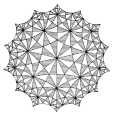
Rev. Mat. Iberoam. 39 (2023), no. 3, 845–896
DOI 10.4171/RMI/1356



Handlebody diagram algebras

Daniel Tubbenhauer and Pedro Vaz

Abstract. In this paper, we study handlebody versions of some classical diagram algebras, most prominently, handlebody versions of Temperley–Lieb, blob, Brauer, BMW, Hecke and Ariki–Koike algebras. Moreover, motivated by Green–Kazhdan–Lusztig's theory of cells, we reformulate the notion of (sandwich, inflated or affine) cellular algebras. We explain this reformulation and how all of the above algebras are part of this theory.

1. Introduction

A large collection of diagram algebras, such as Temperley–Lieb or (type A) Hecke algebras, are interesting from at least two perspectives: they are of fundamental importance in low-dimensional topology and they also have a rich representation theory.

Having an eye on the study of links in 3-manifolds brings the topology of the ambient space into play. In all the classical examples, like Temperley–Lieb or Hecke algebras, the ambient space is the 3-ball, and these algebras are related to spherical Coxeter groups and Artin–Tits braid groups. In the simplest case beyond the 3-ball, when one passes to links in a solid torus, these diagram algebras get replaced by their (extended) affine versions, and the related objects are now affine Coxeter groups and Artin–Tits braid groups. A natural question is what kind of diagrammatics and Coxeter combinatorics one could expect for more general 3-manifolds.

In this paper we consider (three-dimensional) handlebodies of genus g. The 3-ball and the solid torus correspond to $g = 0$ (called classical in this paper) and $g = 1$, respectively. As we will see, the Temperley–Lieb and Hecke algebras, their affine versions as well as algebras along the same lines, can be seen as a low genus class of a more general, higher genus, class of diagram algebras.

In case $g = 0$, these diagram algebras and their associated braid groups are around for donkey's years. For $g = 1$, there is a long history of work on this topic which goes back to at least Brieskorn [7]. For example (with more references to come later on), see [1] for braid pictures, [19] or [52] for connections to knot theory, [24] or [27] for affine diagram algebras. To the best of our knowledge, the first attempts to give a description of braids

2020 Mathematics Subject Classification: Primary 16G99; Secondary 20F36, 18M30, 20C08, 57K14.
Keywords: Diagrammatic algebra, cellular algebras, braid groups, handlebodies.

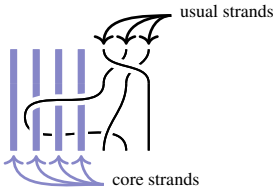
in handlebodies are due to Vershinin [64] and Häring-Oldenburg–Lambropoulou [28,41], while Lambropoulou studied links in 3-manifolds even before that [40].

From the representation theoretical point of view all these algebras share the common feature of being cellular in a certain way that we will make precise. In a nutshell, in the $g = 0$ case, these algebras are often cellular in the sense of Graham–Lehrer [21], and in the $g = 1$ case, they are often affine cellular in the sense of König–Xi [34].

We see this paper as a continuation of these works, focusing on the diagrammatic, algebraic and representation theoretical aspects. That is, we generalize some diagram algebras to higher genus, and we show that they are sandwich cellular. (Sandwich cellular is a notion that generalizes cellularity. Roughly speaking it means that the original algebra can be obtained by sandwiching smaller algebras. One of the upshots of being sandwich cellular is that the classification of simple modules is reduced from the original algebra to the sandwiched algebras.)

1A. What this paper does

Our starting point is a diagrammatic description of handlebody braid groups of genus g, i.e., a diagrammatic description of the configuration space of a disk with g punctures. The pictures hereby are, e.g.,

A handlebody braid for $g = 4$: 

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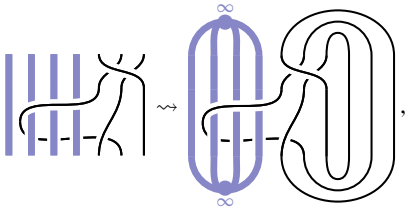
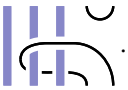
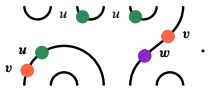
This illustrates a handlebody braid of genus 4: The three strands on the right are usual strands. The four thick and blue/grayish strands on the left are core strands and they correspond to the punctures of the disk respectively the cores of the handlebody. The point is that by an appropriate closure, i.e., merging the core strands at infinity, illustrated by

An Alexander closure: 
,

the core strands correspond to cores of a handlebody as explained in e.g. [55], Section 2, hence the name. All links in such handlebodies can be obtained by this closing procedure, and there is also an associated Markov theorem. In other words, the handlebody braid group gives an algebraic way to study links in handlebodies. The main references here are [64] and [28]. After explaining this setup more carefully, building upon the aforementioned works, in Section 3 we also study an associated handlebody Coxeter group for which we find a basis using versions of Jucys–Murphy elements.

These handlebody braid pictures are also our starting point to define and study various diagram algebras associated to handlebodies:

(1) In Section 4 we study handlebody Temperley–Lieb and blob algebras. The pictures to keep in mind are crossingless matchings and core strands (left, Temperley–Lieb) respectively crossingless matchings decorated with colored blobs (right, blob):

A Temperley–Lieb picture:  . A blob picture:  .

Note that the blobs illustrated in the right picture come in colors, corresponding to the various cores strands. These algebras generalize Temperley–Lieb algebras and blob algebras: If $g = 0$, then these two algebras are the same as the classical Temperley–Lieb algebra. For $g \geq 1$ they are not the same anymore (at least in our formulation), and have a long history of study starting with e.g. [45].

(2) In Section 5 we move on to handlebody versions of Brauer and BMW algebras. These are tangle algebras with core strands and the picture is

A BMW picture:  .

As before, the case $g = 0$ is classical and goes back to Brauer, while $g = 1$ appears in [26]. In the same section we also study their cyclotomic quotients.

(3) Finally, Section 6 studies handlebody versions of Hecke and Ariki–Koike algebras. The picture for handlebody Hecke algebras is the same as for handlebody braid groups, while we choose to illustrate handlebody versions of Ariki–Koike algebras using blobs, e.g.,

An Ariki–Koike picture:  .

The cases $g = 0$ and $g = 1$ are, of course, well-studied and they correspond to Hecke respectively extended affine Hecke algebras or cyclotomic quotients. We learned about the $g > 1$ case from [41] and [5].

We also study a generalization of cellularity in Section 2, giving us a toolkit to parameterize the simples modules of the aforementioned algebras. Note hereby that this generalization heavily builds on and borrows from [23], [36], [25] or [14]. Although it might be known to experts, our exposition is new.

1B. Speculations

Let us mention a number of possible future directions.

- **Quantum topology.** A manifest direction which we do not explore in this work would be to study these algebras in connections to quantum topology and its ramifications.

For example, for $g = 1$, [19] and [52] construct link invariants from Markov traces, and these link invariants admit categorifications [66]. For $g > 1$, [55] takes a few first steps towards categorical handlebody link invariants, but this direction appears to be widely open otherwise.

Moreover, for $g = 0$ most of these diagram algebras are related to representation theory by some form of Schur–Weyl duality. (In fact, this was the reason to define e.g. the Temperley–Lieb algebras to begin with, see [57].) Some work for higher genus on this representation theoretical aspect is done, e.g. in relation to Verma modules [11, 30, 39] or complex reflection groups [48, 58]. Following this track for higher genus seems to be a worthwhile goal.

- **Diagram algebras.** There are plenty of diagram algebras that we do not considered in this paper, but for which (some version of) our discussion goes through.

  Examples of such algebras that come to mind that appear in classical literature are partition algebras [44], rook monoid algebras [60], walled Brauer algebras [35] and alike. Other examples are related to knot theory and categorification such as (type A) webs appearing in a version of Schur–Weyl duality [9, 53, 54, 63].

  Diagrammatic algebras are also important in categorification. After the introduction of the diagrammatic version of the KLR algebra [33] (see also [56]) they have become quite popular, and might admit handlebody extensions. For example, alongside with KLR algebras Webster's tensor product algebras [65], algebras related to Verma categorifications [38, 43, 50, 51], Soergel diagrammatics [16], potentially admit handlebody versions, just to name a few.

  We also expect these handlebody diagram algebras to have "nice" sandwich cellular bases.

- **Categories instead of algebras.** All of our algebras and concepts under study also have appropriate categorical versions.

  For example, it should be fairly straightforward to generalize our discussion of Section 2 to cellular categories [15, 67]. However, let us mention that a reason why we have not touched upon categorical versions of our handlebody diagram algebras is that these do not form monoidal categories for $g > 0$ (at least not in any reasonable sense as far as we are aware), but rather module categories. We think this deserves a thorough treatment, following for example [26] or [59].

1C. How to read this paper

Section 2 explains our generalization of cellularity and is independent of the rest. It can be easily skipped during a first reading. Section 3 treats handlebody braid and Coxeter groups and is fundamental for all sections following it. Moreover, to avoid too much repetition, we decided to construct the remaining sections assuming the reader knows Section 4, in which we define handlebody Temperley–Lieb and blob algebras. So this section is mandatory if one wants to read either Section 5, the BMW part, or Section 6, the Hecke part.

2. A generalization of cellularity

Let $\mathbb{K}$ be a unital, commutative, Noetherian domain, e.g. the integers or a field, or polynomial rings over these. Everything in this paper is linear over $\mathbb{K}$. In particular, algebras are $\mathbb{K}$-algebras.

2A. Inflation by algebras

We start by defining sandwich cellular algebras.

Remark 2.1. Our discussion below is motivated by Green's theory of cells [23], often called Green's relations, and the Clifford–Munn–Ponizovskiĭ theorem (see e.g. [18] for a modern formulation). In fact, we have borrowed part of the terminology from the literature on semigroups.

The following generalization of the notion of cellularity from [21] is well known to experts, see e.g. [36] or [25] for basis-free formulations. Nevertheless, we will state this generalization and some consequences of it.

Definition 2.2. A *sandwich cellular algebra* over $\mathbb{K}$ is an associative, unital algebra A together with a *sandwich cellular datum*, that is,

- a partial ordered set $\Lambda = (\Lambda, \leq_\Lambda)$ (we also write $<_\Lambda$, etc., having the usual meaning);
- finite sets M_λ (bottom) and N_λ (top) for all $\lambda \in \Lambda$;
- an algebra $\mathbb{S}_\lambda$ and a fixed basis B_λ of it for all $\lambda \in \Lambda$;
- a $\mathbb{K}$-basis $\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D \in M_\lambda, U \in N_\lambda, b \in \mathrm{B}_\lambda\}$ of A;

such that we have

(1) for all $x \in \mathrm{A}$, there exist scalars $r(S, D) \in R$ that do not depend on U or on b, such that

$$xc^\lambda_{D,b,U} \equiv \sum_{S \in M_\lambda, a \in \mathrm{B}_\lambda} r(S, D) \cdot c^\lambda_{S,a,U} \pmod{\mathrm{A}^{<_\Lambda \lambda}}, \tag{2.1}$$

where $\mathrm{A}^{<_\Lambda \lambda}$ is the $\mathbb{K}$-submodule of A spanned by the set $\{c^\mu_{D,b,U} \mid \mu \in \Lambda, \mu <_\Lambda \lambda, D \in M_\mu, U \in N_\mu, b \in \mathrm{B}_\mu\}$. We also have a similar condition for right multiplication.

(2) Let $\mathrm{A}(\lambda) = \mathrm{A}^{\leq_\Lambda \lambda}/\mathrm{A}^{<_\Lambda \lambda}$, where $\mathrm{A}^{\leq_\Lambda \lambda}$ is the $\mathbb{K}$-submodule of A spanned by the set $\{c^\mu_{D,b,U} \mid \mu \in \Lambda, \mu \leq_\Lambda \lambda, D \in M_\mu, U \in N_\mu, b \in \mathrm{B}_\mu\}$. Then $\mathrm{A}(\lambda)$ is isomorphic to $\Delta(\lambda) \otimes_{\mathbb{S}_\lambda} (\lambda)\Delta$ for free graded right and left $\mathbb{S}_\lambda$-modules $\Delta(\lambda)$ and $(\lambda)\Delta$, respectively.

The set $\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D \in M_\lambda, U \in N_\lambda, b \in \mathrm{B}_\lambda\}$ is called a (*sandwich*) *cellular basis*.

We very often have $M_\lambda = N_\lambda$, and we will then omit N_λ from the notation. In particular, from Subsection 2D onward we always have $M_\lambda = N_\lambda$.

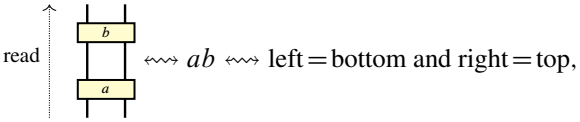
Remark 2.3. One of the advantages of the basis-focused formulation above is that Definition 2.2 works, mutatis mutandis, for relative cellular algebras as in [14] or (strictly object-adapted) cellular categories [15, 67].

We also define:

Definition 2.4. A sandwich cellular algebra A is called *involutive* if $M_\lambda = N_\lambda$ for all $\lambda \in \Lambda$ and A admits an antiinvolution $(\_)^\star \colon \mathrm{A} \to \mathrm{A}$ compatible with the cell structure. That is, $(\_)^\star$ restricts to an antiinvolution $(\_)^\star \colon \mathbb{S}_\lambda \to \mathbb{S}_\lambda$ that is a bijection on B_λ for all $\lambda \in \Lambda$, and we have

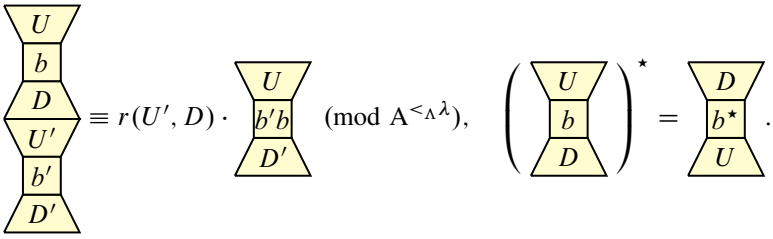
$$(c^\lambda_{D,b,U})^\star = c^\lambda_{U,b^\star,D}. \tag{2.2}$$

Convention 2.5. We will use diagrammatics from now on. Our reading conventions for diagrams are summarized by



which is bottom to top. We omit data, such as a label, if it is not of importance for the situation at hand. Moreover, we use colors in this paper, but they are non-essential and for illustration purpose only. We however still recommend to read the paper in color.

The pictures for (2.1) (with $x = c^{\lambda'}_{D',b',U'}$) and (2.2) are

(2.3)


The picture in (2.3) is an accurate description for all diagram algebras that we use in this paper. However, these pictures should be taken with care as the definition of a sandwich cellular datum is more general.

Throughout the rest of this section we write A for a sandwich cellular algebra with a fixed sandwich cell datum, using the notation from Definition 2.2. We will use the terminology of being a sandwich cellular algebra in the sense that we have fixed a sandwich cell datum. As we will see in e.g. Theorem 2.16, our focus is indeed not on whether an algebra is sandwich cellular but rather whether one can find a useful sandwich cell datum. Here is an explicit example of a not very useful sandwich cell datum.

Example 2.6. For any group G, the group element basis is a sandwich cellular basis in the sense of Definition 2.2. To see this, we let $\Lambda = \{\bullet\} = M_\bullet = N_\bullet$ be trivial, and set $c_{\bullet,b,\bullet} = b$ for $b \in \mathrm{G}$ seen as an element of $\mathbb{S}_\bullet = \mathbb{K}\mathrm{G}$. For this choice, Theorem 2.16 does not reduce the classification problem of finding the simple $\mathbb{K}$G-modules.

Example 2.7. For (important) special cases, the above has appeared in the literature. If A is involutive, then:

(1) If $\mathbb{S}_\lambda = \mathbb{K}$ for all $\lambda \in \Lambda$, then the notion of a sandwich cell datum above is the same as the classical one from [21]. Conversely, any cell datum in the sense of [21] is a sandwich cell datum in the above sense by letting $\mathbb{S}_\lambda = \mathbb{K}$ for all $\lambda \in \Lambda$.

(2) If $\mathbb{S}_\lambda = \mathbb{K}[X, X^{-1}]$ for all $\lambda \in \Lambda$, then a sandwich cellular algebras is affine cellular as in [34]. Allowing any quotient of a finite polynomial ring as $\mathbb{S}_\lambda$, the converse is also true as one can check.

Having Section 3 in mind, we note that $\mathbb{K} \cong \mathbb{K}[\pi_1(D_0)]$ and $\mathbb{K}[X, X^{-1}] \cong \mathbb{K}[\pi_1(D_1)]$, where $\pi_1(D_0)$ respectively $\pi_1(D_1)$ are the fundamental groups of a disc D_0 or a punctured disc D_1.

Example 2.8. The notion of being (involutive) sandwich cellular is a strict generalization of being cellular. An easy, albeit silly, example is to take $\Lambda = \{\bullet\} = M_\bullet = N_\bullet$ and $\mathbb{S}_\bullet$ to be a non-cellular algebra. As an explicit example, consider the set of upper triangular 2×2 matrices over $\mathbb{K}$, and view it as a semigroup so that $\{c^\bullet_{\bullet,b_i,\bullet} = b_i\}_i$ is the semigroup basis of the semigroup ring $\mathbb{S}_\bullet$. See [14] for several examples of non-cellular algebras which one could take as $\mathbb{S}_\bullet$.

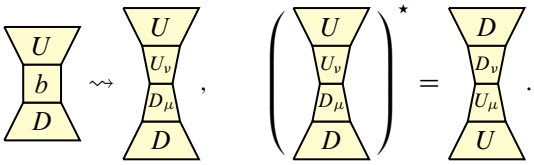
The comparison of sandwich cellular to cellular algebras is:

Proposition 2.9. *An involutive sandwich cellular algebra* A *such that all* $\mathbb{S}_\lambda$ *are cellular (with the same antiinvolution* $(\_)^\star$*) is cellular with a refined sandwich cell datum. Conversely, if at least one* $\mathbb{S}_\lambda$ *is non-cellular, then* A *is non-cellular.*

Note that an algebra can be sandwich cellular without the $\mathbb{S}_\lambda$ being cellular, cf. Example 2.8.

Proof. The trick is to use the sandwich cell datum of the $\mathbb{S}_\lambda$ (let us fix any such datum compatible with $(\_)^\star$) to make Λ finer. The picture is

(2.4)



Precisely, we define $\Lambda' = \{(\lambda, \mu) \mid \lambda \in \Lambda, \mu \in \Lambda_\lambda\}$ with Λ_λ being the poset associated to $\mathbb{S}_\lambda$. The order on Λ' is $(\lambda, \mu) \leq_{\Lambda'} (\lambda', \mu')$ if $(\lambda <_\Lambda \lambda')$ or $(\lambda = \lambda'$ and $\mu \leq_{\Lambda_\lambda} \mu')$. The sets $M_{(\lambda,\mu)}$ are now tuples (D, D_μ) with $D \in M_\lambda$ and $D_\mu \in M_\mu$, while the basis elements are $c^{(\lambda,\mu)}_{(D,D_\mu),(U,U_v)}$, defined as in (2.4), where we also indicated the antiinvolution. By construction, this is a cell datum for A in the sense of [21].

For the converse one can apply (or rather copy) Sections 3 and 4 of [36]: if one inflates along a non-cellular algebra, then the result can not be cellular. ∎

2B. Cell modules

The theory of cellular algebras is particularly nice for finite-dimensional algebras. This is however not always the case in the situation we have in mind, see e.g. Section 4. Nevertheless, parts of the theory still goes through for infinite-dimensional cellular algebras,

see [21], [34] or [14]. In particular, the existence of cell modules, cells and some of their properties, as we will discuss now.

Recall from Convention 2.5 that, in pictures, left actions and left multiplications are stacking from the bottom. For each $\lambda \in \Lambda$ and $U \in N_\lambda$ we have a *left cell* $\mathscr{L}(\lambda, U)$ given by

$$\mathscr{L}(\lambda, U) = \mathbb{K}\{c^\lambda_{D,b,U} \mid D \in M_\lambda, b \in \mathrm{B}_\lambda\}, \quad x \circlearrowright c^\lambda_{D,b,U} = (2.1),$$

which we endow with the left A-module structure given above. (In this paper actions are distinguished from multiplications by using the symbols $\circlearrowright$, respectively $\circlearrowleft$.) In pictures, this means acting on the bottom. There is also a *right cell* $\mathscr{R}(\lambda, D)$ defined verbatim, where the action is from the top.

Lemma 2.10. *We have the following.*

(1) *The left and right cells are* A*-modules.*

(2) *As* A*-modules,* $\mathscr{L}(\lambda, U) \cong \mathscr{L}(\lambda, U')$ *and* $\mathscr{R}(\lambda, D) \cong \mathscr{R}(\lambda, D')$ *for all* U, U' *and* D, D'.

Proof. This is immediate from the definitions. ∎

Using Lemma 2.10, we will write $\Delta(\lambda)$ and $(\lambda)\Delta$ (see also (b) of Definition 2.2) for any choice of $D \in M_\lambda, U \in N_\lambda$ (for the basis elements of these modules we omit the fixed index). In the theory of cellular algebras, these are then also called left, respectively right, cell modules, so we call them *sandwich cell modules*.

The space

$$\mathscr{H}_{\lambda,D,U} = \mathscr{R}(\lambda, D) \otimes_{\mathrm{A}} \mathscr{L}(\lambda, U)$$

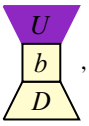
is called an $\mathscr{H}$*-cell*. Moreover, the space

$$\mathscr{J}_\lambda = \mathscr{L}(\lambda, U) \otimes_{\mathbb{S}_\lambda} \mathscr{R}(\lambda, D)$$

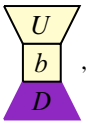
is called a two-sided cell or $\mathscr{J}$*-cell*. As free $\mathbb{K}$-modules we clearly have (cf. part (b) of Definition 2.2)

(2.5)

$$\mathscr{L}(\lambda, U) \cong M_\lambda \otimes_{\mathbb{K}} \mathbb{S}_\lambda \hookleftarrow$$

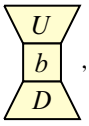


$$, \quad \mathscr{R}(\lambda, D) \cong \mathbb{S}_\lambda \otimes_{\mathbb{K}} N_\lambda \hookleftarrow$$

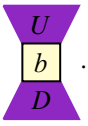


$$,$$

$$\mathscr{J}_\lambda \cong M_\lambda \otimes_{\mathbb{K}} \mathbb{S}_\lambda \otimes_{\mathbb{K}} N_\lambda \hookleftarrow$$



$$, \quad \mathscr{H}_{\lambda,D,U} \cong \mathbb{S}_\lambda \hookleftarrow$$



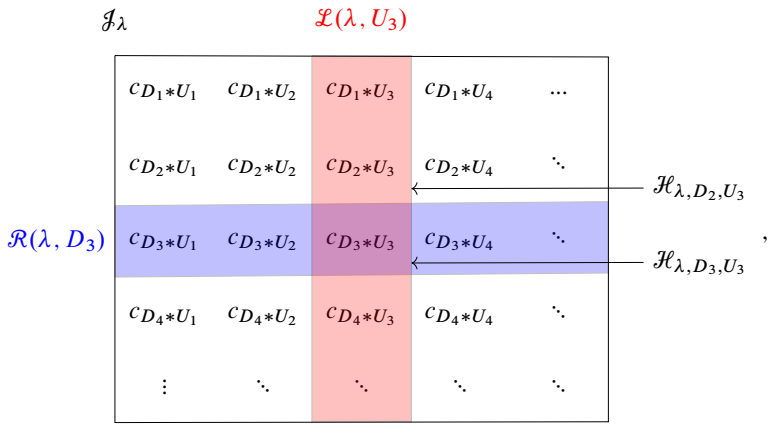
$$.$$

In (2.5) we highlighted the parts which are fixed and do not vary.

Note that $\mathscr{J}$-cells are A-A-bimodules isomorphic to $\Delta(\lambda) \otimes_{\mathbb{S}_\lambda} (\lambda)\Delta$, by definition, and thus, in general non-unital, algebras. In contrast, the $\mathscr{H}$-cells are only $\mathbb{K}$-modules, but are multiplicatively closed, as follows from (2.3) and (2.5) below, so they form, in general non-unital, subalgebras of the $\mathscr{J}$-cells.

The above can be illustrated by

(2.6)



where $*$ runs over all $b \in \mathrm{B}_\lambda$.

Lemma 2.11. *Assume that* $c^{\lambda}_{D',b',U'} c^{\lambda}_{D,b,U} = r(U', D) \equiv c^{\lambda}_{D,b'Fb,U'} \pmod{\mathrm{A}^{<_\Lambda \lambda}}$ *for some* $F \in \mathbb{S}_\lambda$. *If* $r(U', D)$ *is invertible in* $\mathbb{K}$, *then* $\mathcal{H}_{\lambda,D,U} \cong \mathbb{S}_\lambda$ *as algebras.*

Proof. Note that $\mathcal{H}_{\lambda,D,U}$ has a $\mathbb{K}$-basis given by $\{c^{\lambda}_{D,b,U} \mid b \in \mathrm{B}_\lambda\}$. For $F(U, D) = 1$ we have

$$c^{\lambda}_{D,b,U} c^{\lambda}_{D,b',U} = r(U, D) \cdot c^{\lambda}_{D,bb',U},$$

and hence we can scale everything by $1/r(U, D)$ to get the result. For $F(U, D)$ being invertible we calculate

$$c^{\lambda}_{D,bF(U,D)^{-1},U} c^{\lambda}_{D,b'F(U,D)^{-1},U} = r(U, D) \cdot c^{\lambda}_{D,bb'F(U,D)^{-1},U},$$

so the map $b \mapsto bF(U, D)^{-1}$ induces an isomorphism upon division by $r(U, D)$. ∎

Lemma 2.12. *The concatenation multiplication on the* A-A*-bimodule* $\Delta(\lambda) \otimes_{\mathbb{S}_\lambda} (\lambda)\Delta$ *is determined by a bilinear map*

$$\phi^\lambda \colon (\lambda)\Delta \otimes_{\mathrm{A}} \Delta(\lambda) \to \mathbb{S}_\lambda$$

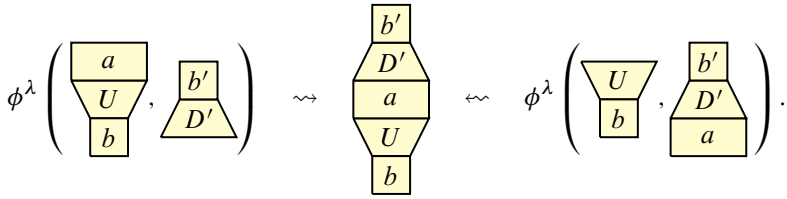
which satisfies, for all $a \in \mathrm{A}$, *that*

$$\phi^\lambda(x, y \circlearrowleft a) = \phi^\lambda(a \circlearrowright x, y).$$

It gives rise to a symmetric bilinear form ϕ^λ *on* $\Delta(\lambda)$. *This symmetric bilinear form extends to a symmetric bilinear form* ϕ^λ_K *on* $\Delta(\lambda, K) = \Delta(\lambda) \otimes_{\mathbb{S}_\lambda} K$ *for any simple* $\mathbb{S}_\lambda$*-module* K.

Proof. In general this follows from a standard lemma in ring theory, see e.g. Lemma 2 in [25]. In the restricted setting of (2.3) one has a nice diagrammatic description: the

map ϕ^λ is clearly bilinear, as it is just the multiplication in $\mathcal{J}_\lambda$. The remaining claim can be illustrated by



The extension to K is then just $\phi_K^\lambda = \phi^\lambda \otimes_{\mathbb{S}_\lambda} \mathrm{id}_K$, having the same properties. ∎

Denote by $\mathrm{rad}(\lambda, K)$ the radical of the bilinear form ϕ_K^λ defined as follows. Let $\bar{\phi}_K^\lambda$ be the associated map from $\Delta(\lambda) \otimes_{\mathbb{S}_\lambda} K \to \mathrm{Hom}_{\mathbb{K}}\big((\lambda)\Delta, \mathbb{K}\big) \otimes_{\mathbb{S}_\lambda} K$. Then the radical $\mathrm{rad}(\lambda, K)$ is defined as the kernel of this map.

Lemma 2.13. *Let $\mathbb{K}$ be a field and K be a simple $\mathbb{S}_\lambda$-module. If ϕ_K^λ is not constant zero, then $L(\lambda, K) = \Delta(\lambda, K)/\mathrm{rad}(\lambda, K)$ is the simple head of $\Delta(\lambda, K)$.*

Proof. The same arguments as in Section 3A of [14] work: any $z \in \Delta(\lambda, K) \setminus \mathrm{rad}(\lambda, K)$ is a generator of $\Delta(\lambda, K)$, while $\mathrm{rad}(\lambda, K)$ is a submodule of $\Delta(\lambda, K)$ by Lemma 2.12. Thus, $L(\lambda, K)$ is a well-defined and simple submodule. It is the head of $\Delta(\lambda, K)$ since $\mathrm{rad}(\lambda, K)$ equals the representation-theoretical radical because all $z \in \Delta(\lambda, K) \setminus \mathrm{rad}(\lambda, K)$ generate. ∎

2C. Classification of simple modules

For an A-module M, let $\mathrm{Ann}_{\mathrm{A}}(M) = \{a \in A \mid a \circlearrowright M = 0\}$ be the annihilator.

Definition 2.14. An *apex* of an A-module M is a $\lambda \in \Lambda$ such that $\mathrm{Ann}_{\mathrm{A}}(M) = \mathcal{J}_{>_\Lambda \lambda} = \bigcup_{\mu >_\Lambda \lambda} \mathcal{J}_\mu$ and ϕ^λ is not constant zero.

Recall that in the setting of non-unital algebras simple modules are defined using the usual assumption of having no non-trivial submodules but one also additionally assumes that at least one element acts not as zero.

Lemma 2.15. *We have the following.*

(a) *Every simple* A*-module L has a unique apex $\lambda \in \Lambda$.*

(b) *If $\mathbb{K}$ is a field, then the simple modules $L(\lambda, K)$ from Lemma 2.13 have apex λ.*

(c) *A simple* A*-module L of apex λ is a simple $\mathcal{J}_\lambda$-module. Conversely, every simple $\mathcal{J}_\lambda$-module L is a simple* A*-module L with apex λ, by inflation.*

Proof. One can reformulate e.g. Corollary 3.2 in [34] to get the claimed results. (Here we use that $\mathbb{K}$ is Noetherian and a domain.) Precisely:

(a) Since $\mathrm{A} \circlearrowright L \neq 0$, there is a $\leq_\Lambda$-minimal λ such that $\mathcal{J}_{\geq_\Lambda \lambda} = \bigcup_{\mu \geq_\Lambda \lambda} \mathcal{J}_\mu$ is not contained in $\mathrm{Ann}_{\mathrm{A}}(M)$. It is not hard to see that $\mathrm{Ann}_{\mathrm{A}}(M)$ is a maximal left ideal of $\mathcal{J}_{\geq_\Lambda \lambda} = \bigcup_{\mu \geq_\Lambda \lambda} \mathcal{J}_\mu$, so it has to be $\mathcal{J}_{>_\Lambda \lambda}$. In particular, $\mathcal{J}_\lambda \circlearrowright L \neq 0$ and actually

$\mathcal{J}_\lambda \circlearrowright L = L$. This can only happen if some linear combination of the $c^\lambda_{D,b,U}$ is an idempotent.

(b) By construction.

(c) By (a), all elements bigger than λ annihilate L which together with the partial ordering implies that $\mathcal{J}_\lambda \circlearrowright L$, by the same formulas, and this action is not zero. Since $\mathrm{Ann}_\mathrm{A}(L)$ is maximal, it follows that $L \cong \mathcal{J}_{\geq_\Lambda \lambda}/\mathrm{Ann}_\mathrm{A}(L)$ stays simple as a $\mathcal{J}_\lambda$-module. Conversely, take a simple $\mathcal{J}_\lambda$-module L and inflate it to a $\mathcal{J}_{\geq_\Lambda \lambda}$-module such that its annihilator is $\mathcal{J}_{>_\Lambda \lambda}$. Note that $\mathcal{J}_{\geq_\Lambda \lambda}$ is an ideal in A, so $\mathrm{A} \circlearrowright L$. Then the same arguments as in Lemma 3.1 of [34] imply that L stays simple. ∎

We get an analog of the Clifford–Munn–Ponizovskiĭ theorem (cf. Remark 2.1).

Theorem 2.16. *Let $\mathbb{K}$ be a field.*

(a) *$\lambda \in \Lambda$ is an apex if and only if the form ϕ^λ is not constant zero if and only if the form ϕ^λ_K is not constant zero for any simple $\mathbb{S}_\lambda$-module K. If the assumptions of Lemma 2.11 hold, then $\lambda \in \Lambda$ is an apex if and only if $r(U, D) \neq 0$ for some $D \in M_\lambda, U \in N_\lambda$.*

(b) *Assume that $\mathbb{S}_\lambda$ is unital and Artinian. For a fixed apex $\lambda \in \Lambda$, the simple* A*-modules of that apex are parameterized by simple modules of $\mathbb{S}_\lambda$. In other words, we have*

$$\{\textit{simple } \mathrm{A}\textit{-modules with apex } \lambda\} \overset{1:1}{\longleftrightarrow} \{\textit{simple } \mathbb{S}_\lambda\textit{-modules}\}.$$

Under this bijection, the simple A*-module $L(\lambda, K)$ associated to the simple $\mathbb{S}_\lambda$-module K is the head of $\Delta(\lambda, K)$.*

(c) *Assume that the assumptions of Lemma 2.11 hold. For a fixed apex $\lambda \in \Lambda$, there exists $\mathcal{H}_{\lambda,D,U}$ such that there is a $1:1$-correspondence*

$$\{\textit{simple } \mathrm{A}\textit{-modules with apex } \lambda\} \overset{1:1}{\longleftrightarrow} \left\{\textit{simple } \mathcal{H}_{\lambda,D,U}\textit{-modules}\right\}.$$

Under this bijection, the simple A*-module $L(\lambda, K)$ for the simple $\mathcal{H}_{\lambda,D,U}$-module K is the head of the induced module* $\mathrm{Ind}^\mathrm{A}_{\mathcal{H}_{\lambda,D,U}}(K)$.

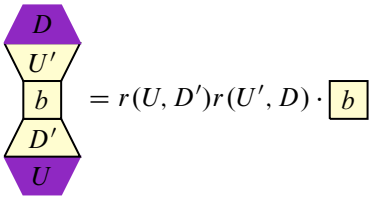
Note that (c) is not a special case of (b) as the sandwiched algebras in (c) need not be unital and Artinian. This is only relevant in the infinite-dimensional world.

Proof. (a) Clearly, ϕ^λ is not constant zero if and only if ϕ^λ_K is not constant zero. Moreover, ϕ^λ being not constant zero implies that $L(\lambda, K)$ exists, see Lemma 2.13, and has apex λ by Lemma 2.15. The converse follows by the definition of an apex.

(c) First, we choose $\mathcal{H}_{\lambda,D,U}$ such that it contains an idempotent $e = \frac{1}{r(U,D)} \cdot c^\lambda_{D,1,U}$, which we can do by (a) and the calculation

$$\frac{1}{r(U,D)} \cdot c^\lambda_{D,1,U} \frac{1}{r(U,D)} \cdot c^\lambda_{D,1,U} = \frac{1}{r(U,D)} \cdot c^\lambda_{D,1,U}.$$

By part (c) of Lemma 2.15, we can reduce the question to matching simple $\mathcal{J}_\lambda$-modules and simple $\mathcal{H}_{\lambda,D,U}$-modules. The picture



shows that $\mathcal{H}_{\lambda,D,U} \cong e\mathcal{J}_\lambda e$ (note that at least $r(U, D)$ is invertible, so $e\mathcal{J}_\lambda e \neq 0$). By a classical theorem of Green, see e.g. Lemma 6 in [18], it remains to show that simple $\mathcal{J}_\lambda$-modules are not annihilated by e. To see this we observe that any two pseudo-idempotents of the form $c^\lambda_{D,1,U}$ are related by appropriate conjugation. That is, for $e = 1/r(U, D) \cdot c^\lambda_{D,1,U}$ and $e' = 1/r(U', D') \cdot c^\lambda_{D',1,U'}$ we have

$$r(U, D)\, r(U', D') \cdot e' = c^\lambda_{D',1,U}\, e\, c^\lambda_{D,1,U'},$$

and all appearing scalars are non-zero, and thus invertible. Hence, e annihilates a simple $\mathcal{J}_\lambda$-module if and only if e' does. All other $\mathcal{H}$-cells do not contain idempotents, so a simple $\mathcal{J}_\lambda$-module can not be annihilated by any e in $\mathcal{J}_\lambda$.

(b) The proof is not much different from the one in (c), and follows by well-established arguments in the theory of cellular algebras, see e.g. [3, 14, 21, 25, 34]. However, the general proof requires the sandwiched algebras to be unital and Artinian. ∎

Remark 2.17. Note a crucial difference to the case $\mathbb{S}_\lambda = \mathbb{K}$, which is (up to having an involution) the cellular case: one apex can have any number of simples associated to it.

2D. The Brauer algebra as an example

Let $\mathrm{Br}_n(\mathsf{c})$ be the *Brauer algebra* in n-strands with circle evaluation parameter $\mathsf{c} \in \mathbb{K}$. The reader unfamiliar with the Brauer algebra is referred to e.g. [21], Section 4. Alternatively, take $g = 0$ in Section 5 below. Let also S_λ denote the symmetric group in λ strands (or on $\{1, \dots, \lambda\}$).

Remark 2.18. The construction we present below is not new, see e.g. [17] or [37]. However, our exposition is new and might be of some use.

The Brauer algebra has a well-known diagrammatic description given by *perfect matchings of* $2n$ *points*, and typical *Brauer diagrams* for $n = 4$ are

We have also illustrated the antiinvolution $(\_)^\star$ on $\mathrm{Br}_n(\mathsf{c})$ given by vertical mirroring. Note that Brauer diagrams also make sense in a categorical setting, meaning with a different number of bottom and top points.

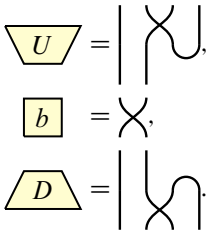
It is known that the Brauer algebra is cellular, see Section 4 in [21] or Section 5 in [3]. To make $\mathrm{Br}_n(\mathsf{c})$ cellular one needs $\mathcal{H}$-cells of size one, which is achieved in Section 4

of [21] by using that $\mathbb{K}\mathrm{S}_n$ is a subalgebra of $\mathrm{Br}_n(\mathtt{c})$. Then they work with the Kazhdan–Lusztig basis of $\mathbb{K}\mathrm{S}_n$. As we will describe now, this is not necessary if one wants to parameterize simple modules.

We let $\Lambda = (\{n, n-2, n-4, ...\}, \leq_{\mathbb{N}}) \subset \mathbb{N}$ with the usual partial order $\leq_{\mathbb{N}}$. The $\lambda \in \Lambda$ are the *through strands* of the Brauer diagrams, that is, we let the set M_λ consist of all Brauer diagrams from n bottom points to λ top points. These are the diagrams of the form D, U below. Moreover, we let $\mathbb{S}_\lambda = \mathbb{K}\mathrm{S}_\lambda$ with the group element basis B_λ. As our $\mathbb{K}$-basis, we choose

$$\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in \mathrm{B}_\lambda\}.$$

The picture for $n = 4$ and $\lambda = 2$ is:

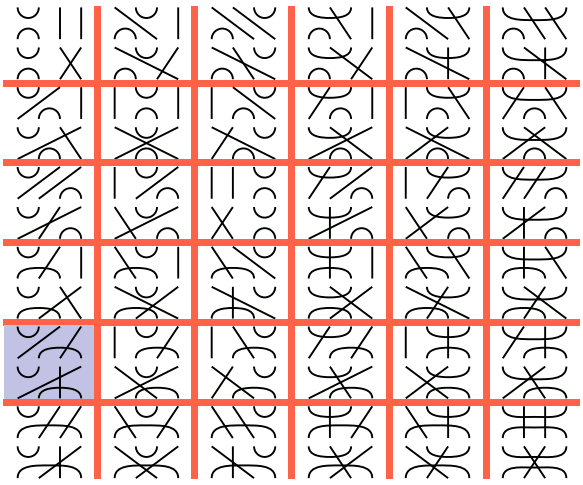


That is, we divide a Brauer diagram into a diagram only containing caps and crossings, a diagram only containing cups and crossings, and a part only containing crossings.

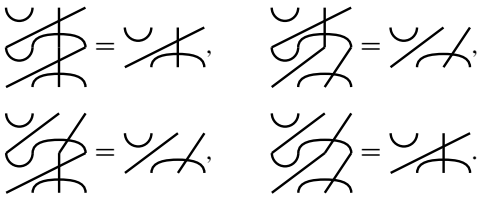
Proposition 2.19. *The above defines an involutive sandwich cell datum for* $\mathrm{Br}_n(\mathtt{c})$.

Proof. Identifying Brauer diagrams with immersed one-dimensional cobordisms (which is a well-known identification), all axioms are easily verified. ∎

Example 2.20. The following illustrates basis elements and the cell structure of the cell $\mathcal{J}_2$ with two through strands for $n = 4$ (using the same conventions as in (2.6)). In particular, the columns are $\mathcal{L}$-cells, the rows are $\mathcal{R}$-cells and the small boxes are $\mathcal{H}$-cells.



multiplication table of the colored box:



Each $\mathcal{H}$-cell is of size two, and the colored box is an $\mathcal{H}$-cell isomorphic to $\mathbb{K}\mathrm{S}_2$ regardless of $\mathtt{c}$. This is illustrated on the right.

We obtain the well-known classification of simple $\mathrm{Br}_n(\mathtt{c})$-modules, cf. Theorem 4.17 in [21].

Theorem 2.21. *Let $\mathbb{K}$ be a field.*

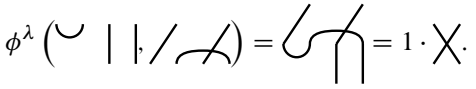
(a) *If $\mathtt{c} \neq 0$, or $\mathtt{c} = 0$ and $\lambda \neq 0$ is odd, then all $\lambda \in \Lambda$ are apexes. In the remaining case, $\mathtt{c} = 0$ and $\lambda = 0$ (this only happens if n is even), all $\lambda \in \Lambda - \{0\}$ are apexes, but $\lambda = 0$ is not an apex.*

(b) *The simple $\mathrm{Br}_n(\mathtt{c})$-modules of apex $\lambda \in \Lambda$ are parameterized by simple $\mathbb{K}\mathrm{S}_\lambda$-modules.*

(c) *The simple $\mathrm{Br}_n(\mathtt{c})$-modules of apex $\lambda \in \Lambda$ can be constructed as the simple heads of $\mathrm{Ind}_{\mathbb{K}\mathrm{S}_\lambda}^{\mathrm{Br}_n(\mathtt{c})}(K)$, where K runs over (equivalence classes of) simple $\mathbb{K}\mathrm{S}_\lambda$-modules.*

Proof. We apply Theorem 2.16 (c) together with the following observations.

Firstly, we are clearly in the situation of Lemma 2.11 for the Brauer algebras. Second, it is easy to see that the $\mathcal{H}$-cells are of a similar form as in Example 2.20, and, if $\mathtt{c} \neq 0$, then all $\mathcal{H}$-cells are isomorphic to $\mathbb{K}\mathrm{S}_\lambda$. For $\mathtt{c} = 0$ and $\lambda \neq 0$, and any two half-diagrams we can find an element such that their pairing is one, by straightening cups-caps. Here is an example that easily generalizes:



This trick works unless $\lambda = 0$, which is clearly degenerate if $\mathtt{c} = 0$. ∎

Remark 2.22. The same strategy works, mutatis mutandis, for the oriented (or walled) Brauer algebra, other diagram algebras in the same spirit, e.g. partition algebras, and the quantum versions of these diagram algebras such as the Birman–Murakami–Wenzl algebra (we will treat this case for higher genus in Section 5 below). We leave the details to the interested reader.

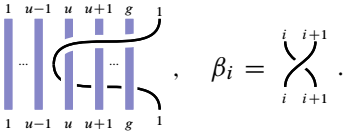
3. Handlebody braid and Coxeter groups

Throughout, we fix the *genus* $g \in \mathbb{N}$ as well as the *number of strands* $n \in \mathbb{N}_{>0}$. As a general conventions, all notions involving g are vacuous if $g = 0$, and similarly all notions involving n are vacuous for $n = 1$.

3A. Handlebody braid diagrams

In this section we consider *handlebody braid diagrams* (in n strands and of genus g). These diagrams are similar to classical braid diagrams with $g+n$ strands in the following sense. We let

(3.1)



Here the numbers indicate the corresponding positions, reading left to right. We have *usual strands*, illustrated in black, and *core strands*, illustrated thick and blue-grayish. We note that all of our diagrams have g core strands and n usual strands, but we tend to illustrate local pictures, as we already did above. The elements of the form τ_u and their inverses are called *positive coils* and *negative coils*, respectively. We also say *coils* for short.

Definition 3.1. We let the *handlebody braid group* (in n strands and of genus g) $\mathrm{B}_{g,n}$ be the group generated by $\{\tau_u, \beta_i \mid 1 \le u \le g, 1 \le i \le n-1\}$ modulo

$$\beta_i\beta_j = \beta_j\beta_i \quad \text{if } |i-j|>1, \quad \beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j \quad \text{if } |i-j|=1, \tag{3.2}$$

$$\tau_u\beta_i = \beta_i\tau_u \quad \text{if } i>1, \quad \tau_v(\beta_1\tau_u\beta_1) = (\beta_1\tau_u\beta_1)\tau_v \quad \text{if } u \le v. \tag{3.3}$$

We think of $\mathrm{B}_{g,n}$ as a handlebody generalization of the extended affine braid group of type A. Note that is $\mathrm{B}_{g,n}$ not attached to a Coxeter group in any straightforward way, see e.g. Remark 4 in [41] and (1-7) in [55].

Remark 3.2. Special cases of Definition 3.1 are:

(1) The case $g=0$ is the classical braid group $\mathrm{B}_n = \mathrm{B}_{0,n}$.

(2) For $g=1$, the handlebody braid group is the braid group of extended affine type A, which is also the braid group of Coxeter type C = B, see [7] or [1].

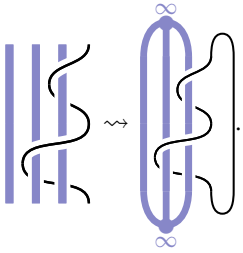
(3) A perhaps surprising fact is that the handlebody braid group for $g=2$ is isomorphic to the braid group of affine Coxeter type C, see [1].

(4) To the best of our knowledge, the case $g>2$ was first studied in [64], and then further in [28].

Remark 3.3. The handlebody braid group describes the configuration space of a disk with g punctures [28,41,64]. Moreover, after taking an appropriate version of an Alexander closure, as explained e.g. in Theorem 2 of [28] or Section 2 of [55] and illustrated in (3.4), these braid groups give an algebraic way to study links in handlebodies.

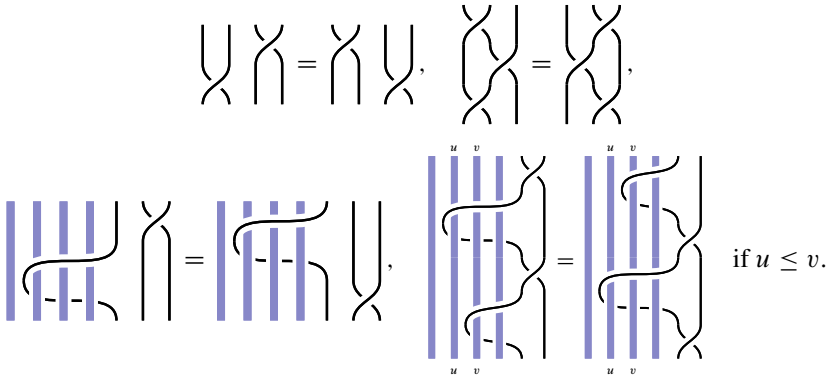
(3.4)

The closure operation merges cores at infinity:



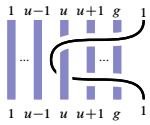
In the topological interpretation, as explained e.g. in Section 2 of [55], the core strands correspond to the cores of the handles of a handlebody. This motivates our nomenclature.

The diagrammatic interpretations of the relations (3.2) and (3.3) are



We will use these diagrammatics whenever appropriate. For completeness, and to make connection to the presentation from Theorem 2 in [28] or Section 2 in [55], for $u = 1, \dots, g$ we defined recursively $\tilde{\tau}_g$ by $\tilde{\tau}_g = \tau_g$, and for $1 \leq u < g$ we let

$$\tilde{\tau}_u = \tilde{\tau}_{u-1}^{-1} \dots \tilde{\tau}_g^{-1} \tau_u =$$



.

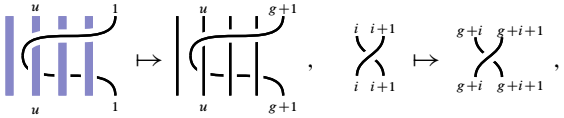
Proposition 3.4. *The handlebody braid group admits the following alternative presentation*:

$$\mathrm{B}_{g,n} \cong \left\langle \begin{matrix} \tilde{\tau}_u,\ u = 1, \dots, g, \\ \beta_i,\ i = 1, \dots, n-1 \end{matrix} \middle| \begin{matrix} \beta_i\beta_j = \beta_j\beta_i \text{ if } |i-j| > 1, \quad \beta_i\beta_j\beta_i = \beta_j\beta_i\beta_j \text{ if } |i-j| = 1, \\ \tilde{\tau}_u\beta_i = \beta_i\tilde{\tau}_u \text{ if } i > 1, \quad \tilde{\tau}_u\beta_1\tilde{\tau}_u\beta_1 = \beta_1\tilde{\tau}_u\beta_1\tilde{\tau}_u, \\ \tilde{\tau}_v(\beta_1\tilde{\tau}_u\beta_1^{-1}) = (\beta_1\tilde{\tau}_u\beta_1^{-1})\tilde{\tau}_v \text{ if } u < v \end{matrix} \right\rangle.$$

Proof. A pleasant exercise (see also e.g. Section 5 in [28]). ∎

The following allows us to use topological arguments and is used several times.

Proposition 3.5. *The rule*



defines an injective group homomorphism $\iota_{g,n} \colon \mathrm{B}_{g,n} \hookrightarrow \mathrm{B}_{g+n}$.

Proof. Using Proposition 3.4, this is Theorem 1 in [64] and Section 5 of [41]. ∎

We use the presentation without the tildes in this paper, but the other presentation can also be chosen, if preferred.

3B. Handlebody Coxeter groups

The appropriate Coxeter groups in this setup are the following.

Definition 3.6. We let the *handlebody Coxeter group* (in n strands and of genus g) $\mathrm{W}_{g,n}$ be the quotient group of $\mathrm{B}_{g,n}$ by the relations

$$\beta_i^2 = 1 \quad \text{for } i = 1, \dots, n-1.$$

We write t_u and s_i for the images of τ_u respectively β_i in the quotient.

Similarly as for the handlebody braid group $\mathrm{B}_{g,n}$, we think of Definition 3.6 as a handlebody generalization of the extended affine Coxeter group of type A.

The asymmetry in (3.3) vanishes and we have the defining relations:

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad \text{if } |i-j| > 1, \quad s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1, \tag{3.5}$$

$$t_u s_i = s_i t_u \quad \text{if } i > 1, \quad t_v(s_1 t_u s_1) = (s_1 t_u s_1) t_v \quad \forall u, v. \tag{3.6}$$

By (3.5), we see that we have an embedding of groups $\mathrm{S}_n \cong \mathrm{W}_{0,n} \hookrightarrow \mathrm{W}_{g,n}$ by identifying the simple transpositions with the s_i. Moreover, the t_u span a free group $\mathrm{F}_g \cong \mathrm{W}_{g,1}$, and F_g is thus also a subgroup of $\mathrm{W}_{g,n}$. We use both below. Note also that F_g, and thus $\mathrm{W}_{g,n}$, is of infinite order unless $g = 0$. Hence, the analog of Proposition 3.5 does not hold for $\mathrm{W}_{g,n}$.

Remark 3.7. Special cases of Definition 3.6 are:

(1) The case $g = 0$ is the symmetric group S_n.

(2) For $g = 1$, the handlebody Coxeter group is not the Coxeter group of type C=B, but rather the extended affine Coxeter (Weyl) group of type A.

For the further study of $\mathrm{W}_{g,n}$, we use $\mathbb{Z}_{\mathsf{z},Y} = \mathbb{Z}[\mathsf{z}^{\pm 1}, Y_1, \dots, Y_g]$ as our ground ring.

Definition 3.8. We define a (right) action $\mathbb{Z}_{\mathsf{z},Y}[X_1, \dots, X_n] \circlearrowleft \mathrm{W}_{g,n}$ by:

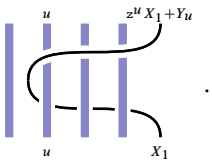
(1) The generators s_i act by the permutation action of S_n.

(2) The generators t_u act by

$$X_i \circlearrowleft t_u = \begin{cases} \mathsf{z}^u X_1 + Y_u & \text{if } i = 1, \\ X_i & \text{otherwise,} \end{cases}$$

$$X_i \circlearrowleft t_u^{-1} = \begin{cases} \mathsf{z}^{-u} X_1 - \mathsf{z}^{-u} Y_u & \text{if } i = 1, \\ X_i & \text{otherwise.} \end{cases} \tag{3.7}$$

The pictorial version of the action (3.7) is



Remark 3.9. Special cases of Definition 3.8 are:

(1) The case $g = 0$ and $\mathtt{z} = 1$ is the permutation representation of S_n on the polynomial ring $\mathbb{Z}[X_1, \dots, X_n]$.

(2) The case $g = 1$ and $\mathtt{z} = 1$ recovers the usual polynomial representation of the extended affine Weyl group of type A.

(3) The case $g = 1$, $\mathtt{z} = -1$ and $Y_1 = 0$ recovers the root-theoretic version of Tits' reflection representation of type $\mathrm{C} = \mathrm{B}$, i.e., the representation where the type A subdiagram acts by permutation and the additional generator acts as -1 on X_1.

Lemma 3.10. *The action in Definition 3.8 is well-defined.*

Proof. Note that t_u and t_u^{-1} act slightly asymmetrically, but this ensures that their actions invert each other since, e.g.,

$$\begin{aligned} X_1 \circlearrowleft t_u t_u^{-1} &= (\mathtt{z}^u X_1 + Y_u) \circlearrowleft t_u^{-1} = \mathtt{z}^u t_u^{-1}(X_1) + Y_u \\ &= \mathtt{z}^u(\mathtt{z}^{-u} X_1 - \mathtt{z}^{-u} Y_u) + Y_u = X_1. \end{aligned}$$

Moreover, by construction of the action, the only non-trivial check is that (3.6) holds. The left equality in (3.6) is immediate, and for the right we compute

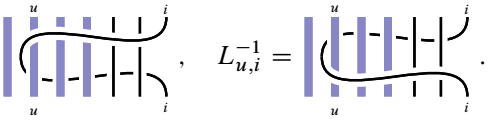
$$\begin{aligned} X_1 \circlearrowleft t_u s_1 t_v s_1 &= \mathtt{z}^u X_2 \circlearrowleft t_v s_1 + Y_u = \mathtt{z}^u X_2 \circlearrowleft s_1 + Y_u = \mathtt{z}^u X_1 + Y_u, \\ X_1 \circlearrowleft s_1 t_v s_1 t_u &= X_2 \circlearrowleft t_v s_1 t_u = X_2 \circlearrowleft s_1 t_u = X_1 \circlearrowleft t_u = \mathtt{z}^u X_1 + Y_u. \end{aligned}$$

Thus, on the value X_1 the actions agree. Furthermore, a very similar calculation shows that they also agree on the value X_2. For all other values the relation (3.6) holds evidently. ∎

The action in Definition 3.8 is actually faithful, and this is what we are going to show next.

Definition 3.11. Define *Jucys–Murphy elements* $L_{u,i}^{\pm 1} \in \mathrm{B}_{g,n}$ as follows. We let $L_{u,1}^{\pm 1} = \tau_u^{\pm 1}$. For $i > 1$, we define $L_{u,i}^{\pm 1} = \beta_{i-1}^{\pm 1} \dots \beta_1^{\pm 1} \tau_u^{\pm 1} \beta_1^{\pm 1} \dots \beta_{i-1}^{\pm 1}$, or diagrammatically,

(3.8)

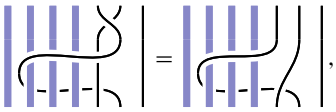


Lemma 3.12. *We have*

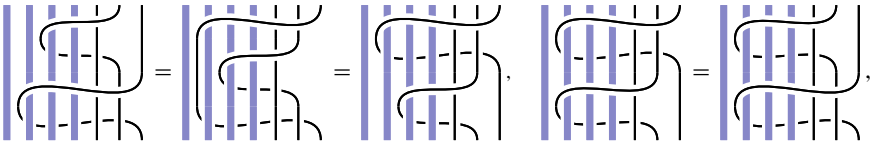
$$\begin{gathered} L_{u,i}^{\pm 1} L_{u,i}^{\mp 1} = 1, \quad \beta_j L_{u,i}^{\pm 1} = L_{u,i}^{\pm 1} \beta_j \quad \text{if } i - 1, i \neq j, \\ \beta_{i-1}^{-1} L_{u,i} = L_{u,i-1} \beta_{i-1}, \quad \beta_i L_{u,i} = L_{u,i+1} \beta_i^{-1}, \\ \beta_{i-1} L_{u,i}^{-1} = L_{u,i-1}^{-1} \beta_{i-1}^{-1}, \quad \beta_i^{-1} L_{u,i}^{-1} = L_{u,i+1}^{-1} \beta_i, \\ L_{u,i} L_{v,j}^{\pm 1} = L_{v,j}^{\pm 1} L_{u,i} \quad \text{and} \quad L_{u,i}^{-1} L_{v,j}^{\pm 1} = L_{v,j}^{\pm 1} L_{u,i}^{-1} \quad \text{if } [v, j] \subset [u, i[, \end{gathered} \tag{3.9}$$

where $[v, j] \subset [u, i[$ *means* $u \leq v$ *and* $j < i$.

Proof. Using topological arguments, all of these are easy to verify. For example, the middle relations in (3.9) are of the form

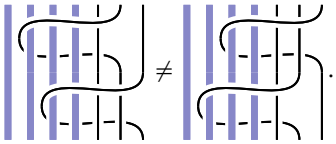


The bottom relations in (3.9) take the form



The other relations can be verified verbatim. ∎

Note that if $[v, j] \not\subset [u, i[$ in the final relation in (3.9), then the displayed elements do not commute (unless $L_{u,i}^{\pm 1} L_{u,i}^{\mp 1} = 1$ applies). For example,



Lemma 3.13. *Any word in the Jucys–Murphy elements can be ordered such that $L_{u,i}^{\pm}$ appears right=above of $L_{v,j}^{\pm}$ only if $j < i$.*

Proof. We use the final relation in (3.9) inductively: first, start with $i = n$ and pull all $L_{u,n}^{\pm}$, for all u, to the right = top, without changing their order (they form a free group). We freeze these elements in their positions and we can thus use induction on the remaining Jucys–Murphy elements as their maximal second index is $n - 1$. ∎

Let us denote the images of the Jucys–Murphy elements in $\mathrm{W}_{g,n}$ by $l_{u,i}^{\pm 1}$.

Lemma 3.14. *The set*

$$\left\{ l_{u_1,i_1}^{a_1} \dots l_{u_m,i_m}^{a_m} w \;\middle|\; \begin{array}{c} w \in \mathrm{S}_n, m \in \mathbb{N}, \boldsymbol{a} \in \mathbb{Z}^m, \\ (\boldsymbol{u}, \boldsymbol{i}) \in (\{1, \dots, g\} \times \{1, \dots, n\})^m, i_1 \le \dots \le i_m \end{array} \right\} \tag{3.10}$$

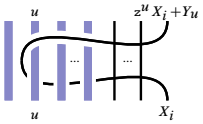
spans $\mathbb{Z}\mathrm{W}_{g,n}$.

Proof. Since $\tau_u = L_{u,1}$, words in $L_{u,i}$ and β_i can clearly generate arbitrary words in $\mathrm{B}_{g,n}$, and thus in $\mathrm{W}_{g,n}$. Further, using the relations (3.9), we see that we can always pull all $s_i \in \mathrm{W}_{g,n}$ to the right = top, since $s_i = s_i^{-1}$. Finally, with the last relation in (3.9) we can order the $L_{u,i}$ as claimed, cf. Lemma 3.13, which of course also works for the handlebody Coxeter group. ∎

Proposition 3.15. *The set in* (3.10) *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}\mathrm{W}_{g,n}$.

Proof. By Lemma 3.14, it only remains to verify that the elements of the set (3.10) are linearly independent. To this end, we first observe that we can let w be trivial since the action of S_n used in Lemma 3.14 is the faithful permutation action. Now, the only non-trivial action of $l_{u,i}$ is on X_i:

$$X_j \circlearrowleft l_{u,i} = \begin{cases} \mathsf{z}^u X_i + Y_u & \text{if } i = j, \\ X_j & \text{otherwise,} \end{cases} \quad \leftrightsquigarrow$$



.

Note that the involved strand i is only relevant for $l_{u,i}$, for all u, and its powers. Moreover, we can distinguish $l_{u,i}$ and $l_{v,i}$ by the appearing variables Y_u respectively Y_v, while their order does not matter due to (3.9). The same argument works mutatis mutandis for the inverses $l_{u,i}^{-1}$, of course. Taking all this together shows that the elements of (3.10) are linearly independent. ∎

Theorem 3.16. *The action of* $\mathrm{W}_{g,n}$ *in Definition* 3.8 *is faithful.*

Proof. Directly from the proof of Proposition 3.15. ∎

4. Handlebody Temperley–Lieb and blob algebras

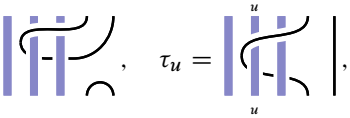
Recall that $\mathbb{K}$ denotes a unital, commutative, Noetherian domain. The example the reader should keep in mind throughout this section is $\mathbb{K} = \mathbb{Z}[\boldsymbol{c}]$, where $\boldsymbol{c} = (\mathsf{c}_\gamma)_\gamma$ is a collection of variables c_γ which are circle evaluations. Recall further that we have fixed $g \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$.

4A. Handlebody Temperley–Lieb algebras

In this section we consider non-topological *crossingless matchings of* $2n$ *points of genus* g, which are all pairings (i,k) of integers from $\{1,\dots,2n\}$ such that $i < j < k < l$ for all (i,k) and (j,l) together with a choice of a reduced word in the free group F_g for each appearing pair.

In slightly misleading pictures, cf. Remark 4.5, these are crossingless matchings of $2n$ points where usual strands can wind around the cores, but not among themselves. Here we use the same conventions as in Section 3 in the sense that all cores are to the left. For example, if $g = 3$ and $n = 2$, then

(4.1)



are examples of such crossingless matchings.

Remark 4.1. We will use a similar notation as in Section 3, e.g., τ_u for elements as illustrated on the right in (4.1).

These crossingless matchings are not allowed to have any circles (circles, by definition, are a connected components of usual strings not touching the bottom or top). But such circles, as usual for these types of algebras, could appear after concatenation. To address this we need to associate circles in such diagrams to conjugacy classes in F_g.

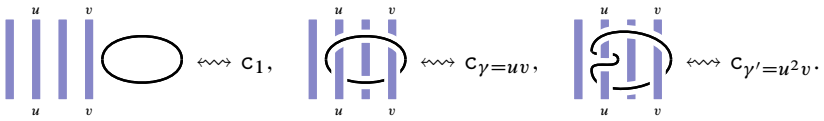
To this end, for each circle in the diagrams we will associate a word in F_g by starting somewhere generic on the circle, say the rightmost point, and read clockwise. This gives an element of F_g, keeping the identification of coils and generators of F_g in mind, associated to each circle which is well-defined up to conjugacy. We then say the circles are F_g*-colored*, meaning tuples of a circle and a representative of a conjugacy class in F_g. These circles will index our parameters momentarily.

For the following definition, we choose a set of parameters $\boldsymbol{c} = (\mathsf{c}_\gamma)_\gamma \in \mathbb{K}$, one for each F_g-colored circle γ. Thus, the parameters are constant on conjugacy classes in F_g.

Definition 4.2. The evaluation of a F_g-colored circle γ is defined to be the removal of a closed component, contributing a factor c_γ. We call this *circle evaluation*.

We could choose all c_γ to be different or equal, there is no restriction on the choice of these parameters (see however Remark 4.5 below). Note that all non-essential circles are associated to the trivial coloring $\gamma = 1$ and are evaluated to c_1.

Example 4.3. Here are a few examples:



The first word in F_g is the trivial word, the second is uv, and the third is u^2v.

To each usual strand in a crossingless matching we associate a word in F_g by starting at the rightmost boundary point of the string and read to the other one, again using the identification of coils and generators of F_g. A F_g*-colored concatenation* of crossingless matchings x and y of $2n$ points of genus g is, up to colors, a concatenation xy in the usual sense and the colors are concatenated reading bottom to top along the strings respectively starting anywhere and moving clockwise along closed components.

Definition 4.4. We let $\mathrm{TL}_{g,n}(\boldsymbol{c})$, the *handlebody Temperley–Lieb algebra* (in n strands and of genus g), be the algebra whose underlying free $\mathbb{K}$-module is the $\mathbb{K}$-linear span of all crossingless matchings of $2n$ points of genus g, and with multiplication given by F_g-colored concatenation of diagrams modulo circle evaluation.

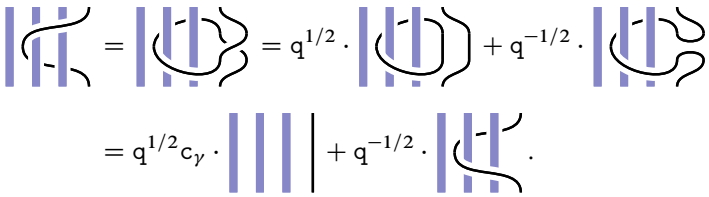
Remark 4.5. There are various ways to define Temperley–Lieb algebras beyond the classical case and some of them are not topological in nature, and the construction in Definition 4.4 is one of those that are not topological. In particular, the *Kauffman skein relation*

(4.2)

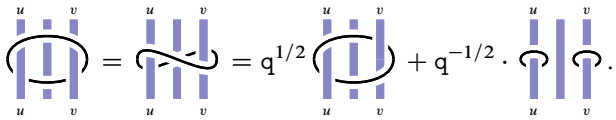

does not behave topologically in $\mathrm{TL}_{g,n}(\boldsymbol{c})$, even for appropriate choices of parameters. We will come back to a topological model of the handlebody Temperley–Lieb algebra in Subsection 4C and for now we just note that:

(1) One reason why we do not want (4.2) for the time being is that this relation implies that coils satisfy an order two relation. This follows from the calculation

(4.3)



(2) In a topological model, (4.2) also implies relations among the circle parameters $\mathtt{c}_\gamma$:



For completeness, we note an easy fact.

Lemma 4.6. *The algebra* $\mathrm{TL}_{g,n}(\boldsymbol{c})$ *is an associative, unital algebra with a* $\mathbb{K}$*-basis given by all crossingless matchings of* $2n$ *points of genus* g.

Proof. Note that the F_g-colored concatenation of closed components ensures that the circle evaluation only depends on the associated word in F_g modulo conjugation. Thus, the only claim which is not immediate is that the F_g-colored concatenation is associative. This however follows by identifying the colors by dots on the strings. ∎

Note that the algebra $\mathrm{TL}_{g,n}(\boldsymbol{c})$ is infinite-dimensional unless $g = 0$, since the group-like elements τ_u are of infinite order and span $\mathrm{F}_g \cong \mathrm{TL}_{g,1}(\boldsymbol{c}) \hookrightarrow \mathrm{TL}_{g,n}(\boldsymbol{c})$.

Remark 4.7. For low genus $\mathrm{TL}_{g,n}(\boldsymbol{c})$ is well-studied (although for $g > 0$ the precise definitions and the nomenclature vary throughout the literature):

(1) For $g = 0$, the algebra $\mathrm{TL}_{0,n}(\boldsymbol{c})$ is the Temperley–Lieb algebra in its crossingless matching definition.

(2) In case $g = 1$, there are the so-called affine Temperley–Lieb algebra or the type C = B Temperley–Lieb algebra (the name comes from the relation of this algebra to the braid group of type C = B as recalled in Remark 3.2), which were studied in many works such as e.g. [22]. The algebra $\mathrm{TL}_{1,n}(\boldsymbol{c})$ is a version of these.

(3) Similarly, for $g = 2$ there is a so-called two-boundary Temperley–Lieb algebra and an affine type C Temperley–Lieb algebra, again independently introduced in many works. The algebra $\mathrm{TL}_{2,n}(\boldsymbol{c})$ is a version of these.

Note that Temperley–Lieb algebras for $g < 2$ are sometimes assumed to satisfy a quadratic relation for the coil elements, but we do not assume that. These algebras are studied in the sections below.

Similarly as in Subsection 2D, we let $\Lambda = \big(\{n, n-2, n-4, ...\}, \leq_{\mathbb{N}}\big) \subset \mathbb{N}$ with the usual partial order. The $\lambda \in \Lambda$ are again the through strands of our diagrams, and we let $\mathbb{S}_\lambda = \mathbb{K}\mathrm{F}_g$, the group ring of the free group in g generators $\mathrm{F}_g \cong \mathrm{Br}_{g,1} \hookrightarrow \mathrm{Br}_{g,\lambda}$. We take the group element basis of F_g as the sandwich basis B_λ.

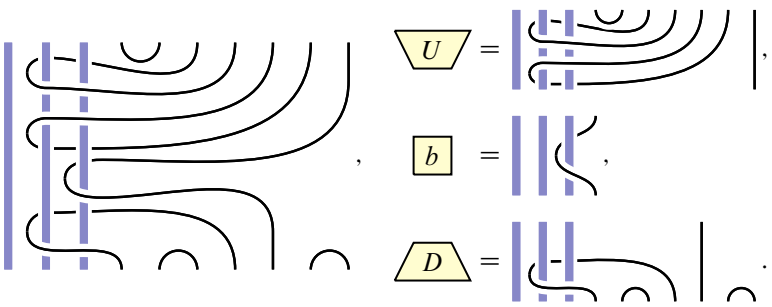
Remark 4.8. We assume that $\mathrm{Br}_{g,\lambda}$ is trivial if $\lambda = 0$. In particular, $\mathbb{K}\mathrm{F}_g \cong \mathbb{K}$ for $\lambda = 0$, since we identify F_g with a subgroup of $\mathrm{Br}_{g,\lambda}$, and the latter is trivial for $\lambda = 0$.

The construction of the basis

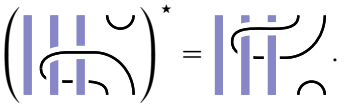
$$\{c^{\lambda}_{D,b,U} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in \mathbb{S}_\lambda\} \tag{4.4}$$

works mutatis mutandis as for the classical Temperley–Lieb algebra, having a concatenation of a cup diagram D, a cup diagram U and through strands b, but now the caps and cups are allowed to wind around the cores (but not the through strands), and we have coils of through strands in the middle. The following picture clarifies the construction:

(4.5)

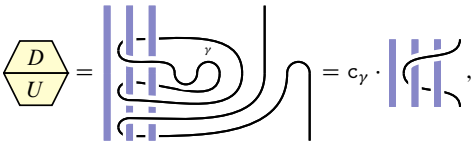


We also have the antiinvolution $(\_)^\star \colon \mathrm{TL}_{g,n}(\boldsymbol{c}) \to \mathrm{TL}_{g,n}(\boldsymbol{c})$ given by flipping pictures upside down but keeping positive coils positive and negative coils negative (that is, one only reverses the words in F_g but does not invert them), e.g.,



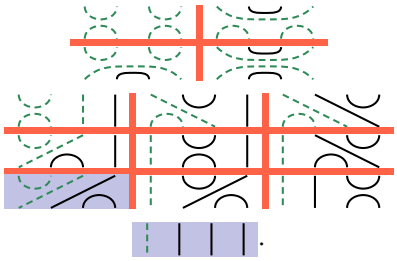
Proposition 4.9. *The above defines an involutive sandwich cell datum for* $\mathrm{TL}_{g,n}(\boldsymbol{c})$.

Proof. It is clear that one can cut any crossingless matching of $2n$ points of genus g uniquely into pieces as illustrated in (4.5), hence that (4.4) is a $\mathbb{K}$-basis follows. So it remains to verify (2.1). That the multiplication is ordered with respect to through strands follows (as for the classical Temperley–Lieb algebra) from the fact that one can not remove cups and caps, they can only be created. All the other requirements are clear by construction; one basically calculates



where we used the elements in (4.5). The property of being involutive also holds since any potential winding is allowed for the diagrams D and U, so $(\_)^\star$ is a 1:1-correspondence between down D and up U diagrams. ∎

Example 4.10. The cells of the algebra $\mathrm{TL}_{n,g}(\boldsymbol{c})$ are of infinite size, so we can not display them here. However, let us illustrate strands of $\mathrm{TL}_{4,g}(\boldsymbol{c})$ dashed (and colored) if they can wind around cores. Using the same conventions as in Example 2.20: then the cells $\mathcal{J}_4$ (bottom) to $\mathcal{J}_0$ (top), in order from smallest (bottom) to biggest (top) are given by



Here we have indicated examples of $\mathcal{H}$-cells that are isomorphic to $\mathbb{K}\mathrm{F}_g$, with F_g corresponding to the leftmost through strand, regardless of the parameters $\boldsymbol{c}$. For the top cell we can only pick up a copy of $\mathbb{K}$ if at least one of the corresponding parameters is invertible in $\mathbb{K}$.

For the following, recall that F_g is trivial by definition for $\lambda = 0$, see Remark 4.8.

Theorem 4.11. *Let $\mathbb{K}$ be a field.*

(1) *If $\boldsymbol{c} \neq 0$, or $\boldsymbol{c} = 0$ and $\lambda \neq 0$ is odd, then all $\lambda \in \Lambda$ are apexes. In the remaining case, $\boldsymbol{c} = 0$ and $\lambda = 0$ (this only happens if n is even), all $\lambda \in \Lambda - \{0\}$ are apexes, but $\lambda = 0$ is not an apex.*

(2) *The simple $\mathrm{TL}_{n,g}(\boldsymbol{c})$-modules of apex $\lambda \in \Lambda$ are parameterized by simple modules of $\mathbb{K}\mathrm{F}_g$.*

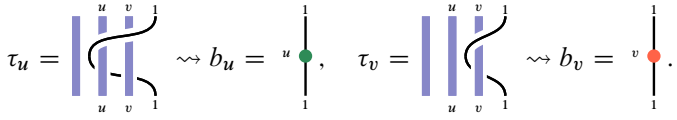
(3) *The simple $\mathrm{TL}_{n,g}(\boldsymbol{c})$-modules of apex $\lambda \in \Lambda$ can be constructed as the simple heads of $\mathrm{Ind}_{\mathbb{K}\mathrm{F}_g}^{\mathrm{TL}_{n,g}(\boldsymbol{c})}(K)$, where K runs over (equivalence classes of) simple $\mathbb{K}\mathrm{F}_g$-modules.*

Proof. Word-by-word as for the Brauer algebra, see the proof of Theorem 2.21. In particular, this is a direct application of Theorem 2.16. ∎

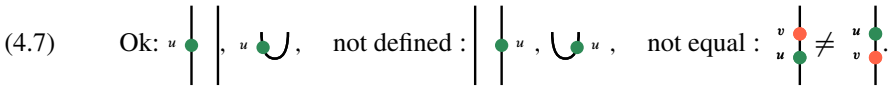
4B. Handlebody blob algebras

An often applied strategy to turn an infinite-dimensional algebra into a finite-dimensional algebra is to impose a cyclotomic condition on generators of infinite order. In our case we will impose relations on the coil generators τ_u.

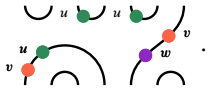
Following history, we will use a slightly different diagrammatic presentation for these algebras, namely using *blob diagrams of $2n$ points of genus g*. First, for everything not involving any core strands the diagrams stay the same. Moreover, we will use

(4.6) 

That is, we use colored *blobs* instead of coils, which clarifies the nomenclature. We denote the elements corresponding to coils by b_u. We also say that a blob labeled u has *type* u. Note that blobs are always reachable from the left by a straight line coming from $-\infty$, and move freely along strands in a vertical direction, but can not pass one another:

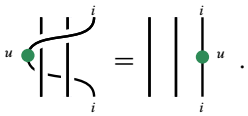
(4.7) 

An example of such a diagram is:



(Some of the blobs in this illustration are strictly speaking not reachable from the left as they are behind cups and caps when drawing a straight line. But here and throughout, to simplify illustrations, we will suppress the relevant height moves since they do not play any role.)

Remark 4.12. The reader might wonder whether (for appropriate parameters $\mathtt{q}^{\pm 1/2}$) one can not use (4.2) to define



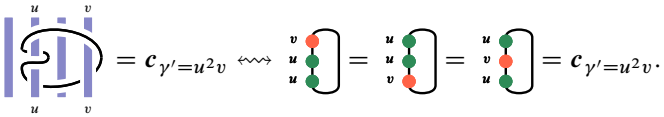
Indeed, in a topological model that is possible. However, as we will explore more carefully in Subsection 4D, these diagrams will not be topological in nature, but have some error terms. So we decided to keep blobs to the left from the start.

Using the same reading conventions as for $\mathrm{TL}_{g,n}(\boldsymbol{c})$ but counting types of blobs instead of coils, we can associated a word in F_g to usual strands and circles. As before, all parameters below will be assumed to be constant on conjugacy classes.

Fix *cyclotomic parameters* $\boldsymbol{b} = (\mathtt{b}_{u,i}) \in \mathbb{K}^{d_1+\cdots+d_g}$. We also fix $\boldsymbol{d} = (\mathtt{d}_u) \in \mathbb{N}^g$, called the *degree vector*. Note that concatenation can create circles with blobs. For each such circle γ with at most $\mathtt{d}_u - 1$ blobs of the corresponding kind we choose a parameter $\mathtt{c}_\gamma \in \mathbb{K}$, whose collection is denoted by $\boldsymbol{c} = (\mathtt{c}_\gamma)_\gamma$.

Definition 4.13. The evaluation of a closed circle γ is defined to be the removal of a closed component, contributing a factor $\mathtt{c}_\gamma$. We call this *blob circle evaluation*.

Example 4.14. The circle evaluation of the handlebody Temperley–Lieb algebra becomes blob circle evaluation, e.g.,



Note that circles with more than $\mathtt{d}_u - 1$ blobs of the corresponding kind can be evaluated by using (4.8) and the above circle evaluation, so we do not need to define their evaluation.

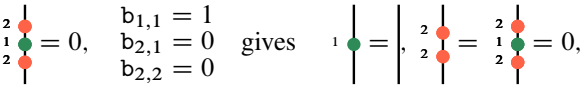
Definition 4.15. We let the (cyclotomic) *handlebody blob algebra* (in n strands and of genus g) $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ be the algebra whose underlying $\mathbb{K}$-vector space is the $\mathbb{K}$-linear span of all blob diagrams on $2n$ points of genus g, with multiplication given by concatenation of diagrams modulo blob circle evaluation, and the two-sided ideal generated by the *cyclotomic relations*

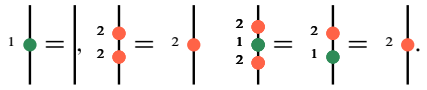
$$(b_u - \mathtt{b}_{u,1})\varpi_1(b_u - \mathtt{b}_{u,2})\varpi_2 \ldots (b_u - \mathtt{b}_{u,\mathtt{d}_u-1})\varpi_{d_u-1}(b_u - \mathtt{b}_{u,\mathtt{d}_u}) = 0, \tag{4.8}$$

where ϖ_j is any finite (potentially empty) word in b_v for $v \neq u$.

In words, any occurrence of $\mathtt{d}_u$ blobs colored g on a strand can be replaced by the corresponding expression in the expansion of (4.8).

Example 4.16. For $g = 2$, let us choose $\mathtt{d}_1 = 1$ and $\mathtt{d}_2 = 2$. We get

$\mathtt{b}_{1,1} = 0$, $\mathtt{b}_{2,1} = 0$, $\mathtt{b}_{2,2} = 0$ gives  $\mathtt{b}_{1,1} = 1$, $\mathtt{b}_{2,1} = 0$, $\mathtt{b}_{2,2} = 0$ gives 

$\mathtt{b}_{1,1} = 1$, $\mathtt{b}_{2,1} = 1$, $\mathtt{b}_{2,2} = 0$ gives 

Note that the final expression can be resolved in two ways: first by removing the blob colored 1, and then by replacing the two blobs colored 2 by one blob colored 2. Second, by applying (4.8) directly, as we did above. Both give the same result.

Remark 4.17. We have not defined $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ as a quotient of $\mathrm{TL}_{g,n}(\boldsymbol{c})$ because blobs would be invertible if we define them as the image of τ_u in the quotient and we want to include the possibility of blobs being nilpotent, cf. Example 4.16.

Note that however that for certain choices of parameters the blobs are invertible, and $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ is often a quotient of $\mathrm{TL}_{g,n}(\boldsymbol{c})$ for these parameters.

Remark 4.18. We again discuss a few instances of Definition 4.15:

(1) For $g = 0$, the blob and the Temperley–Lieb algebra coincide.

(2) The algebra $\mathrm{Bl}_{1,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ is sometimes called the (cyclotomic) blob algebra, see e.g. [45].

(3) In genus $g = 2$, there is a two-boundary blob algebra, see e.g. [12] (beware that the version of $\mathrm{Bl}_{2,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ in that paper is called a two-boundary Temperley–Lieb algebra).

For $g > 0$, the terminology in the literature is not consistent, and Temperley–Lieb algebras and blob algebras might be the same or not. For us, Definition 4.15 generalizes [45].

We need the analog of Lemma 4.6 which reads as follows.

Lemma 4.19. *The algebra* $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ *is an associative, unital algebra with a* $\mathbb{K}$*-basis given by all handlebody blob diagrams on* $2n$ *points of genus* g *where all strands have at most* $\mathtt{d}_u$ *blobs of the corresponding type.*

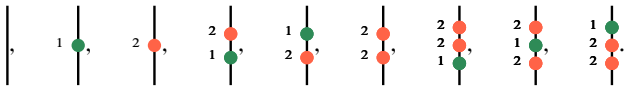
Proof. The only fact to observe is that the cyclotomic condition (4.8) ensures that it suffices to fix an evaluation for any circle whose number of blobs are bounded by the degree vector. ∎

For our fixed genus g and degree vector $\boldsymbol{d}$ we define the (corresponding) *blob numbers* using multinomial coefficients:

$$\mathtt{BN}_{g,\boldsymbol{d}} = \sum_{k\in\mathbb{N}} \sum_{\substack{0\le k_u\le \min(k,\mathtt{d}_u-1)\\ k_1+\cdots+k_g=k}} \binom{k}{k_1,\ldots,k_g}, \tag{4.9}$$

where the sums run over all $k \in \mathbb{N}$ and all $0 \le k_u \le \min(k, \mathtt{d}_u - 1)$, for $u \in \{1,\ldots,g\}$, that sum up to k. Note that this sum is finite.

Example 4.20. We have $\mathtt{BN}_{0,\boldsymbol{d}} = 1$ (by definition), $\mathtt{BN}_{1,\boldsymbol{d}} = \mathtt{d}_1$ and, for $\mathtt{d}_1 = 2$ and $\mathtt{d}_2 = 3$, we get $\mathtt{BN}_{2,\boldsymbol{d}} = \binom{0}{0,0} + \binom{1}{1,0} + \binom{1}{0,1} + \binom{2}{1,1} + \binom{2}{0,2} + \binom{3}{1,2} = 9$. Note that there are nine blob diagrams with one strand and at most one 1 blob and two 2 blobs:



In fact, we have $\mathtt{BN}_{2,\boldsymbol{d}=(\mathtt{d}_1,\mathtt{d}_2)} = \binom{\mathtt{d}_1+\mathtt{d}_2}{\mathtt{d}_1} - 1$.

Lemma 4.21. *For any $\boldsymbol{c}$, we have an isomorphism of algebras*

$$\mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,1}(\boldsymbol{c}) \cong \mathbb{K}\langle b_u \mid u \in \{1,\ldots,g\}\rangle/(4.8).$$

In particular, $\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,1}(\boldsymbol{c}) = \mathtt{BN}_{g,\boldsymbol{d}}$.

Proof. Since blobs do not satisfy any other relation than (4.8), the only claim that is not immediate is the dimension count. To see that that works, we recall that the multinomial coefficient $\binom{k}{a_1,\ldots,a_g}$ counts the appearance of $x_1^{a_1}\ldots x_g^{a_g}$ in the expansion of $(x_1 + \cdots + x_g)^k$, which is the counting problem we need to solve. Finally, note that $b_u^{d_u}$ can be expressed in terms of lower order expressions, which explains our condition on the summation. ∎

We also calculate the dimension of $\mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$.

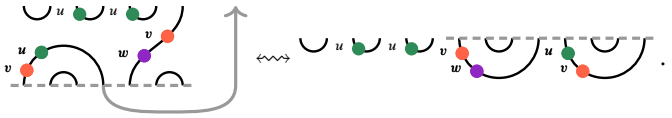
Proposition 4.22. *We have* $\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,0}(\boldsymbol{c}) = 1$, $\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,1}(\boldsymbol{c}) = \mathtt{BN}_{g,\boldsymbol{d}}$ *and*

$$\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c}) = \mathtt{BN}_{g,\boldsymbol{d}} \sum_{k\in\{2,4,6,\ldots,2n\}} C_{k-1} \dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n-k}(\boldsymbol{c}), \tag{4.10}$$

where C_{k-1} is the $(k-1)$th Catalan number.

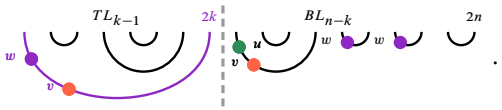
The proof of Proposition 4.22 is an inductive argument which works in quite some generality and that we learned from [62].

Proof. By Lemma 4.19, it suffices to count handlebody blob diagrams of genus g. It actually suffices to study the clapped situation:



We then argue by induction on n. The induction start is clear, so let $n > 1$. First, the strand with the leftmost point can end at position $2k$ for $k \in \mathbb{N}$, and one can divide the diagram into two parts: a part underneath it TL_{k-1} and a part to the right of it BL_{n-k}. For example,

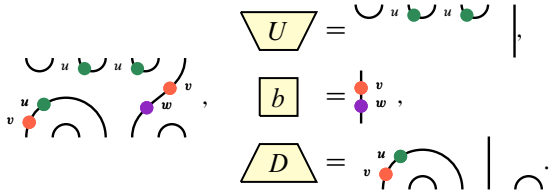
$n = 9, k = 4$:



The part underneath it, denoted TL_{k-1} above, can not carry any blobs, so the number of possible diagrams is the same as for the corresponding Temperley–Lieb algebra, which is the Catalan number appearing in (4.10). The number of possible diagrams on the right, denoted BL_{n-k} above, is the dimension of $\mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n-k}(\boldsymbol{c})$, and we get the corresponding number in (4.10). The remaining number is the number of possible blob placements on the strand with the leftmost point, see Lemma 4.21. ∎

Example 4.23. For $g = 0$ one obtains that $\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{0,n}(\boldsymbol{c}) = C_n$, which is of course expected. For $g = 1$ one can solve the recursion in (4.10) and obtains $\dim_{\mathbb{K}} \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{1,n}(\boldsymbol{c}) = \binom{2n}{n}$, the dimension of the blob algebra, cf. Lemma 5.7 in [24].

Regarding cellular structures, the same strategy as for the handlebody Temperley–Lieb algebra from Subsection 4A works. Precisely, the D part is allowed to have caps with blobs and through strands without blobs, the U part is allowed to have cups with blobs and through strands without blobs, and the middle has blobs on through strands. The picture to keep in mind is



Note that the middle part is $\mathbb{S}_0 = \mathbb{K}$ and

$$\mathbb{S}_\lambda = \mathbb{K}\mathrm{B}^{\boldsymbol{d},\boldsymbol{b}}_{g,1} = \mathbb{K}\langle b_u \mid u \in \{1, \dots, g\}\rangle/(4.8).$$

We choose the monomial basis in the b_u as our sandwich basis. As the antiinvolution we choose the map $(\_)^\star \colon \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c}) \to \mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$ that mirrors diagrams and fixes all blobs.

Proposition 4.24. *The above defines an involutive sandwich cell datum for* $\mathrm{Bl}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$.

Proof. The proof is, mutatis mutandis, as in Proposition 4.9 and omitted. ∎

The cells look similar as in Example 4.10.

Theorem 4.25. *Let $\mathbb{K}$ be a field.*

(1) *If $\boldsymbol{c} \neq 0$, or $\boldsymbol{c} = 0$ and $\lambda \neq 0$ is odd, then all $\lambda \in \Lambda$ are apexes. In the remaining case, $\boldsymbol{c} = 0$ and $\lambda = 0$ (this only happens if n is even), all $\lambda \in \Lambda - \{0\}$ are apexes, but $\lambda = 0$ is not an apex.*

(2) *The simple $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$-modules of apex $\lambda \in \Lambda$ are parameterized by simple modules of $\mathbb{K}\mathrm{B}_{g,1}^{\boldsymbol{d},\boldsymbol{b}}$.*

(3) *The simple $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$-modules of apex $\lambda \in \Lambda$ can be constructed as the simple heads of $\mathrm{Ind}_{\mathbb{K}\mathrm{B}_g^{\boldsymbol{d},\boldsymbol{b}}}^{\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})}(K)$, where K runs over (the equivalence classes of) simple $\mathbb{K}\mathrm{B}_{g,1}^{\boldsymbol{d},\boldsymbol{b}}$-modules.*

Proof. This can be proven verbatim as in the previous cases, see e.g. Theorem 2.21. ∎

The sandwiched algebras are identified in Lemma 4.21. Thus, Theorem 4.25 explicitly gives the desired classification of simple $\mathrm{Bl}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$-modules.

Example 4.26. The low genus cases of Theorem 4.25 are known (for simplicity, we ignore the potential exception for $\lambda = 0$ in the text below):

(1) For $g = 0$, we have $\mathbb{K}\mathrm{B}_{0,1}^{\boldsymbol{d},\boldsymbol{b}} = \mathbb{K}$, so we obtains the classical parametrization of the simple modules of the Temperley–Lieb algebra by through strands.

(2) For $g = 1$, we have that $\mathbb{K}\mathrm{B}_{1,1}^{\boldsymbol{d},\boldsymbol{b}}$ is a finite-dimensional quotient of a polynomial ring in one variable b by an ideal of the form

$$(b - \mathtt{b}_1)...(b - \mathtt{b}_{\mathtt{d}_1}) = 0.$$

In particular, the number of simple modules associated to $\mathbb{K}\mathrm{B}_{1,1}^{\boldsymbol{d},\boldsymbol{b}}$ equals the number of distinct parameters in $\boldsymbol{b}$. For example, if $\mathtt{d}_1 = 2$, then

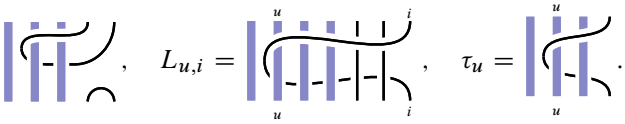
$$\mathtt{b}_1 = \mathtt{b}_2 = 0 \Rightarrow \mathbb{K}\mathrm{B}_{1,1}^{\boldsymbol{d},\boldsymbol{b}} \cong \mathbb{K}[b]/(b^2) \text{ has one simple module,}$$
$$\mathtt{b}_1 = 1, \mathtt{b}_2 = 0 \Rightarrow \mathbb{K}\mathrm{B}_{1,1}^{\boldsymbol{d},\boldsymbol{b}} \cong \mathbb{K}[b]/(b-1)b \text{ has two simple modules.}$$

In the latter case, $b^2 = b$, which is the situation studied in [45]. Thus, we recover the classification of simples of the blob algebra from [45].

4C. Topology of handlebody Temperley–Lieb algebras

Another way of defining the classical Temperley–Lieb algebra would be as the quotient of the algebra of tangles by circle evaluation and the Kauffman skein relation. To discuss a handlebody analog let us define *handlebody (framed) tangle diagrams of $2n$ points of genus g*, using handlebody braid diagrams as in Section 3 and inspired by Proposition 3.5, as the (isotopy classes of) tangle diagrams in $g + n$ strings, which are pure on the first g

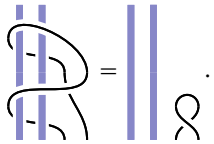
strings, modulo the usual tangle relations. The framing is given by a vector field pointing to the gth core strand from the left. The examples to keep in mind are



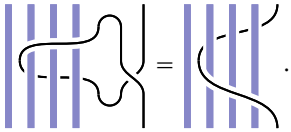
Recall that we call the second diagram a Jucys–Murphy element.

Remark 4.27. A subtle behavior due to our choice of framing occurs under unknotting around a core, as one can see from the example below, see also Figure 3 in [26]:

(4.11)



Let us also mention that these diagrams satisfy the classical Reidemeister relations and other types of isotopy relations, e.g.,



As before we fix circle evaluations $\boldsymbol{c} = (\mathsf{c}_\gamma)_\gamma$, one for any isotopy class of circles γ. We also fix any invertible element $\mathsf{q} \in \mathbb{K}$ that has a square root in $\mathbb{K}$.

Definition 4.28. The evaluation of a circle γ is defined to be the removal of a closed component, contributing a factor c_γ. We call this *circle evaluation*.

Definition 4.29. We let the *topological handlebody Temperley–Lieb algebra* (in n strands and of genus g) $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ be the algebra whose underlying free $\mathbb{K}$-module is the $\mathbb{K}$-linear span of all handlebody tangle diagrams of $2n$ points of genus g, and with multiplication given by concatenation of diagrams modulo circle evaluation and (4.2).

Note that $\mathrm{TL}_{g,n}(\boldsymbol{c})$ and $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ are very different algebras. For example, as we have seen in (4.3), the coils are of finite order in $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ but of infinite order in $\mathrm{TL}_{g,n}(\boldsymbol{c})$.

Remark 4.30.

(1) For $g = 0$, Definition 4.29 is a classical definition, while the case $g = 1$ is related by Schur–Weyl duality to Verma modules, see [30].

(2) Without evaluation of circles, the case $g > 2$ can also be found in [30] under the name multi-polar, but not much appears to be known.

Lemma 4.31. *The algebra* $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ *is an associative, unital algebra with a* $\mathbb{K}$*-spanning set given by all crossingless matchings of* $2n$ *points of genus* g *where each coil is positive and appears at most of order* 1.

Proof. That the claimed set is spanning is clear by circle evaluation and (4.3). ∎

Recall that C_n denotes the nth Catalan number, which is also the dimension of the genus $g = 0$ Temperley–Lieb algebra.

Definition 4.32. We call parameters $\boldsymbol{c} = (\mathtt{c}_\gamma)_\gamma$ such that $\dim_{\mathbb{K}} \mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c}) \geq C_n$ *weakly admissible*.

With Remark 4.5 in mind, the following is in contrast to Lemma 4.6 a non-trivial result. To state it let $[k]_{\mathtt{q}} = \mathtt{q}^{k-1} + \cdots + \mathtt{q}^{-k+1}$ denote the usual quantum number for $k \in \mathbb{N}$, and note that for the choice $\boldsymbol{c} = (-[2]_{\mathtt{q}})$ the algebra $\mathrm{TL}_G(-[2]_{\mathtt{q}}) = \mathrm{TL}^{\mathrm{top}}_{0,G}(\boldsymbol{c})$ is the classical Temperley–Lieb algebra on G strands whose circle evaluates to $-[2]_{\mathtt{q}}$.

Lemma 4.33. *Let $\mathbb{K}$ be a field. For any $\mathtt{q} \in \mathbb{K}$ and $G \geq g$ such that $[k]_{\mathtt{q}} \neq 0$ for all $k < G + 1$, there exist a set $\boldsymbol{c}^G$ and an algebra homomorphism (explicitly given in the proof below)*

$$\iota^G \colon \mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c}^G) \to \mathrm{TL}_{G+n}\big(-[2]_{\mathtt{q}}\big).$$

Moreover, the parameters $\boldsymbol{c}^G$ are weakly admissible.

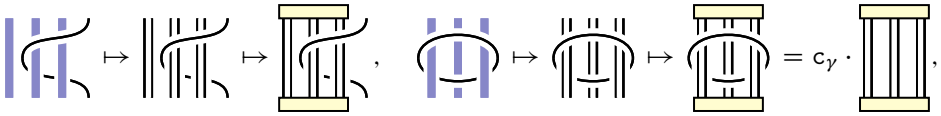
Note that the assumptions in Lemma 4.33 are always satisfied for e.g. $\mathbb{K} = \mathbb{C}(\mathtt{q}^{1/2})$ and $\mathtt{q}$ being the generic variable.

Proof. For the algebra $\mathrm{TL}_{G+n}(-[2]_{\mathtt{q}})$, there exists a Jones–Wenzl idempotent $\mathtt{e}_G$ on G strands, see e.g. Chapter 3 in [32] (strictly speaking, Chapter 3 in [32] uses $\mathbb{K} = \mathbb{C}(\mathtt{q})$, but under the assumption that $[k] \neq 0$ for all $k < G + 1$, the whole discussion therein goes though as long as the number of strands is less than $G + 1$), and we can consider the idempotent truncation $\mathtt{e}_G \mathrm{TL}_G(-[2]_{\mathtt{q}})\mathtt{e}_G$. By the properties of $\mathtt{e}_G$, we have $\mathtt{e}_G \mathrm{TL}_G(-[2]_{\mathtt{q}})\mathtt{e}_G \cong \mathbb{K}\{\mathtt{e}_G\}$, so for any circle γ in $\mathtt{e}_G \mathrm{TL}_G(-[2]_{\mathtt{q}})\mathtt{e}_G$ we get a scalar $\mathtt{c}^G_\gamma$. Fix some partitioning of $G = M_1 + \cdots + M_g$ for $M_i \in \mathbb{N}_{>0}$. With the choices $\mathtt{c}^G_\gamma$ it is easy to see that blowing the ith core strand into M_i parallel strands and flanking them then with $\mathtt{e}_G$ defines an algebra homomorphism

$$\iota \colon \mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c}^G) \to (\mathtt{e}_G \otimes \mathrm{id}_n)\mathrm{TL}_{G+n}(-[2]_{\mathtt{q}})(\mathtt{e}_G \otimes \mathrm{id}_n) \hookrightarrow \mathrm{TL}_{G+n}(-[2]_{\mathtt{q}}).$$

The pictures illustrating the above constructions are

(4.12)



where the boxes represent the Jones–Wenzl idempotent $\mathtt{e}_G$. To see the remaining statement let $\mathrm{Sym}^G_{\mathtt{q}}(\mathbb{K}^2)$ denote the Gth quantum symmetric power of $\mathbb{K}^2$. We use ι and quantum symmetric Howe duality [54], Theorem 2.6 (1) and (2) (which work over $\mathbb{Z}[\mathtt{q}, \mathtt{q}^{-1}]$) in combination with Proposition 2.14 in [54] (which works under the assumptions on $[k]_{\mathtt{q}}$ to the point needed) to define a representation of $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c}^G)$ on $\mathrm{Sym}^G_{\mathtt{q}}(\mathbb{K}^2) \otimes \mathbb{K}^2 \otimes \ldots \otimes \mathbb{K}^2$. Using this representation one can check that $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c}^G)$ can not be trivial since, for example, $\mathrm{TL}^{\mathrm{top}}_{0,n}(\boldsymbol{c}^G)$ acts faithfully by the classical Schur–Weyl duality. ∎

Remark 4.34. Our proof of Lemma 4.33 is directly inspired by [55]: as pointed out in that paper, the handlebody closing (3.4) can be interpreted, in the appropriate algebraic framework, by putting an idempotent on bottom and top. Moreover and alternatively to the usage of (growing) symmetric powers, one might want to associate Verma modules to the core strands as in the $g = 1$ case, see [30]. This approach has however the disadvantage that we do not know analogs of Jones–Wenzl idempotents which one could use to get coherent circle evaluations.

Remark 4.35. For $\mathsf{q} \in \mathbb{K}$ being a root of unity and $\mathbb{K}$ being a field, there are still Jones–Wenzl type idempotents, see e.g. [46] or [61] for general constructions of such idempotents. However, their endomorphism spaces are no longer trivial, so we can not use the argument in the proof of Lemma 4.33.

We do not know any explicit condition to check whether parameter choices are *admissible*, meaning that we do not know a "generic" basis of $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$. We leave it to future work to find such a basis.

4D. Topology of handlebody blob algebras

The purpose of this section is to explain a different presentation for $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$.

Remark 4.36. We will also use a similar construction as presented in this section for cyclotomic handlebody Brauer and BMW algebras later on, so we decided to spell it out here despite Lemma 4.38. However, while the topological handlebody blob algebra is the same as the topological handlebody Temperley–Lieb algebra, this phenomena is no longer true for handlebody Brauer and BMW algebras.

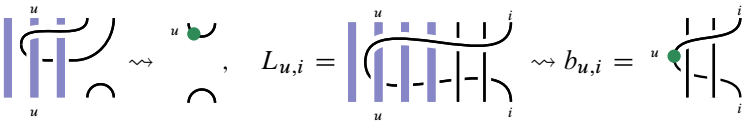
Definition 4.37. Retain the notation and conventions from Definition 4.29. We let the *topological handlebody blob algebra* (in n strands and of genus g) $\mathrm{Bl}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ be the subalgebra of $\mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$ spanned by all elements containing only positive coils.

Lemma 4.38. *We have* $\mathrm{Bl}^{\mathrm{top}}_{g,n}(\boldsymbol{c}) = \mathrm{TL}^{\mathrm{top}}_{g,n}(\boldsymbol{c})$.

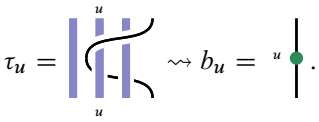
Proof. Clear by (4.3). ∎

To give a different diagrammatic presentation we need the notion of *blobbed presentations*, that will spell out now. The construction of this works mutatis mutandis as in Subsection 4B (so we will be brief) with one main difference: we allow crossings between the usual strands. This gives us the notion of *framed tangled blob diagrams of* $2n$ *points of genus* g, where blobs move freely around its strand, but always keep being reachable from the left, and do not pass each other, cf. (4.7).

Examples of such tangled blob diagrams and how they relate to the diagrammatic used in Subsection 4C are



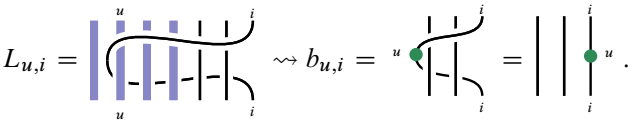
and



where we use the same notation for the blob versions of the coils as in Subsection 4B.

Note that we can introduce blobs on any possible strand by

(4.13)

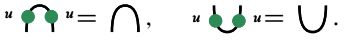


The latter is however just a shorthand notation which is not quite topological in nature. The point to is that the relations among Jucys–Murphy elements in (3.9) give relations among blobs, but not all of them are topological manipulations of blobs. Precisely, we have:

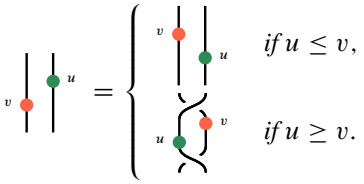
Lemma 4.39. *Blobs satisfy the following relations*:

(4.14)

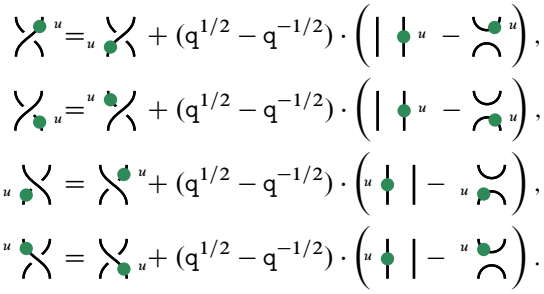
(4.15)



(4.16)



(4.17)

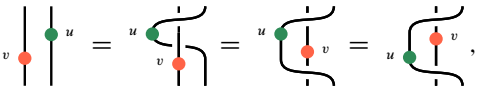


The relations in (4.14) and (4.17) are called *blob slides*. Also note the distorted topology in Lemma 4.39, which is however gets resolved for $\mathtt{q} = 1$.

Proof. Relations (4.14) are a blob version of the equalities $\beta_{i-1}^{-1} L_{u,i} = L_{u,i-1}\beta_{i-1}$ and $\beta_i L_{u,i} = L_{u,i+1}\beta_i^{-1}$ between Jucys–Murphy elements, cf. Lemma 3.12.

To prove the first relation in (4.15) slide down the blob on the right and use (4.13) to write it as a blob on the strand to the left. The rest follows from (4.11). The second relation in (4.15) is proved analogously.

Relation (4.16) in the case $v \geq u$ is the last relation in Lemma 3.12. In the case $v \leq u$ we calculate



where we use the case $v \geq u$ and (4.14).

For the final set of relations in (4.17) we combine (4.14) with the Kauffman skein relation (4.2). Precisely, (4.14) and (4.2) imply e.g.

which in turn prove the claimed formulas. ∎

5. Handlebody Brauer and BMW algebras

We will be brief in this section as it is very similar to the previous discussions. Recall that we have fixed the genus g and the number of strands n.

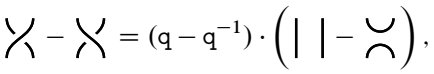
5A. Handlebody Brauer and BMW algebras

We use handlebody (framed) tangle diagrams of $2n$ points of genus g as in Subsection 4C. We fix invertible scalars $\mathtt{q}, \mathtt{a} \in \mathbb{K}$ with $\mathtt{q} \neq \mathtt{q}^{-1}$. Circles that may appear after concatenation of these diagrams are evaluated as in Subsection 4C, and we fix $\mathtt{c}_\gamma \in \mathbb{K}$ as before, except for $\mathtt{c}_1$ which we define as

$$\mathtt{c}_1 = 1 + \frac{\mathtt{a} - \mathtt{a}^{-1}}{\mathtt{q} - \mathtt{q}^{-1}} \cdot$$

Definition 5.1. We let the *handlebody BMW algebra* (in n strands, and of genus g) $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ be the algebra whose underlying free $\mathbb{K}$-module is the $\mathbb{K}$-linear span of all handlebody tangle diagrams of $2n$ points of genus g, and with multiplication given by concatenation of diagrams modulo circle evaluation, and the *skein relation* (5.1) and (5.2):

(5.1)



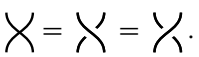
(5.2)

For $\mathtt{q} = \mathtt{q}^{-1}$ we also let $\mathtt{a} = \mathtt{a}^{-1}$ and let $\mathtt{c}_1 \in \mathbb{K}$ be any parameter.

Definition 5.2. We call the specialization $\mathtt{q} = \mathtt{a} = 1$ the *Brauer specialization* and the resulting algebra the *handlebody Brauer algebra* (in n strands and of genus g) $\mathrm{Br}_{g,n}(\boldsymbol{c})$.

Remark 5.3. Note that $\mathrm{Br}_{g,n}(\boldsymbol{c})$ is not a specialization of $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ as the $\mathtt{q} = \mathtt{a} = 1$ limit of $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ has $\mathtt{c}_1 = 2$. This technicality will not play any role for us, so we strategically ignore it.

In the specialization $\mathrm{Br}_{g,n}(\boldsymbol{c})$, we can use a simplified notation for the crossings:



Remark 5.4. Special cases of Definition 5.1 have appeared in the literature:

(1) The case $g = 0$ is the case of the Birman–Murakami–Wenzl algebra, respectively Brauer algebra, and is classical.

(2) For $g = 1$, these algebras appear under the name of affine BMW or Brauer algebras in many works, e.g. in [26] or [52].

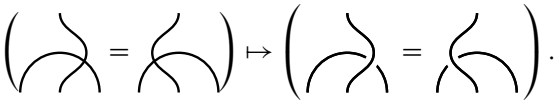
(3) For $g = 2$, we were not able to find a reference, but the definition is easily deduced from the affine type C braid group combinatorics. However, this would give a two-boundary version of the above, with one core strand to the left and one to the right, see e.g. [11] for the corresponding pictures.

In order to define a spanning set for $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$, we need the notion of *perfect matchings of* $2n$ *points of genus* g: these are perfect matchings of $2n$ points where strands can wind around the cores. We keep the conventions of the previous sections and consider that all cores are at the left. For example, if $g = 3$ and $n = 2$, then

are examples of such perfect matchings. These perfect matchings are equal if they describe the same perfect matching with the same winding around cores.

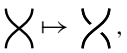
Remark 5.5. With blobs, the situation is trickier because we need to be careful with relation (4.15). Without blobs, we do not have this problem and we can treat these perfect matchings as topological objects.

Forgetting isotopy, each perfect matching as above defines an element of the algebra $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ by lifting crossings. Note however that we have an ambiguity coming from



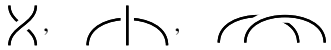
In order to avoid this, we call a *positive lift* a lift such that all through strands form a positive braid monoid, using

(5.3)



where the image is a positive crossing. All caps and cups are assumed to be underneath any through strand, and any caps and cups are also underneath one another, going from left

(lowest) to right (highest) along their left boundary points. For example,

Ok:  , not allowed:  .

Now we can proceed as before:

Lemma 5.6. *The algebra* $\mathrm{BMW}_{g,n}(\mathtt{c},\mathtt{q},\mathtt{a})$ *is an associative, unital algebra with a* $\mathbb{K}$*-spanning set given by positive lifts of perfect matchings of* $2n$ *points of genus* g*.*

Proof. Clear by (5.1) and (5.2), and isotopies. ∎

The dimension bound in the next definition comes from the classical Brauer/BMW algebra. That is, the number $(2n-1)!!$ in the definition of weakly admissible parameters up next is the dimension of the classical BMW and Brauer algebras.

Definition 5.7. We call parameters $(\boldsymbol{c},\mathtt{q},\mathtt{a})$ such that the $\mathbb{K}$-spanning set in Lemma 5.6 is a $\mathbb{K}$-basis *admissible*. We call parameters *weakly admissible* if $\dim_{\mathbb{K}}\mathrm{BMW}_{g,n}(\mathtt{c},\mathtt{q},\mathtt{a})\geq(2n-1)!!$.

To state the analog of Lemma 4.33 denote by $\mathrm{BMW}_{G+n}(\mathtt{c},\mathtt{q},\mathtt{a})=\mathrm{BMW}_{0,G+n}(\mathtt{c},\mathtt{q},\mathtt{a})$ the classical BMW algebra in $G+n$ strands.

Lemma 5.8. *Let* $\mathbb{K}$ *be a field. For any choice of* $\mathtt{q}\in\mathbb{K}$ *and* $G\geq g$ *such that* $[k]_{\mathtt{q}}\neq 0$ *for all* $1\ll k$*, there exist a triple* $(\boldsymbol{c}^G,\mathtt{q}^G,\mathtt{a}^G)$ *and an algebra homomorphism* (*explicitly given in the proof below*)

$$\iota^G\colon\mathrm{BMW}_{g,n}(\boldsymbol{c}^G,\mathtt{q}^G,\mathtt{a}^G)\to\mathrm{BMW}_{G+n}(\tilde{\mathtt{c}},\tilde{\mathtt{q}},\tilde{\mathtt{a}}).$$

Moreover, the parameters $(\boldsymbol{c}^G,\mathtt{q}^G,\mathtt{a}^G)$ *are weakly admissible.*

If $\mathtt{q}$ is not a root of unity, then there is always a corresponding triple $(\boldsymbol{c}^G,\mathtt{q}^G,\mathtt{a}^G)$.

Proof. Denote by $(\tilde{\boldsymbol{c}},\tilde{\mathtt{q}},\tilde{\mathtt{a}})$ a choice of parameters such that Schur–Weyl–Brauer duality (for this duality we refer the reader to, for example, Theorem 3.17 in [2] and Theorem 1.3 in [29]) gives a well-defined algebra homomorphism from $\mathrm{BMW}_{G+n}(\tilde{\mathtt{c}},\tilde{\mathtt{q}},\tilde{\mathtt{a}})$ into the endomorphism space of the $(G+n)$-fold tensor product of the vector representation of an associated quantum group of types BCD, called tensor space. The proof is then essentially the same as the one in Lemma 4.33, using Schur–Weyl–Brauer duality instead of (a special case of) Schur–Weyl duality. Precisely, we define ι^G using the very same pictures as in (4.12), but the boxes represent the pullbacks of the highest weight projectors (splitting off the highest weight summands) in tensor space. Under the appropriate assumptions on the involved quantum numbers these projectors exist. Explicitly:

- In types B and D, one can take e.g. $\tilde{\mathtt{c}}=[2m+1]_{\mathtt{q}}$ and $\tilde{\mathtt{a}}=\mathtt{q}^{2m+1}$ and the quantum group for $\mathfrak{so}_{2m+1}$ to get a well-defined ι^G. To ensure the existence of the idempotent, one takes $m>(G-1)/2$.
- In type C, to define ι^G one can take e.g. $\tilde{\mathtt{c}}=-[2m]_{\mathtt{q}}$ and $\tilde{\mathtt{a}}=-\mathtt{q}^{-2m}$ and the quantum group for $\mathfrak{sp}_{2m}$, and also $m>G$ to ensure the existence of the idempotent.
- In type D, choices that work are e.g. $\tilde{\mathtt{c}}=[2m]_{\mathtt{q}}$, $\tilde{\mathtt{a}}=\mathtt{q}^{2m}$ and the quantum group for $\mathfrak{so}_{2m}$, as well as $m>G+1$.

To see this, we can use the explicit bounds given in e.g. Theorem 3.17 of [2]. ∎

Remark 5.9. The idempotents used in the proof of Lemma 5.8 do not satisfy an easy recursion as the Jones–Wenzl projectors used in the proof of Lemma 4.33. See however [31] or [42] for some work on projectors in Brauer respectively BMW algebras.

As before for the topological handlebody Temperley–Lieb algebra, we do not know any explicit way to construct admissible parameters. However, under the assumption that these exists we conclude this section as follows.

Note that our definition realizes $\mathrm{Br}_{g,n}(\boldsymbol{c})$ as a specialization of $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathsf{q}, \mathsf{a})$, which is not equal to a definition using perfect matchings. So the following needs admissible parameters but is then immediate from Definition 5.7:

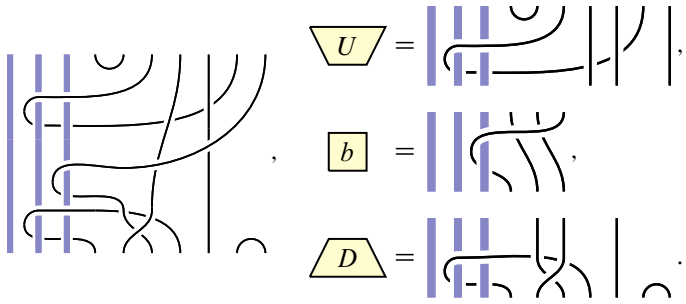
Proposition 5.10. *For admissible parameters, the handlebody Brauer algebra* $\mathrm{Br}_{g,n}(\boldsymbol{c})$ *can be alternatively described as the free* $\mathbb{K}$*-module spanned by perfect matchings of* $2n$ *points of genus* g *with multiplication given by concatenation and circle evaluation.* ∎

For the cellular structure, we fix a sandwich cell datum as in Subsection 2D, and also with a very similar cellular basis. Let us for brevity just stress the differences. First, we only consider positive lifts and we let $\mathrm{B}^+_{g,\lambda}$ denote the handlebody braid monoid, i.e., Definition 3.1 using only positive coils and positive crossings (5.3) for the definition.

Note that there is a map $\mathrm{B}^+_{g,\lambda} \to \mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathsf{q}, \mathsf{a})$ that sends positive coils to positive coils and positive crossings to positive crossings. We let $\mathbb{S}_\lambda = \mathbb{K}\mathrm{B}^+_{g,\lambda}$ with the monoid basis as the sandwiched basis. We then get

$$\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in \mathbb{S}_\lambda\} \tag{5.4}$$

as before, with the D and U part being as for the handlebody Temperley–Lieb algebra, cf. (4.5), namely only caps respectively caps are allowed to wind around the cores. The picture is



The antiinvolution $(\_)^\star \colon \mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathsf{q}, \mathsf{a}) \to \mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathsf{q}, \mathsf{a})$ is defined as for $\mathrm{TL}_{g,n}(\boldsymbol{c})$ with the addition that it maps positive crossings to positive crossings.

Proposition 5.11. *The above defines an involutive sandwich cell datum for the algebra* $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathsf{q}, \mathsf{a})$ *for any admissible parameters.*

Proof. The only claim which is not immediate by construction is that (5.4) is a $\mathbb{K}$-basis, which follows from Definition 5.7. ∎

Not surprisingly, the cell structure is, mutatis mutandis, as in Example 2.20. The next theorem is clear by the previous discussions.

Theorem 5.12. *Let $\mathbb{K}$ be a field, and choose admissible parameters.*

(1) *If $\mathtt{c} \neq 0$, or $\mathtt{c} = 0$ and $\lambda \neq 0$ is odd, then all $\lambda \in \Lambda$ are apexes. In the remaining case, $\mathtt{c} = 0$ and $\lambda = 0$ (this only happens if n is even), all $\lambda \in \Lambda - \{0\}$ are apexes, but $\lambda = 0$ is not an apex.*

(2) *The simple $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$-modules of apex $\lambda \in \Lambda$ are parameterized by simple modules of $\mathbb{K}\mathrm{B}^{+}_{g,\lambda}$.*

(3) *The simple $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$-modules of apex $\lambda \in \Lambda$ can be constructed as the simple heads of $\mathrm{Ind}_{\mathbb{K}\mathrm{B}^{+}_{g,\lambda}}^{\mathrm{BMW}_{g,n}(\boldsymbol{c},\mathtt{q},\mathtt{a})}(K)$, where K runs over (equivalence classes of) simple $\mathbb{K}\mathrm{B}^{+}_{g,\lambda}$-modules.*

Proof. No difference to the Brauer case in Theorem 2.21. ∎

Example 5.13. Let us comment on the parametrization given by Theorem 5.12:

(1) For $g = 0$, the algebras $\mathbb{K}\mathrm{B}^{+}_{0,\lambda}$ are Hecke algebras associated to Coxeter type $A_{\lambda-1}$, so we get the same classification as in Theorem 2.21, but for the BMW algebra. This was of course known, see e.g. Corollary 3.14 in [69].

(2) For $g = 1$, the algebras $\mathbb{K}\mathrm{B}^{+}_{1,\lambda}$ are extended affine Hecke algebras associated to Coxeter type $A_{\lambda-1}$.

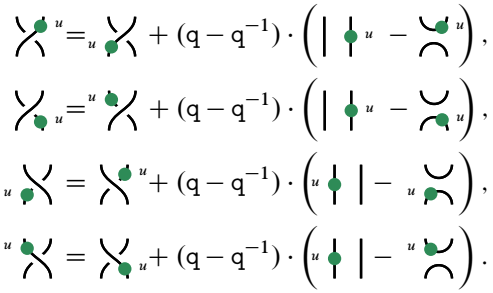
5B. Cyclotomic handlebody Brauer and BMW algebras

Recall the notion of blobbed presentations from Subsection 4D, and retain the notation from that section. One difference to that section is that here we identify blobbed diagrams with a subalgebra of $\mathrm{BMW}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ spanned by all elements containing only positive coils.

We have the following analog of Lemma 4.39. The proof is the same as that of Lemma 4.39 and omitted.

Lemma 5.14. *Blobs satisfy the following relations. First, (4.14), (4.15) and (4.16), and also*

(5.5)



(*Note the difference in the powers of* $\mathtt{q}$ *between (4.17) and (5.5).*)

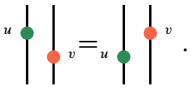
We call the relations collected in Lemma 5.14 *blob sliding relations*. Because of their slightly distorted topology, we introduce cyclotomic handlebody Brauer algebra before their BMW counterparts. That is, we first treat the Brauer specialization $\mathtt{q} = \mathtt{a} = 1$.

In the following lemma we collect some relations that are handy in the Brauer case. In particular, up to (4.15), these *Brauer blobs* move freely along strands, cf. (5.6) and (5.7).

Lemma 5.15. *Brauer blobs satisfy the following relations additionally to* (4.15):

(5.6)

(5.7)



Proof. Relation (5.5) specializes to (5.6), which we then use to derive (5.7) from (4.16). ∎

Fix cyclotomic parameters $\boldsymbol{b} = (\mathtt{b}_{u,i}) \in \mathbb{K}^{d_1+\cdots+d_g}$, a degree vector $\boldsymbol{d} = (\mathtt{d}_u) \in \mathbb{N}^g$. Moreover, let $\mathrm{Br}^+_{g,n}(\boldsymbol{c})$ denote the subalgebra of $\mathrm{Br}_{g,n}(\boldsymbol{c})$ with only positive coils.

Definition 5.16. We let the *cyclotomic handlebody Brauer algebra* (in n strands and of genus g) $\mathrm{Br}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$ be the quotient of $\mathrm{Br}^+_{g,n}(\boldsymbol{c})$ by the two-sided ideal generated by the *cyclotomic relations*

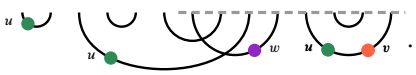
$$(b_u - \mathtt{b}_{u,1})\varpi_1(b_u - \mathtt{b}_{u,2})\varpi_2 \ldots (b_u - \mathtt{b}_{u,\mathtt{d}_u-1})\varpi_{d_u-1}(b_u - \mathtt{b}_{u,\mathtt{d}_u}) = 0, \tag{5.8}$$

where ϖ_j is any finite (potentially empty) expression not involving b_u such that all b_u in (5.8) are on the same strand.

Remark 5.17. Note that the skein relation (5.1) and also (4.15) imply compatibility conditions between parameters. See Definition 5.19, Lemma 5.20 and Proposition 5.22 below.

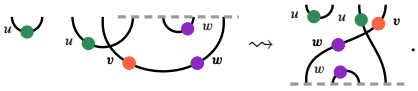
We consider *clapped, blobbed perfect matchings of* $2n$ *points of genus* g which we need for counting purposes. We assume that these perfect matchings points are clapped, similar to the crossingless matchings in the proof of Proposition 4.22. Strands can carry blobs that move freely on the strand they belong but are not allowed to pass each other. In particular, there is no condition on being reachable from the left. This implies that each strand with blobs defines a word in the free monoid $\mathrm{F}^+_g \cong \mathrm{Br}^+_{g,1}(\boldsymbol{c}) \hookrightarrow \mathrm{Br}^+_{g,n}(\boldsymbol{c})$.

Note that we consider them as perfect matchings, so there is an ambiguity in how to illustrate these without further conditions. This problem is however resolved by demanding that each matched pair is connected by a cup with exactly one Morse point, and there is a minimal number of intersection between the cups. Here is an example:



Each such perfect matching defines an element of $\mathrm{Br}^+_{g,n}(\boldsymbol{c})$ by unclapping and simultaneously sliding all blobs to the left that may fall on caps or cups. This procedure is illustrated

in the diagram below:



This operation can described rigorously: if we label the boundary points $1, \dots, 2n$, then the set of perfect matchings is in bijection with the set of all unordered n-tuples of pairs $\{(i_1, j_1), \dots, (i_n, j_n)\}$ that can be formed from the $2n$ distinct elements of $\{1, \dots, 2n\}$ with $i_k < j_k$. Each such pair corresponds to a strand in the diagrammatics. A blob on the strand (i, j) is to be slid to the left if $j \leq n$, to the right if $n \leq i$ and left untouched otherwise in the unclapping process.

The next lemma follows from this discussion.

Lemma 5.18. *The algebra* $\mathrm{Br}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ *has a* $\mathbb{K}$*-spanning set in bijection with clapped, blobbed perfect matchings of* $2n$ *points of genus* g *where each strand has at most* $\mathtt{d}_u - 1$ *blobs of the corresponding type.*

Proof. The above defines a $\mathbb{K}$-module map from the free $\mathbb{K}$-module spanned by all clapped, blobbed perfect matchings to $\mathrm{Br}_{g,n}^{+}(\boldsymbol{c})$. On the other hand, each diagram of $\mathrm{Br}_{g,n}^{+}(\boldsymbol{c})$ defines a clapped, blobbed perfect matching by clapping and then arranging the result until it is of the required form. These are inverse procedures and so the $\mathbb{K}$-module $\mathrm{Br}_{g,n}^{+}(\boldsymbol{c})$ is contained in the free $\mathbb{K}$-module spanned by all clapped, blobbed perfect matchings. The cyclotomic condition then gets rid of having too many blobs on strands. ∎

Definition 5.19. We call parameters $(\boldsymbol{d}, \boldsymbol{b}, \boldsymbol{c})$ such that the $\mathbb{K}$-spanning set in Lemma 5.18 is a $\mathbb{K}$-basis *admissible*.

We do not know any representation theoretical space where $\mathrm{Br}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$ acts on, so we can not copy the arguments from e.g. the proof of Lemma 4.33. We rather do the following (which includes the cases where the parameters are generic):

Lemma 5.20. *Let* $\mathbb{K}$ *be a field and let* $\mathbb{L}$ *be a field extension of* $\mathbb{K}$ *of degree* $[\mathbb{L} : \mathbb{K}] \gg 1$. *For each choice of* $\boldsymbol{d}$ *and* $\boldsymbol{b}$*, there exist* $\boldsymbol{c}$ *and* $\boldsymbol{b}$ *with entries in in* $\mathbb{L}$ *such that* $(\boldsymbol{d}, \boldsymbol{b}, \boldsymbol{c})$ *is admissible.*

Proof. The same arguments as in [20] prove that admissibility is equivalent to the left and right $\mathrm{Br}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c})$-modules

$$\mathrm{Caps} = \left\{ \cap, \ u \ \cap, \ v \ \cap, \ \begin{smallmatrix} v \\ u \end{smallmatrix} \ \cap, \dots \right\}, \quad \mathrm{Cups} = \left\{ \cup, \ u \ \cup, \ v \ \cup, \ \begin{smallmatrix} u \\ v \end{smallmatrix} \ \cup, \dots \right\},$$

of all possible blob placements on a cap respectively cup not affected by the cyclotomic relation (5.8) being free of rank $\mathtt{BN}_{1,\boldsymbol{d}}$. Closing diagrams implies that this happens if and only if the $\mathtt{BN}_{1,\boldsymbol{d}}$-$\mathtt{BN}_{1,\boldsymbol{d}}$ pairing matrix

$$P = \begin{pmatrix} \cap & u \ \cap & v \ \cap & \begin{smallmatrix} v \\ u \end{smallmatrix} \ \cap & \dots \\ u \ \cup & \begin{smallmatrix} u \\ u \end{smallmatrix} \ \cup & \begin{smallmatrix} v \\ u \end{smallmatrix} \ \cup & \begin{smallmatrix} v \\ u \\ u \end{smallmatrix} \ \cup & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is non-degenerate, i.e., of rank $\mathtt{BN}_{1,\boldsymbol{d}}$. Note that each row contains a unique combination of $\mathtt{b}_i$ and $\mathtt{c}_j$. Thus, choosing them all such that they do not satisfy any polynomial equation of low order ensures that the determinant of P is non-zero. ∎

Remark 5.21. For $g = 1$ there is such an effective criterion to get admissible parameters, see Theorem 3.2 in [20] and [68]. It would be interesting to have such a criterion for $g > 1$.

The key players to calculate $\dim_{\mathbb{K}} \mathrm{Br}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$ are again the blob numbers $\mathtt{BN}_{g,\boldsymbol{d}}$ defined in (4.9):

Proposition 5.22. *For admissible parameters we have*

$$\dim_{\mathbb{K}} \mathrm{Br}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c}) = (\mathtt{BN}_{g,\boldsymbol{d}})^n (2n-1)!!. \tag{5.9}$$

Moreover, for admissible parameters, the cyclotomic handlebody Brauer algebra $\mathrm{Br}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$ *can be alternatively described as the free* $\mathbb{K}$*-module spanned by perfect matchings of* $2n$ *points of genus* g *with multiplication given by concatenation and circle evaluation modulo the cyclotomic condition.*

Proof. By Lemma 4.21, the number of ways to put blobs on a single strand is $\mathtt{BN}_{g,\boldsymbol{d}}$. Hence, the number of ways to put these on n strands is $(\mathtt{BN}_{g,\boldsymbol{d}})^n$. Since there are $(2n-1)!!$ corresponding perfect matchings all having n strands, the dimension formula follows from Lemma 5.18. ∎

Example 5.23. For $g = 0$, (5.9) is the dimension of the Brauer algebra. For $g = 1$, (5.9) is the formula in e.g. Section 11 of [27] or Theorem 4.11 in [70].

Remark 5.24. Note also the difference of our construction to [27] or [70]: we define $\mathrm{Br}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c})$ as a tangle algebra to begin with, while those papers start with an algebraic formulation and then prove that they have the suggestive diagrammatic presentation.

We keep the cyclotomic parameters $\boldsymbol{b}$ and degree vector $\boldsymbol{d}$ as before. We are now going to define cyclotomic handlebody BMW algebras, where we as before use the corresponding positive monoid $\mathrm{BMW}^+_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$.

Definition 5.25. We let the *cyclotomic handlebody BMW algebra* (in n strands and of genus g) $\mathrm{BMW}^{\boldsymbol{d},\boldsymbol{b}}_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ be the quotient of $\mathrm{BMW}^+_{g,n}(\boldsymbol{c}, \mathtt{q}, \mathtt{a})$ by the two-sided ideal generated by the *cyclotomic relations*

$$(b_u - \mathtt{b}_{u,1})\varpi_1(b_u - \mathtt{b}_{u,2})\varpi_2 \ldots (b_u - \mathtt{b}_{u,\mathtt{d}_u-1})\varpi_{d_u-1}(b_u - \mathtt{b}_{u,\mathtt{d}_u}) = 0, \tag{5.10}$$

where ϖ_j is any finite (potentially empty) expression not involving b_u such that all b_u in (5.10) are on the same strand.

Remark 5.26. Cyclotomic versions of BMW and Brauer algebras have appeared in the literature:

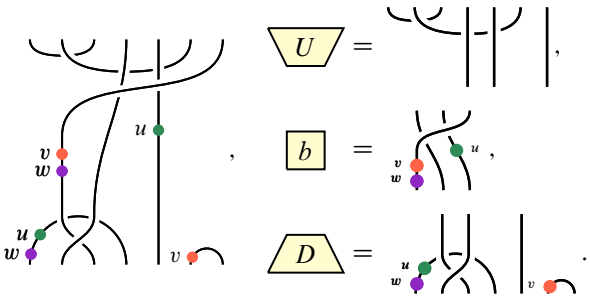
(1) For $g = 0$, they are of course just the BMW respectively Brauer algebra.
(2) For $g = 1$, the definition goes back to [27].

Lemma 5.27. *For an admissible choice of parameters, the algebra* $\mathrm{BMW}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c},\mathtt{q},\mathtt{a})$ *has a* $\mathbb{K}$*-basis in bijection with clapped, blobbed perfect matchings of* $2n$ *points of genus* g*, where each strand has at most* $\mathtt{d}_u$ *blobs of the corresponding type.*

In particular, the dimension of $\mathrm{BMW}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c},\mathtt{q},\mathtt{a})$ is also given by the formula in (5.9), namely it is $\dim_{\mathbb{K}} \mathrm{BMW}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c},\mathtt{q},\mathtt{a}) = (\mathtt{BN}_{g,\boldsymbol{d}})^n(2n-1)!!$.

Proof. By the procedure described in Lemma 5.18, we see that the corresponding images in $\mathrm{BMW}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}(\boldsymbol{c},\mathtt{q},\mathtt{a})$ are linearly independent since they are so in the Brauer specialization. To see that these span, we use Lemma 5.14. ∎

For the cell structure we now combine the one from Subsection 4B with the one from Subsection 5A (in particular, using positive lifts). The two differences worthwhile spelling out are first that (4.15) tell us to demand that blobs on caps and cups are to the right of any of their Morse points. Let $\mathbb{S}_\lambda = \mathbb{K}\mathrm{B}_{g,\lambda}^{+,\boldsymbol{d},\boldsymbol{b}}$ denote the blobbed positive braid monoid in λ strands, i.e., the positive braid monoid with blobs satisfying the relations in Lemma 5.14 and (5.10). As usual we fix the monoid element basis as our sandwiched basis. Because of these relations we need to choose an order of blobs and crossings. We choose to put blobs below all crossings, with blobs on the first strand below blobs on the second strand, etc. We leave the details to the reader, and rather give an example:



The next two statements follow as before. The proofs are omitted.

Proposition 5.28. *For a choice of admissible parameters the above defines an involutive sandwich cell datum for* $\mathrm{BMW}_{g,n}(\boldsymbol{c},\mathtt{q},\mathtt{a})$. ∎

Theorem 5.29. *Let* $\mathbb{K}$ *be a field, and choose admissible parameters.*

(1) *If* $\boldsymbol{c} \neq 0$*, or* $\boldsymbol{c} = 0$ *and* $\lambda \neq 0$ *is odd, then all* $\lambda \in \Lambda$ *are apexes. In the remaining case,* $\boldsymbol{c} = 0$ *and* $\lambda = 0$ *(this only happens if* n *is even), all* $\lambda \in \Lambda - \{0\}$ *are apexes, but* $\lambda = 0$ *is not an apex.*

(2) *The simple* $\mathrm{BMW}_{g,n}(\boldsymbol{c},\mathtt{q},\mathtt{a})$*-modules of apex* $\lambda \in \Lambda$ *are parameterized by simple modules of* $\mathbb{K}\mathrm{B}_{g,\lambda}^{+,\boldsymbol{d},\boldsymbol{b}}$.

(3) *The simple* $\mathrm{BMW}_{g,n}(\boldsymbol{c},\mathtt{q},\mathtt{a})$*-modules of apex* $\lambda \in \Lambda$ *can be constructed as the simple heads of* $\mathrm{Ind}_{\mathbb{K}\mathrm{B}_{g,\lambda}^{+,\boldsymbol{d},\boldsymbol{b}}}^{\mathrm{BMW}_{g,n}(\boldsymbol{c},\mathtt{q},\mathtt{a})}(K)$*, where* K *runs over (equivalence classes of) simple* $\mathbb{K}\mathrm{B}_{g,\lambda}^{+,\boldsymbol{d},\boldsymbol{b}}$*-modules.* ∎

Example 5.30. For $g = 0$, Theorem 5.29 is, of course, a BMW version of Theorem 2.21. For $g = 1$ and the Brauer specialization, this recovers [6], Appendix 6, where $\mathbb{K}\mathrm{B}^{+,\boldsymbol{d},\boldsymbol{b}}_{1,\lambda}$ is a complex reflection group of type $G(\mathtt{d}, 1, \lambda)$. In the semisimple case, the parametrization of simples of $\mathbb{K}\mathrm{B}^{+,\boldsymbol{d},\boldsymbol{b}}_{1,\lambda}$ (and thus, of $\mathrm{Br}_{1,n}(\mathtt{c})$ per apex λ) is given by $\mathtt{d}$-multipartitions of λ.

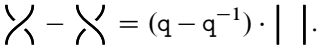
6. Handlebody Hecke and Ariki–Koike algebras

We define handlebody Hecke algebras as quotients of handlebody braid groups. All algebras can alternatively be defined as quotients of an appropriate BMW algebra from Section 5.

6A. Handlebody Hecke algebras

Fix an invertible scalar $\mathtt{q} \in \mathbb{K}$ and recall the definitions regarding the handlebody braid groups $\mathrm{B}_{g,n}$ from Subsection 3A.

Definition 6.1. The *handlebody Hecke algebra* (in n strands and of genus g) $\mathrm{H}_{g,n}$ is the algebra whose underlying free $\mathbb{K}$-module is spanned by isotopy classes of all handlebody braid diagrams, with multiplication given by concatenation of diagrams, modulo the *skein relation*

(6.1) 

Algebraically, $\mathrm{H}_{g,n}$ is the quotient of $\mathbb{K}\mathrm{B}_{g,n}$ by the two-sided ideal generated by the elements

$$(\beta_i + \mathtt{q})(\beta_i - \mathtt{q}^{-1}),$$

for $i = 1, \dots, n-1$. In this interpretation, we write H_i for the image of β_i in the quotient, but keep the notation τ_u for the others. In defining $\mathrm{H}_{g,n}$ we only demand that the H_i satisfy a quadratic relation which is equivalent to either of

$$H_i^2 = (\mathtt{q} - \mathtt{q}^{-1})H_i + 1, \quad H_i^{-1} = H_i - (\mathtt{q} - \mathtt{q}^{-1}).$$

Note that the coils τ_u do not satisfy a polynomial relation. Hence, as for the handlebody Coxeter group, $\mathrm{H}_{g,n}$ does not embed into a Hecke algebra of type A.

Remark 6.2. With respect to Definition 6.1, we note:

(1) In case $g = 0$, the algebra $\mathrm{H}_{0,n}$ is the type A Hecke algebra.

(2) For $g = 1$, the algebra $\mathrm{H}_{1,n}$ is the extended affine Hecke algebra of type A.

(3) The algebra $\mathrm{H}_{g,n}$ has been studied in [5], but not much appears to be known.

For all $w \in \mathrm{S}_n$ we choose a reduced expression $\underline{w} = s_{i_k} \dots s_{i_1} \in \mathrm{S}_n$ once and for all. We define $H_w = H_{\underline{w}} = H_{i_k} \dots H_{i_1}$. We still have the Jucys–Murphy elements $L_{u,i}$ from (3.8), and the analog of Proposition 3.15 is the following.

Proposition 6.3. *The set*

$$\left\{ L_{u_1,i_1}^{a_1} \dots L_{u_m,i_m}^{a_m} H_w \;\middle|\; \begin{array}{c} w \in \mathrm{S}_n, m \in \mathbb{N}, \boldsymbol{a} \in \mathbb{Z}^m, \\ (\boldsymbol{u}, \boldsymbol{i}) \in (\{1,\dots,g\} \times \{1,\dots,n\})^m, i_1 \leq \dots \leq i_m \end{array} \right\} \tag{6.2}$$

is a $\mathbb{K}$*-basis of* $\mathrm{H}_{g,n}$.

Proposition 6.3 can be seen as a higher genus version of equation (3.10) in [4].

Proof. That the set in (6.2) spans can be proven mutatis mutandis as in Lemma 3.14. That is, we use the relations in (3.9) together with the following immediate consequences of (3.9) and the skein relation (6.1):

(6.3)

$$H_{i-1} L_{u,i} = L_{u,i-1} H_{i-1} - (\mathtt{q} - \mathtt{q}^{-1}) L_{u,i}, \quad H_i^{-1} L_{u,i} = L_{u,i+1} H_i^{-1} + (\mathtt{q} - \mathtt{q}^{-1}) L_{u,i},$$
$$H_{i-1}^{-1} L_{u,i}^{-1} = L_{u,i-1}^{-1} H_{i-1}^{-1} + (\mathtt{q} - \mathtt{q}^{-1}) L_{u,i}^{-1}, \quad H_i L_{u,i}^{-1} = L_{u,i+1}^{-1} H_i - (\mathtt{q} - \mathtt{q}^{-1}) L_{u,i}^{-1}.$$

Moreover, Proposition 3.15 shows the elements in (6.2) are linearly independent if $\mathtt{q} = 1$ (note that the handlebody Coxeter group $\mathrm{W}_{g,n}$ can be obtained from $\mathrm{H}_{g,n}$ by specializing $\mathtt{q} = 1$), which implies that they are linearly independent for generic $\mathtt{q}$. ∎

Combining Proposition 6.3 with the respective statement for the handlebody Coxeter group Proposition 3.15, we get the following.

Corollary 6.4. *For* $\mathtt{q} = 1$*, the map given by* $\tau_u \mapsto t_u$, $H_i \mapsto s_i$*, is an isomorphism of* $\mathbb{K}$*-algebras* $\mathrm{H}_{g,n} \xrightarrow{\cong} \mathbb{K}\mathrm{W}_{g,n}$.

In order to define a sandwich cell datum, we first note that there is an antiinvolution $(\_)^\star$ on $\mathrm{H}_{g,n}$ given by mirroring along a horizontal axis, but not changing crossings (e.g. positive crossings as in (5.3) remain positive). Moreover, we now recall the cellular basis of $\mathrm{H}_{0,n}$ from [49], see also [47], Chapter 3.

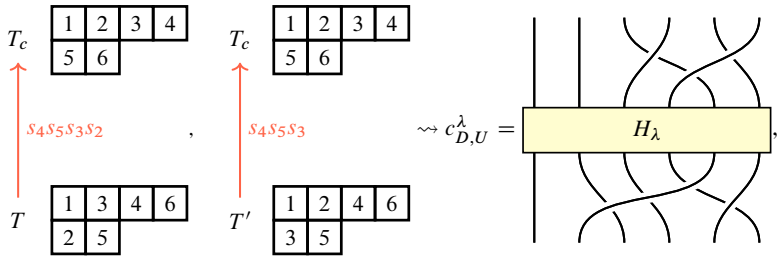
Remark 6.5. The cellular basis we are going to recall is not the Kazhdan–Lusztig basis, but the so-called *Murphy basis* or *standard basis*. This basis has the advantage that it has a known generalization to the case of $g = 1$, see [13]. On the other hand, we are not aware of a generalization of Kazhdan–Lusztig theory to higher genus.

Recall that a *partition* λ of n of length $\ell(\lambda) = k$ is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ of integers adding up to $|\lambda| = n$. Associated to λ is its *Young diagram* $[\lambda]$ which we will illustrate using the English convention. Moreover, any filling T of $[\lambda]$ with non-repeating numbers from $\{1,\dots,n\}$ is called a *tableaux* of shape λ. Such a T is further called *standard* if its entries increase along rows and columns. The *canonical tableaux* T_c is the filling where the numbers increase from left to right and top to bottom.

To a standard tableaux T of shape λ we can associate a down diagram D using the classical *permute entries strategy*. We also need the Young subgroup $\mathrm{S}_\lambda^{row} = \prod_i \mathrm{S}_{r(i)}$ being the row stabilizer of λ, i.e., $r(i)$ is the length of the ith row of $[\lambda]$. Then let $H_\lambda = \prod_i (\sum_{w \in \mathrm{S}_{r(i)}} H_w)$ (note that the bracketed terms commute, so the order is not important) and let H_T be the positive braid lifts of (a choice of) minimal permutation from T to T_c, identifying s_i with the transposition $(i, i+1)$. We then let $D = D(T) = H_T$,

$U = U(T') = D(T')^\star$ and $c^\lambda_{D,U} = DH_\lambda U$. Note that the middle H_λ of $c^\lambda_{D,U}$ is a quasi-idempotent, i.e., $H_\lambda^2 = rH_\lambda$ for some potentially not invertible $r \in \mathbb{K}$. Note also that $H_{0,n}$ is a subalgebra of $H_{g,n}$, so we can use the above construction for $H_{g,n}$ as well.

Example 6.6. The above is best illustrated in an example. For $\lambda = (4,2)$ we have $\mathrm{S}^{row}_\lambda = \mathrm{S}_4 \times \mathrm{S}_2$ and $H_\lambda = (1 + H_1 + \cdots + H_{w_0})(1 + H_5)$, where w_0 is the longest element in S_4 written in terms of the generators s_1, s_2 and s_3. Moreover, for T respectively, T' as below we construct $D = D(T)$ respectively, $U = D(T')^\star$ by



where we illustrated H_λ as a box.

We let $\Lambda = (\Lambda, \leq_d)$ be the set of all partitions of n, which are ordered by the dominance order $\leq_d$ (we use the convention from Section 3.1 in [47]). The set M_λ is the set of all standard tableaux of shape λ. The middle part is $\mathbb{S}_\lambda = \mathrm{L}_{g,n} H_\lambda$, where $\mathrm{L}_{g,n}$ is the subalgebra of $\mathrm{H}_{g,n}$ spanned by the Jucys–Murphy elements. The sandwiched basis of $\mathrm{L}_{g,n} H_\lambda$ that we use is (6.2) restricted to $\mathrm{L}_{g,n}$ and then composed with H_λ. We then let $c^\lambda_{D,b,U} = D(T)bU(T')$, where $b = b(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{i})$ runs over elements of the form $L^{a_1}_{u_1,i_1} \ldots L^{a_m}_{u_m,i_m} H_\lambda$ with the same indices as in (6.2). We then get

$$\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in \mathbb{S}_\lambda\}. \tag{6.4}$$

Proposition 6.7. *The above defines a sandwich cell datum for* $\mathrm{H}_{g,n}$.

Proof. To argue that (6.4) spans $\mathrm{H}_{g,n}$, we use Proposition 3.16 in [47] as follows. Instead of the elements $H_w \in \mathrm{H}_{0,n} \hookrightarrow \mathrm{H}_{g,n}$ in (6.2), we could also use the Murphy basis $DH_\lambda U$ (as explained above, i.e., (6.4) for $g = 0$). Then, by (3.9) and (6.3), we can push the D to the left, that is,

$$L^{a_1}_{u_1,i_1} \ldots L^{a_m}_{u_m,i_m} DH_\lambda U = DL^{a_1}_{u_1,i'_1} \ldots L^{a_m}_{u_m,i'_m} H_\lambda U + \text{error terms},$$

which shows that the $c^\lambda_{D,b,U}$ span. To show that (6.4) is a basis, we observe that, left aside error terms, the process of moving D to the left does not change the number m, the powers $\boldsymbol{a}$, and associated cores $\boldsymbol{u}$, while $\boldsymbol{i}$ changes, but the property $i_1 \leq \cdots \leq i_m$ is preserved. Hence, a counting argument ensures that (6.2) and (6.4) have the same (finite) size per fixed $(m, \boldsymbol{a}, \boldsymbol{u})$, so (6.4) is indeed a basis.

It remains to prove (2.1) as all other claimed properties hold by construction. Let us write $\boldsymbol{L} = L^{a_1}_{u_1,i_1} ... L^{a_m}_{u_m,i_m}$ for short, omitting the precise indices. We calculate:

$$\begin{aligned} D\boldsymbol{L}H_\lambda UD'\boldsymbol{L}'H_\mu U' &= \boldsymbol{L}^\dagger DH_\lambda UD'H_\mu U'(\boldsymbol{L}')^\dagger + \text{error terms} \\ &\equiv_{\leq\Lambda} r(U,D')\boldsymbol{L}^\dagger DH_{\max_{\leq_d}(\lambda,\mu)}U'(\boldsymbol{L}')^\dagger + \text{error terms} \\ &= D\boldsymbol{L}H_{\max_{\leq_d}(\lambda,\mu)}\boldsymbol{L}'U' + \text{error terms} \\ &= D\boldsymbol{L}(\boldsymbol{L}')^\dagger H_{\max_{\leq_d}(\lambda,\mu)}U' + \text{error terms} \\ &\equiv_{\leq\Lambda} D\boldsymbol{L}(\boldsymbol{L}')^\dagger H_{\max_{\leq_d}(\lambda,\mu)}U' = D\boldsymbol{L}F\boldsymbol{L}'H_{\max_{\leq_d}(\lambda,\mu)}U', \end{aligned}$$

where the † indicates a shift of the indices, as before, and F is some product of Jucys–Murphy elements. In this calculation we used (3.9) and (6.3) several times and also: The crucial first congruence follows from the classical case, see e.g. Theorem 3.20 in [47]. The final equality reorders the Jucys–Murphy elements to match the expression in (2.1), while the second congruence uses the observation that the error terms in (6.3) have a lower number of H_i, so correspond to Young diagrams with shorter rows. ∎

Let $e \in \mathbb{N}$ be the smallest number such that $[e]_{\mathsf{q}} = 0$, or let $e = \infty$ if no such e exists. An element $\lambda \in \Lambda$ is called *e-restricted* if $\lambda_i - \lambda_{i+1} < e$.

Theorem 6.8. *Let $\mathbb{K}$ be a field.*

(1) *An element $\lambda \in \Lambda$ is an apex if and only if λ is e-restricted.*

(2) *The simple $\mathrm{H}_{n,g}$-modules of apex $\lambda \in \Lambda$ are parameterized by simple modules of $\mathrm{L}_{g,n}H_\lambda$.*

(3) *The simple $\mathrm{H}_{n,g}$-modules of apex $\lambda \in \Lambda$ can be constructed as the simple heads of $\mathrm{Ind}^{\mathrm{H}_{n,g}}_{\mathrm{L}_{g,n}H_\lambda}(K)$, where K runs over (equivalence classes of) simple $\mathrm{L}_{g,n}H_\lambda$-modules.*

Proof. Claims (b) and (c) follow immediately from the abstract theory, as in the cases discussed before. The statement (a) follows because whether the form ϕ^λ is constant zero or not can be detected on the component where the Jucys–Murphy elements are trivial, meaning those $c^\lambda_{D,b,U}$ with $b = H_\lambda$. Hence, the classical theory applies, see e.g. [47], Section 3.4. ∎

Example 6.9. The cases $g = 0$ and $g = 1$ of Theorem 6.8 are well known:

(1) We have $\mathrm{L}_{0,n}H_\lambda = \mathbb{K}$, so the above is the classical parametrization of simple $\mathrm{H}_{0,n}$-modules, see e.g. Section 3.4 of [47].

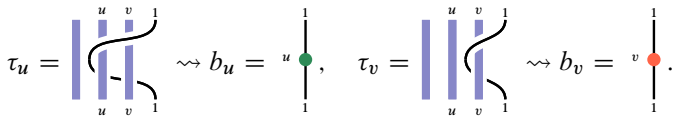
(2) For $g = 1$, Theorem 6.8 can be matched with e.g. Theorem 5.8 in [34].

6B. Cyclotomic handlebody Hecke algebras

We keep the terminology from the previous sections. In order to define and work with cyclotomic quotients, we use *blob diagrams of braids* instead of coils.

Denote by $\mathrm{H}^+_{g,n}$ the subalgebra of $\mathrm{H}_{g,n}$ generated by $\mathrm{H}_{0,n}$ and positive coils. Imitating Subsection 4B we introduce elements $b_1, \dots, b_g$ in $\mathrm{H}^+_{g,n}$ using the same pictures as

in (4.6):

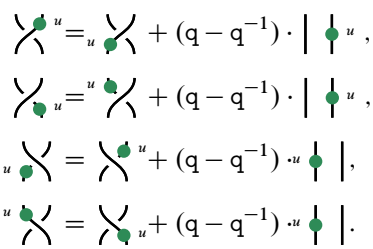


Although blobs are defined on the first strand from the left, one can define blobs on other strands exactly as in (4.13). These are the Jucys–Murphy elements from Definition 3.11 which we denote by $b_{u,i}$ to emphasize that they are not necessarily invertible.

Using the skein relation (6.1), the same calculations as in Lemma 4.39 give:

Lemma 6.10. *Blobs satisfy relations* (4.14) *and* (4.16) *and*

(6.5)



(*In comparison with Lemma* 4.39*, note the missing cup-cap term.*) ∎

Of course, Proposition 6.3 and (6.5) give:

Lemma 6.11. *The set*

$$\left\{ b_{k_1,i_1}^{a_1} \ldots b_{k_m,i_m}^{a_m} H_w \;\middle|\; \begin{array}{c} w \in \mathrm{S}_n, m \in \mathbb{N}, \boldsymbol{a} \in \mathbb{N}^m, \\ (\boldsymbol{u}, \boldsymbol{i}) \in (\{1, \ldots, g\} \times \{1, \ldots, n\})^m, i_1 \leq \cdots \leq i_m \end{array} \right\}$$

is a $\mathbb{K}$*-basis of* $\mathrm{H}_{g,n}^{+}$. ∎

As in the previous sections, we fix cyclotomic parameters $\boldsymbol{b} = (\mathtt{b}_{u,i}) \in \mathbb{K}^{d_1+\cdots+d_g}$, and a degree vector $\boldsymbol{d} = (\mathtt{d}_u) \in \mathbb{N}^g$.

Definition 6.12. We define the *cyclotomic handlebody Hecke algebra* (in n strands and of genus g) $\mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}$ as the quotient of $\mathrm{H}_{g,n}^{+}$ by the two-sided ideal generated by the *cyclotomic relations*

$$(b_u - \beta_{u,1})\varpi_1(b_u - \beta_{u,2})\varpi_2 \ldots (b_u - \beta_{u,d_u-1})\varpi_{d_u-1}(b_u - \beta_{u,d_u}) = 0, \tag{6.6}$$

where ϖ_i is any finite (potentially empty) word in b_v for $v \neq u$.

The relations imply that no strand can carry more than d_u blobs of type u.

Remark 6.13. With respect to Definition 6.12 we note:

(1) In case $g = 0$, the algebra $\mathrm{H}_{0,n}^{\boldsymbol{d},\boldsymbol{b}}$ is the type A Hecke algebra.

(2) For $g = 1$, the algebra $\mathrm{H}_{1,n}^{\boldsymbol{d},\boldsymbol{b}}$ is the Ariki–Koike algebra as defined and studied in e.g. [4], [8] or [10].

(3) For $g = 2$ and $d_1 = d_2 = 2$, the algebra $\mathrm{H}_{2,n}^{\boldsymbol{d},\boldsymbol{b}}$ can be compared to the two boundary Hecke algebra as in [11].

Note also that imposing a single relation involving only τ_g (this is the first coil counting from the right, cf. (3.1)) gives a handlebody version of Ariki–Koike algebras. For $g = 2$, this case can be interpreted as an extended affine version of Ariki–Koike algebras.

Our next aim to find a basis and a dimension formula for $\mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}$.

Lemma 6.14. *If $b_{u,1}$ is of order d_u, then $b_{u,i}$ is also of order d_u for all $1 \leq i \leq n$.*

Proof. Since the relations in Lemma 6.10 preserve the number and the type of the blobs involved, the result follows at once. ∎

Proposition 6.15. *The set*

$$\left\{ b_{k_1,i_1}^{a_1} \dots b_{k_m,i_m}^{a_m} H_w \;\middle|\; \begin{array}{c} w \in \mathrm{S}_n, m \in \mathbb{N}, \boldsymbol{a} \in \mathbb{N}^m, \\ (\boldsymbol{u},\boldsymbol{i}) \in (\{1,\dots,g\} \times \{1,\dots,n\})^m, i_1 \leq \dots \leq i_m, a_j < d_j \end{array} \right\} \tag{6.7}$$

is a $\mathbb{K}$-basis of $\mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}$.

Proof. That the set spans is a consequence of Lemma 6.10, which we use to pull blobs to the bottom, and Lemma 6.14, which gives the restriction $a_j < d_j$. Now let A be the $\mathbb{K}$-span of $\{b_{k_1,i_1}^{a_1} \dots b_{k_m,i_m}^{a_m} j\}$ with the same indexing sets as in (6.7). Let further $J_u \subset \mathrm{H}_{g,n}^{+}$ the two-sided ideal generated by all elements having d_u blobs on the first strand. Linear independence follows from Lemma 6.11 together with the observation that $A \cap J_u = \emptyset$. ∎

Recall the blob numbers $\mathtt{BN}_{g,\boldsymbol{d}}$ from (4.9).

Proposition 6.16. *The dimension of the free $\mathbb{K}$-module $\mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}$ is*

$$\dim_{\mathbb{K}} \mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}} = (\mathtt{BN}_{g,\boldsymbol{d}})^n \, n!.$$

Proof. There are $n!$ elements H_w, each one having n strands. Since every strand can carry up to $\mathtt{BN}_{g,\boldsymbol{d}}$ blobs, the claim follows from Proposition 6.15. ∎

Remark 6.17. Recall that $\mathtt{BN}_{0,\boldsymbol{d}} = 1$ and $\mathtt{BN}_{1,\boldsymbol{d}} = \mathtt{d}_1$, see Example 4.20. Thus, we recover the dimensions

$$\dim_{\mathbb{K}} \mathrm{H}_{0,n}^{\boldsymbol{d},\boldsymbol{b}} = n!, \quad \dim_{\mathbb{K}} \mathrm{H}_{1,n}^{\boldsymbol{d},\boldsymbol{b}} = \mathtt{d}_1^n \, n!,$$

the former being well-known, of course; the latter appears in equation (3.10) in [4].

To construct a sandwich cell datum one can use the same strategy as in Subsection 6A. Keeping the above discussion in mind, e.g. Proposition 6.15, the only difference is that $\mathbb{S}_\lambda$ is now the algebra obtained by taking the quotient of $\mathrm{L}_{g,n} H_\lambda$ by (6.6). The sandwiched basis we choose is the restriction of (6.7). Otherwise nothing changes and we obtain a set

$$\{c_{D,b,U}^{\lambda} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in \mathbb{S}_\lambda\},$$

as well as the two results, where the denote the aforementioned quotient by $\mathrm{L}_{g,n}^{\boldsymbol{d},\boldsymbol{b}} H_\lambda$:

Proposition 6.18. *The above defines a sandwich cell datum for* $\mathrm{H}_{g,n}^{\boldsymbol{d},\boldsymbol{b}}$. ∎

Theorem 6.19. *Let* $\mathbb{K}$ *be a field.*

(1) *A* $\lambda \in \Lambda$ *is an apex if and only if* λ *is e-restricted.*

(2) *The simple* $\mathrm{H}_{n,g}^{\boldsymbol{d},\boldsymbol{b}}$*-modules of apex* $\lambda \in \Lambda$ *are parameterized by simple modules of* $\mathrm{L}_{g,n}^{\boldsymbol{d},\boldsymbol{b}} H_\lambda$.

(3) *The simple* $\mathrm{H}_{n,g}^{\boldsymbol{d},\boldsymbol{b}}$*-modules of apex* $\lambda \in \Lambda$ *can be constructed as the simple heads of* $\mathrm{Ind}_{\mathrm{L}_{g,n}^{\boldsymbol{d},\boldsymbol{b}} H_\lambda}^{\mathrm{H}_{n,g}^{\boldsymbol{d},\boldsymbol{b}}}(K)$*, where* K *runs over (equivalence classes of) simple* $\mathrm{L}_{g,n}^{\boldsymbol{d},\boldsymbol{b}} H_\lambda$*-modules.* ∎

Example 6.20. For $g = 0$, nothing new happens of course compared to Theorem 6.8, while the $g = 1$ version of Theorem 6.19 is a non-explicit version of Theorem 3.30 in [13].

Acknowledgments. We thank a referee, Abel Lacabanne, David Rose, Catharina Stroppel and Arik Wilbert for comments on this paper and discussions related to the diagrammatics of handlebodies.

Funding. D. T. does not deserve to be supported, but was still supported by the Hausdorff Research Institute for Mathematics (HIM) during the Junior Trimester Program New Trends in Representation Theory. Part of this paper were written during that program, which is gratefully acknowledged. P. V. was supported by the Fonds de la Recherche Scientifique-FNRS under Grant no. MIS-F.4536.19.

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Received July 23, 2021; revised April 15, 2022. Published online July 7, 2022.

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