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Uniqueness of Yudovich's solutions to the 2D incompressible Euler equation despite the presence of sources and sinks

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Abstract. In 1962, Yudovich proved the existence and uniqueness of classical solutions to the 2D incompressible Euler equations in the case where the fluid occupies a bounded domain with entering and exiting flows on some parts of the boundary. The normal velocity is prescribed on the whole boundary, as well as the entering vorticity. The uniqueness part of Yudovich's result holds for Hölder vorticity, by contrast with his 1961 result on the case of an impermeable boundary, for which the normal velocity is prescribed as zero on the boundary, and for which the assumption that the initial vorticity is bounded was shown to be sufficient to guarantee uniqueness. Whether or not uniqueness holds as well for bounded vorticities in the case of entering and exiting flows has been left open until 2014, when Weigant and Papin succeeded to tackle the case where the domain is a rectangle. In this paper we adapt Weigant and Papin's result to the case of a smooth domain with several internal sources and sinks.

1. Introduction

This first section is devoted to the presentation of the model and of the mathematical problem which are at stake in this paper.

1.1. Geometry of the domain

We consider a bounded domain $\Omega \subset \mathbb{R}^2$ whose boundary, denoted by Γ, is a C^2 simple curve which can be decomposed as

$$\Gamma = \bigcup_{i \in I} \Gamma_i, \tag{1.1}$$

where I is a finite set of index which admits a partition $I = \{0\} \cup I_{\text{in}} \cup I_{\text{out}}$, the sets Γ_i are the connected components of Γ, with the convention that Γ_0 is the external boundary.

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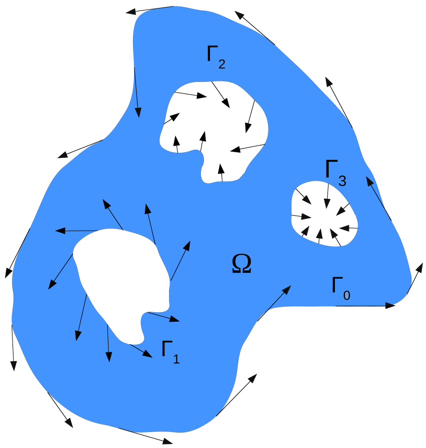

Figure 1. Example of a fluid domain with one source and two sinks.

We set $I^* := I_{\text{in}} \cup I_{\text{out}} = I \setminus \{0\}$, and

$$\Gamma_{\text{in}} := \cup_{i \in I_{\text{in}}} \Gamma_i \quad \text{and} \quad \Gamma_{\text{out}} := \cup_{i \in I_{\text{out}}} \Gamma_i . \tag{1.2}$$

The indexes refer to the fact that, below, Γ_{in} is the zone with entering flux and Γ_{out} is the zone with exiting flux, see (1.6).

1.2. The equations at stake

We assume that the domain Ω is occupied by an incompressible perfect flow whose evolution is driven by the incompressible Euler equation. More precisely, we consider the following transport equation, where the scalar function $\omega(t,x)$ denotes the fluid vorticity:

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{on } \Omega, \tag{1.3a}$$

$$\omega(0,.) = \omega_0 \quad \text{on } \Omega, \tag{1.3b}$$

$$\omega = \omega_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \tag{1.3c}$$

where the vector field $u(t,x)$ is the fluid velocity, given as the solution of the system

$$\text{div } u = 0 \quad \text{on } \Omega, \tag{1.4a}$$

$$\text{curl } u = \omega \quad \text{on } \Omega, \tag{1.4b}$$

$$u \cdot n = g \quad \text{on } \Gamma, \tag{1.4c}$$

$$\int_{\Gamma_i} u \cdot \tau = \mathcal{C}_i \quad \text{for all } i \in I^*, \tag{1.4d}$$

and where n is the outward unit normal vector to the boundary, τ is the counterclockwise tangent vector to the boundary, and the $(\mathcal{C}_i(t))_{i \in I^*}$ are the circulations of the fluids around each connected component Γ_i, for $i \in I^*$, of the boundary, given as the solutions of the following Cauchy problem:

$$\mathcal{C}_i'(t) = -\int_{\Gamma_i} \omega g \quad \text{for all } i \in I^*, \tag{1.5a}$$

$$\mathcal{C}_i(0) = \mathcal{C}_{i,0} \quad \text{for all } i \in I^*. \tag{1.5b}$$

The quantities ω_0, ω_{in}, g and $(\mathcal{C}_{i,0})_{i \in I^*}$, which appear in the right-hand sides of the equations above, are given data. Regarding g, we assume that, at any time t, it has zero average on Γ, which is the compatibility condition associated with the incompressibility, and we also assume, as hinted above, that

$$g \leq 0 \quad \text{on } \Gamma_{\text{in}}, \quad g \geq 0 \quad \text{on } \Gamma_{\text{out}}, \quad g = 0 \quad \text{on } \Gamma_0, \tag{1.6}$$

which means that Γ_{in} is the part of the boundary with entering flux, and Γ_{out} is the part with exiting flux. Thus (1.3c) is a condition on the vorticity which enters in the domain Ω. Since Ω is multiply connected, it is necessary to prescribe the circulations, see (1.4d), to guarantee the uniqueness of the solution u to the system (1.4). These circulations evolve in time according to Kelvin's law (1.5a), where the right-hand side encodes the vorticity flux across Γ_i. We refer to Lemma 1.2 in [33] and to Section 1.3 in [5] for a derivation of (1.5a) from the velocity formulation of the incompressible Euler equation. Observe that for i in I_{in}, the trace of the vorticity on Γ_i, which appears in the right-hand side of (1.5a) is prescribed according to (1.3c), whereas it is part of the solution for i in I_{out}.

1.3. An open problem on the uniqueness of solutions with bounded vorticity

The system (1.3)–(1.4)–(1.5) was proposed by Yudovich in [33], who proved the existence and uniqueness of classical solutions. Later, the existence part of Yudovich's result in the permeable case has been generalized to weaker solutions, see [1, 5, 6, 27]. On the other hand, for a long time, no progress has been obtained on the uniqueness part despite that the conjecture that uniqueness should hold in the case where the vorticity is bounded, was broadly shared. Indeed such a result is well known in the case of impermeable boundaries, for which $u \cdot n = 0$ on Γ instead of (1.4c), as proved by Yudovich in his celebrated result [32]. This problem was recalled for instance in Section 3.2 of [20] and in Section 3.2.1 of [12]. Indeed, this open problem has known a regain of interest in the controllability community after the works on the incompressible Euler system by Coron and Glass in [7–9, 11]. Finally, a breakthrough result was obtained by Weigant and Papin in 2014, see [28], who proved the case where the fluid domain is a rectangle with lateral inlet and outlet, under the assumption that the vorticity is bounded. Their nice proof makes use of two energy estimates on the difference of two solutions, one related to the time-evolution of the kinetic energy, and another one associated with a clever auxiliary function. The two estimates are combined to deduce a stability estimate which in particular guarantees uniqueness of the solutions with bounded vorticities for this particular geometry. In this paper, we extend Weigant and Papin's approach to prove the uniqueness of solutions with bounded vorticity in the case, presented above, of a smooth multiply-connected domain with several interior sources and sinks.

2. Reminder on the definition and existence of solutions with bounded vorticity

This section is devoted to recall a definition and an existence result of weak solutions to the system (1.3)–(1.4)–(1.5) in the case where the vorticity is bounded. This will set up the context to which the uniqueness result of this paper is applied.

Let us start the discussion with the choice of the unknowns. Because of the boundary condition (1.3c), the vorticity ω is a natural choice, for instance compared to the velocity u. Also, as mentioned above, the vorticity which flows through Γ_{out} is unknown; we denote it by ω_{out}. Taking (1.3c) into account for what concerns Γ_{in}, Kelvin's laws (1.5) can therefore be recast as

$$\mathcal{C}_i(t) = \mathcal{C}_{i,0} - \int_0^t \int_{\Gamma_i} \omega_{\text{in}}\, g \quad \text{for } i \in I_{\text{in}}, \tag{2.1a}$$

$$\mathcal{C}_i(t) = \mathcal{C}_{i,0} - \int_0^t \int_{\Gamma_i} \omega_{\text{out}}\, g \quad \text{for } i \in I_{\text{out}}. \tag{2.1b}$$

The fact that the system (1.4) determines a single solution u, the right-hand sides being given, is well known. In particular, we have the following result, see for instance [33], where we omit the time-dependence since the time only plays the role of a parameter in this part of the system.

Proposition 2.1. *Let $(\mathcal{C}_i)_i \in \mathbb{R}^{I^*}$, let g be in the Hölder space $C^{1,\alpha}(\Gamma)$ for some $\alpha \in (0,1)$, with zero average on Γ, and let ω be in $L^\infty(\Omega)$. Then the system (1.4) admits a unique solution*

$$u \in \bigcup_{1 \le p < +\infty} W^{1,p}(\Omega).$$

Moreover, there is a constant $C > 0$ such that, for all $p \geq 2$,

$$\|u\|_{W^{1,p}(\Omega)} \le C\, p \Big(\|\omega\|_{L^p(\Omega)} + \|g\|_{C^{1,\alpha}(\Gamma)} + \sum_{i \in I^*} |\mathcal{C}_i| \Big). \tag{2.2}$$

Remark 2.2. The dependence with respect to p in (2.2) shall be useful in Section 10. Indeed, that the constant in front of the parenthesis in the right-hand side is bounded by p for large p is crucial in Yudovich's proof of uniqueness in the impermeable case, as it allows to bypass the failure of the Lipschitz estimate of u, which is the main difficulty with respect to case of classical solutions. To overcome this difficulty, Yudovich used some *a priori* bounds on the L^p norms of the vorticity. We refer here to [34, 35] for more details.

Finally, it only remains to tackle the transport part (1.3) of the system (1.3)–(1.4)–(1.5). When considering this initial boundary value problem on the time interval $[0, T]$, with $T > 0$, we are naturally led to a weak formulation corresponding to the following family of identities:

$$\int_{t_0}^{t_1} \int_\Omega \omega\, (\partial_t \phi + u \cdot \nabla \phi) = \Big[\int_\Omega \omega \phi \Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\Gamma_{\text{in}}} \omega_{\text{in}} \phi\, g + \int_{t_0}^{t_1} \int_{\Gamma_{\text{out}}} \omega_{\text{out}} \phi\, g, \tag{2.3}$$

for $0 \leq t_0 < t_1 \leq T$, and for some test functions ϕ to determine, and where the notation $[a]_{t_0}^{t_1}$ means

$$[a]_{t_0}^{t_1} := a(t_1) - a(t_0). \tag{2.4}$$

Definition 2.1. Let $T > 0$. Let $\omega_0 \in L^\infty(\Omega)$, $(\mathcal{C}_{i,0})_{i\in I^*} \in \mathbb{R}^{I^*}$, $g \in L^\infty([0,T]; C^{1,\alpha}(\Gamma))$ with zero average on Γ at every time, and let ω_{in} be in $L^\infty([0,T]; L^\infty(\Gamma_{\text{in}}, g))$. We say that

$$(\omega, \omega_{\text{out}}) \in C^0([0,T]; L^\infty(\Omega)\text{-}w_*) \times L^\infty([0,T]; L^\infty(\Gamma_{\text{out}}, |g|)), \tag{2.5}$$

is a weak solution to the system (1.3)–(1.4)–(1.5) when, for every test function ϕ in $H^1([0,T] \times \Omega; \mathbb{R})$, and for every $t_0 < t_1$, the equality (2.3) holds true with u given, at time t in $[0,T]$, as the unique solution of the system (1.4) given by Proposition 2.1, where the circulations are given by (2.1a)–(2.1b).

Let us emphasize that the appearance of g in the notation $L^p(\Gamma, |g|)$ refers to the measure $|g|\mathcal{H}^1$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure on Γ. On the other hand, the notation $L^\infty(\Omega)\text{-}w_*$ in (2.5) stands for the space $L^\infty(\Omega)$ equipped with the weak star topology.

Remark 2.3. The article [4] by Boyer allows to give a sense to the trace of ω on the permeable part of the boundary, that is, where $g \neq 0$, for solutions, in the sense of distributions, of the transport equation (1.3a). Therefore, when it does not lead to confusion, we use the notation ω instead of ω_{in} or ω_{out} on the corresponding parts Γ_{in} and Γ_{out} of the boundary.

It also establishes that the assumption regarding the continuity in time in (2.5) is not restrictive.

The existence of weak solutions to the system (1.3)–(1.4)–(1.5) in the sense of Definition 2.1 has been obtained in Theorem 1 of [27]; see also Theorem 3 in [5] for a slightly different proof, and for some other existence results of weaker solutions.

3. Statement of the main result

This section is devoted to the statement of the main result of the paper. We also make a few remarks about it and finally explain how is organized of the rest of the paper.

3.1. Statement of the main result and a few remarks

The following result establishes a quantitative stability estimate for weak solutions to the system (1.3)–(1.4)–(1.5) in the sense of Definition 2.1, corresponding to different initial and boundary data, which in particular implies uniqueness for solutions corresponding to the same initial and boundary data.

Theorem 3.1. *Let $T > 0$ and two weak solutions to the system* (1.3)–(1.4)–(1.5) *as in Definition* 2.1 *corresponding to T, to the same boundary data g for the normal velocity,* *and to some possibly different data. We denote those initial and boundary data respectively $\omega_{\rm in}^1$ and $\omega_{\rm in}^2$ for the entering vorticity; $(\mathcal{C}_{i,0}^1)_{i\in I^*}$ and $(\mathcal{C}_{i,0}^2)_{i\in I^*}$ for the initial circulations; and ω_0^1 and ω_0^2 for the initial vorticity. Let u^1 and u^2 the corresponding velocities as given by Proposition 2.1. Then there exists a continuous function $F\colon (\mathbb{R}_+)^3 \times \mathbb{R}^{I^*} \mapsto \mathbb{R}_+$ satisfying $F(t,0,0,0) = 0$ for all $t \geq 0$, such that for all t in $[0,T]$,*

$$\begin{aligned}(3.1)\quad \|u^1(t,.) - u^2(t,.)\|_{L^2(\Omega)} + \int_0^t \int_\Gamma |u^1 - u^2|^2\,|g| \\ \leq F\big(t, \|\omega_0^1 - \omega_0^2\|_{L^\infty(\Omega)}, \|\omega_{\rm in}^1 - \omega_{\rm in}^2\|_{L^\infty([0,t];L^\infty(\Gamma_{\rm in},|g|))}, (\mathcal{C}_{i,0}^1 - \mathcal{C}_{i,0}^2)_{i\in I^*}\big).\end{aligned}$$

In particular, for some given initial and boundary data, there exists a unique weak solution to the system (1.3)–(1.4)–(1.5) in the sense of Definition 2.1.

Remark 3.2. The results of Theorem 3.1 can be extended to the slightly more general case where the vorticity is unbounded but with a moderate growth of its L^p norms, as it was done by Yudovich in [34, 35]. Indeed, the limitation in terms of regularity assumption on the vorticity is due to interior terms, which are treated similarly in the permeable and in the impermeable case. As a matter of fact, to extend Theorem 3.1 to the same setting as in [34, 35], it is sufficient to adapt the Osgood argument used in Section 10.

Remark 3.3. Theorem 3.1 can be extended to the case where the boundary datum g for the normal velocity on the boundaries of the sources and of the sinks oscillates in time, as long as the sign is the same, for every time, on each connected component. It is therefore possible to consider some cases where a inner connected component of the boundary is at some time a source and at another time a sink. On the other hand, transitions between an inflow part and an outflow part in space, on the same connected component of a boundary, seem to require careful studies. Let us highlight that such transitions occur in [28], with right angles at the places where transitions occur.

Remark 3.4. Let us insist on the fact that the stability estimate (3.1) concerns two solutions to the system (1.3)–(1.4)–(1.5) corresponding to the same boundary data g for the normal velocity. The analysis performed below does not consider the issue of the stability of the solutions to the system (1.3)–(1.4)–(1.5) with respect to perturbations of g. This issue is of interest with respect to the stabilization issue [9, 11].

Remark 3.5. In the impermeable case, several proofs of uniqueness are available, with quantitative estimates in different topologies for different quantities. More precisely, while the original proof by Yudovich in [32] relies on an energy estimate, that is, on the L^2 norm of the fluid velocity, some alternative proofs have been since found, in particular thanks to a Lagrangian viewpoint in Theorem 3.1 of [21] by Marchioro and Pulvirenti, where the L^1 norm of the flow map is used, and thanks to tools of optimal transportation theory in [19] by Loeper with the Wasserstein distance W_2, and in [14] by Hauray with the Wasserstein distance W_∞. While the proof of Theorem 3.1 given below uses an energy-type argument, it would be interesting to investigate whether an alternative proof of Theorem 3.1 based on the Lagrangian viewpoint could also be carried on.

Remark 3.6. Observe that Theorem 3.1 implies in particular an energy estimate for a (single) weak solution to the system (1.3)–(1.4)–(1.5) in the sense of Definition 2.1, by considering the solution to the system (1.3)–(1.4)–(1.5) corresponding to the same initial circulations and to the same prescribed trace g for the normal velocity but with zero initial and entering vorticity, and therefore with zero vorticity in Ω and on Γ_{out} at any time.

Remark 3.7. Theorem 3.1 implies in particular the uniqueness of the unstationary Euler system with prescribed normal velocity and entering vorticity. Let us recall that, on the other hand, it is known that solutions of the 2D stationary Euler system with prescribed normal velocity and entering vorticity are not in general unique (see e.g. [25]).

Remark 3.8. A natural question is whether the result in Theorem 3.1 and the method used in its proof can be extended to some other nonlinear evolution PDEs with non-conservative boundary conditions, in particular to the ones which share the same features to couple transport and non-local features. A candidate in this direction is the Camassa–Holm equation, set on a finite interval with some inhomogeneous boundary conditions as considered in [22]. An open question left in this paper was the uniqueness of the weak solutions obtained in Theorem 1 of [22].

Remark 3.9. Finally, let us mention that the weak-strong uniqueness property is a natural issue which has not been investigated yet for the system (1.3)–(1.4)–(1.5). In the case of impermeable boundaries, it is associated with the notion of dissipative solutions, see Chapter 4.4 of [18]. Recently, the effect of an impermeable boundary on the issue of the weak-strong uniqueness has been investigated by various authors, we refer here to the survey [29] for more on the subject, and it would be therefore interesting to extend these investigations to the case of permeable boundaries.

3.2. Strategy of the proof of Theorem 3.1 and organization of the rest of the paper

The rest of the paper is devoted to the proof of Theorem 3.1, which compares the difference of two solutions of the Euler equations in presence of sources and sinks. The set-up of the proof of Theorem 3.1, with the derivations of the equations satisfied by the difference of two solutions is done in Section 4. The initial idea is to perform an energy estimate from the weak vorticity formulation by using a stream function of the difference as a test function. In the case where the fluid occupies the whole plane, with nice decay at infinity, this corresponds to the identity

$$\int_{\mathbb{R}^2} u \cdot u = -\int_{\mathbb{R}^2} \psi\, \omega, \quad \text{where } \omega = \operatorname{curl} u = \Delta \psi.$$

This allows to bypass the velocity formulation, which has some unpleasant features in the permeable case, in particular due to the pressure. A technical difficulty in this process is to justify that the stream function associated with the difference of the two solutions at stake is regular enough to be taken as a test function in the weak formulation of the equation. The estimate of the stream function of the difference is performed in Section 5. The most delicate part is to obtain some estimates of the time-derivative of the stream function. This is accomplished thanks to the weak vorticity formulation with some other appropriate particular test functions associated with the geometry. Then the energy estimate of the difference is performed in Section 6. A difficulty is the presence of a "bad" boundary term corresponding to "the energy entering at the sources". Because of its sign, it is needed to bound this term for the energy estimate to be conclusive.

A great idea, first used by Weigant and Papin in the case where the fluid domain is a rectangle with lateral inlet and outlet, see [28], is to couple this energy estimate with a second one, obtained with the help of an appropriate test function. As in [28], we consider a harmonic test function with mixed boundary conditions, more precisely with Neumann inhomogeneous conditions on the boundary of the sources and with zero Dirichlet condition on the rest of the boundary. However, compared to the case of [28], where this test function is constructed and estimated thanks to some Fourier series, the construction and the estimates in the present case requires more work, which is done in Section 7. In Section 8, we state a generalized Lamb lemma which tackles trilinear integrals by some appropriate vector calculus identities and some integrations by parts. This allows to clarify the treatment of some convective terms in the second energy estimate associated with the auxiliary harmonic test function. The auxiliary energy-type estimate is performed in Section 9. Finally, we combine the two energy estimates in Section 10 and conclude the proof of Theorem 3.1 by Osgood's lemma.

4. Equations satisfied by the difference of two solutions

To prove Theorem 3.1, we use an energy method to compare the dynamics of two weak solutions of the Euler equation in the sense of Definition 2.1 corresponding to $T > 0$. To that extent, let us fix two solutions $(\omega^1, \omega^1_{\text{out}})$ and $(\omega^2, \omega^2_{\text{out}})$ of the Euler equation, with initial and boundary conditions

$$(\omega^1_0, (\mathcal{C}^1_{i,0})_{i\in I^*}, g, \omega^1_{\text{in}}) \quad \text{and} \quad (\omega^2_0, (\mathcal{C}^2_{i,0})_{i\in I^*}, g, \omega^2_{\text{in}}).$$

We consider the velocities u^1 and u^2 which are respectively associated to the previous quantities by Proposition 2.1. We use the following notations:

$$\tilde{\omega} := \omega^1 - \omega^2 \quad \text{and} \quad \hat{\omega} := \frac{\omega^1 + \omega^2}{2},$$
$$\tilde{u} := u^1 - u^2 \quad \text{and} \quad \hat{u} := \frac{u^1 + u^2}{2}.$$

By the linearity of (1.4a) and (1.5), the vector field $\tilde{u}$ satisfies, at every time, the system

$$\text{div}\, \tilde{u} = 0 \quad \text{in } \Omega, \tag{4.1a}$$
$$\text{curl}\, \tilde{u} = \tilde{\omega} \quad \text{in } \Omega, \tag{4.1b}$$
$$\tilde{u}\cdot n = 0 \quad \text{on } \Gamma, \tag{4.1c}$$
$$\int_{\Gamma_i} \tilde{u}\cdot \tau = \tilde{\mathcal{C}}_i \quad \text{for all } i \in I^*, \tag{4.1d}$$

where, for i in I^*, the circulation $\tilde{\mathcal{C}}_i$ of $\tilde{u}$ around Γ_i satisfies at any $t \in [0, T]$,

$$\tilde{\mathcal{C}}_i(t) = \tilde{\mathcal{C}}_{i,0} - \int_0^t \int_{\Gamma_i} \tilde{\omega} g, \quad \text{with } \tilde{\mathcal{C}}_{i,0} := \mathcal{C}^1_{i,0} - \mathcal{C}^2_{i,0}, \quad \text{for } i \in I^*. \tag{4.2}$$

Observe that, as hinted in Remark 2.3, we use the notation $\tilde{\omega}$ in the first term of the right-hand side of (4.2) instead of $\tilde{\omega}_{\text{in}} := \omega^1_{\text{in}} - \omega^2_{\text{in}}$ or $\tilde{\omega}_{\text{out}} := \omega^1_{\text{out}} - \omega^2_{\text{out}}$ on the corresponding parts Γ_{in} and Γ_{out} of the boundary Γ.

Using the Hodge–De Rham theory, we decompose, at every time, the vector field $\tilde{u}$ into two types of contributions, respectively corresponding to the circulations and to the vorticity:

- On the one hand, to tackle the effect of the circulations, we consider, for i in I^*, the functions f^i, which are the unique solutions in $C^2(\overline{\Omega})$ of the following boundary value problem:

$$\Delta f^i = 0 \ \text{ on } \Omega, \quad \text{and} \quad f^i = \delta_{i,j} \ \text{ on } \Gamma_j, \tag{4.3}$$

  where $\delta_{i,j}$ is equal to 1 when $i = j$ and to 0 otherwise.

- On the other hand, we consider, for any given bounded function ω, the unique solution $G[\omega]$ of the following boundary value problem:

$$\Delta G[\omega] = \omega \ \text{ on } \Omega, \quad \text{and} \quad G[\omega] = 0 \ \text{ on } \Gamma, \tag{4.4}$$

  and we set

$$K[\omega] := \nabla^\perp G[\omega],$$

  where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ (similarly, we will use below the convention $u^\perp = (-u_2, u_1)$ for any vector $u = (u_1, u_2)$ of $\mathbb{R}^2$).

It is a classical result of the Hodge–De Rham theory, see for instance [16], that there exists some real-valued functions $\tilde{\psi}_i = \tilde{\psi}_i(t)$, for $i \in I^*$, such that, at every time,

$$\tilde{u}(t,\cdot) = \nabla^\perp \tilde{\psi}(t,\cdot) \quad \text{in } \Omega, \tag{4.5}$$

where

$$\tilde{\psi}(t,\cdot) := G[\tilde{\omega}(t,\cdot)] + \sum_{i \in I^*} \tilde{\psi}_i(t) f^i. \tag{4.6}$$

Let us observe that, as a consequence of (4.1a) and (4.5),

$$\Delta \tilde{\psi} = \tilde{\omega} \quad \text{in } \Omega, \tag{4.7}$$

and that, as a consequence of (4.3) and (4.4),

$$\tilde{\psi}_{|\Gamma_i} = \tilde{\psi}_i, \quad \text{for all } i \in I, \tag{4.8}$$

where we set $\tilde{\psi}_0 := 0$. Combining (4.1d) and (4.5), we arrive at

$$\int_{\Gamma_i} \partial_n \tilde{\psi} = \tilde{\mathcal{C}}_i \quad \text{for all } i \in I^*. \tag{4.9}$$

Moreover, for all test function $\phi \in H^1([0,T] \times \Omega)$, and for every $0 \le t_0 < t_1 \le T$,

$$\int_{t_0}^{t_1} \int_\Omega (\tilde{\omega}\, \partial_t \phi + \tilde{\omega}\, \hat{u} \cdot \nabla \phi + \hat{\omega}\, \tilde{u} \cdot \nabla \phi) = \int_{t_0}^{t_1} \int_\Gamma \phi\, \tilde{\omega}\, g + \Big[\int_\Omega \tilde{\omega}\, \phi \Big]_{t_0}^{t_1}. \tag{4.10}$$

On the other hand, the half-sum $\hat{u}$ satisfies

$$\hat{u} \cdot n = g \quad \text{on } \Gamma. \tag{4.11}$$

5. Estimate of the stream function of the difference

This section is devoted to the regularity of the stream function $\tilde{\psi}$. The main result of this section reads as follows.

Proposition 5.1. *The stream function $\tilde{\psi}$ is in the space $C^0([0,T];W^{2,p}(\Omega))$ for all p in $[1,+\infty)$ and in the space $H^1([0,T];H^1(\Omega))$. Moreover, there exists $C>0$ such that for $i\in I$, the time derivative of the trace $\tilde{\psi}_i$ of $\tilde{\psi}$ on Γ_i satisfies, for $0\le t_0<t_1\le T$,*

$$\int_{t_0}^{t_1} |\tilde{\psi}_i'|^2 \le C\Big(\int_{t_0}^{t_1} \|\tilde{u}\|^2_{L^2(\Omega)} + \int_{t_0}^{t_1} \|\tilde{u}\|^2_{L^2(\Gamma,|g|)}\Big). \tag{5.1}$$

Proposition 5.1 will be useful in Section 6 to obtain an energy estimate, see Proposition 6.1, as it allows us to apply (4.10) to the case where the test function is the stream function $\tilde{\psi}$. On the other hand, the estimate (5.1) will be useful in Section 9. Let us highlight that it follows from the regularity of the solutions at stake that the right-hand side of (5.1) is finite.

Proof. The fact that the function $\tilde{\psi}$ is in the space $C^0([0,T];W^{2,p}(\Omega))$ for all for all p in $[1,+\infty)$ can be found in Lemma 1.4 of [33].

We now turn to the estimate of the time derivative of $(\tilde{\psi}_i)_i$, for $i\in I$. For $i=0$, we have by definition that $\tilde{\psi}_0=0$, so that (5.1) holds true in this case with any constant $C>0$ and it is therefore sufficient to deal with the case where $i\in I^*$. To this aim, we recall that there exist some constants $(g^i_j)_{i\in I^*,\, j\in I}$, with $g^i_0=0$, and some smooth functions $(g^i)_{i\in I^*}$ such that

$$\Delta g^i = 0 \qquad \text{on } \Omega, \tag{5.2a}$$

$$g^i = g^i_j \qquad \text{on } \Gamma_j, \text{ for } j\in I, \tag{5.2b}$$

$$\int_{\Gamma_j} \partial_{\mathbf{n}} g^i = -\delta_{i,j} \qquad \text{for } j\in I^*. \tag{5.2c}$$

These functions can be obtained by some appropriate linear combinations of the functions f^i, which are defined in (4.3); we refer here again to [16] for more details.

Lemma 5.2. *For any i in I^*, the time derivative $\tilde{\psi}_i'$ is in the space $L^2(0,T)$, and there exists $C>0$ such that for any $0\le t_0<t_1\le T$, the estimate (5.1) holds true.*

Proof of Lemma 5.2. Let $i\in I^*$. We take g^i as a (time-independent) test function in (4.10), and obtain that for every $0\le t_0<t_1\le T$,

$$\int_{t_0}^{t_1}\int_\Omega \big(\tilde{\omega}\,\hat{u}\cdot\nabla g^i + \hat{\omega}\,\tilde{u}\cdot\nabla g^i\big) = \int_{t_0}^{t_1}\int_\Gamma g^i\,\tilde{\omega}\,g + \Big[\int_\Omega \tilde{\omega}\,g^i\Big]_{t_0}^{t_1}. \tag{5.3}$$

Let us start with the right-hand side of (5.3). By (1.1) and (5.2b), the first term in the right-hand side of (5.3) can be simplified as follows:

$$\int_{t_0}^{t_1}\int_\Gamma g^i\,\tilde{\omega}\,g = \sum_{j\in I^*} g^i_j \int_{t_0}^{t_1}\int_{\Gamma_j}\tilde{\omega}\,g = -\sum_{j\in I^*} g^i_j\,[\tilde{\mathcal{C}}_j]_{t_0}^{t_1},$$

thanks to (4.2).

On the other hand, using (4.7) and two integrations by parts, the second term in the right-hand side of (5.3) can be simplified by observing that

$$\int_\Omega \tilde{\omega}(t,.)g^i = \int_\Omega \tilde{\psi}(t,.)\Delta g^i + \int_\Gamma \partial_{\mathbf{n}}\tilde{\psi}(t,.)g^i - \int_\Gamma \tilde{\psi}(t,.)\partial_{\mathbf{n}} g^i = \sum_{j\in I^*} g^i_j \tilde{\mathcal{C}}_j(t) + \tilde{\psi}_i(t),$$

thanks to (4.8), (4.9) and (5.2). Thus, the right-hand side of (5.3) turns out to be equal to $\tilde{\psi}_i(t_1) - \tilde{\psi}_i(t_0)$.

Regarding the second term in the left-hand side of (5.3), by the Cauchy–Schwarz inequality, we obtain that

$$\left|\int_\Omega \hat{\omega}\tilde{u}\cdot\nabla g^i\right| \le \|\nabla g^i\|_{L^\infty(\Omega)}\,\|\hat{\omega}\|_{L^2(\Omega)}\,\|\tilde{u}\|_{L^2(\Omega)}. \tag{5.4}$$

Since $\|\hat{\omega}\|_{L^\infty([0,T];L^2(\Omega))}$ is bounded, we infer that the left-hand side of (5.4) is in $L^2(0,T)$ and that there exists $C > 0$ such that for $0 \le t_0 < t_1 \le T$,

$$\int_{t_0}^{t_1} \left|\int_\Omega \hat{\omega}\tilde{u}\cdot\nabla g^i\right|^2 \le C\int_{t_0}^{t_1} \|\tilde{u}\|^2_{L^2(\Omega)}. \tag{5.5}$$

On the other hand, regarding the first term in the left-hand side of (5.3), we integrate by parts by observing that $\tilde{\omega} = -\mathrm{div}\,\tilde{u}^\perp$, so that

$$\int_\Omega \tilde{\omega}\hat{u}\cdot\nabla g^i = \int_\Omega \tilde{u}^\perp\cdot\nabla(\hat{u}\cdot\nabla g^i) - \int_\Gamma (\tilde{u}\cdot\tau)(\hat{u}\cdot\nabla g^i). \tag{5.6}$$

Regarding the first term in the right-hand side of (5.6), by the Leibniz rule and the Cauchy–Schwarz inequality, we arrive at

$$\left|\int_\Omega \tilde{u}^\perp\cdot\nabla(\hat{u}\cdot\nabla g^i)\right| \le \|\tilde{u}\|_{L^2(\Omega)}\left(\|\nabla g^i\|_{L^\infty(\Omega)}\,\|\hat{u}\|_{H^1(\Omega)} + \|\nabla^2 g^i\|_{L^2_x}\,\|\hat{u}\|_{L^\infty(\Omega)}\right).$$

Since the term in the parenthesis in the right-hand side is in $L^\infty([0,T])$, the right-hand side is in the space $L^2(0,T)$ and there exists $C > 0$ such that for $0 \le t_0 < t_1 \le T$,

$$\int_{t_0}^{t_1} \left|\int_\Omega \tilde{u}^\perp\cdot\nabla(\hat{u}\cdot\nabla g^i)\right|^2 \le C\int_{t_0}^{t_1} \|\tilde{u}\|^2_{L^2(\Omega)}. \tag{5.7}$$

On the other hand, for the second term in the left-hand side of (5.6), we proceed as follows. Let us set

$$\Upsilon := \int_\Gamma (\tilde{u}\cdot\tau)(\hat{u}\cdot\nabla g^i),$$

which is a time-dependent function. Since the function g^i is constant on each connected component of the boundary Γ,

$$\hat{u}\cdot\nabla g^i = (\hat{u}\cdot n)\,\partial_n g^i = g\,\partial_n g^i,$$

by (4.11). Then we recast Υ into

$$\Upsilon = \int_\Gamma \left((\tilde{u}\cdot\tau)|g|^{1/2}\right)\left((\partial_{\mathbf{n}} g^i)\,\mathrm{sign}(g)\,|g|^{1/2}\right),$$

so that, by the Cauchy–Schwarz inequality,

$$|\Upsilon| \leqslant \Big(\int_\Gamma (\tilde u\cdot\tau)^2\,|g|\Big)^{1/2}\Big(\int_\Gamma (\partial_{\mathbf n} g^i)^2\,|g|\Big)^{1/2}.$$

Therefore,

$$\int_{t_0}^{t_1}|\Upsilon|^2 \le \int_{t_0}^{t_1}\|\tilde u\|^2_{L^2(\Gamma,|g|)}\cdot\int_\Gamma(\partial_{\mathbf n} g^i)^2\,|g| \le C\int_{t_0}^{t_1}\|\tilde u\|^2_{L^2(\Gamma,|g|)}. \tag{5.8}$$

Gathering (5.3), (5.5), (5.7) and (5.8) for $0\le t_0<t_1\le T$, we obtain that $\tilde\psi_i'$ is in $L^2(0,T)$, and the estimate (5.1) holds true. ∎

We now prove the following result on the function $G[\tilde\omega]$, recalling that the operator $G[\cdot]$ is defined in (4.4). For the purpose of proving Proposition 5.1, it would be enough to prove that $G[\tilde\omega]$ is in the space $H^1([0,T];H^1(\Omega))$; however, it does not involve more work to obtain a slightly better regularity in time.

Lemma 5.3. *The function $G[\tilde\omega]$ is in the space* $\mathrm{Lip}([0,T];H^1(\Omega))$.

Proof of Lemma 5.3. Let a in $H^1_0(\Omega)$. By using a as a (constant-in-time) test function in (4.10), we get that for $0\le t_0<t_1\le T$,

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega(\tilde\omega\hat u+\hat\omega\tilde u)\cdot\nabla a &= \int_\Omega(\tilde\omega(t_1,.)-\tilde\omega(t_0,.))\,a\\ &=\int_\Omega(\Delta G[\tilde\omega](t_1,.)-\Delta G[\tilde\omega](t_0,.))\,a = -\int_\Omega\nabla(G[\tilde\omega](t_1,.)-G[\tilde\omega](t_0,.))\cdot\nabla a,\end{aligned}$$

by using the definition of the operator $G[\cdot]$ in (4.4) and an integration by parts. In particular, in the case where

$$a = G[\tilde\omega](t_1,.)-G[\tilde\omega](t_0,.),$$

we infer by the Cauchy–Schwarz inequality as well as the Poincaré inequality, see for example Theorem 13.6.9 in [26], that there exists a constant $C>0$ such that for $0\le t_0<t_1\le T$,

$$\|G[\tilde\omega](t_1,.)-G[\tilde\omega](t_0,.)\|_{H^1(\Omega)}\le C\int_{t_0}^{t_1}\|\tilde\omega\hat u+\hat\omega\tilde u\|_{L^2(\Omega)}, \tag{5.9}$$

which gives

$$\|G[\tilde\omega]\|_{\mathrm{Lip}([0,T];H^1(\Omega))}\le C\|\tilde\omega\hat u+\hat\omega\tilde u\|_{L^\infty([0,T];L^2(\Omega))}. \tag{5.10}$$

Thus by mastering the right-hand side of (5.10) thanks to Proposition 2.1 and (2.5), we conclude the proof of Lemma 5.3. ∎

Using the decomposition (4.6), we deduce from Lemma 5.2, from Lemma 5.3 and from the C^2 regularity on $\overline\Omega$ of the functions f^i defined in (4.3), that the stream function $\tilde\psi$ is in the space $H^1((0,T);H^1(\Omega))$. The proof of Proposition 5.1 is therefore completed. ∎

6. Energy estimate of the difference

This section is devoted to the proof of the following energy inequality.

Proposition 6.1. *For* $0 \leq t_0 < t_1 \leq T$,

$$\frac{1}{2}\left[\|\tilde{u}\|^2_{L^2(\Omega)}\right]^{t_1}_{t_0} + \frac{1}{2}\int_{t_0}^{t_1}\int_\Gamma |\tilde{u}|^2 g + \int_{t_0}^{t_1}\int_\Omega \tilde{u}\cdot((\tilde{u}\cdot\nabla)\,\hat{u}) = 0, \tag{6.1}$$

where we recall the notation (2.4).

Proof. According to Proposition 5.1, we can apply (4.10) to the case where the test function is the stream function $\tilde{\psi}$ defined in (4.6). This entails that for $0 \leq t_0 < t_1 \leq T$,

$$\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\partial_t\tilde{\psi} + \int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\hat{u}\cdot\nabla\tilde{\psi} + \int_{t_0}^{t_1}\int_\Omega \hat{\omega}\,\tilde{u}\cdot\nabla\tilde{\psi} = \int_{t_0}^{t_1}\int_\Gamma \tilde{\psi}\,\tilde{\omega}\,g + \left[\int_\Omega \tilde{\omega}\,\tilde{\psi}\right]^{t_1}_{t_0}. \tag{6.2}$$

Let us successively transform each term of (6.2).

First, by (4.7) and an integration by parts, the first term in the left-hand side of (6.2) can be recast as

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\partial_t\tilde{\psi} &= -\int_{t_0}^{t_1}\int_\Omega \nabla\tilde{\psi}\cdot\partial_t\nabla\tilde{\psi} + \int_{t_0}^{t_1}\int_\Gamma \partial_{\mathbf{n}}\tilde{\psi}\,\partial_t\tilde{\psi} \\ &= -\frac{1}{2}\left[\|\tilde{u}\|^2_{L^2(\Omega)}\right]^{t_1}_{t_0} + \sum_{i\in I^*}\tilde{\psi}_i'\int_{t_0}^{t_1}\tilde{\mathcal{C}}_i,\end{aligned} \tag{6.3}$$

thanks to the boundary decomposition (1.1), (4.5), (4.8) and (4.9).

Next, we recast the second term in the left-hand side of (6.2) by using the following lemma.

Lemma 6.2. *Let u be a divergence-free vector field in $C^0(\overline{\Omega};\mathbb{R}^2)$ with bounded vorticity $\omega := \operatorname{curl} u$. Let w be a divergence-free vector field in $H^1(\Omega)$. Then*

$$\int_\Omega \omega\, u^\perp\cdot w = -\frac{1}{2}\int_\Gamma |u|^2\,(w\cdot n) - \int_\Omega u\cdot((u\cdot\nabla)\,w) + \int_\Gamma (u\cdot w)(u\cdot n). \tag{6.4}$$

We use later on a technical lemma, see Lemma 8.1, which slightly extends Lemma 6.2. Indeed, Lemma 6.2 is the particular case of Lemma 8.1 below where $u = v$ and where w is divergence-free, so that we do not provide here any proof of Lemma 6.2.

Thus, with a preliminary recasting due to (4.5),

$$\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\hat{u}\cdot\nabla\tilde{\psi} = -\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\tilde{u}^\perp\cdot\hat{u} = \int_{t_0}^{t_1}\Big(\frac{1}{2}\int_\Gamma |\tilde{u}|^2 g + \int_\Omega \tilde{u}\cdot((\tilde{u}\cdot\nabla)\,\hat{u})\Big), \tag{6.5}$$

by applying Lemma 6.2 with $(\tilde{u}, \hat{u})$ instead of (u, w), with the observation that the last boundary term of (6.4) vanishes thanks to (4.1c). Observe that we also use (4.11) to deal with the other boundary term.

Regarding the third term in the left-hand side of (6.2), it follows from (4.5) that

$$\int_{t_0}^{t_1}\int_\Omega \hat{\omega}\,\tilde{u}\cdot\nabla\tilde{\psi} = 0. \tag{6.6}$$

Using the decomposition (1.1), (4.8) and (4.2), we get that the first term in the right-hand side of (6.2) satisfies

$$\int_{t_0}^{t_1}\int_{\Gamma}\tilde{\psi}\,\tilde{\omega}\,g = -\int_{t_0}^{t_1}\sum_{i\in I^*}\tilde{\psi}_i\,\tilde{\mathcal{C}}_i'. \tag{6.7}$$

Finally, using again (4.7) and an integration by parts, we observe that the term involved in the bracket of the last term in the right-hand side of (6.2) satisfies, for every time,

$$\int_{\Omega}\tilde{\omega}\,\tilde{\psi} = -\int_{\Omega}\nabla\tilde{\psi}\cdot\nabla\tilde{\psi} + \int_{\Gamma}\tilde{\psi}\,\partial_{\mathbf{n}}\tilde{\psi} = -\|\tilde{u}\|^2_{L^2(\Omega)} + \sum_{i\in I^*}\tilde{\psi}_i\,\tilde{\mathcal{C}}_i, \tag{6.8}$$

by using (4.5), the decomposition (1.1) and (4.8).

Combining (6.2), (6.3), (6.5), (6.6), (6.7) and (6.8), we arrive at (6.1), and this concludes the proof of Proposition 6.1. ∎

From Proposition 6.1, we deduce the following estimate.

Corollary 6.3. *There exists a constant C such that for any $0 \le t_0 < t_1 \le T$, and for every $p \in [2,+\infty)$,*

$$\frac{1}{2}\Big[\|\tilde{u}\|^2_{L^2(\Omega)}\Big]_{t_0}^{t_1} + \frac{1}{2}\int_{t_0}^{t_1}\int_{\Gamma_{\text{out}}}|\tilde{u}|^2 g \le \frac{1}{2}\int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}}|\tilde{u}|^2(-g) + Cp\int_{t_0}^{t_1}\|\tilde{u}\|^{2(p-1)/p}_{L^2(\Omega)}. \tag{6.9}$$

Observe that due to the sign conditions in (1.6), the second term in the left-hand side and the first term in the right-hand side are both non-negative; this is indeed the reason why we display them in this way. Should the latter be discarded, the inequality (6.9) would allow to conclude the proof of Theorem 3.1 by proceeding as in the impermeable case, that is, by optimizing in p and by using Osgood's lemma, see Remark 2.2 and [34, 35]. As already hinted above, see Section 3, the presence of this "bad" boundary term, which is associated with the sources, but not bounded by the left-hand side nor by the data, is an obstacle to this program, which is overcome in the next sections, by making use of a well-chosen auxiliary test function.

Proof of Corollary 6.3. Taking into account the splitting of the second term of (6.1) according to the decomposition of the boundary Γ into $\Gamma = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_0$, and the fact that $g = 0$ on Γ_0 (see (1.1), (1.2) and (1.6)), the proof of Corollary 6.3 amounts to prove that there exists a constant C such that at any non-negative time, and for every $p \in [2,+\infty)$, the third term of (6.1) is bounded as follows:

$$\Big|\int_{\Omega}\tilde{u}\cdot((\tilde{u}\cdot\nabla)\,\hat{u})\Big| \le Cp\,\|\tilde{u}\|^{2(p-1)/p}_{L^2(\Omega)}. \tag{6.10}$$

To that aim, let us first observe that, by Hölder's inequality, we have, for all $p \in (1,+\infty)$, that

$$\Big|\int_{\Omega}\tilde{u}\cdot((\tilde{u}\cdot\nabla)\,\hat{u})\Big| \le \|\tilde{u}\|^2_{L^{2p/(p-1)}(\Omega)}\,\|\hat{u}\|_{W^{1,p}(\Omega)}. \tag{6.11}$$

To bound the first factor in the right-hand side, let us recall that for any p in $(1, +\infty)$, for any vector field w in $L^2(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, we have the following interpolation inequality:

$$\|w\|_{L^{2p/(p-1)}(\Omega)} \leq \|w\|_{L^\infty(\Omega)}^{1/p} \, \|w\|_{L^2(\Omega)}^{(p-1)/p}. \tag{6.12}$$

By Proposition 5.1, the Sobolev embedding theorem and (4.5), we obtain that $\tilde{u}$ is in the space $C^0(\mathbb{R}_+; L^\infty(\Omega))$, so that the first factor of the right-hand side of (6.11) can be bounded by

$$C\,\|\tilde{u}\|_{L^2(\Omega)}^{2(p-1)/p}.$$

Moreover, the second factor of the right-hand side of (6.11) can be bounded thanks to Proposition 2.1 by Cp for a positive constant C independent of time and of p in $[2, +\infty)$. This allows to deduce (6.10) from (6.11), and therefore to conclude the proof of Corollary 6.3. ∎

7. An auxiliary test function

This section is devoted to the existence and to the regularity of an auxiliary test function which is useful in the sequel to establish a second energy estimate. This test function is defined through the following Zaremba-type problem:

$$\Delta\tilde{\varphi} = 0 \quad \text{on } \Omega, \tag{7.1a}$$

$$\tilde{\varphi} = 0 \quad \text{on } \Gamma_{\text{out}} \cup \Gamma_0, \tag{7.1b}$$

$$\partial_{\mathbf{n}}\tilde{\varphi} = -\partial_{\mathbf{n}}\tilde{\psi} \quad \text{on } \Gamma_{\text{in}}. \tag{7.1c}$$

It is instructive to look first at a variational formulation of the system (7.1), associated with the space

$$H^1_{0,\text{out}}(\Omega) := \{a \in H^1(\Omega);\ a_{|\Gamma_{\text{out}}\cup\Gamma_0} = 0\}. \tag{7.2}$$

Thanks to the Poincaré inequality, the space $H^1_{0,\text{out}}(\Omega)$ is a Hilbert space endowed with the homogeneous Sobolev norm associated with the space $\dot{H}^1(\Omega)$. Then, we consider the following variational formulation of the system (7.1):

$$\tilde{\varphi} \text{ is in } H^1_{0,\text{out}}(\Omega) \text{ and } \forall a \in H^1_{0,\text{out}}(\Omega), \quad \int_\Omega \nabla\tilde{\varphi}\cdot\nabla a = -\int_\Omega \nabla\tilde{\psi}\cdot\nabla a - \int_\Omega \tilde{\omega}\, a. \tag{7.3}$$

Indeed, if $\tilde{\varphi}$ is a smooth solution of the system (7.1), then for all a in $H^1_{0,\text{out}}(\Omega)$,

$$0 = \int_\Omega (\Delta\tilde{\varphi})a = -\int_\Omega \nabla\tilde{\varphi}\cdot\nabla a + \int_\Gamma (\partial_{\mathbf{n}}\tilde{\varphi})a, \tag{7.4}$$

by integrating by parts. Moreover, by (7.1c) and the fact that $a \in H^1_{0,\text{out}}(\Omega)$, we deduce that

$$\int_\Gamma (\partial_{\mathbf{n}}\tilde{\varphi})a = -\int_\Gamma (\partial_{\mathbf{n}}\tilde{\psi})a = -\int_\Gamma \nabla\tilde{\psi}\cdot\nabla a - \int_\Omega (\Delta\tilde{\psi})a, \tag{7.5}$$

by integrating by parts. Combining (7.4), (7.5) and recalling (4.7), we arrive at (7.3).

Since the right-hand side of the identity in (7.3) is a linear form with respect to a in $H^1_{0,\mathrm{out}}(\Omega)$ with an operator norm bounded by

$$C\left(\|\tilde{u}\|_{C^0([0,T];L^2(\Omega))} + \|\tilde{\omega}\|_{C^0([0,T];L^2(\Omega))}\right), \tag{7.6}$$

by the Lax–Milgram theorem, there exists a unique function $\tilde{\varphi}$ in $C^0([0,T];H^1_{0,\mathrm{out}}(\Omega))$ which satisfies (7.3) for all non-negative times.

Moreover, the following result establishes that the function $\tilde{\varphi}$ is more regular.

Proposition 7.1. *The function $\tilde{\varphi}$ is in $C^0([0,T];W^{2,p}(\Omega))$ for all p in $(1,+\infty)$. Moreover, for all $T>0$, the function $\tilde{\varphi}$ is also in the space $H^1([0,T];H^1_{0,\mathrm{out}}(\Omega))$.*

Proof. First, by Proposition 2.1, we know that $\tilde{\psi}$ is in $C^0([0,T];W^{2,p}(\Omega))$ for any finite p. Then, it follows from the classical regularity results on the Zaremba-type problems, see e.g. Theorem 2.4.2.6 in [13], that the function $\tilde{\varphi}$ is also in $C^0([0,T];W^{2,p}(\Omega))$ for any p in $(1,+\infty)$.

Thus, it only remains to prove that $\partial_t\tilde{\varphi}$ is in the space $L^2((0,T);H^1_{0,\mathrm{out}}(\Omega))$. To this aim, let us consider ω_ϕ a smooth function on $[0,T]\times\Omega$, compactly supported in $(0,T)$ as a function in time with values in the space of continuous functions on Ω. As argued above, there exists a unique smooth function ϕ which satisfies for all t in $[0,T]$,

$$\Delta\phi = \omega_\phi \quad \text{on } \Omega, \tag{7.7a}$$

$$\phi = 0 \quad \text{on } \Gamma_{\mathrm{out}}\cup\Gamma_0, \tag{7.7b}$$

$$\partial_{\mathbf{n}}\phi = 0 \quad \text{on } \Gamma_{\mathrm{in}}. \tag{7.7c}$$

Moreover, the function ϕ is compactly supported in $(0,T)$. Let us denote by H' the dual space of $H^1(\Omega)$ for the L^2 scalar product, seen as a subset of $H^{-1}(\Omega)$. Using the fact that we have the equality

$$\int_\Omega \phi\,\omega_\phi = -\int_\Omega |\nabla\phi|^2, \tag{7.8}$$

we get that the function ϕ is also in $L^\infty([0,T];H^1(\Omega))$, together with the estimate

$$\|\phi(t,.)\|_{H^1(\Omega)} \le C\|\omega_\phi(t,.)\|_{H'}, \tag{7.9}$$

for every t in $[0,T]$ with some constant C independent of the time.

By integrating by parts, and using that on the one hand $\partial_t\partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}\partial_t\phi = 0$ on Γ_{in} for all t in $[0,T]$, and on the other hand (7.1b), we have that

$$\int_\Omega \tilde{\varphi}\,\partial_t\omega_\phi = -\int_\Omega \nabla\tilde{\varphi}\cdot\nabla\partial_t\phi. \tag{7.10}$$

Since, for all t in $[0,T]$, the function $\partial_t\phi(t,\cdot)$ is in $H^1_{0,\mathrm{out}}(\Omega)$, we have, by applying (7.3) with $\partial_t\phi(t,\cdot)$ instead of a, and integrating over $(0,T)$, that

$$\int_0^T\int_\Omega \nabla\tilde{\varphi}\cdot\nabla\partial_t\phi = -\int_0^T\int_\Omega \nabla\tilde{\psi}\cdot\nabla\partial_t\phi - \int_0^T\int_\Omega \tilde{\omega}\,\partial_t\phi. \tag{7.11}$$

By integrating by parts in space and in time and using (7.7a), we deduce that

$$\int_0^T \int_\Omega \nabla\tilde\psi \cdot \nabla \partial_t \phi = -\int_0^T \int_\Omega \tilde\psi\, \partial_t\, \omega_\phi + \int_0^T \int_\Gamma \tilde\psi\, \partial_{\mathbf{n}} \partial_t \phi$$

$$= \int_0^T \int_\Omega (\partial_t \tilde\psi)\, \omega_\phi - \int_0^T \int_\Gamma (\partial_t \tilde\psi)\, \partial_{\mathbf{n}} \phi. \tag{7.12}$$

On the one hand, by the Cauchy–Schwarz inequality and Proposition 5.1,

$$\Big| \int_0^T \int_\Omega (\partial_t \tilde\psi)\, \omega_\phi \Big| \le C \, \|\omega_\phi\|_{L^2((0,T);H')}. \tag{7.13}$$

On the other hand, by using the boundary decomposition (1.1) and recalling (4.8),

$$\int_0^T \int_\Gamma (\partial_t \tilde\psi)\, \partial_{\mathbf{n}} \phi = \sum_{i \in I} \int_0^T \tilde\psi_i' \int_{\Gamma_i} \partial_{\mathbf{n}} \phi. \tag{7.14}$$

Moreover, for any i in I^*, recalling the definition of f^i in (4.3), we have, for all non-negative times, that

$$\int_{\Gamma_i} \partial_{\mathbf{n}} \phi = \int_\Omega \nabla f^i \cdot \nabla \phi + \int_\Omega f^i \omega_\phi,$$

so that, by the Cauchy–Schwarz inequality and (7.9), we deduce that

$$\Big| \int_{\Gamma_i} \partial_{\mathbf{n}} \phi \Big| \leqslant C \, \|\omega_\phi\|_{H'}.$$

Therefore, by (7.14), the Cauchy–Schwarz inequality with respect to time, Proposition 5.1 and recalling that $\tilde\psi_0$ is identically null, we obtain that

$$\Big| \int_0^T \int_\Gamma (\partial_t \tilde\psi)\, \partial_{\mathbf{n}} \phi \Big| \le C \, \|\omega_\phi\|_{L^2((0,T);H')}. \tag{7.15}$$

Combining (7.12), (7.13) and (7.15), we infer that

$$\Big| \int_0^T \int_\Omega \nabla\tilde\psi \cdot \nabla \partial_t \phi \Big| \le C \, \|\omega_\phi\|_{L^2((0,T);H')}. \tag{7.16}$$

We now turn to the estimate of the last term of (7.11). By (4.10), with $(t_0, t_1) = (0, T)$,

$$-\int_0^T \int_\Omega \tilde\omega\, \partial_t \phi = \int_0^T \int_\Omega \big(\tilde\omega\, \hat u \cdot \nabla\phi + \hat\omega \tilde u \cdot \nabla \phi\big) - \int_0^T \int_\Gamma \phi\, \tilde\omega\, g.$$

Since $\tilde\omega$ and $\hat\omega$ are in $L^\infty((0,T) \times \Omega)$, since $\tilde u$ and $\hat u$ are in $L^2((0,T) \times \Omega)$, since $\tilde\omega$ is in $L^\infty(0,T; L^\infty(\Gamma_{\text{out}}, |g|))$, by the classical trace properties, see for example Section 13.6 in [26], and using (7.9), we obtain that

$$\Big| \int_0^T \int_\Omega \tilde\omega\, \partial_t \phi \Big| \le C \, \|\omega_\phi\|_{L^2((0,T);H')}. \tag{7.17}$$

By gathering (7.10), (7.11), (7.16) and (7.17), we deduce that for any function ω_ϕ which is smooth on $[0, T] \times \Omega$ and compactly supported in $(0, T)$,

$$\Big| \int_0^T \int_\Omega \tilde{\varphi}\, \partial_t \omega_\phi \Big| \leq C \, \|\omega_\phi\|_{L^2((0,T);H')}.$$

This entails that $\partial_t \tilde{\varphi}$ is in the space $L^2((0, T); H^1(\Omega))$ and it follows from the properties of the trace that $\partial_t \tilde{\varphi}$ is indeed in the space $L^2((0, T); H^1_{0,\text{out}}(\Omega))$. This concludes the proof of Proposition 7.1. ∎

8. A generalized Lamb-type lemma

This section is devoted to the following Lamb-type lemma.

Lemma 8.1. *Let u an dv be two divergence-free vector fields in $C^0(\overline{\Omega}; \mathbb{R}^2)$ with bounded vorticity. Let w be a vector field in $H^1(\Omega; \mathbb{R}^2)$. We have the following equality*:

$$\begin{aligned}(8.1)\quad \int_\Gamma (u \cdot v)(w \cdot n) - \int_\Gamma (u \cdot w)(v \cdot n) - \int_\Gamma (v \cdot w)(u \cdot n) \\ = \int_\Omega (u \cdot v) \operatorname{div} w - \int_\Omega (\operatorname{curl} u)\, v^\perp \cdot w - \int_\Omega (\operatorname{curl} v)\, u^\perp \cdot w \\ - \int_\Omega u \cdot ((v \cdot \nabla)\, w) - \int_\Omega v \cdot ((u \cdot \nabla)\, w).\end{aligned}$$

In particular, in the case where $u = v$, then (8.1) *reduces to*

$$\begin{aligned}(8.2)\quad \int_\Gamma |u|^2 (w \cdot n) - 2 \int_\Gamma (u \cdot w)(u \cdot n) \\ = \int_\Omega |u|^2 \operatorname{div} w - 2 \int_\Omega \omega u^\perp \cdot w - 2 \int_\Omega u \cdot ((u \cdot \nabla)\, w),\end{aligned}$$

where $\omega := \operatorname{curl} u$.

Lemma 8.1 implies Lemma 6.2 by considering the particular case where $u = v$ and where w is divergence free. It also implies Lemma 7 in [23] and Lemma 2.15 in [24] in the particular cases where w is a rigid velocity. It will also be useful several times in the sequel.

Proof. It is sufficient to prove (8.1) in the case where the three vector fields u, v and w are smooth, since then the result follows from an approximation process. We start with using Stokes' formula to obtain that

$$(8.3)\quad \int_\Gamma (u \cdot v)(w \cdot n) = \int_\Omega \operatorname{div}((u \cdot v) w) = \int_\Omega \nabla(u \cdot v) \cdot w + \int_\Omega (u \cdot v) \operatorname{div} w.$$

Moreover,

$$(8.4)\quad \nabla(u \cdot v) = (u \cdot \nabla) v + (v \cdot \nabla) u - (\operatorname{curl} u)\, v^\perp - (\operatorname{curl} v)\, u^\perp,$$

and therefore,

$$\int_\Gamma (u\cdot v)(w\cdot n) = \int_\Omega (u\cdot v)\operatorname{div} w - \int_\Omega (\operatorname{curl} u)\, v^\perp \cdot w - \int_\Omega (\operatorname{curl} v)\, u^\perp \cdot w + \int_\Omega ((u\cdot\nabla)v)\cdot w + \int_\Omega ((v\cdot\nabla)u)\cdot w. \tag{8.5}$$

Finally, by some integrations by parts, since u and v are divergence-free,

$$\begin{aligned} \int_\Omega ((u\cdot\nabla)v)\cdot w &= -\int_\Omega v\cdot((u\cdot\nabla)w) + \int_\Omega (v\cdot w)(u\cdot n), \quad \text{and} \\ \int_\Gamma ((v\cdot\nabla)u)\cdot w &= -\int_\Omega u\cdot((v\cdot\nabla)w) + \int_\Gamma (u\cdot w)(v\cdot n). \end{aligned} \tag{8.6}$$

Gathering (8.3), (8.5) and (8.6), we obtain (8.1). ■

9. An auxiliary energy-type estimate

Recall that the function $\tilde\varphi$ is defined in Section 7. We denote by $\tilde v$ its orthogonal gradient

$$\tilde v := \nabla^\perp \tilde\varphi. \tag{9.1}$$

We also set, for i in I^*,

$$\tilde{\mathcal{D}}_i := \int_{\Gamma_i} \partial_{\mathbf{n}}\tilde\varphi. \tag{9.2}$$

Let us observe that, it follows from (7.1c) that, at any time $t \in [0,T]$,

$$\tilde{\mathcal{D}}_i(t) = -\tilde{\mathcal{C}}_i(t) \quad \text{for all } i \in I_{\text{in}}. \tag{9.3}$$

Proposition 9.1. *For* $0 \le t_0 < t_1 \le T$,

$$\begin{aligned} \frac{1}{2}\big[\|\tilde v(t,.)\|^2_{L^2(\Omega)}\big]_{t_0}^{t_1} &+ \int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} |\tilde u|^2(-g) \\ &= \int_{t_0}^{t_1}\int_{\Gamma_{\text{out}}} (\tilde u\cdot\tilde v)(-g) + \int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} (\tilde u\cdot\hat u)(\tilde v\cdot n) \\ &\quad + \int_{t_0}^{t_1}\int_\Omega \big(\tilde u\cdot((\tilde v\cdot\nabla)\hat u) + \tilde v\cdot((\tilde u\cdot\nabla)\hat u) + \hat\omega\,\tilde u\cdot\tilde v^\perp\big) \\ &\quad - \int_{t_0}^{t_1}\sum_{i\in I^*}\tilde\psi_i'\,\tilde{\mathcal{D}}_i + \int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} \tilde\varphi\,\tilde\omega_{\text{in}}\, g. \end{aligned} \tag{9.4}$$

Observe that due to the sign conditions in (1.6), the second term in the left-hand side is non-negative; and is twice the “bad” boundary term in (6.9).

Proof. Thanks to Proposition 7.1, we can apply (4.10) to the case where the test function is the function $\tilde{\varphi}$. This entails that for every t_0, t_1,

$$\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\partial_t\tilde{\varphi} + \int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\hat{u}\cdot\nabla\tilde{\varphi} + \int_{t_0}^{t_1}\int_\Omega \hat{\omega}\,\tilde{u}\cdot\nabla\tilde{\varphi} = \int_{t_0}^{t_1}\int_\Gamma \tilde{\varphi}\,\tilde{\omega}\,g + \Big[\int_\Omega \tilde{\omega}\,\tilde{\varphi}\Big]_{t_0}^{t_1}. \tag{9.5}$$

Let us simplify each term of (9.5).

Let us start with the last one. Taking $\tilde{\varphi}$ as a test function in (7.3), and using (9.1), we get that:

$$-\int_\Omega |\tilde{v}|^2 = \int_\Omega \nabla\tilde{\psi}\cdot\nabla\tilde{\varphi} + \int_\Omega \tilde{\omega}\,\tilde{\varphi}. \tag{9.6}$$

We can simplify the first term of the right-hand side of (9.6) by an integration by parts:

$$\int_\Omega \nabla\tilde{\psi}\cdot\nabla\tilde{\varphi} = -\int_\Omega \tilde{\psi}\,\Delta\tilde{\varphi} + \int_\Gamma \tilde{\psi}\,\partial_{\mathbf{n}}\tilde{\varphi} = \sum_{i\in I^*} \tilde{\psi}_i\,\tilde{\mathcal{D}}_i, \tag{9.7}$$

by (7.1a), (1.1), (4.8) and (9.2). Therefore, by combining (9.6) and (9.7), we obtain that the last term of (9.5) can be recast as

$$\int_\Omega \tilde{\omega}\,\tilde{\varphi} = -\int_\Omega |\tilde{v}|^2 - \sum_{i\in I^*} \tilde{\psi}_i\,\tilde{\mathcal{D}}_i. \tag{9.8}$$

We now turn to the first term in the left-hand side of (9.5) for which we apply similar computations. More precisely, by taking $\partial_t\tilde{\phi}$ as a test function in (7.3), and using again (9.1), we obtain that

$$-\int_\Omega \tilde{v}\cdot\partial_t\tilde{v} = \int_\Omega \nabla\tilde{\psi}\cdot\partial_t\nabla\tilde{\varphi} + \int_\Omega \tilde{\omega}\,\partial_t\tilde{\varphi}. \tag{9.9}$$

Moreover, by an integration by parts, the first term in the right-hand side of (9.9) can be simplified into

$$\int_\Omega \nabla\tilde{\psi}\cdot\partial_t\nabla\tilde{\varphi} = -\int_\Omega \tilde{\psi}\,\partial_t\Delta\tilde{\varphi} + \int_\Gamma \tilde{\psi}\,\partial_t\partial_{\mathbf{n}}\tilde{\varphi} = \sum_{i\in I^*} \tilde{\psi}_i\,\tilde{\mathcal{D}}_i'. \tag{9.10}$$

By combining (9.9) and (9.10) and integrating between t_0 and t_1, we obtain:

$$\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\partial_t\tilde{\varphi} = -\frac{1}{2}\big[\|\tilde{v}\|^2_{L^2(\Omega)}\big]_{t_0}^{t_1} - \int_{t_0}^{t_1}\sum_{i\in I^*} \tilde{\psi}_i\,\tilde{\mathcal{D}}_i'. \tag{9.11}$$

The second term in the left-hand side of (9.5) can be tackled by Lemma 8.1 with $(\hat{u},\tilde{v},\hat{u})$ instead of (u,v,w). Using also that curl $\tilde{v} = 0$ in Ω and (4.1c) this entails that

$$\int_\Omega \tilde{\omega}\,\hat{u}\cdot\nabla\tilde{\varphi} = \int_\Gamma (\tilde{u}\cdot\tilde{v})g - \int_\Gamma (\tilde{u}\cdot\hat{u})(\tilde{v}\cdot n) + \int_\Omega \tilde{u}\cdot((\tilde{v}\cdot\nabla)\hat{u}) + \int_\Omega \tilde{v}\cdot((\tilde{u}\cdot\nabla)\hat{u}).$$

Moreover, using (7.1c), (9.1) and again (4.1c), the first term in the left-hand side above can be decomposed as follows:

$$\int_\Gamma (\tilde{u}\cdot\tilde{v})\,g = \int_{\Gamma_{\text{out}}} (\tilde{u}\cdot\tilde{v})\,g + \int_{\Gamma_{\text{in}}} |\tilde{u}|^2\,(-g), \tag{9.12}$$

and the second one can be simplified into

$$\int_\Gamma (\tilde{u}\cdot\hat{u})\,(\tilde{v}\cdot n) = \int_{\Gamma_{\text{in}}} (\tilde{u}\cdot\hat{u})\,(\tilde{v}\cdot n), \tag{9.13}$$

by (9.1) and (7.1b). We therefore arrive at

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega \tilde{\omega}\,\hat{u}\cdot\nabla\tilde{\varphi} &= \int_{t_0}^{t_1}\int_{\Gamma_{\text{out}}} (\tilde{u}\cdot\tilde{v})\,g + \int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} |\tilde{u}|^2(-g) - \int_{\Gamma_{\text{in}}} (\tilde{u}\cdot\hat{u})(\tilde{v}\cdot n)\\ &\quad + \int_{t_0}^{t_1}\int_\Omega \big(\tilde{u}\cdot((\tilde{v}\cdot\nabla)\,\hat{u}) + \tilde{v}\cdot((\tilde{u}\cdot\nabla)\,\hat{u})\big).\end{aligned} \tag{9.14}$$

The third term of the right-hand side of (9.5) satisfies

$$\int_{t_0}^{t_1}\int_\Omega \hat{\omega}\,\tilde{u}\cdot\nabla\tilde{\varphi} = -\int_{t_0}^{t_1}\int_\Omega \hat{\omega}\,\tilde{u}\cdot\tilde{v}^{\perp}, \tag{9.15}$$

by (9.1).

Finally, the first term in the left-hand side of (9.5) satisfies

$$\int_{t_0}^{t_1}\int_\Gamma \tilde{\varphi}\,\tilde{\omega}\,g = \int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} \tilde{\varphi}\,\tilde{\omega}_{\text{in}}\,g, \tag{9.16}$$

by (7.1b).

Gathering (9.5), (9.8), (9.11), (9.14), (9.15) and (9.16), we arrive at (9.4). This concludes the proof of Proposition 9.1. ■

In the same way as we deduce Corollary 6.3 from Proposition 6.1, we deduce the following corollary from Proposition 9.1.

Corollary 9.2. *There exists a constant* $C > 0$ *such that for every* p *in* $[2, +\infty)$ *and for every* $0 \le t_0 < t_1 \le T$,

$$\begin{aligned}\frac{1}{2}\left[\|\tilde{v}(t,.)\|^2_{L^2(\Omega)}\right]_{t_0}^{t_1} + \frac{7}{8}\int_{t_0}^{t_1}\int_{\Gamma_{\text{in}}} |\tilde{u}|^2(-g) &\le \frac{1}{4}\int_{t_0}^{t_1}\int_{\Gamma_{\text{out}}} |\tilde{u}|^2(-g)\\ &+ C\int_{t_0}^{t_1}\big(\|\tilde{v}\|^2_{L^2(\Omega)} + \|\tilde{u}\|^2_{L^2(\Omega)}\big) + Cp\int_{t_0}^{t_1}\big(\|\tilde{v}\|^{2(p-1)/p}_{L^2(\Omega)} + \|\tilde{u}\|^{2(p-1)/p}_{L^2(\Omega)}\big)\\ &+ C(t_1 - t_0)\,\|\tilde{\omega}_{\text{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))} + (t_1 - t_0)\sum_{i\in I_{\text{in}}} |\tilde{\mathcal{C}}_{i,0}|^2.\end{aligned} \tag{9.17}$$

In the proof of Corollary 9.2, we use the following result on the trace of harmonic functions.

Lemma 9.3. *For all $i \in I$, there exists a constant $C > 0$ such that for any function h which is harmonic on Ω and C^1 on $\overline{\Omega}$,*

$$\int_{\Gamma_i} (\partial_{\mathbf{n}} h)^2 \leq \int_{\Gamma_i} (\partial_\tau h)^2 + C \int_\Omega |\nabla h|^2. \tag{9.18}$$

Lemma 9.3 can be seen for example as local version of Hörmander's trace inequality (see [15]). Let us refer here to [10] for more on the historical context of this trace inequality and its relationship with Rellich and Pohozaev types inequalities. For the sake of completeness, we provide below a proof which uses Lemma 8.1.

Proof of Lemma 9.3. For all $i \in I$, there is a C^1 vector field $N_i : \overline{\Omega} \to \mathbb{R}^2$ which is equal to the outward unit normal vector n on Γ_i, and which is equal to 0 in a neighbourhood of $\Gamma \setminus \Gamma_i$. We apply (8.2) with $(\nabla^\perp h, N_i)$ instead of (u, w). Observe that the first vector field is divergence free and curl free since the function h is harmonic. This entails that

$$\int_{\Gamma_i} |\nabla h|^2 - 2\int_{\Gamma_i} (\partial_\tau h)^2 = \int_\Omega |\nabla h|^2 (\text{div } N_i) - 2\int_\Omega \nabla^\perp h \cdot \big((\nabla^\perp h \cdot \nabla) N_i\big). \tag{9.19}$$

Since N_i is C^1, we deduce from (9.19) that there exists a constant $C > 0$ such that for any function h that is harmonic on Ω and C^1 on $\overline{\Omega}$, the following estimate holds true:

$$\Big| \int_{\Gamma_i} (\partial_{\mathbf{n}} h)^2 - (\partial_\tau h)^2 \Big| \leq C \int_\Omega |\nabla h|^2, \tag{9.20}$$

and thus in particular (9.18). ■

Proof of Corollary 9.2. We successively bound the terms in the right-hand side of (9.4). The first term can be estimated as follows.

Lemma 9.4. *For all $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that*

$$\Big| \int_{\Gamma_{\text{out}}} (\tilde{u} \cdot \tilde{v})\, g \Big| \leq \varepsilon \int_{\Gamma_{\text{out}}} |\tilde{u}|^2 g + M(\varepsilon) \int_\Omega |\tilde{v}|^2. \tag{9.21}$$

Proof of Lemma 9.4. First, since $\tilde{u} \cdot n = 0$,

$$\int_{\Gamma_{\text{out}}} (\tilde{u} \cdot \tilde{v})\, g = \int_{\Gamma_{\text{out}}} (\tilde{u} \cdot \tau)\, (\tilde{v} \cdot \tau)\, g. \tag{9.22}$$

Using (9.1) and the fact that g is bounded on Γ we deduce that for all $\varepsilon > 0$, there exists $\tilde{M}(\varepsilon) > 0$ such that

$$\Big| \int_{\Gamma_{\text{out}}} (\tilde{u} \cdot \tilde{v})\, g \Big| \leq \varepsilon \int_{\Gamma_{\text{out}}} |\tilde{u}|^2\, |g| + \tilde{M}(\varepsilon) \int_{\Gamma_{\text{out}}} |\partial_{\mathbf{n}} \tilde{\varphi}|^2. \tag{9.23}$$

Now we apply Lemma 9.3 for i in I_{out} with $\tilde{\varphi}$ instead of h and we use that $\tilde{\varphi}$ is equal to 0 on Γ_{out} to obtain that there exists $C > 0$ such that

$$\int_{\Gamma_{\text{out}}} |\partial_{\mathbf{n}} \tilde{\varphi}|^2 \leq C \int_\Omega |\tilde{v}|^2. \tag{9.24}$$

Combining (9.23) and (9.24) and setting $M(\varepsilon) := C \tilde{M}(\varepsilon)$, we conclude the proof of Lemma 9.4. ■

The second term in the right-hand side of (9.4) can be estimated as follows.

Lemma 9.5. *There exists a constant $C > 0$ such that for all p in $[2, +\infty)$,*

$$\left| \int_{\Gamma_{\text{in}}} (\hat{u} \cdot \tilde{u})\, (\tilde{v} \cdot n) \right| \leq Cp\, \|\tilde{v}\|_{L^2(\Omega)}^{2(p-1)/p}. \tag{9.25}$$

Proof. First, since $\tilde{u} \cdot n = 0$ and $\tilde{u} \cdot \tau = -\tilde{v} \cdot \tau$ on Γ_{in}, we have:

$$\int_{\Gamma_{\text{in}}} (\hat{u} \cdot \tilde{u})\, (\tilde{v} \cdot n) = -\int_{\Gamma_{\text{in}}} (\hat{u} \cdot \tau)\, (\tilde{v} \cdot n)\, (\tilde{v} \cdot \tau).$$

There exists a vector field $T : \overline{\Omega} \to \mathbb{R}^2$ which is C^2, equal to 0 outside a given neighborhood of Γ_{in} and satisfies $T_{|\Gamma_{\text{in}}} = \tau$. We apply Lemma 8.1 with $(\tilde{v}, \tilde{v}, (\hat{u} \cdot T)\, T)$ instead of (u, v, w), observing that $\tilde{v}$ is divergence free since the function $\tilde{\varphi}$ is harmonic. We arrive at

$$\begin{aligned} 2\int_{\Gamma_{\text{in}}} (\hat{u} \cdot \tau)\, (\tilde{v} \cdot n)\, (\tilde{v} \cdot \tau) &= 2\int_{\Gamma_{\text{in}}} (\tilde{v} \cdot n)\, (\tilde{v} \cdot (\hat{u} \cdot T)\, T) \\ &= -\int_{\Omega} |\tilde{v}|^2 \operatorname{div}\, ((\hat{u} \cdot T)\, T) + 2\int_{\Omega} \tilde{v} \cdot ((\tilde{v} \cdot \nabla)\, (\hat{u} \cdot T)\, T). \end{aligned}$$

By Hölder's inequality, we have that

$$\left| \int_{\Gamma_{\text{in}}} (\hat{u} \cdot \tau)\, (\tilde{v} \cdot n)\, (\tilde{v} \cdot t) \right| \leq C\, \|\tilde{v}\|_{L^{2p/(p-1)}(\Omega)}^2\, \|\hat{u}\|_{W^{1,p}(\Omega)}. \tag{9.26}$$

To conclude, we use the interpolation inequality (6.12) and the estimates of Proposition 2.1 and of Proposition 7.1. ∎

Regarding the second to last term in the right-hand side of (9.4), we first establish the following result on the circulations $\tilde{\mathcal{D}}_i$ of $\tilde{v}$ around each Γ_i.

Lemma 9.6. *For all $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that for every $0 \leq t_0 < t_1 \leq T$,*

$$\begin{aligned} \left| \int_{t_0}^{t_1} \sum_{i \in I^*} \psi_i'\, \tilde{\mathcal{D}}_i \right| \leq M(\varepsilon) \Big((t_1 - t_0) \|\tilde{\omega}_{\text{in}}\|_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))}^2 + (t_1 - t_0) \sum_{i \in I_{\text{in}}} |\tilde{\mathcal{C}}_{i,0}|^2 \\ + \int_{t_0}^{t_1} \|\tilde{v}\|_{L^2(\Omega)}^2 \Big) + \varepsilon \int_{t_0}^{t_1} \left(\|\tilde{u}\|_{L^2(\Omega)}^2 + \|\tilde{u}\|_{L^2(\Gamma,|g|)}^2 \right). \end{aligned} \tag{9.27}$$

Proof. On the one hand, for $i \in I_{\text{in}}$, by (9.3) and (4.2), we obtain that for any $t \geq 0$,

$$|\tilde{\mathcal{D}}_i(t)| \leq |\tilde{\mathcal{C}}_{i,0}| + \int_0^t \int_{\Gamma_i} |\tilde{\omega}_{\text{in}}|\, |g| \leq |\tilde{\mathcal{C}}_{i,0}| + T\, \|\tilde{\omega}_{\text{in}}\|_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))}. \tag{9.28}$$

On the other hand, by the Cauchy–Schwarz inequality and (9.24) we obtain that there exists a constant $C > 0$ such that for any $t \geq 0$, for all $i \in I_{\text{out}}$,

$$|\tilde{\mathcal{D}}_i(t)| \leq C\, \|\tilde{v}(t, .)\|_{L^2(\Omega)}. \tag{9.29}$$

Combining (9.28) and (9.29), we get that there exists a constant $C > 0$ such that for every $0 \leq t_0 < t_1 \leq T$,

$$\begin{aligned}\int_{t_0}^{t_1} \sum_{i \in I^*} |\tilde{\mathscr{D}}_i|^2 \leq C \int_{t_0}^{t_1} \|\tilde{v}\|^2_{L^2(\Omega)} + C(t_1 - t_0)\|\tilde{\omega}_{\text{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))} \\ + C(t_1 - t_0) \sum_{i \in I_{\text{in}}} |\tilde{\mathscr{C}}_{i,0}|^2.\end{aligned} \tag{9.30}$$

Moreover, for any $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that for every $0 \leq t_0 < t_1 \leq T$,

$$\Big| \int_{t_0}^{t_1} \sum_{i \in I^*} \psi_i' \, \tilde{\mathscr{D}}_i \Big| \leq M(\varepsilon) \int_{t_0}^{t_1} \sum_{i \in I^*} |\tilde{\mathscr{D}}_i|^2 + \varepsilon C^{-1} \int_{t_0}^{t_1} \sum_{i \in I^*} |\psi_i'|^2, \tag{9.31}$$

where $C > 0$ is the constant which appears in (5.1). By combining (9.31), (9.30) and (5.1) we obtain (9.27). ∎

Regarding the last term in the right-hand side of (9.4), we have the following estimate.

Lemma 9.7. *There exists a constant $C > 0$ such that for any $0 < t_0 < t_1 < T$:*

$$\Big| \int_{t_0}^{t_1} \int_{\Gamma_{\text{in}}} \tilde{\varphi} \, \tilde{\omega}_{\text{in}} \, g \Big| \leq C \int_{t_0}^{t_1} \|\tilde{v}\|^2_{L^2(\Omega)} + (t_1 - t_0) \, \|\tilde{\omega}_{\text{in}}\|^2_{L^\infty([0,T]\times\Gamma_{\text{in}})}. \tag{9.32}$$

Proof. By classical trace theory, there exists $C > 0$ such that

$$\|\tilde{\varphi}\|_{L^2(\Gamma)} \leq C\|\tilde{\varphi}\|_{H^1(\Omega)} \leq C'\|\tilde{v}\|_{L^2(\Omega)},$$

for another $C' > 0$ by the Poincaré inequality, using that $\tilde{\varphi}$ is equal to 0 on Γ_{out}. Therefore Lemma 9.7 follows by using the Cauchy–Schwarz inequality. ∎

Finally, to bound the fourth term in the right-hand side of (9.4), we proceed as in the proof of Corollary 6.3. More precisely, we first use Hölder's inequality to get that for all p in $(1, +\infty)$,

$$\begin{aligned}\Big| \int_\Omega \big(\hat{\omega} \, \tilde{u} \cdot \tilde{v}^\perp + \tilde{v} \cdot ((\tilde{u} \cdot \nabla) \, \hat{u}) + \tilde{u} \cdot ((\tilde{v} \cdot \nabla) \, \hat{u})\big) \Big| \\ \leq \big(\|\tilde{v}\|^2_{L^{2p/(p-1)}(\Omega)} + \|\tilde{u}\|^2_{L^{2p/(p-1)}(\Omega)}\big) \|\hat{u}\|_{W^{1,p}(\Omega)}.\end{aligned}$$

Then, we observe that both $\tilde{u}$ and $\tilde{v}$ are in $L^\infty([0,T]; L^\infty(\Omega))$, respectively thanks to Proposition 5.1 and Proposition 7.1, the Sobolev embedding theorem and (4.5) and (9.1). Thus, by the interpolation inequality (6.12) and by Proposition 2.1, we deduce that there exists a positive constant C such that for all p in $[2, +\infty)$,

$$\Big| \int_\Omega \big(\hat{\omega} \, \tilde{u} \cdot \tilde{v}^\perp + \tilde{v} \cdot ((\tilde{u} \cdot \nabla) \, \hat{u}) + \tilde{u} \cdot ((\tilde{v} \cdot \nabla) \, \hat{u})\big) \Big| \leq Cp \big(\|\tilde{u}\|^{2(1-1/p)}_{L^2(\Omega)} + \|\tilde{v}\|^{2(1-1/p)}_{L^2(\Omega)}\big). \tag{9.33}$$

Therefore, Corollary 9.2 is a consequence of Lemma 9.4, Lemma 9.7, Lemma 9.5, and Lemma 9.6 with $\varepsilon = 1/8$, and (9.33). ∎

10. Osgood argument and end of the proof

In this section we combine the two energy estimates obtained, respectively, in Section 6 and in Section 9, and use an Osgood argument to conclude the proof of Theorem 3.1. Indeed, by summing (6.9) and (9.17), we obtain that there exists a constant $C > 0$ such that, for $0 \le t_0 < t_1 \le t \le T$, for every $p \ge 2$,

$$\begin{aligned}(10.1)\quad \big[\|\tilde u\|^2_{L^2(\Omega)} + \|\tilde v\|^2_{L^2(\Omega)}\big]^{t_1}_{t_0} + \frac{1}{2}\int_{t_0}^{t_1}\int_\Gamma |\tilde u|^2|g| \\ \le C\int_{t_0}^{t_1}\big(\|\tilde v\|^2_{L^2(\Omega)} + \|\tilde u\|^2_{L^2(\Omega)} + p\|\tilde v\|^{2(p-1)/p}_{L^2(\Omega)} + p\|\tilde u\|^{2(p-1)/p}_{L^2(\Omega)}\big) \\ + C(t_1 - t_0)\Big(\|\tilde\omega_{\mathrm{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\mathrm{in}},|g|))} + \sum_{i\in I_{\mathrm{in}}}|\tilde{\mathcal{C}}_{i,0}|^2\Big).\end{aligned}$$

We first omit the second term on the left-hand side of (10.1) and focus on the estimate of the time-dependent function

$$(10.2)\qquad z := \|\tilde u\|^2_{L^2(\Omega)} + \|\tilde v\|^2_{L^2(\Omega)}.$$

We obtain that for $0 \le t_0 < t_1 \le t \le T$ and for every $p \ge 2$,

$$\begin{aligned}(10.3)\qquad [z]^{t_1}_{t_0} \le C\int_{t_0}^{t_1}\big(z(t) + pz(t)^{1-1/p}\big)\,\mathrm{d}t \\ + C(t_1 - t_0)\Big(\|\tilde\omega_{\mathrm{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\mathrm{in}},|g|))} + \sum_{i\in I_{\mathrm{in}}}|\tilde{\mathcal{C}}_{i,0}|^2\Big).\end{aligned}$$

In the case where $z(t)$ is more than one, the term in the first parenthesis above can be bounded by $(1 + p)z(t)$, and a classical Gronwall argument can be applied. The case where $z(t)$ is less than one, which is of particular interest in view of the uniqueness issue, requires to replace the Gronwall argument by Osgood's lemma, as this was done by Yudovich in [34, 35] in the impermeable case. To this end, we first establish the following result in view of minimizing the right-hand side above with respect to p locally in time.

Lemma 10.1. *Let $a \ge 0$ and let y be a non-negative continuous and non-decreasing function. Let F be a non-negative continuous function on $[2, +\infty) \times \mathbb{R}_+$ which is increasing with respect to the second variable. We assume that for every $0 < t_0 < t_1 < T$, and for every $p \in [2, +\infty)$,*

$$(10.4)\qquad [y]^{t_1}_{t_0} \le a(t_1 - t_0) + \int_{t_0}^{t_1} F(p, y(s))\,\mathrm{d}s.$$

Let μ be a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$. We assume that for any $x \ge 0$, there exists p_x in $[2, +\infty)$ such that

$$(10.5)\qquad F(p_x, x) \le \mu(x).$$

Then, for all $0 < t < T$,

$$(10.6)\qquad [y]^t_0 \le at + \int_0^t \mu(y(s))\,\mathrm{d}s.$$

Proof of Lemma 10.1. Let t be in $(0,T)$ and n in $\mathbb{N}^*$. Using (10.4) and (10.5), we get that there are some $(p_{y((k+1)t/n)})_{0\le k\le n-1}$ in $[2,+\infty)^n$ and $C>0$ such that

$$[y]_0^t = \sum_{k=0}^{n-1}\Big(y\Big(\frac{(k+1)t}{n}\Big)-y\Big(\frac{kt}{n}\Big)\Big) \le at + \sum_{k=0}^{n-1}\int_{kt/n}^{(k+1)t/n} F\big(p_{y((k+1)t/n)}, y(s)\big)\,\mathrm{ds}, \tag{10.7}$$

and for $0\le k\le n-1$,

$$F\Big(p_{y((k+1)t/n)}, y\Big(\frac{(k+1)t}{n}\Big)\Big) \le C\mu\Big(y\Big(\frac{(k+1)t}{n}\Big)\Big). \tag{10.8}$$

Then, as y is an non-decreasing function of t and F is an increasing function of y, the function $t\mapsto F(p,y(t))$ is increasing, which leads to

$$\int_{kt/n}^{(k+1)t/n} F\big(p_{y((k+1)t/n)}, y(s)\big)\,\mathrm{ds} \le \frac{t}{n}F\Big(p_{y((k+1)t/n)}, y\Big(\frac{(k+1)t}{n}\Big)\Big). \tag{10.9}$$

Combining (10.7), (10.8) and (10.9), we obtain that

$$[y]_0^t \le at + \frac{Ct}{n}\sum_{k=0}^{n-1}\mu\Big(y\Big(\frac{(k+1)t}{n}\Big)\Big). \tag{10.10}$$

As this is true for every n, we conclude by Riemann summations, using the fact that μ is continuous. ∎

We define the function y by

$$y(t) := \max_{s\in[0,t]} z(s), \tag{10.11}$$

where z is the function defined in (10.2), and we set

$$F(p,x) := x + px^{1-1/p}, \quad \mu(x) := Cx(1+|\ln(x)|) \tag{10.12}$$

and

$$a := C\Big(\|\tilde\omega_{\mathrm{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\mathrm{in}},|g|))} + \sum_{i\in I_{\mathrm{in}}}|\tilde{\mathcal{C}}_{i,0}|^2\Big), \tag{10.13}$$

where C is the constant which appears in (10.3). The condition (10.5) is fulfilled with $p_x = |\ln(x)|$ for x in $[0,e^{-2}]$ and $p_x=2$ for x in $(e^{-2},+\infty)$. Moreover, according to (10.3), the function y satisfies (10.4). Therefore, by Lemma 10.1, the function y satisfies (10.6) for all $t\in[0,T]$.

Let us now recall Osgood's lemma, see for example Lemma 3.4 in [2].

Lemma 10.2. *Let* $\mu\colon [0,1] \to [0,+\infty)$ *be an increasing continuous function with* $\mu(0) = 0$, $T > 0$ *and le* $tc \geq 0$. *Let* $y\colon \mathbb{R}_+ \to [0,1]$ *be a function satisfying, for* t *in* $[0,T]$,

$$y(t) \leq c + \int_0^t \mu(y(s))\, ds, \tag{10.14}$$

In the case where $c > 0$, *then, for* t *in* $[0,T]$,

$$\int_c^{y(t)} \frac{dx}{\mu(x)} \leq t.$$

In the case where $c = 0$, *if the function* μ *also satisfies*

$$\int_0^1 \frac{dt}{\mu(t)} = +\infty,$$

then for t *in* $[0,T]$, $y(t) = 0$.

We now apply Osgood's lemma to the function y defined in (10.11), with μ and a defined in (10.12), and $c = y(0) + aT$. Focusing on the first case, this yields that for $0 \leq t \leq T$,

$$\int_{y(0)+aT}^{y(t)} \frac{dx}{\mu(x)} \leq t,$$

and thus in particular that for any $t \geq 0$,

$$\int_{y(0)+at}^{y(t)} \frac{dx}{\mu(x)} \leq t,$$

which leads to

$$y(t) \leq e\,(y(0) + ta)^{e^{-Ct}}. \tag{10.15}$$

Using that, similarly to (7.6), there holds

$$\|\tilde{v}(0,\cdot)\|_{L^2(\Omega)} \leq C\big(\|\tilde{u}(0,\cdot)\|_{L^2(\Omega)} + \|\tilde{\omega}(0,\cdot)\|_{L^2(\Omega)}\big), \tag{10.16}$$

and recalling (10.2) and (10.11), we deduce from (10.15) that

$$\|\tilde{u}(t)\|^2_{L^2(\Omega)} \leq C\Big(\|\tilde{u}(0,\cdot)\|^2_{L^2(\Omega)} + \|\tilde{\omega}(0,\cdot)\|^2_{L^2(\Omega)} + t\Big(\|\tilde{\omega}_{\text{in}}\|^2_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))} + \sum_{i\in I_{\text{in}}} |\tilde{\mathcal{C}}_{i,0}|^2\Big)\Big)^{e^{-Ct}}.$$

Now, by using Proposition 2.1, we conclude that there exists a continuous function $F\colon (\mathbb{R}_+)^3 \times \mathbb{R}^{I^*} \mapsto \mathbb{R}_+$ satisfying $F(\tau,0,0,0) = 0$ for all $\tau \geq 0$, and such that for every $t \in [0,T]$,

$$\|\tilde{u}(t)\|^2_{L^2(\Omega)} \leq F\big(t, \|\omega_0^1 - \omega_0^2\|_{L^\infty(\Omega)}, \|\omega_{\text{in}}^1 - \omega_{\text{in}}^2\|_{L^\infty([0,t];L^\infty(\Gamma_{\text{in}},|g|))}, (\mathcal{C}^1_{i,0} - \mathcal{C}^2_{i,0})_{i\in I^*}\big).$$

Moreover, going back to (10.1), we deduce a similar bound of the second term on the left-hand side, and we arrive at (3.1). The uniqueness result corresponds to the case where $y(0)$ and a are both zero, for which we apply the second case of Osgood's lemma.

Remark 10.3. With some bookkeeping, we observe that formally the inequality (10.1) is obtained by applying the weak formulation (4.10) with a test function which is a combination of $\tilde{\psi}$, of $\tilde{\varphi}$ and of $\tilde{\mathcal{D}}_i g^i$, for i in I^*, and that such a test function is a non-local operator of order 0 acting on $\tilde{\psi}$. This is reminiscent of the Kreiss symmetrizer technics in hyperbolic theory, see for example [3, 17].

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