Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1353

Published by EMS Press

C^2 interpolation with range restriction

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Abstract. Given $-\infty < \lambda < \Lambda < \infty$, $E \subset \mathbb{R}^n$ finite, and $f: E \to [\lambda, \Lambda]$, how can we extend f to a $C^m(\mathbb{R}^n)$ function F such that $\lambda \le F \le \Lambda$ and $\|F\|_{C^m(\mathbb{R}^n)}$ is within a constant multiple of the least possible, with the constant depending only on m and n? In this paper, we provide the solution to the problem for the case $m = 2$. Specifically, we construct a (parameter-dependent, nonlinear) $C^2(\mathbb{R}^n)$ extension operator that preserves the range $[\lambda, \Lambda]$, and we provide an efficient algorithm to compute such an extension using $O(N \log N)$ operations, where $N = \#(E)$.

Contents

1. Introduction

For integers $m \ge 0$, $n \ge 1$, we write $C^m(\mathbb{R}^n)$ to denote the Banach space of m-times continuously differentiable real-valued functions such that the following norm is finite:

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \le m} |\partial^\alpha F(x)|.$$

We use $C(m,n)$, $k(m,n)$, etc., to denote controlled constants that depend only on m and n. If E is a finite subset of $\mathbb{R}^n$, we write $\#E$ to denote the number of elements in E.

2020 Mathematics Subject Classification: Primary 41A05; Secondary 26B05.
Keywords: Whitney problems, interpolation, range restriction, non-negative.

We consider the following interpolation problem with lower and upper bounds λ, Λ, respectively.

Problem 1. Let $E \subset \mathbb{R}^n$ be a finite set. Let $-\infty < \lambda < \Lambda < \infty$. Let $f : E \to [\lambda, \Lambda]$. Compute a function $F : \mathbb{R}^n \to [\lambda, \Lambda]$ such that

(A) $F = f$ on E, and

(B) $\|F\|_{C^m(\mathbb{R}^n)} \leq C(m,n) \cdot \inf\{\|\tilde{F}\|_{C^m(\mathbb{R}^n)} : \tilde{F} = f \text{ on } E, \text{and } \lambda \leq \tilde{F} \leq \Lambda\}$.

By "computing a function F" from (E, λ, Λ, f), we mean the following: After processing the input (E, λ, Λ, f), we are able to accept a query consisting of a point $x \in \mathbb{R}^n$, and produce a list of numbers $(F_\alpha(x) : |\alpha| \leq m)$. The algorithm "computes the function F" if for each $x \in \mathbb{R}^n$, we have $\partial^\alpha F(x) = F_\alpha(x)$ for $|\alpha| \leq m$. In other words, for each $x \in \mathbb{R}^n$, we want to produce the coefficients of the m-th degree Taylor polynomial of F at x. See Theorem 1.2 below.

We call the function F in Problem 1 a C-optimal interpolant of f (see Condition (B)).

Problem 1 is closely related to a common theme in data visualization, where one wants to present some given three-dimensional data as a surface or a contour map. Moreover, one may want to preserve some crucial inherent properties of the data, such as nonnegativity or convexity. This occurs when the data arises as some physical quantities and we want to preserve the physical meaning of the interpolant. For instance, nonnegative constraint is natural when the data represents a probability distribution or (absolute) temperature. More generally, it is sometimes desirable to impose both upper and lower bounds on the interpolants, commonly referred to as "range-restricted interpolants". See e.g., [12, 26, 27, 31–33]. These problems arise, for example, when the predicted trajectory must avoid collision with prescribed obstacles. We refer the readers to the aforementioned references for further background and related topics on range-restricted interpolation.

By letting $\tau := (\Lambda - \lambda)/2$ and replacing f by $f - (\Lambda + \lambda)/2$, we see that Problem 1 admits the following symmetric formulation.

Problem 2. Let $E \subset \mathbb{R}^n$ be a finite set. Let $\tau > 0$. Let $f : E \to [-\tau, \tau]$. Compute a function $F : \mathbb{R}^n \to [-\tau, \tau]$ such that

(A) $F = f$ on E, and

(B) $\|F\|_{C^m(\mathbb{R}^n)} \leq C(m,n) \cdot \inf\{\|\tilde{F}\|_{C^m(\mathbb{R}^n)} : \tilde{F} = f \text{ on } E, \text{and } -\tau \leq \tilde{F} \leq \tau\}$.

Since translating a $C^m(\mathbb{R}^n)$ function by a constant does not affect its (nonzero-th order) derivatives, Problem 2 captures all the difficulties of Problem 1. To recover Problem 1 from Problem 2, we simply modify the zero-th order estimate by the translated amount. From now on, we abuse language and say that Problems 1 and 2 are "equivalent".

Formally letting $\tau \to \infty$ in Problem 2, we recover the classical Whitney interpolation problem.

Problem 3 (Classical Whitney interpolation problem). Let $E \subset \mathbb{R}^n$ be a finite set. Let $f : E \to \mathbb{R}$. Compute a function $F : \mathbb{R}^n \to \mathbb{R}$ such that

(A) $F = f$ on E, and

(B) $\|F\|_{C^m(\mathbb{R}^n)} \leq C(m,n) \cdot \inf\{\|\tilde{F}\|_{C^m(\mathbb{R}^n)} : \tilde{F} = f \text{ on } E\}$.

Problems 1–3 for the cases $m = 0, 1$ can be immediately solved by the classical Whitney extension theorem. Recall that the classical C^m Whitney extension operator for a finite set (see Theorem 3.1 (B) below) is a continuous linear operator that smoothly averages the given $(m-1)$-jets and can be efficiently constructed (in the sense of Definition 1.1 below). For $m = 1$ (or the more trivial case $m = 0$), the Whitney extension operator simply averages the given function values, and thus preserves the prescribed range. For $m \geq 2$, the operator fails to do the job for two reasons. One is that we are not given $(m-1)$-jets but only function values, and the other one is that averaging $(m-1)$-jets will no longer preserve the prescribed range in general.

Further alternatives for the case $m = 0$ include Urysohn's lemma and Kirszbraun's formula.

The classical Whitney interpolation Problem 3 is well-understood thanks to the works of Brudnyi and Shvartsman [6,8,9], Fefferman and Klartag [14,17,20,21,24]. In [20,21], the authors provide an efficient algorithm for solving the classical Whitney interpolation Problem 3. Their algorithm pre-processes the set E using at most $C(m,n)N \log N$ operations (on a von Neumann machine that can operate with exact numbers) and $C(m,n)N$ storage with $N = \#E$. Then, after reading f, the algorithm is ready to answer queries. A query consists of a point $x \in \mathbb{R}^n$, and the answer to a query is the m-th order Taylor polynomial of an interpolant F with the least norm up to a constant factor $C(m,n)$. The number of operations to answer a query is $C(m,n) \log N$. The complexity of the Fefferman–Klartag algorithm is most likely the best possible.

Problem 2 (or equivalently Problem 1) and the classical Whitney interpolation Problem 3 are related to the following smooth selection problem.

Problem 4. Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, let $K(x) \subset \mathbb{R}^d$ be convex. Find a function $\vec{F} = (F_1, \ldots, F_d): \mathbb{R}^n \to \mathbb{R}^d$ such that $F(x) \in K(x)$ for all $x \in E$ and $\|\vec{F}\|_{C^m(\mathbb{R}^n,\mathbb{R}^d)}$ as small as possible, up to a constant factor depending only on m, n, and d.

If we specialize $K(x) \subset \mathbb{R}$ (hence $d = 1$) in Problem 4 to a singleton, we obtain the classical Whitney interpolation Problem 3.

Here we note a subtle but crucial difference between Problems 1, 2 and Problem 4 (with $d = 1$ and $K(x)$ being a fixed compact interval for each $x \in E$). The lower and upper bounds, $[\lambda, \Lambda]$ or $[-\tau, \tau]$, for F are global in Problem 1 or 2. On the other hand, these bounds are only imposed on the set E in Problem 4.

Problem 4 and the related "finiteness principles" (see e.g. Theorem 1.4 below) have been extensively studied by Y. Brudnyi and P. Shvartsman [5,9], C. Fefferman, A. Israel, and G. K. Luli[19], C. Fefferman and P. Shvartsman [23].

In this paper, inspired by [17,20,21], building on the work of [28–30], we solve Problem 2 for the case $n \in \mathbb{N}$ and $m = 2$.

To facilitate the discussion on algorithms, we introduce the following concepts.

Definition 1.1. Let $N_0 \geq 1$ be an integer. Let $B = \{\xi_1, \ldots, \xi_{N_0}\}$ be a basis of $\mathbb{R}^{N_0}$. Let $\Omega \subset \mathbb{R}^{N_0}$ be a subset. Let X be a set. Let $\Xi: \Omega \to X$ be a map.

- We say Ξ *has depth* D (with respect to the basis B) if D is the smallest integer such that the following holds: There exists a D-dimensional subspace V spanned by $\xi_{i_1}, \ldots, \xi_{i_D} \in B$, such that for all $z_1, z_2 \in \Omega$ with $\pi_V(z_1) = \pi_V(z_2)$, we have

$\Xi(z_1) = \Xi(z_2)$. Here, $\pi_V : \mathbb{R}^{N_0} \to V$ is the natural projection. We call the set of indices $\{i_1, \ldots, i_D\}$ the *source* of Ξ (with respect to the basis B).

- Suppose Ξ has depth D. Let $V = \mathrm{span}(\xi_{i_1}, \ldots, \xi_{i_D})$ and let π_V be as above. By an *efficient representation* of Ξ, we mean a specification of the index set $\{i_1, \ldots, i_D\} \subset \{1, \ldots, N_0\}$ and a map $\tilde{\Xi} : \pi_V(\Omega) \to X$ with $\Xi = \tilde{\Xi} \circ \pi_V$ on Ω, such that given $v \in \pi_V(\Omega)$, $\tilde{\Xi}(v)$ can be computed using at most C_D operations. Here, C_D is a constant depending only on D.

Remark 1.1. In [21], the authors introduced the notion of a "compact representation" of a linear functional $\Xi : \mathbb{R}^{\bar{N}} \to \mathbb{R}$, which consists of a list of indices $\{i_1, \ldots, i_D\} \subset \{1, \ldots, \bar{N}\}$ and a list of coefficients $\chi_{i_1}, \ldots, \chi_{i_D}$, so that the action of Ξ is characterized by

$$\Xi : (\xi_1, \ldots, \xi_{\bar{N}}) \mapsto \sum_{\Delta=1}^{D} \chi_{i_\Delta} \cdot \xi_{i_\Delta}.$$

Therefore, given $v \in \mathrm{span}(\xi_{i_1}, \ldots, \xi_{i_D})$, we can compute $\Xi(v)$ by the dot product of two vectors of length D, which requires C_D operations. The present notion of "efficient representation" is a natural generalization of the "compact representation" in [21] adapted to the nonlinear nature of constrained interpolation (see also [28–30]).

We write $C^2(E, \tau)$ to denote the collection of functions $f : E \to [-\tau, \tau]$, which can be identified with $[-\tau, \tau]^{\#E}$. We define

$$\|f\|_{C^2(E,\tau)} := \inf\{\|F\|_{C^2(\mathbb{R}^n)} : F = f \text{ on } E \text{ and } -\tau \le F \le \tau\}.$$

Our first main theorem is the following.

Theorem 1.1. *Let n be a positive integer. Let $\tau > 0$. Let $E \subset \mathbb{R}^n$ be a finite set. There exist controlled constants $C(n)$, $D(n)$, and a map $\mathcal{E}_\tau : C^2(E, \tau) \times [0, \infty) \to C^2(\mathbb{R}^n)$ such that the following hold*:

(A) *Let $M \ge 0$. Then for all $f \in C^2(E, \tau)$ with $\|f\|_{C^2(E,\tau)} \le M$, we have $\mathcal{E}_\tau(f, M) \in C^2(\mathbb{R}^n, \tau)$, $\mathcal{E}_\tau(f, M) = f$ on E, and $\|\mathcal{E}_\tau(f, M)\|_{C^2(\mathbb{R}^n)} \le CM$.*

(B) *For each $x \in \mathbb{R}^n$, there exists a set $S(x) \subset E$, independent of τ, with $\#S(x) \le D$, such that for all $M \ge 0$ and $f, g \in C^2(E, \tau)$ with $f|_{S(x)} = g|_{S(x)}$, we have*

$$\partial^\alpha \mathcal{E}_\tau(f, M)(x) = \partial^\alpha \mathcal{E}_\tau(g, M)(x) \quad \text{for } |\alpha| \le 2.$$

In fact, we can prove a stronger version of Theorem 1.1. Let $\mathcal{P}^+$ denote the vector space of polynomials on $\mathbb{R}^n$ with degree no greater than two, and let $\mathcal{J}_x^+ F$ denote the two-jet of F at x.

Theorem 1.2. *Let n be a positive integer. Let $E \subset \mathbb{R}^n$ be a finite set with $\#(E) = N$. Then there exists a collection of maps $\{\Xi_{\tau,x} : \tau \in [0, \infty),\ x \in \mathbb{R}^n\}$, where*

$$\Xi_{\tau,x} : C^2(E, \tau) \times [0, \infty) \to \mathcal{P}^+$$

for each $x \in \mathbb{R}^n$, such that the following hold:

(A) *There exists a controlled constant $D(n)$ such that for each $x \in \mathbb{R}^n$, the map $\Xi_{\tau,x}(\cdot,\cdot)\colon C^2(E,\tau) \times [0,\infty) \to \mathcal{P}^+$ is of depth D. Here, we view $C^2(E,\tau) \subset \mathbb{R}^{\#E}$ with the standard basis. Moreover, the source of $\Xi_{\tau,x}$ (in the sense of Definition 1.1) is independent of τ.*

(B) *Suppose we are given $(f, M) \in C^2(E,\tau) \times [0,\infty)$ with $\|f\|_{C^2(E,\tau)} \le M$. Then there exists a function $F \in C^2(\mathbb{R}^n,\tau)$ such that*

$$\mathfrak{J}_x^+ F \equiv \Xi_{\tau,x}(f, M) \quad \text{for all } x \in \mathbb{R}^n,$$

$$\|F\|_{C^2(\mathbb{R}^n)} \le CM \quad \text{and} \quad F(x) = f(x) \quad \text{for } x \in E.$$

Here, C depends only on n.

(C) *There is an algorithm that takes the given data set E, performs one-time work, and then responds to queries. A query consists of a pair $(\tau, x) \in [0,\infty) \times \mathbb{R}^n$, and the response to the query is the map $\Xi_{\tau,x}$, given in an efficient representation. The one-time work takes $C_1 N \log N$ operations and $C_2 N$ storage. The work to answer a query is $C_3 \log N$. Here, C_1, C_2, C_3 depend only on n.*

We briefly explain the strategy for the proofs of Theorem 1.1 and Theorem 1.2, sacrificing accuracy for the ease of understanding.

We will prove Theorem 1.1 and Theorem 1.2 by inducting on the dimension n. The base case for the induction is given by

Theorem 1.3. *Theorems 1.1 and 1.2 are true for $n = 1$.*

Assume the validity of Theorems 1.1 and 1.2 for $n-1$. Let $E \subset \mathbb{R}^n$ be a finite set. We perform a Calderón–Zygmund decomposition of $\mathbb{R}^n$ into dyadic cubes $\{Q : Q \in CZ^0\}$, such that near each Q, E lies on a hypersurface with curvature bounded by $C\delta_Q^{-1}$, where δ_Q is the sidelength of Q and C is some constant depending only on n. As such, E can be locally straightened to lie within a hyperplane by a C^2-diffeomorphism, and the local interpolation problem is readily solvable by the induction hypothesis. We then construct the global interpolation map by patching together these local interpolation maps via a partition of unity. To avoid large derivatives caused by the partition functions supported on small cubes, we introduce a collection of "transition jets" that guarantee Whitney compatibility among neighboring cubes, and construct our local interpolants in accordance with these transition jets.

We have given an overly simplified account of our strategy. In practice, we have to take great care to preserve the range restriction $-\tau \le F \le \tau$, and control the derivative contribution from hypersurface with large curvature.

The proof for Theorem 1.3 will be given in Section 4. The proofs for Theorem 1.1 and Theorem 1.2 will be presented in Sections 5–9.

Using Theorem 1.2, together with the "well separated pairs decomposition" technique from computational geometry, we obtain the following.

Theorem 1.4. *Let $E \subset \mathbb{R}^n$ be a finite set with $\#E = N < \infty$. Then there exist controlled constants $C_1, \ldots, C_5$, depending only on n, and a list of subsets $S_1, \ldots, S_L \subset E$ satisfying the following:*

(A) *We can compute the list $\{S_\ell : \ell = 1, \ldots, L\}$ from E using one-time work of at most $C_1 N \log N$ operations and using storage at most $C_2 N$.*

Algorithm 1 Algorithm for $C^2(\mathbb{R}^n)$ interpolation with range restriction.

DATA: $E \subset \mathbb{R}^n$ finite with $\#E = N$, $\tau > 0$, $f : E \to [-\tau, \tau]$, $M \geq 0$.

ORACLE: $\|f\|_{C^2(E,\tau)} \leq M$.

RESULT: A query function that accepts a point $x \in \mathbb{R}^n$ and produces a list of numbers $(F_\alpha(x) : |\alpha| \leq 2)$ that guarantees the following: There exists a function $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \leq CM$ and $F|_E = f$, such that $\partial^\alpha F(x) = F_\alpha(x)$ for $|\alpha| \leq 2$. The function F is independent of the query point x, and is uniquely determined by (E, τ, f, M).

COMPLEXITY: • Preprocessing (E, τ): at most $CN \log N$ operations and CN storage.

- Processing f: CN operations and CN storage.
- Answering a query: at most $C \log N$ operations.

Algorithm 2 Algorithm for approximate $C^2(\mathbb{R}^n)$ norm with range restriction.

DATA: $E \subset \mathbb{R}^n$ finite with $\#E = N$, $\tau > 0$.

QUERY: $f : E \to [-\tau, \tau]$.

RESULT: The order of magnitude of $\|f\|_{C^2(E,\tau)}$. More precisely, the algorithm outputs a number $M \geq 0$ such that both of the following hold:

- We guarantee the existence of a function $F \in C^2(\mathbb{R}^n, \tau)$ such that $F|_E = f$ and $\|F\|_{C^2(\mathbb{R}^n)} \leq CM$.
- We guarantee there exists no $F \in C^2(\mathbb{R}^n, \tau)$ with norm at most $C^{-1}M$ satisfying $F|_E = f$.

COMPLEXITY: • Preprocessing E: at most $CN \log N$ operations and CN storage.

- Answer a query: at most CN operations.

(B) *$\#S_\ell \leq C_3$ for each $\ell = 1, \ldots, L$.*

(C) *$L \leq C_4 N$.*

(D) *Given any $\tau > 0$ and $f : E \to [-\tau, \tau]$, we have*

$$\max_{1 \leq \ell \leq L} \|f\|_{C^2(S_\ell,\tau)} \leq \|f\|_{C^2(E,\tau)} \leq C_5 \max_{1 \leq \ell \leq L} \|f\|_{C^2(S_\ell,\tau)}.$$

The proof for Theorem 1.4 will be given in Section 10.

The above theoretical results allow us to produce efficient algorithms to solve Problem 2 (or Problem 1) in the case $m = 2$. In this paper, we content ourselves with an idealized computer with standard von Neumann architecture that is able to process exact real numbers. We refer the readers to [21] for discussion on finite-precision computing.

Theorem 1.2 guarantees the existence of Algorithm 1. Theorem 1.4 guarantees the existence of Algorithm 2 for approximating $\|f\|_{C^2(E,\tau)}$.

Finally, we mention that the techniques developed in this paper can readily be adapted to treat $C^2(\mathbb{R}^n)$ nonnegative interpolation; for comparison, see [28–30].

This paper is a part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney's seminal works [44–46], and including fundamental contributions by G. Glaeser [25], Y. Brudnyi and P. Shvartsman [5–10,34–42], and E. Bierstone, P. Milman, and W. Pawłucki [1–3], as well as our own papers [13–17, 20–22, 24, 28–30]. See, e.g., [18] for the history of the problem, as well as N. Zobin [47, 48] for a related problem.

2. Notations

We use c_*, C_*, C', etc., to denote constants depending only on n, referred to as "controlled constants". They may be different quantities in different occurrences. We will label them to avoid confusion when necessary.

Let M and M' be two nonnegative quantities determined by E, f and n. We say that M and M' have the *same order of magnitude*, provided that there exists a controlled constant $C(n)$ such that $C^{-1}M \le M' \le CM$. In this case we write $M \approx M'$. To compute the order of magnitude of M' is to compute a number M such that $M \approx M'$.

We assume that we are given an ordered orthogonal coordinate system on $\mathbb{R}^n$, specified by an ordered list of unit vectors $[e_1, \dots, e_n]$. We use $|\cdot|$ to denote the Euclidean distance. We use $B(x, r)$ to denote the ball of radius r centered at x. For nonempty $X, Y \subset \mathbb{R}^n$, we write $\operatorname{dist}(X, Y) := \inf_{x \in X, y \in Y} |x - y|$.

We use $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, etc., to denote multi-indices. We write ∂^α to denote $\partial_{e_1}^{\alpha_1} \cdots \partial_{e_n}^{\alpha_n}$. We adopt the partial ordering $\alpha \le \beta$ if and only if $\alpha_i \le \beta_i$ for $i = 1, \dots, n$.

By a cube, we mean a set of the form $Q = \prod_{i=1}^n [a_i, a_i + \delta)$ for some $a_1, \dots, a_n \in \mathbb{R}$ and $\delta > 0$. If Q is a cube, we write δ_Q to denote its sidelength. For $r > 0$, we use rQ to denote the cube whose center is that of Q and whose sidelength is $r\delta_Q$. Given two cubes Q, Q', we write $Q \leftrightarrow Q'$ if either $Q = Q'$, or if $\operatorname{closure}(Q) \cap \operatorname{closure}(Q') \neq \varnothing$ and $\operatorname{interior}(Q) \cap \operatorname{interior}(Q') = \varnothing$.

A dyadic cube is a cube of the form $Q = \prod_{i=1}^n [2^{-k} \cdot p_i, 2^{-k} \cdot (p_i + 1))$ for some $p_1, \dots, p_n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Let $\mathcal{D}_0$ be the collection of dyadic cubes with unit sidelength. For $k \ge 1$, we form $\mathcal{D}_k$ by bisecting each cube in $\mathcal{D}_{k-1}$ into 2^n congruent dyadic cubes with sidelength 2^{-k}. If $Q \in \mathcal{D}_k$ for some $k \ge 1$, then there exists a unique cube in $\mathcal{D}_{k-1}$ containing Q, and we denote this cube by Q^+.

Let $n \ge 1$. Let X be a C^2-diffeomorphic image of a cube or all of $\mathbb{R}^n$. We use $C^2(X)$ to denote the vector space of twice continuously differentiable real-valued functions up to the closure of X, whose derivatives up to order two are bounded. Let X_0 be the interior of X. For $F \in C^2(X)$, we define

$$\|F\|_{C^2(X)} := \sup_{x \in X_0} \max_{|\alpha| \le 2} |\partial^\alpha F(x)|.$$

We write $C^2(X, \tau)$ to denote the collection of functions $F \in C^2(X)$ such that $-\tau \le F \le \tau$ on X.

Let $E \subset \mathbb{R}^n$ be finite.

- Define $C^2(E) := \{f : E \to \mathbb{R}\}$, which can be (non-uniquely) identified with $\mathbb{R}^{\#(E)}$.

- Define $\|f\|_{C^2(E)} := \inf\{\|F\|_{C^2(\mathbb{R}^n)} : F|_E = f\}$.
- Define $C^2(E,\tau) := \{f\colon E \to [-\tau,\tau]\}$, which can be (non-uniquely) identified with $[-\tau,\tau]^{\#(E)}$.
- For $f \in C^2(E,\tau)$, define $\|f\|_{C^2(E,\tau)} := \inf\{\|F\|_{C^2(\mathbb{R}^n)} : F|_E = f \text{ and } -\tau \le F \le \tau\}$.

Note that both infima above are finite, since we can always interpolate f using bump functions, each of which is supported near one and only one point in E.

We write $\mathcal{P}$ and $\mathcal{P}^+$, respectively, to denote the vector spaces of polynomials with degree no greater than one and two.

For $x \in \mathbb{R}^n$ and a function F twice continuously differentiable at x, we write $\mathfrak{J}_x F$ and $\mathfrak{J}_x^+ F$ to denote the one-jet and two-jet of F at x, respectively, which we identify with the first and second-order Taylor polynomials, respectively:

$$\mathfrak{J}_x F(y) := \sum_{|\alpha|\le 1} \frac{\partial^\alpha F(x)}{\alpha!}(y-x)^\alpha \quad \text{and} \quad \mathfrak{J}_x^+ F(y) := \sum_{|\alpha|\le 2} \frac{\partial^\alpha F(x)}{\alpha!}(y-x)^\alpha. \tag{2.1}$$

We use $\mathcal{R}_x$, $\mathcal{R}_x^+$ to denote the rings of one-jets, two-jets at x, respectively. The multiplications on $\mathcal{R}_x$ and $\mathcal{R}_x^+$ are defined in the following way:

$$P \odot_x R :\equiv \mathfrak{J}_x(PR) \quad \text{and} \quad P^+ \odot_x^+ R^+ :\equiv \mathfrak{J}_x^+(P^+R^+),$$

for $P, R \in \mathcal{R}_x$ and $P^+, R^+ \in \mathcal{R}_x^+$.

Definition 2.1. Let $S \subset \mathbb{R}^n$ be a finite set. A *Whitney field* on S is an array of polynomials $\vec{P} = (P^x)_{x\in S}$ parameterized by points in S, such that $P^x \in \mathcal{P}$ for each $x \in S$. The collection of Whitney fields will be denoted by $W(S)$. It is a finite dimensional vector space equipped with a norm

$$\|(P^x)_{x\in S}\|_{W(S)} := \max_{x,y\in S, x\ne y, |\alpha|\le 1} \Big\{ |\partial^\alpha P^x(x)|, \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}} \Big\}.$$

It is a vector space of dimension $(\#S)\cdot \dim \mathcal{P}$.

Given a Whitney field $\vec{P} = (P^x)_{x\in S}$, we sometimes use the notation

$$(\vec{P}, x) := P^x.$$

3. Essential polynomials

3.1. Jets with range restriction

The following object captures the effect of the range restriction on jets.

Definition 3.1. Let $\tau > 0$. Let $x \in \mathbb{R}^n$ and $M \ge 0$. We define $\mathcal{K}_\tau(x, M)$ to be the collection of polynomials $P \in \mathcal{P}$ such that

$$|P(x)| \le \min\{M, \tau\}, \quad |\nabla P| \le M, \tag{3.1}$$

$$|\nabla P| \le M^{1/2} \cdot \min\{\sqrt{\tau - P(x)}, \sqrt{\tau + P(x)}\}. \tag{3.2}$$

Lemma 3.1. *Let $P \in \mathcal{P}$ and $x \in \mathbb{R}^n$ be such that $-\tau \le P(x) \le \tau$, and that if $P(x) = \pm\tau$, then $\nabla P = 0$. Let $\mu := \frac{|\nabla P|^2}{(\min\{\sqrt{\tau - P(x)}, \sqrt{\tau + P(x)}\})^2}$, where we use the convention $\frac{0}{0} = 0$. Then*

$$\mathrm{dist}(\{P = P(x)\}, \{P = \pm\tau\}) = \mu^{-1/2} \cdot \min\{\sqrt{\tau - P(x)}, \sqrt{\tau + P(x)}\}. \tag{3.3}$$

In particular, given $P \in \mathcal{K}_\tau(x, M)$ as in Definition 3.1, we have

$$\mathrm{dist}(\{P = P(x)\}, \{P = \pm\tau\}) \ge M^{-1/2} \cdot \min\{\sqrt{\tau - P(x)}, \sqrt{\tau + P(x)}\}. \tag{3.4}$$

Proof. Suppose $P(x) = \pm\tau$ or $\nabla P = 0$. Then (3.3) obviously holds.

Suppose $-\tau < P(x) < \tau$ and $\nabla P \ne 0$. Since P is an affine function, the level sets of P are parallel hyperplanes. We have

$$\begin{aligned} \frac{\tau + P(x)}{\mathrm{dist}(\{P = P(x)\}, \{P = -\tau\})} &= |\nabla(\tau + P)| \\ &= |\nabla(\tau - P)| = \frac{\tau - P(x)}{\mathrm{dist}(\{P = P(x)\}, \{P = \tau\})}. \end{aligned} \tag{3.5}$$

Combining the definition of μ and (3.5), we see that (3.3) holds.

Since $\mu \le M$ for any given $P \in \mathcal{K}_\tau(x, M)$, as in Definition 3.1, we see that (3.4) follows. ∎

Lemma 3.2. *There exists a controlled constant $C(n)$ such that the following holds. Let $x_0 \in \mathbb{R}^n$.*

(A) *Assume that there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le M$. Then we have $\mathcal{J}_{x_0} F \in \mathcal{K}_\tau(x_0, CM)$.*

(B) *There exists a map $\mathcal{T}_*^{x_0} : \bigcup_{M \ge 0} \mathcal{K}_\tau(x_0, M) \to C^2(\mathbb{R}^n, \tau)$ such that the following holds: Suppose $P \in \mathcal{K}_\tau(x_0, M)$. Then $\mathcal{T}_*^{x_0}(P)$ satisfies $\|\mathcal{T}_*^{x_0}(P)\|_{C^2(\mathbb{R}^n)} \le CM$ and $\mathcal{J}_{x_0} \mathcal{T}_*^{x_0}(P) \equiv P$.*

Proof. We write C, c, etc., to denote constants that depend only on n.

Without loss of generality, we may assume that $x_0 = \vec{0} \in \mathbb{R}^n$.

(A) Let $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le M$. Let $P :\equiv \mathcal{J}_0 F$. By Taylor's theorem,

$$-CM|x|^2 \le F(x) - P(x) \le CM|x|^2 \quad \text{for all } x \in \mathbb{R}^n. \tag{3.6}$$

Since $F \in C^2(\mathbb{R}^n, \tau)$ and $\|F\|_{C^2(\mathbb{R}^n)} \le M$, we have

$$|F(x)| \le \min\{M, \tau\} \quad \text{for all } x \in \mathbb{R}^n. \tag{3.7}$$

Combining (3.6) and (3.7), we see that

$$(\tau + P(0)) + \nabla P \cdot x + CM|x|^2 \ge 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{3.8}$$

$$(\tau - P(0)) - \nabla P \cdot x + CM|x|^2 \ge 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{3.9}$$

By restricting to each line and computing the discriminant in both (3.8) and (3.9), we see that $|\nabla P| \le CM^{1/2} \cdot \min\{\sqrt{\tau - P(0)}, \sqrt{\tau + P(0)}\}$. Moreover, since $\|F\|_{C^2(\mathbb{R}^n)} \le M$, we see that (3.1) also holds for P with CM in placed of M. Hence, $P \in \mathcal{K}_\tau(0, CM)$.

(B) Let $P \in \mathcal{K}_\tau(x_0, M)$ be given. We write $\mathcal{T}_*$ instead of $\mathcal{T}_*^0$.

Suppose that $P(0) = \pm\tau$. Then property (3.2) of $\mathcal{K}_\tau$ in Definition 3.1 implies that $P \equiv \pm\tau$. We may simply take $\mathcal{T}_*(P) :\equiv \pm\tau$.

Suppose $-\tau < P(0) < \tau$. We define the following quantities:

- $\mu := |\nabla P|^2 \cdot (\min\{\sqrt{\tau - P(0)}, \sqrt{\tau + P(0)}\})^{-2}$ (see Lemma 3.1). It is clear from (3.2) that $\mu \le M$.
- $\delta := \mu^{-1/2} \cdot \min\{\sqrt{\tau - P(0)}, \sqrt{\tau + P(0)}\}$.

We note that the definitions of μ and δ depend only on the polynomial P; in particular, they are independent of M.

If $\delta \ge 1$, then the size of P does exceed τ within the unit ball. Let θ_1 be a nonnegative C^2 function supported in the unit ball, such that $\theta_1 \equiv 1$ near $x_0 = \vec{0}$ and $|\partial^\alpha \theta_1| \le C$ for $|\alpha| \le 2$. We immediately verify that

$$\mathcal{T}_*(P) := \theta_1 \cdot P$$

satisfies the conclusion of Lemma 3.2 (B).

For the rest of the proof, we assume $\delta < 1$.

Let $c_0 \in (0, 1)$ be a small universal constant. For concreteness, we can take $c_0 = 1/100$. We consider the following two cases:

Case I. Either $\mu \ge c_0\tau$ or $|P(0)| \ge c_0\tau$.

Case II. Both $\mu < c_0\tau$ and $|P(0)| < c_0\tau$.

Proof for Case I. By (3.1) and the observation that $M \ge \max\{\mu, |P(0)|\}$, we see that

$$\tau \le c_0^{-1} M. \tag{3.10}$$

For convenience, we set

$$u := \frac{\nabla P}{|\nabla P|}.$$

Consider the auxiliary functions

$$R^-(x) := P(x) + \frac{\mu}{4}|x \cdot u|^2 \quad \text{and} \quad R^+(x) := P(x) - \frac{\mu}{4}|x \cdot u|^2.$$

Note that the graphs of R^- and R^+ are parabolic cylinders that are constant along each direction orthogonal to u. By construction,

$$\mathfrak{J}_0 R^- \equiv \mathfrak{J}_0 R^+ \equiv P. \tag{3.11}$$

We see from the definition of μ that

$$R^-(x) \ge -\tau, \quad R^+(x) \le \tau \quad \text{for all } x \in \mathbb{R}^n. \tag{3.12}$$

By computing the root along the u-direction, we also have

$$\begin{aligned} R^-(x) &\le \tau \qquad \text{for } 0 \ge x \cdot u \ge -2(\sqrt{2}+1)\delta, \\ R^+(x) &\ge -\tau \quad \text{for } 0 \le x \cdot u \le 2(\sqrt{2}+1)\delta. \end{aligned} \tag{3.13}$$

Consider the following regions in $\mathbb{R}^n$:

$$A_{0,-\tau} = \{x \in \mathbb{R}^n : -\delta \le x \cdot u \le 0\},$$
$$A_{0,\tau} = \{x \in \mathbb{R}^n : 0 \le x \cdot u \le \delta\}$$

(it follows from construction that $-\tau \le P(x) \le P(0)$ on $A_{0,-\tau}$ and $P(0) \le P(x) \le \tau$ on $A_{0,\tau}$),

$$A_{1,-\tau} := \{x \in \mathbb{R}^n : -2(\sqrt{2}+1)\delta \le x \cdot u \le -\delta\},$$
$$A_{1,\tau} := \{x \in \mathbb{R}^n : \delta \le x \cdot u \le 2(\sqrt{2}+1)\delta\},$$
$$A_{2,-\tau} := \{x \in \mathbb{R}^n : x \cdot u \le -2(\sqrt{2}+1)\delta\},$$
$$A_{2,\tau} := \{x \in \mathbb{R}^n : x \cdot u \ge 2(\sqrt{2}+1)\delta\}.$$

We define a C^2 partition of unity $\{\theta^{[0]}, \theta^{[1]}_{-\tau}, \theta^{[1]}_{\tau}, \theta^{[2]}_{-\tau}, \theta^{[2]}_{\tau}\}$ with the following properties:

($\theta 1$) $\theta^{[0]} + \theta^{[1]}_{-\tau} + \theta^{[1]}_{\tau} + \theta^{[2]}_{-\tau} + \theta^{[2]}_{\tau} \equiv 1$ on $\mathbb{R}^n$.

($\theta 2$) $0 \le \theta^{[0]} \le 1$, $\operatorname{supp}(\theta^{[0]}) \subset A_{0,-\tau} \cup A_{0,\tau}$, $\theta^{[0]} \equiv 1$ near $A_{0,-\tau} \cap A_{0,\tau}$, and $|\partial^\alpha \theta^{[0]}| \le C\delta^{-|\alpha|}$ for $|\alpha| \le 2$.

($\theta 3$) $0 \le \theta^{[1]}_{-\tau} \le 1$, $\operatorname{supp}(\theta^{[1]}_{-\tau}) \subset A_{0,-\tau} \cup A_{1,-\tau}$, $\theta^{[1]}_{-\tau} \equiv 1$ on $A_{0,-\tau} \cap A_{1,-\tau}$, and $|\partial^\alpha \theta^{[1]}_{-\tau}| \le C\delta^{-|\alpha|}$ for $|\alpha| \le 2$.

($\theta 4$) $0 \le \theta^{[1]}_{\tau} \le 1$, $\operatorname{supp}(\theta^{[1]}_{\tau}) \subset A_{0,\tau} \cup A_{1,\tau}$, $\theta^{[1]}_{\tau} \equiv 1$ on $A_{0,\tau} \cap A_{1,\tau}$, and $|\partial^\alpha \theta^{[1]}_{\tau}| \le C\delta^{-|\alpha|}$ for $|\alpha| \le 2$.

($\theta 5$) $0 \le \theta^{[2]}_{-\tau} \le 1$, $\operatorname{supp}(\theta^{[2]}_{-\tau}) \subset A_{1,-\tau} \cup A_{2,-\tau}$, $\theta^{[2]}_{-\tau} \equiv 1$ on $A_{2,-\tau}$, and $|\partial^\alpha \theta^{[2]}_{-\tau}| \le C\delta^{-|\alpha|}$ for $|\alpha| \le 2$.

($\theta 6$) $0 \le \theta^{[2]}_{\tau} \le 1$, $\operatorname{supp}(\theta^{[2]}_{\tau}) \subset A_{1,\tau} \cup A_{2,\tau}$, $\theta^{[2]}_{\tau} \equiv 1$ on $A_{2,\tau}$, and $|\partial^\alpha \theta^{[2]}_{\tau}| \le C\delta^{-|\alpha|}$ for $|\alpha| \le 2$.

We define $\mathcal{T}_*(P) \in C^2(\mathbb{R}^n)$ by

$$\mathcal{T}_*(P) := \theta^{[0]} P + \theta^{[1]}_{-\tau} R^- + R^-(-2\delta u)\theta^{[2]}_{-\tau} + \theta^{[1]}_{\tau} R^+ + R^+(2\delta u)\theta^{[2]}_{\tau}.$$

We see from ($\theta 1$)–($\theta 6$), (3.12), and (3.13) that $-\tau \le \mathcal{T}_*(P)(x) \le \tau$ for all $x \in \mathbb{R}^n$. Hence, $\mathcal{T}_*(P) \in C^2(\mathbb{R}^n, \tau)$. Moreover, thanks to (3.10), we have

$$|\mathcal{T}_*(P)(x)| \le CM \quad \text{for all } x \in \mathbb{R}^n.$$

We now estimate the derivatives of $\mathcal{T}_*(P)$. Since all the θ sum to one everywhere, we have, for $\alpha \ne 0$,

$$\partial^\alpha \mathcal{T}_*(P)(x) = \begin{cases} \sum_{\beta \le \alpha} C_{\alpha,\beta} \cdot \partial^\beta \theta^{[0]}(x) \cdot \partial^{\alpha-\beta}(P - R^-)(x) & \text{for } x \in A_{0,-\tau}, \\ \sum_{\beta \le \alpha} C_{\alpha,\beta} \cdot \partial^\beta \theta^{[0]}(x) \cdot \partial^{\alpha-\beta}(P - R^+)(x) & \text{for } x \in A_{0,\tau}, \\ \sum_{\beta \le \alpha} C_{\alpha,\beta} \cdot \partial^\beta \theta^{[1]}_{-\tau}(x) \cdot \partial^{\alpha-\beta}(R^-(x) + \tau) & \text{for } x \in A_{1,-\tau}, \\ \sum_{\beta \le \alpha} C_{\alpha,\beta} \cdot \partial^\beta \theta^{[1]}_{\tau}(x) \cdot \partial^{\alpha-\beta}(R^+(x) - \tau) & \text{for } x \in A_{1,-\tau}, \\ 0 & \text{for } x \in A_{2,-\tau} \cup A_{2,\tau}. \end{cases}$$

We analyze the first and third sums. The analysis for the second is similar to the first, and the fourth to the third.

From the definitions of P, δ, R^-, $A_{0,-\tau}$, and the fact that $\mu \leq M$, we see that

$$|\partial^\alpha(P - R^-)(x)| \leq C\mu\delta^{2-|\alpha|} \leq CM\delta^{2-|\alpha|} \quad \text{for } x \in A_{0,-\tau}, 0 < |\alpha| \leq 2. \tag{3.14}$$

Combining (3.14) and ($\theta 2$), we see that $|\partial^\alpha \mathcal{T}_* P(x)| \leq CM$ for $x \in A_{0,-\tau}$, $0 < |\alpha| \leq 2$.

From the definitions of δ, R^-, and $A_{1,-\tau}$, we see that

$$|\partial^\alpha(R^-(x) - R^-(-2\delta u))| \leq C\mu\delta^{2-|\alpha|} \leq C'M\delta^{2-|\alpha|} \quad \text{for } x \in A_{1,-\tau}, 0 < |\alpha| \leq 2. \tag{3.15}$$

Combining (3.15) and ($\theta 4$), we see that $|\partial^\alpha \mathcal{T}_* P(x)| \leq CM$ for $x \in A_{1,-\tau}$, $0 < |\alpha| \leq 2$.

Thus, we have shown that $\|\mathcal{T}_* P\|_{C^2(\mathbb{R}^n)} \leq CM$. This concludes our treatment of Case I.

Proof for Case II. Since both $\mu < c_0\tau$ and $|P(0)| < c_0\tau$, there exists a universal constant c_1 such that

$$\delta = \mu^{-1/2} \cdot \min\{\sqrt{\tau - P(0)}, \sqrt{\tau + P(0)}\} \geq c_1.$$

We fix such c_1. Thanks to Lemma 3.1, we have

$$\operatorname{dist}(\{P = P(0)\}, \{P = \pm\tau\}) \geq c_1. \tag{3.16}$$

As before, we set $u := \nabla P/|\nabla P|$. Note that u is orthogonal to the level sets of P. Let θ be a nonnegative C^2 function such that $\theta \equiv 1$ near $\{P = P(0)\}$, $\operatorname{supp}(\theta) \subset \{x \in \mathbb{R}^n : |x \cdot u| \leq c_1/2\}$, and $|\partial^\alpha \theta| \leq C$. We define

$$\mathcal{T}_* P(x) := \theta(x) \cdot P(x).$$

Thanks to (3.16) and the support of θ, we have $\mathcal{T}_* P(x) \in [-\tau, \tau]$ for all $x \in \mathbb{R}^n$, so $\mathcal{T}_* P \in C^2(\mathbb{R}^n, \tau)$.

By the fundamental theorem of calculus, we see that

$$|P(x)| \leq |P(0)| + \frac{c_1}{2}|\nabla P| \leq CM \quad \text{for all } x \in \operatorname{supp}(\theta). \tag{3.17}$$

From (3.1) and (3.17), we have, for $0 < |\alpha| \leq 2$,

$$|\partial^\alpha \mathcal{T}_* P(x)| \leq \sum_{0 < \beta \leq \alpha} |C_{\alpha,\beta} \cdot \partial^\beta \theta(x) \cdot \partial^{\alpha-\beta} P(x)| \leq CM \quad \text{for all } x \in \operatorname{supp}(\theta). \tag{3.18}$$

Since $\mathcal{T}_* P$ vanishes outside $\operatorname{supp}(\theta)$, we can conclude from (3.17) and (3.18) that $\|\mathcal{T}_* P\|_{C^2(\mathbb{R}^n)} \leq CM$. This concludes the treatment of the second case and the proof of the lemma. ∎

3.2. Whitney's extension theorem for finite sets

Recall the notion of a Whitney field in Definition 2.1. Let $\tau > 0$. Recall $\mathcal{K}_\tau$ in Definition 3.1. We use $W(S, \tau)$ to denote the sub-collection of Whitney fields $(P^x)_{x \in S}$ such that for any $x \in S$, $P^x \in \mathcal{K}_\tau(x, M)$ for some $M \geq 0$. We define

$$\|(P^x)_{x\in S}\|_{W(S,\tau)} := \|(P^x)_{x\in S}\|_{W(S)} + \inf\{M \geq 0 : P^x \in \mathcal{K}_\tau(x, M) \text{ for all } x \in S\}. \tag{3.19}$$

Recall the following classical result.

Theorem 3.1. *There exists a controlled constant $C(n)$ such that the following holds. Let $S \subset \mathbb{R}^n$ be a finite set.*

(A) (Taylor's theorem). *Let $F \in C^2(\mathbb{R}^n)$. Then $\|(\mathcal{J}_x F)_{x\in S}\|_{W(S)} \le C\|F\|_{C^2(\mathbb{R}^n)}$.*

(B) (Whitney's extension theorem). *There exists a linear map $\mathcal{T}_w \colon W(S) \to C^2(\mathbb{R}^n)$ such that given any $\vec{P} = (P^x)_{x\in S} \in W(S)$, we have $\|\mathcal{T}_w(\vec{P})\|_{C^2(\mathbb{R}^n)} \le C\|\vec{P}\|_{W(S)}$ and $\mathcal{J}_x \mathcal{T}_w(\vec{P}) \equiv P^x$ for each $x \in S$.*

A proof of Theorem 3.1 can be found in standard textbooks, see, for instance, [43].

Theorem 3.2. *There exists a controlled constant $C(n)$ such that the following holds. Let $S \subset \mathbb{R}^n$ be a finite set.*

(A) (Taylor's theorem for $C^2(\mathbb{R}^n, \tau)$). *Let $F \in C^2(\mathbb{R}^n, \tau)$. Then $\|(\mathcal{J}_x F)_{x\in S}\|_{W(S,\tau)} \le C\|F\|_{C^2(\mathbb{R}^n)}$.*

(B) (Whitney's extension theorem for $C^2(\mathbb{R}^n, \tau)$). *There exists a map $\mathcal{T}_{w,\tau} :$ $W(S,\tau) \to C^2(\mathbb{R}^n, \tau)$ such that given any $\vec{P} = (P^x)_{x\in S} \in W(S,\tau)$, we have $\|\mathcal{T}_{w,\tau}(\vec{P})\|_{C^2(\mathbb{R}^n)} \le C\|\vec{P}\|_{W(S,\tau)}$ and $\mathcal{J}_x \mathcal{T}_{w,\tau}(\vec{P}) \equiv P^x$ for each $x \in S$.*

Theorem 3.2 (A) is an immediate consequence of Lemma 3.2 (A) and the definition of $\|\cdot\|_{W(S,\tau)}$ in (3.19). To prove Theorem 3.2 (B), we may proceed as in the proof of Theorem 3.1, but instead of pasting together P^x using a Whitney partition of unity, we paste together $\mathcal{T}_*^x(P^x)$, with $\mathcal{T}_*^x$ as in Lemma 3.2 (B). Note that the operator $\mathcal{T}_*^x$ in Lemma 3.2 (B) is nonlinear, resulting in the nonlinearity of $\mathcal{T}_{w,\tau}$.

3.3. Norms on small subsets

Throughout this section, we fix a finite set

$$S \subset \mathbb{R}^n, \quad \#S \le k_0, \quad \text{where } k_0 = k_0(n) \text{ is a controlled constant.}$$

We define a norm $\mathcal{L}$ on $W(S)$ by

$$\begin{aligned} &\mathcal{L} \colon W(S) \to [0,\infty), \\ &(P^x)_{x\in S} \mapsto \sum_{x\in S, |\alpha|\le 1} |\partial^\alpha P(x)| + \sum_{x,y\in S, x\ne y, |\alpha|\le 1} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}}. \end{aligned} \tag{3.20}$$

Lemma 3.3. *There exists a controlled constant C such that given any $\vec{P} \in W(S)$, we have*

$$C^{-1}\mathcal{L}(\vec{P}) \le \|\vec{P}\|_{W(S)} \le C\mathcal{L}(\vec{P}). \tag{3.21}$$

Proof. Recall Definition 2.1. Recall the assumption that $\#S \le k_0$ where k_0 is a controlled quantity. For a given $\vec{P} \in W(S)$, it is clear that $\|\vec{P}\|_{W(S)} \le \mathcal{L}(\vec{P})$. For the reverse inequality, we have

$$\begin{aligned} \mathcal{L}(\vec{P}) &\le \left[k_0(n+1) + k_0(k_0-1)(n+1)\right] \cdot \max_{\substack{x,y\in S, x\ne y \\ |\alpha|\le 1}} \left\{ |\partial^\alpha P^x(x)|, \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}} \right\} \\ &\le C(n)\|\vec{P}\|_{W(S)}. \end{aligned}$$

This proves (3.21). ∎

For $\tau > 0$, we define a function that measures the effect of the range restriction on the optimal Whitney extension.

$$\mathcal{M}_\tau\colon W(S,\tau) \to [0,\infty), \quad (P^x)_{x\in S} \mapsto \sum_{x\in S, |\alpha|=1} \frac{|\partial^\alpha P^x|^2}{\tau - P(x)} + \frac{|\partial^\alpha P^x|^2}{\tau + P(x)}. \tag{3.22}$$

In (3.22), we adopt the convention $\frac{0}{0} = 0$.

Lemma 3.4. *There exists a controlled constant $C(n, k_0)$ such that given any $\vec{P} \in W(S,\tau)$, we have*

$$C^{-1}(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}) \le \|\vec{P}\|_{W(S,\tau)} \le C(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}). \tag{3.23}$$

Proof. We adopt the notation $A \approx B$ if there exists a controlled constant $C(n)$ such that $C^{-1}A \le B \le CA$.

Comparing (3.2) and (3.22), we see that

$$\mathcal{M}_\tau(\vec{P}) \approx \inf\{M \ge 0 : P^x \in \mathcal{K}_\tau(x,M) \text{ for all } x \in S\}.$$

Thanks to Lemma 3.3, we have

$$\mathcal{L}(\vec{P}) \approx \|\vec{P}\|_{W(S,\tau)}.$$

Adding the two equivalences above, we see that equivalence in (3.23) follows. ■

We now explain how to compute the order of magnitude of $\|f\|_{C^2(S,\tau)}$. To be more specific, using at most a bounded number of operations, we compute a Whitney field $(P^x)_{x\in S} \in W(S,\tau)$ such that $P^x(x) = f(x)$ for each $x \in S$ and $\|(P^x)_{x\in S}\|_{W(S,\tau)} \approx \|f\|_{C^2(S,\tau)}$.

Consider the affine subspace $\mathbb{A}_{\tau,f} \subset W(S)$ defined by

$$\begin{aligned}\mathbb{A}_{\tau,f} := \big\{\vec{P} = (P^x)_{x\in S} \in W(S) : \; & P^x(x) = f(x) \text{ for } x \in S,\\ & \text{if } f(x) = \tau, \text{ then } P^x \equiv \tau,\\ & \text{if } f(x) = -\tau, \text{ then } P^x \equiv -\tau\big\}.\end{aligned}$$

Equivalently,

$$\mathbb{A}_{\tau,f} = \big\{\vec{P} = (P^x)_{x\in S} \in W(S,\tau) : P^x(x) = f(x) \text{ for } x \in S\big\}.$$

Note that $\mathbb{A}_{\tau,f}$ has dimension $n \cdot \#(S \setminus f^{-1}(\{\pm\tau\}))$, as we can thin of $\mathbb{A}_{\tau,f}$ as a collection of n-dimensional gradients, one for each point in S for which f is not $\pm\tau$.

Let $\mathcal{L}$ and $\mathcal{M}_\tau$ be as in (3.20) and (3.22). Thanks to Whitney's extension theorem (Theorem 3.2 (B)) and Lemma 3.4, we have

$$\|f\|_{C^2(E,\tau)} \approx \inf\{(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}) : \vec{P} \in \mathbb{A}_{\tau,f}\}. \tag{3.24}$$

Let $d := \dim W(S) = \#S \cdot \dim \mathcal{P} \le k_0(n+1)$. We identify $W(S) \cong \mathbb{R}^d$ via

$$(P^x)_{x\in S} \mapsto (P^x(x), \partial_1 P^x, \ldots, \partial_n P^x)_{x\in S}.$$

We define the ℓ_1 and ℓ_2 norms, respectively, on $\mathbb{R}^d$ by

$$\|v\|_{\ell_1} := \sum_{i=1}^{d} |v_i| \quad \text{and} \quad \|v\|_{\ell_2} := \left(\sum_{i=1}^{d} |v_i|^2\right)^{1/2}, \quad v = (v_1, \ldots, v_d) \in \mathbb{R}^d.$$

Let $L_w : W(S) \to \mathbb{R}^d$ be a linear isomorphism that maps $\vec{P} \in W(S)$ to a vector in $\mathbb{R}^d$ with components

$$\frac{\partial^\alpha (P^y - P^z)(y)}{|y-z|^{2-|\alpha|}}, \quad \partial^\alpha P^{x_S}(x_S), \quad |\alpha| \leq 1,$$

for suitable $x_S, y, z \in S$ in certain order, such that

$$\|L_w(\vec{P})\|_{\ell^1(\mathbb{R}^d)} \approx \mathcal{L}(\vec{P}) \quad \text{for } \vec{P} \in W(S). \tag{3.25}$$

One possible construction of such an L_w is based on the technique of "clustering" introduced in [1]. See Remark 3.3 of [1]. We can compute L_w from S using at most C operations, since $\#S$ is controlled.

For the rest of this section, we identify $W(S)$ with $\mathbb{R}^d$ via L_w.

Let $V_{\tau,f} \subset W(S)$ be the subset defined by

$$V_{\tau,f} := \big\{(P^x)_{x\in S} : P^x(x) = 0 \text{ for } x \in S \setminus f^{-1}(\{-\tau,\tau\}),\\ P^x \equiv 0 \text{ for } x \in f^{-1}(\{-\tau,\tau\})\big\}.$$

Let $\Pi_{\tau,f} = (\Pi^x_{\tau,f})_{x\in S} : W(S) \to V_{\tau,f}$ be the orthogonal projection. Let $\vec{P}_f$ denote the vector $(f(x), 0, 0)_{x\in S}$. It is clear that $\mathbb{A}_{\tau,f} = \vec{P}_f + V_{\tau,f}$.

Let $L_{\tau,f} = (L^x_{\tau,f})_{x\in S} : W(S) \to W(S)$ be a linear endomorphism defined by

$$L^x_{\tau,f}(P^x) = \begin{cases} P^x \cdot \left(\min\{\sqrt{\tau - f(x)}, \sqrt{\tau + f(x)}\}\right)^{-1/2} & \text{for } x \in S \setminus f^{-1}(\{-\tau,\tau\}), \\ 0 & \text{for } x \in f^{-1}(\{-\tau,\tau\}), \end{cases}$$

for $(P^x)_{x\in S} \in W(S)$.

We see from the definition of $\mathcal{M}_\tau$ that

$$\mathcal{M}_\tau(\vec{P}) \approx \|L_{\tau,f}\Pi_{\tau,f}(\vec{P})\|^2_{\ell^2(\mathbb{R}^d)} \quad \text{for } \vec{P} \in \mathbb{A}_{\tau,f}. \tag{3.26}$$

Combining Lemma 3.4, (3.25), and (3.26), we have

$$\|\vec{P}\|_{W(S,\tau)} \approx \|L_{\tau,f}\Pi_{\tau,f}(\vec{P})\|^2_{\ell^2(\mathbb{R}^d)} + \|L_w(\vec{P})\|_{\ell^1(\mathbb{R}^d)} \quad \text{for } \vec{P} \in \vec{P}_f + V_{\tau,f}. \tag{3.27}$$

Setting $\beta := L_w(\vec{P})$ and $X := (L_{\tau,f}\Pi_{\tau,f})^\dagger (L_{\tau,f}\Pi_{\tau,f})$, we see from (3.24) and (3.27) that computing the order of magnitude of $\|f\|_{C^2(S,\tau)}$ amounts to solving the following minimization problem:

$$\text{Minimize } \langle \beta, X\beta\rangle + \|\beta\|_{\ell^1(\mathbb{R}^d)} \quad \text{subject to } L_w^{-1}\beta \in \vec{P}_f + V_{\tau,f}. \tag{3.28}$$

In particular, an optimal (feasible) solution to (3.28) is a Whitney field $\vec{P}_* = (P_*^x)_{x\in S} \in W(S,\tau)$ with $P_*^x(x) = f(x)$ for $x \in S$ and $\|\vec{P}_*\|_{W(S,\tau)} \approx \|f\|_{C^2(S,\tau)}$.

Finally, we note that (3.28) is a convex quadratic programming problem with affine constraints. We can find the exact solution to (3.28) by solving for its Karush–Kuhn–Tucker conditions, which consist of a bounded system of linear equations and inequalities, see [4]. We can solve such a system, for instance via the simplex method or elimination, using at most $C(n)$ operations, since the system size is controlled. We refer the readers to the appendix of [30] for an elementary discussion, and [4] for a detailed treatment of convex programming.

3.4. Homogeneous convex sets

For $x \in \mathbb{R}^n$, $S \subset E$ and $k \geq 0$, we define

$$\begin{aligned}\sigma(x,S) &:= \big\{P \in \mathcal{P} : \text{There exists } \varphi^S \in C^2(\mathbb{R}^n), \text{ with } \|\varphi^S\|_{C^2(\mathbb{R}^n)} \leq 1,\\ &\qquad\qquad \text{such that } \varphi^S = 0 \text{ on } S \text{ and } \mathcal{J}_x\varphi^S \equiv P\big\},\\ \sigma^\sharp(x,k) &:= \bigcap_{S\subset E,\, \#S\leq k} \sigma(x,S).\end{aligned} \tag{3.29}$$

Theorem 3.3 (Finiteness principle). *There exist controlled constants $k^\sharp_{n,\text{old}}$ and $C(n)$ for which the following holds. Let $E \subset \mathbb{R}^n$ be a finite set.*

(A) *Let $f\colon E \to \mathbb{R}$. Suppose that, for every $S \subset E$ with $\#S \leq k^\sharp_{n,\text{old}}$, there exists $F^S \in C^2(\mathbb{R}^n)$ with $\|F^S\|_{C^2(\mathbb{R}^n)} \leq M$ and $F^S = f$ on S. Then there exists $F \in C^2(\mathbb{R}^n)$ with $\|F\|_{C^2(\mathbb{R}^n)} \leq CM$ and $F = f$ on E.*

(B) *Let σ and $\sigma^\sharp$ be as in (3.29). Then for all $k \geq k^\sharp_{n,\text{old}}$,*

$$C^{-1}\sigma^\sharp(x,k) \subset \sigma(x,E) \subset C\cdot\sigma^\sharp(x,k) \quad \textit{for all } x \in \mathbb{R}^n.$$

For each $n \in \mathbb{N}$, we fix a choice of $k^\sharp_{n,\text{old}}$. We further assume that $k^\sharp_{n+1,\text{old}} > k^\sharp_{n,\text{old}}$ for all n.

See [9,20,21,24] for a proof of Theorem 3.3. For the special case $n = 2$, see also [28].

3.5. Main convex sets

For the rest of the section, we assume we are given a finite set $E \subset \mathbb{R}^n$.

For $x \in \mathbb{R}^n$, $S \subset E$, $f\colon E \to [-\tau,\tau]$, and $M \geq 0$, we define

$$\begin{aligned}\Gamma_\tau(x,S,f,M) := \big\{P \in \mathcal{P} : &\text{ There exists } F^S \in C^2(\mathbb{R}^n,\tau),\\ &\text{with } \|F^S\|_{C^2(\mathbb{R}^n)} \leq M,\\ &\text{such that } F^S = f \text{ on } S \text{ and } \mathcal{J}_x F^S \equiv P.\big\}.\end{aligned} \tag{3.30}$$

For $x \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $f\colon E \to [-\tau,\tau]$, and $M \geq 0$, we define

$$\Gamma^\sharp_\tau(x,k,f,M) := \bigcap_{S\subset E,\, \#S\leq k} \Gamma_\tau(x,S,f,M). \tag{3.31}$$

Note that Γ_τ and $\Gamma_\tau^\sharp$ are (possibly empty) bounded convex subsets of $\mathcal{P}$. With respect to set inclusion, Γ_τ is decreasing in S and increasing in M; $\Gamma_\tau^\sharp$ is decreasing in k and increasing in M.

It follows from Lemma 3.2 that

$$\mathcal{K}_\tau(x, C^{-1}M) \subset \Gamma_\tau(x, \varnothing, f, M) \subset \mathcal{K}_\tau(x, CM) \tag{3.32}$$

for some controlled constant $C(n)$.

Theorem 3.4 (Helly's theorem). *Let $D \in \mathbb{N}_0$. Let $\mathcal{F}$ be a finite family of convex subsets of $\mathbb{R}^D$. If every $D+1$ members of $\mathcal{F}$ have nonempty intersection, then all members of $\mathcal{F}$ have nonempty intersection.*

Lemma 3.5. *Let $x, x' \in \mathbb{R}^n$. Let $k, k' > 0$ be such that $k \geq (n+2)k'$. Then given $P \in \Gamma_\tau^\sharp(x, k, f, M)$, there exists $P' \in \Gamma_\tau^\sharp(x', k', f, M)$ satisfying*

$$|\partial^\alpha(P - P')(x)|, |\partial^\alpha(P - P')(x')| \leq C(n) \cdot M|x - x'|^{2-|\alpha|} \quad \textit{for } |\alpha| \leq 2. \tag{3.33}$$

Proof. For $S \subset E$, we define

$$K(S) := \left\{\mathcal{J}_{x'} F^S : F^S \in C^2(\mathbb{R}^n, \tau) \text{ with } \|F^S\|_{C^2(\mathbb{R}^n)} \leq M,\ F^S|_S = f \text{ and } \mathcal{J}_x F^S \equiv P\right\}.$$

Note that $K(S)$ is convex, and that $K(S') \subset K(S)$ whenever $S \subset S'$. It also follows from the definition of $\Gamma_\tau^\sharp$ in (3.31) that if $\#S \leq k$, then $K(S) \neq \varnothing$.

Let $S_1, \dots, S_{n+2} \subset E$ be given with $\#S_i \leq k'$ for each i. Setting $S := \bigcup_{i=1}^{n+2} S_i$, we see that $\#S \leq (n+2)k' \leq k$, so that $K(S) \neq \varnothing$. Therefore,

$$\bigcap_{i=1}^{n+2} K(S_i) \supset K(S) \neq \varnothing.$$

Since $S_1, \dots, S_{n+2} \subset E$ are arbitrary, Helly's theorem (Theorem 3.4) applied to the convex sets $K(S_i) \subset \mathcal{P}$ (with $\dim \mathcal{P} = n+1$) yields

$$\Gamma_\tau^\sharp(x', k', f, M) = \bigcap_{S' \subset E, \#S' \leq k'} K(S') \neq \varnothing.$$

Pick $P' \in \bigcap_{S' \subset E, \#S' \leq k'} K(S')$. Estimate (3.33) then follows from Taylor's theorem. ∎

Lemma 3.6. *Let $Q \subset \mathbb{R}^n$ be a cube, let $k \geq 2$, and let $f : E \to [-\tau, \tau]$. Suppose that $\Gamma_\tau^\sharp(x, k, f, M) \neq \varnothing$ for each $x \in E \cap 5Q$. Given any $A_{\text{centric}} > 0$, there exists $A_{\text{polar}} = C(n) \cdot (A_{\text{centric}}^{1/2} + 1)^2 > 0$ such that the following hold:*

(A) *Either $\tau + f(x) \geq A_{\text{centric}} M\delta_Q^2$ for all $x \in E \cap 5Q$, or $\tau + f(x) \leq A_{\text{polar}} M\delta_Q^2$ for all $x \in E \cap 5Q$.*

(B) *Either $\tau - f(x) \geq A_{\text{centric}} M\delta_Q^2$ for all $x \in E \cap 5Q$, or $\tau - f(x) \leq A_{\text{polar}} M\delta_Q^2$ for all $x \in E \cap 5Q$.*

Proof. We prove (A) here. The proof of (B) is similar.

Fix $A_{\text{centric}} > 0$. If $\tau + f(x) \geq A_{\text{centric}} M\delta_Q^2$ for all $x \in E \cap 5Q$, then there is nothing to prove.

Suppose not, namely, there exists $x_0 \in E \cap 5Q$ such that

$$\tau + f(x_0) < A_{\text{centric}} M\delta_Q^2. \tag{3.34}$$

If $\#(E \cap 5Q) = 1$, then there is nothing to prove. We assume that $\#(E \cap 5Q) \geq 2$.

Let $P_0 \in \Gamma_\tau^\sharp(x_0, k, f, M)$. Let $S \subset E \cap 5Q$ with $\#(S) \leq 2$ and $x_0 \in S$. By (3.31), there exists a function $F^S \in C^2(\mathbb{R}^n, \tau)$ with $\|F^S\|_{C^2(\mathbb{R}^n)} \leq M$, $F^S|_S = f$, and $\mathfrak{J}_{x_0} F^S \equiv P_0$. In particular, $P_0(x_0) = f(x_0)$. By (3.34), we have

$$\min\{\sqrt{\tau - P_0(x_0)}, \sqrt{\tau + P_0(x_0)}\} < A_{\text{centric}}^{1/2} M^{1/2} \delta_Q. \tag{3.35}$$

Recall $\mathcal{K}_\tau$ in Definition 3.1. Since $P_0 \in \Gamma_\tau^\sharp(x_0, k, f, M)$, we have $P_0 \in \mathcal{K}_\tau(x_0, CM)$ by Lemma 3.2. Therefore, by (3.2) and (3.35),

$$|\nabla F^S(x_0)| = |\nabla P_0| \leq C A_{\text{centric}}^{1/2} M\delta_Q.$$

Since $\|F^S\|_{C^2(\mathbb{R}^n)} \leq M$, Taylor's theorem implies

$$|\nabla F^S(x)| \leq C(1 + A_{\text{centric}}^{1/2}) M\delta_Q \quad \text{for } x \in 5Q.$$

From (3.34), we see that $\tau + F^S(x_0) < A_{\text{centric}} M\delta_Q^2$. Writing

$$F^S(x) - F^S(x_0) = \int_{\text{seg}(x_0 \to x)} \nabla F^S,$$

we see that

$$\begin{aligned} \tau + F^S(x) \leq |\tau + F^S(x)| &\leq \tau + F^S(x_0) + \int_{\text{seg}(x_0 \to x)} |\nabla F^S| \\ &\leq C(A_{\text{centric}} + A_{\text{centric}}^{1/2} + 1) M\delta_Q^2 \leq C'(A_{\text{centric}}^{1/2} + 1)^2 M\delta_Q^2. \end{aligned}$$

In particular, for each $x \in S$,

$$\tau + f(x) = \tau + F^S(x) \leq C(A_{\text{centric}}^{1/2} + 1)^2 M\delta_Q^2.$$

Since S is any arbitrary subset of $E \cap 5Q$ containing two points, conclusion (A) follows. ∎

For $x \in \mathbb{R}^n$ and $\delta > 0$, we define

$$\mathcal{B}(x, \delta) := \{P \in \mathcal{P} : |\partial^\alpha P(x)| \leq \delta^{2-|\alpha|} \text{ for } |\alpha| \leq 1\}. \tag{3.36}$$

The significance of $\mathcal{B}$ is that given $F \in C^2(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, Taylor's theorem implies

$$\mathfrak{J}_x F - \mathfrak{J}_y F \in C(n)\|F\|_{C^2(\mathbb{R}^n)} \cdot \mathcal{B}(x, |x - y|).$$

Lemma 3.7. *Let $k \geq 2$. Let $Q \subset \mathbb{R}^n$ be a dyadic cube ($\delta_Q \leq 1$) with $E \cap 5Q \neq \varnothing$. Suppose $x_Q \in Q$ satisfy $\operatorname{dist}(x_Q, E) \geq c_0\delta_Q$ for some controlled constant $c_0(n)$. Let $f: E \to [-\tau, \tau]$ be given. Suppose $\Gamma^\sharp_\tau(x_Q, k, f, M) \neq \varnothing$. The following are true:*

(A) *There exists a number A_{perturb} exceeding a large controlled constant such that the following holds. Suppose $\min\{\tau - f(x), \tau + f(x)\} \geq A_{\text{perturb}} M\delta_Q^2$ for each $x \in E \cap 5Q$. Then*

$$\Gamma^\sharp_\tau(x_Q, k, f, M) + M \cdot \mathcal{B}(x_Q, \delta_Q) \subset \Gamma^\sharp_\tau(x_Q, k, f, AM), \quad A = A(n, A_{\text{perturb}}).$$

(B) *Suppose $\tau + f(x) \leq A_{\text{flat}} M\delta_Q^2$ for some $x \in E \cap 5Q$ and $A_{\text{flat}} \geq 0$. Then*

$$-\tau \in \Gamma^\sharp_\tau(x_Q, k, f, AM), \quad A = A(n, A_{\text{flat}}).$$

Here, $-\tau$ is the constant polynomial. Similarly, suppose $\tau - f(x) \leq A_{\text{flat}} M\delta_Q^2$ for some $x \in E \cap 5Q$ and $A_{\text{flat}} \geq 0$. Then

$$\tau \in \Gamma^\sharp_\tau(x_Q, k, f, AM), \quad A = A(n, A_{\text{flat}}).$$

Here, τ is the constant polynomial.

Proof. We write C, C', etc., to denote controlled constants depending only on n. Recall $\mathcal{K}_\tau$ in Definition 3.1.

Proof of (A). We claim that under the hypothesis of (A), given any $P \in \Gamma^\sharp_\tau(x_Q, k, f, M)$, we have

$$\min\{\tau - P(x_Q), \tau + P(x_Q)\} \geq c(n) \cdot (A_{\text{perturb}}^{1/2} - 1)^2 \cdot M\delta_Q^2. \tag{3.37}$$

To see this, we fix $P \in \Gamma^\sharp_\tau(x_Q, k, f, M)$ and $x \in E \cap 5Q$. By the definition of $\Gamma^\sharp_\tau$, there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \leq M$, $\mathfrak{J}_{x_Q} F \equiv P$, and $F(x) = f(x)$. In particular,

$$\min\{\tau - F(x), \tau + F(x)\} \geq A_{\text{perturb}} M\delta_Q^2. \tag{3.38}$$

Suppose toward a contradiction that

$$\tau - |F(x_Q)| \leq A_0 M\delta_Q^2 \tag{3.39}$$

for some to-be-determined A_0 depending only on n and A_{perturb}. Since we have $\mathfrak{J}_{x_Q} F \in \mathcal{K}_\tau(x_Q, CM)$, (3.2) and (3.39) imply

$$|\nabla F(x_Q)| \leq CA_0^{1/2} M\delta_Q. \tag{3.40}$$

Applying Taylor's theorem to (3.39) and (3.40), we see that

$$\tau - |F(x)| \leq C(A_0 + A_0^{1/2} + 1)M\delta_Q^2 \leq C'(A_0^{1/2} + 1)^2 M\delta_Q^2 \quad \text{for } x \in 5Q. \tag{3.41}$$

If we pick A_0 to be so small that $A_0^{1/2} < A_{\text{perturb}}^{1/2}/C' - 1$, with C' as in (3.41), we see that (3.41) will contradict (3.38). Hence, (3.37) holds.

Now we fix a jet $P \in \Gamma_\tau^\sharp(x_Q, k, f, M)$. We know that P satisfies (3.37). Let $\tilde{P} \in M \cdot \mathcal{B}(x_Q, \delta_Q)$. By definition (3.36), we have $|\partial^\alpha \tilde{P}(x_Q)| \leq M\delta_Q^{2-|\alpha|}$ for $|\alpha| \leq 2$. We want to show that $P + \tilde{P} \in \Gamma_\tau^\sharp(x_Q, k, f, CM)$.

By the definition of $\Gamma_\tau^\sharp$, we want to show that given $S \subset E$ with $\#S \leq k$, there exists $F^S \in C^2(\mathbb{R}^n, \tau)$ with $\|F^S\|_{C^2(\mathbb{R}^n)} \leq CM$, $F^S(x) = f(x)$ for $x \in S$, and $\mathfrak{J}_{x_Q} F^S \equiv P + \tilde{P}$.

Fix $S \subset E$ with $\#S \leq k$. We define $S^+ := S \cup \{x_Q\}$. Since $P \in \Gamma_\tau^\sharp(x_Q, k, f, M)$, there exists $F^S \in C^2(\mathbb{R}^n, \tau)$ with

$$\|F^S\|_{C^2(\mathbb{R}^n)} \leq M, \quad F^S(x) = f(x) \text{ for } x \in S, \quad \text{and} \quad \mathfrak{J}_{x_Q} F^S \equiv P. \tag{3.42}$$

Consider the Whitney field $\vec{P} \in W(S^+)$ defined by

- $P^x \equiv \mathfrak{J}_x F^S$ for $x \in S$, and
- $P^{x_Q} \equiv \mathfrak{J}_{x_Q} F^S + \tilde{P} \equiv P + \tilde{P}$.

Thanks to Whitney's extension theorem (Theorem 3.2 (B)), it suffices to show that $\vec{P} \in W(S^+, \tau)$ and $\|\vec{P}\|_{W(S^+,\tau)} \leq CM$.

Thanks to (3.42), we have

$$P^x \in \mathcal{K}_\tau(x, CM) \qquad \text{for } x \in S, \tag{3.43}$$

$$|\partial^\alpha (P^x - P^y)(x)| \leq CM|x-y|^{2-|\alpha|} \qquad \text{for } x, y \in S,\ x \neq y,\ |\alpha| \leq 1. \tag{3.44}$$

On the other hand, using (3.37), we have

$$\tau - |P^{x_Q}(x_Q)| = \tau - |P(x_Q) + \tilde{P}(x_Q)| \geq (C(A_{\text{perturb}}^{1/2} - 1)^2 - 1)M\delta_Q^2. \tag{3.45}$$

Since $P \equiv \mathfrak{J}_{x_Q} F^S$, (3.42) implies that $P \in \mathcal{K}_\tau(x_Q, CM)$. Using property (3.2) of $\mathcal{K}_\tau$ and (3.37), we have

$$|\nabla P^{x_Q}| \leq |\nabla P| + |\nabla \tilde{P}| \leq (C(A_{\text{perturb}}^{1/2} - 1) - 1)M\delta_Q. \tag{3.46}$$

Combining (3.45) and (3.46), we see that

$$P^{x_Q} \in \mathcal{K}_\tau(x_Q, AM), \quad \text{with } A = A(n, A_{\text{perturb}}). \tag{3.47}$$

Combining (3.43) and (3.47), we see that $\vec{P} \in W(S^+, \tau)$.

Now we estimate $\|\vec{P}\|_{W(S^+,\tau)}$. By assumption, $x_Q \in Q$ satisfies

$$|x_Q - x| \geq c_0\delta_Q \quad \text{for } x \in E.$$

As a consequence,

$$|\partial^\alpha \tilde{P}(x_Q)| \leq CM\delta_Q^{2-|\alpha|} \leq M|x - x_Q|^{2-|\alpha|} \quad \text{for } x \in E,\ |\alpha| \leq 1. \tag{3.48}$$

By Taylor's theorem and (3.42), we have

$$|\partial^\alpha (P^x - P)(x)|, |\partial^\alpha (P^x - P)(x_Q)| \leq CM|x - x_Q|^{2-|\alpha|} \quad \text{for } x \in S,\ |\alpha| \leq 1. \tag{3.49}$$

Combining (3.48) and (3.49), we see that

(3.50)
$$\begin{aligned}|\partial^\alpha(P^x - P^{x_Q})(x_Q)| &\le |\partial^\alpha(P^x - P)(x_Q)| + |\partial^\alpha \tilde{P}(x_Q)| \\ &\le CM|x - x_Q|^{2-|\alpha|} \quad \text{for } x \in S,\ |\alpha| \le 1.\end{aligned}$$

By Taylor's theorem and (3.50), we also have

(3.51)
$$|\partial^\alpha(P^x - P^{x_Q})(x)| \le CM|x - x_Q|^{2-|\alpha|} \quad \text{for } x \in S, |\alpha| \le 1.$$

Finally, we see from (3.44), (3.50), and (3.51) that $\|\vec{P}\|_{W(S^+,\tau)} \le CM$.

This proves Lemma 3.7 (A).

Proof of (B). We prove the case when $\tau + f(x) \le A_{\text{flat}} M\delta_Q^2$. The case $\tau - f(x)$ is similar.

We claim that under the assumption of (B), given any $P \in \Gamma_\tau^\sharp(x_Q, k, f, M)$, we have

(3.52)
$$\tau + P(x_Q) \le C(A_{\text{flat}}^{1/2} + 1)^2 M\delta_Q^2.$$

The proof is similar to that of Lemma 3.6 (B). We provide the proof here for completeness.

Fix $x_0 \in E \cap 5Q$ such that $\tau + f(x_0) \le A_{\text{flat}} M\delta_Q^2$. Let $P \in \Gamma_\tau^\sharp(x_Q, k, f, M)$. By the definition of $\Gamma_\tau^\sharp$, there exists a function $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le M$, $\mathfrak{J}_{x_Q} F \equiv P$, and $F(x_0) = f(x_0)$. Since $\mathfrak{J}_{x_0} F \in \mathcal{K}_\tau(x_0, CM)$, property (3.2) of $\mathcal{K}_\tau$ implies $|\nabla F(x_0)| \le CA_{\text{flat}}^{1/2} M\delta_Q$. Inequality (3.52) then follows from Taylor's theorem and the estimates immediately above.

Now we need to show that the constant polynomial $-\tau \in \Gamma_\tau^\sharp(x_Q, k, f, AM)$ for some A depending only on n and A_{flat}. We write A, A', etc., to denote quantities that depend only on n and A_{flat}.

Let $P \in \Gamma_\tau^\sharp(x_Q, k, f, M)$. Then $P \in \mathcal{K}_\tau(x_Q, AM)$ and satisfies (3.52). By (3.2) and (3.52),

(3.53)
$$|\partial^\alpha(\tau + P)(x_Q)| \le AM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 2.$$

Fix $S \subset E$ with $\#S \le k$. We want to show that $-\tau \in \Gamma_\tau(x_Q, S, f, AM)$.

By the definition of $\Gamma_\tau^\sharp$, there exists $F^S \in C^2(\mathbb{R}^n, \tau)$ with

(3.54)
$$\|F^S\|_{C^2(\mathbb{R}^n)} \le M, \quad F^S(x) = f(x) \ \text{ for } x \in S, \quad \mathfrak{J}_{x_Q} F^S \equiv P.$$

From Taylor's theorem, together with (3.53) and (3.54), we see that

(3.55)
$$|\partial^\alpha(\tau + F^S)(x)| \le AM\delta_Q^{2-|\alpha|} \quad \text{for } x \in B(x_Q, c_0\delta_Q), |\alpha| \le 2.$$

Here, c_0 is the constant in the hypothesis of Lemma 3.7.

Let $\chi \in C^2(\mathbb{R}^n)$ be a cutoff function such that

(χ1) $0 \le \chi \le 1$ on $\mathbb{R}^n$,

(χ2) $\chi \equiv 1$ near x_Q and $\text{supp}(\chi) \subset B(x_Q, c_0\delta_Q)$,

(χ3) $|\partial^\alpha \chi| \le C\delta_Q^{2-|\alpha|}$ for $|\alpha| \le 2$.

We define

$$\tilde{F}^S := \chi \cdot (-\tau) + (1-\chi) \cdot F^S.$$

It is clear that $\tilde{F}^S \in C^2(\mathbb{R}^n)$. Thanks to ($\chi$1), $\tilde{F}^S$ is defined as the convex combination of two functions with range $[-\tau, \tau]$. Therefore, $\tilde{F}^S \in C^2(\mathbb{R}^n, \tau)$. Thanks to ($\chi$2) and the fact that $\mathrm{dist}(x_Q, E) \geq c_0 \delta_Q$, we have $\tilde{F}^S(x) = f(x)$ for $x \in S$. Thanks to (χ2) again, $\mathcal{J}_{x_Q} \tilde{F}^S \equiv -\tau$. Finally, thanks to (3.55) and (χ3), we have $\|\tilde{F}^S\|_{C^2(\mathbb{R}^n)} \leq AM$.

Hence, $-\tau \in \Gamma_\tau(x_Q, S, f, AM)$. Since S is chosen arbitrarily, we can conclude that $-\tau \in \Gamma_\tau(x_Q, k, f, AM)$. This proves Lemma 3.7 (B). ∎

4. Base case of the induction

In this section, we prove a stronger version of Theorem 1.3. We use $\mathcal{P}$ and $\mathcal{P}^+$, respectively, to denote the vector space of single-variable polynomials with degree no greater than one and two. We use $\mathcal{J}_x$ and $\mathcal{J}_x^+$, respectively, to denote the one-jet and two-jet of a single-variable function twice continuously differentiable near $x \in \mathbb{R}$.

Lemma 4.1. *Suppose we are given a finite set $E_* \subset \mathbb{R}$ with $\#E_* \leq 3$. Then there exists a collection of maps $\{\Xi_{\tau,x} : \tau \in [0,\infty),\ x \in \mathbb{R}\}$, where*

$$\Xi_{\tau,x} \colon C^2(E_*, \tau) \to \mathcal{P}^+$$

for each $x \in \mathbb{R}$, such that the following hold:

(A) *Let $f \in C^2(E_*, \tau)$ be given. Then there exists a function $F \in C^2(\mathbb{R}, \tau)$ such that*

$$\mathcal{J}_x^+ F \equiv \Xi_{\tau,x}(f) \quad \textit{for all } x \in \mathbb{R},$$
$$\|F\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2(E_*,\tau)} \quad \textit{and} \quad F(x) = f(x) \ \textit{ for } x \in E_*.$$

Here, C depends only on n.

(B) *There is an algorithm that takes the given data set E_*, performs one-time work, and then responds to queries. A query consists of a pair $(\tau, x) \in [0,\infty) \times \mathbb{R}$, and the response to the query is the map $\Xi_{\tau,x}$, given in its efficient representation. The one-time work takes C_1 operations and C_2 storage. The work to answer a query is C_3. Here, C_1, C_2, C_3 are universal constants.*

Proof. Let $\tau > 0$ and $f \in C^2(E_*, \tau)$ be given. Let $\mathcal{L}$ and $\mathcal{M}_\tau$, respectively, be as in (3.20) and (3.22), with $S = E_*$. Consider the affine subspace $\mathbb{A}_{\tau,f} \subset W(E_*)$ given by

$$\mathbb{A}_{\tau,f} := \Big\{ \vec{P} = (P^x)_{x\in E} \in W(E_*) : P^x(x) = f(x) \text{ for } x \in E_*, \\ \text{if } f(x) = \tau, \text{ then } P^x \equiv \tau, \\ \text{if } f(x) = -\tau, \text{ then } P^x \equiv -\tau \Big\}. \tag{4.1}$$

Consider the optimization problem

$$\text{Minimize } \mathcal{L} + \mathcal{M}_\tau \text{ over } \mathbb{A}_{\tau,f}. \tag{4.2}$$

By Section 3.3, we can find an approximate minimizer $\vec{P}_0 \in W(E_*, \tau)$ of (4.2) using C operations. Namely, $(\mathcal{L} + \mathcal{M})(\vec{P}_0) \leq C \cdot \inf\{(\mathcal{L} + \mathcal{M})(\vec{P}) : \vec{P} \in \mathbb{A}_{\tau,f}\}$ for some universal constant C. We denote the solution procedure by

$$\widetilde{\mathrm{Min}}_\tau : C^2(E_*, \tau) \to W(E_*, \tau).$$

It follows from Lemma 3.4 that

$$\|\widetilde{\mathrm{Min}}_\tau(f)\|_{W(E_*,\tau)} \leq C\|f\|_{C^2(E_*,\tau)}. \tag{4.3}$$

Let $\mathcal{T}_{w,\tau}$ be as in Whitney's extension theorem (Theorem 3.2 (B)) with $S = E_*$. We define an extension operator

$$\mathcal{E}_{\tau,*} := \mathcal{T}_{w,\tau} \circ \widetilde{\mathrm{Min}}_\tau : C^2(E_*, \tau) \to C^2(\mathbb{R}, \tau). \tag{4.4}$$

We then define

$$\Xi_{\tau,x} :\equiv \mathcal{J}_x^+ \circ \mathcal{E}_{\tau,*}$$

Lemma 4.1 (A) follows from the conclusion of Whitney's extension theorem (Theorem 3.2 (B)) and (4.3).

Recall from Section 3.3 that constructing the operator $\widetilde{\mathrm{Min}}_\tau$ amounts to solving a convex quadratic programming problem with affine constraint. Since $\#E \leq 3$, this procedure, as well as constructing a one-dimensional Whitney extension operator, requires at most C operations and C' storage. Lemma 4.1 (B) follows. ∎

First sorting the set $E \subset \mathbb{R}$, and then patching together adjacent maps, we arrive at the following Theorem 4.1. Note that Theorem 4.1 is in fact stronger than Theorem 1.3, since the extension maps do not depend on the parameter M.

Theorem 4.1. *Suppose we are given a finite set $E \subset \mathbb{R}$ with $\#(E) = N$. Then there exists a collection of maps $\{\Xi_{\tau,x} : \tau \in [0, \infty),\ x \in \mathbb{R}\}$, where*

$$\Xi_{\tau,x} : C^2(E, \tau) \to \mathcal{P}^+$$

for each $x \in \mathbb{R}$, such that the following hold:

(A) *There exists a universal constant D such that for each $x \in \mathbb{R}$, the map $\Xi_{\tau,x}(\cdot) : C^2(E, \tau) \to \mathcal{P}^+$ is of depth at most D. Moreover, the source of $\Xi_{\tau,x}$ (in the sense of Definition 1.1) is independent of τ.*

(B) *Let $f \in C^2(E, \tau)$ be given. Then there exists a function $F \in C^2(\mathbb{R}, \tau)$ such that*

$$\mathcal{J}_x^+ F \equiv \Xi_{\tau,x}(f) \quad \text{for all } x \in \mathbb{R},$$
$$\|F\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2(E,\tau)} \quad \text{and} \quad F(x) = f(x) \ \text{ for } x \in E.$$

Here, C depends only on n.

(C) *There is an algorithm that takes the given data set E, performs one-time work, and then responds to queries. A query consists of a pair $(\tau, x) \in [0, \infty) \times \mathbb{R}$, and the response to the query is the map $\Xi_{\tau,x}$, given in its efficient representation. The one-time work takes $C_1 N \log N$ operations and $C_2 N$ storage. The work to answer a query is $C_3 \log N$. Here, C_1, C_2, C_3 are universal constants.*

Proof. If $\#E \le 3$, then Theorem 4.1 reduces to Lemma 4.1. Suppose $\#E \ge 4$. We sort E into an order list $E = \{x_1 < \cdots < x_N\}$. For convenience, we set $x_0 := -\infty$ and $x_{N+1} := +\infty$. We set:

- $E_\nu := \{x_{\nu-1}, x_\nu, x_{\nu+1}\}$ for $\nu = 2, \dots, N-1$, $E_1 := E_2$ and $E_N := E_{N-1}$.
- $J_\nu := (x_{\nu-1}, x_{\nu+1})$ for $\nu = 1, \dots, N$.

Let $\{\theta_\nu\}_{\nu=1}^N$ be a nonnegative C^2-partition of unity subordinate to $\{J_\nu\}_{\nu=1}^N$ that satisfies the following:

(4.5) For each $\nu = 1, \dots, N$, $\theta_\nu \equiv 1$ near x_ν, and $\mathrm{supp}(\theta_\nu) \subset J_\nu$. Note that each $x \in \mathbb{R}$ lies in the support of at most two θ_ν.

(4.6) For each $\nu = 1, \dots, N$,

$$\left|\frac{d^m}{dx^m}\theta_\nu(x)\right| \le \begin{cases} C|x_\nu - x_{\nu-1}|^{-m} & \text{for } x \in (x_{\nu-1}, x_\nu), \\ C|x_{\nu+1} - x_\nu|^{-m} & \text{for } x \in (x_\nu, x_{\nu+1}), \end{cases}$$

for $m = 0, 1, 2$, and some universal constant C. For $\nu = 1, N$, we use the convention $\infty^0 = 1$ and $\infty^{-m} = 0$ for $m \ge 1$.

We can construct each θ_ν using the standard trick with summing and dividing cutoff functions.

For each $\nu = 2, \dots, N-1$, we define $\mathcal{E}_{\tau,\nu}$ as in (4.4) of Lemma 4.1 with $* = \nu$. We define an extension operator $\mathcal{E}_\tau \colon C^2(E,\tau) \to C^2(\mathbb{R},\tau)$ by

$$\mathcal{E}_\tau(f)(x) := \sum_{\nu=1}^N \theta_\nu(x) \cdot \mathcal{E}_{\tau,\nu}(f|_{E_\nu}) \quad \text{for } f \in C^2(E,\tau).$$

It is clear that $\mathcal{E}_\tau(f) \in C^2(\mathbb{R})$ since each $\mathcal{E}_{\tau,\nu}(f|_{E_\nu}) \in C^2(\mathbb{R})$. Moreover, since the range of each $\mathcal{E}_{\tau,\nu}(f|_{E_\nu})$ is $[-\tau,\tau]$ and $\mathcal{E}_\tau$ is a convex combination of $\mathcal{E}_{\tau,\nu}$, we have that $\mathcal{E}_\tau(f) \in C^2(\mathbb{R},\tau)$.

Now we show that $\mathcal{E}_\tau$ is bounded.

Let $M := \max_{\nu=1,\dots,N} \|f\|_{C^2(E_\nu,\tau)}$. By the definition of the trace norm, we have $\|f\|_{C^2(E,\tau)} \ge M$.

Let $F_\nu := \mathcal{E}_{\tau,\nu}(f|_{E_\nu})$ for each ν. By Lemma 4.1 (A), we have

$$\|F_\nu\|_{C^2(\mathbb{R})} \le CM \quad \text{for } \nu = 1, \dots, N. \tag{4.7}$$

Since $\{\theta_\nu : \nu = 1, \dots, N\}$ is a partition of unity, we see from (4.7) that

$$|\mathcal{E}_\tau(f)(x)| \le CM \quad \text{for } x \in \mathbb{R}. \tag{4.8}$$

On J_1 or J_N, we see, from the definitions of E_1, E_N and the support condition (4.5), that $\mathcal{E}_\tau(f) \equiv \mathcal{E}_{\tau,1}(f)$ or $\mathcal{E}_\tau(f) \equiv \mathcal{E}_{\tau,N}(f)$. Therefore,

$$\left|\frac{d^m}{dx^m}\mathcal{E}_\tau(f)(x)\right| \le CM \quad \text{for } x \in J_1 \cup J_N,\ m \le 2. \tag{4.9}$$

Suppose that $x \in [x_2, x_{N-1}]$. Let $\nu(x)$ be the least integer such that $x \in J_{\nu(x)}$. Then the only partition functions that are possibly nonzero at x are $\theta_{\nu(x)}$ and $\theta_{\nu(x)+1}$. Since

$\frac{d^m}{dx^m}\theta_{\nu(x)} = -\frac{d^m}{dx^m}\theta_{\nu(x)+1}(x)$ for $m = 1$ and 2, we have

$$\frac{d^m}{dx^m}\mathcal{E}_\tau(f)(x) = \sum_{0\le m'\le m} C_{m,m'}\frac{d^{m'}}{dx^{m'}}(F_{\nu(x)} - F_{\nu(x)+1})(x)\frac{d^{m-m'}}{dx^{m-m'}}\theta_{\nu(x)}(x). \tag{4.10}$$

We claim that

$$\left|\frac{d^{m'}}{dx^{m'}}(F_{\nu(x)} - F_{\nu(x)+1})(x)\right| \le CM|x_{\nu(x)+1} - x_{\nu(x)}|^{2-m'} \quad \text{for } 0 \le m' \le 2. \tag{4.11}$$

To see this, observe that $F_{\nu(x)} = F_{\nu(x)+1}$ at $x_{\nu(x)}$ and $x_{\nu(x)+1}$. Therefore, by Rolle's theorem, there exists $\hat{x}_{\nu(x)} \in (x_{\nu(x)}, x_{\nu(x)+1})$ such that

$$\frac{d}{dx}(F_{\nu(x)} - F_{\nu(x)+1})(\hat{x}_{\nu(x)}) = 0. \tag{4.12}$$

Now, (4.11) follows from (4.12) and Taylor's theorem.

Using (4.6) and (4.11) to estimate (4.10), we can conclude that

$$\left|\frac{d^m}{dx^m}\mathcal{E}_\tau(f)(x)\right| \le CM \quad \text{for } x \in [x_2, x_{N-1}]. \tag{4.13}$$

From (4.8), (4.9), and (4.13), we see that

$$\|\mathcal{E}_\tau(f)\|_{C^2(\mathbb{R})} \le CM \le C\|f\|_{C^2(E,\tau)}.$$

Namely, $\mathcal{E}_\tau : C^2(E,\tau) \to C^2(\mathbb{R},\tau)$ is bounded.

Finally, we set

$$\Xi_{\tau,x} :\equiv \mathcal{J}_x^+ \circ \mathcal{E}_\tau.$$

Theorem 4.1 (A)–(B), then follows from the boundedness of $\mathcal{E}_\tau$.

The one-time work consists of sorting the set E and computing E_ν, J_ν, and θ_ν for $\nu = 1, \dots, N$. This requires at most $CN \log N$ operations and CN storage.

Now we discuss query work. Let $(\tau, x) \in [0,\infty) \times \mathbb{R}$ and let $f \in C^2(E,\tau)$ be given. It requires at most $C \log N$ operations to locate $\nu(x)$, where $\nu(x)$ is the least integer that $x \in J_{\nu(x)}$. Note that $\Xi_{\tau,x}(f)$ is a linear combination of

$$\mathcal{J}_x^+\mathcal{E}_{\tau,\nu(x)}(f|_{E_{\nu(x)}}),\quad \mathcal{J}_x^+\mathcal{E}_{\tau,\nu(x)+1}(f|_{E_{\nu(x)+1}}),\quad \mathcal{J}_x^+\theta_{\nu(x)},\quad \mathcal{J}_x^+\theta_{\nu(x)+1}.$$

From Lemma 4.1, we can compute $\mathcal{J}_x^+\mathcal{E}_{\tau,\nu(x)}(f|_{E_{\nu(x)}})$ and $\mathcal{J}_x^+\mathcal{E}_{\tau,\nu(x)+1}(f|_{E_{\nu(x)+1}})$ from $(E_{\nu(x)}, f)$ and $(E_{\nu(x)}, f)$, respectively, using at most C operations. On the other hand, we can compute $\mathcal{J}_x^+\theta_{\nu(x)}$ and $\mathcal{J}_x^+\theta_{\nu(x)+1}$ from $J_{\nu(x)}$ and $J_{\nu(x)+1}$ using at most C operations. Therefore, the work to answer a query is $C \log N$. ∎

5. Set up for the induction

We write $\overline{x} \in \mathbb{R}^{n-1}$ to denote points in $\mathbb{R}^{n-1}$. We write $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}^+$ to denote the vector spaces of polynomials on $\mathbb{R}^{n-1}$ with degree no greater than one and two, respectively. We write $\overline{\mathcal{J}}_{\overline{x}}$ and $\overline{\mathcal{J}}_{\overline{x}}^+$ to denote the one-jet and two-jet of a function (twice) differentiable near $\overline{x} \in \mathbb{R}^{n-1}$.

Given any finite set $\bar{E} \subset \mathbb{R}^{n-1}$ with $\#\bar{E} = \bar{N}$, we assume the following.

(5.1) There exists a collection of maps $\{\bar{\Xi}_{\tau,\bar{x}} : \tau \in [0,\infty),\ \bar{x} \in \mathbb{R}^{n-1}\}$, where

$$\bar{\Xi}_{\tau,\bar{x}} : C^2(\bar{E},\tau) \times [0,\infty) \to \bar{\mathcal{P}}^+$$

for each $\bar{x} \in \mathbb{R}^{n-1}$, such that the following hold:

(A) There exists a controlled constant $\bar{D}$, depending only on $n-1$, such that for each $\bar{x} \in \mathbb{R}^{n-1}$, the map $\bar{\Xi}_{\tau,\bar{x}}$ is of depth at most $\bar{D}$, and the source of $\bar{\Xi}_{\tau,\bar{x}}$ (in the sense of Definition 1.1) is independent of τ.

(B) Suppose we are given $(\bar{f}, M) \in C^2(\bar{E},\tau) \times [0,\infty)$ with $\|\bar{f}\|_{C^2(\bar{E},\tau)} \le M$. Then there exists a function $\bar{F} \in C^2(\mathbb{R}^{n-1},\tau)$ such that

- $\bar{\mathfrak{J}}^+_{\bar{x}} \bar{F} \equiv \bar{\Xi}_{\tau,x}(\bar{f}, M)$ for all $\bar{x} \in \mathbb{R}^{n-1}$,
- $\|\bar{F}\|_{C^2(\mathbb{R}^{n-1})} \le C(n-1) \cdot M$, and
- $\bar{F}(\bar{x}) = \bar{f}(\bar{x})$ for all $\bar{x} \in \bar{E}$.

(C) There exists an algorithm that takes the given data set $\bar{E}$, and then responds to queries. A query consists of a pair $(\tau, \bar{x}) \in [0,\infty) \times \mathbb{R}^{n-1}$, and the response to the query is the map $\bar{\Xi}_{\tau,x}$, given in its efficient representation. The one-time work takes $\bar{C}_1 \bar{N} \log \bar{N}$ operations and $\bar{C}_2 \bar{N}$ storage. The work to answer a query is $\bar{C}_3 \log \bar{N}$. Here, $\bar{C}_1, \bar{C}_2, \bar{C}_3$ depend only on $n-1$.

(D) For each $\tau > 0$, we use $\bar{\mathcal{E}}_\tau : C^2(\bar{E},\tau) \times [0,\infty) \to C^2(\mathbb{R}^n)$ to denote the operator associated with $\{\Xi_{\tau,\bar{x}} : \bar{x} \in \mathbb{R}^{n-1}\}$ determined by the relation

$$\bar{\mathfrak{J}}^+_{\bar{x}} \circ \bar{\mathcal{E}}_\tau(\bar{f}, M) \equiv \bar{\Xi}_{\tau,\bar{x}}(\bar{f}, M) \quad \text{for all } (\bar{f}, M) \in C^2(E,\tau) \times [0,\infty).$$

Thanks to (A), for each $\bar{x} \in \mathbb{R}^n$, there exists $\bar{S}(\bar{x}) \subset \bar{E}$ with $\#\bar{S}(\bar{x}) \le \bar{D}$, independent of τ, such that for all $\bar{f}, \bar{g} \in C^2(E,\tau)$ with $\bar{f} = \bar{g}$ on $\bar{S}(\bar{x})$, we have

$$\partial^\alpha \bar{\mathcal{E}}_\tau(\bar{f}, M)(\bar{x}) = \partial^\alpha \bar{\mathcal{E}}_\tau(\bar{g}, M)(\bar{x}) \quad \text{for } |\alpha| \le 2 \text{ and } M \ge 0.$$

(5.2) We also assume that we are given the Fefferman–Klartag interpolation maps, i.e., a collection of linear maps $\{\bar{\Psi}_{\bar{x}} : \bar{x} \in \mathbb{R}^{n-1}\}$, where

$$\bar{\Psi}_{\bar{x}} : \mathbb{R}^{\bar{N}} \to \bar{\mathcal{P}}^+$$

for each $\bar{x} \in \mathbb{R}^{n-1}$, such that the following hold:

(A) There exists a controlled constant $\bar{D}$, depending only on $n-1$, such that for each $\bar{x} \in \mathbb{R}^{n-1}$, the map $\bar{\Psi}_{\bar{x}}$ is of depth at most $\bar{D}$.

(B) Suppose we are given $\bar{\varphi} \in \mathbb{R}^{\bar{N}}$. Then there exists a function $\bar{\Phi} \in C^2(\mathbb{R}^{n-1})$ such that

- $\bar{\mathfrak{J}}^+_{\bar{x}} \bar{\Phi} \equiv \bar{\Psi}_x(\varphi)$ for all $\bar{x} \in \mathbb{R}^{n-1}$,
- $\|\bar{\Phi}\|_{C^2(\mathbb{R}^{n-1})} \le C(n-1)\|\bar{\varphi}\|_{C^2(\bar{E})}$, and
- $\bar{\Phi}(\bar{x}) = \bar{\varphi}(\bar{x})$ for all $\bar{x} \in \bar{E}$.

Here, $\|\bar{\varphi}\|_{C^2(\bar{E})} := \inf\{\|\tilde{\Phi}\|_{C^2(\mathbb{R}^{n-1})} : \tilde{\Phi} \in C^2(\mathbb{R}^{n-1}) \text{ and } \tilde{\Phi}|_{\bar{E}} = \bar{\varphi}\}$.

(C) There exists an algorithm that takes the given data set $\bar{E}$, and then responds to queries. A query consists of a point $\bar{x} \in \mathbb{R}^{n-1}$, and the response to the query is the map $\bar{\Psi}_{\bar{x}}$, given in its efficient representation. The one-time work takes $\bar{C}_1 \bar{N} \log \bar{N}$ operations and $\bar{C}_2 \bar{N}$ storage. The work to answer a query is $\bar{C}_3 \log \bar{N}$. Here, $\bar{C}_1, \bar{C}_2, \bar{C}_3$ depend only on $n-1$.

(D) We use $\bar{\mathcal{E}}_\infty$ to denote the operator associated with $\{\bar{\Psi}_{\bar{x}} : \bar{x} \in \mathbb{R}^{n-1}\}$, mapping from $\{\bar{\varphi}: \bar{E} \to \mathbb{R}\}$ into $C^2(\mathbb{R}^{n-1})$, determined by the relation

$$\bar{\mathscr{J}}^+_{\bar{x}} \circ \bar{\mathcal{E}}_\infty(\bar{\varphi}) \equiv \bar{\Psi}_{\bar{x}}(\bar{\varphi}) \quad \text{for all } \bar{\varphi}: \bar{E} \to \mathbb{R}.$$

Thanks to (A), for each $\bar{x} \in \mathbb{R}^n$, there exists $\bar{S}(\bar{x}) \subset \bar{E}$ with $\#\bar{S}(\bar{x}) \leq \bar{D}$, such that for all $\bar{\varphi}, \bar{\gamma}: \bar{E} \to \mathbb{R}$ with $\bar{\varphi} = \bar{\gamma}$ on $\bar{S}(\bar{x})$, we have

$$\partial^\alpha \bar{\mathcal{E}}_\infty(\bar{\varphi})(\bar{x}) = \partial^\alpha \bar{\mathcal{E}}_\infty(\bar{\gamma})(\bar{x}) \quad \text{for } |\alpha| \leq 2.$$

For the construction of the Fefferman–Klartag maps, see [20, 21]. Such maps can also be constructed using the techniques in [28–30], which are adapted from [20, 21].

We will be working with the finiteness constants $k^\sharp_{n-1,\text{old}}$ and $k^\sharp_{n,\text{old}}$ in Theorem 3.3. Their precise values do not matter. We will also be working with a constant $k^\sharp_{\text{LIP}}$ associated with the local interpolation problems, where we take $k^\sharp_{\text{LIP}} \geq (n+2)^2 k^\sharp_{n,\text{old}}$. We will remind the readers of these quantities when necessary.

6. Preliminary data structure

Recall Theorem 3.3. We begin by reviewing some key objects introduced in [20,21], which we will use to effectively approximate $\sigma^\sharp$ for $x \in E$.

We will be working with $C^2(\mathbb{R}^n)$ functions instead of $C^2(\mathbb{R}^n, \tau)$ functions.

Let $E \subset \mathbb{R}^n$ be a finite set with $\#E = N$. We assume that E is labeled, that is, $E = \{x_1, \ldots, x_N\}$. We write $C(E)$ to denote the collection of functions $\varphi: E \to \mathbb{R}$, which we can identify (non-uniquely) with $\mathbb{R}^N$.

6.1. Parameterized approximate linear algebra problems (PALP)

We equip $\mathbb{R}^N$ with the standard coordinate basis $\{\xi_1, \ldots, \xi_N\}$. The following definition was introduced in Section 6 of [21].

Definition 6.1. A *parameterized approximate linear algebra problem (PALP for short)* is an object of the form:

$$\underline{\mathcal{A}} = \left[(\underline{\lambda}_1, \ldots, \underline{\lambda}_{i_{\max}}), (\underline{b}_1, \ldots, \underline{b}_{i_{\max}}), (\epsilon_1, \ldots, \epsilon_{i_{\max}})\right], \tag{6.1}$$

where

- each $\underline{\lambda}_i$ is a linear functional on $\mathcal{P}$, which we will refer to as a “linear functional”;
- each $\underline{b}_i$ is a linear functional on $C(E)$, which we will refer to as a “target functional”;
- each $\epsilon_i \in [0, \infty)$, which we will refer to as a “tolerance”.

Given a PALP $\underline{\mathcal{A}}$ in the form (6.1), we introduce the following terminologies:

- We call $i_{\max}$ the *length* of $\underline{\mathcal{A}}$;
- We say $\underline{\mathcal{A}}$ *has depth* D if D is the smallest integer such that each of the linear functionals $\underline{b}_i$ on $\mathbb{R}^N$ has depth at most D with respect to the basis $\{\xi_1, \ldots, \xi_N\}$ (see Definition 1.1).

Recall Definition 1.1. We assume that every PALP is "efficiently stored", namely, each of the target functionals are stored in its efficient representation. In particular, given a PALP $\underline{\mathcal{A}}$ of the form (6.1) and a target $\underline{b}_i$ of $\underline{\mathcal{A}}$, we have access to a set of indices $\{i_1, \ldots, i_D\} \subset \{1, \ldots, N\}$ such that $\underline{b}_i$ is completely determined by its action on the subset $\{\xi_{i_1}, \ldots, \xi_{i_D}\} \subset \{\xi_1, \ldots, \xi_N\}$. Here $i_D = \operatorname{depth}(\underline{b}_i)$. We define

$$S(\underline{b}_i) := \{x_{i_1}, \ldots, x_{i_D}\} \subset E. \tag{6.2}$$

Given a PALP of the form (6.1), we define

$$S(\underline{\mathcal{A}}) := \bigcup_{i=1}^{i_{\max}} S(\underline{b}_i) \subset E \tag{6.3}$$

with $S(\underline{b}_i)$ as in (6.2).

6.2. Blobs and PALPs

Definition 6.2. A *blob* in $\mathcal{P}$ is a family $\vec{\Omega} = (\Omega_M)_{M\geq 0}$ of (possibly empty) convex subsets $\Omega_M \subset \mathcal{P}$ parameterized by $M \in [0,\infty)$, such that $M < M'$ implies $\Omega_M \subseteq \Omega_{M'}$. We say two blobs $\vec{\Omega} = (\Omega_M)_{M\geq 0}$ and $\vec{\Omega}' = (\Omega'_M)_{M\geq 0}$ are C*-equivalent* if $\Omega_{C^{-1}M} \subset \Omega'_M \subset \Omega_{CM}$ for each $M \in [0,\infty)$.

Let $\underline{\mathcal{A}}$ be a PALP of the form (6.1). For each $\varphi \in C(E)$, we have a blob defined by

$$\vec{\Omega}_\varphi(\underline{\mathcal{A}}) = (\Omega_\varphi(\underline{\mathcal{A}}, M))_{M\geq 0}, \tag{6.4}$$

where

$$\Omega_\varphi(\underline{\mathcal{A}}, M) := \left\{P \in \mathcal{P} : |\underline{\lambda}_i(P) - \underline{b}_i(\varphi)| \leq M\epsilon_i \text{ for } i = 1, \ldots, i_{\max}\right\} \subset V.$$

In this paper, we will be mostly interested in the centrally symmetric (called "homogeneous" in [21]) polytope defined by setting $\varphi \equiv 0$:

$$\sigma(\underline{\mathcal{A}}) := \Omega_0(\underline{\mathcal{A}}, 1). \tag{6.5}$$

Note that $\sigma(\underline{\mathcal{A}})$ is never empty, since it contains the zero polynomial.

6.3. Essential PALPs and blobs

Definition 6.3. Let $E \subset \mathbb{R}^n$ be finite. For each $x \in \mathbb{R}^n$ and $\varphi\colon E \to \mathbb{R}$, we define a blob

$$\vec{\Sigma}_\varphi(x) = (\Sigma_\varphi(x, M))_{M\geq 0}, \tag{6.6}$$

where

$$\Sigma_\varphi(x, M) := \left\{P \in \mathcal{P} : \exists G \in C^2(\mathbb{R}^n) \text{ with } \|G\|_{C^2(\mathbb{R}^n)} \leq M, G|_E = \varphi \text{ and } \mathfrak{J}_x G \equiv P\right\}$$

It is clear from the definition of σ in (3.29) that

$$\sigma(x, E) = \Sigma_0(x, 1).$$

The zero in the formula above denotes the zero polynomial. Therefore, thanks to Theorem 3.3, we have, for $k \geq k^{\sharp}_{n,\text{old}}$ and $x \in E$,

$$C^{-1} \cdot \sigma^{\sharp}(x, k) \subset \Sigma_0(x, 1) \subset C \cdot \sigma^{\sharp}(x, k), \tag{6.7}$$

for some controlled constants $k^{\sharp}_{n,\sigma}(n)$ and $C(n)$.

We summarize some relevant results from [21].

Lemma 6.1. *Let $E \subset \mathbb{R}^n$ be finite with $\#E = N$. Using at most $C(n) N \log N$ operations and $C(n)N$ storage, we can compute a list of PALPs $\{\underline{\mathcal{A}}(x) : x \in E\}$ such that the following hold*:

(A) *There exists a controlled constant $D_0(n)$ such that for each $x \in E$, $\underline{\mathcal{A}}(x)$ has length no greater than $(n+1) = \dim \mathcal{P}$ and has depth at most D_0.*

(B) *For each given $x \in \mathbb{R}^n$ and $\varphi \in C^2(E)$, the blobs $\vec{\Omega}_{\varphi}(\underline{\mathcal{A}}(x))$ in (6.4) and $\vec{\Sigma}_{\varphi}(x)$ in (6.6) are $C(n)$-equivalent.*

See Section 11 of [21] for Lemma 6.1 (A), and Sections 10, 11, and Lemma 34.3 of [21] for Lemma 6.1 (B).

The main lemma of this section is the following.

Lemma 6.2. *Let $k^{\sharp}_{n,\text{old}}$ be as in Theorem 3.3. There exists a controlled constant $C(n)$ such that the following holds. Let $E \subset \mathbb{R}^n$ be given. Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 6.1. Recall the definitions of σ and $S(\underline{\mathcal{A}}(x))$ as in (3.29) and (6.3). Then for $k \geq k^{\sharp}_{n,\text{old}}$ and $x \in E$,*

$$C^{-1} \cdot \sigma(x, S(\underline{\mathcal{A}}(x))) \subset \sigma^{\sharp}(x, k) \subset C \cdot \sigma(x, S(\underline{\mathcal{A}}(x))).$$

Proof. For centrally symmetric $\sigma, \sigma' \subset \mathcal{P}$, we write $\sigma \approx \sigma'$ if there exists a controlled constant $C(n)$ such that $C^{-1} \cdot \sigma \subset \sigma' \subset C \cdot \sigma$. Thus, we need to show $\sigma(x, \underline{\mathcal{A}}(x)) \approx \sigma^{\sharp}(x, k^{\sharp}_{n,\text{old}})$ for $x \in E$.

Thanks to Theorem 3.3, Lemma 6.1 (B) (applied to $\varphi \equiv 0$), (6.5), and (6.7), we have

$$\sigma^{\sharp}(x, k) \approx \sigma^{\sharp}(x, E) \approx \Omega_0(\underline{\mathcal{A}}(x), 1) = \sigma(\underline{\mathcal{A}}(x)) \quad \text{for } x \in E. \tag{6.8}$$

Therefore, it suffices to show that

$$\sigma(x, S(\underline{\mathcal{A}}(x))) \approx \sigma(\underline{\mathcal{A}}(x)) \quad \text{for } x \in E.$$

From (6.8) and the definition of σ in (3.29), we see that

$$\sigma(\underline{\mathcal{A}}(x)) \subset C \cdot \sigma(x, E) \subset C \cdot \sigma(x, S(\underline{\mathcal{A}}(x))).$$

It remains to show that

$$\sigma(x, S(\underline{\mathcal{A}}(x))) \subset C \cdot \sigma(\underline{\mathcal{A}}(x)).$$

Let $x \in E$ and let $P \in \sigma(x, S(\underline{\mathcal{A}}(x)))$. Then there exists $\varphi \in C^2(\mathbb{R}^n)$ such that $\|\varphi\|_{C^2(\mathbb{R}^n)} \leq 1$, $\varphi(x) = 0$ for all $x \in S(\underline{\mathcal{A}}(x))$, and $\mathcal{J}_x(\varphi) \equiv P$. Note that $\varphi|_E \in C^2(\mathbb{R}^n)$. We abuse notation and write φ in place of $\varphi|_E$ when there is no possibility of confusion.

It is clear from the definition of $\Sigma_\varphi(x, M)$ in (6.6) that

$$P \in \Sigma_\varphi(x, 1).$$

By Lemma 6.1 (B), we have

$$P \in \Omega_\varphi(\underline{\mathcal{A}}(x), C)$$

with $\Omega_\varphi(\underline{\mathcal{A}}(x), C)$ as in (6.4). In particular, we have

$$|\underline{\lambda}_i(P) - \underline{b}_i(\varphi)| \le C\epsilon_i \quad \text{for } i = 1, \dots, L = \text{length}(\underline{\mathcal{A}}(x)). \tag{6.9}$$

Here, the $\underline{\lambda}_1, \dots, \underline{\lambda}_L$, $\underline{b}_1, \dots, \underline{b}_L$, and $\epsilon_1, \dots, \epsilon_L$, respectively, are the linear functionals, target functionals, and the tolerance of $\underline{\mathcal{A}}(x)$. However, by the definition of $S(\underline{\mathcal{A}}(x))$ in (6.3) and the fact that $\varphi \equiv 0$ on $S(\underline{\mathcal{A}}(x))$, we see that (6.9) simplifies to

$$|\underline{\lambda}_i(P)| \le C\epsilon_i \quad \text{for } i = 1, \dots, L = \text{length}(\underline{\mathcal{A}}(x)).$$

This is equivalent to the statement

$$P \in \Omega_0(\underline{\mathcal{A}}(x), C) = C \cdot \sigma(\underline{\mathcal{A}}(x)). \qquad \blacksquare$$

7. Calderón–Zygmund cubes

Let $\tilde{\sigma} \subset \mathbb{R}^n$ be a convex set symmetric about the origin. We define

$$\operatorname{diam} \tilde{\sigma} := 2 \cdot \sup_{u \in \mathbb{R}^n, |u|=1} p_{\tilde{\sigma}}(u), \tag{7.1}$$

where $p_{\tilde{\sigma}}(u)$ is a gauge function given by

$$p_{\tilde{\sigma}}(u) := \sup\{r \ge 0 : ru \subset \tilde{\sigma}\}. \tag{7.2}$$

Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 6.1, and let $\sigma(\underline{\mathcal{A}}(x)) \subset \mathcal{P}$ be as in (6.5). Note that for each $x \in E$, $\sigma(\underline{\mathcal{A}}(x)) \subset \mathcal{P}$ is n-dimensional. Indeed, thanks to Lemma 6.1 (B) (with $\varphi \equiv 0$), any $P \in \sigma(\underline{\mathcal{A}}(x))$, $x \in E$, must have $P(x) = 0$. Thus, for each $x \in E$, we can identify $\sigma(\underline{\mathcal{A}}(x))$ as a subset of $\mathbb{R}^n$ via the map

$$\sigma(\underline{\mathcal{A}}(x)) \ni P \mapsto (\nabla P \cdot e_1, \dots, \nabla P \cdot e_n), \tag{7.3}$$

where $\{e_1, \dots, e_n\}$ is the chosen orthonormal system.

7.1. OK cubes

Definition 7.1. Let $A_1, A_2 > 0$ be sufficiently large dyadic numbers to be fixed later. Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 6.1. Let Q be a dyadic cube. We say Q *is OK* if the following hold:

- Either $\#(E \cap 5Q) \le 1$, or $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x)) \ge A_1\delta_Q$ for all $x \in E \cap 5Q$. Here and below, the diam $(\sigma(\underline{\mathcal{A}}(x)))$ is defined using the formula (7.1) via the identification (7.3).
- $\delta_Q \le A_2^{-1}$.

The importance of OK cubes is illustrated in the following lemma. Roughly speaking if Q is OK, then E lies on a hypersurface near Q with controlled curvature. Moreover, this hypersurface can be realized as the null set of a C^2 function.

Lemma 7.1. *Let Q be OK. Suppose $E \cap 5Q \neq \varnothing$. Let $x_0 \in E \cap 5Q$. Let $u_0 \in \mathbb{R}^n$ be a unit vector such that*

$$\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0)) = p_{\sigma(\underline{\mathcal{A}}(x_0))}(u_0), \tag{7.4}$$

with $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0))$ *and* $p_{\sigma(\underline{\mathcal{A}}(x_0))}$ *as in* (7.1) *and* (7.3)*, respectively. Let ρ be a rigid motion of $\mathbb{R}^n$ given by the simple rotation*

$$\begin{cases} u_0 \mapsto e_n, \\ \textit{identity on } (\mathbb{R}u_0 \oplus \mathbb{R}e_n)^\perp, \end{cases}$$

and the translation $x \mapsto x - x_0$. Then there exists $\varphi \in C^2(\mathbb{R}^{n-1})$ satisfying the following:

$$\rho(E \cap 5Q) \subset \{(\bar{x}, \varphi(\bar{x})) : \bar{x} \in \mathbb{R}^{n-1}\}, \tag{7.5}$$

$$|\nabla_{\bar{x}}^m \varphi(\bar{x})| \leq C A_1^{-1} \delta_Q^{1-m} \textit{ for } \bar{x} \in \mathbb{R}^{n-1},\ m = 1, 2,\ \textit{with } A_1 \textit{ as in Definition 7.1}, \tag{7.6}$$

$$x_0 = (\bar{0}, \varphi(\bar{0})),\ \textit{where } \bar{0} \textit{ is the origin of } \mathbb{R}^{n-1}. \tag{7.7}$$

Moreover, let $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\Phi \circ \rho(\bar{x}, t) := (\bar{x}, t - \varphi(\bar{x})).$$

Then Φ is a C^2 diffeomorphism of $\mathbb{R}^n$ satisfying $\Phi(E \cap 5Q) \subset \mathbb{R}^{n-1} \times \{t = 0\}$, and $|\nabla^m \Phi|, |\nabla^m \Phi^{-1}| \leq C A_1^{-1} \delta_Q^{1-m}$ for $m = 1, 2$ and A_1 as in Definition 7.1.

Proof. If $\#(E \cap 5Q) \leq 1$, then we may take φ to be the constant function. The conclusions of the lemma are trivially satisfied.

Assume $\#(E \cap 5Q) > 1$. Since Q is OK, we see that $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x)) \geq A_1 \delta_Q$ for all $x \in E \cap 5Q$. By Lemma 6.2, we see that

$$\operatorname{diam} \sigma^\sharp(x, k^\sharp_{n,\mathrm{old}}) \geq c A_1 \delta_Q \quad \text{for all } x \in E \cap 5Q, \tag{7.8}$$

with $k^\sharp_{n,\mathrm{old}}$ as in Theorem 3.3.

Let x_0 and u_0 be as in the hypothesis. Without loss of generality, we may assume that $x_0 = 0$, $u_0 = e_n$, and ρ is the identity map. Let $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the natural projection that eliminates the last coordinate.

By (7.8) and the symmetry of $\sigma^\sharp$, there exists $P_0 \in \sigma^\sharp(x_0, k^\sharp_{n,\mathrm{old}})$ such that

$$\partial_n P_0 \geq c A_1 \delta_Q \quad \text{and} \quad \partial_i P_0 = 0 \quad \text{for } i = 1, \dots, n-1. \tag{7.9}$$

Claim 7.1. Under the assumption of Lemma 7.1, for any $S \subset E \cap 5Q$ with $\#S \leq k^\sharp_{n,\mathrm{old}} - 1$, there exists $\varphi^S \in C^2(\mathbb{R}^{n-1})$ such that

$$S \subset \{(\bar{x}, \varphi^S(\bar{x})) : \bar{x} \in \mathbb{R}^{n-1}\}, \text{ and} \tag{7.10}$$

$$|\nabla_{\bar{x}}^m \varphi^S(\bar{x})| \leq C A_1^{-1} \delta_Q^{1-m} \text{for } \bar{x} \in \mathbb{R}^{n-1}, m = 1, 2, \text{ with } A_1 \text{ as in Definition 7.1.} \tag{7.11}$$

Proof of Claim 7.1. Fix $S \subset E \cap 5Q$ with $\#S \le k^\sharp_{n-1,\text{old}} \le k^\sharp_{n,\text{old}} - 1$. Here, $k^\sharp_{n-1,\text{old}}$ and $k^\sharp_{n,\text{old}}$ are as in Theorem 3.3. Let $S_0 := S \cup \{x_0\}$. Then $\#S_0 \le k^\sharp_{n-1,\text{old}} + 1$. Since $P_0 \in \sigma^\sharp(x_0, k^\sharp_{n,\text{old}}) \subset \sigma^\sharp(x_0, k^\sharp_{n-1,\text{old}} + 1)$, by the definition of $\sigma^\sharp$ in (3.29), there exists $\Psi \in C^2(\mathbb{R}^n)$ such that $S \subset \{\Psi = 0\}$, $\|\Psi\|_{C^2(\mathbb{R}^n)} \le 1$, and $\mathcal{J}_{x_0}\Psi \equiv P_0$. For A_1 sufficiently large, from Taylor's theorem and (7.9), we see that

$$\partial_n \Psi(x) \ge c A_1 \delta_Q \quad \text{and} \quad |\partial_i \Psi(x)| \le C\delta_Q \quad \text{for } x \in 5Q. \tag{7.12}$$

Thanks to (7.12) and the implicit function theorem, there exists a well-defined function $\varphi^S \in C^2_{\text{loc}}(\mathbb{R}^n)$ such that $S \subset \{(\bar{x}, \varphi^S(\bar{x})) : \bar{x} \in \mathbb{R}^{n-1}\}$. This proves (7.10).

Let $i, j \in \{1, \dots, n-1\}$ and $\bar{x} \in \pi(5Q)$. We have

$$\begin{aligned} \partial_i \varphi^S(\bar{x}) &= \frac{\partial_i \Psi}{\partial_n \Psi}(x), \\ \partial^2_{ij}\varphi^S(\bar{x}) &= \frac{-(\partial_n\Psi)^2\partial^2_{ij}\Psi + (\partial^2_{ni}\Psi\partial_j\Psi + \partial^2_{nj}\Psi\partial_i\Psi)(\partial_n\Psi)^2 - \partial^2_n\Psi\partial_i\Psi\partial_j\Psi}{(\partial_n\Psi)^3}(\bar{x}). \end{aligned} \tag{7.13}$$

We see that (7.11) follows from (7.12) and (7.13). Claim 7.1 is proved. ■

Consider the function $\varphi_0 \colon \pi(E \cap 5Q) \to \mathbb{R}$ defined by

$$\varphi_0(\bar{x}) := t \quad \text{for } x = (\bar{x}, t) \in E \cap 5Q.$$

By Claim 7.1, given $\bar{S} \subset \pi(E \cap 5Q)$ with $\#\bar{S} \le k^\sharp_{n-1,\text{old}}$, there exists $\varphi^{\bar{S}} \in C^2_{\text{loc}}(\mathbb{R}^{n-1})$ such that $\varphi^{\bar{S}}|_{\bar{S}} = \varphi_0$, $|\nabla^m_{\bar{x}}\varphi^{\bar{S}}(\bar{x})| \le CA_1^{-1}\delta_Q^{1-m}$ for $\bar{x} \in \pi(5Q)$ and $m = 1, 2$.

By Theorem 3.3 (A) together with the rescaling $\tilde{\varphi}(\bar{x}) \mapsto \delta_Q^{-1}\tilde{\varphi}(\delta_Q \cdot \bar{x})$, there exists $\varphi \in C^2(\mathbb{R}^{n-1})$ such that $\varphi|_{\pi(E\cap 5Q)} = \varphi_0$ and $|\nabla^m_{\bar{x}}\varphi(\bar{x})| \le CA_1^{-1}\delta_Q^{1-m}$ for $\bar{x} \in \mathbb{R}^{n-1}$ and $m = 1, 2$.

The properties of Φ follow immediately from those of φ. ■

7.2. CZ cubes

Definition 7.2. We write CZ^0 to denote the collection of dyadic cubes Q such that both of the following hold:

(A) Q is OK (see Definition 7.1).

(B) Suppose $\delta_Q < A_2^{-1}$. Then Q^+ is not OK. Recall that Q^+ is the unique dyadic parent of Q defined in Section 2

We recall the following results from [21].

Lemma 7.2 (Lemma 21.2 of [21]). *CZ^0 forms a cover of $\mathbb{R}^n$. Moreover, if $Q, Q' \in CZ^0$ with $(1 + 2c_G)Q \cap (1 + 2c_G)Q' \neq \varnothing$, then*

$$C^{-1}\delta_Q \le \delta_{Q'} \le C\delta_Q.$$

As a consequence, for each $Q \in CZ^0$,

$$\#\{Q' \in CZ^0 : (1 + c_G Q') \cap (1 + c_G)Q \neq \varnothing\} \le C'.$$

Here, C, C' are controlled constants depending only on n, and c_G is a fixed small dyadic number, say $c_G = 2^{-5}$.

Lemma 7.3. *After one-time work using at most $CN \log N$ operations and CN storage, we can perform each of the the following tasks using at most $C \log N$ operations.*

(A) (Section 26 of [21]). *Given a point $x \in \mathbb{R}^n$, we compute a list*

$$\Lambda(x) := \{Q \in CZ^0 : (1 + c_G)Q \ni x\}.$$

(B) (Section 27 of [21]). *Given a dyadic cube $Q \subset \mathbb{R}^n$, we can compute* $\mathrm{Empty}(Q)$, *with* $\mathrm{Empty}(Q) = \mathrm{True}$ *if $E \cap 25Q = \emptyset$, and* $\mathrm{Empty}(Q) = \mathrm{False}$ *if $E \cap 25Q \neq \emptyset$.*

(C) (Section 27 of [21]). *Given a dyadic cube $Q \subset \mathbb{R}^n$ with $E \cap 25Q \neq \emptyset$, we can compute* $\mathrm{Rep}(Q) \in E \cap 25Q$, *with the property that* $\mathrm{Rep}(Q) \in E \cap 5Q$ *if $E \cap 5Q \neq \emptyset$.*

We define the following subcollections of CZ^0:

(7.14) $$CZ^{\sharp\sharp} := \left\{Q \in CZ^0 : E \cap (1 + c_G)Q \neq \emptyset\right\}, \quad \text{with } c_G \text{ as in Lemma 7.2},$$

(7.15) $$CZ^{\sharp} := \{Q \in CZ^0 : E \cap 5Q \neq \emptyset\},$$

(7.16) $$CZ^{\mathrm{empty}} := \{Q \in CZ^0 \setminus CZ^{\sharp} : \delta_Q < A_2^{-1}\}, \quad \text{with } A_2 \text{ as in Definition 7.2}.$$

Lemma 7.4. *After one-time work using at most $CN \log N$ operations and CN storage, we can perform the following task using at most $C \log N$ operations. Given $Q \in CZ^0$, we can decide if $Q \in CZ^{\sharp}$, $Q \in CZ^{\mathrm{empty}}$, or $Q \in CZ^0 \setminus (CZ^{\sharp} \cup CZ^{\mathrm{empty}})$.*

Proof. This is a direct application of Lemma 7.3 (B), (C) to Q. ∎

Lemma 7.5. *We can compute a map*

(7.17) $$\mu\colon CZ^{\mathrm{empty}} \to CZ^{\sharp}$$

that satisfies

(7.18) $$(1 + c_G)\mu(Q) \cap 25Q \neq \emptyset \quad \text{for any } Q \in CZ^{\mathrm{empty}}.$$

The one-time work uses at most $CN \log N$ operations and CN storage. After that, we can answer queries using at most $C \log N$ operations. A query consists of a cube $Q \in CZ^{\mathrm{empty}}$, and the response to the query is a cube $\mu(Q)$ that satisfies (7.18).

Proof. Suppose that $Q \in CZ^{\mathrm{empty}}$. Then we have $E \cap 5Q^+ \neq \emptyset$. On the other hand, $5Q^+ \subset 25Q$. Hence, $E \cap 25Q \neq \emptyset$. Therefore, the map Rep in Lemma 7.3 (C) is defined for Q.

We set

(7.19) $$x := \mathrm{Rep}(Q) \subset E \cap 25Q,$$

with Rep as in Lemma 7.3. Note that $x \notin 5Q$, since $Q \in CZ^{\mathrm{empty}}$.

Let $\Lambda(x) \subset CZ^0$ be as in Lemma 7.3 (A). Pick $Q' \in \Lambda(x)$. Note that the choice of Q' may not be unique. By the defining property of $\Lambda(x)$ and the fact that $x \in E$, we have $Q' \in CZ^{\sharp}$. Set

$$\mu(Q) := Q' \in CZ^{\sharp}.$$

By the previous comment, we have

$$(1 + c_G)\mu(Q) \ni x. \tag{7.20}$$

Combining (7.19) and (7.20), we see that $(1 + c_G)\mu(Q) \cap 25Q \neq \varnothing$. (7.18) is satisfied.

By Lemma 7.3 (A), (C), the tasks $\Lambda(\cdot)$ and $\mathrm{Rep}(\cdot)$ require at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage. Therefore, computing $\mu(Q)$ requires at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage. ∎

Lemma 7.6. *After one-time work using at most $CN \log N$ operations and CN storage, we can perform the following task using at most $C \log N$ operations. Given $Q \in CZ^\sharp$, compute an orthonormal frame $[u_1, \dots, u_{n-1}, u_Q]$ of $\mathbb{R}^n$, such that the following hold:*

(A) *The orthonormal frame $[u_1, \dots, u_{n-1}, u_Q]$ has the same orientation as*

$$[e_1, \dots, e_{n-1}, e_n].$$

(B) *Let ρ be the rigid motion given by the simple rotation*

$$\begin{cases} u_Q \mapsto e_n, \\ \textit{identity on } (\mathbb{R}u_Q \oplus \mathbb{R}e_n)^\perp, \end{cases}$$

and the translation $x \mapsto x - \mathrm{Rep}(Q)$. Then there exists a function $\varphi \in C^2(\mathbb{R}^{n-1})$ that satisfies (7.5) and (7.7) with this particular ρ.

Proof. Fix $Q \in CZ^\sharp$. This means that $E \cap 5Q \neq \varnothing$. In particular, $\mathrm{Rep}(Q)$ is defined, and by Lemma 7.3 (C),

$$x_0 := \mathrm{Rep}(Q) \in E \cap 5Q.$$

Computing x_0 requires at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage.

Let $\underline{\mathcal{A}}(x_0)$ be as in Lemma 6.1, and let $\sigma(\underline{\mathcal{A}}(x_0))$ be as in (6.5). By Lemma 6.1 (B) (with $\varphi \equiv 0$), any $P \in \sigma(\underline{\mathcal{A}}(x_0))$ must satisfy $P(x_0) = 0$. By Lemma 6.1 (A) and definitions (6.4), (6.5) of $\sigma(\underline{\mathcal{A}}(x_0))$, we see that $\sigma(\underline{\mathcal{A}}(x_0))$ is an n-dimensional parallelepiped in $\mathcal{P}$ centered at the zero polynomial. Therefore, we have

$$\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0)) = \mathrm{length}(\Delta_0),$$

where diam is defined in (7.1) and Δ_0 is one of the longest diagonals of $\sigma(\underline{\mathcal{A}}(x_0))$.

Set u_Q to be a unit vector parallel to Δ_0. Lemma 7.6 (B) then follows from Lemma 7.1.

Using the Gram–Schmidt process, we can compute the rest of the vectors $u_1, \dots, u_{n-1}$ such that $[u_1, \dots, u_{n-1}, u_Q]$ satisfies Lemma 7.6 (A). Computing $[u_1, \dots, u_{n-1}, u_Q]$ from $\sigma(\underline{\mathcal{A}}(x_0))$ uses elementary linear algebra, and requires at most C operations. ∎

Lemma 7.7. *After one-time work using at most $CN \log N$ operations and CN storage, we can perform the following task using at most $C \log N$ operations. Given $Q \in CZ^0$, we can compute a point $x_Q^\sharp \in Q$ such that*

$$\operatorname{dist}(x_Q^\sharp, E) \geq a_0 \delta_Q \tag{7.21}$$

for some $a_0 = a_0(n, A_1)$

Proof. Let $Q \in CZ^0$ be given.

Suppose $\text{Empty}(Q) = \text{True}$, with $\text{Empty}(\cdot)$ as in Lemma 7.3 (B). We set

$$x_Q^\sharp := \text{center}(Q).$$

It is clear that $x_Q^\sharp \in Q$ and (7.21) holds with $a_0 = 1/2$.

Suppose $\text{Empty}(Q) = \text{False}$. Let $x_0 := \text{Rep}(Q) \in E \cap 25Q$.

Suppose $x_0 \notin 5Q$. Then $E \cap 5Q = \varnothing$ by Lemma 7.3 (C). Again, we set

$$x_Q^\sharp := \text{center}(Q).$$

Suppose $x_0 \in 5Q$. This means that $Q \in CZ^\sharp$ with $CZ^\sharp$ as in (7.15). Let u_Q be as in Lemma 7.6.

By Lemma 7.1, we have $E \cap 5Q \subset \{(\bar{x}, \varphi(\bar{x})) : \bar{x} \in \mathbb{R}^{n-1}\}$ up to the rotation $u_Q \mapsto e_n$, and the function φ satisfies $|\nabla_{\bar{x}}^m \varphi| \le CA_1^{-1}\delta_Q^{1-m}$ for $m = 1, 2$, with A_1 as in Definition 7.2. Therefore, by the defining property of u_Q in Lemma 7.6, we have

$$E \cap 5Q \subset \left\{ y \in \mathbb{R}^n : |(y - x_0) \cdot u_Q| \le CA_1^{-1}|y - x_0| \right\} =: Z(x_0).$$

Suppose $\text{dist}(\text{center}(Q), Z(x_0)) \ge \delta_Q/1024$. We set

$$x_Q^\sharp := \text{center}(Q).$$

In this case, it is clear that $x_Q^\sharp \in Q$ and (7.21) holds with $a_0 = 2^{-10}$.

Suppose $\text{dist}(\text{center}(Q), Z(x_0)) < \delta_Q/1024$. We set

$$x_Q^\sharp := \text{center}(Q) + \frac{\delta_Q}{4} \cdot u_Q.$$

It is clear that $x_Q^\sharp \in Q$. For sufficiently large A_1, we also have $\text{dist}(x_Q^\sharp, Z(x_0)) \ge 2^{-10}\delta_Q$. Thus, (7.21) holds with $a_0 = 2^{-10}$, if we pick A_1 to be sufficiently large.

After one-time work using at most $CN \log N$ operations and CN storage, the procedure $\text{Empty}(Q)$ requires at most $C \log N$ operations by Lemma 7.3 (B); the procedure $\text{Rep}(Q)$ requires at most $C \log N$ operations by Lemma 7.3 (C); computing the vector u_Q requires at most $C \log N$ operations; and computing the distance between $\text{center}(Q)$ and $Z(x_0)$ is a routine linear algebra problem, and requires at most C operations. ∎

We now turn our attention to $CZ^{\sharp\sharp}$ as in (7.14).

Lemma 7.8. *Using at most $CN \log N$ operations and CN storage, we can compute the list $CZ^{\sharp\sharp}$ as in* (7.14).

Proof. This is a direct application of Lemma 7.3 (A) to each $x \in E$. ∎

The next lemma states that we can efficiently sort the data contained in cubes in $CZ^{\sharp\sharp}$.

Lemma 7.9. *Using at most $CN \log N$ operations and CN storage, we can do the following. For each $Q \in CZ^{\sharp\sharp}$ with $CZ^{\sharp\sharp}$ as in* (7.14)*, we can compute a list of points*

$$\text{Proj}_{u_Q^\perp}(E \cap (1 + c_G)Q - \text{Rep}(Q)) \subset \mathbb{R}^{n-1}.$$

Here, u_Q is as in Lemma 7.6, $u_Q^\perp$ is the subspace orthogonal to u_Q, $\text{Proj}_{u_Q^\perp}$ is the orthogonal projection onto $u_Q^\perp$, and $\text{Rep}(Q)$ is as in Lemma 7.3 (C).

Proof. By the bounded intersection property in Lemma 7.2, we have

(7.22) $$\#(CZ^{\sharp\sharp}) \le CN.$$

From the definitions of $CZ^{\sharp\sharp}$ and $CZ^{\sharp}$ in (7.14) and (7.15), we see that $CZ^{\sharp\sharp} \subset CZ^{\sharp}$. Therefore, we can compute $\text{Rep}(Q)$ and $u_Q^{\perp}$ for each $Q \in CZ^{\sharp\sharp}$ using at most $C \log N$ operations, by Lemma 7.3 (B) and Lemma 7.6.

Recall the proof of Lemma 7.8 that we can compute the list $CZ^{\sharp\sharp}$ by computing each $\Lambda(x)$ for $x \in E$, with $\Lambda(x)$ as in Lemma 7.3 (A). During this procedure, we can store the information $Q \ni x$ for $Q \in \Lambda(x)$.

By the bounded intersection property in Lemma 7.2, we have

(7.23) $$\sum_{Q \in CZ^{\sharp\sharp}} \#(E \cap (1 + c_G)Q) \le CN.$$

By Lemma 7.3 (A) and (7.23), we can compute the list

$$\{E \cap (1 + c_G)Q : Q \in CZ^{\sharp\sharp}\}$$

using at most $CN \log N$ operations and CN storage. Then, by Lemma 7.3 (C), Lemma 7.6, and (7.22), we can compute the list

(7.24) $$\text{Proj}_{u_Q^{\perp}}(E \cap (1 + c_G)Q - \text{Rep}(Q))$$

for each $Q \in CZ^{\sharp\sharp}$ using at most $CN \log N$ operations and CN storage. ■

Lemma 7.10. *Suppose we are given*

- $Q \in CZ^{\sharp\sharp}$,
- $E \cap (1 + c_G)Q$,
- u_Q *as in Lemma 7.6, and*
- $\text{Proj}_{u_Q^{\perp}}(E \cap (1 + c_G)Q - \text{Rep}(Q)) \subset \mathbb{R}^{n-1}$.

Let $\overline{N} := \#(E \cap (1 + c_G)Q)$. *After one-time work using at most* $C\overline{N} \log \overline{N}$ *operations and* $C\overline{N}$ *storage, we can compute a function* $\varphi \in C^2(\mathbb{R}^{n-1})$ *and a query algorithms with the following properties*:

(A) $\rho(E \cap (1 + c_G)Q) \subset \{(\overline{x}, \varphi(\overline{x})) : \overline{x} \in \mathbb{R}^{n-1}\}$. *Here,* ρ *is the rigid motion given by the simple rotation*

$$\begin{cases} u_Q \mapsto e_n, \\ \text{identity on } (\mathbb{R}u_Q \oplus \mathbb{R}e_n)^{\perp}, \end{cases}$$

and the translation $x \mapsto x - \text{Rep}(Q)$, *with* u_Q *as in Lemma 7.6.*

(B) $|\nabla_{\overline{x}}^m \varphi(\overline{x})| \le CA_1^{-1}\delta_Q^{1-m}$ *for* $m = 1, 2$, *with* A_1 *as in Definition 7.1.*

(C) $\text{Rep}(Q) = (\overline{0}, \varphi(\overline{0}))$, *where* $\text{Rep}(\cdot)$ *is the map in Lemma 7.3 (C) and* $\overline{0}$ *is the origin of* $\mathbb{R}^{n-1}$.

A query consists of a point $\overline{x} \in \text{Proj}_{u_Q^{\perp}}(1 + c_G)Q \subset \mathbb{R}^{n-1}$. *An answer to a query is the two-jet* $\mathfrak{J}_{\overline{x}}^{+}\varphi$. *The work to answer a query is* $C \log \overline{N}$.

Proof. We set $\bar{E} := \mathrm{Proj}_{u_Q^\perp}(E \cap (1+c_G)Q - \mathrm{Rep}(Q))$. We define a function $\varphi_0 \colon \bar{E} \to \mathbb{R}$ by

$$\varphi_0(\bar{x}) := (x - \mathrm{Rep}(Q)) \cdot u_Q, \quad \text{where } x = (\bar{x}, x_n) \in E.$$

For each $\bar{x} \in \mathbb{R}^{n-1}$, let $\bar{\Psi}_{\bar{x}} : \mathbb{R}^{\bar{N}} \to \bar{\mathcal{P}}^+$ be as in (5.2) in Section 5. We define the function φ by specifying

$$\mathcal{J}_{\bar{x}}^+ \varphi :\equiv \bar{\Psi}_{\bar{x}} \varphi_0.$$

We see from (5.2) (B) that $\varphi \in C^2(\mathbb{R}^{n-1})$. By construction, φ satisfies Lemma 7.10 (A) and (C). From (5.2) (B), Lemma 7.1, and an obvious rescaling, we see that Lemma 7.10 (B) follows.

By (5.2) (C), the one-time work uses at most $C\bar{N}\log\bar{N}$ operations and $C\bar{N}$ storage, and the work to answer a query is $C\log\bar{N}$. ∎

7.3. Compatible jets on CZ cubes

Recall that $\mathcal{B}(x,\delta) = \{P \in \mathcal{P} : |\partial^\alpha P(x)| \le \delta^{2-|\alpha|} \text{for } |\alpha| \le 1\}$.

Lemma 7.11. *Let $Q \subset CZ^0$ and let $x_Q^\sharp \in Q$ be as in Lemma 7.7. Let $k_{n,\mathrm{old}}^\sharp$ be as in Theorem 3.3 and let $k \ge (n+2)k_{n,\mathrm{old}}^\sharp$. Then*

$$\sigma^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp) \subset A \cdot \mathcal{B}(x_Q^\sharp, \delta_Q) \tag{7.25}$$

with $A = A(n, A_1, A_2)$.

Proof. First we note that (7.25) holds when $\delta_Q = A_2^{-1}$ with A_2 as in Definition 7.2.

Suppose $\delta_Q < A_2^{-1}$. Then Q^+ is not OK (see Definition 7.1). Combining this with Lemma 6.2, we see that $\#(E \cap 5Q) > 1$ and there exists $\hat{x} \in E \cap 5Q$ such that

$$\mathrm{diam}\,\sigma^\sharp(\hat{x}, k_{n,\mathrm{old}}^\sharp) < CA_1\delta_Q. \tag{7.26}$$

Using Helly's theorem (Theorem 3.4) and a similar argument as in Lemma 3.5, we see that given $P_0 \in \sigma^\sharp(x_Q^\sharp, k)$, there exists $P \in \sigma^\sharp(\hat{x}, k_{n,\mathrm{old}}^\sharp)$ such that

$$|\partial^\alpha(P_0 - P)(x_Q^\sharp)|,\ |\partial^\alpha(P_0 - P)(\hat{x})| \le C\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 1. \tag{7.27}$$

Combining (7.26) and (7.27), we see that any $P \in \sigma^\sharp(x_Q^\sharp, k)$ satisfies $|\partial^\alpha P(x_Q^\sharp)| \le C\delta_Q^{2-|\alpha|}$ for $|\alpha| \le 1$, which is precisely (7.25). ∎

The next lemma is crucial in controlling the derivatives when we patch together nearby local solutions.

Lemma 7.12. *Let $Q, Q' \in CZ^0$ and $x_Q^\sharp, x_{Q'}^\sharp$ be as Lemma 7.7. Let $f \colon E \to [-\tau, \tau]$ be given, and let $\Gamma_\tau^\sharp$ be as in (3.31). Let $k_{\mathrm{LIP}}^\sharp \ge (n+2)^2 k_{n,\mathrm{old}}^\sharp$ with $k_{n,\mathrm{old}}^\sharp$ as in Theorem 3.3. Let $P \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, M)$ and $P' \in \Gamma_\tau^\sharp(x_{Q'}^\sharp, k_{\mathrm{LIP}}^\sharp, f, M)$. Then for $x \in 25Q \cup 25Q'$,*

$$|\partial^\alpha(P - P')(x)| \le AM(|x_Q^\sharp - x_{Q'}^\sharp| + \delta_Q + \delta_{Q'}^\sharp)^{2-|\alpha|} \quad \textit{for } |\alpha| \le 2, \tag{7.28}$$

with $A = A(n, A_1, A_2)$.

Proof. We write A, A', etc., to denote constants depending only on n, A_1, A_2.

Set
$$\delta := \max\{|x^\sharp_Q - x^\sharp_{Q'}|, \delta_Q, \delta_{Q'}\}.$$
By Lemma 7.2 and Lemma 7.7, we see that

$$|x - x^\sharp_Q|, |x - x^\sharp_{Q'}| \leq A\delta \quad \text{for } x \in 25Q \cup 25Q'. \tag{7.29}$$

Lemma 3.5 applied to P yields $P_0 \in \Gamma^\sharp_\tau(x^\sharp_{Q'}, (n+1)k^\sharp_{n,\text{old}}, f, M)$ such that

$$|\partial^\alpha(P - P_0)(x^\sharp_{Q'})| \leq CM|x^\sharp_Q - x^\sharp_{Q'}| \leq AM\delta^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2. \tag{7.30}$$

Observe that[1]

$$\Gamma^\sharp_\tau(x^\sharp_Q, k^\sharp_{n,\text{old}}, f, M) - \Gamma^\sharp_\tau(x^\sharp_Q, k^\sharp_{n,\text{old}}, f, M) \subset CM \cdot \sigma^\sharp(x^\sharp_Q, (n+1)k^\sharp_{n,\text{old}}).$$

Therefore, we have

$$P' - P_0 \in CM \cdot \sigma^\sharp(x^\sharp_{Q'}, (n+1)k^\sharp_{n,\text{old}}). \tag{7.31}$$

By Lemma 7.11, we see that

$$|\partial^\alpha(P' - P_0)(x^\sharp_{Q'})| \leq CM\delta_{Q'}^{2-|\alpha|} \leq AM\delta^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2. \tag{7.32}$$

Taylor's theorem, combined with (7.29), (7.30) and (7.32), yields

$$|\partial^\alpha(P - P')(x')| \leq AM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2 \text{ and } x' \in 25Q'. \tag{7.33}$$

By Taylor's theorem, (7.29), and (7.33), we have

$$|\partial^\alpha(P - P')(x)| \leq AM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2 \text{ and } x \in 25Q. \tag{7.34}$$

Estimate (7.28) follows from (7.33) and (7.34). ∎

8. Local interpolation problem

8.1. Distortion estimate

Lemma 8.1. *Let $0 < \delta \leq 1$. Let $\Psi: \mathbb{R}^n \to \mathbb{R}^n$ be a C^2-diffeomorphism such that*

$$|\nabla^m \Psi(x)| \leq A\delta^{1-m} \quad \text{for } m = 1, 2 \text{ and } x \in \mathbb{R}^n.$$

Let $\Omega \subset \mathbb{R}^n$ be a C^2 diffeomorphic image of a dyadic cube and let $F \in C^2(\bar{\Omega})$. Suppose

$$|\partial^\alpha F(x)| \leq M\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2 \text{ and } x \in \Omega.$$

Then

$$|\partial^\alpha(F \circ \Psi)(x)| \leq C(n)AM\delta^{2-|\alpha|} \quad \text{for } |\alpha| \leq 2 \text{ and } x \in \Psi^{-1}(\bar{\Omega}). \tag{8.1}$$

[1]For $X, Y \subset V$ with V a vector space, we write $X - Y$ to denote the set $\{z : z = x - y, x \in X, y \in Y\}$.

Proof. We expand $\Psi = (\Psi_1, \dots, \Psi_n)$ in coordinates. Then

$$\partial_i (F \circ \Psi) = \sum_{k=1}^{n} \partial_i \Psi_k \cdot \partial_k F \circ \Psi,$$
$$\partial_{ij} (F \circ \Psi) = \sum_{k,l=1}^{n} c_{k,l} \cdot \partial_i \Psi_k \cdot \partial_j \Psi_l \cdot \partial^2_{kl} F \circ \Psi + \sum_{k=1}^{n} \partial^2_{ij} \Psi_k \cdot \partial_k F \circ \Psi.$$

Then (8.1) follows from the derivative estimates on F and Ψ. ∎

8.2. Local clusters

The next lemma shows how to relay local information to the point $x^\sharp_Q$.

Lemma 8.2. *Let $Q \in CZ^\sharp$. Let $x^\sharp_Q$ be as in Lemma 7.7. Let $x \in E \cap 5Q$. Let $\underline{\mathcal{A}}(x)$ be as in Lemma 6.1. Let $S(\underline{\mathcal{A}}(x))$ be as in (6.3). Let $k^\sharp_{n,\mathrm{old}}$ be as in Theorem 3.3. Then*

$$\sigma(x^\sharp_Q, S(\underline{\mathcal{A}}(x))) \subset A \cdot \sigma^\sharp(x^\sharp_Q, k^\sharp_{n,\mathrm{old}}) \tag{8.2}$$

with $A = A(n, A_1, A_2)$.

Proof. We write A, a, etc., to denote quantities depending only on n, A_1, A_2.

Fix x as in the hypothesis. By our choice of $x^\sharp_Q$ in Lemma 7.7, we have

$$|x^\sharp_Q - x| \geq a\delta_Q. \tag{8.3}$$

Let $P_0 \in \sigma(x^\sharp_Q, S(\underline{\mathcal{A}}(x)))$. By the definition of σ, there exists $\varphi \in C^2(\mathbb{R}^n)$ with $\|\varphi\|_{C^2(\mathbb{R}^n)} \leq 1$, $\varphi|_{S(\underline{\mathcal{A}}(x))} = 0$, and $\mathcal{J}_{x^\sharp_Q}\varphi \equiv P_0$. Set $P :\equiv \mathcal{J}_x \varphi$. Then

$$P \in \sigma(x, S(\underline{\mathcal{A}}(x))).$$

Since $x \in E$, by Lemma 6.2, we have

$$P \in C \cdot \sigma^\sharp(x, k^\sharp_{n,\mathrm{old}}).$$

Let $S \subset E$ with $\#(S) \leq k^\sharp_{n,\mathrm{old}}$. By the definition of $\sigma^\sharp$ in (3.29) and Taylor's theorem (Theorem 3.1 (A)), there exists a Whitney field $\vec{P} = (P, (P^y)_{y \in S}) \in W(S \cup \{x\})$, with $\|\vec{P}\|_{W(S \cup \{x\})} \leq C$ and $P^y(y) = 0$ for $y \in S$.

Consider another Whitney field $\vec{P}_0 = (P_0, (P^y)_{y \in S}) \in W(S \cup \{x^\sharp_Q\})$ defined by replacing P by P_0 in $\vec{P}$ and leaving other entries unchanged. By Whitney's extension theorem (Theorem 3.2 (B)), it suffices to show that $\vec{P}_0$ satisfies

$$P^y(y) = 0 \quad \text{for } y \in S, \tag{8.4}$$
$$\|\vec{P}_0\|_{W(S \cup \{x^\sharp_Q\})} \leq C. \tag{8.5}$$

Note that (8.4) is obvious by construction.

We turn to (8.5). Since $P_0 \equiv \mathcal{J}_{x_Q^\sharp}\varphi$ and $P \equiv \mathcal{J}_x\varphi$, Taylor's theorem implies

$$(8.6) \qquad |\partial^\alpha(P - P_0)(x_Q^\sharp)|, |\partial^\alpha(P - P_0)(x)| \le C|x - x_Q^\sharp|^{2-|\alpha|} \quad \text{for } |\alpha| \le 1.$$

Since the Whitney field $\vec{P} = (P, (P^y)_{y\in S})$ satisfies $\|\vec{P}\|_{W(S\cup\{x\})} \le C$, we have

$$(8.7) \qquad \|(P^y)_{y\in S}\|_{W(S)} \le C$$

and

$$(8.8) \qquad |\partial^\alpha(P - P^y)(x)|, |\partial^\alpha(P - P^y)(y)| \le C|x - y|^{2-|\alpha|} \quad \text{for } |\alpha| \le 2, y \in S.$$

Applying the triangle inequality to (8.6) and (8.8), and using (8.3), we see that

$$(8.9) \qquad |\partial^\alpha(P_0 - P^y)(x_Q^\sharp)|, |\partial^\alpha(P_0 - P^y)(y)| \le A|x_Q^\sharp - y|^{2-|\alpha|} \quad \text{for } |\alpha| \le 1.$$

Moreover, since $P_0 \in \sigma(x_Q^\sharp, S(\underline{\mathcal{A}}(x)))$, we have

$$(8.10) \qquad |\partial^\alpha P_0(x_Q^\sharp)| \le 1 \quad \text{for } |\alpha| \le 1.$$

Then (8.5) follows from (8.7), (8.9) and (8.10). ■

Let $Q \in CZ^\sharp$ with $CZ^\sharp$ as in (7.15). Let $\underline{\mathcal{A}}(x)$, $x \in E$ be as in Lemma 6.1. Let $S(\underline{\mathcal{A}}(x))$ be as in (6.3). Let $\text{Rep}(Q)$ be as in Lemma 7.3 (C). Let $x_Q^\sharp$ be as in Lemma 7.7. We set

$$(8.11) \qquad S^\sharp(Q) := S(\underline{\mathcal{A}}(\text{Rep}(Q))) \cup \{x_Q^\sharp\}.$$

Thanks to Lemma 6.1 (A), we have

$$(8.12) \qquad \#S^\sharp(Q) \le C(n).$$

8.3. Transition jet

Recall Section 3.3. For $S \subset E$ and $\tau > 0$, we define the following functions:

$$(8.13) \qquad \begin{aligned} &\mathcal{L}\colon W(S) \to [0,\infty),\\ &(P^x)_{x\in S} \mapsto \sum_{x\in S, |\alpha|\le 1} |\partial^\alpha P(x)| + \sum_{x,y\in S, x\ne y, |\alpha|\le 1} \frac{|\partial^\alpha(P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}},\\ &\mathcal{M}_\tau\colon W(S,\tau) \to [0,\infty),\\ &(P^x)_{x\in S} \mapsto \sum_{x\in S, |\alpha|=1} \frac{|\partial^\alpha P^x|^2}{\tau - P(x)} + \frac{|\partial^\alpha P^x|^2}{\tau + P(x)}. \end{aligned}$$

We adopt the conventions that $\frac{0}{0} = 0$ and $\frac{a}{0} = \infty$ for $a > 0$. Note that $\mathcal{L}$ is a norm on $W(S)$.

Let $Q \in CZ^\sharp$ with $CZ^\sharp$ as in (7.15). Let $x_Q^\sharp$ be as in Lemma 7.7. Let $S^\sharp(Q)$ be as in (8.11). We set $x_0 := \text{Rep}(Q)$ with Rep as in Lemma 7.3 (C). Note that $x_0 \in S^\sharp(Q)$, thanks to Lemma 6.2 and the definition of σ.

Let $f\colon E \to [-\tau, \tau]$ be given. We consider the following spaces:

(8.14)
$$\begin{aligned}
\mathbb{A}_f^{-\tau} &:= \big\{(P^y)_{y\in S^\sharp(Q)} \in W(S^\sharp(Q),\tau) : \\
&\qquad P^{x_Q^\sharp} \equiv -\tau \text{ and } P^x(x) = f(x) \text{ for } x \in S^\sharp(Q)\cap E\big\},\\
\mathbb{A}_f^{\tau} &:= \big\{(P^y)_{y\in S^\sharp(Q)} \in W(S^\sharp(Q),\tau) : \\
&\qquad P^{x_Q^\sharp} \equiv \tau \text{ and } P^x(x) = f(x) \text{ for } x \in S^\sharp(Q)\cap E\big\},\\
\mathbb{A}_f^{0} &:= \big\{(P^y)_{y\in S^\sharp(Q)\cap E} \in W(S^\sharp(Q)\cap E,\tau) : \\
&\qquad P^x(x) = f(x) \text{ for } x \in S^\sharp(Q)\cap E\big\}.
\end{aligned}$$

Note that $\mathbb{A}_f^{-\tau}$ and $\mathbb{A}_f^{\tau}$ are affine subspaces of $W(S^\sharp(Q))$, and $\mathbb{A}_f^{0}$ is an affine subspace of $W(S^\sharp(Q)\cap E)$. All three depend only on τ and $f|_{S^\sharp(Q)\cap E}$.

We will be considering the following minimization problems:

MP$(-\tau)$: Let $S = S^\sharp(Q)$ in (8.13). Minimize $\mathcal{L} + \mathcal{M}_\tau$ over $\mathbb{A}_f^{-\tau}$.

MP(τ): Let $S = S^\sharp(Q)$ in (8.13). Minimize $\mathcal{L} + \mathcal{M}_\tau$ over $\mathbb{A}_f^{\tau}$.

MP(0): Let $S = S^\sharp(Q)\cap E$ in (8.13). Minimize $\mathcal{L} + \mathcal{M}_\tau$ over $\mathbb{A}_f^{0}$.

For $\star = -\tau, \tau, 0$, we say a Whitney field $\vec{P} \in \mathbb{A}_f^{\star}$ is an *approximate minimizer* of **MP**$(\star)$ if

$$(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}) \le C(n)\cdot \inf\{(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}') : \vec{P}' \in \mathbb{A}_f^{\star}\}.$$

We make the following important remark that will be referenced in various later places.

Remark 8.1. By Section 3.3, **MP**$(\star)$ can be reformulated as a convex quadratic programming problem with affine constraint and can be solved efficiently using at most $C(n)$ operations, since the size of $S^\sharp(Q)$ is controlled. Thus, we can find the approximate minimizers for **MP**$(\star)$ using at most $C(n)$ operations. Computing $S^\sharp(Q)$ requires at most $C \log N$ operations after one-time work using $CN \log N$ operations and CN storage, since it involves computing the point $x_Q^\sharp$ as in Lemma 7.7.

For future reference, we fix these approximate minimizers:

(8.15) For $\star = -\tau, \tau, 0$, let $\vec{P}[Q,\star]$ be the approximate minimizer of **MP**$(\star)$ solved via the method in Section 3.3.

Notice that the approximate minimizer for **MP**(0) contains no information at $x_Q^\sharp$. The next lemma takes care of this gap.

Lemma 8.3. *Let $Q \in CZ^\sharp$. Let $x_Q^\sharp$ be as in Lemma 7.7. Let $f\colon E \to [-\tau,\tau]$, with $\|f\|_{C^2(E,\tau)} \le M$. Let $\vec{P} = \vec{P}[Q,0]$ be as in (8.15) above with $\star = 0$. Let $x_0 :=$ Rep(Q) with* Rep(Q) *as in Lemma 7.3* (C). *Let $\mathcal{T}_{w,\tau}$ be the τ-constrained Whitney extension operator as in Theorem 3.2 associated with the singleton $S = \{x_0\}$. Then*

$$\mathcal{J}_{x_Q^\sharp} \circ \mathcal{T}_{w,\tau}(P^{x_0}) \in \Gamma_\tau(x_Q^\sharp, S^\sharp(Q)\cap E, f, CM).$$

Here, C depends only on n.

Proof. We set $P^\sharp :\equiv \mathcal{J}_{x_Q^\sharp} \circ \mathcal{T}_{w,\tau}(P^{x_0})$. We adjoin $P^\sharp$ to $\vec{P}$ to form

$$\vec{P}^\sharp := (P^\sharp, \vec{P}) \in W(S^\sharp(Q)).$$

By Theorem 3.2 (B), it suffices to show that $\vec{P}^\sharp \in W(S^\sharp(Q), \tau)$ and $\|\vec{P}^\sharp\|_{W(S^\sharp(Q),\tau)} \le CM$.

Since $\vec{P}$ is an approximate minimizer of **MP**(0) and $\|f\|_{C^2(E,\tau)} \le M$, we have

$$\|\vec{P}\|_{W(S^\sharp(Q)\cap E,\tau)} \le CM. \tag{8.16}$$

Since P^{x_0} is a component of $\vec{P}$,

$$\mathcal{T}_{w,\tau}(P^{x_0}) \in C^2(\mathbb{R}^n, \tau) \quad \text{and} \quad \|\mathcal{T}_{w,\tau}(P^{x_0})\| \le CM.$$

Therefore, we have (recall Definition 3.1)

$$P^\sharp \in \mathcal{K}_\tau(x_Q^\sharp, CM). \tag{8.17}$$

Thus, $\vec{P}^\sharp \in W(S^\sharp(Q), \tau)$.

For $x \in S^\sharp(Q) \cap E$ and $|\alpha| \le 1$, we have

$$|\partial^\alpha(P^x - P^\sharp)(x)| \le |\partial^\alpha(P^x - P^{x_0})(x)| + |\partial^\alpha(P^{x_0} - \mathcal{J}_{x_Q^\sharp} \circ \mathcal{T}_{w,\tau}(P^{x_0}))(x)|.$$

Using (8.16) to estimate the first term and Taylor's theorem to the second, we have

$$|\partial^\alpha(P^x - P^\sharp)(x)| \le CM(|x - x_0| + |x_0 - x_Q^\sharp|)^{2-|\alpha|} \le C'M|x - x_Q^\sharp|^{2-|\alpha|}. \tag{8.18}$$

For the last inequality, we use the fact that $\operatorname{dist}(x_Q^\sharp, E) \ge c\delta_Q$, thanks to Lemma 7.7. Applying Taylor's theorem to estimate (8.18), we have

$$|\partial^\alpha(P^x - P^\sharp)(x_Q^\sharp)| \le CM|x - x_Q^\sharp|^{2-|\alpha|}. \tag{8.19}$$

Combining (8.16)–(8.19), we see that $\|\vec{P}^\sharp\|_{W(S^\sharp(Q),\tau)} \le CM$. Lemma 8.3 is proved. ∎

We fix a large parameter A_T exceeding a constant depending only on n. For $Q \in CZ^\sharp$ and $x_Q^\sharp$ as in Lemma 7.7, we define a map

$$\mathcal{T}_{\tau,Q} : C^2(E,\tau) \times [0,\infty) \to \mathcal{P} \tag{8.20}$$

via the following rules. Let $(f, M) \in C^2(E,\tau) \times [0,\infty)$ be given.

(8.21) Let $\mathcal{L}$ and $\mathcal{M}_\tau$ be as in (8.13) with $S = S^\sharp(Q)$, and let $\vec{P} = \vec{P}[Q, -\tau]$ be as in (8.15). Suppose $(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}) \le A_T M$. Then we set $\mathcal{T}_{\tau,Q}(f, M) :\equiv -\tau$.

(8.22) Let $\vec{P} = \vec{P}[Q, \tau]$ be as in (8.15). Suppose $(\mathcal{L} + \mathcal{M}_\tau)(\vec{P}) \le A_T M$. Then we set $\mathcal{T}_{\tau,Q}(f, M) :\equiv \tau$.

(8.23) Suppose both conditions (8.21) and (8.22) above fail. Let $\vec{P} = \vec{P}[Q, 0]$ be as in (8.15). We set $\mathcal{T}_{\tau,Q}(f, M) :\equiv \mathcal{J}_{x_Q^\sharp} \circ \mathcal{T}_{w,\tau}(P^{\operatorname{Rep}(Q)})$. Here, $P^{\operatorname{Rep}(Q)}$ is the component of $\vec{P}$ corresponding to the point $\operatorname{Rep}(Q)$, with $\operatorname{Rep}(Q)$ as in Lemma 7.3 (C), and $\mathcal{T}_{w,\tau}$ is the τ-constrained Whitney extension operator in Theorem 3.2 associated with the singleton $S = \{\operatorname{Rep}(Q)\}$.

To wit, we first test if the putative solution is close to the lower range threshold, and then we test for the upper threshold if the first test fails. Lastly, we conclude that the solution is situated away from both thresholds if both of the previous tests fail.

The main lemma of this section is the following.

Lemma 8.4. *Let $Q \in CZ^\sharp$ and $x_Q^\sharp$ be as in Lemma 7.7. Let $k_{\mathrm{LIP}}^\sharp = (n+2)^2 k_{n,\mathrm{old}}^\sharp$ with $k_{n,\mathrm{old}}^\sharp$ as in Theorem 3.3. Let $\mathcal{T}_{\tau,Q}$ be as in (8.20). Let $(f, M) \in C^2(E,\tau) \times [0,\infty)$ with $\|f\|_{C^2(E,\tau)} \le M$. Then*

$$\mathcal{T}_{\tau,Q}(f, M) \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, AM)$$

with $A = A(n, A_T)$.

Proof. We write A, A', etc., to denote quantities depending only on n and A_T.

Since $\|f\|_{C^2(E,\tau)} \le M$, we see that $\Gamma_\tau^\sharp(x, k_{\mathrm{LIP}}^\sharp, f, 2M) \neq \varnothing$. Therefore, the hypothesis of Lemma 3.7 is satisfied with $k = k_{\mathrm{LIP}}^\sharp$.

Recall that $\mathcal{T}_{\tau,Q}$ is defined in terms of a series of rules (8.21), (8.22), (8.23). We analyze them in this order.

Rule for (8.21). Suppose $\mathcal{T}_{\tau,Q}(f, M)$ is defined in terms of (8.21). By Theorem 3.2, there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le AM$, $F|_{S^\sharp(Q)\cap E} = f$, and $\mathcal{J}_{x_Q^\sharp} \equiv -\tau$. Let Rep be the map in Lemma 7.3 (C), and recall that $\mathrm{Rep}(Q) \in S^\sharp(Q) \cap 5Q$. By Taylor's theorem, we have

$$\tau + f(\mathrm{Rep}(Q)) = \tau + F(\mathrm{Rep}(Q)) \le AM\delta_Q^2.$$

Lemma 3.7 then implies $-\tau \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, AM)$.

Rule for (8.22). Suppose $\mathcal{T}_{\tau,Q}(f, M)$ is defined in terms of (8.22). By Theorem 3.2, there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le AM$, $F|_{S^\sharp(Q)\cap E} = f$, and $\mathcal{J}_{x_Q^\sharp} \equiv \tau$. By Taylor's theorem, we have

$$\tau - f(\mathrm{Rep}(Q)) = \tau - F(\mathrm{Rep}(Q)) \le AM\delta_Q^2.$$

Lemma 3.7 then implies $\tau \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, AM)$.

Rule for (8.23). Suppose $\mathcal{T}_{\tau,Q}(f, M)$ is defined in terms of (8.23). Recall that we have chosen A_T to be sufficiently large in (8.21) and (8.22). Taylor's theorem then implies, with A_{perturb} as in Lemma 3.7,

$$\min\{\tau - f(x), \tau + f(x)\} \ge CA_{\mathrm{perturb}} M\delta_Q^2 \quad \text{for } x \in E \cap 5Q.$$

Thus, the hypothesis of Lemma 3.7 (A) is satisfied.

Since $\|f\|_{C^2(E,\tau)} \le M$, there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $\|F\|_{C^2(\mathbb{R}^n)} \le 2M$, $F|_E = f$, and

$$\mathcal{J}_{x_Q^\sharp} F \in \Gamma_\tau(x_Q^\sharp, E, f, 2M).$$

By Lemma 8.3,

$$\mathcal{T}_{\tau,Q}(f, M) \in \Gamma_\tau(x_Q^\sharp, S^\sharp(Q) \cap E, f, CM).$$

Thus, by Lemma 6.2, Lemma 8.2, and the definition of $S^\sharp(Q)$ in (8.11), we see that

$$\mathcal{J}_{x_Q^\sharp} F - \mathcal{T}_{\tau,Q}(f,M) \in CM \cdot \sigma(x_Q^\sharp, S^\sharp(Q) \cap E) \subset C'M \cdot \sigma^\sharp(x_Q^\sharp, k_{n,\mathrm{old}}^\sharp) \subset C''M \cdot \sigma^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp).$$

By Lemma 7.11, we see that

$$\mathcal{J}_{x_Q^\sharp} F - \mathcal{T}_{\tau,Q}(f,M) \in CM \cdot \mathcal{B}(x_Q^\sharp, \delta_Q).$$

For sufficiently large A_{perturb}, Lemma 3.7 (A) implies

$$\mathcal{T}_{\tau,Q}(f,M) \in \mathcal{J}_{x_Q^\sharp} F + CM \cdot \mathcal{B}(x_Q^\sharp, \delta_Q) \subset \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, C'M).$$

Lemma 8.4 is proved. ■

8.4. Fixing the parameters A_*

(8.24) In Definitions 7.1 and 7.2, we fix $A_1, A_2 \gg C(n)$ so that Lemma 7.1 holds.

(8.25) Let A_{perturb} and A_{flat} be as in Lemma 3.7. We fix A_{perturb} so that Lemma 3.7 (A) holds. We then fix $A_{\mathrm{flat}} \gg C(n) A_{\mathrm{perturb}}$.

(8.26) Let A_T be the parameter associated with the map (8.20). We fix $A_T = c(n) A_{\mathrm{flat}}$.

Henceforth, we treat all the parameters A_*, a_* appeared in the previous sections as controlled constants and write C_*, c_* instead.

8.5. Local interpolation problem with a prescribed jet

Recall $CZ^{\sharp\sharp}$ from (7.14). Also recall that c_G is a small dyadic number fixed in Lemma 7.2.

Lemma 8.5. *Let $Q \in CZ^{\sharp\sharp}$. There exists a map*

$$\mathcal{E}_{\tau,Q} : C^2(E,\tau) \times [0,\infty) \to C^2((1+c_G)Q)$$

such that the following hold:

(A) *Given $(f,M) \in C^2(E,\tau) \times [0,\infty)$ with $\|f\|_{C^2(E,\tau)} \le M$, we have*

(1) $-\tau \le \mathcal{E}_{\tau,Q}(f,M) \le \tau$ *on* $(1+c_G)Q$,

(2) $\mathcal{E}_{\tau,Q}(f,M)(x) = f(x)$ *for* $x \in E \cap (1+c_G)Q$,

(3) $\|\mathcal{E}_{\tau,Q}(f,M)\|_{C^2((1+c_G)Q)} \le C(n)M$, *and*

(4) $\mathcal{J}_{x_Q^\sharp} \circ \mathcal{E}_{\tau,Q}(f,M) \equiv \mathcal{T}_{\tau,Q}(f,M)$, *with $x_Q^\sharp$ as in Lemma 7.7 and $\mathcal{T}_{\tau,Q}$ as in* (8.20).

(B) *For each $x \in (1+c_G)Q$, there exists a set $S_Q(x) \subset E$ with $S_Q(x) \le D(n)$, such that given $f, g \in C^2(E,\tau)$ with $f|_{S_Q(x)} = g|_{S_Q(x)}$, we have*

$$\partial^\alpha \mathcal{E}_{\tau,Q}(f,M)(x) = \partial^\alpha \mathcal{E}_{\tau,Q}(g,M)(x) \quad \text{for } |\alpha| \le 2 \text{ and } M \ge 0. \tag{8.27}$$

Proof. We fix $k_{\mathrm{LIP}}^\sharp = (n+2)^2 k_{n,\mathrm{old}}^\sharp$ with $k_{n,\mathrm{old}}^\sharp$ as in Theorem 3.3.

The essential ingredients in the construction of the map $\mathcal{E}_{\tau,Q}$ are as follows.

- Let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be the C^2-diffeomorphism associated with Q defined by $\Phi \circ \rho^{-1} = (\bar{x}, x_n - \varphi(\bar{x}))$, with φ and ρ as in Lemma 7.10. See also Lemma 7.1. In particular, Φ satisfies the estimate

$$(8.28) \qquad |\nabla^m \Phi(x)| \le C\delta_Q^{1-m} \quad \text{for } x \in \mathbb{R}^n \text{ and } m = 1, 2.$$

- Let $x_Q^\sharp$ and $c_0 = a_0$ be as in Lemma 7.7. Let $\psi \in C^2(\mathbb{R}^n)$ be a cutoff function such that

$$(8.29) \qquad \begin{gathered} 0 \le \psi \le 1, \quad \psi \equiv 1 \text{ near } x_Q^\sharp, \quad \operatorname{supp}(\psi) \subset B(x_Q^\sharp, c_0\delta_Q), \\ |\partial^\alpha \psi| \le C(n)\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 2. \end{gathered}$$

- Define an indicator function

$$(8.30) \qquad \Delta(f, M, Q) := \begin{cases} 1 & \text{if } \mathcal{T}_{\tau,Q}(f, M) \text{ is not the constant polynomial } \pm\tau, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\overline{\mathcal{E}}_\tau$ and $\overline{\mathcal{E}}_\infty$ be as in (5.1) and (5.2) associated with the set $\Phi(E \cap (1 + c_G)Q) \subset \mathbb{R}^{n-1} \times \{0\}$. We identify $\mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$.
- Let $\mathcal{V}$ be the vertical extension map defined by $\mathcal{V}(\overline{F})(\bar{x}, x_n) := \overline{F}(\bar{x})$ for an $(n-1)$-variable function $\overline{F}$, $(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

We begin with conclusion (A). We define

$$(8.31) \qquad \mathcal{E}_{\tau,Q}(f, M) := \Delta\mathcal{T}_{\tau,Q}(f, M) + (1 - \psi) \cdot \widetilde{\mathcal{E}}_{\tau,Q}(f, M),$$

where

$$\widetilde{\mathcal{E}}_{\tau,Q}(f, M) :=$$

$$\underbrace{\overbrace{\Big(\mathcal{V} \circ \big[(\Delta\overline{\mathcal{E}}_\infty + (1-\Delta)\overline{\mathcal{E}}_\tau(\cdot, C_0M))\ \underbrace{\big((f - \Delta\mathcal{T}_{\tau,Q}(f, M))|_{E\cap(1+c_G)Q} \circ \Phi^{-1}\big)\ \big]}_{\text{local flattening}}}^{\text{vertical extension}}}_{(n-1)\text{-dimensional extension}}\Big) \circ \Phi.$$

In the formula above, $\Delta = \Delta(f, M, Q)$ and C_0 is some large controlled constant depending only on n. We also identify $\mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$.

First we note that the map $\widetilde{\mathcal{E}}_{\tau,Q}(f, M)$ is well defined. Indeed, when $\Delta = 1$, the operator in effect is $\overline{\mathcal{E}}_\infty$, which can be applied to any $f\colon \mathbb{R}^{n-1} \times \{0\} \supset \Phi(E \cap (1 + c_G)Q) \to \mathbb{R}$; when $\Delta = 0$, the operator in effect is $\overline{\mathcal{E}}_\tau(\cdot, C_0M)$, and the argument is $f|_{E\cap(1+c_G)Q} \circ \Phi^{-1}$, which has domain $\mathbb{R}^{n-1} \times \{0\}$ and range $[-\tau, \tau]$.

We proceed to verify (A1)–(A4) in the following four claims.

Verification of (A1). Suppose $\mathcal{T}_{\tau,Q}(f, M) \equiv \pm\tau$. By (8.30), $\Delta = 0$. Formula (8.31) simplifies to

$$(8.32) \qquad \mathcal{E}_{\tau,Q}(f, M) = (1 - \psi) \cdot \big(\mathcal{V} \circ \overline{\mathcal{E}}_\tau(f|_{E\cap(1+c_G)Q} \circ \Phi^{-1})\big) \circ \Phi.$$

By the induction hypothesis (5.1), $-\tau \leq \overline{\mathcal{E}}_\tau(f|_{E\cap(1+c_G)Q} \circ \Phi^{-1}) \leq \tau$. On the other hand, left composition with $\mathcal{V}$ and right composition with Φ do not alter the range, and $0 \leq 1-\psi \leq 1$. Therefore, we have $-\tau \leq \mathcal{E}_{\tau,Q}(f, M) \leq \tau$.

Now we analyze the more delicate case when $\mathcal{T}_{\tau,Q}(f, M)$ is not the constant polynomial $\pm\tau$. In this case, formula (8.31) becomes

$$\begin{aligned}(8.33)\quad \mathcal{E}_{\tau,Q}(f, M) &= \mathcal{T}_{\tau,Q}(f, M)\\ &\quad + (1-\psi)\cdot\big(\mathcal{V}\circ\overline{\mathcal{E}}_\infty((f-\mathcal{T}_{\tau,Q}(f,M))|_{E\cap(1+c_G)Q}\circ\Phi^{-1})\big)\circ\Phi.\end{aligned}$$

By Lemma 8.4, we have

$$\mathcal{T}_{\tau,Q}(f, M) \in \Gamma^\sharp_\tau(x^\sharp_Q, k^\sharp_{\mathrm{LIP}}, f, CM). \tag{8.34}$$

By the assumption $\|f\|_{C^2(E,\tau)} \leq M$, we know that there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $F = f$ on E, $\|F\|_{C^2(\mathbb{R}^n)} \leq CM$, and

$$\mathcal{J}_{x^\sharp_Q} F \in \Gamma_\tau(x^\sharp_Q, E, f, CM) \subset \Gamma^\sharp_\tau(x^\sharp_Q, k^\sharp_{\mathrm{LIP}}, f, CM). \tag{8.35}$$

Thanks to Lemma 7.12, Taylor's theorem, (8.34), and (8.35), we see that

$$|\partial^\alpha(F-\mathcal{T}_{\tau,Q}(f,M))(x)| \leq CM\delta_Q^{2-|\alpha|} \quad \text{for all } |\alpha|\leq 2,\ x\in(1+c_G)Q. \tag{8.36}$$

Using Lemma 8.1, (8.28), and (8.36), we have

$$|\partial^\alpha((F-\mathcal{T}_{\tau,Q}(f,M))\circ\Phi^{-1})(x)| \leq CM\delta_Q^{2-|\alpha|} \quad \text{for all } |\alpha|\leq 2,\ x\in\Phi((1+c_G)Q). \tag{8.37}$$

Restricting $(F-\mathcal{T}_{\tau,Q}(f,M))\circ\Phi^{-1}$ to $\mathbb{R}^{n-1}\times\{0\}\cong\mathbb{R}^{n-1}$, we see that

$$\|(f-\mathcal{T}_{\tau,Q}(f,M))\circ\Phi^{-1}\|_{C^2(\Phi(E\cap(1+c_G)Q))} \leq C_* M. \tag{8.38}$$

Here, C_* is a constant depending only on n, and the trace norm is taken in $\mathbb{R}^{n-1}$.

Note that the vertical extension map $\mathcal{V}$ does not increase the C^2 norm. Therefore, by taking $C_0 \geq C_*$ in (8.31) and (8.38), the induction hypothesis (5.1) implies

$$\|G\|_{C^2(\Phi((1+c_G)Q))} \leq CM, \tag{8.39}$$

where

$$G := \mathcal{V}\circ\overline{\mathcal{E}}_\infty((f-\mathcal{T}_{\tau,Q}(f,M))\circ\Phi^{-1}, C_0M)$$

and the norm is evaluated on $\mathbb{R}^n$. In fact, by using (8.37), (8.38), and a standard rescaling, we have a stronger estimate

$$|\partial^\alpha G(x)| \leq CM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha|\leq 2 \text{ and } x\in\Phi((1+c_G)Q). \tag{8.40}$$

Lemma 8.1, (8.28), and (8.40) yield

$$|\partial^\alpha(G\circ\Phi)(x)| \leq CM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha|\leq 2 \text{ and } x\in(1+c_G)Q. \tag{8.41}$$

Now thanks to (8.41) and the fact that $0 \le \psi \le 1$, if $\tau - |\mathcal{T}_{\tau,Q}(f, M)| \ge AM\delta_Q^2$ on $(1 + c_G)Q$ for some sufficiently large parameter A, then we can conclude that $-\tau \le \mathcal{E}_{\tau,Q}(f, M) \le \tau$ on $(1 + c_G)Q$. We proceed to examine the value of $\mathcal{T}_{\tau,Q}(f, M)$ on $(1 + c_G)Q$.

Since we assume that $\mathcal{T}_{\tau,Q}(f, M)$ is not the constant polynomial $\pm\tau$, $\mathcal{T}_{\tau,Q}(f, M)$ must be defined according to (8.23) (recall (8.20)). In particular, both assumptions in (8.21) and (8.22) fail. Lemma 3.6 then implies that

$$\tau - |f(x)| \ge A_0 M\delta_Q^2 \quad \text{for } x \in E \cap (1 + c_G)Q. \tag{8.42}$$

Here, $A_0 \ge c(n) \cdot (A_T^{1/2} - 1)$ with A_T as in the definition (8.20) of $\mathcal{T}_{\tau,Q}$.

Recall from Lemma 8.4 that $\mathcal{T}_{\tau,Q}(f, M) \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\mathrm{LIP}}^\sharp, f, CM)$. We claim that

$$\tau - |\mathcal{T}_{\tau,Q}(f, M)(x_Q^\sharp)| \ge c(n) \cdot (\sqrt{A_0} - 1)M\delta_Q^2. \tag{8.43}$$

Suppose toward a contradiction, that $\tau - |\mathcal{T}_{\tau,Q}(f, M)(x_Q^\sharp)| < A_{\mathrm{bad}} M\delta_Q^2$ for some A_{bad} to be determined. For any $x \in E \cap (1 + c_G)Q$, there exists $F \in C^2(\mathbb{R}^n, \tau)$ such that $\|F\|_{C^2(\mathbb{R}^n)} \le CM$, $F(x) = f(x)$, and $\mathcal{J}_{x_Q^\sharp} F \equiv \mathcal{T}_{\tau,Q}(f, M) \in \mathcal{K}_\tau(x_Q^\sharp, C'M)$, with $\mathcal{K}_\tau$ as in Definition 3.1. Thus, (3.6) and Taylor's theorem imply

$$\begin{aligned} |\nabla F(x)| &\le |\nabla F(x_Q^\sharp)| + C\|F\|_{C^2(\mathbb{R}^n)}\delta_Q \\ &\le C'(\sqrt{A_{\mathrm{bad}}} + 1)M\delta_Q^2 \quad \text{for } x \in (1 + c_G)Q \end{aligned} \tag{8.44}$$

and

$$\begin{aligned} \tau - |F(x)| &\le (\tau - |F(x_Q^\sharp)|) + C\delta_Q \cdot \sup_{y\in(1+c_G)Q}|\nabla F| \\ &\le C_{\mathrm{bad}}(\sqrt{A_{\mathrm{bad}}} + 1)^2 M\delta_Q^2 \quad \text{for } x \in (1 + c_G)Q. \end{aligned} \tag{8.45}$$

If $A_{\mathrm{bad}} < C_{\mathrm{bad}}(\sqrt{A_0} - 1)$, with A_0 as in (8.42) and C_{bad} as in (8.44) and (8.45), we see that (8.44) and (8.45) contradict (8.42). Therefore, (8.43) holds.

Thanks to Lemma 3.1 and (8.43), we have

$$\mathrm{dist}(x_Q^\sharp, \{\mathcal{T}_{\tau,Q}(f, M) = 0\}) \ge c(\sqrt{A_0} - 1)\delta_Q. \tag{8.46}$$

For sufficiently large A_0, i.e., sufficiently large A_T chosen in (8.25) and (8.26), we have

$$\tau - |\mathcal{T}_{\tau,Q}(f, M)(x)| \ge CM(\sqrt{A_0} - 1)\delta_Q^2 \quad \text{for } x \in (1 + c_G)Q. \tag{8.47}$$

Combining (8.41) and (8.47), we see that $-\tau \le \mathcal{E}_{\tau,Q}(f, M) \le \tau$ on $(1 + c_G)Q$, thus (A1) is established.

Verification of (A2). Conclusion (A2) follows from the following observation.

- The support of ψ is disjoint from E by (8.29).
- When $\Delta = 0$, formula (8.31) takes the form of (8.32). Here, the $(n-1)$-dimensional extension operator in effect is $\overline{\mathcal{E}}_\tau$, which interpolates the values of f.
- When $\Delta = 1$, formula (8.31) takes the form of (8.33). Here, the $(n-1)$-dimensional extension operator in effect is $\overline{\mathcal{E}}_\infty$, which interpolates the values of $f - \mathcal{T}_{\tau,Q}(f, M)$. Note that we added $\mathcal{T}_{\tau,Q}(f, M)$ back to the final extension.

Verification of (A3). We first deal with the easy case when $\mathcal{T}_{\tau,Q}(f,M)$ is not the constant polynomial $\pm\tau$. Then formula (8.31) becomes (8.33). By Lemma 8.4, $\mathcal{T}_{\tau,Q}(f,M) \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\text{LIP}}^\sharp, f, CM)$. Thus,

$$\|\mathcal{T}_{\tau,Q}(f,M)\|_{C^2((1+c_G)Q)} \le CM. \tag{8.48}$$

Recall from (8.29) that ψ satisfies the estimate $|\partial^\alpha \psi| \le C\delta_Q^{-|\alpha|}$ for $|\alpha| \le 2$. Using Lemma 8.1, (8.41), and (8.48) to estimate (8.33), we have $\|\mathcal{E}_{\tau,Q}(f,M)\|_{C^2((1+c_G)Q)} \le CM$.

We now move on to the case where $\mathcal{T}_{\tau,Q}(f,M) \equiv \pm\tau$. We will analyze the case $\mathcal{T}_{\tau,Q}(f,M) \equiv -\tau$. The case $\mathcal{T}_{\tau,Q}(f,M) \equiv \tau$ is similar.

In the present setting, formula (8.31) is simplified to (8.32), $\mathcal{T}_{\tau,Q}(f,M)$ in (8.20) is defined using (8.21), and by Lemma 3.7 we have $-\tau \equiv \mathcal{T}_{\tau,Q}(f,M) \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\text{LIP}}^\sharp, f, CM)$. Thus,

$$f(x) + \tau \le CM\delta_Q^2 \quad \text{for } x \in E \cap (1+c_G)Q. \tag{8.49}$$

Since $\|f\|_{C^2(E,\tau)} \le M$, there exists $F \in C^2(\mathbb{R}^n, \tau)$ with $F|_E = f$ and $\|F\|_{C^2(\mathbb{R}^n)} \le CM$. In particular, $\mathcal{J}_x F \in \mathcal{K}_\tau(x, CM)$ for each $x \in E \cap (1+c_G)Q$, with $\mathcal{K}_\tau$ as in Definition 3.1. Using Taylor's theorem and property (3.2) of $\mathcal{K}_\tau$, we see that

$$|\partial^\alpha F(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for } x \in (1+c_G)Q. \tag{8.50}$$

Using Lemma 8.1, (8.28), and (8.50), we have

$$|\partial^\alpha (F \circ \Phi^{-1})(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 2 \text{ and } x \in \Phi((1+c_G)Q). \tag{8.51}$$

Restricting $F \circ \Phi^{-1}$ to $\mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$, we see that

$$\|f \circ \Phi^{-1}\|_{C^2(\Phi(E \cap (1+c_G)Q))} \le C_* M. \tag{8.52}$$

Here, C_* is a constant depending only on n, and the trace norm is taken in $\mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$.

The vertical extension map does not increase the C^2 norm. By taking $C_0 \ge C_*$ in (8.31) and (8.52), the induction hypothesis (5.1) implies

$$\|H\|_{C^2((1+c_G)Q)} \le CM, \tag{8.53}$$

where

$$H := \mathcal{V} \circ \overline{\mathcal{E}}_\tau(f \circ \Phi^{-1}, C_0 M).$$

In fact, by using (8.51) and (8.52), together with a standard rescaling, we arrive at the stronger estimate

$$|\partial^\alpha H(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 2 \text{ and } x \in \Phi((1+c_G)Q). \tag{8.54}$$

Lemma 8.1, (8.28), and (8.54) then imply

$$|\partial^\alpha (G \circ \Phi)(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for } |\alpha| \le 2 \text{ and } x \in (1+c_G)Q. \tag{8.55}$$

Recall that the cutoff function φ satisfies $|\partial^\alpha \psi| \le C\delta_Q^{-|\alpha|}$ for $|\alpha| \le 2$. Combining this with (8.55), we can conclude that $\|\mathcal{E}_{\tau,Q}(f,M)\|_{C^2((1+c_G)Q)} \le CM$.

We have established (A3).

Verification of (A4). Since $\psi \equiv 1$ near $x_Q^\sharp$ by (8.29), we have, by Lemma 8.4,

$$\mathcal{J}_{x_Q^\sharp} \circ \mathcal{E}_{\tau,Q}(f, M) \equiv \mathcal{T}_{\tau,Q}(f, M) \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\text{LIP}}^\sharp, f, CM).$$

This prove conclusion (A4).

Therefore, Lemma 8.5 (A) holds.

Verification of (B). Now we turn to Lemma 8.5 (B).

Fix $x \in (1 + c_G)Q$. We begin by defining the set $S_Q(x)$ to be

$$S_Q(x) := (S^\sharp(Q) \cap E) \cup \overline{S}(\text{Proj}_{u_Q^\perp}(x - \text{Rep}(Q))), \tag{8.56}$$

with $S^\sharp(Q)$ as in (8.11), $\overline{S}(\cdot)$ as in (5.1) (D), and u_Q as in Lemma 7.6. Thanks to (8.12) and (5.1) (D), we have $\#S_Q(x) \le D(n)$.

Let $M \ge 0$. Let $f, g \in C^2(E, \tau)$ with $f = g$ on $S_Q(x)$. Since $f = g$ on $S^\sharp(Q) \cap E$, we see from the definition of the map $\mathcal{T}_{\tau,Q}(\cdot, M)$ that

$$\mathcal{T}_{\tau,Q}(f, M) \equiv \mathcal{T}_{\tau,Q}(g, M) \quad \text{for } M \ge 0. \tag{8.57}$$

As a consequence, we have

$$\Delta_f := \Delta(f, M, Q) = \Delta(g, M, Q) =: \Delta_g. \tag{8.58}$$

The assumption that $f = g$ on $\overline{S}(\text{Proj}_{u_Q^\perp}(x - \text{Rep}(Q)))$ along with (8.57) implies

$$\begin{aligned} &(f - \Delta_f \mathcal{T}_{\tau,Q}(f, M)) \circ \Phi^{-1} \\ &\quad = (g - \Delta_g \mathcal{T}_{\tau,Q}(g, M)) \circ \Phi^{-1} \quad \text{on } \overline{S}(\text{Proj}_{u_Q^\perp}(x - \text{Rep}(Q))). \end{aligned} \tag{8.59}$$

It is easy to see that (8.27) follows from substituting (8.57)–(8.59) into (8.31). This proves Lemma 8.5 (B).

Lemma 8.5 is proved. ∎

Next, we analyze the algorithmic complexity of Lemma 8.5. Recall that $\mathcal{P}^+$ denotes the vector space of polynomials with degree no greater than two and $\mathcal{J}_x^+$ denotes the two-jet at x.

Lemma 8.6. *Let $Q \in CZ^{\sharp\sharp}$ with $CZ^{\sharp\sharp}$ as in (7.14). Then there exists a collection of maps*

$$\{\Xi_{\tau,x,Q} : \tau \in [0, \infty),\ x \in (1 + c_G)Q\},$$

where

$$\Xi_{\tau,x,Q} : C^2(E, \tau) \times [0, \infty) \to \mathcal{P}^+$$

for each $x \in (1 + c_G)Q$, such that the following hold:

(A) *There exists a controlled constant $D(n)$ such that for each $x \in \mathbb{R}^n$, the map $\Xi_{\tau,x,Q}(\cdot, \cdot)\colon C^2(E, \tau) \times [0, \infty) \to \mathcal{P}^+$ is of depth at most D. Moreover, the source of $\Xi_{\tau,x,Q}$ is independent of τ.*

(B) *Suppose we are given $(f, M) \in C^2(E, \tau) \times [0, \infty)$ with $\|f\|_{C^2(E,\tau)} \le M$. Then there exists a function $F_Q \in C^2((1 + c_G)Q, \tau)$ such that:*

(1) *$\mathcal{J}_x^+ F_Q \equiv \Xi_{\tau,x,Q}(f, M)$ for all x in the interior of $(1 + c_G)Q$,*

(2) $\|F_Q\|_{C^2((1+c_G)Q)} \le CM$,
(3) $F_Q(x) = f(x)$ *for* $x \in E \cap (1+c_G)Q$, *and*
(4) $\mathcal{J}_{x_Q^\sharp} F_Q \in \Gamma_\tau^\sharp(x_Q^\sharp, k_{\text{LIP}}^\sharp, f, CM)$, *with* $x_Q^\sharp$ *as in Lemma 7.7.*

Here, C *depends only on* n.

(C) *There is an algorithm that takes the given data set* E*, performs one-time work, and then responds to queries. A query consists of a pair* (τ, x, Q)*, and the response to the query is the map* $\Xi_{\tau,x,Q}$*, given in its efficient representation. The one-time work takes* $C_1 N \log N$ *operations and* $C_2 N$ *storage. The work to answer a query is* $C_3 \log N$*. Here,* C_1, C_2, C_3 *depend only on* n.

Proof. Let $\mathcal{E}_{\tau,Q}$ be as in Lemma 8.5, defined by the formula (8.31). For convenience, we repeat the formula here. We set

$$\mathcal{E}_{\tau,Q}(f,M) := \Delta\mathcal{T}_{\tau,Q}(f,M) + (1-\psi)\cdot\widetilde{\mathcal{E}}_{\tau,Q}(f,M), \tag{8.60}$$

where

$$\widetilde{\mathcal{E}}_{\tau,Q}(f,M) := \underbrace{\overbrace{\big(\mathcal{V}\circ\big[(\Delta\overline{\mathcal{E}}_\infty + (1-\Delta)\overline{\mathcal{E}}_\tau(\cdot, C_0M))\,\underbrace{((f-\Delta\mathcal{T}_{\tau,Q}(f,M))\big|_{E\cap(1+c_G)Q}\circ\Phi^{-1})}_{\text{local flattening}}\big]}^{\text{vertical extension}}}_{(n-1)\text{-dimensional extension}}\big)\circ\Phi.$$

Recall that $\mathcal{J}_x^+$ denotes the two-jet of a function at x. We would like to define

$$\Xi_{\tau,x,Q} :\equiv \mathcal{J}_x^+ \circ \mathcal{E}_{\tau,Q} \quad \text{for } x \in (1+c_G)Q. \tag{8.61}$$

Lemma 8.6 (A) follows from Lemma 8.5 (B), and Lemma 8.6 (B) follows from Lemma 8.5 (A).

Now we examine Lemma 8.6 (C). Suppose we have performed the necessary one-time work using at most $CN\log N$ operations and CN storage. See Remark 8.2 below. Fix Q as in the hypothesis, and let $x \in (1+c_G)Q$ be given.

- We compute the point
$$\bar{y}(x) := \text{Proj}_{u_Q^\perp}(x - \text{Rep}(Q)).$$
Here, $\text{Proj}_{u_Q^\perp}$ is the orthogonal projection onto the hyperplane $u_Q^\perp$ orthogonal to u_Q; u_Q is the vector as in Lemma 7.6; $\text{Rep}(Q)$ is the map in Lemma 7.3 (C). Thanks to Lemma 7.6 and Lemma 7.9, Step 1 requires at most $C \log N$ operations.
- Let ρ be the rotation specified by $e_n \to u_Q$. We can compute $\mathcal{J}_x^+\rho$ using at most C operations.
- Let u_Q and $\text{Proj}_{u_Q^\perp}$ be as in Step 1. We set
$$\bar{E}_Q := \text{Proj}_{u_Q^\perp}(E \cap (1+c_G)Q - \text{Rep}(Q)) \subset \mathbb{R}^{n-1}.$$

Recall from Lemma 7.9 that we can compute all of the unsorted lists $\bar{E}_Q$ associated with each $Q \in CZ^{\sharp\sharp}$ using at most $CN \log N$ operations and CN storage, which is included in the one-time work.

- Let $C^2(\bar{E}_Q, \tau)$ be the $(n-1)$-dimensional trace class. Let $\bar{\Xi}_\tau$ and $\bar{\Psi}$ be the maps as in (5.1) and (5.2) associated with the set $\bar{E}_Q$. The one-time work to pre-process $\bar{E}_Q$ involves $CN_Q \log N_Q$ operations and CN_Q storage, with $N_Q := \#(E \cap (1 + c_G)Q)$. The work to answer a query is $C \log N_Q$. Recall that an answer to a query is the map $\bar{\Xi}_\tau$ or $\bar{\Psi}$, given its efficient representation (Definition 1.1).

Remark 8.2. Here, we make the crucial remark that the one-time work to pre-process *all* the $\bar{E}_Q$ uses at most

$$\sum_{Q \in CZ^{\sharp\sharp}} CN_Q \log N_Q \leq C' N \log N$$

operations and

$$\sum_{Q \in CZ^{\sharp\sharp}} CN_Q \leq C'' N$$

storage, thanks to the bounded intersection property of CZ^0 in Lemma 7.2.

- We can compute the map $\mathcal{T}_{\tau,Q}(f, M)$ in (8.20) from $S^\sharp(Q)$ using at most C operations (Remark 8.1). Computing $S^\sharp(Q)$ requires at most $C \log N$ operations, thanks to Lemma 7.3 (C) and Lemma 7.7.
- The jets of Φ and Φ^{-1} can be computed from the jets of φ, with φ as in Lemma 7.10, in $C \log N_Q$ operations, with $N_Q = \#(E \cap (1 + c_G)Q)$.
- For appropriate choice of cutoff function ψ, we can compute the jets of ψ using at most C operations. See Section 9.1 below.

Summarizing all of the above, we see that Lemma 8.6 (C) holds. ■

9. Theorems 1.1 and 1.2 and Algorithm 1

9.1. Partitions of unity

Recall that $\mathcal{J}_x^+$ denotes the two-jet of a function twice continuously differentiable near $x \in \mathbb{R}^n$.

We can construct a partition of unity $\{\theta_Q : Q \in CZ^0\}$ that satisfies the following properties:

- $0 \leq \theta_Q \leq 1$ for each $Q \in CZ^0$.
- $\sum_{Q \in CZ^0} \theta_Q \equiv 1$.
- $\operatorname{supp}(\theta_Q) \subset (1 + c_G/2)Q$ for each $Q \in CZ^0$.
- For each $Q \in CZ^0$, $|\partial^\alpha \theta_Q| \leq C\delta_Q^{2-|\alpha|}$ for $|\alpha| \leq 2$.
- After one-time work using at most $CN \log N$ operations and CN storage, we can answer queries as follows: Given $x \in \mathbb{R}^n$ and $Q \in CZ^0$, we return $\mathcal{J}_x^+ \theta_Q$. The work to answer a query is $C \log N$.

See Section 28 of [21] for details.

9.2. Proof of Theorem 1.1

Proof. Let $CZ^0, CZ^\sharp, CZ^{\sharp\sharp}, CZ^{\text{empty}}$ be as in Definition 7.2, (7.14)–(7.16). We have

$$CZ^{\sharp\sharp} \subset CZ^\sharp \subset CZ^0 \quad \text{and} \quad CZ^{\text{empty}} = \{Q \in CZ^0 \setminus CZ^\sharp : \delta_Q \le A_2^{-1}\}.$$

For $Q \in CZ^0$, we define a map $\mathcal{E}^0_{\tau,Q}: C^2(E,\tau) \times [0,\infty) \to C^2((1+c_G)Q)$ via the following rules:

- Suppose $Q \in CZ^{\sharp\sharp}$. We set $\mathcal{E}^0_{\tau,Q} := \mathcal{E}_{\tau,Q}$, with $\mathcal{E}_{\tau,Q}$ as in Lemma 8.5.
- Suppose $Q \in CZ^\sharp \setminus CZ^{\sharp\sharp}$. We set $\mathcal{E}^0_{\tau,Q} := \mathcal{T}_*^{x_Q^\sharp} \circ \mathcal{T}_{\tau,Q}$, where $\mathcal{T}_*^{x_Q^\sharp}$ is as in Lemma 3.2 (B) with $x_0 = x_Q^\sharp$ and $\mathcal{T}_{\tau,Q}$ is as in (8.20).
- Suppose $Q \in CZ^{\text{empty}}$. We set $\mathcal{E}^0_{\tau,Q} := \mathcal{T}_*^{x^\sharp_{\mu(Q)}} \circ \mathcal{T}_{\tau,\mu(Q)}$, where μ is as in Lemma 7.5, $\mathcal{T}_*^{x^\sharp_{\mu(Q)}}$ is as in Lemma 3.2 (B) with $x_0 = x^\sharp_{\mu(Q)}$, and $\mathcal{T}_{\tau,\mu(Q)}$ is as in (8.20).
- Suppose $Q \in CZ^0 \setminus (CZ^\sharp \cup CZ^{\text{empty}})$. We set $\mathcal{E}^0_{\tau,Q} := 0$.

Let $\{\theta_Q : Q \in CZ^0\}$ be a partition of unity subordinate to CZ^0 as in Section 9.1 above. We define $\mathcal{E}_\tau$ by the formula

$$\mathcal{E}_\tau(f,M)(x) := \sum_{Q \in CZ^0} \theta_Q(x) \cdot \mathcal{E}^0_{\tau,Q}(f,M)(x). \tag{9.1}$$

Since $-\tau \le \mathcal{E}^0_{\tau,Q}(f,M) \le \tau$ for each Q, we see that $-\tau \le \mathcal{E}_\tau(f,M) \le \tau$.

Since $\mathcal{E}^0_{\tau,Q}(f,M) = f$ on $E \cap (1+c_G)Q$ for each $Q \in CZ^0$, we see that $\mathcal{E}_\tau(f,M) = f$ on E.

To estimate the C^2 norm of $\mathcal{E}_\tau(f,M)$, we need the following lemma.

Lemma 9.1. *Let $\mathcal{E}^0_{\tau,Q}$ be defined as above for each $Q \in CZ^0$. Let $x \in Q \in CZ^0$ and $Q' \in CZ^0$ such that $Q' \leftrightarrow Q$. Let $(f,M) \in C^2(E,\tau) \times [0,\infty)$ with $\|f\|_{C^2(E,\tau)} \le M$. We have*

$$|\partial^\alpha(\mathcal{E}^0_{\tau,Q}(f,M) - \mathcal{E}^0_{\tau,Q'}(f,M))(x)| \le CM\delta_Q^{2-|\alpha|} \quad \textit{for } |\alpha| \le 2. \tag{9.2}$$

We proceed with the proof of Theorem 1.1 assuming the validity of Lemma 9.1, postponing the latter's rather tedious proof till the end of the section.

Now we estimate the C^2 norm of $\mathcal{E}_\tau(f,M)$. Fix $x \in \mathbb{R}^n$. Let $Q(x) \in CZ^0$ denote the cube such that $Q \ni x$. We can write

$$\begin{aligned} \partial^\alpha \mathcal{E}_\tau(f,M)(x) = &\sum_{Q' \leftrightarrow Q(x)} \theta_{Q'}(x) \cdot \partial^\alpha \mathcal{E}^\sharp_{Q'}(f,M)(x) \\ &+ \sum_{Q' \leftrightarrow Q(x), 0 < \beta \le \alpha} \partial^\beta \theta_{Q'}(x) \cdot \partial^{\alpha-\beta}(\mathcal{E}^0_{\tau,Q}(f,M) - \mathcal{E}^0_{\tau,Q'}(f,M))(x). \end{aligned} \tag{9.3}$$

Now, using the bounded intersection property in Lemma 7.2 and Lemma 8.5 to estimate the first sum in (9.3), and Lemma 9.1 to estimate the second sum, we can conclude that

$$\|\mathcal{E}_\tau(f,M)\|_{C^2(\mathbb{R}^n)} \le CM.$$

This proves Theorem 1.1 (A).

Now we turn to Theorem 1.1 (B). Fix $x \in \mathbb{R}^n$. Let $Q(x) \in CZ^0$ be such that $Q(x) \ni x$. We define

$$S(x) := \Big(\bigcup_{Q' \leftrightarrow Q(x), Q' \in CZ^{\sharp\sharp}} S_{Q'}(x) \Big) \cup \Big(\bigcup_{Q' \leftrightarrow Q(x), Q' \in CZ^{\text{empty}}} S^{\sharp}(Q') \Big).$$

Here, $S_{Q'}(x)$ is as in Lemma 8.5 (B) and $S^{\sharp}(Q')$ is as in (8.11).

Thanks to Lemma 7.2, Lemma 8.5 (B), and (8.12), we have

$$\#S(x) \le D(n).$$

Given $f, g \in C^2(E, \tau)$ with $f = g$ on $S(x)$, we see from the construction of $S(x)$ and $\mathcal{E}^0_{\tau,Q}$ that

$$\partial^\alpha \mathcal{E}^0_{\tau,Q}(f, M) = \partial^\alpha \mathcal{E}^0_{\tau,Q}(g, M) \quad \text{for } |\alpha| \le 2 \text{ and } M \ge 0.$$

From formula (9.1), we see that

$$\partial^\alpha \mathcal{E}_{\tau}(f, M) = \partial^\alpha \mathcal{E}_{\tau}(g, M) \quad \text{for } |\alpha| \le 2 \text{ and } M \ge 0.$$

Thus, we have established Theorem 1.1 (B).

The proof of Theorem 1.1 is complete once we prove Lemma 9.1.

We return to Lemma 9.1.

Proof of Lemma 9.1. We fix a number $k^{\sharp}_{\text{LIP}} = k^{\sharp}_{n,\text{old}}$ with $k^{\sharp}_{n,\text{old}}$ as in Theorem 3.3. Fix α with $|\alpha| \le 2$ and expand

$$\begin{aligned}
(9.4)\qquad & |\partial^\alpha (\mathcal{E}^0_{\tau,Q}(f, M) - \mathcal{E}^0_{\tau,Q'}(f, M))(x)| \\
&\le |\partial^\alpha (\mathcal{E}^0_{\tau,Q}(f, M) - \mathcal{J}_{x^{\sharp}_Q} \mathcal{E}^0_{\tau,Q}(f, M))(x)| \\
&\quad + |\partial^\alpha (\mathcal{E}^0_{\tau,Q'}(f, M) - \mathcal{J}_{x^{\sharp}_{Q'}} \mathcal{E}^0_{\tau,Q'}(f, M))(x)| \\
&\quad + |\partial^\alpha (\mathcal{J}_{x^{\sharp}_Q} \mathcal{E}^0_{\tau,Q}(f, M) - \mathcal{J}_{x^{\sharp}_{Q'}} \mathcal{E}^0_{\tau,Q'}(f, M))(x)| \\
&=: \eta_1 + \eta_2 + \eta_3.
\end{aligned}$$

By Taylor's theorem,

$$(9.5)\qquad \eta_1, \eta_2 \le CM\delta_Q^{2-|\alpha|}.$$

It remains to show that

$$(9.6)\qquad \eta_3 \le CM\delta_Q^{2-|\alpha|}.$$

Recall from Lemma 7.2 that $C^{-1}\delta_Q \le \delta_{Q'} \le C\delta_Q$, and observe that η_3 is symmetric with respect to $Q \leftrightarrow Q'$. It then suffices to analyze the following cases.

Case 1. Suppose either Q or Q' belongs to $CZ^0 \setminus (CZ^{\sharp} \cup CZ^{\text{empty}})$. Then (9.6) follows from Lemmas 7.2, 8.4, 8.5, and Taylor's theorem. For the rest of the analysis, we assume that neither Q nor Q' belongs to $CZ^0 \setminus (CZ^{\sharp} \cup CZ^{\text{empty}})$.

Case 2. Suppose both $Q, Q' \in CZ^{\sharp\sharp}$. Recall from Lemmas 8.4 and 8.5 that

$$\mathcal{J}_{x^\sharp_Q}\mathcal{E}^0_{\tau,Q}(f,M) \in \Gamma^\sharp_\tau(x^\sharp_Q, k^\sharp_{\mathrm{LIP}}, f, CM),$$
$$\mathcal{J}_{x^\sharp_{Q'}}\mathcal{E}^0_{\tau,Q'}(f,M) \in \Gamma^\sharp_\tau(x^\sharp_{Q'}, k^\sharp_{\mathrm{LIP}}, f, CM).$$

Then (9.6) follows from Lemma 7.12.

Case 3. Suppose $Q \in CZ^{\sharp\sharp}$ and $Q' \in CZ^\sharp \setminus CZ^{\sharp\sharp}$. This means that

- $\mathcal{E}^0_{\tau,Q} = \mathcal{E}_{\tau,Q}$ as in Lemma 8.5, and
- $\mathcal{E}^0_{\tau,Q'} = \mathcal{T}^{x^\sharp_{Q'}}_* \circ \mathcal{T}_{\tau,Q'}$, with $\mathcal{T}^{x^\sharp_{Q'}}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,Q'}$ as in (8.20).

We expand

$$\begin{aligned}\eta_3 &\le |\partial^\alpha(\mathcal{J}_{x^\sharp_Q}\mathcal{E}^0_{\tau,Q}(f,M) - \mathcal{T}_{\tau,Q'}(f,M))(x)| \\ &\quad + |\partial^\alpha(\mathcal{T}_{\tau,Q'}(f,M) - \mathcal{T}^{x^\sharp_{Q'}}_* \circ \mathcal{T}_{\tau,Q'}(f,M))(x)| \\ &=: \eta_{3,1} + \eta_{3,2}.\end{aligned} \tag{9.7}$$

Taylor's theorem implies

$$\eta_{3,2} \le CM\delta_Q^{2-|\alpha|}. \tag{9.8}$$

On the other hand, we have (by Lemma 8.5 (A4) and Lemmas 7.12, 8.4, respectively)

$$\eta_{3,1} = |\partial^\alpha(\mathcal{T}_{\tau,Q}(f,M) - \mathcal{T}_{\tau,Q'}(f,M))(x)| \le CM\delta_Q^{2-|\alpha|}. \tag{9.9}$$

We see that (9.6) follows from (9.7)–(9.9).

Case 4. Suppose $Q \in CZ^{\sharp\sharp}$ and $Q' \in CZ^{\mathrm{empty}}$. This means that

- $\mathcal{E}^0_{\tau,Q} = \mathcal{E}_{\tau,Q}$ as in Lemma 8.5, and
- $\mathcal{E}^0_{\tau,Q'} = \mathcal{T}^{x^\sharp_{\mu(Q')}}_* \circ \mathcal{T}_{\tau,\mu(Q')}$, with $\mathcal{T}^{x^\sharp_{\mu(Q')}}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,\mu(Q')}$ as in (8.20).

Thanks to Lemma 7.5 (B), we have

$$|x^\sharp_Q - x^\sharp_{\mu(Q')}|,\ |x - x^\sharp_Q|,\ |x - x^\sharp_{\mu(Q')}| \le C\delta_Q. \tag{9.10}$$

We expand

$$\begin{aligned}\eta_3 &\le |\partial^\alpha(\mathcal{J}_{x^\sharp_Q}\mathcal{E}^0_{\tau,Q}(f,M) - \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \\ &\quad + |\partial^\alpha(\mathcal{T}_{\tau,\mu(Q')}(f,M) - \mathcal{T}^{x^\sharp_{Q'}}_* \circ \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \\ &=: \eta_{3,1} + \eta_{3,2}.\end{aligned} \tag{9.11}$$

Taylor's theorem and (9.10) implies

$$\eta_{3,2} \le CM\delta_Q^{2-|\alpha|}. \tag{9.12}$$

On the other hand, we have (by Lemma 8.5 (A4) and Lemmas 7.12, 8.4, respectively)

$$\eta_{3,1} = |\partial^\alpha(\mathcal{T}_{\tau,Q}(f,M) - \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \le CM\delta_Q^{2-|\alpha|}. \tag{9.13}$$

We see that (9.6) follows from (9.11)–(9.13).

Case 5. Suppose $Q \in CZ^\sharp \setminus CZ^{\sharp\sharp}$ and $Q' \in CZ^{\text{empty}}$. This means that

- $\mathcal{E}^0_{\tau,Q} = \mathcal{T}^{x^\sharp_Q}_* \circ \mathcal{T}_{\tau,Q}$, with $\mathcal{T}^{x^\sharp_Q}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,Q}$ as in (8.20),
- $\mathcal{E}^0_{\tau,Q'} = \mathcal{T}^{x^\sharp_{\mu(Q')}}_* \circ \mathcal{T}_{\tau,\mu(Q')}$, with $\mathcal{T}^{x^\sharp_{\mu(Q')}}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,\mu(Q')}$ as in (8.20).

Notice that $\mathcal{J}_{x^\sharp_Q} \circ \mathcal{E}^0_{\tau,Q} = \mathcal{T}_{\tau,Q}$, so that Taylor's theorem, Lemma 7.12, and (9.10) imply

$$\eta_3 = |\partial^\alpha(\mathcal{T}_{\tau,Q}(f,M) - \mathcal{T}^{x^\sharp_{Q'}}_* \circ \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \le CM\delta_Q^{2-|\alpha|}.$$

This is precisely (9.6).

Case 6. Suppose both $Q, Q' \in CZ^{\text{empty}}$. By construction,

- $\mathcal{E}^0_{\tau,Q} = \mathcal{T}^{x^\sharp_{\mu(Q)}}_* \circ \mathcal{T}_{\tau,\mu(Q)}$, with $\mathcal{T}^{x^\sharp_{\mu(Q)}}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,\mu(Q)}$ as in (8.20),
- $\mathcal{E}^0_{\tau,Q'} = \mathcal{T}^{x^\sharp_{\mu(Q')}}_* \circ \mathcal{T}_{\tau,\mu(Q')}$, with $\mathcal{T}^{x^\sharp_{\mu(Q')}}_*$ as in Lemma 3.2 and $\mathcal{T}_{\tau,\mu(Q')}$ as in (8.20).

Thanks to Lemma 7.5 (B) and the assumption that $Q' \leftrightarrow Q$, we have

$$|x^\sharp_{\mu(Q)} - x^\sharp_{\mu(Q')}|,\ |x - x^\sharp_{\mu(Q)}|,\ |x - x^\sharp_{\mu(Q')}| \le C\delta_Q. \tag{9.14}$$

We expand

$$\begin{aligned}\eta_3 &\le |\partial^\alpha(\mathcal{T}^{x^\sharp_{\mu(Q)}}_* \circ \mathcal{T}_{\tau,\mu(Q)}(f,M) - \mathcal{T}_{\tau,\mu(Q)}(f,M))(x)| \\ &\quad + |\partial^\alpha(\mathcal{T}^{x^\sharp_{\mu(Q')}}_* \circ \mathcal{T}_{\tau,\mu(Q')}(f,M) - \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \\ &\quad + |\partial^\alpha(\mathcal{T}_{\tau,\mu(Q)}(f,M) - \mathcal{T}_{\tau,\mu(Q')}(f,M))(x)| \\ &=: \eta_{3,1} + \eta_{3,2} + \eta_{3,3}.\end{aligned} \tag{9.15}$$

It follows from Taylor's theorem, Lemma 3.2, and (9.14) that

$$\eta_{3,1}, \eta_{3,2} \le CM\delta_Q^{2-|\alpha|}. \tag{9.16}$$

On the other hand, Lemma 7.12, Lemma 8.5 (A4), and (9.14) imply

$$\eta_{3,3} \le CM\delta_Q^{2-|\alpha|}. \tag{9.17}$$

Therefore, (9.6) follows from (9.15)–(9.17).

We have analyzed all the possible cases. Therefore, (9.6) holds.

Finally, (9.2) follows from (9.5) and (9.6). The proof of Lemma 9.1 is complete. ■

The proof of Theorem 1.1 is complete. ■

9.3. Proof of Theorem 1.2

Proof. We put ourselves in the setting of the proof of Theorem 1.1 in Section 9.2. In particular, recall formula (9.1) and the assignment rules for $\mathcal{E}^0_{\tau,Q}$.

For each $x \in \mathbb{R}^n$, we define

$$\Xi_{\tau,x}(f,M) :\equiv \mathcal{J}^+_x \circ \mathcal{E}_\tau(f,M) = \sum_{Q \in \Lambda(x)} \mathcal{J}^+_x \theta_Q \odot^+_x \mathcal{J}^+_x \circ \mathcal{E}^0_{\tau,Q}(f,M). \tag{9.18}$$

In the formula above, $\mathcal{J}^+_x$ denotes the two-jet at x, $\odot^+_x$ is the multiplication on the ring of two-jets $\mathcal{R}^+_x$, and $\Lambda(x)$ is as in Lemma 7.3 (A), i.e., $\Lambda(x) = \{Q \in CZ^0 : (1+c_G)Q \ni x\}$.

Theorem 1.2 (A) follows from Theorem 1.1 (B). Theorem 1.2 (B) follows from Theorem 1.1 (A).

We now turn to Theorem 1.2 (C). Recall from the proof of Theorem 1.1 in Section 9.2 that $\mathcal{E}^0_{\tau,Q}$ can take the following forms:

- $\mathcal{E}^0_{\tau,Q} = \mathcal{E}_{\tau,Q}$ as in Lemma 8.5 for $Q \in CZ^{\sharp\sharp}$.
- $\mathcal{E}^0_{\tau,Q} = \mathcal{T}_*^{x_Q^\sharp} \circ \mathcal{T}_{\tau,Q}$, with $\mathcal{T}_*^{x^\sharp_{\mu(Q)}}$ as in Lemma 3.2 and $\mathcal{T}_{\tau,Q}$ as in (8.20), for $Q \in CZ^\sharp \setminus CZ^{\sharp\sharp}$.
- $\mathcal{E}^0_{\tau,Q} = \mathcal{T}_*^{x^\sharp_{\mu(Q)}} \circ \mathcal{T}_{\tau,\mu(Q)}$, with μ as in Lemma 7.5, for $Q \in CZ^{\text{empty}}$.
- $\mathcal{E}^0_{\tau,Q} \equiv 0$ for $Q \in CZ^0 \setminus CZ^{\text{empty}}$.

Suppose we have performed the necessary one-time work using at most $CN \log N$ operations and CN storage. Note that this includes the work and storage involved in Lemma 6.1, Lemma 7.3, and Remark 8.2 in the proof of Lemma 8.6.

- By Lemma 7.3 (A) and Section 9.1, we can compute $\Lambda(x)$ and $\{\mathcal{J}_x^+\theta_Q : Q \in \Lambda(x)\}$ using at most $C \log N$ operations.
- By Lemma 8.6, we can compute
$$\{\mathcal{J}_x^+ \circ \mathcal{E}_{\tau,Q} : Q \in CZ^{\sharp\sharp} \cap \Lambda(x)\}$$
using at most $C \log N$ operations. Recall Remark 8.2 in the proof of Lemma 8.6.
- By Lemma 7.7 and Remark 8.1, we can compute
$$\left\{\mathcal{J}_x^+ \circ \mathcal{T}_*^{x_Q^\sharp} \circ \mathcal{T}_{\tau,Q}(f, M) : Q \in \Lambda(x) \cap (CZ^\sharp \setminus CZ^{\sharp\sharp})\right\},$$
using at most $C \log N$ operations.
- By Lemma 7.5, Lemma 7.7, and Remark 8.1, we can compute
$$\left\{\mathcal{J}_x^+ \circ \mathcal{T}_*^{x^\sharp_{\mu(Q)}} \circ \mathcal{T}_{\tau,\mu(Q)}(f, M) : Q \in CZ^{\text{empty}} \cap \Lambda(x)\right\},$$
using at most $C \log N$ operations.

Thus, given $(f, M) \in C^2(E,\tau) \times [0,\infty)$, we can compute $\Xi_{\tau,x}(f, M)$ in $C \log N$ operations. Theorem 1.2 (C) follows.

This completes the proof of Theorem 1.2. ■

10. Proof of Theorem 1.4 and Algorithm 2

10.1. Callahan–Kosaraju decomposition

We will use the data structure introduced by Callahan and Kosaraju [11].

Lemma 10.1 (Callahan–Kosaraju decomposition). *Let $E \subset \mathbb{R}^n$ with $\#E = N < \infty$. Let $\kappa > 0$. We can partition $E \times E \setminus \text{diagonal}(E)$ into subsets $E'_1 \times E''_1, \dots, E'_L \times E''_L$ satisfying the following*:

(A) $L \le C(\kappa, n)N$.

(B) *For each* $\ell = 1, \dots, L$, *we have*

$$\operatorname{diam} E'_\ell, \operatorname{diam} E''_\ell \le \kappa \cdot \operatorname{dist}(E'_\ell, E''_\ell)\,.$$

(C) *Moreover, we may pick representatives* $x'_\ell \in E'_\ell$ *and* $x''_\ell \in E''_\ell$ *for each* $\ell=1,\dots,L$, *such that* x'_ℓ, x''_ℓ *for* $\ell=1,\dots,L$ *can all be computed using at most* $C(\kappa,n)N\log N$ *operations and* $C(\kappa,n)N$ *storage.*

Here, $C(\kappa,n)$ *is a constant that depends only on* κ *and* n.

The same argument in the proof of Lemma 3.1 in [17], with obvious modifications, yields the following.

Lemma 10.2. *Let* $\tau > 0$. *Let* $E \subset \mathbb{R}^n$ *be a finite set. Let* $\kappa_0 > 0$ *be a constant that is sufficiently small. Let* $E'_\ell, E''_\ell, x'_\ell, x''_\ell$ *be as in Lemma* 10.1 *with* $\kappa = \kappa_0$. *Suppose* $\vec{P} = (P^x)_{x\in E} \in W(E,\tau)$ *satisfies the following*:

(A) $P^x \in \mathcal{K}_\tau(x, M)$ *for each* $x \in E$, *with* $\mathcal{K}_\tau$ *as in Definition* 3.1.

(B) $|\partial^\alpha(P^{x'_\ell} - P^{x''_\ell})(x''_\ell)| \le M|x'_\ell - x''_\ell|^{2-|\alpha|}$ *for* $|\alpha| \le 1$, $\ell = 1, \dots, L$.

Then $\|\vec{P}\|_{W(E,\tau)} \le CM$.

Lemma 10.3 (Lemma 3.2 of [17]). *Let* $E \subset \mathbb{R}^2$ *be a finite set. Let* E'_ℓ *and* E''_ℓ *be as in Lemma* 10.1 *with* $\ell = 1, \dots, L$. *Then every* $x \in E$ *arises as an* x'_ℓ *for some* $\ell \in \{1, \dots, L\}$.

10.2. Proof of Theorem 1.4

Proof of Theorem 1.4 *assuming Theorem* 1.2. Let $E \subset \mathbb{R}^n$ be a finite set, and let $\{\Xi_{\tau,x} : x \in \mathbb{R}^n\}$ be as in Theorem 1.2. For each $x \in E$, let $S(x)$ be the source of $\Xi_{\tau,x}$ (see Definition 1.1). Note that $S(x)$ is independent of τ, thanks to Theorem 1.2 (A).

Let κ_0 be as in Lemma 10.2. Let $(x'_\ell, x''_\ell) \in E \times E$, $\ell = 1, \dots, L$, be as in Lemma 10.1 with $\kappa = \kappa_0$.

We set

$$S_\ell := \{x'_\ell, x''_\ell\} \cup S(x'_\ell) \cup S(x''_\ell), \quad \ell = 1, \dots, L. \tag{10.1}$$

Conclusion (A) follows from Theorem 1.2 (B), (C) and Lemma 10.1.

Conclusion (B) follows from Theorem 1.2 (B), (C).

Conclusion (C) follows from Lemma 10.1 (C).

Now we verify conclusion (D). We modify the argument in [17]. Fix $\tau > 0$ and $f : E \to [-\tau, \tau]$. Set

$$M := \max_{\ell=1,\dots,L} \|f\|_{C^2(S_\ell,\tau)}. \tag{10.2}$$

Thanks to (10.2), we see that $\|f\|_{C^2(S_\ell,\tau)} \le M$ for $\ell = 1, \dots, L$. Thus, for each $\ell = 1, \dots, L$, there exists $F_\ell \in C^2(\mathbb{R}^n, \tau)$ such that

$$\|F_\ell\|_{C^2(\mathbb{R}^n)} \le 2M \quad \text{and} \quad F_\ell(x) = f(x) \quad \text{for } x \in S_\ell. \tag{10.3}$$

Fix such F_ℓ. For $\ell = 1, \dots, L$, we define

$$f_\ell \colon E \to [0,\infty), \quad f_\ell(x) := \begin{cases} f(x) & \text{for } x \in S_\ell, \\ F_\ell(x) & \text{for } x \in E \setminus S_\ell. \end{cases} \tag{10.4}$$

From (10.3) and (10.4), we see that

(10.5) $$\|f_\ell\|_{C^2(E,\tau)} \le 2M \quad \text{for } \ell = 1,\dots,L.$$

For each $\ell = 1,\dots,L$, we define

(10.6) $$P'_\ell :\equiv \mathfrak{J}_{x'_\ell}(\Xi_{\tau,x'_\ell}(f_\ell, 2M)) \quad \text{and} \quad P''_\ell :\equiv \mathfrak{J}_{x''_\ell}(\Xi_{\tau,x''_\ell}(f_\ell, 2M)).$$

We will show that assignment (10.6) unambiguously defines a Whitney field over E.

Claim 10.1. Let $\ell_1, \ell_2 \in \{1,\dots,L\}$.

(a) Suppose $x'_{\ell_1} = x'_{\ell_2}$. Then $P'_{\ell_1} = P'_{\ell_2}$.
(b) Suppose $x''_{\ell_1} = x''_{\ell_2}$. Then $P''_{\ell_1} = P''_{\ell_2}$.
(c) Suppose $x'_{\ell_1} = x''_{\ell_2}$. Then $P'_{\ell_1} = P''_{\ell_2}$.

Proof of Claim 10.1. We prove (a). The proofs for (b) and (c) are similar.

Suppose $x'_{\ell_1} = x'_{\ell_2} =: x_0$. Let $S(x_0)$ be the source of Ξ_{τ,x_0}. By (10.1), we see that

$$S(x_0) \subset S_{\ell_1} \cap S_{\ell_2}.$$

Therefore, we have

$$f_{\ell_1}(x) = f_{\ell_2}(x) \quad \text{for } x \in S(x_0).$$

Thanks to Theorem 1.2 (A) and (10.5), we see that

$$\Xi_{\tau,x_0}(f_{\ell_1}, 2M) = \Xi_{\tau,x_0}(f_{\ell_2}, 2M).$$

By (10.6), we see that $P_{\ell_1} = P_{\ell_2}$. This proves (a). ∎

By Lemma 10.3, there exists a pair of maps:

(10.7) A surjection $\pi: \{1,\dots,L\} \to E$ such that $\pi(\ell) = x'_\ell$ for $\ell = 1,\dots,L$, and an injection $\rho: E \to \{1,\dots,L\}$ such that $x'_{\rho(x)} = x$ for $x \in E$, i.e., $\pi \circ \rho = \mathrm{id}_E$.

The surjection π is determined by the Callahan–Kosaraju decomposition (Lemma 10.1), but the choice of ρ is not necessarily unique.

Thanks to Claim 10.1 and the fact that $E'_\ell \times E''_\ell \subset E \times E \setminus \mathrm{diagonal}(E)$, assignment (10.6) produces for each $x \in E$ a uniquely defined polynomial

(10.8) $$P^x \equiv \mathfrak{J}_x(\Xi_{\tau,x}(f_{\rho(x)}, 2M)),$$

with Ξ_x as in Theorem 1.2 and $\rho(x)$ as in (10.7). Note that, as shown in Claim 10.1, the polynomial P^x in (10.8) is independent of the choice of ρ as a right-inverse of π in (10.7).

Thanks to Theorem 1.2 (B) and (10.5)–(10.8), for each $\ell = 1,\dots,L$, there exists a function $\tilde{F}_\ell \in C^2(\mathbb{R}^n)$ such that

(10.9) $\|\tilde{F}_\ell\|_{C^2(\mathbb{R}^n)} \le CM$ and $-\tau \le \tilde{F}_\ell \le \tau$ on $\mathbb{R}^n$,

(10.10) $\tilde{F}_\ell = f_\ell(x) = f(x)$ for $x \in S_\ell$,

(10.11) $\mathfrak{J}_{x'_\ell}\tilde{F}_\ell \equiv P^{x'_\ell} \equiv \mathfrak{J}_{x'_\ell}(\Xi_{\tau,x'_\ell}(f_\ell, 2M))$, $\mathfrak{J}_{x''_\ell}\tilde{F}_\ell \equiv P^{x''_\ell} \equiv \mathfrak{J}_{x''_\ell}(\Xi_{\tau,x''_\ell}(f_\ell, 2M))$.

Thanks to (10.9) and (10.10), we have

$$P^{x'_\ell} \in \Gamma_\tau(x'_\ell, \{x'_\ell\}, f, CM) \quad \text{for } \ell = 1, \dots, L. \tag{10.12}$$

Thanks to (10.9) and (10.11), we have

$$|\partial^\alpha (P^{x'_\ell} - P^{x''_\ell})(x''_\ell)| \le CM |x'_\ell - x''_\ell|^{2-|\alpha|} \quad \text{for } |\alpha| \le 1, \ell = 1, \dots, L. \tag{10.13}$$

Therefore, by Lemma 10.2, (10.12), and (10.13), the Whitney field $\vec{P} = (P^x)_{x \in E}$, with P^x as in (10.8), satisfies

$$\vec{P} \in W(E,\tau), \quad P^x(x) = f(x) \text{ for } x \in E, \quad \|\vec{P}\|_{W(E,\tau)} \le CM.$$

By Whitney's extension theorem (Theorem 3.2 (B)), there exists a function $F \in C^2(\mathbb{R}^n, \tau)$ such that $\|F\|_{C^2(\mathbb{R}^n)} \le CM$ and $\mathcal{J}_x F \equiv P^x$ for each $x \in E$. In particular, $F(x) = P^x(x) = f(x)$ for each $x \in E$. Thus, $\|f\|_{C^2(E,\tau)} \le CM$. This proves conclusion (D).

Theorem 1.4 is proved. ∎

10.3. Explanation of Algorithm 2

Let $E \subset \mathbb{R}^n$ be given. We compute $S_1, \dots, S_L$ from E as in Theorem 1.4. This uses one-time work using at most $CN \log N$ operations and CN storage.

For each $\ell = 1, \dots, L$, we compute a number M_ℓ that has the order of magnitude as $\|f\|_{C^2(S_\ell,\tau)}$. This can be reformulated as a collection of convex quadratic programming problems as in Section 3.3, and requires at most CN operations, since $\#S_\ell \le C$ and $L \le CN$. Finally, $\|f\|_{C^2(E,\tau)}$ has the same order of magnitude as $\max\{M_\ell : \ell = 1, \dots, L\}$.

Acknowledgments. We are indebted to Jesús A. De Loera, Kevin O'Neill, Naoki Saito, and the referees for their valuable comments. We also thank all the participants in the 11th Whitney workshop for fruitful discussions, and Trinity College Dublin for hosting the workshop.

Funding. The first author is supported by the Air Force Office of Scientific Research (AFOSR) under award FA9550-18-1-0069, the National Science Foundation (NSF) under grant DMS-1700180, and the US-Israel Binational Science Foundation (BSF) under grant 2014055. The second author is supported by the Alice Leung Scholarship, the UC Davis Summer Graduate Student Researcher (GSR) Award and the Yueh-Jing Lin Fund. The third author is supported by NSF Grant DMS-1554733 and the UC Davis Chancellor's Fellowship.

References

[1] Bierstone, E. and Milman, P. D.: $\mathcal{C}^m$-norms on finite sets and $\mathcal{C}^m$ extension criteria. *Duke Math. J.* **137** (2007), no. 1, 1–18.

[2] Bierstone, E., Milman, P. D. and Pawłucki, W.: Differentiable functions defined in closed sets. A problem of Whitney. *Invent. Math.* **151** (2003), no. 2, 329–352.

[3] Bierstone, E., Milman, P. D. and Pawłucki, W.: Higher-order tangents and Fefferman's paper on Whitney's extension problem. *Ann. of Math. (2)* **164** (2006), no. 1, 361–370.

[4] Boyd, S. and Vandenberghe, L.: *Convex optimization*. Cambridge University Press, Cambridge, 2004.

[5] Brudnyi, Y. and Shvartsman, P.: Generalizations of Whitney's extension theorem. *Internat. Math. Res. Notices* (1994), no. 3, 129–139.

[6] Brudnyi, Y. and Shvartsman, P.: The traces of differentiable functions to subsets of $\mathbb{R}^n$. In *Linear and complex analysis. Problem book 3*, pp. 279–281, Lecture Notes in Mathematics 1574, Springer, Berlin, 1994.

[7] Brudnyi, Y. and Shvartsman, P.: The Whitney problem of existence of a linear extension operator. *J. Geom. Anal.* **7** (1997), no. 4, 515–574.

[8] Brudnyi, Y. and Shvartsman, P.: The trace of jet space $J^k\Lambda^\omega$ to an arbitrary closed subset of $\mathbb{R}^n$. *Trans. Amer. Math. Soc.* **350** (1998), no. 4, 1519–1553.

[9] Brudnyi, Y. and Shvartsman, P.: Whitney's extension problem for multivariate $C^{1,\omega}$-functions. *Trans. Amer. Math. Soc.* **353** (2001), no. 6, 2487–2512.

[10] Brudnyi, Y. and Shvartsman, P.: A linear extension operator for a space of smooth functions defined on a closed subset in $\mathbb{R}^n$. *Dokl. Akad. Nauk SSSR* **280** (1985), no. 2, 268–272.

[11] Callahan, P. B. and Kosaraju, S. R.: A decomposition of multidimensional point sets with applications to k-nearest-neighbors and n-body potential fields. *J. Assoc. Comput. Mach.* **42** (1995), no. 1, 67–90.

[12] Chan, E. S. and Ong, B. H.: Range restricted scattered data interpolation using convex combination of cubic Bézier triangles. *J. Comput. Appl. Math.* **136** (2001), no. 1-2, 135–147.

[13] Fefferman, C.: A generalized sharp Whitney theorem for jets. *Rev. Mat. Iberoam.* **21** (2005), no. 2, 577–688.

[14] Fefferman, C.: Interpolation and extrapolation of smooth functions by linear operators. *Rev. Mat. Iberoam.* **21** (2005), no. 1, 313–348.

[15] Fefferman, C.: Whitney's extension problem for C^m. *Ann. of Math. (2)* **164** (2006), no. 1, 313–359.

[16] Fefferman, C.: C^m extension by linear operators. *Ann. of Math. (2)* **166** (2007), no. 3, 779–835.

[17] Fefferman, C.: Fitting a C^m-smooth function to data. III. *Ann. of Math. (2)* **170** (2009), no. 1, 427–441.

[18] Fefferman, C.: Whitney's extension problems and interpolation of data. *Bull. Amer. Math. Soc. (N.S.)* **46** (2009), no. 2, 207–220.

[19] Fefferman, C., Israel, A. and Luli, G. K.: Finiteness principles for smooth selection. *Geom. Funct. Anal.* **26** (2016), no. 2, 422–477.

[20] Fefferman, C. and Klartag, B.: Fitting a C^m-smooth function to data. I. *Ann. of Math. (2)* **169** (2009), no. 1, 315–346.

[21] Fefferman, C. and Klartag, B.: Fitting a C^m-smooth function to data. II. *Rev. Mat. Iberoam.* **25** (2009), no. 1, 49–273.

[22] Fefferman, C. and Luli, G. K.: The Brenner–Hochster–Kollár and Whitney problems for vector-valued functions and jets. *Rev. Mat. Iberoam.* **30** (2014), no. 3, 875–892.

[23] Fefferman, C. and Shvartsman, P.: Sharp finiteness principles for Lipschitz selections. *Geom. Funct. Anal.* **28** (2018), no. 6, 1641–1705.

[24] Fefferman, C. L.: A sharp form of Whitney's extension theorem. *Ann. of Math. (2)* **161** (2005), no. 1, 509–577.

[25] Glaeser, G.: Étude de quelques algèbres tayloriennes. *J. Analyse Math.* **6** (1958), 1–124; erratum, insert to 6 (1958), no. 2.

[26] Goodman, T. N. T., Ong, B. H. and Unsworth, K.: Constrained interpolation using rational cubic splines. In *NURBS for curve and surface design (Tempe, AZ, 1990)*, pp. 59–74, SIAM, Philadelphia, PA, 1991.

[27] Herrmann, M., Mulansky, B. and Schmidt, J. W.: Scattered data interpolation subject to piecewise quadratic range restrictions. *J. Comput. Appl. Math.* **73** (1996), no. 1-2, 209–223.

[28] Jiang, F. and Luli, G. K.: Nonnegative $C^2(\mathbb{R}^2)$ interpolation. *Adv. Math.* **375** (2020), Article ID 107364.

[29] Jiang, F. and Luli, G. K.: $C^2(\mathbb{R}^2)$ nonnegative extension by bounded-depth operators. *Adv. Math.* **375** (2020), Article ID 107391, 28.

[30] Jiang, F. and Luli, G. K.: Algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation. *Adv. Math.* **385** (2021), Article ID 107756, 43.

[31] Mulansky, B. and Schmidt, J. W.: Powell–Sabin splines in range restricted interpolation of scattered data. *Computing* **53** (1994), no. 2, 137–154.

[32] Ong, B. H. and Unsworth, K.: On nonparametric constrained interpolation. In *Mathematical methods in computer aided geometric design, II (Biri, 1991)*, pp. 419–430, Academic Press, Boston, MA, 1992.

[33] Opfer, G. and Oberle, H. J.: The derivation of cubic splines with obstacles by methods of optimization and optimal control. *Numer. Math.* **52** (1988), no. 1, 17–31.

[34] Shvartsman, P.: Lipschitz sections of set-valued mappings and traces of functions from the Zygmund class on an arbitrary compactum. *Dokl. Akad. Nauk SSSR* **276** (1984), no. 3, 559–562.

[35] Shvartsman, P.: Lipschitz sections of multivalued mappings. In *Studies in the theory of functions of several real variables (Russian)*, pp. 121–132, Yaroslav. Gos. Univ., Yaroslavl', 1986.

[36] Shvartsman, P.: Traces of functions of Zygmund class. *Sibirsk. Mat. Zh.* **28** (1987), no. 5, 203–215.

[37] Shvartsman, P.: Traces of functions of the BMO space on a class of nonregular subsets of $\mathbb{R}^n$. In *Studies in the theory of functions of several real variables (Russian)*, pp. 145–158, Matematika, Yaroslav. Gos. Univ., Yaroslavl', 1988.

[38] Shvartsman, P.: K-functionals of weighted Lipschitz spaces and Lipschitz selections of multivalued mappings. In *Interpolation spaces and related topics (Haifa, 1990)*, pp. 245–268, Israel Math. Conf. Proc. 5, Bar-Ilan Univ., Ramat Gan, 1992.

[39] Shvartsman, P.: On Lipschitz selections of affine-set valued mappings. *Geom. Funct. Anal.* **11** (2001), no. 4, 840–868.

[40] Shvartsman, P.: Lipschitz selections of set-valued mappings and Helly's theorem. *J. Geom. Anal.* **12** (2002), no. 2, 289–324.

[41] Shvartsman, P.: Barycentric selectors and a Steiner-type point of a convex body in a Banach space. *J. Funct. Anal.* **210** (2004), no. 1, 1–42.

[42] Shvartsman, P.: The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces. *Trans. Amer. Math. Soc.* **360** (2008), no. 10, 5529–5550.

[43] Stein, E. M.: *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series 30, Princeton University Press, Princeton, N.J., 1970.

[44] Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), no. 1, 63–89.

[45] Whitney, H.: Differentiable functions defined in closed sets. I. *Trans. Amer. Math. Soc.* **36** (1934), no. 2, 369–387.

[46] Whitney, H.: Functions differentiable on the boundaries of regions. *Ann. of Math. (2)* **35** (1934), no. 3, 482–485.

[47] Zobin, N.: Whitney's problem on extendability of functions and an intrinsic metric. *Adv. Math.* **133** (1998), no. 1, 96–132.

[48] Zobin, N.: Extension of smooth functions from finitely connected planar domains. *J. Geom. Anal.* **9** (1999), no. 3, 491–511.

Received July 21, 2021; revised April 13, 2022. Published online June 25, 2022.

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