$, then $w \in \mathrm{RH}_{s+\delta}(dx)$ for some $\delta > 0$. Hereafter, given $w \in A_p(dx)$, let

$$r_w = \inf\{p : w \in A_p(dx)\} \quad \text{and} \quad s_w = \inf\{q : w \in \mathrm{RH}_{q'}(dx)\}. \tag{2.4}$$

Note that, according to our definition, s_w is the conjugated exponent of the one defined in Lemma 4.1 of [8]. Given $0 \le p_0 < q_0 \le \infty$ and $w \in A_\infty(dx)$, Lemma 4.1 in [8] implies that

$$\mathcal{W}_w(p_0,q_0) := \Big\{p \in (p_0,q_0) : w \in A_{p/p_0}(dx) \cap \mathrm{RH}_{(q_0/p)'}(dx)\Big\} = \Big(p_0 r_w, \frac{q_0}{s_w}\Big). \tag{2.5}$$

In the case $p_0 = 0$ and $q_0 < \infty$, it is understood that the only condition that stays is $w \in \mathrm{RH}_{(q_0/p)'}(dx)$. Analogously, if $0 < p_0$ and $q_0 = \infty$, the only assumption is $w \in A_{p/p_0}(dx)$. Finally, $\mathcal{W}_w(0,\infty) = (0,\infty)$.

Furthermore, given $p \in (0,\infty)$ and a weight $w \in A_\infty(dx)$, we define the following Sobolev exponents with respect to w:

$$(p)_{w,*} := \frac{n r_w p}{n r_w + p}, \tag{2.6}$$

and, for $k \in \mathbb{N}$,

$$p_w^{k,*} := \begin{cases} \frac{n r_w p}{n r_w - kp} & \text{if } n r_w > kp, \\ \infty & \text{otherwise.} \end{cases} \tag{2.7}$$

We write $p_w^* := p_w^{1,*}$.

2.1.2. $A_\infty(w)$ weights. Fix now $w \in A_\infty(dx)$. As mentioned above, (2.3) says that w is a doubling measure, hence $(\mathbb{R}, dw, |\cdot|)$ is a space of homogeneous type (here and elsewhere, $|\cdot|$ stands for the ordinary Euclidean distance). One can then introduce the weighted maximal operator

$$\mathcal{M}^w f(x) := \sup_{B \ni x} \fint_B |f(y)|\, dw(y). \tag{2.8}$$

Much as before, $\mathcal{M}^w$ is bounded on $L^p(v\,dw)$, $1 < p < \infty$, if and only if $v \in A_p(w)$, which means that

$$[v]_{A_p(w)} = \sup_B \Big(\fint_B v(x)\, dw\Big)\Big(\fint_B v(x)^{1-p'}\, dw\Big)^{p-1} < \infty. \tag{2.9}$$

Analogously, we can define the classes $\mathrm{RH}_s(w)$ by replacing the Lebesgue measure in the definitions above with dw: $v \in \mathrm{RH}_s(w)$, $1 < s < \infty$, if

$$[v]_{\mathrm{RH}_s(w)} = \sup_B \Big(\fint_B v(x)\, dw\Big)^{-1}\Big(\fint_B v(x)^s\, dw\Big)^{1/s} < \infty. \tag{2.10}$$

From these definitions, it follows at once that there is a "duality" relationship between the weighted and unweighted $A_p(dx)$ and $\mathrm{RH}_s(dx)$ conditions: $w^{-1} \in A_p(w)$ if and only if $w \in \mathrm{RH}_{p'}(dx)$, and $w^{-1} \in \mathrm{RH}_s(w)$ if and only if $w \in A_{s'}(dx)$.

For every measurable set $E \in \mathbb{R}^n$, we write $vw(E) = \int_E d(vw) = \int_E v\,dw = (v\,dw)(E)$ and $L^p(v\,dw) = L^p(\mathbb{R}^n, v(x)\,w(x)\,dx)$. In this direction, for every $w \in A_p(dx)$, $v \in A_q(w)$, $1 \le p, q < \infty$, it follows that

$$\Big(\frac{|E|}{|B|}\Big)^{pq} \le [w]_{A_p(dx)}^q \Big(\frac{w(E)}{w(B)}\Big)^q \le [w]_{A_p(dx)}^q [v]_{A_q(w)} \frac{vw(E)}{vw(B)}, \quad \forall E \subset B. \tag{2.11}$$

Analogously, if $w \in \mathrm{RH}_p(dx)$ and $v \in \mathrm{RH}_q(w)$, $1 < p, q \le \infty$, one has

$$(2.12)\quad \frac{vw(E)}{vw(B)} \le [v]_{\mathrm{RH}_q(w)} \Big(\frac{w(E)}{w(B)}\Big)^{1/q'} \le [v]_{\mathrm{RH}_q(w)} [w]_{\mathrm{RH}_p(dx)}^{1/q'} \Big(\frac{|E|}{|B|}\Big)^{1/(p'q')}, \quad \forall E \subset B.$$

As before, for a weight $v \in A_\infty(w)$ (recall that $w \in A_\infty(dx)$ is fixed) we set

$$(2.13)\qquad \mathrm{r}_v(w) := \inf\{r : v \in A_r(w)\} \quad \text{and} \quad \mathfrak{s}_v(w) := \inf\{s : v \in \mathrm{RH}_{s'}(w)\}.$$

For $0 \le p_0 < q_0 \le \infty$ and $v \in A_\infty(w)$, by a similar argument to that of Lemma 4.1 in [8], we have

$$(2.14)\qquad \mathcal{W}_v^w(p_0, q_0) := \{p \in (p_0, q_0) : v \in A_{p/p_0}(w) \cap \mathrm{RH}_{(q_0/p)'}(w)\} = \Big(p_0 \mathrm{r}_v(w), \frac{q_0}{\mathfrak{s}_v(w)}\Big).$$

If $p_0 = 0$ and $q_0 < \infty$, as before, it is understood that the only condition that stays is $v \in \mathrm{RH}_{(q_0/p)'}(w)$. Analogously, if $0 < p_0$ and $q_0 = \infty$, the only assumption is $v \in A_{p/p_0}(w)$. Finally, $\mathcal{W}_v^w(0, \infty) = (0, \infty)$.

Remark 2.1. The proof of our main result will use the Calderón–Zygmund decomposition from Lemma 2.13 with respect to the underlying measure $v(x)\,dw(x) = v(x)w(x)\,dx$, where $w \in A_\infty(dx)$ and $v \in A_\infty(w)$. In that scenario, it was shown in [33], Remark 2.15, that $wv \in A_\infty(dx)$ and moreover $r_{vw} \le r_w \mathrm{r}_v(w)$. The converse inequality is false in general: let $w(x) := |x|^n$ and $v := w^{-1}$; then one can easily see that $r_w \mathrm{r}_v(w) = r_w s_w = 2$ and $r_{vw} = 1$.

We state a lemma which will be useful in the sequel.

Lemma 2.2 (Remark 2.16 in [33]). *Let $B \subset \mathbb{R}^n$ be a ball and let $j \ge 1$. Given $0 < p \le q < \infty$, the following holds.*

(a) *If $v \in A_{q/p}(w)$, then*

$$\Big(\fint_{C_j(B)} |f(x)|^p\, dw(x)\Big)^{1/p} \lesssim \Big(\fint_{C_j(B)} |f(x)|^q\, d(vw)(x)\Big)^{1/q}.$$

(b) *If $v \in \mathrm{RH}_{(q/p)'}(w)$, then*

$$\Big(\fint_{C_j(B)} |f(x)|^p\, d(vw)(x)\Big)^{1/p} \lesssim \Big(\fint_{C_j(B)} |f(x)|^q\, dw(x)\Big)^{1/q}.$$

2.2. Square functions and non-tangential maximal functions

In this section, we introduce several auxiliary operators (vertical and conical square functions, non-tangential maximal functions) which will be needed at various points along the proofs.

Consider, for $\kappa \ge 1$, the non-tangential maximal function $\mathcal{N}^{\kappa,w}$ defined as

$$(2.15)\qquad \mathcal{N}^{\kappa,w} F(x) := \sup_{t>0} \Big(\int_{B(x,\kappa t)} |F(y,t)|^2 \frac{dw(y)}{w(B(x,t))}\Big)^{1/2}.$$

We write $\mathcal{N}^w$ when $\kappa = 1$. We are particularly interested in the non-tangential maximal functions associated with the heat or Poisson semigroup. For $f \in L^2(w)$, define

$$\mathcal{N}_{\mathrm{H}}^{\kappa,w} f(x) := \sup_{t>0} \Big(\int_{B(x,\kappa t)} \big|e^{-t^2 L_w} f(y)\big|^2 \frac{dw(y)}{w(B(x,t))} \Big)^{1/2}, \tag{2.16}$$

$$\mathcal{N}_{\mathrm{P}}^{\kappa,w} f(x) := \sup_{t>0} \Big(\int_{B(x,\kappa t)} \big|e^{-t\sqrt{L_w}} f(y)\big|^2 \frac{dw(y)}{w(B(x,t))} \Big)^{1/2}. \tag{2.17}$$

Again, when $\kappa = 1$, we write $\mathcal{N}_{\mathrm{P}}^w$ and $\mathcal{N}_{\mathrm{H}}^w$. We shall obtain weighted boundedness of these operators in Section 3.2.

We also consider several variants of the vertical square functions associated with the heat semigroup which were studied in [16], Sections 5 and 10:

$$\mathrm{g}_{\mathrm{H}}^w f(x) := \Big(\int_0^\infty \big|t^2 L_w e^{-t^2 L_w} f(x)\big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{2.18}$$

$$\mathrm{G}_{1/2,\mathrm{H}}^w f(x) := \Big(\int_0^\infty \big|t\nabla (t^2 L_w)^{1/2} e^{-t^2 L_w} f(x)\big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{2.19}$$

$$\mathrm{G}_{\mathrm{H}}^w f(x) := \Big(\int_0^\infty \big|t\nabla t^2 L_w e^{-t^2 L_w} f(x)\big|^2 \frac{dt}{t} \Big)^{1/2}. \tag{2.20}$$

Proceeding as in Propositions 5.1 and 10.1 of [16], by a standard argument, we have the following lemma.

Lemma 2.3. *Let L_w be a degenerate elliptic operator with $w \in A_2(dx)$ and let $v \in A_\infty(w)$. Then*

(a) *$\mathrm{g}_{\mathrm{H}}^w$ is bounded on $L^p(v dw)$ for all $p \in \mathcal{W}_v^w\,(p_-(L_w), p_+(L_w))$,*

(b) *$\mathrm{G}_{1/2,\mathrm{H}}^w$ and $\mathrm{G}_{\mathrm{H}}^w$ are bounded on $L^p(v dw)$ for all $p \in \mathcal{W}_v^w\,(q_-(L_w), q_+(L_w))$.*

Now we recall the following conical square functions studied by the authors in [12]:

$$\mathcal{S}_{\mathrm{H}}^{\alpha,w} f(x) := \Big(\iint_{\Gamma^\alpha(x)} \big|t^2 L_w e^{-t^2 L_w} f(y)\big|^2 \frac{dw(y)\,dt}{t\,w(B(y,t))} \Big)^{1/2}, \tag{2.21}$$

where $\Gamma^\alpha(x) := \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < \alpha t\}$ is the cone with vertex at x and aperture $\alpha > 0$. When $\alpha = 1$, we write $\Gamma(x)$ and $\mathcal{S}_{\mathrm{H}}^w$. According to Proposition 3.1 in [12], we have that $\mathcal{S}_{\mathrm{H}}^w$ is bounded on $L^p(v dw)$ for all $p \in \mathcal{W}_v^w\,(p_-(L_w), \infty)$.

Finally, we introduce the following “inhomogeneous” vertical and conical square functions:

$$\widetilde{\mathrm{G}}_{\mathrm{H}}^w f(x) := \Big(\int_0^\infty \big|\nabla t^2 L_w e^{-t^2 L_w} f(x)\big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{2.22}$$

$$\widetilde{\mathcal{S}}_{\mathrm{H}}^w f(x) := \Big(\iint_{\Gamma(x)} \big|t^{-1}(t^2 L_w) e^{-t^2 L_w} f(y)\big|^2 \frac{dw(y)\,dt}{t\,w(B(y,t))} \Big)^{1/2}. \tag{2.23}$$

By inhomogeneity we mean that the power rule of t inside the square functions is not in accordance with that of the operator L_w: we are modifying respectively $\mathrm{G}_{\mathrm{H}}^w$ and $\mathcal{S}_{\mathrm{H}}^w$ by removing one power of t which makes the modified square functions applied to f homogeneous instead to the gradient of f, so that we expect bounds in terms on ∇f only for them. The analogues of the above two square functions in other settings turn to be very useful in the study of Riesz transform and Hardy space theory, see for instance [14, 27]. Sections 2.6 and 2.7 below study the boundedness of $\widetilde{\mathrm{G}}_{\mathrm{H}}^{w}$ and $\widetilde{\mathcal{S}}_{\mathrm{H}}^{w}$ on weighted Sobolev spaces, which plays an essential role in the proof of our main results.

We finish this subsection by recalling the results about the reverse inequality of the Riesz transform associated with the operator L_w proved in [16]. The Riesz transform $\nabla L_w^{-1/2}$ associated with the operator L_w can be written as

$$\nabla L_w^{-1/2} = \frac{2}{\sqrt{\pi}} \int_0^\infty t \nabla e^{-t^2 L_w} \frac{dt}{t},$$

Consider also the following square root representation (see for instance [10, 19]):

$$\sqrt{L_w} = \frac{2}{\sqrt{\pi}} \int_0^\infty t L_w e^{-t^2 L_w} \frac{dt}{t}. \tag{2.24}$$

Proposition 2.4 (Proposition 6.1 in [16]). *Let* $\max\{r_w, (p_-(L_w))_{w,*}\} < p < p_+(L_w)$. *Then for all* $f \in \mathcal{S}$,

$$\|\sqrt{L_w} f\|_{L^p(w)} \lesssim \|\nabla f\|_{L^p(w)}.$$

Furthermore, if $p \in \mathcal{W}_v^w(\max\{r_w, p_-(L_w)\}, p_+(L_w))$, *then for all* $f \in \mathcal{S}$,

$$\|\sqrt{L_w} f\|_{L^p(v dw)} \lesssim \|\nabla f\|_{L^p(v dw)}.$$

2.3. Off-diagonal estimates

Definition 2.5. Let $\{T_t\}_{t>0}$ be a family of sublinear operators, and let $1 \le p \le \infty$. Given a doubling measure μ, we say that $\{T_t\}_{t>0}$ satisfies $L^p(\mu) - L^p(\mu)$ full off-diagonal estimates, denoted by $T_t \in \mathcal{F}(L^p(\mu) - L^p(\mu))$, if there exist constants $C, c > 0$ such that for all closed sets E and F, all $f \in L^p(\mathbb{R}^n)$, and all $t > 0$, we have

$$\Big(\int_F |T_t(\mathbf{1}_E f)|^p d\mu\Big)^{1/p} \le C\, e^{-cd(E,F)^2/t} \Big(\int_E |f|^p d\mu\Big)^{1/p}, \tag{2.25}$$

where $d(E, F) = \inf\{|x - y| : x \in E, y \in F\}$.

In the previous definition, when $p = \infty$ one has to change the L^p-norms by the corresponding essential suprema.

Set $\Upsilon(s) = \max\{s, s^{-1}\}$ for $s > 0$. Recall that, given a ball B, we use the notation $C_j(B) = 2^{j+1}B \backslash 2^j B$ for $j \ge 2$, and for any doubling measure μ,

$$\fint_B h\, d\mu = \frac{1}{\mu(B)} \int_B h\, d\mu, \quad \fint_{C_j(B)} h\, d\mu = \frac{1}{\mu(2^{j+1}(B))} \int_{C_j(B)} h\, d\mu.$$

Definition 2.6. Given $1 \le p \le q \le \infty$ and any doubling measure μ, we say that a family of sublinear operators $\{T_t\}_{t>0}$ satisfies $L^p(\mu) - L^q(\mu)$ off-diagonal estimates on balls, denoted by $T_t \in \mathcal{O}(L^p(\mu) - L^q(\mu))$, if there exist $\theta_1, \theta_2 > 0$ and $c > 0$ such that for all $t > 0$ and for all ball B with radius r_B,

$$\Big(\fint_B |T_t(f\, \mathbf{1}_B)|^q d\mu\Big)^{1/q} \lesssim \Upsilon\Big(\frac{r_B}{\sqrt{t}}\Big)^{\theta_2} \Big(\fint_B |f|^p d\mu\Big)^{1/p}, \tag{2.26}$$

and for $j \ge 2$,

$$\Big(\fint_B \big|T_t(f\,\mathbf{1}_{C_j(B)})\big|^q\,d\mu\Big)^{1/q} \lesssim 2^{j\theta_1}\Upsilon\Big(\frac{2^j r_B}{\sqrt{t}}\Big)^{\theta_2} e^{-c4^j r_B^2/t}\Big(\fint_{C_j(B)} |f|^p\,d\mu\Big)^{1/p}, \tag{2.27}$$

and

$$\Big(\fint_{C_j(B)} |T_t(f\,\mathbf{1}_B)|^q\,d\mu\Big)^{1/q} \lesssim 2^{j\theta_1}\,\Upsilon\Big(\frac{2^j r_B}{\sqrt{t}}\Big)^{\theta_2} e^{-c4^j r_B^2/t}\Big(\fint_B |f|^p\,d\mu\Big)^{1/p}. \tag{2.28}$$

Again when $q = \infty$, or $p = \infty$, one has to change the L^q-norms, or L^p-norms, by the corresponding essential suprema.

Let us recall some results about off-diagonal estimates on balls for the heat semigroup associated with L_w.

Lemma 2.7 ([9], Section 2, and [16], Sections 3 and 7). *Let L_w be a degenerate elliptic operator with $w \in A_2(dx)$.*

(a) *If $p_-(L_w) < p \le q < p_+(L_w)$, then e^{-tL_w} and $(tL_w)^m e^{-tL_w}$, for every $m \in \mathbb{N}$, belong to $\mathcal{O}(L^p(w) - L^q(w))$.*

(b) *Let $p_-(L_w) < p \le q < p_+(L_w)$. If $v \in A_{p/p_-(L_w)}(w) \cap \mathrm{RH}_{(p_+(L_w)/q)'}(w)$, then e^{-tL_w} and $(tL_w)^m e^{-tL_w}$, for every $m \in \mathbb{N}$, belong to $\mathcal{O}(L^p(v dw) - L^q(v dw))$.*

(c) *There exists an interval $\mathcal{K}(L_w)$ such that if $p, q \in \mathcal{K}(L_w)$, with $p \le q$, then $\sqrt{t}\nabla e^{-tL_w}$ belongs to $\mathcal{O}(L^p(w) - L^q(w))$. Moreover, denoting by $q_-(L_w)$ and $q_+(L_w)$ the left and right endpoints of $\mathcal{K}(L_w)$, then $q_-(L_w) = p_-(L_w)$ and $2 < q_+(L_w) \le (q_+(L_w))^*_w \le p_+(L_w)$.*

(d) *Let $q_-(L_w) < p \le q < q_+(L_w)$. If $v \in A_{p/q_-(L_w)}(w) \cap \mathrm{RH}_{(q_+(L_w)/q)'}(w)$, it follows that $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}(L^p(v dw) - L^q(v dw))$.*

(e) *If $p = q$ and μ is a doubling measure, then $\mathcal{F}(L^p(\mu) - L^p(\mu))$ and $\mathcal{O}(L^p(\mu) - L^p(\mu))$ are equivalent.*

Remark 2.8. Since off-diagonal estimates on balls are stable under composition (see Theorem 2.3 in [9]), it follows from (b) and (d) that $\sqrt{t}\nabla t L_w e^{-tL_w} \in \mathcal{O}(L^p(v dw) - L^q(v dw))$ for $q_-(L_w) < p \le q < q_+(L_w)$ and $v \in A_{p/q_-(L_w)}(w) \cap \mathrm{RH}_{(q_+(L_w)/q)'}(w)$.

Moreover, in the following result, which is a weighted version of [30], (5.12) (see also [27]), and whose proof can be found in Proposition 2.42 of [33], we have off-diagonal estimates for the family $\{\mathcal{T}_{t,s}\}_{s,t>0} := \{(e^{-t^2 L_w} - e^{-(t^2+s^2)L_w})^M\}_{s,t>0}$, for all $M \in \mathbb{N}$.

Proposition 2.9. *Let $p \in (p_-(L_w), p_+(L_w))$ and let $0 < t, s < \infty$. Then, for all sets $E_1, E_2 \subset \mathbb{R}^n$ and $f \in L^p(w)$ such that $\operatorname{supp}(f) \subset E_1$, we have that $\{\mathcal{T}_{t,s}\}_{s,t>0}$ satisfies the following $L^p(w) - L^p(w)$ off-diagonal estimate*:

$$\|\mathbf{1}_{E_2}\mathcal{T}_{t,s} f\|_{L^p(w)} \lesssim \Big(\frac{s^2}{t^2}\Big)^M e^{-c\frac{d(E_1,E_2)^2}{t^2+s^2}}\,\|f\,\mathbf{1}_{E_1}\|_{L^p(w)}. \tag{2.29}$$

In particular, there holds

$$\|\mathcal{T}_{t,s} f\|_{L^p(w)} \lesssim \Big(\frac{s^2}{t^2}\Big)^M \|f\|_{L^p(w)}. \tag{2.30}$$

We conclude this section by introducing the following off-diagonal estimates on Sobolev spaces (for non-degenerate elliptic operators, see [1]).

Lemma 2.10. *Let $q \in (q_-(L_w), q_+(L_w))$ and $\alpha > 0$. Assume that p satisfies*

$$\max\{r_w, (q_-(L_w))_{w,*}\} < p \le q.$$

Then, there exists $\theta > 0$ such that for every $(x,t) \in \mathbb{R}^{n+1}_+$,

$$\Big(\fint_{B(x,\alpha t)} |\nabla e^{-t^2 L_w} f|^q dw\Big)^{1/q} \lesssim \Upsilon(\alpha)^\theta \sum_{j=1}^\infty e^{-c\alpha 4^j} \Big(\fint_{B(x,2^{j+1}\alpha t)} |\nabla f|^p dw\Big)^{1/p}. \tag{2.31}$$

Proof. For simplicity, we write $B := B(x,\alpha t)$ and $h := f - f_{4B,w}$, where, for every $\lambda > 0$, $f_{\lambda B,w}$ is the average of f in λB with respect to the measure dw. By the conservation property, that is, $e^{-t^2 L_w} 1 = 1$,

$$\nabla e^{-t^2 L_w} f = \nabla e^{-t^2 L_w}(f - f_{4B,w}) = \sum_{j=1}^\infty \nabla e^{-t^2 L_w} h_j,$$

with

$$h_j := h \mathbf{1}_{C_j(B)}.$$

By Lemma 2.7, for any $q_-(L_w) < q_0 < q$, we have that $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}(L^{q_0}(w) - L^q(w))$, and then

$$\begin{aligned}\Big(\fint_B |\nabla e^{-t^2 L_w} f|^q dw\Big)^{1/q} &\le \sum_{j=1}^\infty \Big(\fint_B |\nabla e^{-t^2 L_w} h_j|^q dw\Big)^{1/q} \\ &\lesssim \Upsilon(\alpha)^{\theta_2} \sum_{j=1}^\infty \frac{2^{j(\theta_1+\theta_2)} e^{-c\alpha 4^j}}{t} \Big(\fint_{C_j(B)} |h|^{q_0} dw\Big)^{1/q_0}.\end{aligned}$$

Using the weighted Poincaré–Sobolev inequality (see Theorem 2.1 in [16] and also Theorem 1.6 in [24]), we obtain that for any $p > \max\{r_w, (q_0)_{w,*}\}$,

$$\begin{aligned}\Big(\fint_{C_j(B)} |h|^{q_0} dw\Big)^{1/q_0} &\le \Big(\fint_{2^{j+1}B} |f - f_{2^{j+1}B,w}|^{q_0} dw\Big)^{1/q_0} + \sum_{l=2}^j |f_{2^{l+1}B,w} - f_{2^l B,w}| \\ &\lesssim \sum_{l=2}^j \Big(\fint_{2^{l+1}B} |f - f_{2^{l+1}B,w}|^{q_0} dw\Big)^{1/q_0} \lesssim \sum_{l=2}^j 2^l \alpha t \Big(\fint_{2^{l+1}B} |\nabla f|^p dw\Big)^{1/p}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big(\fint_B |\nabla e^{-t^2 L_w} f|^q dw\Big)^{1/q} &\lesssim \Upsilon(\alpha)^\theta \sum_{j=1}^\infty \frac{e^{-c\alpha 4^j}}{t} \sum_{l=2}^j 2^l t \Big(\fint_{2^{l+1}B} |\nabla f|^p dw\Big)^{1/p} \\ &\lesssim \Upsilon(\alpha)^\theta \sum_{j=1}^\infty e^{-c\alpha 4^j} \Big(\fint_{2^{j+1}B} |\nabla f|^p dw\Big)^{1/p}.\end{aligned}$$

This completes the proof. ∎

2.4. Change of angles

We shall use two change of angles results. The first one is a version of Proposition 3.30 in [29] in the weighted degenerate case.

Proposition 2.11 (Proposition A.2 in [12]). *Let $w \in A_{\widehat{r}}(dx)$ and $v \in \mathrm{RH}_{r'}(w)$ with $1 \le \widehat{r}, r < \infty$. Given a non-negative measurable function h, for every $1 \le q \le \widehat{r}$, $0 < \alpha \le 1$ and $t > 0$, there holds*

$$\int_{\mathbb{R}^n} \Big(\int_{B(x,\alpha t)} |h(y,t)| \frac{dw(y)}{w(B(y,\alpha t))} \Big)^{1/q} v(x)\, dw(x) \lesssim \alpha^{n\widehat{r}(1/r-1/q)} \int_{\mathbb{R}^n} \Big(\int_{B(x,t)} |h(y,t)| \frac{dw(y)}{w(B(y,t))} \Big)^{1/q} v(x)\, dw(x). \tag{2.32}$$

The second result was proved for the unweighted non-degenerate case in [2] and for the weighted non-degenerate case in Proposition 3.2 of [29]. Consider, for $\alpha > 0$, the following operator acting over measurable functions F defined in $\mathbb{R}^{n+1}_+$:

$$\mathcal{A}^{\alpha}_{w} F(x) := \Big(\iint_{\Gamma^{\alpha}(x)} |F(y,t)|^2 \frac{dw(y)\, dt}{t w(B(y,t))} \Big)^{1/2}, \quad x \in \mathbb{R}^n,$$

where $\Gamma^{\alpha}(x)$ the cone with vertex at x and aperture $\alpha > 0$ defined right below (2.21).

Proposition 2.12 (Proposition 4.9 in [12]). *Let $0 < \alpha \le \beta < \infty$.*

(a) *For every $w \in A_{\widetilde{r}}(dx)$ and $v \in A_r(w)$, $1 \le r, \widetilde{r} < \infty$, there holds*

$$\|\mathcal{A}^{\beta}_{w} F\|_{L^p(v dw)} \le C \Big(\frac{\beta}{\alpha}\Big)^{n\widetilde{r}r/p} \|\mathcal{A}^{\alpha}_{w} F\|_{L^p(v dw)} \quad \text{for all } 0 < p \le 2r, \tag{2.33}$$

where $C \ge 1$ depends on n, p, r, $\widetilde{r}$, $[w]_{A_{\widetilde{r}}(dx)}$, and $[v]_{A_r(w)}$, but it is independent of α and β.

(b) *For every $w \in \mathrm{RH}_{\tilde{s}'}(dx)$ and $v \in \mathrm{RH}_{s'}(w)$, $1 \le s, \tilde{s} < \infty$, there holds*

$$\|\mathcal{A}^{\alpha}_{w} F\|_{L^p(v dw)} \le C \Big(\frac{\alpha}{\beta}\Big)^{\frac{n}{s\tilde{s}p}} \|\mathcal{A}^{\beta}_{w} F\|_{L^p(v dw)} \quad \text{for all } 2/s \le p < \infty, \tag{2.34}$$

where $C \ge 1$ depends on n, p, s, $\tilde{s}$, $[w]_{\mathrm{RH}_{\tilde{s}'}(dx)}$, and $[v]_{\mathrm{RH}_{s'}(w)}$, but it is independent of α and β.

2.5. Calderón–Zygmund decomposition on Sobolev spaces

Our proofs rely on the following Calderón–Zygmund decomposition on Sobolev spaces.

Lemma 2.13 (Lemma 6.6 in [7]). *Let $n \ge 1$, $\alpha > 0$, $\varpi \in A_\infty(dx)$, and let $1 \le p < \infty$ be such that $\varpi \in A_p(dx)$. Assume that $f \in \mathcal{S}$ is such that $\|\nabla f\|_{L^p(\varpi)} < \infty$. Then, there exist a collection of balls $\{B_i\}_i$ with radii r_{B_i}, smooth functions $\{b_i\}_i$, and a function $g \in L^1_{\mathrm{loc}}(\varpi)$, such that*

$$f = g + \sum_i b_i \tag{2.35}$$

and the following properties hold:

$$|\nabla g(x)| \leq C\alpha, \quad \textit{for } \mu\textit{-a.e. } x, \tag{2.36}$$

$$\operatorname{supp} b_i \subset B_i \quad \textit{and} \quad \int_{B_i} |\nabla b_i|^p \, d\varpi \leq C\alpha^p \varpi(B_i), \tag{2.37}$$

$$\sum_i \varpi(B_i) \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, d\varpi, \tag{2.38}$$

$$\sum_i \mathbf{1}_{4B_i} \leq N, \tag{2.39}$$

where C and N depend only on n, p, and ϖ.

In addition, for $1 \leq q < p^*_\varpi$, where p^*_ϖ is defined in (2.7),

$$\Big(\fint_{B_i} |b_i|^q \, d\varpi \Big)^{1/q} \lesssim \alpha \, r_{B_i}. \tag{2.40}$$

2.6. Non-homogeneous vertical square function

In this section, we study the weighted boundedness of $\widetilde{\mathrm{G}}^w_{\mathrm{H}}$, see (2.22). Our result is the following.

Theorem 2.14. *Let $w \in A_2(dx)$ and let L_w be a degenerate elliptic operator. Given $v \in A_\infty(w)$, assume that $\mathcal{W}^w_v(\max\{r_w, q_-(L_w)\}, q_+(L_w)) \neq \emptyset$. Then, for every $f \in \mathcal{S}$ and $p \in \mathcal{W}^w_v(\max\{r_w, (q_-(L_w))_{w,*}\}, q_+(L_w))$, it holds*

$$\|\widetilde{\mathrm{G}}^w_{\mathrm{H}} f\|_{L^p(v dw)} \lesssim \|\nabla f\|_{L^p(v dw)}. \tag{2.41}$$

Before starting with the proof, we make some remarks and prove Lemma 2.17 (stated below). These results will be useful in this proof and also in the remainder of the paper.

Remark 2.15. Let $w \in A_2(dx)$ and $v \in A_\infty(w)$. Take $0 < q_- < q_+ < \infty$ and suppose that

$$p > \mathrm{r}_v(w) \max\{r_w, (q_-)_{w,*}\}.$$

Assuming that

$$\big(\mathrm{r}_v(w) \max\{r_w, q_-\}, q_+/\mathfrak{s}_v(w)\big) = \mathcal{W}^w_v(\max\{r_w, q_-\}, q_+) \neq \emptyset,$$

we claim that

$$\big(\mathrm{r}_v(w) \max\{r_w, q_-\}, \min\{q_+/\mathfrak{s}_v(w), p^*_{vw}\}\big) \neq \emptyset, \tag{2.42}$$

where we recall that by Remark 2.1, $vw \in A_\infty(dx)$, and where p^*_{vw} is defined in (2.7). Indeed, since by hypothesis $\mathrm{r}_v(w) \max\{r_w, q_-\} < q_+/\mathfrak{s}_v(w)$, this can be seen from the fact that

$$\mathrm{r}_v(w) \max\{r_w, q_-\} < p^*_{vw}. \tag{2.43}$$

To prove (2.43), we distinguish two cases. If $\mathrm{r}_v(w)\max\{r_w,q_-\} = \mathrm{r}_v(w)r_w$, since we are taking p such that $p > \mathrm{r}_v(w)\max\{r_w,(q_-)_{w,*}\}$ and since $(q_-)_{w,*} \le q_-$ (see (2.6)), then

$$\mathrm{r}_v(w)\max\{r_w,q_-\} = \mathrm{r}_v(w)\max\{r_w,(q_-)_{w,*}\} < p < p^*_{vw}.$$

If now

$$\mathrm{r}_v(w)\max\{r_w,q_-\} = \mathrm{r}_v(w)q_-,$$

then we can assume that $nr_{vw} > p$ (otherwise $p^*_{vw} = \infty$ and the inequality is trivial). Hence, by hypothesis and by (2.7),

$$\frac{1}{p^*_{vw}} = \frac{1}{p} - \frac{1}{nr_{vw}} < \frac{1}{\mathrm{r}_v(w)(q_-)_{w,*}} - \frac{1}{nr_{vw}} = \frac{nr_w + q_-}{\mathrm{r}_v(w)q_- nr_w} - \frac{1}{nr_{vw}}$$

$$(2.44)\qquad = \frac{1}{\mathrm{r}_v(w)q_-} - \frac{1}{nr_{vw}}\Big(1 - \frac{r_{vw}}{r_w\mathrm{r}_v(w)}\Big) \le \frac{1}{\mathrm{r}_v(w)q_-} = \frac{1}{\mathrm{r}_v(w)\max\{r_w,q_-\}}.$$

Remark 2.16. Let $\{B_i\}_i$ be a collection of balls with bounded overlap, $w \in A_\infty(dx)$, and $v \in A_\infty(w)$. Besides, consider $1 < \tilde p < \infty$, $u \in L^{\tilde p'}(vdw)$ such that $\|u\|_{L^{\tilde p'}(vdw)} = 1$, and $\mathcal{M}^{vw}$ the weighted maximal operator defined as

$$\mathcal{M}^{vw} f(x) := \sup_{B \ni x} \fint_B |f(y)|\, d(vw)(y).$$

Then, by Kolmogorov's inequality, we have that

$$(2.45)\qquad \Big(\sum_i \int_{B_i} \big(\mathcal{M}^{vw}(|u|^{\tilde p'})\big)^{1/\tilde p'} vdw\Big)^{\tilde p} \lesssim \Big(\int_{\cup_i B_i} \big(\mathcal{M}^{vw}(|u|^{\tilde p'})\big)^{1/\tilde p'} vdw\Big)^{\tilde p}$$
$$\lesssim vw\Big(\bigcup_i B_i\Big)\|u\|^{\tilde p}_{L^{\tilde p'}(vdw)} \lesssim vw\Big(\bigcup_i B_i\Big).$$

We next state a technical lemma which will be used several times. We notice that the statement, which may appear slightly clumsy, is written so that it can be easily invoked in some of our proofs.

Lemma 2.17. *Given $w \in A_2(dx)$ and $v \in A_\infty(w)$, fix $1 < p_1 < \infty$, and a collection $\{B_i\}_i$ of balls in $\mathbb{R}^n$ with bounded overlap. Assume that there is a sequence of positive numbers $\{\mathcal{I}_{ij}\}_{i,j}$ (whose significance will become clear when applying the result) so that*

$$(2.46)\qquad \mathcal{I}_{ij} \le \widehat C\, vw(2^{j+1}B_i)^{1/p_1}\, 2^{-j(2M-\widetilde C)}, \quad j \ge 4,$$

where $\widehat C, \widetilde C$ are fixed constants, and $2M > \widetilde C + nr_w\mathrm{r}_v(w)$. Then

$$\sup_{\|u\|_{L^{p_1'}(vdw)}=1} \sum_i \sum_{j\ge 4} \mathcal{I}_{ij}\, \|u\,\mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} \lesssim vw\Big(\bigcup_i B_i\Big)^{1/p_1}.$$

Proof. Fix u so that $\|u\|_{L^{p_1'}(vdw)} = 1$. Note that we can find $p > r_w$, $q > \mathrm{r}_v(w)$ so that $2M > \widetilde C + nr$ with $r = pq$. In particular, $w \in A_p(dx)$, $v \in A_q(w)$ and we have (2.11) at our disposal. This, together with (2.46) and (2.45) with $\tilde{p} = p_1$, allows us to show that

$$\begin{aligned}\sum_i \sum_{j\ge 4} \mathcal{I}_{ij} \|u\,\mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} &\lesssim \sum_i \sum_{j\ge 4} vw(B_i)\, 2^{-j(2M-\widetilde{C}-nr)} \Big(\fint_{C_j(B_i)} |u(x)|^{p_1'}\, d(vw)(x)\Big)^{1/p_1'} \\ &\lesssim \sum_i vw(B_i) \inf_{x\in B_i} (\mathcal{M}^{vw}(|u|^{p_1'})(x))^{1/p_1'} \\ &\lesssim \sum_i \int_{B_i} (\mathcal{M}^{vw}(|u|^{p_1'})(x))^{1/p_1'}\, v(x)\, dw(x) \lesssim vw\Big(\bigcup_i B_i\Big)^{1/p_1}.\end{aligned}$$

This readily leads to the desired estimate. ■

Proof of Theorem 2.14. Throughout the proof, fix $w \in A_2(dx)$ and denote $q_- := q_-(L_w)$ and $q_+ := q_+(L_w)$.

If $p \in \mathcal{W}_v^w(\max\{r_w, q_-\}, q_+)$, then (2.41) follows easily from Lemma 2.3 and Proposition 2.4. Indeed, we have

$$(2.47)\quad \begin{aligned}\|\widetilde{\mathrm{G}}_{\mathrm{H}}^w f\|_{L^p(vdw)} &= \Big\| \Big(\int_0^\infty \big| t\nabla (t^2 L_w)^{1/2} e^{-t^2 L_w} \big(\sqrt{L_w} f\big)\big|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(vdw)} \\ &\lesssim \|\sqrt{L_w} f\|_{L^p(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)}.\end{aligned}$$

In accordance with (2.14), to go below $\mathfrak{r}_v(w)\max\{r_w, q_-\}$, we shall show that if p satisfies

$$(2.48)\qquad \mathfrak{r}_v(w)\max\{r_w, (q_-)_{w,*}\} < p < \mathfrak{r}_v(w)\max\{r_w, q_-\},$$

then for any $\alpha > 0$ and $f \in \mathcal{S}$ it follows that

$$(2.49)\qquad vw\big(\{x \in \mathbb{R}^n : \widetilde{\mathrm{G}}_{\mathrm{H}}^w f(x) > \alpha\}\big) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p\, v\, dw.$$

Hence, using interpolation between Sobolev spaces (see [11]), we shall conclude the desired estimate.

In order to prove (2.49), we apply to f the Calderón–Zygmund decomposition in Lemma 2.13 at height $\alpha > 0$ for the product weight vw (recall that $r_{vw} \le r_w \mathfrak{r}_v(w) < p$, see Remark 2.1). Thus, by (2.35),

$$\begin{aligned} vw\big(\{x \in \mathbb{R}^n : \widetilde{\mathrm{G}}_{\mathrm{H}}^w f(x) > \alpha\}\big) &\le vw\Big(\Big\{x \in \mathbb{R}^n : \widetilde{\mathrm{G}}_{\mathrm{H}}^w g(x) > \frac{\alpha}{3}\Big\}\Big) \\ &\quad + vw\Big(\Big\{x \in \mathbb{R}^n : \widetilde{\mathrm{G}}_{\mathrm{H}}^w \Big(\sum_i b_i\Big)(x) > \frac{2\alpha}{3}\Big\}\Big) =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Note that by Remark 2.15 we can pick q such that

$$(2.50)\qquad \mathfrak{r}_v(w)\max\{r_w, q_-\} < q < \min\Big\{\frac{q_+}{\mathfrak{s}_v(w)}, p_{vw}^*\Big\}.$$

Keeping this choice of q, by (2.47) we have

$$\|\widetilde{\mathrm{G}}_{\mathrm{H}}^w f\|_{L^q(vdw)} \lesssim \|\nabla f\|_{L^q(vdw)}.$$

Besides, since $p < q$ (see (2.48)), properties (2.36)–(2.39) yield

$$\mathrm{I} \lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n} |\widetilde{\mathrm{G}}_{\mathrm{H}}^w g|^q \, v \, dw \lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n} |\nabla g|^q \, v \, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v \, dw.$$

To estimate term II, for every $k \in \mathbb{Z}$, let $r_i := 2^k$ if $2^k \le r_{B_i} < 2^{k+1}$. Then,

$$\begin{aligned}
\mathrm{II} &\le vw\Big(\bigcup_i 16B_i\Big) \\
&\quad + vw\bigg(\bigg\{x \in \mathbb{R}^n : \bigg(\int_0^\infty \Big|t^2 \nabla L_w e^{-t^2 L_w}\Big(\sum_{i: r_i \le t} b_i\Big)(x)\Big|^2 \frac{dt}{t}\bigg)^{1/2} > \frac{\alpha}{3}\bigg\}\bigg) \\
&\quad + vw\bigg(\bigg\{x \in \mathbb{R}^n \setminus \bigcup_i 16B_i : \bigg(\int_0^\infty \Big|t^2 \nabla L_w e^{-t^2 L_w}\Big(\sum_{i: r_i > t} b_i\Big)(x)\Big|^2 \frac{dt}{t}\bigg)^{1/2} > \frac{\alpha}{3}\bigg\}\bigg) \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p v dw + \mathrm{II}_1 + \mathrm{II}_2,
\end{aligned}$$

where we have used (2.3) and (2.38).

In order to estimate term II_1, write

$$\begin{aligned}
&\bigg(\int_0^\infty \Big|\nabla t^2 L_w e^{-t^2 L_w}\Big(\sum_{i: r_i \le t} b_i\Big)(x)\Big|^2 \frac{dt}{t}\bigg)^{1/2} \\
(2.51) \quad &= \bigg(\int_0^\infty \Big|t^3 \nabla L_w e^{-t^2 L_w}\Big(\frac{1}{t}\sum_{i: r_i \le t} b_i\Big)(x)\Big|^2 \frac{dt}{t}\bigg)^{1/2} = \bigg(\int_0^\infty |T_t f_t(x)|^2 \frac{dt}{t}\bigg)^{1/2},
\end{aligned}$$

where

$$T_t := t^3 \nabla L_w e^{-t^2 L_w} \quad \text{and} \quad f_t(x) := \frac{1}{t} \sum_{i: r_i \le t} b_i(x).$$

Moreover, note that $2 \in (\max\{r_w, q_-\}, q_+)$, then, by Remark 2.8, for every $v_0 \in A_{2/\max\{r_w,q_-\}}(w) \cap RH_{(q_+/2)'}(w)$ we have $t^{3/2} \nabla L_w e^{-t L_w} \in \mathcal{O}(L^2(v_0 dw) - L^2(v_0 dw))$. In particular, T_t is bounded from $L^2(v_0 dw)$ to $L^2(v_0 dw)$. Consequently,

$$\begin{aligned}
\bigg\|\bigg(\int_0^\infty |T_t f_t|^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^2(v_0 dw)}^2 &= \int_0^\infty \int_{\mathbb{R}^n} |T_t f_t|^2 v_0 \, dw \, \frac{dt}{t} \\
&\lesssim \int_0^\infty \int_{\mathbb{R}^n} |f_t|^2 v_0 \, dw \, \frac{dt}{t} = \bigg\|\bigg(\int_0^\infty |f_t|^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^2(v_0 dw)}^2.
\end{aligned}$$

Now, by extrapolation (see Theorem A.1 in [12] and also Theorem 3.31 in [15]), we obtain that for $\tilde{v} \in A_\infty(w)$ and any $\tilde{q} \in \mathcal{W}_{\tilde{v}}^w(\max\{r_w, q_-\}, q_+)$,

$$(2.52) \qquad \bigg\|\bigg(\int_0^\infty |T_t f_t|^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^{\tilde{q}}(\tilde{v} dw)} \lesssim \bigg\|\bigg(\int_0^\infty |f_t|^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^{\tilde{q}}(\tilde{v} dw)}.$$

In particular the above inequality holds for our choices of q and v.

Next, the proof follows much as in [3], p. 543, but we write the details for the sake of completeness. Consider the following sum:

$$\beta_k := \sum_{i: r_i = 2^k} \frac{b_i}{r_i},$$

and note that

$$f_t = \frac{1}{t} \sum_{i:r_i \le t} b_i = \sum_{k:2^k \le t} \frac{2^k}{t} \sum_{i:r_i = 2^k} \frac{b_i}{r_i} = \sum_{k:2^k \le t} \frac{2^k}{t} \beta_k.$$

By the Cauchy–Schwarz inequality, for every $t > 0$,

$$|f_t|^2 \le \Big(\sum_{k:2^k \le t} \frac{2^k}{t} |\beta_k|^2 \Big) \Big(\sum_{k:2^k \le t} \frac{2^k}{t} \Big) \lesssim \sum_{k:2^k \le t} \frac{2^k}{t} |\beta_k|^2 = \sum_{k \in \mathbb{Z}} \frac{2^k}{t} |\beta_k|^2 \mathbf{1}_{[2^k,\infty)}(t),$$

and hence,

$$\int_0^\infty |f_t|^2 \frac{dt}{t} \lesssim \sum_{k \in \mathbb{Z}} \int_{2^k}^\infty \frac{2^k}{t} \frac{dt}{t} |\beta_k|^2 = \sum_{k \in \mathbb{Z}} |\beta_k|^2.$$

Using the bounded overlap property (2.39), the fact that $r_i \approx r_{B_i}$, and also (2.40), we have

$$\begin{aligned}
\Big\| \Big(\int_0^\infty |f_t|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^q(v\,dw)}^q &\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \Big)^{1/2} \Big\|_{L^q(v\,dw)}^q \lesssim \int_{\mathbb{R}^n} \sum_i \frac{|b_i|^q}{r_i^q} v\,dw \\
&\lesssim \alpha^q \sum_i vw(B_i) \lesssim \alpha^{q-p} \int_{\mathbb{R}^n} |\nabla f|^p v\,dw.
\end{aligned}$$

This estimate, (2.51), and (2.52) with q and v, yield, as desired,

$$\begin{aligned}
\mathrm{II}_1 &\lesssim \frac{1}{\alpha^q} \Big\| \Big(\int_0^\infty |T_t f_t|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^q(v\,dw)}^q \lesssim \frac{1}{\alpha^q} \Big\| \Big(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \Big)^{1/2} \Big\|_{L^q(v\,dw)}^q \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p v\,dw.
\end{aligned}$$

In order to estimate term II_2, notice that

$$\begin{aligned}
\Big(\int_0^\infty \Big| t^2 \nabla L_w e^{-t^2 L_w} \Big(\sum_{i:r_i > t} b_i \Big) \Big|^2 \frac{dt}{t} \Big)^{1/2} &\le \sum_i \Big(\int_0^{r_i} \big| t^2 \nabla L_w e^{-t^2 L_w} b_i \big|^2 \frac{dt}{t} \Big)^{1/2} \\
&=: \sum_i T_i b_i.
\end{aligned}$$

Then, by duality, we have

$$\begin{aligned}
\mathrm{II}_2 &\lesssim \frac{1}{\alpha^q} \int_{\mathbb{R}^n \setminus \bigcup_i 16B_i} \Big| \sum_i T_i b_i(x) \Big|^q v(x)\,dw(x) \\
&\le \frac{1}{\alpha^q} \Big(\sup_{\|u\|_{L^{q'}(v\,dw)}=1} \sum_i \int_{\mathbb{R}^n \setminus \bigcup_i 16B_i} |T_i b_i(x)|\,|u(x)|\,v(x)\,dw(x) \Big)^q \\
&\le \frac{1}{\alpha^q} \Big(\sup_{\|u\|_{L^{q'}(v\,dw)}=1} \sum_i \sum_{j \ge 4} \int_{C_j(B_i)} |T_i b_i(x)|\,|u(x)|\,v(x)\,dw(x) \Big)^q \\
&\lesssim \frac{1}{\alpha^q} \Big(\sup_{\|u\|_{L^{q'}(v\,dw)}=1} \sum_i \sum_{j \ge 4} \|T_i b_i\|_{L^q(C_j(B_i),v\,dw)} \|u\|_{L^{q'}(C_j(B_i),v\,dw)} \Big)^q.
\end{aligned}$$

To estimate $\|T_i b_i\|_{L^q(C_j(B_i),v dw)}$, we pick p_0 close enough to q_-, and $q_0 > q$ close enough to q_+ such that

$$q_- < p_0 < 2 < q_0 < q_+ \quad \text{and} \quad v \in A_{q/p_0}(w) \cap \mathrm{RH}_{(q_0/q)'}(w). \tag{2.53}$$

Note that $\mathcal{W}_v^w(q_-, q_+) \neq \emptyset$ since by assumption $\mathcal{W}_v^w(\max\{r_w, q_-\}, q_+) \neq \emptyset$ and additionally $\mathcal{W}_v^w(\max\{r_w, q_-\}, q_+) \subset \mathcal{W}_v^w(q_-, q_+)$. Notice also that applying Remark 2.8 with $v \equiv 1$, we have $t^{3/2}\nabla L_w e^{-tL_w} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$. Then, by Minkowski's integral inequality, Lemma 2.2 (a) and (b) (see (2.53)), (2.40) (see (2.50)), and recalling that $r_i \approx r_{B_i}$, for $j \geq 2$,

$$\begin{aligned}
&\|T_i b_i\|_{L^q(C_j(B_i),v dw)} \\
&\quad = vw(2^{j+1}B)^{1/q}\Big(\fint_{C_j(B_i)}\Big(\int_0^{r_i} \big|t^3\nabla L_w e^{-t^2 L_w} b_i\big|^2 \frac{dt}{t^3}\Big)^{q/2} d(vw)\Big)^{1/q} \\
&\quad \lesssim vw(2^{j+1}B_i)^{1/q}\Big(\fint_{C_j(B_i)}\Big(\int_0^{r_i} \big|t^3\nabla L_w e^{-t^2 L_w} b_i\big|^2 \frac{dt}{t^3}\Big)^{q_0/2} dw\Big)^{1/q_0} \\
&\quad \leq vw(2^{j+1}B_i)^{1/q}\Big(\int_0^{r_i}\Big(\fint_{C_j(B_i)} \big|t^3\nabla L_w e^{-t^2 L_w} b_i\big|^{q_0} dw\Big)^{2/q_0} \frac{dt}{t^3}\Big)^{1/2} \\
&\quad \lesssim 2^{j\theta_1} vw(2^{j+1}B_i)^{1/q}\Big(\fint_{B_i} |b_i|^{p_0}\, dw\Big)^{1/p_0}\Big(\int_0^{r_i}\Big(\frac{2^j r_{B_i}}{t}\Big)^{2\theta_2} e^{-c\,4^j r_{B_i}^2/t^2}\frac{dt}{t^3}\Big)^{1/2} \\
&\quad \lesssim e^{-c4^j}\, vw(2^{j+1}B_i)^{1/q}\Big(\fint_{B_i}\Big|\frac{b_i}{r_{B_i}}\Big|^q d(vw)\Big)^{1/q} \lesssim e^{-c4^j}\alpha\, vw(2^{j+1}B_i)^{1/q}.
\end{aligned}$$

Now we use Lemma 2.17 with $p_1 = q$, $\mathcal{I}_{ij} = \alpha^{-1}\|T_i b_i\|_{L^q(C_j(B_i),v dw)}$, $\{B_i\}_i$ the collection of balls given by Lemma 2.13, and with e^{-c4^j} replacing $2^{-j(2M-\widetilde{C})}$ (consequently M and $\widetilde{C}$ do not play any role here). Therefore, Lemma 2.17 and (2.38) imply

$$\mathrm{II}_2 \lesssim vw\Big(\bigcup_i B_i\Big) \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |\nabla f|^p\, v\, dw.$$

Collecting the above estimates, we get the desired result. ■

2.7. Non-homogeneous conical square function

In this section, we shall prove weighted boundedness in Sobolev spaces for the inhomogeneous conical square function $\widetilde{\mathcal{S}}_{\mathrm{H}}^w$ defined in (2.23). The analogous result for elliptic operators was studied in [27] for the Riesz transform characterization of Hardy spaces. See also [32] for the Riesz transform characterization of weighted Hardy spaces. Our result is stated as follows.

Theorem 2.18. *Given $w \in A_2(dx)$, $v \in A_\infty(w)$, assume that*

$$\mathcal{W}_v^w\left(\max\{r_w, q_-(L_w)\}, q_+(L_w)\right) \neq \emptyset. \tag{2.54}$$

Then, for every $h \in \mathcal{S}$ and $p \in \mathcal{W}_v^w\left(\max\{r_w, (p_-(L_w))_{w,*}\}, p_+(L_w)\right)$, it holds

$$\|\widetilde{\mathcal{S}}_{\mathrm{H}}^w h\|_{L^p(w)} \lesssim \|\nabla h\|_{L^p(w)}. \tag{2.55}$$

To prove this theorem, we shall use Lemma 2.19 and Proposition 2.20. Lemma 2.19 will be also useful in the proof of Proposition 3.2 (all these results are stated below).

Lemma 2.19. *Let $w \in A_2(dx)$ and $v \in A_\infty(w)$ be such that $\mathcal{W}_v^w(q_-(L_w), q_+(L_w)) \neq \emptyset$, and let*

$$p \in (\mathrm{r}_v(w) \max\{r_w, (q_-(L_w))_{w,*}\}, \mathrm{r}_v(w) \max\{r_w, q_-(L_w)\}).$$

Given $\alpha > 0$ and $f \in \mathcal{S}$ such that $\|\nabla f\|_{L^p(vdw)} < \infty$, let $\{b_i\}_i$ be the collection of smooth functions from Lemma 2.13 (applied to f, p, α, and $\varpi = vw$). Write

$$\widetilde{b} := \sum_{i=1}^{\infty} A_{r_{B_i}} b_i, \quad \text{where } A_{r_{B_i}} := I - (I - e^{-r_{B_i}^2 L_w})^M,$$

and where $M \in \mathbb{N}$ is sufficiently large. Then, for $p_1 \in \mathcal{W}_v^w(q_-(L_w), q_+(L_w))$ such that $1 \leq p_1 < p_{vw}^*$ (note that following (2.44) we get that $\mathrm{r}_v(w) q_-(L_w) < p_{vw}^*$), there holds

$$\|\nabla \widetilde{b}\|_{L^{p_1}(vdw)}^{p_1} \lesssim \alpha^{p_1 - p} \|\nabla f\|_{L^p(vdw)}^p.$$

Proof. First of all, denote $q_- := q_-(L_w)$ and $q_+ := q_+(L_w)$. By duality and expanding $A_{r_{B_i}}$, we have

$$\begin{aligned}
\|\nabla \widetilde{b}\|_{L^{p_1}(vdw)}^{p_1} &= \int_{\mathbb{R}^n} \Big| \nabla \Big(\sum_i \sum_{k=1}^M C_{k,M} e^{-k r_{B_i}^2 L_w} b_i \Big) \Big|^{p_1} v dw \\
&\lesssim \sup_{\|u\|_{L^{p_1'}(vdw)}=1} \Big(\sum_{k=1}^M \sum_i \int_{\mathbb{R}^n} \Big| r_{B_i} \nabla e^{-k r_{B_i}^2 L_w} \Big(\frac{b_i}{r_{B_i}} \Big) \Big| |u| v dw \Big)^{p_1}.
\end{aligned}$$

By hypothesis, $v \in A_{p_1/q_-}(w) \cap \mathrm{RH}_{(q_+/p_1)'}(w)$ (see (2.14)), hence

$$\sqrt{\tau} \nabla e^{-\tau L_w} \in \mathcal{O}(L^{p_1}(vdw) - L^{p_1}(vdw)).$$

Using this, (2.40), and also (2.3),

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big| r_{B_i} \nabla e^{-k r_{B_i}^2 L_w} \Big(\frac{b_i}{r_{B_i}} \Big) \Big| |u| \, v \, dw \\
&\quad \lesssim \sum_{j \geq 1} vw(2^{j+1} B_i) \Big(\fint_{C_j(B_i)} \Big| r_{B_i} \nabla e^{-k r_{B_i}^2 L_w} \Big(\frac{b_i}{r_{B_i}} \Big) \Big|^{p_1} d(vw) \Big)^{1/p_1} \\
&\qquad \times \Big(\fint_{C_j(B_i)} |u|^{p_1'} d(vw) \Big)^{1/p_1'} \\
&\quad \lesssim \sum_{j \geq 1} e^{-c 4^j} vw(B_i) \Big(\fint_{B_i} \Big| \frac{b_i}{r_{B_i}} \Big|^{p_1} d(vw) \Big)^{1/p_1} \inf_{x \in B_i} \big(\mathcal{M}^{vw}(|u|^{p_1'})(x) \big)^{1/p_1'} \\
&\quad \lesssim \alpha \int_{B_i} \big(\mathcal{M}^{vw}(|u|^{p_1'}) \big)^{1/p_1'} v dw.
\end{aligned}$$

Consequently, (2.45) with $\tilde{p} = p_1$ and (2.38) imply

$$\|\nabla\widetilde{b}\|^{p_1}_{L^{p_1}(vdw)} \lesssim \alpha^{p_1} \sup_{\|u\|_{L^{p_1'}(vdw)}=1} \Big(\sum_i \int_{B_i} \Big(\mathcal{M}^{vw}(|u|^{p_1'})\Big)^{1/p_1'} vdw\Big)^{p_1}$$
$$\lesssim \alpha^{p_1} vw\Big(\bigcup_i B_i\Big) \lesssim \alpha^{p_1-p} \int_{\mathbb{R}^n} |\nabla f|^p vdw.$$

This completes the proof. ∎

The next result is the particular case of Proposition 4.5 in [33], taking $m = 1$. In order to formulate it (proceeding similarly as in [27, 32]), we introduce the following conical square function:

$$\mathcal{S}^w_{1/2,\mathrm{H}} f(x) := \Big(\int_{B(x,t)} \int_0^\infty \big|t\sqrt{L_w} e^{-t^2 L_w} f(y)\big|^2 \frac{dw(y)\,dt}{tw(B(y,t))}\Big)^{1/2}.$$

Observe that $\widetilde{\mathcal{S}}^w_{\mathrm{H}} f = \mathcal{S}^w_{1/2,\mathrm{H}}\sqrt{L_w} f$. Our goal is to see that $\mathcal{S}^w_{1/2,\mathrm{H}} f$ compares with $\mathcal{S}^w_{\mathrm{H}} f$ (defined in (2.21)) in some weighted spaces (see Proposition 4.5 [33] for a general version of this result). For the following statement, we recall that $p_+(L_w)^{k,*}_w$ was defined in (2.7).

Proposition 2.20. *Given $w \in A_2(dx)$, $v \in A_\infty(w)$, and $f \in L^2(w)$, there hold*

(a) $\|\mathcal{S}^w_{\mathrm{H}} f\|_{L^p(vdw)} \lesssim \|\mathcal{S}^w_{1/2,\mathrm{H}} f\|_{L^p(vdw)}$, *for all* $p \in \mathcal{W}^w_v(0, p_+(L_w)^{2,*}_w)$,

(b) $\|\mathcal{S}^w_{1/2,\mathrm{H}} f\|_{L^p(vdw)} \lesssim \|\mathcal{S}^w_{\mathrm{H}} f\|_{L^p(vdw)}$, *for all* $p \in \mathcal{W}^w_v(0, p_+(L_w)^{*}_w)$.

In particular, if $p \in \mathcal{W}^w_v(0, p_+(L_w)^*_w)$, we have

$$\|\mathcal{S}^w_{1/2,\mathrm{H}} f\|_{L^p(vdw)} \approx \|\mathcal{S}^w_{\mathrm{H}} f\|_{L^p(vdw)}.$$

Proof. We observe that $\mathcal{S}^w_{\mathrm{H}}$ and $\mathcal{S}^w_{1/2,\mathrm{H}}$ respectively correspond to $\mathcal{S}^w_{2,\mathrm{H}}$ and $\mathcal{S}^w_{1,\mathrm{H}}$ in [33]. Then the proof follows from that of Proposition 4.5 in [33] taking $m = 1$. ∎

Proof of Theorem 2.18. First of all, fix $w \in A_2(dx)$, and denote $q_- := p_-(L_w) = q_-(L_w)$ (see Lemma 2.7), $q_+ := q_+(L_w)$, and $p_+ := p_+(L_w)$.

We claim that for all $p \in \mathcal{W}^w_v(\max\{r_w, q_-\}, p_+)$ and $h \in \mathcal{S}$,

$$\|\widetilde{\mathcal{S}}^w_{\mathrm{H}} h\|_{L^p(vdw)} \lesssim \|\nabla h\|_{L^p(vdw)}. \tag{2.56}$$

Indeed, applying Proposition 2.20, Theorem 3.1 in [12], and Proposition 2.4, we have that

$$\|\widetilde{\mathcal{S}}^w_{\mathrm{H}} h\|_{L^p(vdw)} = \|\mathcal{S}^w_{1/2,\mathrm{H}}\sqrt{L_w} h\|_{L^p(vdw)} \approx \|\mathcal{S}^w_{\mathrm{H}}\sqrt{L_w} h\|_{L^p(vdw)}$$
$$\lesssim \|\sqrt{L_w} h\|_{L^p(vdw)} \lesssim \|\nabla h\|_{L^p(vdw)}.$$

Note that $\mathcal{W}^w_v(\max\{r_w, q_-\}, p_+) = \big(\mathfrak{r}_v(w)\max\{r_w, q_-\}, p_+/\mathfrak{s}_v(w)\big)$. Therefore, for every p satisfying

$$\mathfrak{r}_v(w)\max\{r_w, (q_-)_{w,*}\} < p < \mathfrak{r}_v(w)\max\{r_w, q_-\}, \tag{2.57}$$

if we show that

$$\|\widetilde{\mathcal{S}}_{\mathrm{H}}^{w} h\|_{L^{p,\infty}(v dw)} \lesssim \|\nabla h\|_{L^p(v dw)}, \quad \forall h \in \mathcal{S}, \tag{2.58}$$

then, by interpolation (see [11] and Remark 2.1) we would conclude (2.55).

Now fix p as in (2.57), and note that $vw \in A_p(dx)$, since $r_{vw} \leq r_w \mathfrak{r}_v(w) < p$ (see Remark 2.1). Given $\alpha > 0$, we apply Lemma 2.13 to $h \in \mathcal{S}$, α, the product weight $\varpi = vw$ and p. Let $\{B_i\}_i$ be the collection of balls given by Lemma 2.13. Consider for $M \in \mathbb{N}$ arbitrarily large,

$$B_{r_{B_i}} := (I - e^{-r_{B_i}^2 L_w})^M \quad \text{and} \quad A_{r_{B_i}} := I - B_{r_{B_i}} = \sum_{k=1}^{M} C_{k,M} e^{-k r_{B_i}^2 L_w}.$$

Then

$$h = g + \sum_i A_{r_{B_i}} b_i + \sum_i B_{r_{B_i}} b_i =: g + \widetilde{b} + \widehat{b}.$$

It follows that

$$\begin{aligned} vw\big(\{x \in \mathbb{R}^n : \widetilde{\mathcal{S}}_{\mathrm{H}}^{w} h(x) > \alpha\}\big) &\leq vw\Big(\Big\{x \in \mathbb{R}^n : \widetilde{\mathcal{S}}_{\mathrm{H}}^{w} g(x) > \frac{\alpha}{3}\Big\}\Big) \\ &\quad + vw\Big(\Big\{x \in \mathbb{R}^n : \widetilde{\mathcal{S}}_{\mathrm{H}}^{w} \widetilde{b}(x) > \frac{\alpha}{3}\Big\}\Big) \\ &\quad + vw\Big(\Big\{x \in \mathbb{R}^n : \widetilde{\mathcal{S}}_{\mathrm{H}}^{w} \widehat{b}(x) > \frac{\alpha}{3}\Big\}\Big) =: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{2.59}$$

Now, since $\mathcal{W}_v^w(\max\{r_w, q_-\}, q_+) \neq \emptyset$ by assumption and $p > \mathfrak{r}_v(w) \max\{r_w, (q_-)_{w,*}\}$ (see (2.57)), by Remark 2.15 we can pick p_1 such that

$$\mathfrak{r}_v(w) \max\{r_w, q_-\} < p_1 < \min\Big\{\frac{q_+}{\mathfrak{s}_v(w)}, p_{vw}^*\Big\}. \tag{2.60}$$

Observe that

$$p_1 > p \quad \text{and} \quad p_1 \in \mathcal{W}_v^w(\max\{r_w, q_-\}, q_+).$$

Note also that in particular $\mathfrak{r}_v(w)\, q_- < p_1 < q_+/\mathfrak{s}_v(w)$, that is,

$$v \in A_{p_1/q_-}(w) \cap \mathrm{RH}_{(q_+/p_1)'}(w). \tag{2.61}$$

Now we are ready to estimate I. Applying Chebyshev's inequality with p_1, (2.56), and the properties (2.36)–(2.39), we obtain

$$\mathrm{I} \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\widetilde{\mathcal{S}}_{\mathrm{H}}^{w} g|^{p_1} v\, dw \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g|^{p_1} v\, dw \lesssim \frac{1}{\alpha^{p}} \int_{\mathbb{R}^n} |\nabla h|^{p} v\, dw. \tag{2.62}$$

In order to estimate II, apply Chebyshev's inequality, (2.56), and Lemma 2.19 (with $f = h$). Then

$$\mathrm{II} \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} \big|\widetilde{\mathcal{S}}_{\mathrm{H}}^{w} \widetilde{b}\big|^{p_1} v\, dw \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} \big|\nabla \widetilde{b}\big|^{p_1} v\, dw \lesssim \frac{1}{\alpha^{p}} \int_{\mathbb{R}^n} |\nabla h|^{p} v\, dw. \tag{2.63}$$

Next, we estimate III. Note that, by (2.38)

$$\text{III} \lesssim vw\Big(\bigcup_i 16B_i\Big) + vw\Big(\Big\{x \in \mathbb{R}^n \setminus \bigcup_i 16B_i : \widetilde{S}^w_{\mathrm{H}} \widehat{b}(x) > \frac{\alpha}{3}\Big\}\Big) \tag{2.64}$$

$$\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla h|^p \, v \, dw + \text{III}_1,$$

where

$$\text{III}_1 := vw\Big\{\Big(x \in \mathbb{R}^n \setminus \bigcup_i 16B_i : \widetilde{S}^w_{\mathrm{H}} \widehat{b}(x) > \frac{\alpha}{3}\Big\}\Big).$$

By Chebyshev's inequality, duality, splitting the integral in x, and applying Hölder's inequality,

$$\text{III}_1 \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n \setminus \cup_i 16B_i} \big|\widetilde{S}^w_{\mathrm{H}} \widehat{b}\big|^{p_1} \, v \, dw \tag{2.65}$$

$$\lesssim \frac{1}{\alpha^{p_1}} \Bigg(\sup_{\|u\|_{L^{p_1'}(vdw)}=1} \sum_i \sum_{j \geq 4} \Big(\int_{C_j(B_i)} \big|\widetilde{S}^w_{\mathrm{H}}(B_{r_{B_i}} b_i)\big|^{p_1} v \, dw \Big)^{1/p_1}$$

$$\times \|u \, \mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} \Bigg)^{p_1}$$

$$=: \frac{1}{\alpha^{p_1}} \sup_{\|u\|_{L^{p_1'}(vdw)}=1} \Big(\sum_i \sum_{j \geq 4} \mathcal{III}_{ij} \, \|u \, \mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} \Big)^{p_1}.$$

Splitting the integral in t (recall that $j \geq 4$), we have

$$\mathcal{III}_{ij} \lesssim \Bigg(\int_{C_j(B_i)} \Big(\int_0^{2^{j-2} r_{B_i}} \int_{B(x,t)} |t L_w e^{-t^2 L_w}(B_{r_{B_i}} b_i)(y)|^2 \frac{dw(y) \, dt}{t w(B(y,t))} \Big)^{p_1/2}$$

$$\times v(x) \, dw(x) \Bigg)^{1/p_1}$$

$$+ \Bigg(\int_{C_j(B_i)} \Big(\int_{2^{j-2} r_{B_i}}^{\infty} \int_{B(x,t)} \Big| t^2 L_w e^{-t^2 L_w} \Big(B_{r_{B_i}} \Big(\frac{b_i}{r_{B_i}} \Big) \Big)(y) \Big|^2 \frac{dw(y) \, dt}{t w(B(y,t))} \Big)^{p_1/2}$$

$$\times v(x) \, dw(x) \Bigg)^{1/p_1}$$

$$=: \mathcal{III}^1_{ij} + \mathcal{III}^2_{ij}. \tag{2.66}$$

Before estimating $\mathcal{III}^1_{ij}$ and $\mathcal{III}^2_{ij}$, we take p_0 close enough to q_-, and q_0 close enough to q_+ so that

$$q_- < p_0 < \min\{2, p_1\}, \quad \max\{2, p_1\} < q_0 < q_+, \quad v \in A_{p_1/p_0}(w) \cap RH_{(q_0/p_1)'}(w). \tag{2.67}$$

Hence, by Lemma 2.2 (b),

$$\mathcal{III}^1_{ij} \lesssim vw(2^{j+1} B_i)^{1/p_1} \tag{2.68}$$

$$\times \Bigg(\fint_{C_j(B_i)} \Big(\int_0^{2^{j-2} r_{B_i}} \int_{B(x,t)} |t L_w e^{-t^2 L_w}(B_{r_{B_i}} b_i)(y)|^2 \frac{dw(y) \, dt}{t w(B(y,t))} \Big)^{q_0/2} dw(x) \Bigg)^{1/q_0}.$$

Besides, note that for $x \in C_j(B_i)$ and $0 < t \leq 2^{j-2} r_{B_i}$ we have that $B(x,t) \subset 2^{j+2} B_i \setminus 2^{j-1} B_i$. Then, by (2.3), recalling that $q_0 > 2$, applying Jensen's inequality with respect to $dw(y)\,dt$, and Fubini's theorem, we get

$$\begin{aligned}
&\Big(\fint_{C_j(B_i)} \Big(\int_0^{2^{j-2} r_{B_i}} \int_{B(x,t)} \big| t L_w e^{-t^2 L_w} (B_{r_{B_i}} b_i)(y) \big|^2 \frac{dw(y)\,dt}{t\, w(B(y,t))} \Big)^{q_0/2} dw(x) \Big)^{1/q_0} \\
&\quad \lesssim (2^j r_{B_i})^{1/2} \Big(\fint_{C_j(B_i)} \Big(\fint_0^{2^{j-2} r_{B_i}} \fint_{B(x,t)} \frac{1}{t} \big| t L_w e^{-t^2 L_w} (B_{r_{B_i}} b_i)(y) \big|^2 \\
&\qquad \times dw(y)\,dt \Big)^{q_0/2} dw(x) \Big)^{1/q_0} \\
&\quad \lesssim \Big(\fint_{C_j(B_i)} \int_0^{2^{j-2} r_{B_i}} \Big(\frac{2^j r_{B_i}}{t} \Big)^{q_0/2-1} \fint_{B(x,t)} \big| t L_w e^{-t^2 L_w} (B_{r_{B_i}} b_i)(y) \big|^{q_0} \\
&\qquad \times \frac{dw(y)\,dt}{t} dw(x) \Big)^{1/q_0} \\
&\quad \lesssim \Big(\fint_{C_j(B_i)} \int_0^{2^{j-2} r_{B_i}} \Big(\frac{2^j r_{B_i}}{t} \Big)^{q_0/2-1} \int_{B(x,t)} \big| t L_w e^{-t^2 L_w} (B_{r_{B_i}} b_i)(y) \big|^{q_0} \\
&\qquad \times \frac{dw(y)\,dt}{t\, w(B(y,t))} dw(x) \Big)^{1/q_0} \\
&\quad \lesssim \Big(\int_0^{2^{j-2} r_{B_i}} \Big(\frac{2^j r_{B_i}}{t} \Big)^{q_0/2-1} t^{-q_0} \fint_{2^{j+2} B_i \setminus 2^{j-1} B_i} \big| t^2 L_w e^{-t^2 L_w} (B_{r_{B_i}} b_i)(y) \big|^{q_0} \\
&\qquad \times \frac{dw(y)\,dt}{t} \Big)^{1/q_0}.
\end{aligned} \tag{2.69}$$

We estimate the integral in y by using functional calculus. The notation is taken from [1] and Section 7 of [7]. We write $\vartheta \in [0, \pi/2)$ for the supremum of $|\arg(\langle L_w f, f \rangle_{L^2(w)})|$ over all f in the domain of L_w. Let $0 < \vartheta < \theta < \nu < \mu < \pi/2$ and note that, for a fixed $t > 0$, $\phi(z,t) := e^{-t^2 z} (1 - e^{-r_{B_i}^2 z})^M$ is holomorphic in the open sector $\Sigma_\mu = \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu \}$ and satisfies $|\phi(z,t)| \lesssim |z|^M (1 + |z|)^{-2M}$ (with implicit constant depending on μ, $t > 0$, r_{B_i}, and M) for every $z \in \Sigma_\mu$. Hence, we can write

$$\phi(L_w, t) = \int_\Gamma e^{-z L_w} \eta(z,t)\, dz, \quad \text{where} \quad \eta(z,t) = \int_\gamma e^{\zeta z} \phi(\zeta, t)\, d\zeta. \tag{2.70}$$

Here $\Gamma = \partial \Sigma_{\pi/2-\theta}$ with positive orientation (although orientation is irrelevant for our computations) and $\gamma = \mathbb{R}_+ e^{i\, \mathrm{sign}(\mathrm{Im}(z))\, \nu}$. It is not difficult to see that for every $z \in \Gamma$,

$$|\eta(z,t)| \lesssim \frac{r_{B_i}^{2M}}{(|z| + t^2)^{M+1}}.$$

Moreover, observe that $2^{j+2} B_i \setminus 2^{j-1} B_i = \bigcup_{l=1}^3 C_{l+j-2}(B_i)$, $\forall j \geq 4$. Also, our choices of p_0 and q_0 in (2.67) yield that $z L_w e^{-z L_w} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$. Thus, by these facts and Minkowski's integral inequality, we obtain

$$\begin{aligned}
&\Big(\fint_{2^{j+2}B_i\setminus 2^{j-1}B_i} \big|t^2 L_w e^{-t^2 L_w}(B_{r_{B_i}} b_i)\big|^{q_0}\,dw\Big)^{1/q_0}\\
&\quad\lesssim \int_\Gamma \Big(\fint_{2^{j+2}B_i\setminus 2^{j-1}B_i} |zL_w e^{-zL_w} b_i|^{q_0}\,dw\Big)^{1/q_0} \frac{t^2}{|z|}\,\frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}\,|dz|\\
&\quad\lesssim 2^{j\theta_1}\Big(\fint_{B_i}|b_i|^{p_0}\,dw\Big)^{1/p_0}\int_\Gamma \Upsilon\Big(\frac{2^j r_{B_i}}{|z|^{1/2}}\Big)^{\theta_2} e^{-c\,4^j r_{B_i}^2/|z|}\,\frac{t^2}{|z|}\,\frac{r_{B_i}^{2M}}{(|z|+t^2)^{M+1}}\,|dz|\\
&\quad\lesssim 2^{j\theta_1} t^2\Big(\fint_{B_i}|b_i|^{p_1}\,d(vw)\Big)^{1/p_1}\int_0^\infty \Upsilon\Big(\frac{2^j r_{B_i}}{s^{1/2}}\Big)^{\theta_2} e^{-c\,4^j r_{B_i}^2/s}\,\frac{r_{B_i}^{2M}}{s^{M+1}}\,\frac{ds}{s}\\
&\quad\lesssim \alpha\, r_{B_i}^{-1}\,2^{-j(2M+2-\theta_1)}\,t^2\int_0^\infty \Upsilon(s)^{\theta_2} e^{-cs^2} s^{2M+2}\,\frac{ds}{s}\\
&\quad\lesssim \alpha\, r_{B_i}^{-1}\,2^{-j(2M+2-\theta_1)}\,t^2,
\end{aligned}$$

where, for the last inequality, we need to take $M \in \mathbb{N}$ large enough so that $2M+2>\theta_2$. Besides, we use Lemma 2.2 (a) in the third inequality; and the fourth inequality follows from (2.40) (see (2.61)) and the change of variable s into $4^j r_{B_i}^2/s^2$.

Plugging the above estimate into (2.69) and changing the variable t into $2^j r_{B_i} t$, allows us to obtain

$$\begin{aligned}
&\Big(\fint_{C_j(B_i)}\Big(\int_0^{2^{j-2}r_{B_i}}\int_{B(x,t)}\big|tL_w e^{-t^2L_w}(B_{r_{B_i}} b_i)(y)\big|^2\,\frac{dw(y)\,dt}{tw(B(y,t))}\Big)^{q_0/2} dw(x)\Big)^{1/q_0}\\
&\qquad\lesssim \alpha\, r_{B_i}^{-1}\,2^{-j(2M+2-\theta_1)}\Big(\int_0^{2^{j-2}r_{B_i}}\Big(\frac{2^j r_{B_i}}{t}\Big)^{q_0/2-1} t^{q_0}\,\frac{dt}{t}\Big)^{1/q_0}\lesssim \alpha\,2^{-j(2M+1-\theta_1)}.
\end{aligned}$$

This and (2.68) yield, for $M\in\mathbb{N}$ such that $2M+2>\theta_2$,

$$\mathcal{I}\mathcal{I}\mathcal{I}_{ij}^1 \lesssim \alpha\, vw(2^{j+1}B_i)^{1/p_1}\,2^{-j(2M+1-\theta_1)}. \tag{2.71}$$

In order to estimate $\mathcal{I}\mathcal{I}\mathcal{I}_{ij}^2$, we first change the variable t into $t\theta_M := t\sqrt{M+1}$. Then

$$\begin{aligned}
&\Big(\int_{2^{j-2}r_{B_i}}^\infty\int_{B(x,t)}\Big|t^2L_w e^{-t^2L_w}\Big(B_{r_{B_i}}\Big(\frac{b_i}{r_{B_i}}\Big)\Big)(y)\Big|^2\,\frac{dw(y)\,dt}{tw(B(y,t))}\Big)^{1/2}\\
&\qquad\lesssim\Big(\int_{2^{j-2}r_{B_i}}^\infty\fint_{B(x,t)}\Big|t^2L_w e^{-t^2L_w}\Big(B_{r_{B_i}}\Big(\frac{b_i}{r_{B_i}}\Big)\Big)(y)\Big|^2\,\frac{dw(y)\,dt}{t}\Big)^{1/2}\\
&\qquad\lesssim\Big(\int_{2^{j-2}r_{B_i}/\theta_M}^\infty\fint_{B(x,\theta_M t)}\Big|\mathcal{T}_{t,r_{B_i}}\,t^2L_w e^{-t^2L_w}\Big(\frac{b_i}{r_{B_i}}\Big)(y)\Big|^2\,\frac{dw(y)\,dt}{t}\Big)^{1/2},
\end{aligned}\tag{2.72}$$

where we recall that

$$\mathcal{T}_{t,r_{B_i}} := \big(e^{-t^2L_w}-e^{-(t^2+r_{B_i}^2)L_w}\big)^M.$$

Next, fix $x \in C_j(B_i)$ and $t > 2^{j-2} r_{B_i}/\theta_M$. In this case, $B_i \subset B(x, 12\theta_M t)$. Thus, by (2.30) and the fact that $\tau L_w e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^2(w))$, for $\tau > 0$, we get

$$\begin{aligned}
&\fint_{B(x,\theta_M t)} \Big| \mathcal{T}_{t,r_{B_i}} t^2 L_w e^{-t^2 L_w} \Big(\frac{b_i}{r_{B_i}}\Big)(y)\Big|^2 dw(y) \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \frac{1}{w(B(x,\theta_M t))} \int_{\mathbb{R}^n} \Big| t^2 L_w e^{-t^2 L_w} \Big(\mathbf{1}_{B(x,12\theta_M t)} \frac{b_i}{r_{B_i}}\Big)(y)\Big|^2 dw(y) \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \sum_{l \ge 1} 2^{2nl} \fint_{C_l(B(x,12\theta_M t))} \Big| t^2 L_w e^{-t^2 L_w} \Big(\mathbf{1}_{B(x,12\theta_M t)} \frac{b_i}{r_{B_i}}\Big)(y)\Big|^2 dw(y) \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \sum_{l \ge 1} e^{-c4^l} \Big(\fint_{B(x,12\theta_M t)} \Big|\frac{b_i(y)}{r_{B_i}}\Big|^{p_0} dw(y)\Big)^{2/p_0} \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \Big(\frac{w(B_i)}{w(B(x,12\theta_M t))}\Big)^{2/p_0} \Big(\fint_{B_i} \Big|\frac{b_i}{r_{B_i}}\Big|^{p_0} dw\Big)^{2/p_0} \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \Big(\fint_{B_i} \Big|\frac{b_i}{r_{B_i}}\Big|^{p_1} d(vw)\Big)^{2/p_1} \lesssim \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \alpha^2.
\end{aligned}$$

The next-to-last inequality is due to Lemma 2.2 (a) and the fact that $B_i \subset B(x, 12\theta_M t)$, and the last inequality follows from (2.40).

Plugging the above estimate into (2.72) and recalling the definition of $\mathcal{III}_{ij}^2$ in (2.66) allows us to obtain

$$\begin{aligned}
\mathcal{III}_{ij}^2 &\lesssim \alpha \Big(\int_{C_j(B_i)} \Big(\int_{2^{j-2} r_{B_i}/\theta_M}^{\infty} \Big(\frac{r_{B_i}^2}{t^2}\Big)^{2M} \frac{dt}{t}\Big)^{p_1/2} v\, dw\Big)^{1/p_1} \\
&\lesssim \alpha\, vw(2^{j+1}B_i)^{1/p_1}\, 2^{-j2M}.
\end{aligned}$$

By this and (2.71), for $M \in \mathbb{N}$ such that $2M > \theta_2 - 2$, we have

$$\mathcal{III}_{ij} \lesssim \alpha\, vw(2^{j+1}B_i)^{1/p_1}\, 2^{-j(2M-\theta_1)}.$$

Then, by Lemma 2.17 with $\mathcal{I}_{ij} = \alpha^{-1}\mathcal{III}_{ij}$, $\widetilde{C} = \theta_1$, and $\{B_i\}_i$ the collection of balls given by Lemma 2.13, and by (2.38) and (2.65), for $M \in \mathbb{N}$ so that $2M > \max\{\theta_2 - 2, \theta_1 + r_w \mathfrak{r}_v(w) n\}$, we conclude that

$$\mathrm{III}_1 \lesssim vw\Big(\bigcup_i B_i\Big) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla h|^p\, v\, dw.$$

This, together with (2.62)–(2.64) and (2.59), yields (2.58). ∎

3. Proof of Theorem 1.1

Fix $w \in A_2(dx)$, $v \in A_\infty(w)$ and $f \in C_c^\infty(\mathbb{R}^n)$, and note that for every $(x,t) \in \mathbb{R}^{n+1}_+$ and $u(x,t) := \nabla_{x,t} e^{-t\sqrt{L_w}} f(x)$,

$$|u(x,t)|^2 = |\nabla e^{-t\sqrt{L_w}} f(x)|^2 + |\partial_t e^{-t\sqrt{L_w}} f(x)|^2,$$

where we define the Poisson semigroup $\{e^{-t\sqrt{L_w}}\}_{t>0}$ using the classical subordination formula, or the functional calculus for L_w (see [1, 16]):

$$e^{-t\sqrt{L_w}} = C \int_0^\infty e^{-\lambda} \lambda^{1/2} e^{-\frac{t^2}{4\lambda} L_w} \frac{d\lambda}{\lambda}. \tag{3.1}$$

Therefore, it suffices to see that if $\mathcal{W}_v^w(\max\{r_w, q_-(L_w)\}, q_+(L_w)) \neq \emptyset$, then

$$\|\mathcal{N}_w(\nabla e^{-t\sqrt{L_w}} f)\|_{L^p(v dw)} + \|\mathcal{N}_w(\partial_t e^{-t\sqrt{L_w}} f)\|_{L^p(v dw)} \lesssim \|\nabla f\|_{L^p(v dw)},$$

for all $p \in \mathcal{W}_v^w(\max\{r_w, (q_-(L_w))_{w,*}\}, q_+(L_w))$. We shall see this in Propositions 3.1 and 3.2 below.

3.1. Non-tangential maximal function estimate for the spatial derivatives

Proposition 3.1. *Let $w \in A_2(dx)$ and $v \in A_\infty(w)$ be such that*

$$\mathcal{W}_v^w(\max\{r_w, q_-(L_w)\}, q_+(L_w)) \neq \emptyset.$$

Then, for all $f \in \mathcal{S}$ and $p \in \mathcal{W}_v^w(\max\{r_w, (q_-(L_w))_{w,*}\}, q_+(L_w))$, we have

$$\left\|\mathcal{N}_w(\nabla e^{-t\sqrt{L_w}} f)\right\|_{L^p(v dw)} \lesssim \|\nabla f\|_{L^p(v dw)}. \tag{3.2}$$

Proof. First of all, fix $w \in A_2(dx)$ and define $q_- := q_-(L_w)$ and $q_+ := q_+(L_w)$.

In the context of (1.7), we set $\alpha := c_0 c_1$. We claim that

$$\mathcal{N}_w(\nabla e^{-t\sqrt{L_w}} f)(x) \lesssim \sup_{t>0} \left(\fint_{B(x,\alpha t)} \left|\nabla e^{-t\sqrt{L_w}} f(z)\right|^2 dw(z)\right)^{1/2}. \tag{3.3}$$

Indeed, by (2.3),

$$\begin{aligned}
\mathcal{N}_w(\nabla e^{-t\sqrt{L_w}} f)(x) &= \sup_{t>0} \left(\fint_{c_0^{-1}t<s<c_0 t} \fint_{B(x,c_1 t)} \left|\nabla e^{-s\sqrt{L_w}} f(z)\right|^2 dw(z)\, ds\right)^{1/2} \\
&\lesssim \sup_{t>0} \sup_{c_0^{-1}t<s<c_0 t} \left(\fint_{B(x,\alpha s)} \left|\nabla e^{-s\sqrt{L_w}} f(z)\right|^2 dw(z)\right)^{1/2} \\
&\lesssim \sup_{t>0} \left(\fint_{B(x,\alpha t)} \left|\nabla e^{-t\sqrt{L_w}} f(z)\right|^2 dw(z)\right)^{1/2}.
\end{aligned}$$

Besides, by the subordination formula (3.1) and Minkowski's integral inequality,

$$\begin{aligned}
&\left(\fint_{B(x,\alpha t)} \left|\nabla e^{-t\sqrt{L_w}} f(z)\right|^2 dw(z)\right)^{1/2} \\
&\qquad \lesssim \int_0^{1/4} e^{-\lambda} \lambda^{1/2} \left(\fint_{B(x,\alpha t)} \left|\nabla e^{-\frac{t^2}{4\lambda} L_w} f(z)\right|^2 dw(z)\right)^{1/2} \frac{d\lambda}{\lambda} \\
&\qquad\quad + \int_{1/4}^\infty e^{-\lambda} \lambda^{1/2} \left(\fint_{B(x,\alpha t)} \left|\nabla e^{-\frac{t^2}{4\lambda} L_w} f(z)\right|^2 dw(z)\right)^{1/2} \frac{d\lambda}{\lambda} =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Dealing first with term I, note that

$$\mathrm{I} \le \int_0^{1/4} \lambda^{1/2} \Big(\fint_{B(x,\alpha t)} \big|\nabla e^{-t^2 L_w} f(z)\big|^2 \, dw(z)\Big)^{1/2} \frac{d\lambda}{\lambda} + \int_0^{1/4} \lambda^{1/2} \Big(\fint_{B(x,\alpha t)} \big|\big(\nabla e^{-\frac{t^2}{4\lambda} L_w} - \nabla e^{-t^2 L_w}\big) f(z)\big|^2 \, dw(z)\Big)^{1/2} \frac{d\lambda}{\lambda} =: \mathrm{I}_1 + \mathrm{I}_2.$$

In order to estimate term I_1, for any $p \in \mathcal{W}_v^w(\max\{r_w, (q_-)_{w,*}\}, q_+)$, we pick p_0 in the interval $(\max\{r_w, (q_-)_{w,*}\}, \min\{2, p\})$, close enough to $\max\{r_w, (q_-)_{w,*}\}$ so that $v \in A_{p/p_0}(w)$ (see (2.13) and (2.14)). Therefore $\mathcal{M}_{p_0}^w(f) := (\mathcal{M}_{p_0}^w(|f|^{p_0}))^{1/p_0}$ is bounded on $L^p(v dw)$. This and Lemma 2.10, with $\alpha = c_0 c_1$, yield

$$\begin{aligned} &\Big\| \sup_{t>0} \Big(\fint_{B(\cdot,\alpha t)} \big|\nabla e^{-t^2 L_w} f(z)\big|^2 \, dw(z)\Big)^{1/2} \Big\|_{L^p(v dw)} \\ &\qquad \lesssim \Big\| \sup_{t>0} \sum_{j\ge 1} e^{-c 4^j} \Big(\fint_{B(\cdot,2^{j+1}\alpha t)} |\nabla f(z)|^{p_0} \, dw(z)\Big)^{1/p_0} \Big\|_{L^p(v dw)} \\ &\qquad \lesssim \sum_{j\ge 1} e^{-c 4^j} \big\| \mathcal{M}_{p_0}^w(\nabla f)\big\|_{L^p(v dw)} \lesssim \|\nabla f\|_{L^p(v dw)}. \end{aligned}$$

Consequently, by Minkowski's integral inequality,

$$\begin{aligned} \big\|\sup_{t>0} I_1\big\|_{L^p(v dw)} &\le \int_0^{1/4} \lambda^{1/2} \Big\| \sup_{t>0} \Big(\fint_{B(\cdot,\alpha t)} \big|\nabla e^{-t^2 L_w} f(z)\big|^2 \, dw(z)\Big)^{1/2} \Big\|_{L^p(v dw)} \frac{d\lambda}{\lambda} \\ &\lesssim \|\nabla f\|_{L^p(v dw)}. \end{aligned}$$

Now we turn to the estimate of term I_2. Write

$$\nabla e^{-\frac{t^2}{4\lambda} L_w} - \nabla e^{-t^2 L_w} = \nabla e^{-\frac{t^2}{2} L_w} \big(e^{-(\frac{1}{4\lambda} - \frac{1}{2}) t^2 L_w} - e^{-\frac{t^2}{2} L_w}\big)$$

and use again Lemma 2.10 and (2.3). Then,

$$\begin{aligned} \mathrm{I}_2 &= \int_0^{1/4} \lambda^{1/2} \Big(\fint_{B(x,\alpha t)} \big|\nabla e^{-\frac{t^2}{2} L_w} \big(e^{-(\frac{1}{4\lambda} - \frac{1}{2}) t^2 L_w} - e^{-\frac{t^2}{2} L_w}\big) f(z)\big|^2 \, dw(z)\Big)^{1/2} \frac{d\lambda}{\lambda} \\ &\lesssim \sum_{j=1}^{\infty} e^{-c 4^j} \int_0^{1/4} \lambda^{1/2} \Big(\fint_{B(x,2^{j+2}\alpha t)} \big(\nabla \big(e^{-(\frac{1}{4\lambda} - \frac{1}{2}) t^2 L_w} - e^{-\frac{t^2}{2} L_w}\big) f(z)\big|^{p_0} dw(z)\Big)^{1/p_0} \frac{d\lambda}{\lambda}. \end{aligned}$$

Since $0 < \lambda \le 1/4$, there holds

$$\begin{aligned} &\big|\big(\nabla e^{-(\frac{1}{4\lambda} - \frac{1}{2}) t^2 L_w} - \nabla e^{-\frac{t^2}{2} L_w}\big) f(z)\big| \\ &\quad \le \int_{t/\sqrt{2}}^{t\sqrt{1/(4\lambda) - 1/2}} \big|\partial_s \nabla e^{-s^2 L_w} f(z)\big| \, ds \lesssim \int_{t/\sqrt{2}}^{t\sqrt{1/(4\lambda) - 1/2}} \big|s^2 \nabla L_w e^{-s^2 L_w} f(z)\big| \frac{ds}{s} \\ &\quad \lesssim \Big(\int_0^\infty \big|s^2 \nabla L_w e^{-s^2 L_w} f(z)\big|^2 \frac{ds}{s}\Big)^{1/2} (\log(2\lambda)^{-1/2})^{1/2} \lesssim (\log \lambda^{-1})^{1/2} \, \widetilde{\mathrm{G}}_{\mathrm{H}}^w f(z), \end{aligned}$$

where $\widetilde{\mathrm{G}}_{\mathrm{H}}^{w}$ is the vertical square function defined in (2.22). Then, we get

$$\sup_{t>0} \mathrm{I}_2 \lesssim \sum_{j\geq 1} e^{-c4^j} \int_0^{1/4} \lambda^{1/2}(\log \lambda^{-1})^{1/2} \sup_{t>0} \Big(\fint_{B(x,2^{j+2}\alpha t)} \big|\widetilde{\mathrm{G}}_{\mathrm{H}}^{w} f(z)\big|^{p_0} dw(z) \Big)^{1/p_0} \frac{d\lambda}{\lambda}$$
$$\lesssim \mathcal{M}_{p_0}^{w}(\widetilde{\mathrm{G}}_{\mathrm{H}}^{w} f)(x).$$

Then, since $\mathcal{M}_{p_0}^{w}$ is bounded on $L^p(v dw)$, the above computation and Theorem 2.14 imply

$$\big\| \sup_{t>0} I_2 \big\|_{L^p(vdw)} \leq \big\| \mathcal{M}_{p_0}^{w} \widetilde{\mathrm{G}}_{\mathrm{H}}^{w} f) \big\|_{L^p(vdw)} \lesssim \big\| \widetilde{\mathrm{G}}_{\mathrm{H}}^{w} f \big\|_{L^p(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)}.$$

We finally estimate term II. Applying Lemma 2.10, we have that, for every $t > 0$,

$$\Big(\fint_{B(x,\alpha 2\sqrt{\lambda} t)} |\nabla e^{-t^2 L_w} f(z)|^2 \, dw(z) \Big)^{1/2}$$
$$\lesssim \Upsilon(\sqrt{\lambda})^{\theta} \sum_{j=1}^{\infty} e^{-c4^j} \Big(\fint_{B(x,2^{j+2}\alpha\sqrt{\lambda} t)} |\nabla f(z)|^{p_0} \, dw(z) \Big)^{1/p_0} \lesssim \Upsilon(\sqrt{\lambda})^{\theta} \mathcal{M}_{p_0}^{w}(\nabla f)(x).$$

Hence,

$$\big\| \sup_{t>0} \mathrm{II} \big\|_{L^p(vdw)} \lesssim \int_{1/4}^{\infty} e^{-\lambda} \, \Upsilon(\sqrt{\lambda})^{\theta} \, \frac{d\lambda}{\lambda} \, \|\mathcal{M}_{p_0}^{w}(\nabla f)\|_{L^p(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)}.$$

Collecting the above estimates, we conclude (3.2). ∎

3.2. Non-tangential maximal function estimate for the time derivative

Proposition 3.2. *Given $w \in A_2(dx)$ and $v \in A_\infty(w)$, assume that*

$$\mathcal{W}_v^w \left(\max\{r_w, q_-(L_w)\}, q_+(L_w) \right) \neq \emptyset.$$

Then, for all $p \in \mathcal{W}_v^w \left(\max\{r_w, (p_-(L_w))_{w,*}\}, p_+(L_w) \right)$ and $f \in \mathcal{S}$, we have

$$\big\| \mathcal{N}_w(\partial_t e^{-t\sqrt{L_w}} f) \big\|_{L^p(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)}. \tag{3.4}$$

To prove this result, we need Theorem 2.18, a change of angles result in $L^p(vdw)$ for the operator defined in (2.15), and the boundedness of the non-tangential maximal square functions defined in (2.16) and (2.17). We obtain these results in Lemma 3.3, and Proposition 3.4 below.

Our next result is an extension of Lemma 6.2 in [27] (see also Lemma 7.3 in [30]).

Lemma 3.3. *Given $w \in A_r(dx)$ and $v \in A_{\widehat{r}}(w)$, $1 \leq r, \widehat{r} < \infty$, let $0 < p < \infty$ and $\kappa \geq 1$. There hold*

$$\|\mathcal{N}^{\kappa,w} F\|_{L^{p,\infty}(vdw)} \lesssim \kappa^{n((r+1)/2 + r\widehat{r}/p)} \, \|\mathcal{N}^{w} F\|_{L^{p,\infty}(vdw)} \tag{3.5}$$

and

$$\|\mathcal{N}^{\kappa,w} F\|_{L^p(vdw)} \lesssim \kappa^{n((r+1)/2 + r\widehat{r}/p)} \, \|\mathcal{N}^{w} F\|_{L^p(vdw)}. \tag{3.6}$$

Proof. We will just prove (3.5); the proof of (3.6) follows analogously by writing the $L^p(vdw)$-norm as an integral of the level sets. Details are left to the interested reader.

Consider, for any $\lambda > 0$,

$$O_\lambda := \{x \in \mathbb{R}^n : \mathcal{N}^w F(x) > \lambda\} \quad \text{and} \quad E_\lambda := \mathbb{R}^n \setminus O_\lambda,$$

and, for $\gamma = 1 - \frac{1}{[w]_{A_r(dx)}(11\kappa)^{rn}}$, the set of γ-density

$$E_\lambda^* := \Big\{x \in \mathbb{R}^n : \forall\, r > 0,\ \frac{w(E_\lambda \cap B(x,r))}{w(B(x,r))} \geq \gamma\Big\}.$$

Note that

$$O_\lambda^* := \mathbb{R}^n \setminus E_\lambda^* = \Big\{x \in \mathbb{R}^n : \mathcal{M}^w(\mathbf{1}_{O_\lambda})(x) > \frac{1}{[w]_{A_r(dx)}(11\kappa)^{rn}}\Big\}.$$

We claim that for every $\lambda > 0$,

$$\mathcal{N}^{\kappa,w} F(x) \leq [w]_{A_r(dx)}\, 2^{nr/2}\, (9\kappa)^{n(r+1)/2}\, \lambda, \quad \forall x \in E_\lambda^*. \tag{3.7}$$

Assuming this momentarily, let $0 < p < \infty$. Since $\mathcal{M}^w : L^{\hat{r}}(vdw) \to L^{\hat{r},\infty}(vdw)$, as we are assuming that $v \in A_{\hat{r}}(w)$, we get

$$\begin{aligned}
\|\mathcal{N}^{\kappa,w} F\|_{L^{p,\infty}(vdw)}^p &= \sup_{\lambda>0} \lambda^p vw(\{x \in \mathbb{R}^n : \mathcal{N}^{\kappa,w} F(x) > \lambda\}) \\
&= \sup_{\lambda>0} \big([w]_{A_r(dx)}\, 2^{nr/2}\, (9\kappa)^{n(r+1)/2}\, \lambda\big)^p \\
&\qquad \times vw\big(\{x \in \mathbb{R}^n : \mathcal{N}^{\kappa,w} F(x) > [w]_{A_r(dx)}\, 2^{nr/2}\, (9\kappa)^{n(r+1)/2}\, \lambda\}\big) \\
&\leq [w]_{A_r(dx)}^p\, 2^{pnr/2}\, (9\kappa)^{n(r+1)p/2} \sup_{\lambda>0} \lambda^p vw(O_\lambda^*) \lesssim \kappa^{n((r+1)p/2 + r\hat{r})} \sup_{\lambda>0} \lambda^p vw(O_\lambda) \\
&= \kappa^{n((r+1)p/2 + r\hat{r})}\, \|\mathcal{N}^w F\|_{L^{p,\infty}(vdw)}^p,
\end{aligned}$$

which would finish the proof.

It remains to show (3.7). First, note that if $x \in E_\lambda^*$ and $t > 0$, for every $y \in B(x, 2\kappa t)$ we have $B(y,t/2) \cap E_\lambda \neq \emptyset$. To prove this, suppose by way of contradiction that $B(y,t/2) \subset O_\lambda$. Then, by (2.2), since $B(y,t/2) \subset B(x,3\kappa t)$ and $B(x,3\kappa t) \subset B(y,5\kappa t)$,

$$\mathcal{M}^w(\mathbf{1}_{O_\lambda})(x) \geq \frac{w(B(y,t/2))}{w(B(x,3\kappa t))} \geq \frac{w(B(y,t/2))}{w(B(y,5\kappa t))} \geq \frac{1}{[w]_{A_r(dx)}(10\kappa)^{rn}} > \frac{1}{[w]_{A_r(dx)}(11\kappa)^{rn}}.$$

This implies that $x \in O_\lambda^*$, which contradicts our assumption.

Let us fix now $x \in E_\lambda^*$ and $t > 0$, and note that if $y \in B(x, 2\kappa t)$ there exists $x_0 \in B(y,t/2) \cap E_\lambda$, hence $\mathcal{N}^w F(x_0) \leq \lambda$. Besides, since $B(y,t/2) \subset B(x_0,t)$ and by (2.2), for every $y \in B(x, 2\kappa t)$,

$$\begin{aligned}
&\Big(\int_{B(y,t/2)} |F(z,t)|^2 \frac{dw(z)}{w(B(y,t/2))}\Big)^{1/2} \\
&\qquad\qquad \leq [w]_{A_r(dx)}^{1/2}\, 2^{nr/2} \sup_{s>0} \Big(\int_{B(x_0,s)} |F(z,s)|^2 \frac{dw(z)}{w(B(x_0,s))}\Big)^{1/2} \\
&\qquad\qquad = [w]_{A_r(dx)}^{1/2}\, 2^{nr/2}\, \mathcal{N}^w F(x_0) \leq [w]_{A_r(dx)}^{1/2}\, 2^{nr/2}\, \lambda.
\end{aligned} \tag{3.8}$$

On the other hand, for every $x \in \mathbb{R}^n$ and $t > 0$, we have that $B(x, \kappa t) \subset \bigcup_i B(x_i, t/2)$, where $\{B(x_i, t/2)\}_i$ is a collection of at most $(9\kappa)^n$ balls such that, for every i, we have that $x_i \in B(x, 2\kappa t)$. In particular, $B(x_i, t/2),\ B(x, t) \subset B(x_i, 3\kappa t)$.

Therefore, by the above observations and (2.2), we conclude that

$$\begin{aligned}\int_{B(x,\kappa t)} |F(y,t)|^2 \frac{dw(y)}{w(B(x,t))} &\le [w]_{A_r(dx)} (3\kappa)^{nr} \sum_i \int_{B(x_i,t/2)} |F(y,t)|^2 \frac{dw(y)}{w(B(x_i,t/2))} \\ &\le (9\kappa)^{n(r+1)} [w]^2_{A_r(dx)} 2^{nr} \lambda^2,\end{aligned}$$

where we have used (3.8), since $x_i \in B(x, 2\kappa t)$. Finally, taking the supremum over all $t > 0$, we obtain

$$\mathcal{N}^{\kappa,w} F(x)^2 \le [w]^2_{A_r(dx)} 2^{nr} (9\kappa)^{n(r+1)} \lambda^2, \quad \forall\, x \in E^*_\lambda.$$

This readily gives (3.7) and the proof is complete. ■

Proposition 3.4. *Let L_w be a degenerate elliptic operator with $w \in A_2(dx)$ and let $v \in A_\infty(w)$. Then*

(a) *$\mathcal{N}^w_{\mathrm{H}}$ is bounded on $L^p(v\,dw)$ for all $p \in \mathcal{W}^w_v(p_-(L_w), \infty)$,*

(b) *$\mathcal{N}^w_{\mathrm{P}}$ is bounded on $L^p(v\,dw)$ for all $p \in \mathcal{W}^w_v(p_-(L_w), p_+(L_w)^*_w)$.*

Proof. Part (b) is proved in Theorem 3.7 of [33].

In order to prove part (a), fix $p \in \mathcal{W}^w_v(p_-(L_w), \infty)$ and choose p_0 close enough to $p_-(L_w)$ so that

$$p_-(L_w) < p_0 < \min\{2, p\} \quad \text{and} \quad v \in A_{p/p_0}(w). \tag{3.9}$$

Then $e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^2(w))$. This fact and (2.3) yield

$$\begin{aligned}\mathcal{N}^w_{\mathrm{H}} f(x) &\lesssim \sup_{t>0} \sum_{j\ge1} \Big(\fint_{B(x,t)} \big|e^{-t^2 L_w} (\mathbf{1}_{C_j(B(x,t))} f)(z)\big|^2 dw(z) \Big)^{1/2} \\ &\lesssim \sup_{t>0} \sum_{j\ge1} 2^{j\theta_1} \Upsilon(2^{j+1})^{\theta_2} e^{-c4^j} \Big(\fint_{C_j(B(x,t))} |f(z)|^{p_0} dw(z) \Big)^{1/p_0} \lesssim \mathcal{M}^w_{p_0} f(x).\end{aligned}$$

Consequently,

$$\|\mathcal{N}^w_{\mathrm{H}} f\|_{L^p(v dw)} \lesssim \|\mathcal{M}^w_{p_0} f\|_{L^p(v dw)} \lesssim \|f\|_{L^p(v dw)},$$

since $\mathcal{M}^w_{p_0}$ is bounded on $L^p(v\,dw)$ by our choice of p_0. ■

Proof of Proposition 3.2. First of all, fix $w \in A_2(dx)$ and denote

$$q_- := p_-(L_w) = q_-(L_w), \quad q_+ := q_+(L_w), \quad p_+ := p_+(L_w),$$

and

$$u(x,t) := \partial_t e^{-t\sqrt{L_w}} f(x) = -\sqrt{L_w}\, e^{-t\sqrt{L_w}} f(x).$$

From the definitions of $\mathcal{N}_w$ and $\mathcal{N}_{\mathrm{P}}^w$ (see (1.7) and (2.17)), proceeding as in the proof of (3.3) we have that

$$\mathcal{N}_w u(x) \lesssim \mathcal{N}_{\mathrm{P}}^{\alpha,w}(\sqrt{L_w} f)(x), \quad \forall x \in \mathbb{R}^n, \tag{3.10}$$

with $\alpha = c_0 c_1$. Consequently, Lemma 3.3, and Propositions 3.4 (b) and 2.4 imply

$$\|\mathcal{N}_w u\|_{L^p(vdw)} \lesssim \left\|\mathcal{N}_{\mathrm{P}}^w(\sqrt{L_w} f)\right\|_{L^p(vdw)} \lesssim \|\sqrt{L_w} f\|_{L^p(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)},$$

for all $p \in \mathcal{W}_v^w(\max\{r_w, q_-\}, p_+) = (\mathrm{r}_v(w)\max\{r_w, q_-\}, p_+/\mathfrak{s}_v(w))$ and $f \in \mathcal{S}$.

Our goal is to obtain (3.4) for all

$$p \in \mathcal{W}_v^w(\max\{r_w, (q_-)_{w,*}\}, p_+) = (\mathrm{r}_v(w)\max\{r_w, (q_-)_{w,*}\}, p_+/\mathfrak{s}_v(w)).$$

Recall that $(q_-)_{w,*} < q_-$ (see (2.6)). Hence, fix p such that

$$\mathrm{r}_v(w)\max\{r_w, (q_-)_{w,*}\} < p < \mathrm{r}_v(w)\max\{r_w, q_-\}. \tag{3.11}$$

Then, in view of inequality (3.10) and Lemma 3.3, if we show that, for all $f \in \mathcal{S}$,

$$\left\|\mathcal{N}_{\mathrm{P}}^w\left(\sqrt{L_w} f\right)\right\|_{L^{p,\infty}(vdw)} \lesssim \|\nabla f\|_{L^p(vdw)}, \tag{3.12}$$

by interpolation, see [11] and Remark 2.1, we would conclude the desired estimate.

Given $\alpha > 0$, take a function $f \in \mathcal{S}$. We apply Lemma 2.13 to f, α, and the product weight $\varpi = vw$ (note that $vw \in A_p(dx)$ since $r_{vw} \le r_w \mathrm{r}_v(w) < p$, see Remark 2.1). Let $\{B_i\}_i$ be the collection of balls given by Lemma 2.13. Consider for $M \in \mathbb{N}$ arbitrarily large,

$$B_{r_{B_i}} := \left(I - e^{-r_{B_i}^2 L_w}\right)^M \quad \text{and} \quad A_{r_{B_i}} := I - B_{r_{B_i}} = \sum_{k=1}^{M} C_{k,M}\, e^{-k r_{B_i}^2 L_w}.$$

Hence,

$$f = g + \sum_i A_{r_{B_i}} b_i + \sum_i B_{r_{B_i}} b_i =: g + \widetilde{b} + \widehat{b}. \tag{3.13}$$

To prove the weak-type estimates for g, $\widetilde{b}$ and $\widehat{b}$, we need some preparations. On the one hand, since we assume that $\mathcal{W}_v^w(\max\{r_w, q_-\}, q_+) \neq \emptyset$, by (3.11) and (2.42) we can take p_1 satisfying

$$\mathrm{r}_v(w)\max\{r_w, q_-\} < p_1 < \min\left\{\frac{q_+}{\mathfrak{s}_v(w)}, p_{vw}^*\right\}. \tag{3.14}$$

In particular, $\mathrm{r}_v(w) q_- < p_1 < q_+/\mathfrak{s}_v(w)$, that is, $p_1 \in \mathcal{W}_v^w(q_-, q_+)$. This can be written as

$$v \in A_{p_1/q_-}(w) \cap \mathrm{RH}_{(q_+/p_1)'}(w). \tag{3.15}$$

On the other hand, take p_0 satisfying $q_- < p_0 < \min\{2, p_1\}$ close enough to q_-, and q_0 satisfying $\max\{2, p_1\} < q_0 < q_+$ close enough to q_+, so that

$$v \in A_{p_1/p_0}(w) \cap \mathrm{RH}_{(q_0/p_1)'}(w). \tag{3.16}$$

Next, note that by (2.3) and from the proof of Theorem 4.20 in [33], for

$$\mathrm{g}^w_{\mathrm{H},t} f(z) := \Big(\int_{t/2}^{\infty} \big|s^2 L_w e^{-s^2 L_w} f(z)\big|^2 \frac{ds}{s}\Big)^{1/2}$$

and for any function $h \in L^2(w)$, we have that

$$\mathcal{N}^w_{\mathrm{P}} h(x) \lesssim \mathcal{N}^w_{\mathrm{H}} h(x) + \sum_{l\ge 1} e^{-c4^l} \sup_{t>0} \Big(\fint_{B(x,2^{l+1}t)} \big|\mathrm{g}^w_{\mathrm{H},t} h(y)\big|^{p_0} dw(y)\Big)^{1/p_0}$$
$$+ \int_{1/4}^{\infty} e^{-c\lambda}\, \mathcal{S}^{2\sqrt{\lambda},w}_{\mathrm{H}} h(x) \frac{d\lambda}{\lambda} =: \mathcal{N}^w_{\mathrm{H}} h(x) + \sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{2,l} h(x) + \mathfrak{D}_3 h(x).$$

Besides, note that, the fact that $e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^2(w))$, and (2.3) yield

$$\mathcal{N}^w_{\mathrm{H}} h(x) = \sup_{t>0} \Big(\fint_{B(x,t)} \big|e^{-t^2 L_w} h(y)\big|^2 dw(y)\Big)^{1/2}$$
$$\lesssim \sum_{l\ge 1} e^{-c4^l} \sup_{t>0} \Big(\fint_{B(x,2^{l+1}t)} \big|e^{-\frac{t^2}{2} L_w} h(y)\big|^{p_0} dw(y)\Big)^{1/p_0}$$
$$\lesssim \sum_{l\ge 1} e^{-c4^l} \sup_{t>0} \Big(\fint_{B(x,2^{l+2}t)} \big|e^{-t^2 L_w} h(y)\big|^{p_0} dw(y)\Big)^{1/p_0} =: \sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{1,l} h(x).$$

Therefore, for any function $h \in L^2(w)$, we have that

$$\mathcal{N}^w_{\mathrm{P}} h(x) \le C\Big(\sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{1,l} h(x) + \sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{2,l} h(x) + \mathfrak{D}_3 h(x)\Big), \quad \forall x \in \mathbb{R}^n.$$

Using this and (3.13), we get

$$\begin{aligned}
(3.17)\quad vw\Big(\Big\{x \in \mathbb{R}^n : \mathcal{N}^w_{\mathrm{P}}\big(\sqrt{L_w} f\big)(x) > \alpha\Big\}\Big)
&\le vw\Big(\Big\{x \in \mathbb{R}^n : \mathcal{N}^w_{\mathrm{P}}\big(\sqrt{L_w} g\big)(x) > \frac{\alpha}{5}\Big\}\Big)\\
&\quad + vw\Big(\Big\{x \in \mathbb{R}^n : \mathcal{N}^w_{\mathrm{P}}\big(\sqrt{L_w}\, \widetilde{b}\big)(x) > \frac{\alpha}{5}\Big\}\Big)\\
&\quad + \sum_{m=1}^{2} vw\Big(\Big\{x \in \mathbb{R}^n : C\sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{m,l}\big(\sqrt{L_w}\, \widehat{b}\big)(x) > \frac{\alpha}{5}\Big\}\Big)\\
&\quad + vw\Big(\Big\{x \in \mathbb{R}^n : C\mathfrak{D}_3\big(\sqrt{L_w}\, \widehat{b}\big)(x) > \frac{\alpha}{5}\Big\}\Big)\\
&=: \mathrm{I} + \mathrm{II} + \sum_{m=1}^{2} \mathrm{III}_m + \mathrm{IV}.
\end{aligned}$$

In order to estimate I, first note that $p < p_1$ (see (3.11) and (3.14)). Then, apply Chebyshev's inequality, Propositions 3.4 (b) and 2.4, and properties (2.35)–(2.39), to get

$$(3.18)\quad \mathrm{I} \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} \big|\mathcal{N}^w_{\mathrm{P}}\big(\sqrt{L_w} g\big)\big|^{p_1} v\,dw \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g|^{p_1} v\,dw \lesssim \frac{1}{\alpha^{p}} \int_{\mathbb{R}^n} |\nabla f|^{p} v\,dw.$$

Now we estimate II. To this end, apply Chebyshev's inequality, Propositions 3.4 (b) and 2.4, and Lemma 2.19. Then,

$$\begin{aligned}\text{II} &\lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} \big|\mathcal{N}_{\mathrm{P}}^{w}\big(\sqrt{L_w}\,\widetilde{b}\big)\big|^{p_1} v\,dw \lesssim \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla \widetilde{b}|^{p_1} v\,dw \\ &\lesssim \frac{1}{\alpha^{p}} \int_{\mathbb{R}^n} |\nabla f|^{p} v\,dw.\end{aligned} \tag{3.19}$$

We next estimate IV. With this aim, we write $b = \sum_i b_i$ so that $\widehat{b} = b - \widetilde{b}$, and note that

$$\begin{aligned}\text{IV} \le vw\Big(\Big\{x \in \mathbb{R}^n : C\mathfrak{D}_3\big(\sqrt{L_w}\,b\big)(x) > \frac{\alpha}{10}\Big\}\Big) \\ + vw\Big(\Big\{x \in \mathbb{R}^n : C\mathfrak{D}_3\big(\sqrt{L_w}\,\widetilde{b}\big)(x) > \frac{\alpha}{10}\Big\}\Big) =: \text{IV}_1 + \text{IV}_2.\end{aligned}$$

In order to estimate IV_1, apply Chebyshev's inequality, Minkowski's integral inequality, and Proposition 2.12, to get

$$\begin{aligned}\text{IV}_1 &\lesssim \frac{1}{\alpha^p}\Big(\int_{1/4}^{\infty} e^{-cu} \big\|\mathcal{S}_{\mathrm{H}}^{2\sqrt{u},w}\big(\sqrt{L_w}\,b\big)\big\|_{L^p(vdw)} \frac{du}{u}\Big)^p \\ &\lesssim \frac{1}{\alpha^p} \big\|\mathcal{S}_{\mathrm{H}}^{w}\big(\sqrt{L_w}\,b\big)\big\|_{L^p(vdw)}^p \lesssim \frac{1}{\alpha^p} \big\|\mathcal{S}_{1/2,\mathrm{H}}^{w}\big(\sqrt{L_w}\,b\big)\big\|_{L^p(vdw)}^p \\ &= \frac{1}{\alpha^p} \|\widetilde{\mathcal{S}}_{\mathrm{H}}^{w} b\|_{L^p(vdw)}^p \lesssim \frac{1}{\alpha^p} \|\nabla b\|_{L^p(vdw)}^p \\ &\lesssim \frac{1}{\alpha^p} \sum_i \int_{B_i} |\nabla b_i|^p\, v\,dw \lesssim \sum_i vw(B_i) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p\, v\,dw,\end{aligned}$$

where we have used Proposition 2.20 in the third inequality, Theorem 2.18 in the fourth inequality, and the last two inequalities follow from (2.37) and (2.38).

As for the estimate of IV_2, apply again Chebyshev's inequality, Minkowski's integral inequality and Proposition 2.12. Then, Theorem 3.1 in [12], Proposition 2.4, and Lemma 2.19 readily give

$$\begin{aligned}\text{IV}_2 &\lesssim \frac{1}{\alpha^{p_1}}\Big(\int_{1/4}^{\infty} e^{-cu} \big\|\mathcal{S}_{\mathrm{H}}^{2\sqrt{u},w}\big(\sqrt{L_w}\,\widetilde{b}\big)\big\|_{L^{p_1}(vdw)} \frac{du}{u}\Big)^{p_1} \\ &\lesssim \frac{1}{\alpha^{p_1}} \big\|\mathcal{S}_{\mathrm{H}}^{w}\big(\sqrt{L_w}\,\widetilde{b}\big)\big\|_{L^{p_1}(vdw)}^{p_1} \lesssim \frac{1}{\alpha^{p_1}} \|\nabla \widetilde{b}\,\|_{L^{p_1}(vdw)}^{p_1} \\ &\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p\, v\,dw.\end{aligned}$$

Therefore, we conclude that

$$\text{IV} \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p\, v\,dw. \tag{3.20}$$

Now, it remains to estimate III_m, for $m = 1, 2$. Note that by (2.38),

$$\begin{aligned}
\mathrm{III}_m &\le vw\Big(\bigcup_i 16B_i\Big) \\
&\quad + vw\Big(\Big\{x \in \mathbb{R}^n \setminus \cup_i 16B_i : C \sum_{l\ge 1} e^{-c4^l} \mathfrak{D}_{m,l}\big(\sqrt{L_w}\,\widehat{b}\big)(x) > \frac{\alpha}{5}\Big\}\Big) \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v \, dw \\
&\quad + \sum_{l\ge 1} vw\Big(\Big\{x \in \mathbb{R}^n \setminus \cup_i 16B_i : \mathfrak{D}_{m,l}\big(\sqrt{L_w}\,\widehat{b}\big)(x) > \frac{e^{c4^l}\alpha}{C2^l}\Big\}\Big) \\
&=: \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v \, dw + \sum_{l\ge 1} \mathrm{III}_{m,l}. \qquad (3.21)
\end{aligned}$$

Applying Chebyshev's inequality, duality, and Hölder's inequality, it follows that

$$\begin{aligned}
\mathrm{III}_{m,l} &\lesssim \frac{e^{-c4^l}}{\alpha^{p_1}} \int_{\mathbb{R}^n\setminus \cup_i 16B_i} \big|\mathfrak{D}_{m,l}\big(\sqrt{L_w}\,\widehat{b}\big)\big|^{p_1} v \, dw \\
&\lesssim \frac{e^{-c4^l}}{\alpha^{p_1}} \Big(\sup_{\|u\|_{L^{p_1'}(vdw)}=1} \sum_i \sum_{j\ge 4} \Big(\int_{C_j(B_i)} \big|\mathfrak{D}_{m,l}\big(\sqrt{L_w}(B_{r_{B_i}} b_i)\big)\big|^{p_1} v \, dw \Big)^{1/p_1} \\
&\qquad \times \|u \mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} \Big)^{p_1} \\
&=: \frac{e^{-c4^l}}{\alpha^{p_1}} \Big(\sup_{\|u\|_{L^{p_1'}(vdw)}=1} \sum_i \sum_{j\ge 4} I_{m,l}^{ij} \, \|u \mathbf{1}_{C_j(B_i)}\|_{L^{p_1'}(vdw)} \Big)^{p_1}. \qquad (3.22)
\end{aligned}$$

Then, for $m = 1$, we have that

$$\begin{aligned}
I_{1,l}^{ij} &\lesssim \Big(\int_{C_j(B_i)} \Big(\sup_{0<t<2^{j-l-3} r_{B_i}} \Big(\fint_{B(x,2^{l+2}t)} \big| e^{-t^2 L_w} \sqrt{L_w}(B_{r_{B_i}} b_i)(y) \big|^{p_0} dw(y) \Big)^{1/p_0} \Big)^{p_1} \\
&\qquad \times v \, dw(x) \Big)^{1/p_1} \\
&\quad + \Big(\int_{C_j(B_i)} \Big(\sup_{t\ge 2^{j-l-3} r_{B_i}} \Big(\fint_{B(x,2^{l+2}t)} \big| e^{-t^2 L_w} \sqrt{L_w}(B_{r_{B_i}} b_i)(y) \big|^{p_0} dw(y) \Big)^{1/p_0} \Big)^{p_1} \\
&\qquad \times v \, dw(x) \Big)^{1/p_1} \\
&=: \mathfrak{C}_1 + \mathfrak{C}_2.
\end{aligned}$$

In order to estimate $\mathfrak{C}_1$, we use functional calculus as in the proof of Theorem 2.18. Recall (2.70) and take

$$\phi(z,t) := t\, z^{1/2} \, e^{-t^2 z} (1 - e^{-r_{B_i}^2 z})^M.$$

Then $\phi(z,t)$ is holomorphic in the open sector $\Sigma_\mu = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu\}$ and satisfies $|\phi(z,t)| \lesssim |z|^M (1+|z|)^{-2M}$ (with implicit constant depending on μ, $t>0$, r_{B_i}, and M) for every $z \in \Sigma_\mu$. We can check that for every $z \in \Gamma = \partial\Sigma_{\pi/2-\theta}$,

$$|\eta(z,t)| \lesssim \frac{t\, r_{B_i}^{2M}}{(|z|+t^2)^{M+3/2}}.$$

Now fix $x \in C_j(B_i)$, $j \geq 4$, and $0 < t < 2^{j-l-3} r_{B_i}$, so $B(x, 2^{l+2}t) \subset 2^{j+2}B_i \setminus 2^{j-1}B_i$. This and Minkowski's integral inequality imply

$$\begin{aligned}
&\Big(\fint_{B(x,2^{l+2}t)} \big|e^{-t^2 L_w} \sqrt{L_w}(B_{r_{B_i}} b_i)(y)\big|^{p_0} dw(y)\Big)^{1/p_0} \\
&\quad = \Big(\fint_{B(x,2^{l+2}t)} \Big|\phi(L_w,t)\Big(\frac{b_i}{t}\Big)(y)\Big|^{p_0} dw(y)\Big)^{1/p_0} \\
&\quad \lesssim \int_\Gamma \Big(\fint_{B(x,2^{l+2}t)} \Big|e^{-zL_w}\Big(\frac{b_i}{t}\Big)(y)\Big|^{p_0} dw(y)\Big)^{1/p_0} \frac{t\, r_{B_i}^{2M}}{(|z|+t^2)^{M+3/2}}\, |dz| \\
&\quad \lesssim \int_\Gamma \Big(\fint_{B(x,2^{l+2}t)} \big|\mathbf{1}_{2^{j+2}B_i\setminus 2^{j-1}B_i} e^{-zL_w} b_i(y)\big|^{p_0} dw(y)\Big)^{1/p_0} \frac{r_{B_i}^{2M}}{|z|^{M+3/2}}\, |dz| \\
&\quad \lesssim \int_\Gamma \mathcal{M}_{p_0}^w \big(\mathbf{1}_{2^{j+2}B_i\setminus 2^{j-1}B_i} e^{-zL_w} b_i\big)(x)\, \frac{r_{B_i}^{2M}}{|z|^{M+3/2}}\, |dz|.
\end{aligned}$$

Recalling that $\mathcal{M}_{p_0}^w$ on $L^{p_1}(v\,dw)$ since $v \in A_{p_1/p_0}(w)$, and applying again Minkowski's integral inequality, we get

$$\begin{aligned}
\mathfrak{C}_1 &\lesssim \int_\Gamma \Big(\int_{C_j(B_i)} \big|\mathcal{M}_{p_0}^w \big(\mathbf{1}_{2^{j+2}B_i\setminus 2^{j-1}B_i} e^{-zL_w} b_i\big)(x)\big|^{p_1} v(x)dw(x)\Big)^{1/p_1} \frac{r_{B_i}^{2M}}{|z|^{M+3/2}}\, |dz| \\
&\lesssim \int_\Gamma \Big(\int_{2^{j+2}B_i\setminus 2^{j-1}B_i} \big|e^{-zL_w} b_i(y)\big|^{p_1} v(y)dw(y)\Big)^{1/p_1} \frac{r_{B_i}^{2M}}{|z|^{M+3/2}}\, |dz|.
\end{aligned}$$

Observe that $2^{j+2}B_i \setminus 2^{j-1}B_i = \cup_{l=1}^3 C_{l+j-2}(B_i)$.

We next use that $e^{-zL_w} \in \mathcal{O}(L^{p_1}(v\,dw) - L^{p_1}(v\,dw))$, (2.40), and change the variable s into $4^j r_{B_i}^2/s^2$:

$$\begin{aligned}
\mathfrak{C}_1 &\lesssim vw(2^{j+1}B_i)^{1/p_1}\, 2^{j\theta_1} \Big(\fint_{B_i} |b_i|^{p_1}\, d(vw)\Big)^{1/p_1} \\
&\qquad \times \int_0^\infty \Upsilon\Big(\frac{2^j r_{B_i}}{s^{1/2}}\Big)^{\theta_2} e^{-c\,4^j r_{B_i}^2/s}\, \frac{s\, r_{B_i}^{2M}}{s^{M+3/2}}\, \frac{ds}{s} \\
&\lesssim \alpha\, v\, w(2^{j+1}B_i)^{1/p_1}\, 2^{-j(2M+1-\theta_1)} \int_0^\infty \Upsilon(s)^{\theta_2} e^{-cs^2} s^{2M+1}\, \frac{ds}{s} \\
&\lesssim \alpha\, v\, w(2^{j+1}B_i)^{1/p_1}\, 2^{-j(2M+1-\theta_1)},
\end{aligned}$$

provided $2M+1 > \theta_2$.

We continue by estimating $\mathfrak{C}_2$. To this end, first change the variable t into $t\sqrt{M+1} =: t\theta_M$. Next, for any $x \in C_j(B_i)$ and $t \geq \frac{2^{j-1} r_{B_i}}{2^{l+2}\theta_M}$, note that

$$B_i \subset B(x_{B_i}, \theta_M 2^{l+2} t) =: B_i^l \subset B(x, \theta_M 2^{l+2} 5t)$$

(x_{B_i} denotes the center of B_i). Then,

$$\begin{aligned}
\mathfrak{C}_2 &\lesssim \Big(\int_{C_j(B_i)} \Big(\sup_{t \geq 2^{j-l-3} r_{B_i}/\theta_M} \fint_{B(x, \theta_M 2^{l+2} t)} \big| \mathcal{T}_{t, r_{B_i}} \sqrt{L_w}\, e^{-t^2 L_w} (\mathbf{1}_{B_i^l} b_i)(y) \big|^{p_0} \\
&\qquad \times dw(y) \Big)^{p_1/p_0} d(vw)(x) \Big)^{1/p_1} \\
&\lesssim \Big(\int_{C_j(B_i)} \Big(\sup_{t \geq 2^{j-l-3} r_{B_i}/\theta_M} w(B(x, \theta_M 2^{l+2} t))^{-1} \\
&\qquad \times \int_{\mathbb{R}^n} \big| \mathcal{T}_{t, r_{B_i}} \sqrt{L_w} e^{-t^2 L_w} (\mathbf{1}_{B_i^l} b_i)(y) \big|^{p_0} dw(y) \Big)^{p_1/p_0} d(vw)(x) \Big)^{1/p_1},
\end{aligned}$$

where $\mathcal{T}_{t, r_{B_i}} := (e^{-t^2 L_w} - e^{-(t^2 + r_{B_i}^2) L_w})^M$.

In the above setting, (2.30), Proposition 2.4, the fact that $\sqrt{\tau}\nabla e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^{p_0}(w))$, (2.3), Lemma 2.2 (a) (see (3.16)), and (2.40) imply

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} \big| \mathcal{T}_{t, r_{B_i}} \sqrt{L_w} e^{-t^2 L_w} (\mathbf{1}_{B_i^l} b_i) \big|^{p_0} dw \Big)^{1/p_0} \\
&\quad \lesssim \Big(\frac{r_{B_i}^2}{t^2} \Big)^M \Big(\int_{\mathbb{R}^n} \big| \nabla e^{-t^2 L_w} (\mathbf{1}_{B_i^l} b_i) \big|^{p_0} dw \Big)^{1/p_0} \\
&\quad \lesssim 2^l \sum_{N \geq 1} w(C_N(B_i^l))^{1/p_0} \Big(\frac{r_{B_i}^2}{t^2} \Big)^M \Big(\fint_{C_N(B_i^l)} \Big| t \nabla e^{-t^2 L_w} \Big(\mathbf{1}_{B_i^l} \frac{b_i}{r_{B_i}} \Big) \Big|^{p_0} dw \Big)^{1/p_0} \\
&\quad \lesssim 2^{l\theta}\, w(B_i^l)^{1/p_0} \sum_{N \geq 1} e^{-c 4^N} \Big(\frac{r_{B_i}^2}{t^2} \Big)^M \Big(\fint_{B_i^l} \Big| \frac{b_i}{r_{B_i}} \Big|^{p_0} dw \Big)^{1/p_0} \\
&\quad \lesssim 2^{l\theta}\, w(B_i^l)^{1/p_0} \Big(\frac{r_{B_i}^2}{t^2} \Big)^M \Big(\fint_{B_i} \Big| \frac{b_i}{r_{B_i}} \Big|^{p_1} d(vw) \Big)^{1/p_1} \\
&\quad \lesssim 2^{l\theta}\, w(B_i^l)^{1/p_0} \alpha \Big(\frac{r_{B_i}^2}{t^2} \Big)^M. \qquad (3.23)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathfrak{C}_2 &\lesssim 2^{l\theta} \alpha \Big(\int_{C_j(B_i)} \Big(\sup_{t \geq 2^{j-l-3} r_{B_i}/\theta_M} \Big(\frac{r_{B_i}^2}{t^2} \Big)^M \Big)^{p_1} v\, dw \Big)^{1/p_1} \\
&\lesssim \alpha\, vw(2^{j+1} B_i)^{1/p_1}\, 2^{-j2M}\, 2^{l(2M+\theta)},
\end{aligned}$$

where in the first inequality we have used that $w(B(x, \theta_M 2^{l+2} t))^{-1} w(B_i^l) \leq C$, since $B_i^l \subset B(x, \theta_M 2^{l+2} 5t)$.

Collecting the estimates obtained for $\mathfrak{C}_1$ and $\mathfrak{C}_2$, we conclude that, for $M \in \mathbb{N}$ such that $2M + 1 > \theta_2$,

$$I_{1,l}^{ij} \lesssim \alpha\, vw(2^{j+1} B_i)^{1/p_1}\, 2^{-j(2M-\theta_1)}\, 2^{l(2M+\theta)}. \tag{3.24}$$

Next, let us estimate term $I_{2,l}^{ij}$. Splitting the supremum in t, we have

$$\begin{aligned} I_{2,l}^{ij} &\lesssim \Big(\int_{C_j(B_i)} \sup_{0<t<2^{j-l-2} r_{B_i}} \Big(\fint_{B(x,2^{l+1}t)} \big(\mathrm{g}_{\mathrm{H}}^{w} \sqrt{L_w}(B_{r_{B_i}} b_i)(y)\big)^{p_0} dw(y)\Big)^{p_1/p_0} \\ &\qquad \times v\, dw(x)\Big)^{1/p_1} \\ &\quad + \Big(\int_{C_j(B_i)} \sup_{t \geq 2^{j-l-2} r_{B_i}} \Big(\fint_{B(x,2^{l+1}t)} \big(\mathrm{g}_{\mathrm{H},t}^{w} \sqrt{L_w}(B_{r_{B_i}} b_i)(y)\big)^{p_0} dw(y)\Big)^{p_1/p_0} \\ &\qquad \times v\, dw(x)\Big)^{1/p_1} \\ &=: D_1^{ij} + D_2^{ij}. \end{aligned}$$

Regarding D_1^{ij}, we claim that

$$D_1^{ij} \lesssim \alpha\, v\, w(2^{j+1} B_i)^{1/p_1}\, 2^{-j(2M+1-\widetilde{\theta}_1)}. \tag{3.25}$$

To this end, first note that for $0 < t < 2^{j-l-2} r_{B_i}$ and $x \in C_j(B_i)$, then

$$B(x, 2^{l+1}t) \subset 2^{j+2} B_i \setminus 2^{j-1} B_i.$$

Next recall that $\mathcal{M}_{p_0}^{w}$ is $L^{p_1}(v dw)$ bounded since $v \in A_{p_1/p_0}(w)$ (see (3.16)). Hence

$$\begin{aligned} D_1^{ij} &\lesssim \Big(\int_{C_j(B_i)} \big| \mathcal{M}_{p_0}^{w} \big(\mathbf{1}_{2^{j+2} B_i \setminus 2^{j-1} B_i}\, \mathrm{g}_{\mathrm{H}}^{w} \sqrt{L_w}(B_{r_{B_i}} b_i)\big)\big|^{p_1} v\, dw \Big)^{1/p_1} \\ &\lesssim vw(2^{j+1} B_i)^{1/p_1} \Big(\fint_{2^{j+2} B_i \setminus 2^{j-1} B_i} \big| \mathrm{g}_{\mathrm{H}}^{w} \sqrt{L_w}(B_{r_{B_i}} b_i)\big|^{p_1} d(vw)\Big)^{1/p_1}. \end{aligned}$$

In view of (3.16), we can apply Lemma 2.2 (b) and the Minkowski integral inequality to get

$$\begin{aligned} D_1^{ij} &\lesssim v\, w(2^{j+1} B_i)^{1/p_1} \Big(\int_0^\infty \Big(\fint_{2^{j+2} B_i \setminus 2^{j-1} B_i} \big| r L_w r \sqrt{L_w} e^{-r^2 L_w} \big(B_{r_{B_i}} b_i(x)\big)\big|^{q_0} \\ &\qquad \times dw(x)\Big)^{2/q_0} \frac{dr}{r}\Big)^{1/2}. \end{aligned} \tag{3.26}$$

In order to estimate the integral in x, we use functional calculus as in the estimate of $\mathfrak{C}_1$.

Apply the fact that $zL_w e^{-zL_w} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$, Lemma 2.2 (a), and (2.40), to get

$$\Big(\fint_{2^{j+2}B_i\setminus 2^{j-1}B_i} \big|rL_w r\sqrt{L_w}e^{-r^2L_w}(B_{r_{B_i}}b_i)\big|^{q_0}dw\Big)^{1/q_0}$$
$$\lesssim \int_\Gamma \Big(\fint_{2^{j+2}B_i\setminus 2^{j-1}B_i} |zL_w e^{-zL_w}b_i|^{q_0}dw\Big)^{1/q_0} \frac{r^2 r_{B_i}^{2M}}{(|z|+r^2)^{M+3/2}}\frac{|dz|}{|z|}$$
$$\lesssim 2^{j\widetilde{\theta}_1}\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2}\frac{r^2 r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+3/2}}\frac{ds}{s}\Big(\fint_{B_i}|b_i|^{p_0}dw\Big)^{1/p_0}$$
$$\lesssim 2^{j\widetilde{\theta}_1}\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2}\frac{r^2 r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+3/2}}\frac{ds}{s}\Big(\fint_{B_i}|b_i|^{p_1}d(vw)\Big)^{1/p_1}$$
$$\lesssim \alpha r_{B_i}2^{j\widetilde{\theta}_1}\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2}\frac{r^2 r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+3/2}}\frac{ds}{s}.$$

Plugging this into (3.26) and changing the variable r into $2^j r_{B_i} r$, we obtain, for $M \in \mathbb{N}$ such that $2M > \widetilde{\theta}_2$,

$$D_1^{ij} \lesssim \alpha r_{B_i}2^{j\widetilde{\theta}_1}vw(2^{j+1}B_i)^{1/p_1}\Big(\int_0^\infty\Big(\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2} \times \frac{r^2 r_{B_i}^{2M}}{(4^j r_{B_i}^2/s^2+r^2)^{M+3/2}}\frac{ds}{s}\Big)^2\frac{dr}{r}\Big)^{1/2}$$
$$\lesssim \alpha v\,w(2^{j+1}B_i)^{1/p_1}2^{-j(2M+1-\widetilde{\theta}_1)}\Big(\int_0^\infty r^4\Big(\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2} \times \frac{1}{(1/s^2+r^2)^{M+3/2}}\frac{ds}{s}\Big)^2\frac{dr}{r}\Big)^{1/2}$$
$$\lesssim \alpha v\,w(2^{j+1}B_i)^{1/p_1}2^{-j(2M+1-\widetilde{\theta}_1)}\Big(\Big(\int_0^1 r^4\frac{dr}{r}\Big)^{1/2}\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2}s^{2M+3}\frac{ds}{s} + \Big(\int_1^\infty r^{-2}\frac{dr}{r}\Big)^{1/2}\int_0^\infty \Upsilon(s)^{\widetilde{\theta}_2}e^{-cs^2}s^{2M}\frac{ds}{s}\Big)$$
$$\lesssim \alpha v\,w(2^{j+1}B_i)^{1/p_1}2^{-j(2M+1-\widetilde{\theta}_1)}.$$

Now turning to the estimate of D_2^{ij}, we claim that

$$D_2^{ij} \lesssim 2^{l(2M+\widetilde{\theta})}\alpha vw(2^{j+1}B_i)^{1/p_1}2^{-2jM}. \tag{3.27}$$

For any $t \ge 2^{j-l-2}r_{B_i}$ and $f \in L^2(w)$, we have that

$$\mathrm{g}_{\mathrm{H},t}^w f(x) = \Big(\int_{t/2}^\infty |r^2L_w e^{-r^2L_w}f(x)|^2\frac{dr}{r}\Big)^{1/2}$$
$$\le \Big(\int_{2^{j-l-3}r_{B_i}}^\infty |r^2L_w e^{-r^2L_w}f(x)|^2\frac{dr}{r}\Big)^{1/2}.$$

Moreover, recall that $p_0 < q_0$ (see (3.16)), this implies the boundedness of the maximal operator $\mathcal{M}_{p_0}^w$ on $L^{q_0}(w)$. This, together with Lemma 2.2 (b) and Minkowski's integral inequality, allows us to obtain

$$D_2^{ij} \lesssim vw(2^{j+1}B_i)^{1/p_1} \Big(\fint_{C_j(B_i)} \big|\mathcal{M}_{p_0}^w \big(\mathrm{g}^w_{\mathrm{H},2^{j-l-2}r_{B_i}} \big(\sqrt{L_w}(B_{r_{B_i}} b_i)\big)\big)\big|^{q_0} dw \Big)^{1/q_0} \tag{3.28}$$

$$\lesssim \frac{vw(2^{j+1}B_i)^{1/p_1}}{w(2^{j+1}B_i)^{1/q_0}} \Big(\int_{\mathbb{R}^n} \Big(\int_{2^{j-l-3}r_{B_i}}^{\infty} \big| r^2 L_w e^{-r^2 L_w} \big(\sqrt{L_w}(B_{r_{B_i}} b_i)\big)\big|^2 \frac{dr}{r}\Big)^{q_0/2} dw \Big)^{1/q_0}$$

$$\lesssim \frac{vw(2^{j+1}B_i)^{1/p_1}}{w(2^{j+1}B_i)^{1/q_0}} \Big(\int_{2^{j-l-3}r_{B_i}/\theta_M}^{\infty} \Big(\int_{\mathbb{R}^n} \big|\mathcal{T}_{r,r_{B_i}} \sqrt{L_w} r^2 L_w e^{-r^2 L_w} (\mathbf{1}_{B_i^l} b_i)\big|^{q_0} \times dw \Big)^{2/q_0} \frac{dr}{r} \Big)^{1/2},$$

where in the last inequality we have changed the variable r into $r\theta_M := r\sqrt{M+1}$, used that $B_i \subset B(x_{B_i}, \theta_M 2^{l+1} r) =: B_i^l$, for $r > 2^{j-l-3} r_{B_i}/\theta_M$ and $j \geq 4$ (x_{B_i} denotes the center of B_i), and we recall that

$$\mathcal{T}_{r,r_{B_i}} := \big(e^{-r^2 L_w} - e^{-(r^2 + r_{B_i}^2) L_w}\big)^M.$$

Proceeding as in the estimate of (3.23), but using now the fact that $\sqrt{\tau}\nabla \tau L_w e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^{q_0}(w))$ instead of $\sqrt{\tau}\nabla e^{-\tau L_w} \in \mathcal{O}(L^{p_0}(w) - L^{p_0}(w))$, we get

$$\Big(\int_{\mathbb{R}^n} \big|\mathcal{T}_{r,r_{B_i}} \sqrt{L_w} r^2 L_w e^{-r^2 L_w} (\mathbf{1}_{B_i^l} b_i)\big|^{q_0} dw \Big)^{1/q_0}$$

$$\lesssim 2^{l\widetilde{\theta}} \alpha w(B_i^l)^{1/q_0} \Big(\frac{r_{B_i}^2}{r^2}\Big)^M \lesssim 2^{l(\widetilde{\theta} + 2n/q_0)} \alpha w(2^{j+1}B_i)^{1/q_0} 2^{-2jn/q_0} \Big(\frac{r_{B_i}^2}{r^2}\Big)^{M - n/q_0},$$

where in the last inequality we have used that for $r > 2^{j-l-3} r_{B_i}/\theta_M$ and $j \geq 4$, $2^{j+1}B_i \subset 2^3 B_i^l$, and (2.2). Plugging this into (3.28) leads to

$$D_2^{ij} \lesssim 2^{l(\widetilde{\theta} + 2n/q_0)} \alpha 2^{-2jn/q_0} vw(2^{j+1}B_i)^{1/p_1} \Big(\int_{2^{j-l-3}r_{B_i}/\theta_M}^{\infty} \Big(\frac{r_{B_i}^2}{r^2}\Big)^{2M - 2n/q_0} \frac{dr}{r} \Big)^{1/2}$$

$$\lesssim 2^{l(2M + \widetilde{\theta})} \alpha vw(2^{j+1}B_i)^{1/p_1} 2^{-2jM},$$

provided $2M > 2n/q_0$.

Gather (3.25) and (3.27); then, for $M \in \mathbb{N}$ such that $2M > \max\{\widetilde{\theta}_2, 2n/q_0\}$,

$$I_{2,l}^{ij} \lesssim 2^{l(2M + \widetilde{\theta})} \alpha vw(2^{j+1}B_i)^{1/p_1} 2^{-j(2M - \widetilde{\theta}_1)}.$$

This and (3.24) yield, for $2M > \max\{\widetilde{\theta}_2, 2n/q_0, \theta_2 - 1\}$,

$$I_{m,l}^{ij} \leq C_1 \alpha vw(2^{j+1}B_i)^{1/p_1} 2^{-j(2M - C_2)}, \quad m = 1, 2,$$

with $C_2 := \max\{\theta_1, \widetilde{\theta}_1\}$ and $C_1 := C\,2^{lC_M}$. Then, in view of (3.22), applying Lemma 2.17 with $\mathcal{I}_{ij} = \alpha^{-1} I^{ij}_{m,l}$ and $\{B_i\}_i$ the collection of balls given by Lemma 2.13, and (2.38), for $2M > \max\{C_2 + n r_w \mathrm{r}_v(w), \widetilde{\theta}_2, 2n/q_0, \theta_2 - 1\}$, we get

$$\mathrm{III}_{m,l} \lesssim e^{-c4^l} vw\Big(\bigcup_i B_i\Big) \lesssim e^{-c4^l} \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v\, dw, \quad m = 1, 2.$$

Therefore, by (3.21),

$$\mathrm{III}_m \lesssim \sum_{l \geq 1} e^{-c4^l} \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v\, dw \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f|^p \, v\, dw.$$

Collecting this estimate and (3.17)–(3.20), the proof is complete. ∎

4. The regularity problem in unweighted Lebesgue spaces

Our main result, Theorem 1.1, establishes the solvability of the regularity problem in $L^p(v\,dw)$ of the block operator $\mathbb{L}_w$. Recall that $w \in A_2(dx)$ is fixed and controls the degeneracy of the operator and that $v \in A_\infty(w)$. This means that we can establish the solvability of the regularity problem in unweighted Lebesgue spaces by taking $v = w^{-1}$. In this section, our goal is to explore this idea and study ranges for which we can solve the regularity problem in terms of the weight w. A particular case of interest, where we can be more explicit, is that of power weights.

To start with, fix $w \in A_2(dx)$ and recall the definitions of r_w and s_w in (2.4). As just mentioned, we let $v = w^{-1}$ and observe that from the definitions it is clear that for every $1 \leq r < \infty$ one has $w^{-1} \in A_r(w)$ if and only if $w \in \mathrm{RH}_{r'}(dx)$, and $w^{-1} \in \mathrm{RH}_{r'}(w)$ if and only if $w \in A_r(dx)$. Hence, according to (2.13) we have $\mathrm{r}_{w^{-1}}(w) = s_w$ and $\mathfrak{s}_{w^{-1}}(w) = r_w$. Then looking at Theorem 1.1 and using (2.14), we see that (1.9) is equivalent to

$$\max\{r_w, q_-(L_w)\}\, s_w < \frac{q_+(L_w)}{r_w}, \tag{4.1}$$

and if that holds we have $(R^{\mathbb{L}_w})_{L^p(dx)}$ solvability for p so that

$$\max\Big\{r_w, \frac{n r_w\, q_-(L_w)}{n r_w + q_-(L_w)}\Big\}\, s_w < p < \frac{q_+(L_w)}{r_w}. \tag{4.2}$$

It is important to note that $q_-(L_w)$ and $q_+(L_w)$ are defined in an abstract way and depend intrinsically on w. From Propositions 3.1 and 7.1 in [16] and recalling that $n \geq 2$, we know that $q_-(L_w) = p_-(L_w) \leq 2n r_w/(n r_w + 2)$, hence we have an estimate for $q_-(L_w)$ in terms of n and r_w. On the other hand $q_+(L_w) > 2$ and can be arbitrarily close to 2 (even in the case $w \equiv 1$), and we do not have an explicit bound in terms of w (see the proof of Theorem 11.8 in [16] in this regard). Taking this into account and in order to check that (4.1) holds, we will replace its right-hand side with $2/r_w$.

Our first result for general weights is as follows:

Corollary 4.1. *Let $w \in A_2(dx)$ and let $\mathbb{L}_w$ be a block degenerate elliptic operator in $\mathbb{R}^{n+1}_+$ as in (1.5). Associated with $\mathbb{L}_w$, consider the regularity problem $(R^{\mathbb{L}_w})_{L^p(dx)}$ as in Section 1. Given $f \in C_c^\infty(\mathbb{R}^n)$, if one sets $u(x,t) = e^{-t\sqrt{L_w}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, then*

$$\|\mathcal{N}_w(\nabla_{x,t} u)\|_{L^p(dx)} \le C \|\nabla f\|_{L^p(dx)} \tag{4.3}$$

in any of the following scenarios:

(a) *If $w \in A_1(dx) \cap \mathrm{RH}_{1+n/2}(dx)$ and*

$$\max\left\{1, \frac{n\, q_-(L_w)}{n + q_-(L_w)}\right\} s_w < p < q_+(L_w),$$

in particular, in the range

$$\max\left\{1, \frac{2n}{n+4}\right\} s_w < p \le 2.$$

(b) *If $w \in A_{r_0}(dx) \cap \mathrm{RH}_\infty(dx)$ with $r_0 := \min\left\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\right\}$ and*

$$\max\left\{r_w, \frac{n r_w\, q_-(L_w)}{n r_w + q_-(L_w)}\right\} < p < \frac{q_+(L_w)}{r_w},$$

in particular, in the range

$$\max\left\{r_w, \frac{2n r_w}{n r_w + 4}\right\} < p \le \frac{2}{r_w}.$$

(c) *If $w \in A_r(dx) \cap \mathrm{RH}_{s(r)'}(dx)$ with $1 < r < r_0$ and $s(r) = \min\{\frac{2}{r^2}, \frac{nr+2}{nr^2}\}$, and*

$$\max\left\{r_w, \frac{n r_w\, q_-(L_w)}{n r_w + q_-(L_w)}\right\} s_w < p < \frac{q_+(L_w)}{r_w},$$

in particular, in the range

$$\max\left\{r_w, \frac{2n r_w}{n r_w + 4}\right\} s_w < p \le \frac{2}{r_w}.$$

(d) *Given $\Theta \ge 1$, there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda) \in (0, \frac{1}{2n}]$, such that for every $w \in A_{1+\varepsilon}(dx) \cap \mathrm{RH}_{\max\{\frac{2}{1-\varepsilon}, 1+(1+\varepsilon)\frac{n}{2}\}}(dx)$, with $0 \le \varepsilon < \varepsilon_0$, and $[w]_{A_2(dx)} \le \Theta$, (4.3) holds with $p = 2$, or equivalently $(R^{\mathbb{L}_w})_{L^2(dx)}$ is solvable.*

Proof. We first consider (a). Let $w \in A_1(dx) \cap \mathrm{RH}_{1+n/2}(dx)$; then $r_w = 1$ and $s_w < (1+n/2)' = 1 + 2/n$. Using that $q_-(L_w) \le \frac{2n}{n+2}$ (since $n \ge 2$) we have

$$\max\{r_w, q_-(L_w)\}\, s_w < \max\left\{1, \frac{2n}{n+2}\right\}\left(1 + \frac{2}{n}\right) = 2 < q_+(L_w) = \frac{q_+(L_w)}{r_w}.$$

That is, (4.1) holds and according to (4.2) we have $(R^{\mathbb{L}_w})_{L^p(dx)}$-solvability for p so that

$$\max\left\{1, \frac{n\, q_-(L_w)}{n + q_-(L_w)}\right\} s_w < p < q_+(L_w)$$

and, in particular, in the range

$$\max\left\{1, \frac{2n}{n+4}\right\} s_w < p \le 2.$$

To prove (b) and (c), let us assume that $w \in A_r(dx) \cap \mathrm{RH}_{s(r)'}(dx)$ with $1 < r \leq \min\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\}$ and $s(r) = \min\{\frac{2}{r^2}, \frac{nr+2}{nr^2}\}$, and note that the restriction on r gives $s(r) \in [1, \infty)$. In particular, $r_w < r$, $s_w \leq s(r)$, and

$$r_w \max\{r_w, q_-(L_w)\} s_w \leq r_w \max\left\{r_w, \frac{2nr_w}{nr_w+2}\right\} s_w < \max\left\{r^2, \frac{2nr^2}{nr+2}\right\} s(r) = 2 < q_+(L_w).$$

This implies (4.1) and we have $(R^{\mathbb{L}_w})_{L^p(dx)}$-solvability for p in the range given by (4.2), and in particular for those p's satisfying

$$\max\left\{r_w, \frac{2nr_w}{nr_w+4}\right\} s_w < p \leq \frac{2}{r_w}.$$

All these show (b) by taking $r = r_0$ so that $s(r) = 1$ and hence $s_w = 1$. Also (c) follows from the case $1 < r < r_0$.

To deal with (d), we proceed as in [16], pp. 654–655. There, it is shown that given $\Theta \geq 1$ there exists $\varepsilon_0 = \varepsilon_0(\Theta, n, \Lambda/\lambda) \in (0, \frac{1}{2n}]$ such that if $w \in A_{1+\varepsilon}(dx)$ with $0 \leq \varepsilon < \varepsilon_0$ so that $[w]_{A_2(dx)} \leq \Theta$ then $2 r_w < q_+(L_w)$. That is, $2 < q_+(L_w)/r_w$. On the other hand, if we additionally assume that $w \in \mathrm{RH}_{\max\{\frac{2}{1-\varepsilon}, 1+(1+\varepsilon)\frac{n}{2}\}}(dx)$, then

$$s_w < \left(\max\left\{\frac{2}{1-\varepsilon}, 1 + (1+\varepsilon)\frac{n}{2}\right\}\right)' = \min\left\{\frac{2}{1+\varepsilon}, 1 + \frac{2}{n(1+\varepsilon)}\right\}$$
$$< \min\left\{\frac{2}{r_w}, 1 + \frac{2}{nr_w}\right\} = \frac{2}{\max\left\{r_w, \frac{2nr_w}{nr_w+2}\right\}},$$

that is,

$$\max\left\{r_w, \frac{2nr_w}{nr_w+2}\right\} s_w < 2.$$

Altogether we have obtained that $\max\left\{r_w, \frac{2nr_w}{nr_w+2}\right\} s_w < 2 < \frac{q_+(L_w)}{r_w}$. This implies (4.1) and also that $p = 2$ satisfies (4.2). Consequently, $(R^{\mathbb{L}_w})_{L^2(dx)}$ is solvable as desired. ∎

Concerning power weights, we have the following result.

Corollary 4.2. *Consider the power weight $w_\beta(x) = |x|^{n(\beta-1)}$ with $0 < \beta < 2$, and let $\mathbb{L}_{w_\beta}$ be the associated block operator*

$$\mathbb{L}_{w_\beta} u(x,t) = -|x|^{-n(\beta-1)} \operatorname{div}_x \left(|x|^{n(\beta-1)} A(x) \nabla_x u(x,t)\right) - \partial_t^2 u(x,t), \tag{4.4}$$

where A is an $n \times n$ matrix of complex L^∞-valued coefficients defined on $\mathbb{R}^n$, $n \geq 2$, satisfying the uniform ellipticity condition (1.1).

Assume that

$$\frac{n}{n+2} \leq \beta \leq \min\left\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\right\}.$$

Then for every $f \in C_c^\infty(\mathbb{R}^n)$, if one sets $u(x,t) = e^{-t\sqrt{L_\beta}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, then

$$\|\mathcal{N}_{w_\beta}(\nabla_{x,t} u)\|_{L^p(dx)} \leq C \|\nabla f\|_{L^p(dx)}, \tag{4.5}$$

for every p satisfying

$$\max\left\{1, \beta, \frac{n\, q_-(L_w)}{n + q_-(L_w)}, \frac{n\beta\, q_-(L_w)}{n\beta + q_-(L_w)}\right\} \max\{1, \beta^{-1}\} < p < \frac{q_+(L_w)}{\max\{1,\beta\}}.$$

In particular, in the non-empty range

$$\max\left\{1, \beta, \frac{2n}{n+4}, \frac{2n\beta}{n\beta+4}\right\} \max\{1, \beta^{-1}\} < p \le \frac{2}{\max\{1,\beta\}}$$

provided

$$\frac{n}{n+2} < \beta < \min\left\{\sqrt{2}, \tfrac{1+\sqrt{1+8/n}}{2}\right\}.$$

Moreover, there exists $\varepsilon_1 = \varepsilon_1(n, \Lambda/\lambda) \in (0, \frac{1}{2n})$ such that if

$$\frac{n}{n+2} < \beta < 1 + \varepsilon_1,$$

then (4.5) holds with $p = 2$, or equivalently $(R^{\mathbb{L}_{w_\beta}})_{L^2(dx)}$ is solvable.

Proof. Write $w_\beta(x) = |x|^{n(\beta-1)}$ with $0 < \beta < 2$ so that $w_\beta \in A_2(dx)$. It is not difficult to see that

$$r_{w_\beta} = \max\{1, \beta\} \quad \text{and} \quad s_{w_\beta} = \max\{1, \beta^{-1}\}.$$

Consider first the case $0 < \beta \le 1$, so that $w_\beta \in A_1(dx)$, $r_{w_\beta} = 1$, and $s_{w_\beta} = \beta^{-1}$. If $\beta \ge \frac{n}{n+2}$, then

$$\max\{r_{w_\beta}, q_-(L_{w_\beta})\}\, s_w \le \frac{2n}{n+2}\,\frac{1}{\beta} \le 2 < q_+(L_{w_\beta}) = \frac{q_+(L_{w_\beta})}{r_{w_\beta}}.$$

Thus (4.1) holds and if $\frac{n}{n+2} \le \beta \le 1$ we have $(R^{\mathbb{L}_{w_\beta}})_{L^p(dx)}$-solvability for p such that

$$\max\left\{1, \frac{n\, q_-(L_{w_\beta})}{n + q_-(L_{w_\beta})}\right\} \frac{1}{\beta} < p < q_+(L_{w_\beta}).$$

In particular, if $\frac{n}{n+2} < \beta \le 1$, the solvability holds in the range $\max\{1, \frac{2n}{n+4}\}\, \beta^{-1} < p \le 2$.

Let us treat the case $1 < \beta < 2$, so that we have $r_{w_\beta} = \beta$ and $s_{w_\beta} = 1$. If $1 < \beta \le \min\left\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\right\}$, then

$$r_{w_\beta}\ \max\{r_{w_\beta}, q_-(L_{w_\beta})\} \le \max\left\{\beta^2, \frac{2n\beta^2}{n\beta+2}\right\} \le 2 < q_+(L_{w_\beta}).$$

This implies that (4.1) holds. Thus, (4.2) yields that if $1 < \beta \le \min\left\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\right\}$, then $(R^{\mathbb{L}_{w_\beta}})_{L^p(dx)}$ is solvable in the range

$$\max\left\{\beta, \frac{n\beta\, q_-(L_w)}{n\beta + q_-(L_w)}\right\} < p < \frac{q_+(L_w)}{\beta}.$$

In particular, if $1 < \beta < \min\{\sqrt{2}, \frac{1+\sqrt{1+8/n}}{2}\}$, one can solve $(R^{\mathbb{L}_{w_\beta}})_{L^p(dx)}$ for p satisfying

$$\max\left\{\beta, \frac{2n\beta}{n\beta+4}\right\} < p \le \frac{2}{\beta}.$$

Let us finally focus on the $(R^{\mathbb{L}_{w_\beta}})_{L^2}$-solvability. Consider first the case when $\frac{n}{n+2} < \beta \le 1$. Then

$$\max\left\{1, \beta, \frac{2n}{n+4}, \frac{2n\beta}{n\beta+4}\right\} \max\{1, \beta^{-1}\} = \max\left\{1, \frac{2n}{n+4}\right\} \frac{1}{\beta} < 2 = \frac{2}{\max\{1, \beta\}}.$$

Hence what we have proved so far gives the $(R^{\mathbb{L}_{w_\beta}})_{L^2(dx)}$-solvability. To consider the case $\beta > 1$, we first assume that $\beta < \frac{2n+1}{2n}$ so that $w_\beta \in A_{1+1/(2n)}(dx)$. Note that one can easily see that there exists $\Theta \ge 1$ depending just on n (and independent of β) such that $[w_\beta]_{A_2(dx)} \le \Theta$. Next we repeat the argument given in the proof of Corollary 4.1 to find the corresponding $\varepsilon_0 \in (0, 1/(2n)]$, which depends only on n and Λ/λ. Set $\varepsilon_1 = \varepsilon_0$ and assume that $1 < \beta < 1 + \varepsilon_1 \le \frac{2n+1}{2n}$. Pick $\varepsilon' > 0$ so that $1 < \beta < 1 + \varepsilon' < 1 + \varepsilon_1$. Hence $w_\beta \in A_{1+\varepsilon'}(dx)$ with $0 < \varepsilon' < \varepsilon_1 = \varepsilon_0$ and we can invoke (d) in Corollary 4.3 to conclude the $(R^{\mathbb{L}_{w_\beta}})_{L^2(dx)}$-solvability. ■

Proof of Corollary 1.2. It suffices to observe that the first part is just item (d) in Corollary 4.1. Regarding power weights, setting $\alpha = -n(\beta - 1)$ and with a slight abuse of notation the desired estimate follows at once from Corollary 4.2. ■

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