Rev. Mat. Iberoam. 39 (2023), no. 2, 711–729
DOI 10.4171/RMI/1391

Real semisimple Lie groups and balanced metrics

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Abstract. Given any non-compact real simple Lie group G_o of inner type and even dimension, we prove the existence of an invariant complex structure J and a Hermitian balanced metric on G_o and on any compact quotient $M = \Gamma \backslash G_o$, with Γ a cocompact lattice. We also prove that (M, J) does not carry any pluriclosed metric, in contrast to the case of even dimensional compact Lie groups, which admit pluriclosed but not balanced metrics.

1. Introduction

Given a complex non-Kähler n-dimensional manifold (M, J), it is a natural and meaningful problem to find special Hermitian metrics which might help in understanding the geometry of M. Great effort has been spent in the last decades in this research topic and among special metrics the pluriclosed and the balanced conditions have proved to be highly significant.

The balanced condition can be defined saying that the fundamental form $\omega = h(\cdot, J\cdot)$ of a Hermitian metric h satisfies the non-linear condition $d\omega^{n-1} = 0$, or equivalently, $-J\theta = \delta\omega = 0$, where δ denotes the codifferential and θ the torsion 1-form (see e.g. [20]). While this concept appears in [19] under the name of semi-Kähler (see also [23]), in [26] the balanced condition was started to be thoroughly investigated, highlighting also the duality with the Kähler condition and establishing necessary and sufficient conditions for the existence of these metrics in terms of currents. While Kähler metrics are obviously balanced and share with these the important relation among Laplacians $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ acting on functions (see [19]), there are many examples of non-Kähler manifolds carrying balanced metrics. Basic examples are given by compact complex parallelizable manifolds, which are covered by complex unimodular Lie groups G and every left invariant Hermitian metric turns out to be balanced (see [1, 19, 22, 25]). Further examples of balanced metrics are provided by any Hermitian invariant metric on a compact homogeneous flag manifold (see also [13] for a characterization of compact homogeneous complex manifolds carrying balanced metrics) as well as by twistor spaces of certain self-dual 4-manifolds ([26]) and more generally ([33]) by twistor spaces of compact hypercomplex manifolds (see also [15] for other examples on toric bundles over hyper-Kähler manifolds). Contrary to

2020 Mathematics Subject Classification: Primary 53C55; Secondary 53C25.
Keywords: Hermitian manifolds, semisimple Lie algebras, balanced metrics.

the Kählerness condition, being balanced is a birational invariant (see [4], so that e.g. Moishezon manifolds are balanced) and compact complex manifolds X which can be realized as the base of a holomorphic proper submersion $f: Y \to X$ inherit the balanced condition whenever Y has it ([26]), while the balanced property is not stable under small deformations of the complex structure (see [3,6,17]). On the other hand, the balanced condition is obstructed, as on compact manifolds with balanced metrics no compact complex hypersurface is homologically trivial, so that for instance Calabi–Eckmann manifolds do not carry balanced metrics. This is in contrast with the fact that Gauduchon metrics, which satisfy the weaker condition $\partial\bar{\partial}\omega^{n-1} = 0$, always exist on a compact complex manifold.

In more recent years, the rising interest in the Strominger system (see [18] and [12], where invariant solutions of this system of equations are constructed on complex Lie groups) has given balanced metrics a really central role in non-Kähler geometry, as the equivalence between the dilatino equation (i.e., one of the equations of the system) and the conformally balanced equation requires the manifolds admitting solutions of the system to be necessarily balanced. We refer also to the work [16], where new examples of balanced metrics are constructed on some Calabi Yau non-Kähler threefolds, as well as to the results in [7], where a new balanced flow is introduced and investigated.

The main goal of this paper is to search for invariant special Hermitian, in particular balanced, metrics in the class of semisimple real non-compact Lie groups and on their compact (non-Kähler) quotients by a cocompact lattice; actually it appears that, despite invariant complex structures on semisimple (reductive) Lie algebras being fully classified in [32] (after the special case of compact Lie algebras had been considered by Samelson [30] and later in [28]), they have never been deeply investigated from this point of view. In contrast, the case where K is compact is fully understood, as in this case it is very well known that every invariant complex structure can be deformed to an invariant one for which the opposite of the Cartan–Killing form is a pluriclosed Hermitian metric h, i.e., it satisfies $dd^c\omega_h = 0$. Moreover it has been proved in [13] that K does not carry *any* balanced metric at all, fueling the conjecture ([14]) that a compact complex manifold carrying two Hermitian metrics, one balanced and the other pluriclosed, must be actually Kähler.

More specifically, in this work we focus on a large class of simple non-compact real Lie algebras $\mathfrak{g}_o$ of even dimension, namely those which are of inner type, i.e., when the maximal compactly embedded subalgebra $\mathfrak{k}$ in a Cartan decomposition of $\mathfrak{g}_o$ contains a Cartan subalgebra. In these algebras we construct standard invariant complex structures (regular in [32]) and write down the balanced condition for invariant Hermitian metrics. A careful analysis of the resulting equation together with some general argument on root systems allows us to show the existence of a suitable invariant complex structure and a corresponding Hermitian metric satisfying the balanced equation. Our main result is the following.

Theorem 1.1. *Every non-compact simple Lie group* G_o *of even dimension, and of inner type, admits an invariant complex structure* J *and an invariant balanced* J*-Hermitian metric.*

By Borel's theorem, every semisimple Lie group G_o admits a cocompact lattice Γ so that the compact quotient $\Gamma\backslash\mathrm{G}_o$ inherits the invariant balanced structure from G_o. We note here that the resulting metrics come in families and moreover the same kind of arguments can be applied to show the existence of balanced structures on quotients G_o/S, where G_o is any simple non-compact Lie group of inner type of any dimension and S is a suitable abelian closed subgroup.

Our second result concerns the non-existence of pluriclosed metrics on the compact quotients of the complex manifolds we have constructed in the main theorem 1.1. Namely, we prove the following

Theorem 1.2. *Let* G_o *be a non-compact simple even-dimensional Lie group of inner type endowed with the invariant complex structure* J *as in Theorem* 1.1. *If* Γ *is a co-compact lattice of* G_o*, then the complex manifold* (M, J) *with* $\mathrm{M} = \Gamma\backslash\mathrm{G}_o$ *does not carry any pluriclosed metric.*

This result is in accordance with the above mentioned conjecture by Fino and Vezzoni ([14]), that has been already verified in several cases, and in some sense reflects a kind of duality between the compact and non-compact case, switching the existence of balanced/pluriclosed Hermitian metrics.

In the last section, we prove some geometric properties of the complex balanced manifolds we have constructed. In particular, we show that the Chern Ricci form of these balanced manifolds M never vanishes, although their first Chern class and their Chern scalar curvature do. Using then the fact that no power of the canonical bundle K_{M} is holomorphically trivial, we deduce using [35] that the Kodaira dimension satisfies $\kappa(\mathrm{M}) = -\infty$.

The paper is structured as follows. In Section 2, we review basic facts on simple real non-compact Lie algebras with invariant complex structures and we consider a class of invariant Hermitian metrics for which we write down the balanced condition in terms of roots. In Section 3 we prove our main result, namely Theorem 1.1, in several steps. We first rewrite the balanced equation in terms of simple roots and then the key Lemma 3.2 allows us to select an invariant complex structure so that the relative balanced equation admits solutions. In Section 4 we prove Theorem 1.2, and in the last section, Proposition 5.1 collects some properties of these balanced manifolds concerning the Chern Ricci form and the Kodaira dimension.

2. Preliminaries

Let $\mathfrak{g}_o$ be a real simple $2n$-dimensional Lie algebra and let G_o be a connected Lie group with Lie algebra $\mathfrak{g}_o$. It is well known that either the complexification $\mathfrak{g}_o^c$ is a complex simple Lie algebra (and in this case $\mathfrak{g}_o$ is called *absolutely simple*), or $\mathfrak{g}_o$ is the realification $\mathfrak{g}_{\mathbb{R}}$ of a complex simple Lie algebra $\mathfrak{g}$ (see e.g. [24]).

When $\mathfrak{g}_o$ is even dimensional, it is known ([27], see also [31]) that $\mathfrak{g}_o$ admits an invariant complex structure, namely an endomorphism $\mathrm{J} \in \mathrm{End}(\mathfrak{g}_o)$, with $\mathrm{J}^2 = -\mathrm{Id}$, which extends by left translation to an almost complex structure on G_o with vanishing Nijenhuis tensor. This last condition can be written at the level of the Lie algebra $\mathfrak{g}_o^c$ as

$$\mathfrak{g}_o^c = \mathfrak{g}_o^{1,0} \oplus \mathfrak{g}_o^{0,1} \quad \text{and} \quad [\mathfrak{g}_o^{1,0}, \mathfrak{g}_o^{1,0}] \subseteq \mathfrak{g}_o^{1,0},$$

where $\mathfrak{g}_o^{1,0}$ and $\mathfrak{g}_o^{0,1}$ are the $+i$ and $-i$-eigenspace of J on $\mathfrak{g}_o^c$, respectively.

When G_o is non-compact, a result due to Borel ([10]), guarantees the existence of a discrete, torsion-free cocompact lattice $\Gamma \subset G_o$ so that $M := \Gamma\backslash G_o$ is compact and the left-invariant complex structure J on G_o descends to a complex structure J on M.

When G_o is compact and even-dimensional, i.e., $\mathfrak{g}_o$ is of compact type, we recall that the existence of an invariant complex structure was already established by Samelson ([30]), while in [28] it was shown that every invariant complex structure on G_o is obtained by means of Samelson's construction.

If we now consider an even-dimensional simple Lie group G_o and a compact quotient M endowed with an invariant complex structure J, we are interested in the existence of special Hermitian metrics h. The following proposition states a known fact, namely the non-existence of (invariant) Kähler structures.

Proposition 2.1. *Let G_o be a semisimple Lie group endowed with a left invariant complex structure J, and let $\Gamma \subset G_o$ be a cocompact lattice so that $M = \Gamma\backslash G_o$ is compact. Then the group G_o does not admit any invariant Kähler metric and M is not Kähler.*

Proof. The first assertion is contained in [11], but we give here an elementary proof. If ω is an invariant symplectic form on $\mathfrak{g}_o$, then the closedness condition $d\omega = 0$ can be written as follows, for $x, y, z \in \mathfrak{g}_o$:

$$\omega([x,y],z) + \omega([z,x],y) + \omega([y,z],x) = 0. \tag{2.1}$$

If B denotes the non-degenerate Cartan–Killing form of $\mathfrak{g}_o$, then we can define the endomorphism $F \in \mathrm{End}(\mathfrak{g}_o)$ by $B(Fx,y) = \omega(x,y)$ $(x, y \in \mathfrak{s})$, so that using the biinvariance of B, (2.1) can be written as

$$B(F([x,y]),z) - B([Fx,y],z) - B([x,Fy],z) = 0,$$

hence F turns out to be a derivation of $\mathfrak{g}_o$. As $\mathfrak{g}_o$ is semisimple, there exists a unique $u \in \mathfrak{g}_o$ with $F = \mathrm{ad}(u)$, so that for $x, y \in \mathfrak{g}_o$, $\omega(x,y) = B([u,x],y)$, and therefore $u \in \ker\omega$, a contradiction.

We now suppose that the compact complex manifold M has a Kähler metric with Kähler form ω. Using ω and a symmetrization procedure that goes back to [8], we are going to construct an *invariant* Kähler form on G_o, obtaining a contradiction. We fix a basis $x_1, \ldots, x_{2n}$ of $\mathfrak{g}_o$ and we extend each vector as a left invariant vector fields on G_o; these vector fields can be projected down to M as vector fields $x_1^*, \ldots, x_{2n}^*$ that span the tangent space TM at each point. As G_o is semisimple, we can find a biinvariant volume form $d\mu$, that also descends to a volume form on M. We now define a left-invariant non-degenerate 2-form ϕ on G_o by setting

$$\phi_e(x_i, x_j) := \int_M \omega(x_i^*, x_j^*)\, d\mu.$$

As ϕ is left invariant and ω is closed, we have for $i, j, k = 1, \ldots, 2n$,

$$\begin{aligned} 3\, d\phi(x_i, x_j, x_k) &= - \sum_{\text{cyclic } (i,j,k)} \phi([x_i, x_j], x_k) = - \int_M \sum_{\text{cyclic } (i,j,k)} \omega([x_i^*, x_j^*], x_k^*)\, d\mu \\ &= - \int_M \sum_{\text{cyclic } (i,j,k)} x_i^* \omega(x_j^*, x_k^*)\, d\mu. \end{aligned}$$

As $\mathcal{L}_{x_i^*} d\mu = 0$ for every i, we have

$$\int_M x_i^* \omega(x_j^*, x_k^*)\, d\mu = \int_M \mathcal{L}_{x_i^*}(\omega(x_j^*, x_k^*)\, d\mu) = 0$$

by Stokes' theorem, and therefore we obtain that $d\phi = 0$, hence ϕ is invariant and symplectic, a contradiction. ∎

Therefore we are interested in the existence of special Hermitian metrics on the complex manifold (M, J), in particular balanced and pluriclosed metrics, when the group G_o is of non-compact type.

The case of a simple Lie algebra $\mathfrak{g}_o$, which is the realification of a complex simple Lie algebra $\mathfrak{g}$, can be easily treated and will be dealt with in Subsection 2.3.

We will now focus on some subclasses of simple real algebras, namely those which are absolutely simple and of inner type.

2.1. Simple Lie algebras of inner type

Let $\mathfrak{g}_o$ be an absolutely simple real algebra (i.e., $\mathfrak{g}_o^c$ is a simple Lie algebra) of non-compact type. It is well known that $\mathfrak{g}_o$ admits a Cartan decomposition

$$\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ is a maximal compactly embedded subalgebra and

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k},$$

so that $(\mathfrak{g}_o, \mathfrak{k})$ is a symmetric pair. Moreover, the algebra $\mathfrak{g}_o$ is said to be *of inner type* when the symmetric pair $(\mathfrak{g}_o, \mathfrak{k})$ is of inner type, i.e., when a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}_o$ or equivalently its complexification $\mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g}_o^c$. Using the notation as in [24], p. 126, we obtain the list of all inner symmetric pairs $(\mathfrak{g}_o, \mathfrak{k})$ of non-compact type with $\mathfrak{g}_o$ simple and even dimensional (Table 1).

2.2. Invariant complex structures

In this section we will describe how to construct invariant complex structures on even-dimensional absolutely simple non-compact Lie algebras $\mathfrak{g}_o$ of inner type.

We fix a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, so that $\mathfrak{h} := \mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g} := \mathfrak{g}_o^c$. Note that if $\mathfrak{g}_o$ is even dimensional, the same holds for $\mathfrak{t}$. The corresponding root system is denoted by R, and we have the following decompositions:

$$\mathfrak{k}^c = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in R_{\mathfrak{k}}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{p}^c = \bigoplus_{\alpha \in R_{\mathfrak{p}}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space relative to $\alpha \in R$. A root α will be called *compact* (respectively, *non-compact*), when $\mathfrak{g}_\alpha \subseteq \mathfrak{k}^c$ (respectively, $\mathfrak{g}_\alpha \subseteq \mathfrak{p}^c$) and the set of all compact (respectively, non-compact) roots is denoted by $R_{\mathfrak{k}}$ (respectively, $R_{\mathfrak{p}}$). It is a standard

Type	$\mathfrak{g}$	$\mathfrak{k}$	conditions
A	$\mathfrak{su}(p,q)$	$\mathfrak{su}(p)+\mathfrak{su}(q)+\mathbb{R}$	$p \geq q \geq 1,\ p+q$ odd
B	$\mathfrak{so}(2p+1,2q)$	$\mathfrak{so}(2p+1)+\mathfrak{so}(2q)$	$p \geq 0, q \geq 1,\ p+q$ even
C	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(2n)+\mathbb{R}$	$n \geq 1$
C	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p)+\mathfrak{sp}(q)$	$p,q \geq 1,\ p+q$ even
D	$\mathfrak{so}(4n)^*$	$\mathfrak{su}(2n)+\mathbb{R}$	$n \geq 2$
D	$\mathfrak{so}(2p,2q)$	$\mathfrak{so}(2p)+\mathfrak{so}(2q)$	$p,q \geq 1, p+q$ even ≥ 4
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(2)$	
F	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	
F	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2)+\mathfrak{sp}(3)$	
E	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(6)$	
E	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)+\mathbb{R}$	
E	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	
E	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2)+\mathfrak{e}_7$	

Table 1. Inner symmetric pairs $(\mathfrak{g},\mathfrak{k})$ of non-compact type with $\mathfrak{g}$ simple and even dimensional.

fact that $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{g}$ is a compact real form of $\mathfrak{g}$ and that we can choose the standard Weyl basis $\{E_\alpha\}_{\alpha\in R}$ of root spaces so that

$$\tau(E_\alpha) = -E_{-\alpha}, \quad B(E_\alpha, E_{-\alpha}) = 1 \quad \text{and} \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

where τ denotes the anticomplex involution defining $\mathfrak{u}$, B is the Cartan Killing form of $\mathfrak{g}$, and H_α is the B-dual of α (see e.g. [24]). If σ is the involutive anticomplex map defining $\mathfrak{g}_o$, we then have that

$$\sigma(E_\alpha) = -E_{-\alpha}, \quad \alpha \in R_{\mathfrak{k}},$$
$$\sigma(E_\alpha) = E_{-\alpha}, \quad \alpha \in R_{\mathfrak{p}}.$$

If we fix an ordering , namely a splitting $R = R^+ \cup R^-$ with $R^- = -R^+$ and $(R^+ + R^+) \cap R \subseteq R^+$, we can define a subalgebra

$$\mathfrak{q} := \mathfrak{h}_1 \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}_1 \subset \mathfrak{h}$ is a subspace so that $\mathfrak{h}_1 \oplus \sigma(\mathfrak{h}_1) = \mathfrak{h}$. The subalgebra $\mathfrak{q} \subset \mathfrak{g}$ defined in this way satisfies

$$\mathfrak{g} = \mathfrak{q} \oplus \sigma(\mathfrak{q})$$

and therefore it defines a complex structure J on $\mathfrak{g}_o$ with the property that $\mathfrak{q} = \mathfrak{g}_o^{10}$. This complex structure depends on the arbitrary choice of $\mathfrak{h}_1$, i.e., on the arbitrary choice of a complex structure on $\mathfrak{t}$.

We remark that the complex structure J enjoys the further property of being ad($\mathfrak{t}$)-invariant, namely

$$[\mathrm{ad}(x), \mathrm{J}] = 0, \quad x \in \mathfrak{t}.$$

Therefore, if G_o is a Lie group with Lie algebra $\mathfrak{g}_o$, then J extends to a left-invariant complex structure on G_o, and it will be also right-invariant with respect to right translations by elements $h \in T := \exp(\mathfrak{t})$ (note that T might be non-compact, unless G_o has finite center).

We will call such an invariant complex structure *standard*.

Remark 2.2. In [32], the class of (simple) real Lie algebras of inner type is called "Class I" and it is then proved that *every* invariant complex structure in these algebras are standard with respect to a suitable choice of a Cartan subalgebra (such complex structures are called regular in [32]).

2.3. Invariant metrics and the balanced condition

Let M be a compact complex manifold of the form $\Gamma\backslash G_o$, endowed with a complex structure J which is induced by a standard invariant complex structure J on G_o, as in the previous section. It is clear that any left invariant J-Hermitian metric h on G_o induces an Hermitian metric $\bar{h}$ on M, and that $\bar{h}$ is balanced or pluriclosed if and only if h is so. For the converse, we prove the following.

Proposition 2.3. *If* (M, J) *admits a balanced* (*pluriclosed*) *Hermitian metric, there exists a left invariant and right* T*-invariant Hermitian metric on* G_o *which is balanced* (*pluriclosed, respectively*).

Proof. Suppose we have a balanced metric h on M with associated fundamental form ω. Then using the same notation and arguments as in the proof of Proposition 2.1, we define a left-invariant positive $(n-1, n-1)$-form ϕ on G_o as follows:

$$\phi_e(x_{i_1}, \ldots, x_{i_{2n-2}}) := \int_M \omega^{n-1}(x^*_{i_1}, \ldots, x^*_{i_{2n-2}})\, d\mu.$$

As $d\omega^{n-1} = 0$, we obtain that also $d\phi = 0$. Therefore, we can find an unique $(1,1)$-form $\hat{\omega}$ so that $\hat{\omega}^{n-1} = \phi$ (see [26]), and the metric given by $\hat{\omega}$ is balanced. As ϕ is left invariant, so is $\hat{\omega}$ by uniqueness. Now, the group Ad(T) is compact, and using a standard averaging process, we can make ϕ_e also Ad(T)-invariant. This means that ϕ is also invariant under right T-translations. Again, by the uniqueness, the same will hold true for $\hat{\omega}$.

As for the pluriclosed condition, the lifted metric from M to G_o is clearly pluriclosed and can be made T-invariant by a standard averaging. ∎

Remark 2.4. We can now deal with the case when $\mathfrak{g}_o$ is the realification of a simple Lie algebra $\mathfrak{g}$. In this case, the complex structure J commutes with $\mathrm{ad}(\mathfrak{g}_o)$ and $\mathfrak{g}_o = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition, where $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$. Let G_o be a real group with algebra $\mathfrak{g}_o$, and let U be the compact subgroup with algebra $\mathfrak{u}$. Then the metric h which coincides with $-B$ on $\mathfrak{u}$, with B on $i\mathfrak{u}$ and such that $h(\mathfrak{u}, i\mathfrak{u}) = 0$, is a Hermitian metric which is balanced. Indeed, h is Ad(U)-invariant and therefore the corresponding $\delta\omega$ is Ad(U)-invariant 1-form, hence it vanishes identically. This is consistent with the fact that complex parallelizable manifolds carry balanced metrics as they carry Chern-flat metrics, as noted in [19], p. 121 (see also [1, 22]).

On the other hand, G_o admits no invariant pluriclosed metric. Indeed, any such metric h can be averaged to produce an Ad(U)-invariant pluriclosed metric, which would be balanced by the previous argument. This is not possible, as a metric which is balanced and pluriclosed at the same time has to be Kähler (see e.g. [5]), contrary to Proposition 2.1.

We now focus on the case where $\mathfrak{g}_o$ is absolutely simple of inner type, endowed with an invariant standard complex structure. We fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ with corresponding root system $R = R_{\mathfrak{k}} \cup R_{\mathfrak{p}}$ as in Section 2.1, and we consider an ordering $R = R^+ \cup R^-$ giving an invariant complex structure J_o on $\mathfrak{g}_o/\mathfrak{t}$. We extend J_o to an invariant complex structure J on $\mathfrak{g}_o$.

We also fix a basis of a complement of $\mathfrak{t}$ in $\mathfrak{g}_o$:

$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \quad w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha}), \quad \alpha \in R_{\mathfrak{k}}^+,$$
$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}), \quad w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \quad \alpha \in R_{\mathfrak{p}}^+,$$

so that $v_\alpha, w_\alpha \in \mathfrak{g}_o$ for every $\alpha \in R^+$, and moreover,

$$J v_\alpha = w_\alpha, \quad J w_\alpha = -v_\alpha,$$
$$[H, v_\alpha] = -i\alpha(H) w_\alpha, \quad H \in \mathfrak{h},$$
$$[v_\alpha, w_\alpha] = i H_\alpha, \quad \alpha \in R_{\mathfrak{k}}^+,$$
$$[v_\alpha, w_\alpha] = -i H_\alpha, \quad \alpha \in R_{\mathfrak{p}}^+.$$

We now construct invariant Hermitian metrics h on $\mathfrak{g}_o$. First, we define h on $\mathfrak{t}$ by choosing a J-Hermitian metric $h_{\mathfrak{t}}$ on $\mathfrak{t}$. If we set $\mathfrak{m}_\alpha := \mathrm{Span}\{v_\alpha, w_\alpha\}_{\alpha \in R^+}$, we define for $\alpha \neq \beta \in R^+$,

$$h(\mathfrak{t}, \mathfrak{m}_\alpha) = 0, \quad h(\mathfrak{m}_\alpha, \mathfrak{m}_\beta) = 0,$$
$$h(v_\alpha, v_\alpha) = h(w_\alpha, w_\alpha) = h_\alpha^2, \quad h(v_\alpha, w_\alpha) = 0$$

for $h_\alpha \in \mathbb{R}^+$.

In particular, we are interested in constructing balanced Hermitian metrics, namely Hermitian metrics whose associated (1, 1)-form $\omega = h(\cdot, \mathrm{J}\cdot)$ satisfies $d\omega^{n-1} = 0$, or equivalently $\delta\omega = 0$, where δ denotes the codifferential.

We use the standard expression of the codifferential in terms of the Levi Civita connection ∇ of h (see e.g. [9], p. 34):

$$\delta\omega(x) = -\mathrm{Tr}\nabla.\omega(\cdot, x) = -\sum_{i=1}^{2n} \nabla_{e_i}\omega(e_i, x) = \sum_{i=1}^{2n} \left(\omega(\nabla_{e_i} e_i, x) + \omega(e_i, \nabla_{e_i} x)\right),$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}_o$ with respect to h. Note that both h and J are ad($\mathfrak{t}$)-invariant and therefore $\delta\omega$ is ad($\mathfrak{t}$)-invariant too. This last fact implies that $\delta\omega$ vanishes identically if and only if $\delta\omega(x) = 0$ for every $x \in \mathfrak{t}$.

We have the following expression for the Levi Civita connection: for $x, y, z \in \mathfrak{g}_o$,

$$2h(\nabla_x y, z) = h([x, y], z) + h([z, x], y) + h([z, y], x).$$

Then for every $x \in \mathfrak{t}$, $y \in \mathfrak{g}_o$,

$$h(\nabla_y y, x) = h([x, y], y) = 0.$$

Therefore, for $x \in \mathfrak{t}$ we have

$$\delta\omega(x) = \sum_i \omega(e_i, \nabla_{e_i} x) = -\sum_i h(Je_i, \nabla_{e_i} x) \tag{2.2}$$
$$= -\frac{1}{2}\left(h([e_i, x], Je_i) + h([Je_i, e_i], x) + h([Je_i, x], e_i)\right).$$

We now observe that J is ad($\mathfrak{t}$)-invariant and hence

$$h([Je_i, x], e_i) = -h([e_i, x], Je_i) \quad \text{for every } i = 1, \ldots, 2n,$$

so that (2.2) can be written as

$$-\delta\omega(x) = \frac{1}{2}\sum_i h([Je_i, e_i], x)$$
$$= \frac{1}{2}\cdot 2\Big(\sum_{\alpha\in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} h([w_\alpha, v_\alpha], x) + \sum_{\alpha\in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} h([w_\alpha, v_\alpha], x)\Big)$$
$$= \sum_{\alpha\in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} h(-iH_\alpha, x) + \sum_{\alpha\in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} h(iH_\alpha, x),$$

whence $\delta\omega|_{\mathfrak{t}} = 0$ if and only if

$$-\sum_{\alpha\in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} H_\alpha + \sum_{\alpha\in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} H_\alpha = 0.$$

Summing up, the metric h is balanced when the following equation is satisfied:

$$\sum_{\alpha\in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha}\alpha = \sum_{\alpha\in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha}\alpha. \tag{2.3}$$

Note that this does *not* depend on the choice of the metric along the toral part $\mathfrak{t}$.

3. Proof of the main result

In this section we will prove our main result Theorem 1.1.

We keep the same notation as in the previous sections, and we start noting that equation (2.3) involves the unknowns $\{h_\alpha\}_{\alpha\in R^+}$ and also a choice of positive roots, i.e., an ordering or equivalently a complex structure on $\mathfrak{g}_o$. We will always fix a complex structure on $\mathfrak{t}$ once for all. It is known that giving an ordering on the root system R is equivalent to the choice of a system of simple roots Π, and that two systems of simple roots are conjugate under the action of the Weyl group W. We may fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and put $\Pi = \Pi_{\mathrm{c}} \cup \Pi_{\mathrm{nc}}$, where $\Pi_{\mathrm{c/nc}}$ denotes the set of simple roots which are compact or noncompact. We set $\Pi_{\mathrm{c}} = \{\phi_1, \ldots, \phi_k\}$, $\Pi_{\mathrm{nc}} = \{\psi_1, \ldots, \psi_l\}$, $k + l = r = \mathrm{rank}(\mathfrak{g}_o)$.

Each root $\alpha \in R^+$ can be written as

$$\alpha = \sum_{i=1}^{k} n_i(\alpha)\,\phi_i + \sum_{j=1}^{l} m_j(\alpha)\,\psi_j$$

for $n_i(\alpha), m_j(\alpha) \in \mathbb{N}$ nonnegative integers. If we set $g_\alpha := 1/h_\alpha^2$ and $g_j := g_{\phi_j}, h_j := g_{\psi_j}$, equation (2.3) can be written as

$$\sum_{\alpha\in R_{\mathfrak{k}}^+,\alpha\notin\Pi} g_\alpha\Big(\sum_{j=1}^{k} n_j(\alpha)\,\phi_j + \sum_{j=1}^{l} m_j(\alpha)\,\psi_j\Big) + \sum_{j=1}^{k} g_j\,\phi_j$$
$$= \sum_{\alpha\in R_{\mathfrak{p}}^+,\alpha\notin\Pi} g_\alpha\Big(\sum_{j=1}^{k} n_j(\alpha)\,\phi_j + \sum_{j=1}^{l} m_j(\alpha)\,\psi_j\Big) + \sum_{j=1}^{l} h_j\,\psi_j,$$

and therefore,

$$\text{(3.1)}\qquad \begin{cases} g_j = \displaystyle\sum_{\alpha\in R_{\mathfrak{p}}^+,\alpha\notin\Pi} g_\alpha\, n_j(\alpha) - \sum_{\alpha\in R_{\mathfrak{k}}^+,\alpha\notin\Pi} g_\alpha\, n_j(\alpha), & j = 1,\dots,k,\\ h_j = \displaystyle\sum_{\alpha\in R_{\mathfrak{k}}^+,\alpha\notin\Pi} g_\alpha\, m_j(\alpha) - \sum_{\alpha\in R_{\mathfrak{p}}^+,\alpha\notin\Pi} g_\alpha\, m_j(\alpha), & j = 1,\dots,l. \end{cases}$$

Remark 3.1. If we consider for instance the case $\mathfrak{g}_o = \mathfrak{su}(p,q)$ (with $p,q \geq 2$ and $p+q$ even) and the standard system of simple roots $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{p-1} - \varepsilon_p, \varepsilon_p - \varepsilon_{p+1}, \dots, \varepsilon_{p+q-1} - \varepsilon_{p+q}\}$ of $\mathfrak{sl}(p+q,\mathbb{C})$, then $\Pi_{\mathrm{nc}} = \{\varepsilon_p - \varepsilon_{p+1}\}$ and Π_{c} gives a system of simple roots for the semisimple part $\mathfrak{k}_{ss}$ of $\mathfrak{k}$. This means that every root $\alpha \in R_{\mathfrak{k}}^+, \alpha \notin \Pi$ is a linear combination of roots in Π_{c} and therefore the right-hand side of the last equation in (3.1) is nonpositive, so that (3.1) has no solution. This shows that the choice of the invariant complex structure might not be straightforward.

The following lemma provides key tools in our argument.

Lemma 3.2. *For each symmetric pair $(\mathfrak{g}_o,\mathfrak{k})$ as in Table 1, $(\mathfrak{g}_o,\mathfrak{k}) \ncong (\mathfrak{so}(1,2n),\mathfrak{so}(2n))$, and given a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ with corresponding root system R, there exists an ordering of the roots, hence a system of simple roots Π, such that*

$$\text{(3.2)}\qquad \forall\psi \in \Pi_{\mathrm{nc}},\ \exists\psi' \in \Pi_{\mathrm{nc}} \text{ with } \psi + \psi' \in R.$$

This implies that, if $\Pi_{\mathrm{nc}} = \{\psi_1,\dots,\psi_l\}$, then for every $\psi_j \in \Pi_{\mathrm{nc}}$ there exists $\alpha \in R_{\mathfrak{k}}^+$ with $m_j(\alpha) \neq 0$ and $\alpha \in \mathrm{Span}\{\Pi_{\mathrm{nc}}\}$.

Remark 3.3. Note that $\mathfrak{sp}(1,1) \cong \mathfrak{so}(1,4)$ is also not admissible in the above lemma. In general, for $\mathfrak{g}_o = \mathfrak{so}(1,2n)$ we have the standard system $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n,\ i = 1,\dots,n-1\}$ with $\Pi_{\mathrm{nc}} = \{\varepsilon_n\}$. As R_c consists precisely of all the short roots, it is clear that for any element σ of the Weyl group $W \cong \mathbb{Z}_2^n \ltimes S_n$ we have that $\sigma(\Pi)_{\mathrm{nc}}$ consists of one element. We will deal with this case later on.

Proof of Lemma 3.2. We first deal with the classical case. We start with the standard system of simple roots Π, following the notation as in [24]. It is immediate to check that in this case Π_{nc} consists of a single root ψ.

Assume first to be in the case where ψ is a short root. Let Λ be the set of all simple roots which are connected to ψ in the Dynkin diagram relative to Π. If $s \in W$ denotes the reflection around ψ, then s leaves every element $\Pi \setminus \Lambda$ pointwise fixed. We observe that Λ consists of either at most three short roots or it contains a long root. In the first case, $s(\Lambda) = \{\psi + \lambda |\ \lambda \in \Lambda\} \subseteq R_{\mathfrak{p}}$ so that $s(\Pi)_{\mathrm{nc}} = \{-\psi, s(\Lambda)\}$ and therefore the system of simple roots $s(\Pi)$ satisfies (3.2). If Λ contains a long root, then it also contains a short root, unless $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{so}(2,3), \mathbb{R} + \mathfrak{so}(3))$, that is isomorphic to $(\mathfrak{sp}(2), \mathfrak{u}(2))$; this case will be dealt with in the second part of the proof. Therefore $\Lambda = \{\phi_1, \phi_2\}$, with ϕ_1 short and ϕ_2 long. Again the reflection s around ψ gives $s(\phi_1) = \psi + \phi_1$ and $s(\phi_2) = \phi_2 + 2\psi \in R_{\mathfrak{k}}$ or $s(\phi_2) = \psi + \phi_2 \in R_{\mathfrak{p}}$. This implies that the system of simple roots $s(\Pi)$ has $s(\Pi)_{\mathrm{nc}} = \{-\psi, \psi + \phi_1\}$ or $\{-\psi, \psi + \phi_1, \psi + \phi_2\}$ and in both cases it satisfies (3.2).

We are then left with the case where ψ is a long root, namely the case where $\mathfrak{g}_o = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{u}(2n)$. A standard system of simple roots is given by $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{2n-1} - \varepsilon_{2n}, 2\varepsilon_{2n}\}$ and $\Pi_{\mathrm{nc}} = \{\psi = 2\varepsilon_{2n}\}$. Again using s_β, we see that $s_\beta(\Pi)_{\mathrm{nc}} = \{-2\varepsilon_{2n}, \varepsilon_{2n-1} + \varepsilon_{2n}\}$ so that condition (3.2) is satisfied.

We may now deal with the exceptional cases. Starting with the standard system of simple roots Π, we list the set Π_{nc}, that turns out to consist of a single root β. For each case, using the symmetry s_β, we obtain the system of simple roots $\Pi' := s_\beta(\Pi)$ that satisfies condition (3.2).

(1) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{su}(2) + \mathfrak{su}(2))$. Here $\Pi = \{\alpha, \beta\}$, with β long. We have $\Pi_{\mathrm{nc}} = \{\beta\}$ and $\Pi' = \{-\beta, \alpha + \beta\}$.

(2) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$. According to [24], the standard system of simple roots is $\Pi = \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ so that $\Pi_{\mathrm{nc}} = \{\alpha_4\}$ and therefore $\Pi'_{\mathrm{nc}} = \{-\alpha_4, \alpha_4 + \alpha_3\}$.

(3) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(4)}, \mathfrak{su}(2) + \mathfrak{sp}(3))$. In this case, $\Pi_{\mathrm{nc}} = \{\alpha_1\}$ and therefore $\Pi'_{\mathrm{nc}} = \{-\alpha_1, \alpha_1 + \alpha_2\}$.

(4) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$. For $\mathfrak{e}_8$, we have the standard system of simple roots

$$\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \quad \alpha_2 = \varepsilon_1 + \varepsilon_2,$$
$$\alpha_j = \varepsilon_{j-1} - \varepsilon_{j-2}, \quad j = 3, \ldots, 8.$$

Then $\Pi_{\mathrm{nc}} = \{\alpha_1\}$ and $\Pi'_{\mathrm{nc}} = \{-\alpha_1, \alpha_1 + \alpha_3\}$.

(5) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(-24)}, \mathfrak{su}(2) + \mathfrak{e}_7)$. Keeping the same notation for simple roots as above, we have $\Pi_{\mathrm{nc}} = \{\alpha_8\}$ and $\Pi'_{\mathrm{nc}} = \{-\alpha_8, \alpha_8 + \alpha_7\}$.

(6) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(2)}, \mathfrak{su}(2) + \mathfrak{su}(6))$. As the system root system Π can be taken to be composed of the simple roots $\{\alpha_1, \ldots, \alpha_6\}$ of $\mathfrak{e}_8$, we have $\Pi_{\mathrm{nc}} = \{\alpha_2\}$ and $\Pi'_{\mathrm{nc}} = \{-\alpha_2, \alpha_2 + \alpha_4\}$.

(7) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(-14)}, \mathbb{R} + \mathfrak{so}(10))$. In this case, we have $\Pi_{\mathrm{nc}} = \{\alpha_1\}$ and $\Pi'_{\mathrm{nc}} = \{-\alpha_1, \alpha_1 + \alpha_3\}$. ∎

Lemma 3.4. *For every system of simple roots* $\Pi = \Pi_c \cup \Pi_{nc}$ *with* $\Pi_c = \{\phi_1, \dots, \phi_k\}$*, we have*

$$\forall j = 1, \dots, k, \ \exists \alpha \in R^+_{\mathfrak{p}}, \ \alpha \notin \Pi : \ n_j(\alpha) \neq 0,$$

where $n_j(\alpha)$ *denotes the coordinate of* α *along the root* ϕ_j*.*

Proof. We start noting that the centralizer $C_{\mathfrak{k}^c}(\mathfrak{p}^c) = C_{\mathfrak{k}}(\mathfrak{p})^c = \{0\}$. It then follows that $[E_{\phi_j}, \mathfrak{p}^c] \neq \{0\}$, hence there exists $\gamma \in R_{\mathfrak{p}}$ with $[E_{\phi_j}, E_\gamma] \neq 0$, i.e., $\phi_j + \gamma \in R_{\mathfrak{p}}$. Now, if $\gamma > 0$, then $\alpha := \phi_j + \gamma \in R^+_{\mathfrak{p}} \setminus \Pi$ and $n_j(\alpha) \geq 1$. Suppose now $\gamma < 0$. We write $\gamma = c_j \phi_j + \sum_{\theta \in \Pi \setminus \phi_j} c_\theta \theta$, for some nonpositive integers c_j, c_θ. As $\gamma \neq -\phi_j$, there exists at least one negative coefficient $c_\theta < 0$, for some $\theta \in \Pi, \theta \neq \phi_j$. Therefore the root $\gamma + \phi_j$ must be negative and $1 + c_j \leq 0$, i.e., $\alpha := -\gamma \in R^+_{\mathfrak{p}} \setminus \Pi$ and $n_j(\alpha) = -c_j \geq 1$. ∎

We now fix a system of simple roots Π as in Lemma 3.2. In order to solve the corresponding system of equations (3.1) for the positive unknowns $\{g_i, h_j, g_\alpha\}$, we will show how to choose the positive values $\{g_\alpha\}_{\alpha \in R^+ \setminus \Pi}$ in such a way to guarantee that the constants $\{g_i, h_j\}$, defined to satisfy (3.1), are positive.

We set

$$\Sigma_{\mathfrak{k}} := \{\alpha \in R^+_{\mathfrak{k}} \,|\, \alpha \notin \Pi, \alpha \in \mathrm{Span}\{\Pi_{nc}\}\}, \qquad A_{\mathfrak{k}} = (R^+_{\mathfrak{k}} \setminus \Pi_c) \setminus \Sigma_{\mathfrak{k}}.$$

Then the system of equations (3.1) can be written as

$$\text{(3.3)} \qquad \begin{cases} g_j = \displaystyle\sum_{\alpha \in R^+_{\mathfrak{p}},\, \alpha \notin \Pi} g_\alpha n_j(\alpha) - \sum_{\alpha \in A_{\mathfrak{k}}} g_\alpha n_j(\alpha), & j = 1, \dots, k, \quad (1) \\ h_j = \displaystyle\sum_{\alpha \in R^+_{\mathfrak{k}},\, \alpha \notin \Pi} g_\alpha m_j(\alpha) - \sum_{\alpha \in R^+_{\mathfrak{p}},\, \alpha \notin \Pi} g_\alpha m_j(\alpha), & j = 1, \dots, l. \quad (2) \end{cases}$$

We start assigning $g_\alpha = 1$ for every $\alpha \in A_{\mathfrak{k}}$.

Then, for every $j = 1, \dots, k$, we use Lemma 3.4 selecting a root $\alpha \in R^+_{\mathfrak{p}}$ with $n_j(\alpha) \neq 0$, $\alpha \notin \Pi$. This root α, which depends on j, contributes to the first sum in the right-hand side of equation (1) in (3.3), and the value g_α can be chosen big enough so that g_j is strictly positive. Summing up, we can assign values $\{g_\alpha\}_{\alpha \in R^+_{\mathfrak{p}} \setminus \Pi_{nc}}$ so that all g_j, $j = 1, \dots, k$, can be defined as in (1) of (3.3), and are strictly positive.

We now turn to equation (2) in (3.3), which can now be written as

$$\text{(3.4)} \qquad h_j = \sum_{\alpha \in \Sigma_{\mathfrak{k}}} g_\alpha m_j(\alpha) + \sum_{\alpha \in A_{\mathfrak{k}}} m_j(\alpha) - \sum_{\alpha \in R^+_{\mathfrak{p}},\, \alpha \notin \Pi} g_\alpha m_j(\alpha),$$

where in the right-hand side the last two sums have a fixed value. Now, by Lemma 3.2, we know that for every $j = 1, \dots, l$, we can find $\alpha \in \Sigma_{\mathfrak{k}}$ with $m_j(\alpha) \neq 0$. These roots can be used to choose the coefficients $\{g_\beta\}_{\beta \in \Sigma_{\mathfrak{k}}}$ big enough to guarantee that h_j, when defined to satisfy (3.4), is strictly positive.

In order to complete the proof of our main result Theorem 1.1, we are left with the case $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{so}(1, 2n), \mathfrak{so}(2n))$, with standard system of simple roots $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n, i = 1, \dots, n-1\}$, $\Pi_{nc} = \{\varepsilon_n\}$. We see that

$$R^+_{\mathfrak{k}} = \{\varepsilon_i \pm \varepsilon_j, \ i < j\}, \qquad R_{\mathfrak{p}} = \{\varepsilon_1, \dots, \varepsilon_n\}.$$

Now, we use equation (2.3) and search for positive real numbers $\{x, y, z_i,\ i = 1, \dots, n\}$ so that

$$x \cdot \sum_{i<y} \varepsilon_i - \varepsilon_j + y \cdot \sum_{i<j} \varepsilon_i + \varepsilon_j = \sum_{i=1}^{n} z_i\, \varepsilon_i,$$

i.e.,

$$\sum_{i=1}^{n} [(x+y)(n-i) + (x-y)(i-1)]\, \varepsilon_i = \sum_{i=1}^{n} z_i\, \varepsilon_i.$$

It is clear that the above equation has positive solutions by simply choosing $x > y > 0$. This concludes the proof of Theorem 1.1. ■

Remark 3.5. We can consider the metric h_o which coincides with $-B$ on the compact part $\mathfrak{k}$, with B on $\mathfrak{p}$ and such that $h_o(\mathfrak{k}, \mathfrak{p}) = 0$. This metric is easily seen to depend only on $\mathfrak{g}_o$ and *not* on the Cartan decomposition $\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p}$. We could then ask whether there exists a suitable complex structure such that the metric h_o turns out to be balanced. The resulting equation has been already treated in [2] and has a solution if and only if $\mathfrak{g}_o = \mathfrak{su}(p, p+1) \cong \mathfrak{su}(p+1, p)$ for $p \geq 1$.

4. Non-existence of pluriclosed metrics

In this section we prove our result Theorem 1.2 concerning the non-existence of pluriclosed metrics on the compact quotients M we have constructed in the previous sections; we will keep the same notation used above.

Suppose now that h is a pluriclosed metric on $\mathrm{M} = \Gamma\backslash \mathrm{G}_o$. Then we can obtain a pluriclosed invariant metric h on G_o which is also invariant under right T-translations. It follows that on $\mathfrak{g}$ we have

$$h(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{if } \beta \neq -\alpha.$$

In order to write down the condition $dd^c\omega = 0$, where ω is the fundamental form of h, we recall the standard formula for the differential of invariant forms (see e.g. [24], p. 136). If ϕ is any invariant k-form on G_o or equivalently on $\mathfrak{g}_o$, then for every $v_o, \dots, v_k$ in $\mathfrak{g}_o$,

$$(k+1) \cdot d\phi(v_o, \dots, v_k) = \sum_{i<j} (-1)^{i+j}\, \phi([v_i, v_j], v_o, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k).$$

We set $\phi := d^c\omega$ and compute $d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta})$ for $\alpha, \beta \in R^+$. We have

$$\begin{aligned} 4\, d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = &-\phi(H_\alpha, E_\beta, E_{-\beta}) + \phi(N_{\alpha\beta} E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) \\ &- \phi(N_{\alpha,-\beta} E_{\alpha-\beta}, E_{-\alpha}, E_\beta) - \phi(N_{-\alpha,\beta} E_{\beta-\alpha}, E_\alpha, E_{-\beta}) \\ &+ \phi(N_{-\alpha,-\beta} E_{-\alpha-\beta} E_\alpha, E_\beta) - \phi(H_\beta, E_\alpha, E_{-\alpha}), \end{aligned}$$

where we use the standard notation $[E_\gamma, E_\varepsilon] = N_{\gamma,\varepsilon} E_{\gamma+\varepsilon}$ for every $\gamma, \varepsilon \in R$. Using the known identities for the Weyl basis (see [24], pp. 172 and 176), we can write that

$$\begin{aligned} 4\, d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = &-\phi(H_\alpha, E_\beta, E_{-\beta}) - \phi(H_\beta, E_\alpha, E_{-\alpha}) \\ &+ 2\phi(N_{\alpha\beta} E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) - 2\phi(N_{\alpha,-\beta} E_{\alpha-\beta}, E_{-\alpha}, E_\beta). \end{aligned}$$

We also introduce the notation $JE_\gamma = i\varepsilon_\gamma E_\gamma$ for every $\gamma \in R$, where $\varepsilon_\gamma = \pm 1$ according to $\gamma \in R^\pm$. Then

$$\begin{aligned} 4\,dd^c\omega(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) &= -d\omega(JH_\alpha, E_\beta, E_{-\beta}) - d\omega(JH_\beta, E_\alpha, E_{-\alpha}) \\ &\quad - 2iN_{\alpha,\beta}\,d\omega(E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) - 2iN_{\alpha,-\beta}\varepsilon_{\alpha-\beta}\,d\omega(E_{\alpha-\beta}, E_{-\alpha}, E_\beta). \end{aligned}$$

Now we easily compute

$$3\,d\omega(JH_\alpha, E_\beta, E_{-\beta}) = -\omega(H_\beta, JH_\alpha)$$

and

$$3\,d\omega(E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) = N_{\alpha,\beta}(\omega(E_\alpha, E_{-\alpha}) + \omega(E_\beta, E_{-\beta}) - \omega(E_{\alpha+\beta}, E_{-\alpha-\beta})),$$

where we have used the fact that $N_{\alpha,\beta} = N_{\alpha+\beta,-\beta} = -N_{\alpha+\beta,-\alpha}$ (see [24], p. 172). Similarly,

$$3\,d\omega(E_{\alpha-\beta}, E_{-\alpha}, E_\beta) = N_{\alpha,-\beta}(-\omega(E_\beta, E_{-\beta}) + \omega(E_\alpha, E_{-\alpha}) - \omega(E_{\alpha-\beta}, E_{\beta-\alpha})).$$

Summing up, we have

$$\begin{aligned} 12\,dd^c\,\omega(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) &= -2\omega(JH_\alpha, H_\beta) \\ &\quad - 2iN^2_{\alpha,\beta}(\omega(E_\alpha, E_{-\alpha}) + \omega(E_\beta, E_{-\beta}) - \omega(E_{\alpha+\beta}, E_{-\alpha-\beta})) \\ &\quad - 2iN^2_{\alpha,-\beta}\,\varepsilon_{\alpha-\beta}(-\omega(E_\beta, E_{-\beta}) + \omega(E_\alpha, E_{-\alpha}) - \omega(E_{\alpha-\beta}, E_{\beta-\alpha})). \end{aligned} \tag{4.1}$$

If we now set $a_\alpha := h(E_\alpha, E_{-\alpha})$, the pluriclosed condition and (4.1) imply that

$$\begin{aligned} 0 = -h(H_\alpha, H_\beta) &- iN^2_{\alpha,\beta}(-ia_\alpha - ia_\beta + ia_{\alpha+\beta}) \\ &- iN^2_{\alpha,-\beta}\,\varepsilon_{\alpha-\beta}(-i\varepsilon_{\beta-\alpha}a_{\alpha-\beta} + ia_\beta - ia_\alpha), \end{aligned}$$

hence

$$h(H_\alpha, H_\beta) = N^2_{\alpha,\beta}(a_{\alpha+\beta} - a_\alpha - a_\beta) + N^2_{\alpha,-\beta}\,\varepsilon_{\alpha-\beta}(\varepsilon_{\alpha-\beta}a_{\alpha-\beta} + a_\beta - a_\alpha). \tag{4.2}$$

We recall that

$$\begin{aligned} a_\alpha = h(E_\alpha, E_{-\alpha}) = -h(v_\alpha, v_\alpha) < 0, &\quad \alpha \in R^+_{\mathfrak{k}}, \\ a_\alpha = h(E_\alpha, E_{-\alpha}) = h(v_\alpha, v_\alpha) > 0, &\quad \alpha \in R^+_{\mathfrak{p}}, \\ h(H_\alpha, H_\beta) = -h(iH_\alpha, iH_\beta) \in \mathbb{R}, &\quad h(H_\alpha, H_\alpha) < 0. \end{aligned}$$

Now, we note that the existence of the complex structure J, which we constructed in Section 3, relies on Lemma 3.2. In particular, when $\mathfrak{g}_o \neq \mathfrak{so}(1, 2n)$, we have the existence of two simple roots $\psi_1, \psi_2 \in \Pi_{\text{nc}}$ with $\psi_1 + \psi_2 = \phi \in R_{\mathfrak{k}}$. The following lemma is elementary.

Lemma 4.1. *Either $\psi_1 + 2\psi_2 \notin R$ or $\psi_2 + 2\psi_1 \notin R$.*

Proof. As ψ_1, ψ_2 are simple, we have $\pm(\psi_1 - \psi_2) \notin R$. Now, $\psi_i + n\psi_j \in R$ if and only if $0 \leq n \leq q_j$ with $q_j = -2\frac{\langle \psi_1, \psi_2 \rangle}{\|\psi_j\|^2} \in \mathbb{N}$ for $i \neq j$. It is then clear that $q_1, q_2 \geq 2$ is impossible, as $\psi_1 \neq \psi_2$ implies $q_1 \cdot q_2 < 4$. ∎

Suppose then that $\phi + \psi_1 = \psi_2 + 2\psi_1 \notin R$. We now apply (4.2) with two possible choices for α, β, namely:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1,\psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}).$$

(2) $\alpha = \phi, \beta = \psi_1$. Then

$$h(H_\phi, H_{\psi_2}) = N^2_{\phi,-\psi_1}(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

Subtracting (1) from (2) we get

$$h(H_{\psi_2}, H_{\psi_2}) = \left(N^2_{\phi,-\psi_1} + N^2_{\psi_1,\psi_2}\right)(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

This is a contradiction, as $h(H_{\psi_2}, H_{\psi_2}) < 0$, while $a_{\psi_i} > 0$ for $i = 1, 2$ and $a_\phi < 0$.

We are left with the case $\mathfrak{g}_o = \mathfrak{so}(1, 2n)$, that we have dealt with separately in Section 3. In this case, the complex structure J is defined by the standard system of positive roots, namely $R^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_i,\ 1 \leq i \neq j \leq n\}$. In particular, $R^+_{\mathfrak{k}} = \{\varepsilon_i \pm \varepsilon_j\}_{i \neq j}$ and $R^+_{\mathfrak{p}} = \{\varepsilon_i\}_{i=1,\dots,n}$. We now consider $\psi_i = \varepsilon_i$, $i = 1, 2$, $\phi_1 = \psi_1 + \psi_2 \in R^+_{\mathfrak{k}}$ and $\phi_2 = \psi_1 - \psi_2 \in R^+_{\mathfrak{k}}$. We apply (4.2) in two different ways:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1,\psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}) + N^2_{\psi_1,-\psi_2}(a_{\phi_2} + a_{\psi_2} - a_{\psi_1}).$$

(2) $\alpha = \phi_1, \beta = \psi_2$. Note that $\phi_1 + \psi_1 \notin R$. Then

$$h(H_{\phi_1}, H_{\psi_1}) = N^2_{\phi_1,-\psi_1}(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}).$$

Therefore,

$$h(H_{\psi_1}, H_{\psi_1}) = (N^2_{\phi_1,-\psi_1} + N^2_{\psi_1,\psi_2})(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}) + N^2_{\psi_1,-\psi_2}(a_{\psi_1} - a_{\phi_2} - a_{\psi_2}).$$

We now recall that, if $\gamma, \delta \in R$, then

$$N^2_{\gamma,\delta} = \frac{q(1-p)}{2}\,\|\gamma\|^2,$$

where $\delta + n\gamma$, $p \leq n \leq q$, is the γ-series containing δ (see [24], p.176). We then immediately see that $N^2_{\psi_1,\psi_2} = N^2_{\psi_1,-\psi_2}$ and noting furthermore that $N^2_{\phi_1,-\psi_1} = N^2_{\psi_1,\psi_2}$, we can write

$$h(H_{\psi_1}, H_{\psi_1}) = N^2_{\psi_1,\psi_2}(a_{\psi_2} + 3a_{\psi_1} - 2a_{\phi_1} - a_{\phi_2}),$$

hence the contradiction $h(H_{\psi_1}, H_{\psi_1}) > 0$. This concludes the proof of Theorem 1.2.

5. Geometric properties

In this section, we collect some properties of the complex Hermitian manifolds we have constructed in the previous section.

Proposition 5.1. *Let* G_o *be a non-compact simple group of even dimension and of inner type together with a co-compact lattice* $\Gamma \subset \mathrm{G}_o$*. If* $\mathrm{M} = \Gamma\backslash\mathrm{G}_o$ *is endowed with a standard complex structure and a Hermitian balanced metric h, then the Chern Ricci form* ρ *of h never vanishes, and the Kodaira dimension* $\kappa(M) = -\infty$*.*

Proof. We consider a standard complex structure J on a manifold $\mathrm{M} = \Gamma\backslash\mathrm{G}_o$. We denote by D the Chern connection relative to a Hermitian metric h which is induced by an invariant metric on G_o, again denoted by h. We can moreover suppose that h is invariant by the right T-translations.

If $x \in \mathfrak{g}_o$, we define the endomorphism $D_x \in \mathrm{End}(\mathfrak{g}_o)$ as follows: given $y \in \mathfrak{g}_o$, we extend x and y as left invariant vector fields x^* and y^* on G_o, and we put $D_x y := D_{x^*} y^*|_e$. Clearly, $D_x \in \mathfrak{so}(\mathfrak{g}_o, h)$ and $[D_x, \mathrm{J}] = 0$. Moreover,

$$D_x y = [x, y]^{10}, \quad \forall x \in \mathfrak{g}_o^{01},\ y \in \mathfrak{g}_o^{10}, \tag{5.1}$$

that follows from the fact that $T^{1,1} = 0$, where T is the torsion of D.

If R denotes the curvature of the Chern connection, where $R_{xy} = [D_x, D_y] - D_{[x,y]}$, we are interested in the first Ricci form ρ given by

$$\rho(x, y) = -\frac{1}{2}\mathrm{Tr}(J \circ R_{xy}).$$

As the complex structure and the metric are both invariant under the adjoint action of the group $T = \exp(\mathfrak{t})$, we see that

$$\rho(\mathfrak{t}, E_\alpha) = 0, \quad \forall \alpha \in R,$$
$$\rho(E_\alpha, E_\beta) \neq 0 \quad \text{implies} \quad \beta = -\alpha,\ \alpha, \beta \in R.$$

Therefore we can compute

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2}\mathrm{Tr}(JD_{H_\alpha}).$$

Lemma 5.2. *For every* $x \in \mathfrak{h}$*,*

$$D_x = \mathrm{ad}(x).$$

Proof. We use similar arguments as in [29]. It will suffice to consider the case where $x \in \mathfrak{h}^{10}$; then for every $\alpha \in R^+$ we have

$$D_x E_{-\alpha} = [x, E_{-\alpha}]^{01} = [x, E_{-\alpha}] \quad \text{and} \quad D_x \mathfrak{h}^{01} = 0,$$

by (5.1). Then if $\beta \in R^+$, we have

$$h(D_x E_\alpha, E_{-\beta}) = -h(E_\alpha, D_x E_{-\beta}) = -\beta(x)h(E_\alpha, E_{-\beta}) = 0 \quad \text{if } \alpha \neq \beta,$$

so that $D_x E_\alpha = \alpha(x) E_\alpha = [x, E_\alpha] \pmod{\mathfrak{h}}$. As $h(D_x E_\alpha, \mathfrak{h}^{01}) = -h(E_\alpha, D_x \mathfrak{h}^{01}) = 0$, we conclude that

$$D_x E_\alpha = [x, E_\alpha].$$

Finally, $h(D_x \mathfrak{h}^{10}, \mathfrak{h}^{01}) = 0$ and $h(D_x \mathfrak{h}^{10}, E_{-\alpha}) = -h(\mathfrak{h}^{10}, [x, E_{-\alpha}]) = 0$, so that $D_x \mathfrak{h} = 0 = [x, \mathfrak{h}]$. ∎

It follows that

$$\rho(\mathfrak{h}, \mathfrak{h}) = 0$$

and

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2}\Big(2 \sum_{\beta \in R^+} i\beta(H_\alpha)\Big) = B(H_\alpha, \delta), \tag{5.2}$$

where

$$\delta = \sum_{\beta \in R^+} iH_\beta \in \mathfrak{t} \neq 0,$$

hence ρ never vanishes.

We now show that the tensor powers $K_{\mathrm{M}}^{\otimes m}$ are holomorphically nontrivial for every $m \geq 1$. Indeed, the metric h induces a Hermitian metric on the line bundles $K_{\mathrm{M}}^{\otimes m}$ with curvature form $m\rho$. If Ω is a nowhere vanishing holomorphic section of $K_{\mathrm{M}}^{\otimes m}$, then $m\rho = -i\partial\bar{\partial}\log(\|\Omega\|^2)$. If we denote by $\widehat{\ }$ the result of the symmetrization process, which commutes with the operators ∂ and $\bar{\partial}$, we obtain on G_o that

$$\hat{\rho} = -i\partial\bar{\partial}\widehat{\log(\|\Omega\|^2)} = 0.$$

As ρ is invariant, $\hat{\rho} = \rho = 0$ and we get a contradiction as $\delta \neq 0$.

The claim $\kappa(\mathrm{M}) = -\infty$ now follows from Theorem 1.4 in [35]. ∎

Remark 5.3. We note that the manifold M is parallelizable and therefore $c_1(M) = 0$, hence the Chern Ricci form ρ is exact. Moreover the Chern scalar curvature s^{Ch} vanishes identically, as it can be deduced from the expression (5.2) of ρ or in a simpler way[1] since $d\omega^{n-1} = 0$ and

$$0 = \int_M \rho \wedge \omega^{n-1} = \frac{1}{n}\int_M s^{Ch}\omega^n = s^{Ch}\int_M \omega^n.$$

We also remark here that the balanced condition implies that the two scalar curvatures that one can obtain tracing the Chern curvature tensor coincide (see [21], p. 501).

We finally note that also for a compact group K endowed with an invariant complex structure we have $h^{n,0}(\mathrm{K}) = 0$, see [28], Proposition 3.7.

Acknowledgements. The authors would like to thank Daniele Angella for useful conversations and the anonymous referees for their valuable remarks and suggestions.

Funding. The first author is supported by Villum Young Investigator 0019098. The second author is supported by GNSAGA of INdAM and by the project PRIN 2017 "Real and complex manifolds: topology, geometry and holomorphic dynamics", no. 2017JZ2SW5.

[1] We are indebted to an anonymous referee for this remark

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Received July 21, 2021; revised September 2, 2022. Published online October 19, 2022.

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