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A Fubini type theorem for rough integration

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Abstract. Jointly controlled paths as used in Hairer and Gerasimovičs (2019), are a class of two-parameter paths Y controlled by a p-rough path X for $2 \le p < 3$ in each time variable, and serve as a class of paths twice integrable with respect to X. We extend the notion of jointly controlled paths to two-parameter paths Y controlled by p-rough and $\tilde{p}$-rough paths X and $\tilde{X}$ (on finite dimensional spaces) for arbitrary p and $\tilde{p}$, and develop the corresponding integration theory for this class of paths. In particular, we show that for paths Y jointly controlled by X and $\tilde{X}$, they are integrable with respect to X and $\tilde{X}$, and moreover we prove a rough Fubini type theorem for the double rough integrals of Y via the construction of a third integral analogous to the integral against the product measure in the classical Fubini theorem. Additionally, we also prove a stability result for the double integrals of jointly controlled paths, and show that signature kernels, which have seen increasing use in data science applications, are jointly controlled paths.

1. Introduction

Rough paths provide a rich theory of integration, beginning with Lyons' seminal work [15] featuring the integration of one-forms with respect to geometric rough paths. Gubinelli then introduced a class of controlled paths, for which a theory of integration is developed in [10] for the $2 \le p < 3$ case, and is extended to a more general framework of branched rough paths as the driving paths in [11]. These works and subsequent ones provide a solid core foundation for rough integration theory in one time variable.

In the multivariable case, previous works such as [3] and [9] have succeeded in defining two-parameter rough integrals in the $2 \le p < 3$ case via rough sheets and jointly controlled paths, respectively. The jointly controlled paths of [9] are a class of two-parameter paths which are twice rough integrable with respect to a controlling p-rough path X for $2 \le p < 3$, with the basic idea being that a jointly controlled path Y is an X-controlled path in both time variables, and the Gubinelli derivatives of Y are also X-controlled paths in the other time variable. For the double integrals of these jointly controlled paths, a rough Fubini type theorem is developed under some assumptions of smooth approximability. These results primarily serve as a tool to prove a Hörmander theorem for SPDEs in that paper, and are not the main focus, which motivates us to expand

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upon these ideas here in this paper. Given the existence of the double rough integrals, a natural question to ask would be whether we could construct a two-parameter rough integral which serves as an analogue to the integral with respect to a product measure in the classical Fubini theorem.

Here we look to develop the work on rough Fubini type theorems in [9] in several ways. Firstly, we extend the notion of jointly controlled paths to consider two different driving rough paths (on finite dimensional spaces) of arbitrary p- and $\tilde{p}$-variation. Secondly, we provide a definition of a two-parameter rough integral of a jointly controlled path, corresponding to the integral against a product measure in the classical case. We give conditions under which this integral is well-defined and establish bounds on the integral which naturally extend those obtained in the one-parameter setting. Using this new integral, which we will call the joint integral, we are able to establish a Fubini type theorem in the arbitrary p- and $\tilde{p}$-variation case. In the process, we are also able to relax the assumption that the integrand is a smoothly approximable path which had been made in the preceding work [9].

The argument made to prove existence of the joint integral is in essence a Young argument for integral existence, in which a maximal inequality on the discrete integrals is applied to show convergence of these discrete integrals over increasingly fine meshes. The two-dimensional maximal inequality on the rough discrete integrals here is heavily inspired by the two-dimensional Young–Towghi maximal inequality of [19] as it was presented in [8]. Once existence of all three integrals is shown, we are then able to equate the other two double integrals to the third under the same regularity conditions used for existence.

We also show that double rough integration satisfies some form of continuity: if two jointly controlled paths are close to each other in some sense and their driving rough paths are close to each other in p-variation and $\tilde{p}$-variation respectively, then their double integrals are close. This stability result is analogous to the one in Theorem 4.17 of [6], which treats the stability of one-parameter controlled path rough integration.

Two examples of jointly controlled paths satisfying the regularity conditions necessary for integration are provided, the first being paths induced by smooth functions of the traces X^1 and $\tilde{X}^1$ of geometric p-rough and $\tilde{p}$-rough paths

$$X = (1, X^1, \dots, X^{\lfloor p \rfloor}) \quad \text{and} \quad \tilde{X} = (1, \tilde{X}^1, \dots, \tilde{X}^{\lfloor \tilde{p} \rfloor}).$$

The second is the signature kernel, which has seen data science applications in works such as [4, 13, 14, 16–18]. This also gives us an alternative account of giving meaning to the rough integral equation

$$\text{“}\, K_{(s,t),(u,v)}(X, \tilde{X}) = 1 + \int_{r=s}^{t} \int_{r'=u}^{v} K_{(s,r),(u,r')}(X, \tilde{X})\, d\tilde{X}\, dX \,\text{”},$$

which was shown in [17] for geometric rough paths X and $\tilde{X}$ using rough integration of one-forms. Here, we will be able to extend this integral equation to non-geometric X and $\tilde{X}$.

2. Preliminaries

2.1. Basic definitions and notation

While we will define rough paths here, we are assuming some familiarity with rough paths and will not be discussing the motivations and intricacies here. We refer to [1, 6, 15] for comprehensive introductions to rough path theory. There are several different conventions when discussing rough paths, and so this section primarily serves to familiarise the reader with the choice of notation and language used in this paper. When discussing rough paths throughout the paper, we will assume that they are over the time interval $[0, T]$ (for some $T > 0$) and on finite dimensional Banach spaces, typically denoted by V and $\tilde{V}$.

For vector spaces U, V and W, we let $\mathrm{Hom}(V, W)$ denote the space of linear maps from V to W, and we let $\mathrm{Bi}(U \times V \to W)$ denote the space of bilinear maps $U \times V \to W$. If U and V are finite dimensional and $j, k \in \mathbb{N}$, for functions $f \in \mathrm{Hom}(U^{\otimes j}, \mathrm{Hom}(U, W)) \cong \mathrm{Hom}(U^{\otimes j+1}, W)$, we will use the convention

$$f(a_1 \otimes \cdots \otimes a_{j+1}) = f(a_1 \otimes \cdots \otimes a_j)(a_{j+1}), \quad \text{for } a_1, \ldots, a_{j+1} \in U, \tag{2.1}$$

and similarly, for $g\colon \mathrm{Bi}(U^{\otimes j} \times V^{\otimes k} \to \mathrm{Bi}(U \times V \to W)) \cong \mathrm{Bi}(U^{\otimes j+1} \times V^{\otimes k+1} \to W)$, for any $a_1, \ldots, a_{j+1} \in U$ and $b_1, \ldots, b_{k+1} \in V$, we will use

$$\begin{aligned} &g(a_1 \otimes \cdots \otimes a_{j+1}, b_1 \otimes \cdots \otimes b_{k+1}) \\ &\qquad = g(a_1 \otimes \cdots \otimes a_j, b_1 \otimes \cdots \otimes b_k)(a_{j+1}, b_{k+1}). \end{aligned} \tag{2.2}$$

Denote by Δ_T the 2-simplex on $[0, T]$, that is, $\Delta_T := \{(s, t) \in [0, T]^2 \mid s \leq t\}$. A control $\omega\colon \Delta_T \to \mathbb{R}^+$ is a continuous superadditive function, by which we mean that $\omega_{s,t} \geq \omega_{s,s'} + \omega_{s',t}$ for any $0 \leq s \leq s' \leq t \leq T$.

We will be dealing with partitions and sums over them frequently, so we use the following shorthand: given a partition $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\} \subset [0, T]$ and a function $\Xi\colon \Delta_T \to E$, we write

$$\sum_{\mathcal{D}} \Xi = \sum_{[u,v] \in \mathcal{D}} \Xi_{u,v} = \sum_{m=0}^{m_0 - 1} \Xi_{s_m, s_{m+1}}.$$

Let $\Delta_T^3 = \{(s, s', t) \in [0, T]^3 \mid s \leq s' \leq t\}$ denote the 3-simplex over $[0, T]$. We will use δ to denote an operator on functions $\Xi\colon \Delta_T \to E$ such that $\delta\Xi\colon \Delta_T^3 \to E$, where

$$\delta\Xi(s, s', t) = \Xi_{s,t} - \Xi_{s,s'} - \Xi_{s',t}, \quad (s, s', t) \in \Delta_T^3. \tag{2.3}$$

It is clear that, for additive Ξ, we have $\delta\Xi \equiv 0$. As we shall see later, this quantity can be viewed as a way of measuring how far a function Ξ is from being additive. For $\Xi_1\colon \Delta_T \times \Lambda \to E$ and $\Xi_2\colon \Lambda \times \Delta_T \to E$, for some set Λ (typically either Δ_T or Δ_T^3), define δ_1 and δ_2 to be such that for $\lambda \in \Lambda$,

$$\delta_1 \Xi_1\begin{pmatrix} s, s', t \\ \lambda \end{pmatrix} = \delta\Big(\Xi_1\begin{pmatrix} \cdot, \cdot \\ \lambda \end{pmatrix}\Big)(s, s', t) \quad \text{and} \quad \delta_2 \Xi_2\begin{pmatrix} \lambda \\ u, u', v \end{pmatrix} = \delta\Big(\Xi_2\begin{pmatrix} \lambda \\ \cdot, \cdot \end{pmatrix}\Big)(u, u', v).$$

Definition 2.1. Let ω be a control on $[0,T]$, let E be a real Banach space, and let $p \geq 1$ be a real number. For functions $f\colon \Delta_T \to E$, define the norm

$$\|f\|_{p,\omega} := \inf\{C \geq 0 \,:\, |f_{s,t}| \leq \omega(s,t)^{1/p} \ \text{ for } 0 \leq s \leq t \leq T\},$$

and define $C^p_\omega([0,T];E)$ as the space of additive functions with finite $\|\cdot\|_{p,\omega}$-norm. For a function on the interval, $g\colon [0,T] \to E$, we also write $g \in C^p_\omega([0,T];E)$ if the associated function $g_{s,t} = g_t - g_s$ has finite $\|\cdot\|_{p,\omega}$-norm.

Similarly, for functions on the 3-simplex, we define a norm for $F\colon \Delta^3_T \to E$ and for any $\beta > 0$, as follows:

$$|||F|||_{\beta,\omega} := \inf\{C \geq 0 \,:\, |F_{s,s',t}| \leq \omega(s,t)^{1/\beta} \ \text{ for } 0 \leq s \leq t \leq T\}.$$

Denote by $C^{p,\beta}_\omega([0,T];E)$ the space containing all $\Xi\colon \Delta_T \to E$ such that $\|\Xi\|_{p,\omega} + |||\delta\Xi|||_{\beta,\omega} < \infty$.

For both norms, we will drop the control ω from the subscript when the choice of control is clear from the context.

Remark 2.2. For any $(s,t) \in \Delta_T$ and $q > p > 0$, we can write

$$\omega(s,t)^{1/p} = \omega(s,t)^{1/q}\,\omega(s,t)^{1/p-1/q} \leq \omega(0,T)^{1/p-1/q}\,\omega(s,t)^{1/q},$$

and so there exists $C(p,q,T,\omega) = \omega(0,T)^{1/p-1/q}$ such that

$$\|f\|_p \leq C(p,q,T,\omega)\|f\|_q \quad \text{and} \quad |||F|||_p \leq C(p,q,T,\omega)|||F|||_q.$$

For a real Banach space V, we denote $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ and $T((V)) = \prod_{n \geq 0} V^{\otimes n}$, where $V^{\otimes 0} := \mathbb{R}$. We will identify the truncated tensor algebra over V of order N, denoted by $T^N(V)$, with the space $\bigoplus_{n=0}^N V^{\otimes n}$. For any two elements $a = (a_0, a_1, \dots, a_N)$ and $b = (b_0, b_1, \dots, b_N)$ of $T^N(V)$, we will use the tensor product $a \otimes b$ to mean $a \otimes b = c$, where $c = (c_0, c_1, \dots, c_N) \in T^N(V)$ is defined by

$$c_n = \sum_{m=0}^{N-n} a_m \otimes b_{n-m}, \quad n = 0, \dots, N.$$

Definition 2.3 (Rough paths). For $p \geq 1$, $X = (1, X^1, \dots, X^{\lfloor p \rfloor})$ is a p-rough path on V with control ω if $X\colon \Delta_T \to T^{\lfloor p \rfloor}(V)$ satisfies:

(1) (Regularity) For every $j = 1, \dots, \lfloor p \rfloor$,

$$\left\|X^j\right\|_{p/j,\omega} \leq \frac{1}{\beta_p\,\Gamma(j/p+1)}, \tag{2.4}$$

where Γ is the Gamma function and

$$\beta_p = p^2\Big(1 + \sum_{r=3}^{\infty}\Big(\frac{2}{r-2}\Big)^{(\lfloor p \rfloor+1)/p}\Big).$$

(2) (Multiplicity) For every $0 \le s \le s' \le t \le T$,

$$X_{s,t} = X_{s,s'} \otimes X_{s',t}. \tag{2.5}$$

This is also known as *Chen's identity*.

The form of the regularity bound (2.4) is tied to the extension theorem (Theorem 2.2.1 of [15], Theorem 3.7 of [1]), which is one of the foundational results of rough path theory. For the majority of this paper, this exact form is not important, although it will be useful when looking at the example of the signature kernel.

When talking about geometric rough paths, we will use the term to mean weakly geometric rough paths, that is, a weakly geometric p-rough path on V is a p-rough path on V that takes values in the step-$\lfloor p \rfloor$ free nilpotent group $G^{\lfloor p \rfloor}(V)$. The difference between weakly geometric rough paths and geometric rough paths (as the p-variation limit of lifted smooth paths) is explored in [7] and is likened to the difference between the space of α-Hölder paths and the α-Hölder closure of smooth paths. For our purposes, we only use the property that, for a weakly geometric p-rough path $X = (1, X^1, \dots, X^{\lfloor p \rfloor})$, the symmetric part of X is determined solely by the trace X^1: for any $j = 0, \dots, \lfloor p \rfloor$, and any $(s,t) \in \Delta_T$,

$$\mathrm{Sym}(X^j_{s,t}) = \frac{1}{j!}(X^1_{s,t})^{\otimes j}. \tag{2.6}$$

2.2. Rough integration of controlled paths

The notion of controlled paths was first introduced by Gubinelli in [10] in the case $p < 3$, and has since been extended to a more general framework involving branched rough paths in [11]. The definition used here parallels the notion of a $\mathrm{Lip}(\gamma)$ function which is used in the rough integration of one-forms as introduced in [15].

Definition 2.4 (Controlled paths in one variable). Let $X = (1, X^1, X^2, \dots, X^{\lfloor p \rfloor})$ be an ω-controlled p-rough path on V. Suppose that $Y : [0,T] \to E$, for some Banach space E. Let $N := \lfloor p \rfloor - 1$ and set $J = \{0, 1, \dots, N\}$. For $j \in J$, there exists $Y^{(j)} : [0,T] \to \mathrm{Hom}(V^{\otimes j}, E)$, with $Y^{(0)} = Y$, such that $Y^{(j)} \in C^p_\omega([0,T]; E)$, and for all $\xi \in V^{\otimes j}$,

$$Y^{(j)}_t(\xi) = \sum_{l=0}^{N-j} Y^{(j+l)}_s(X^l_{s,t} \otimes \xi) + R^{(j)}_{s,t}(\xi), \tag{2.7}$$

for remainders $R^{(j)}$ satisfying the regularity condition $\|R^{(j)}\|_{p/(\lfloor p \rfloor - j)} < \infty$.

Then we say that the tuple $(Y^0, Y^{(1)}, \dots, Y^{(N)})$ is an X-controlled path over $[0,T]$ with values in E. Denote this space of X-controlled paths by $\mathcal{D}^p_X([0,T]; E)$. For $j \in J \setminus \{0\}$, we call $Y^{(j)}$ the Gubinelli derivative of order j of Y.

Remark 2.5. An equivalent formulation is given in Definition 2.4 of [2] for geometric rough paths. While we do not enforce that X is geometric here, it can be difficult to construct families of controlled paths without geometricity for $p > 3$. Moreover, geometricity allows the integrals of these controlled paths to be lifted to rough paths, see [12] and [2]. Since we do not consider the lifts of integrals here, it will remain omitted from the definition here.

Example 2.6 (Smooth functions of geometric rough paths, Example 2.5 in [2]). Suppose that X is a geometric p-rough path on V and write $X_t^1 = x_t$. Let W be a Banach space and for $f \in C^\infty(V, W)$, let $F_t^{(0)} = f(x_t)$ and $F_t^{(j)} = D^j f(x_t)$. By the symmetry of higher differentials, for $\xi \in \mathrm{Sym}(V^{\otimes j})$,

$$\begin{aligned} R_{s,t}^{(F;j)}(\xi) &:= D^j f(x_t)(\xi) - \sum_{l=0}^{N-j} D^{j+l} f(x_s)(X_{s,t}^l \otimes \xi) \\ &= D^j f(x_t)(\xi) - \sum_{l=0}^{N-j} D^{j+l} f(x_s)\big(\mathrm{Sym}(X_{s,t}^l \otimes \xi)\big) \\ &= D^j f(x_t)(\xi) - \sum_{l=0}^{N-j} \frac{1}{l!} D^{j+l} f(x_{s,t}^{\otimes l} \otimes \xi), \end{aligned}$$

and so, by Taylor's theorem, these remainders have finite $p/(\lfloor p \rfloor - j)$-norm. Thus the collection $(F^{(j)})_{j\in J}$ is an X-controlled path.

An important case is when $E = \mathrm{Hom}(V, W)$ for some Banach space W. In this case, the definition of controlled paths leads naturally to considering *enhanced Riemann sums*: for a controlled path $Y \in \mathcal{D}_X^p([0,T], \mathrm{Hom}(V, W))$, consider $\Xi^Y : \Delta_T \to W$ defined by

$$\Xi_{s,t}^Y = \sum_{j=0}^{N} Y_s^{(j)}(X_{s,t}^{j+1}), \tag{2.8}$$

where $Y_s^{(j)}(X_{s,t}^{j+1})$ is defined using the convention (2.1),

$$Y_s^{(j)}(a_1 \otimes \cdots \otimes a_{j+1}) = Y_s^{(j)}(a_1 \otimes \ldots \otimes a_j)(a_{j+1}),$$

for $a_1, \ldots, a_{j+1} \in V$. The quantity Ξ^Y has corresponding enhanced Riemann sum over a partition $\mathcal{D} = \{s_0 < \cdots < s_{m_0}\} \subset [s,t]$,

$$\sum_{\mathcal{D}} \Xi^Y = \sum_{m=0}^{m_0-1} \sum_{j=0}^{N} Y_{s_m}^{(j)}(X_{s_m,s_{m+1}}^{j+1}).$$

The local approximations Ξ^Y are such that $\delta\Xi^Y$ can be expressed in terms of the remainders of Y.

Lemma 2.7. *Let W be a Banach space and let $(Y^{(0)}, \ldots, Y^{(N)})$ be an X-controlled path on $E = \mathrm{Hom}(V, W)$ with remainders $\{R^{(j)}\}_{j\in J}$. Define the process Ξ^Y on the 2-simplex over $[0,T]$ by (2.8). Then for $0 \le s \le s' \le t \le T$,*

$$-\delta\Xi^Y(s,s',t) = \sum_{j=0}^{N} \big(Y_s^{(j)}(X_{s,s'}^{j+1}) + Y_{s'}^{(j)}(X_{s',t}^{j+1}) - Y_s^{(j)}(X_{s,t}^{j+1})\big) = \sum_{j=0}^{N} R_{s,s'}^{(j)}(X_{s',t}^{j+1}). \tag{2.9}$$

Proof. It follows from a straightforward applications of Chen's identity written as

$$X_{s,t}^{j+1} - X_{s,s'}^{j+1} - X_{s',t}^{j+1} = \sum_{l=1}^{j} X_{s,s'}^{j+1-l} \otimes X_{s',t}^{l} = \sum_{l=0}^{j-1} X_{s,s'}^{j-l} \otimes X_{s',t}^{l+1},$$

for $j \in J$. Using this identity, we then write

$$
\begin{aligned}
-\delta \Xi^Y(s,s',t) &= \sum_{j=0}^{N} \big((Y^{(j)}_{s'} - Y^{(j)}_{s})(X^{j+1}_{s',t}) + Y^{(j)}_{s}(X^{j+1}_{s,s'} + X^{j+1}_{s',t} - X^{j+1}_{s,t})\big) \\
&= \sum_{j=0}^{N}\Big(\sum_{m=1}^{N-j} Y^{(j+m)}_{s}(X^{m}_{s,s'}) + R^{(j)}_{s,s'}\Big)(X^{j+1}_{s',t}) - \sum_{j=1}^{N}\sum_{l=0}^{j-1} Y^{(j)}_{s}(X^{j-l}_{s,s'})(X^{l+1}_{s',t}) \\
&= \sum_{j=0}^{N}\Big(\sum_{m=1}^{N-j} Y^{(j+m)}_{s}(X^{m}_{s,s'}) + R^{(j)}_{s,s'}\Big)(X^{j+1}_{s',t}) - \sum_{l=0}^{N-1}\sum_{j=l+1}^{N} Y^{(j)}_{s}(X^{j-l}_{s,s'})(X^{l+1}_{s',t}) \\
&= \sum_{j=0}^{N}\Big(\sum_{m=1}^{N-j} Y^{(j+m)}_{s}(X^{m}_{s,s'}) + R^{(j)}_{s,s'}\Big)(X^{j+1}_{s',t}) - \sum_{l=0}^{N-1}\sum_{m=1}^{N-l} Y^{(l+m)}_{s}(X^{m}_{s,s'})(X^{l+1}_{s',t}) \\
&= \sum_{j=0}^{N} R^{(j)}_{s,s'}(X^{j+1}_{s',t}). \qquad \blacksquare
\end{aligned}
$$

A key result for integration of controlled paths is the sewing lemma, which is used to establish the existence, uniqueness and the regularity of this underlying additive function under suitable conditions on Ξ and $\delta\Xi$.

Lemma 2.8 (Sewing lemma). *Let $p > 1$ and $0 < \beta < 1$. There exists a unique map $\mathcal{I}\colon C^{p,\beta}_{\omega}([0,T];E) \to C^{p}_{\omega}([0,T];E)$ such that $(\mathcal{I}\,\Xi)_0 = 0$ and*

$$
|(\mathcal{I}\,\Xi)_{s,t} - \Xi_{s,t}| \le 2^{1/\beta}\,\zeta(1/\beta)\,\omega(s,t)^{1/\beta}\,|||\delta\Xi|||_{\beta}, \tag{2.10}
$$

where ζ is the Riemann zeta function and $(\mathcal{I}\,\Xi)_{s,t} = (\mathcal{I}\,\Xi)_t - (\mathcal{I}\,\Xi)_s$. Moreover, $\mathcal{I}\,\Xi$ is such that

$$
\mathcal{I}\,\Xi_{s,t} = \lim_{|\mathcal{D}|\to 0} \sum_{\mathcal{D}} \Xi
$$

for partitions $\mathcal{D}$ over $[s,t]$.

The sewing map $\mathcal{I}$ is a special case of the argument used for rough integral existence in [15], limited to the first level component only, whereby we take a sufficiently good local approximation and "sew" these together with the map $\mathcal{I}$. The proof of the lemma in the form given here follows a Young type argument based on the presentation given in Lemma 4.2 of [6] with minor modifications. The proof here also serves as a simpler example of the strategy used to prove the existence and uniqueness of two-parameter rough integrals in later parts.

Proof. We begin this by establishing Young's maximal inequality in the one-parameter case. Let $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\} \subset [s,t]$ be an arbitrary partition. The idea here is to selectively remove points from $\mathcal{D}$ until we are left with the trivial partition $\{s,t\}$. It is evident that

$$
\sum_{m=1}^{m_0-1} \omega(s_{m-1}, s_{m+1}) \le \sum_{m \text{ even}} \omega(s_{m-1}, s_{m+1}) + \sum_{m \text{ odd}} \omega(s_{m-1}, s_{m+1}) \le 2\omega(s,t).
$$

Thus there exists $0 < m^* < m_0$ such that

$$\omega(s_{m^*-1}, s_{m^*+1}) \le \frac{2}{m_0 - 1}\, \omega(s,t),$$

since assuming otherwise leads to the contradiction $\sum_{m=1}^{m_0-1} \omega(s_{m-1}, s_{m+1}) > 2\omega(s,t)$. So for any $\Xi \in C_\omega^{p,\beta}([0,T]; E)$, we have

$$\begin{aligned}\Big|\sum_{\mathcal{D}} \Xi - \sum_{\mathcal{D}\setminus\{s_{m^*}\}} \Xi\Big| &= |\delta\Xi(s_{m^*-1}, s_{m^*}, s_{m^*+1})| \le \omega(s_{m^*-1}, s_{m^*+1})^{1/\beta} \|\!|\delta\Xi|\!\|_{\beta,\omega} \\ &\le \Big(\frac{2}{m_0-1}\,\omega(s,t)\Big)^{1/\beta} \|\!|\delta\Xi|\!\|_{\beta,\omega}.\end{aligned}$$

Repeatedly choosing and removing points in the same manner then gives us

$$\Big|\sum_{\mathcal{D}} \Xi - \Xi_{s,t}\Big| \le \sum_{l=1}^{m_0-1} \Big(\frac{2}{m_0-l}\,\omega(s,t)\Big)^{1/\beta} \|\!|\delta\Xi|\!\|_{\beta,\omega} \le 2^{1/\beta}\zeta\Big(\frac{1}{\beta}\Big)\,\omega(s,t)^{1/\beta} \|\!|\delta\Xi|\!\|_{\beta,\omega}.$$

With this maximal bound on the sums over partitions, we now look to prove existence and uniqueness of the limit of these sums as the partition mesh size goes to zero. The control ω is continuous, and thus uniformly continuous on Δ_T. In particular, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|t-s| < \delta_\varepsilon$ then $|\omega(s,t) - \omega(s,s)| = |\omega(s,t)| < \varepsilon$.

Let $\varepsilon > 0$ and let $\mathcal{D}$ and $\mathcal{D}'$ be any two partitions of $[s,t]$ such that $|\mathcal{D}| < \delta_\varepsilon$ and $|\mathcal{D}'| < \delta_\varepsilon$. First we will consider the case where $\mathcal{D} \subset \mathcal{D}'$. Writing $\mathcal{D} = \{s_0 < \cdots < s_{m_0}\}$, we now view $\mathcal{D}'$ as the union of partitions of $[s_m, s_{m+1}]$. Define $\mathcal{D}'_m = \mathcal{D}' \cap [s_m, s_{m+1}]$. Applying the earlier maximal bound,

$$\begin{aligned}\Big|\sum_{\mathcal{D}'} \Xi - \sum_{\mathcal{D}} \Xi\Big| &\le \sum_{m=0}^{m_0-1} \Big|\sum_{\mathcal{D}'_m} \Xi - \Xi_{s_m,s_{m+1}}\Big| \\ &\le \sum_{m=0}^{m_0-1} 2^{1/\beta}\zeta\Big(\frac{1}{\beta}\Big)\,\omega(s_m,s_{m+1})^{1/\beta} \|\!|\delta\Xi|\!\|_{\beta,\omega} \le 2^{1/\beta}\zeta\Big(\frac{1}{\beta}\Big)\,\omega(s,t)\|\!|\delta\Xi|\!\|_{\beta,\omega}\,\varepsilon^{1/\beta-1}.\end{aligned}$$

In the general case, where the two partitions may not be nested, we then write

$$\begin{aligned}\Big|\sum_{\mathcal{D}'} \Xi - \sum_{\mathcal{D}} \Xi\Big| &\le \Big|\sum_{\mathcal{D}\cup\mathcal{D}'} \Xi - \sum_{\mathcal{D}} \Xi\Big| + \Big|\sum_{\mathcal{D}\cup\mathcal{D}'} \Xi - \sum_{\mathcal{D}'} \Xi\Big| \\ &\le 2^{(\beta+1)/\beta}\zeta(1/\beta)\,\omega(s,t)\,\|\!|\delta\Xi|\!\|_{\beta,\omega}\,\varepsilon^{1/\beta-1}.\end{aligned}$$

Taking ε arbitrarily small, we have existence and uniqueness of the limit

$$(\mathcal{I}\Xi)_{s,t} = \lim_{|\mathcal{D}|\to 0} \sum_{\mathcal{D}} \Xi,$$

where $\mathcal{I}\Xi$ satisfies the bound (2.10) and is such that $(\mathcal{I}\Xi)_0 = 0$. ∎

A straightforward application of the sewing lemma on Ξ^Y as defined in (2.8) then yields existence of the rough integral as the limit of enhanced Riemann sums.

Theorem 2.9. *Let W be a Banach space, $p > 1$, $N = \lfloor p \rfloor - 1$, and $\theta = (\lfloor p \rfloor + 1)/p$. If X is a p-rough path on V and $Y = \big(Y^{(0)}, \dots, Y^{(N)}\big) \in \mathcal{D}_X^p([0,T]; \mathrm{Hom}(V,W))$, with remainders $(R^{(0)}, \dots, R^{(N)})$, then for $[s,t] \subset [0,T]$, the rough integral*

$$\int_s^t Y\, dX := \lim_{|\mathcal{D}|\to 0} \sum_{[s_m, s_{m+1}]\subset \mathcal{D}} \Big(\sum_{j=0}^N Y_{s_m}^{(j)} (X_{s_m, s_{m+1}}^{j+1}) \Big) \tag{2.11}$$

exists and is such that

$$\Big| \int_s^t Y\, dX - \sum_{j=0}^N Y_s^{(j)} (X_{s,t}^{j+1}) \Big| \le 2^\theta \zeta(\theta) \Big(\sum_{j=0}^N \| X^{j+1} \|_{p/(j+1)} \, \| R^{(j)} \|_{p/(\lfloor p \rfloor - j)} \Big) \, \omega(s,t)^\theta. \tag{2.12}$$

Remark 2.10. More generally, if we let $p_j = p/(j+1)$ and assume there exist q_j and θ_j such that $1/p_j + 1/q_j = \theta_j > 1$ and $\| R^{(j)} \|_{q_j}$ are finite, then we can change (2.12) into the bound

$$\Big| \int_s^t Y\, dX - \sum_{j=0}^N Y_s^{(j)} (X_{s,t}^{j+1}) \Big| \le C(\omega, \theta) \sup_{j \in J} \Big(\sum_{j=0}^N \| X^{j+1} \|_{p_j} \, \| R^{(j)} \|_{q_j} \Big) \, \omega(s,t)^{\theta_*}, \tag{2.13}$$

where $\theta_* = \min \theta_j$ and $C(\omega, \theta) = 2^{\theta_*} \zeta(\theta_*) \sup_{j \in J} \omega(0,T)^{\theta_* - \theta_j}$.

3. Jointly controlled paths and their double integrals

3.1. Paths controlled by two rough paths

In order to study two-parameter rough integrals, we first need to find a class of two-parameter paths which are twice integrable under one-parameter rough integration. Hairer and Gerasimovičs do this (see [9], Section 5) by introducing the notion of jointly controlled paths, for which they show that the one-parameter rough integrals of these jointly controlled paths are themselves controlled paths. An alternative method of constructing double rough integrals using rough sheets is done in [3], although we will not discuss this here.

We will adopt the approach of [9], which handles the case where X is a p-rough path for $2 \le p < 3$ with Hölder control, which we will extend to the case where X is rough path of arbitrary p-variation and arbitrary control. The following definition is a natural extension of Definition 5.1 in [9]. Recall that $\mathrm{Bi}(V \times \tilde{V} \to E)$ is the space of bilinear maps from $V \times \tilde{V}$ to E.

Definition 3.1. Let $p, \tilde{p} > 1$, with p and $\tilde{p}$ not integers, let $X = (1, X^1, \dots, X^{\lfloor p \rfloor})$ be an ω-controlled p-rough path on V, and let $\tilde{X} = (1, \tilde{X}^1, \dots, \tilde{X}^{\lfloor \tilde{p} \rfloor})$ be an $\tilde{\omega}$-controlled $\tilde{p}$-rough path on $\tilde{V}$. Let $N = \lfloor p \rfloor - 1$, $\tilde{N} = \lfloor \tilde{p} \rfloor - 1$, $J = \{0, 1, \dots, N\}$ and $K = \{0, 1, \dots, \tilde{N}\}$.

A two parameter path $Y: [0,T] \times [0,T] \to E$ on a Banach space E is jointly $(X, \tilde{X})$-controlled if the following condition is satisfied: let $Y^{(0,0)}_{s;u} = Y_{s;u}$ and suppose for every $j \in J$, $k \in K$, there exists $Y^{(j,k)}$ defined on $[0,T]^2$ with

$$Y^{(j,k)}_{s;u} \in \mathrm{Bi}(V^{\otimes j} \times \tilde{V}^{\otimes k} \to E)$$

such that for every $u \in [0,T]$, $(Y^{(0,k)}_{\cdot;u}, \ldots, Y^{(N,k)}_{\cdot;u})$ is an X-controlled path, and for every $s \in [0,T]$, $(Y^{(j,0)}_{s;\cdot}, \ldots, Y^{(j,\tilde{N})}_{s;\cdot})$ is an $\tilde{X}$-controlled path. The collection $\{Y^{(j,k)} \mid (j,k) \in J \times K\}$ defines the jointly $(X, \tilde{X})$-controlled path, and we denote this class of jointly controlled paths by $\mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T] \times [0,T]; E)$.

From here onwards, we will assume that X is a p-rough path on V with control ω, and that $\tilde{X}$ is a $\tilde{p}$-rough path on $\tilde{V}$ with control $\tilde{\omega}$ for finite dimensional Banach spaces V and $\tilde{V}$. The idea behind jointly controlled paths is that Y is both an X-controlled path and an $\tilde{X}$-controlled path, with Gubinelli derivatives $(Y^{(j,0)})_{j \in J}$ and $(Y^{(0,k)})_{k \in K}$ respectively, and that these order j and order k Gubinelli derivatives are themselves $\tilde{X}$-controlled and X-controlled paths, respectively.

Example 3.2 (Smooth functions of two geometric rough paths). Suppose that X and $\tilde{X}$ are geometric, let E be a Banach space, and let $\gamma: V \times \tilde{V} \to E$ be a smooth function. Writing $X^1_t = x_t$ and $\tilde{X}^1_t = \tilde{x}_t$, consider the induced path $Y_{s;u} = \gamma(x_s, \tilde{x}_u)$, and auxiliary paths $Y^{(j,k)}$ defined for $\xi \in V^{\otimes j}$ and $\eta \in \tilde{V}^{\otimes k}$ by

$$Y^{(j,k)}_{s;u}(\xi, \eta) := \big(D^j_1 D^k_2 \gamma\big)(x_s, \tilde{x}_u)(\xi, \eta),$$

where D^j_1 is the j-th differential in the first variable and D^k_2 is the k-th differential in the second variable. Using Example 2.6 and that the partial differentials commute, it is easy to see that $\{Y^{(j,k)}\}$ constitutes a jointly controlled path.

In the case $E = \mathrm{Bi}(V \otimes \tilde{V} \to W)$ for some Banach space W, for $(\xi, \eta) \in V^{\otimes j} \times \tilde{V}^{\otimes k}$ and $(a,b) \in V \times \tilde{V}$, we write out $Y^{(j,k)}$ as

$$Y^{(j,k)}_{s;u}(\xi \otimes a, \eta \otimes b) := \big(D^j_1 D^k_2 \gamma_{(a,b)}\big)(x_s, \tilde{x}_u)(\xi, \eta),$$

where $\gamma_{(a,b)}(\mu, \nu) := \gamma(\mu, \nu)(a,b)$ for $(\mu, \nu) \in V \times \tilde{V}$.

From the definition of controlled paths, we have remainders $R^{(j,k)}_{\tilde{X}}$ and $R^{(j,k)}_X$ defined, for $(s,t), (u,v) \in \Delta^2_T$, as

$$R^{(j,k)}_X\binom{s,t}{u}(\xi, \eta) = Y^{(j,k)}_{t;u}(\xi, \eta) - \sum_{m=0}^{N-j} Y^{(j+m,k)}_{s;u}(X^m_{s,t} \otimes \xi, \eta), \tag{3.1}$$

$$R^{(j,k)}_{\tilde{X}}\binom{s}{u,v}(\xi, \eta) = Y^{(j,k)}_{s;v}(\xi, \eta) - \sum_{l=0}^{\tilde{N}-k} Y^{(j,k+l)}_{s;u}(\xi, \tilde{X}^l_{u,v} \otimes \eta), \tag{3.2}$$

and that satisfy

$$\Big\| R^{(j,k)}_X\binom{\cdot,\cdot}{u} \Big\|_{p/(\lfloor p \rfloor - j)} < \infty \quad \text{and} \quad \Big\| R^{(j,k)}_{\tilde{X}}\binom{s}{\cdot,\cdot} \Big\|_{p/(\lfloor p \rfloor - j)} < \infty.$$

One of the key observations of [9] is that, in the case $2 \le p < 3$, these remainders are also controlled paths. As the following lemma will show, this is also true in the more general case here.

Lemma 3.3. *Let* $R_X^{(j,k)}$ *and* $R_{\tilde{X}}^{(j,k)}$ *be the remainders of* $Y \in \mathcal{D}_{X,\tilde{X}}^{p,\tilde{p}}([0,T]\times[0,T];E)$ *as presented above. Let* $\mathbf{R}^{(j,k)}\colon \Delta_T \times \Delta_T \to \mathrm{Bi}(V^{\otimes j} \times \tilde{V}^{\otimes k} \to E)$ *be defined by*

$$\mathbf{R}^{(j,k)}\binom{s,t}{u,v}(\xi,\eta) := Y_{t;v}^{(j,k)}(\xi,\eta) - \sum_{m=0}^{N-j} Y_{s;v}^{(j+m,k)}(X_{s,t}^m \otimes \xi, \eta)$$

$$- \sum_{l=0}^{\tilde{N}-k} Y_{t;u}^{(j,k+l)}(\xi, \tilde{X}_{u,v}^l \otimes \eta) + \sum_{m=0}^{N-j}\sum_{l=0}^{\tilde{N}-k} Y_{s;u}^{(j+m,k+l)}(X_{s,t}^m \otimes \xi, \tilde{X}_{u,v}^l \otimes \eta), \tag{3.3}$$

for $(\xi,\eta) \in V^{\otimes j} \times \tilde{V}^{\otimes k}$. *Then*

$$\begin{aligned}\mathbf{R}^{(j,k)}\binom{s,t}{u,v}(\xi,\eta) &= R_{\tilde{X}}^{(j,k)}\binom{t}{u,v}(\xi,\eta) - \sum_{m=0}^{N-j} R_{\tilde{X}}^{(j+m,k)}\binom{s}{u,v}(X_{s,t}^m \otimes \xi, \eta) \\ &= R_X^{(j,k)}\binom{s,t}{v}(\xi,\eta) - \sum_{l=0}^{\tilde{N}-k} R_X^{(j,k+l)}\binom{s,t}{u}(\xi, \tilde{X}_{u,v}^l \otimes \eta).\end{aligned} \tag{3.4}$$

In particular, for every $j \in J$, $k \in K$, *and* $(s,t),(u,v) \in \Delta_T$, *the remainder tuples*

$$\left(R_{\tilde{X}}^{(0,k)}\binom{\cdot}{u,v}, \ldots, R_{\tilde{X}}^{(N,k)}\binom{\cdot}{u,v}\right) \quad \text{and} \quad \left(R_X^{(j,0)}\binom{s,t}{\cdot}, \ldots, R_X^{(j,\tilde{N})}\binom{s,t}{\cdot}\right)$$

are X*-controlled and* $\tilde{X}$*-controlled paths, respectively, with remainders*

$$\left(\mathbf{R}^{(0,k)}\binom{\cdot,\cdot}{u,v}, \ldots, \mathbf{R}^{(N,k)}\binom{\cdot,\cdot}{u,v}\right) \quad \text{and} \quad \left(\mathbf{R}^{(j,0)}\binom{s,t}{\cdot,\cdot}, \ldots, \mathbf{R}^{(j,\tilde{N})}\binom{s,t}{\cdot,\cdot}\right),$$

respectively.

Proof. The fact that the remainders are controlled paths follows immediately from (3.4). The equality in (3.4) follows rather simply from appropriate factorisation of $\mathbf{R}^{(j,k)}$,

$$\begin{aligned}\mathbf{R}^{(j,k)}\binom{s,t}{u,v}(\xi,\eta) &= Y_{t;v}^{(j,k)}(\xi,\eta) - \sum_{l=0}^{\tilde{N}-k} Y_{t;u}^{(j,k+l)}(\xi, \tilde{X}_{u,v}^l \otimes \eta) \\ &\quad - \sum_{m=0}^{N-j}\Big[Y_{s;v}^{(j+m,k)}(X_{s,t}^m \otimes \xi, \eta) \\ &\quad - \sum_{l=0}^{\tilde{N}-k} Y_{s;u}^{(j+m,k+l)}(X_{s,t}^m \otimes \xi, \tilde{X}_{u,v}^l \otimes \eta)\Big] \\ &= R_{\tilde{X}}^{(j,k)}\binom{t}{u,v}(\xi,\eta) - \sum_{m=0}^{N-j} R_{\tilde{X}}^{(j+m,k)}\binom{s}{u,v}(X_{s,t}^m \otimes \xi, \eta)\end{aligned}$$

and

$$\mathbf{R}^{(j,k)}\binom{s,t}{u,v}(\xi,\eta) = Y^{(j,k)}_{t;v}(\xi,\eta) - \sum_{m=0}^{N-j} Y^{(j+m,k)}_{s;v}(X^m_{s,t}\otimes\xi,\eta)$$

$$- \sum_{l=0}^{\tilde{N}-k}\Big[Y^{(j,k+l)}_{t;u}(\xi,\tilde{X}^l_{u,v}\otimes\eta)$$

$$- \sum_{m=0}^{N-j} Y^{(j+m,k+l)}_{s;u}(X^m_{s,t}\otimes\xi,\tilde{X}^l_{u,v}\otimes\eta)\Big]$$

$$= R^{(j,k)}_X\binom{s,t}{v}(\xi,\eta) - \sum_{l=0}^{\tilde{N}-k} R^{(j,k+l)}_X\binom{s,t}{u}(\xi,\tilde{X}^l_{u,v}\otimes\eta). \qquad \blacksquare$$

Remark 3.4. With the above in mind, we will drop the subscripts X and $\tilde{X}$ from the remainders $R^{(j,k)}_X$ and $R^{(j,k)}_{\tilde{X}}$, and relabel them as simply $R^{(j,k)}$, with the correct interpretation being clear from the arguments taken. We will call the remainders $R^{(j,k)}$ *first order remainders* of Y, and $\mathbf{R}^{(j,k)}$ will be the *second order remainders* of Y.

3.2. Integrals of jointly controlled paths

When we take $E = \mathrm{Bi}(V \otimes \tilde{V} \to W)$ for some Banach space W, we may perform rough integration on the jointly controlled paths with respect to both X and $\tilde{X}$. Moreover, under some conditions on the mixed variation of second order remainders (see [5,8,19] for more on mixed variation), we can show that the integral with respect to X is an $\tilde{X}$-controlled path, and likewise that the integral with respect to $\tilde{X}$ is an X-controlled path.

Definition 3.5. Let $p,q \geq 1$, let $F\colon \Delta^2_T \times \Delta^2_T \to E$ on a Banach space E, and let ω and $\tilde{\omega}$ be controls. If the quantity

$$\|F\|_{(p,q),(\omega,\tilde{\omega})} := \inf\left\{C \geq 0 : \left|F\binom{s,t}{u,v}\right| \leq \omega(s,t)^{1/p}\tilde{\omega}(u,v)^{1/q} \text{ for } (s,t),(u,v)\in\Delta_T\right\}$$

is finite, we will say that F has finite $(\omega,\tilde{\omega})$-controlled (p,q)-variation over $[0,T]\times[0,T]$.

Condition 3.6. Suppose that $\{Y^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]\times[0,T];E)$ with remainders $R^{(j,k)}$. For $j \in J$ and $k \in K$, let $p_j = p/(j+1)$ and $\tilde{p}_k = \tilde{p}/(k+1)$. Assume there exist some q_j and $\tilde{q}_k$ such that

$$\frac{1}{p_j} + \frac{1}{q_j} = \theta_j > 1 \quad \text{and} \quad \frac{1}{\tilde{p}_k} + \frac{1}{\tilde{q}_k} = \tilde{\theta}_k > 1,$$

and assume that, for every $s,u \in [0,T]$, the remainders $R^{(j,k)}\binom{s}{\cdot,\cdot}$ have finite $\tilde{q}_k$-variation, the $R^{(j,k)}\binom{\cdot,\cdot}{u}$ have finite q_j-variation, and the $\mathbf{R}^{(j,k)}$ have finite $(\omega,\tilde{\omega})$-controlled $(q_j,\tilde{q}_k)$-variation. Let $\overline{\boldsymbol{\theta}}$ denote the collection

$$\overline{\boldsymbol{\theta}} := \{p,\tilde{p},q_0,\ldots,q_N,\tilde{q}_0,\ldots,\tilde{q}_{\tilde{N}}\},$$

and define

$$\theta^* := \max_{(j,k)\in J\times K}\{\theta_j, \tilde{\theta}_k\} \quad \text{and} \quad \theta_* := \min_{(j,k)\in J\times K}\{\theta_j, \tilde{\theta}_k\}.$$

Example 3.7. An example of a class of paths that satisfies this mixed variation condition are those introduced in Example 3.2. Letting $q_j = p/(\lfloor p \rfloor - j)$ and $\tilde{q}_k = \tilde{p}/(\lfloor \tilde{p} \rfloor - k)$, the assumptions on first order remainders holds automatically and we only need to check the second order remainders $\mathbf{R}^{(j,k)}$, which are given by

$$\begin{aligned}\mathbf{R}^{(j,k)}\begin{pmatrix} s, t \\ u, v \end{pmatrix}(\xi,\eta) = \big(D_1^j D_2^k \gamma\big)(x_t, \tilde{x}_v)(\xi,\eta) - \sum_{m=0}^{N-j} \big(D_1^{j+m} D_2^k \gamma\big)(x_s, \tilde{x}_v)(X_{s,t}^m \otimes \xi, \eta) \\ - \sum_{l=0}^{\tilde{N}-k} \big(D_1^j D_2^{k+l} \gamma\big)(x_t, \tilde{x}_u)(\xi, \tilde{X}_{u,v}^l \otimes \eta) \\ + \sum_{m=0}^{N-j} \sum_{l=0}^{\tilde{N}-k} \big(D_1^{j+m} D_2^{k+l} \gamma\big)(x_s, \tilde{x}_u)(X_{s,t}^m \otimes \xi, \tilde{X}_{u,v}^l \otimes \eta).\end{aligned}$$

To proceed, we use the following version of Taylor's theorem.

Lemma 3.8. *Suppose that $V, \tilde{V}$ and W are Banach spaces, where V and $\tilde{V}$ are finite dimensional, and let $f : V \times \tilde{V} \to W$ be a smooth function. Let $L, M \in \mathbb{N}$. For $a, \Delta a \in V$ and $b, \Delta b \in \tilde{V}$, define*

$$\begin{aligned}R := f(a + \Delta a,\ b + \Delta b) - \sum_{m=0}^{M} \frac{1}{m!} D_1^m f(a,\ b + \Delta b)\ \Delta a^{\otimes m} \\ - \sum_{l=0}^{L} \frac{1}{l!} D_2^l f(a + \Delta a,\ b)\ \Delta b^{\otimes l} + \sum_{m=0}^{M} \sum_{l=0}^{L} \frac{1}{m!\,l!} D_1^m D_2^l f(a,b)(\Delta a^m \otimes \Delta b^l).\end{aligned}$$

Then

$$\begin{aligned}R = \frac{1}{M!\,L!} \int_0^1 \int_0^1 D_1^{M+1} D_2^{L+1} f(a + r\Delta a,\ b + \lambda \Delta b) \\ \cdot [\Delta a^{\otimes M+1} \otimes \Delta b^{\otimes L+1}](1-r)^M (1-\lambda)^L\, dr\, d\lambda.\end{aligned} \tag{3.5}$$

Using the symmetry of derivatives, the relation (2.6) from the geometricity of X and $\tilde{X}$, and that partial derivatives commute in combination with Lemma 3.8, we have that for $\xi \in \mathrm{Sym}(V^{\otimes j})$, $\eta \in \mathrm{Sym}(\tilde{V}^{\otimes k})$,

$$\begin{aligned}\mathbf{R}^{(j,k)}\begin{pmatrix} s, t \\ u, v \end{pmatrix}(\xi,\eta) = \big(D_1^j D_2^k \gamma\big)(x_t, \tilde{x}_v)(\xi,\eta) \\ - \sum_{m=0}^{N-j} \frac{1}{m!} \big(D_1^{j+m} D_2^k \gamma\big)(x, \tilde{x}_u + \tilde{x}_{u,v})(x_{s,t}^{\otimes m} \otimes \xi, \eta) \\ - \sum_{l=0}^{\tilde{N}-k} \frac{1}{l!} \big(D_1^j D_2^{k+l} \gamma\big)(x_s + x_{s,t}, \tilde{x}_u)(\xi, \tilde{x}_{u,v}^{\otimes l} \otimes \eta)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=0}^{N-j}\sum_{l=0}^{\tilde N-k}\frac{1}{m!\,l!}\left(D_1^{j+m}D_2^{k+l}\gamma\right)(x_s,\tilde x_u)(x_{s,t}^{\otimes m}\otimes\xi,\ \tilde x_{u,v}^{\otimes l}\otimes\eta)\\
&=\frac{1}{(N-j)!(\tilde N-k)!}\int_0^1\int_0^1\left(D_1^{j+m}D_2^{k+l}\gamma\right)(x_s+rx_{s,t},\tilde x_u+\lambda\tilde x_{u,v})\\
&\qquad\cdot\left(x_{s,t}^{\otimes\lfloor p\rfloor-j}\otimes\xi,\ \tilde x_{u,v}^{\otimes\lfloor\tilde p\rfloor-k}\otimes\eta\right)(1-r)^{N-j}(1-\lambda)^{\tilde N-k}\,dr\,d\lambda.
\end{aligned}
$$

In this form, it is easy to see that $\|\mathbf{R}^{(j,k)}\|_{q_j,\tilde q_k}$ is finite for $q_j=p/(\lfloor p\rfloor-j)$ and $\tilde q_k=\tilde p/(\lfloor\tilde p\rfloor-k)$, and thus $\{Y^{(j,k)}\}$ satisfies Condition 3.6.

Proof of Lemma 3.8. Define

$$
g(b):=f(a+\Delta a,b)-\sum_{m=0}^{M}\frac{1}{m!}D_1^m f(a,b)\,\Delta a^{\otimes m}.
$$

The use of Taylor's theorem with integral remainder tells us that

$$
g(b)=\frac{1}{M!}\int_0^1 D_1^{M+1}f(a+r\Delta a,b)\,\Delta a^{\otimes M+1}(1-r)^M\,dr.
$$

Rewriting R and applying this version of Taylor's theorem again,

$$
\begin{aligned}
R&=f(a+\Delta a,\ b+\Delta b)-\sum_{m=0}^{M}\frac{1}{m!}D_1^m f(a,b+\Delta b)\ \Delta a^{\otimes m}\\
&\quad-\sum_{l=0}^{L}\frac{1}{l!}\Big(D_2^l f(a+\Delta a,b)\ \Delta b^{\otimes l}-\sum_{m=0}^{M}\frac{1}{m!}D_2^lD_1^m f(a,b)\big(\Delta a^m\otimes\Delta b^l\big)\Big)\\
&=g(b+\Delta b)-\sum_{l=0}^{L}\frac{1}{l!}D_2^l g(b)\ \Delta b^{\otimes l}\\
&=\frac{1}{L!}\int_0^1 D_2^{L+1}g(a,b+\lambda\Delta b)\ \Delta b^{\otimes L+1}(1-\lambda)^L\,d\lambda\\
&=\frac{1}{M!\,L!}\int_0^1\int_0^1 D_1^{M+1}D_2^{L+1}f(a+r\Delta a,\ b+\lambda\Delta b)\big[\Delta a^{\otimes M+1}\otimes\Delta b^{\otimes L+1}\big]\\
&\qquad\cdot(1-r)^M(1-\lambda)^L\,dr\,d\lambda.
\end{aligned}
$$
■

Under this condition of mixed variation, when $E=\mathrm{Bi}(V\times\tilde V\to W)$ we see that the rough integrals of jointly controlled paths with respect to X are controlled paths with respect to $\tilde X$ and the integrals with respect to $\tilde X$ are controlled paths with respect to X, which then proves existence of double integrals of jointly controlled paths. Going forward, it will be helpful to recall the tensor conventions (2.1) (2.2) that we use in this paper.

Lemma 3.9. *Suppose that $E=\mathrm{Bi}(V\times\tilde V\to W)$ for some Banach space W and that Condition 3.6 holds. For $j\in J$, $k\in K$, define $I_{\tilde X}^{(Y;j)}:[0,T]\times\Delta_T\to\mathrm{Hom}(V^{\otimes j},\mathrm{Hom}(V,W))$*

and $I_X^{(Y;k)}\colon \Delta_T \times [0,T] \to \mathrm{Hom}(\tilde{V}^{\otimes k}, \mathrm{Hom}(\tilde{V}, W))$ *by*

$$\mathcal{I}_{\tilde{X}}^{(Y;j)}\binom{r}{u,v}(\xi)(a) := \int_u^v Y_{r;\cdot}^{(j,\cdot)}\, d\tilde{X}(\xi)(a) = \lim_{|\tilde{\mathcal{D}}|\to 0} \sum_{[u_n,u_{n+1}]\subset \tilde{\mathcal{D}}} \Big(\sum_{k=0}^{\tilde{N}} Y_{r;u_n}^{(j,k)}(\xi \otimes a, \tilde{X}_{u_n,u_{n+1}}^{k+1})\Big), \tag{3.6}$$

and

$$\mathcal{I}_X^{(Y;k)}\binom{s,t}{r'}(\eta)(b) := \int_s^t Y_{\cdot;r'}^{(\cdot,k)}\, dX(\eta)(b) = \lim_{|\mathcal{D}|\to 0} \sum_{[s_m,s_{m+1}]\subset \mathcal{D}} \Big(\sum_{j=0}^{N} Y_{s_m;r'}^{(j,k)}(X_{s_m,s_{m+1}}^{j+1}, \eta \otimes b)\Big), \tag{3.7}$$

for $(\xi,\eta) \in V^{\otimes j} \times \tilde{V}^{\otimes k}$, *and* $(a,b) \in V \times \tilde{V}$. *Then, for every* $(s,t),(u,v) \in \Delta_T$,

$$\Big(\mathcal{I}_{\tilde{X}}^{(Y;0)}\binom{\cdot}{u,v},\dots,\mathcal{I}_{\tilde{X}}^{(Y;N)}\binom{\cdot}{u,v}\Big) \quad \text{and} \quad \Big(\mathcal{I}_X^{(Y;0)}\binom{s,t}{\cdot},\dots,\mathcal{I}_X^{(Y;\tilde{N})}\binom{s,t}{\cdot}\Big)$$

are X*-controlled and* $\tilde{X}$*-controlled paths on* $\mathrm{Hom}(V,W)$ *and* $\mathrm{Hom}(\tilde{V},W)$*, respectively.*

Proof. For $\xi \in V^{\otimes J}$ and $\eta \in \tilde{V}^{\otimes k}$, we first write

$$Y_{t;r'}^{(j,k)}(\xi,\eta) = \sum_{m=0}^{N-j} Y_{s;r'}^{(j+m,k)}(X_{s,t}^m \otimes \xi, \eta) + R^{(j,k)}\binom{s,t}{r'}(\xi,\eta).$$

For each $j \in J$, we decompose the $\tilde{X}$-controlled path $(Y_{t;\cdot}^{(j,k)}(\xi))_{k\in K}$ into the sum of $\tilde{X}$-controlled paths $(R^{(j,k)}\binom{s,t}{\cdot}(\xi))_{k\in K}$ and $(Y^{(j+m,k)}(X_{s,t}^m \otimes \xi))_{k\in K}$ for $m = 0,\dots,N-j$. Under this decomposition, we write

$$\begin{aligned}
\mathcal{I}_{\tilde{X}}^{(Y;j)}\binom{t}{u,v}(\xi)(a) &= \Big[\int_u^v \Big(\sum_{m=0}^{N-j} Y_{s;\cdot}^{(j+m,\cdot)}(X_{s,t}^m \otimes \xi) + R^{(j,\cdot)}\binom{s,t}{\cdot}(\xi)\Big)\, d\tilde{X}\Big](a) \\
&= \sum_{m=0}^{N-j} \Big[\int_u^v (Y_{s;\cdot}^{(j+m,\cdot)})\, d\tilde{X}\Big](X_{s,t}^m \otimes \xi)(a) + \Big[\int_u^v R^{(j,\cdot)}\binom{s,t}{\cdot}\, d\tilde{X}\Big](\xi)(a) \\
&= \sum_{m=0}^{N-j} \mathcal{I}_{\tilde{X}}^{(Y;j)}\binom{s}{u,v}(X_{s,t}^m \otimes \xi)(a) + \Big[\int_u^v R^{(j,\cdot)}\binom{s,t}{\cdot}\, d\tilde{X}\Big](\xi)(a).
\end{aligned}$$

All that is left now is to establish the regularity of the remainder integral above. Consider the quantity $\Xi_{Y,R_{\tilde{X}}}^{(j)}\colon \Delta_T \times \Delta_T \to \mathrm{Hom}(V^{\otimes j}, \mathrm{Hom}(V,W))$ defined by

$$\Xi_{Y,R_{\tilde{X}}}^{(j)}\binom{s,t}{u,v}(\xi)(a) = \sum_{k=0}^{\tilde{N}} R^{(j,k)}\binom{s,t}{u}(\xi, \tilde{X}_{u,v}^{k+1})(a).$$

From Lemmas 2.7 and 3.3, we can then write

$$\delta_2 \Xi^{(j)}_{Y,R_{\tilde X}}\binom{s,t}{u,u',v}(\xi)(a) = -\sum_{k=0}^{\tilde N} \mathbf{R}^{(j,k)}\binom{s,s'}{u,v}(\xi, \tilde X^{k+1}_{u',v})(a),$$

from which it follows that

$$\begin{aligned}\left\|\delta_2 \Xi^{(j)}_{Y,R_{\tilde X}}\binom{s,t}{\cdot,\cdot,\cdot}\right\|_{1/\theta_*,\tilde\omega} &\le \sum_{k=0}^{\tilde N} \left\|\mathbf{R}^{(j,k)}\binom{s,t}{\cdot,\cdot}\right\|_{\tilde q_k} \|\tilde X^{k+1}\|_{\tilde p_k}\, \tilde\omega(u,v)^{\tilde\theta_k-\theta_*} \\ &\le \sum_{k=0}^{\tilde N} \|\mathbf{R}^{(j,k)}\|_{q_j,\tilde q_k} \|\tilde X^{k+1}\|_{\tilde p_k}\, \omega(s,t)^{1/q_j}\, \tilde\omega(u,v)^{\tilde\theta_k-\theta_*}.\end{aligned}$$

We can now apply Lemma 2.8 to get the bound

$$\begin{aligned}\left|\int_u^v R^{(j,\cdot)}\binom{s,t}{\cdot}\, d\tilde X\right| &\le \left|\Xi^{(j)}_{Y,R_{\tilde X}}\binom{s,t}{u,v}\right| + 2^{\theta_*}\zeta(\theta_*)\, \omega(u,v)^{\theta_*}\left\|\delta_2 \Xi^{(j)}_{Y,R_{\tilde X}}\binom{s,t}{\cdot,\cdot,\cdot}\right\|_{1/\theta_*,\tilde\omega} \\ &\le \omega(s,t)^{1/q_j}\sum_{k=0}^{\tilde N}\left(\left\|R^{(j,k)}\binom{\cdot,\cdot}{u}\right\|_{q_j} \tilde\omega(u,v)^{1/\tilde q_k} \right. \\ &\qquad \left. +\ 2^{\theta_*}\zeta(\theta_*)\, \tilde\omega(u,v)^{\theta_k}\|\mathbf{R}^{(j,k)}\|_{q_j,\tilde q_k}\right)\|\tilde X^{k+1}\|_{\tilde p_k}.\end{aligned} \tag{3.8}$$

Thus this integral has finite (q_j,ω)-norm, and

$$\left(\mathcal{I}^{(Y;0)}_{\tilde X}\binom{\cdot}{u,v}, \dots, \mathcal{I}^{(Y;N)}_{\tilde X}\binom{\cdot}{u,v}\right)$$

is an X-controlled path. The proof follows similarly for

$$\left(\mathcal{I}^{(Y;0)}_{X}\binom{s,t}{\cdot}, \dots, \mathcal{I}^{(Y;\tilde N)}_{X}\binom{s,t}{\cdot}\right)$$

being an $\tilde X$-controlled path. ∎

Corollary 3.10. *Under the conditions of Lemma 3.9, we have existence of the double integrals* $\mathcal{I}^Y_{\tilde X,X}\colon \Delta_T\times\Delta_T \to W$ *and* $\mathcal{I}^Y_{X,\tilde X}\colon \Delta_T\times\Delta_T\to W$ *defined by*

$$\mathcal{I}^Y_{\tilde X,X}\binom{s,t}{u,v} := \int_s^t \mathcal{I}^{(Y;\cdot)}_{\tilde X}\binom{\cdot}{u,v}\, dX \quad \text{and} \quad \mathcal{I}^Y_{X,\tilde X}\binom{s,t}{u,v} := \int_u^v \mathcal{I}^{(Y;\cdot)}_{X}\binom{s,t}{\cdot}\, d\tilde X. \tag{3.9}$$

These double integrals justify the definition of jointly controlled paths as a class of two-parameter paths that are integrable with respect to two rough paths. When compared to the double integrals in [9], we have arbitrary p-variation and control, but have an additional condition on the mixed variation of second order remainders.

3.3. Stability of double integrals

Before we move on to work on a Fubini type theorem, we show that if two jointly controlled paths are close in some regard, then their double integrals are also close. For this section, we will use the same conventions as in Condition 3.6: for p and $\tilde{p}$, we let $p_j = p/(j+1)$ and $\tilde{p}_k = \tilde{p}/(k+1)$, and we consider $q_j, \tilde{q}_k, \theta_j$ and $\tilde{\theta}_k$ such that

$$\frac{1}{p_j} + \frac{1}{q_j} = \theta_j > 1 \quad \text{and} \quad \frac{1}{\tilde{p}_k} + \frac{1}{\tilde{q}_k} = \tilde{\theta}_k > 1.$$

We define

$$\theta^* := \max_{(j,k)\in J\times K} \{\theta_j, \tilde{\theta}_k\} \quad \text{and} \quad \theta_* := \min_{(j,k)\in J\times K} \{\theta_j, \tilde{\theta}_k\},$$

and we denote by $\bar{\boldsymbol{\theta}}$ the collection of $p, \tilde{p}, q_j, \tilde{q}_k$. The following estimate will be useful.

Lemma 3.11. *Let X and Z be p-rough paths on V with control ω. Suppose that $(F^{(j)})$ is an X-controlled path and $(G^{(j)})$ is a Z-controlled path whose remainders satisfy*

$$\|R^{(F;j)}\|_{q_j} < \infty \quad \text{and} \quad \|R^{(G;j)}\|_{q_j} < \infty.$$

There exists $C = C(\omega, T, p, \{q_j\})$ such that for any $(s,t) \in \Delta_T$,

$$\begin{aligned}\left|\mathcal{I}_X^F(s,t) - \mathcal{I}_Z^G(s,t)\right| \le C \sum_{j=0}^{N} \Big(&\|X^{j+1} - Z^{j+1}\|_{p_j}(|F_s^{(j)}| + \|R^{(F;j)}\|_{q_j}) \\ &+ (|G_s^{(j)} - F_s^{(j)}| + \|R^{(F;j)} - R^{(G;j)}\|_{q_j})\|Z^{j+1}\|_{p_j}\Big).\end{aligned}$$

Proof. Let $\Xi^{F,G}\colon \Delta_T \to W$ be defined by

$$\Xi^{F,G}(s,t) := \sum_{j=0}^{N} \big(F_s^{(j)}(X_{s,t}^{j+1}) - G_s^{(j)}(Z_{s,t}^{j+1})\big),$$

Elementary estimates of the form $|a_1 b_1 - a_2 b_2| \le |a_1||b_1 - b_2| + |a_1 - a_2||b_2|$ lead to

$$\begin{aligned}\left|\Xi^{F,G}(s,t)\right| &= \Big|\sum_{j=0}^{N} \big(F_s^{(j)}(X_{s,t}^{j+1} - Z_{s,t}^{j+1}) - (G_s^{(j)} - F_s^{(j)})(Z_{s,t}^{j+1})\big)\Big| \\ &\le \sum_{j=0}^{N} \big(|F_s^{(j)}|\,|X_{s,t}^{j+1} - Z_{s,t}^{j+1}| + |G_s^{(j)} - F_s^{(j)}|\,|Z_{s,t}^{j+1}|\big) \\ &\le \sum_{j=0}^{N} \big(|F_s^{(j)}|\,\|X^{j+1} - Z^{j+1}\|_{p_j} + |G_s^{(j)} - F_s^{(j)}|\,\|Z^{j+1}\|\big)\,\omega(s,t)^{1/p_j}.\end{aligned}$$

By (2.9) and similar estimates to above,

$$
\begin{aligned}
|\delta \Xi^{F,G}(s,s',t)| &\le \Big| \sum_{j=0}^{N} \big(R^{(F;j)}_{s,s'}(X^{j+1}_{s',t}) - R^{(G;j)}_{s,s'}(Z^{j+1}_{s',t}) \big) \Big| \\
&\le \sum_{j=0}^{N} |R^{(F;j)}_{s,s'}|\,|X^{j+1}_{s',t} - Z^{j+1}_{s',t}| + |R^{(F;j)}_{s,s'} - R^{(G;j)}_{s,s'}|\,|Z^{j+1}_{s',t}| \\
&\le \sum_{j=0}^{N} \big(\|R^{(F;j)}\|_{q_j} \|X^{j+1} - Z^{j+1}\|_{p_j} + \|R^{(F;j)} - R^{(G;j)}\|_{q_j} \|Z^{j+1}\|_{p_j} \big)\, \omega(s,t)^{\theta_j}.
\end{aligned}
$$

Combining the above two estimates with Lemma 2.8, we have

$$
\begin{aligned}
|\mathcal{I}^F_X(s,t) - \mathcal{I}^G_Z(s,t)| &\le |\Xi^{F,G}(s,t)| + 2^{\theta_*}\zeta(\theta_*)\,\omega(s,t)^{\theta_*} \|\!|\!|\delta \Xi^{F,G}|\!|\!|_{1/\theta_*} \\
&\le \sum_{j=0}^{N} \Big[\|X^{j+1} - Z^{j+1}\|_{p_j} |F^{(j)}_s|\, \omega(s,t)^{1/p_j} \\
&\quad + \|X^{j+1} - Z^{j+1}\|_{p_j} \|R^{(F;j)}\|_{q_j}\, \omega(s,t)^{\theta_j} \\
&\quad + \|Z^{j+1}\|_{p_j} |G^{(j)}_s - F^{(j)}_s|\, \omega(s,t)^{1/p_j} \\
&\quad + \|Z^{j+1}\|_{p_j} \|R^{(F;j)} - R^{(G;j)}\|_{q_j} \omega(s,t)^{\theta_j} \Big] \\
&\le C \sum_{j=0}^{N} \Big(\|X^{j+1} - Z^{j+1}\|_{p_j} \big(|F^{(j)}_s| + \|R^{(F;j)}\|_{q_j} \big) \\
&\quad + \big(|G^{(j)}_s - F^{(j)}_s| + \|R^{(F;j)} - R^{(G;j)}\|_{q_j} \big) \|Z^{j+1}\|_{p_j} \Big). \qquad \blacksquare
\end{aligned}
$$

We are now equipped to show a stability result akin to Theorem 4.17 of [6] for double rough integrals instead. Similarly to the result it is inspired by, we introduce a sort of distance between two jointly controlled paths, however this is not a true metric and in fact compares objects which generally live in different spaces.

Theorem 3.12 (Stability of double integrals). *Let X and Z be p-rough paths on V with control ω, and let $\tilde{X}$ and $\tilde{Z}$ be $\tilde{p}$-rough paths on $\tilde{V}$ with control $\tilde{\omega}$. Consider jointly controlled paths $\{Y^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]^2; \mathrm{Bi}(V \otimes \tilde{V} \to W))$ and $\{A^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{Z,\tilde{Z}}([0,T]^2; \mathrm{Bi}(V \otimes \tilde{V} \to W))$ with remainders $R^{(Y;j,k)}, \mathbf{R}^{(Y;j,k)}$ and $R^{(A;j,k)}, \mathbf{R}^{(A;j,k)}$, respectively. Suppose that Y and A both satisfy Condition 3.6 for the same $\overline{\boldsymbol{\theta}}$.*

For any $(s,t), (u,v) \in \Delta_T$, define the following distance between the two jointly controlled paths:

$$
\begin{aligned}
d^{(X,\tilde{X}),(Z,\tilde{Z}),\overline{\boldsymbol{\theta}}}_{[s,t],[u,v]}(Y,A) := \sum_{(j,k)\in J\times K} \Big[& |Y^{(j,k)}_{s;u} - A^{(j,k)}_{s;u}| + \Big\| R^{(Y;j,k)}\binom{\cdot,\cdot}{u} - R^{(A;j,k)}\binom{\cdot,\cdot}{u} \Big\|_{q_j} \\
&+ \Big\| R^{(Y;j,k)}\binom{s}{\cdot,\cdot} - R^{(A;j,k)}\binom{s}{\cdot,\cdot} \Big\|_{\tilde{q}_k} + \|\mathbf{R}^{(Y;j,k)} - \mathbf{R}^{(A;j,k)}\|_{q_j,\tilde{q}_k} \\
&+ \|X^{j+1} - Z^{j+1}\|_{q_j} + \|\tilde{X}^{k+1} - \tilde{Z}^{k+1}\|_{\tilde{q}_k} \Big].
\end{aligned}
$$

Let $M = M(s,u)$ be such that

$$\sup_{(j,k)\in J\times K}\left\{|Y^{(j,k)}_{s;u}|, \left\|R^{(Y;j,k)}\binom{\cdot,\cdot}{u}\right\|_{q_j}, \left\|R^{(Y;j,k)}\binom{s}{\cdot,\cdot}\right\|_{\tilde{q}_k}, \|\mathbf{R}^{(Y;j,k)}\|_{q_j,\tilde{q}_k}\right\} \le M,$$

$$\sup_{(j,k)\in J\times K}\left\{|A^{(j,k)}_{s;u}|, \left\|R^{(A;j,k)}\binom{\cdot,\cdot}{u}\right\|_{q_j}, \left\|R^{(A;j,k)}\binom{s}{\cdot,\cdot}\right\|_{\tilde{q}_k}, \|\mathbf{R}^{(A;j,k)}\|_{q_j,\tilde{q}_k}\right\} \le M,$$

$$\sup_{(j,k)\in J\times K}\left\{\|X^{j+1}\|_{p_j}, \|Z^{j+1}\|_{p_j}, \|\tilde{X}^{k+1}\|_{\tilde{p}_k}, \|\tilde{Z}^{k+1}\|_{\tilde{p}_k}\right\} \le M.$$

There exists $C(T,\overline{\boldsymbol{\theta}},M)$ such that

$$\left|\mathcal{I}^{Y}_{\tilde{X},X}\binom{s,t}{u,v} - \mathcal{I}^{A}_{\tilde{Z},Z}\binom{s,t}{u,v}\right| \le C(T,\overline{\boldsymbol{\theta}},M)\, d^{(X,\tilde{X}),(Z,\tilde{Z}),\overline{\boldsymbol{\theta}}}_{[s,t],[u,v]}(Y,A), \tag{3.10}$$

$$\left|\mathcal{I}^{Y}_{X,\tilde{X}}\binom{s,t}{u,v} - \mathcal{I}^{A}_{Z,\tilde{Z}}\binom{s,t}{u,v}\right| \le C(T,\overline{\boldsymbol{\theta}},M)\, d^{(X,\tilde{X}),(Z,\tilde{Z}),\overline{\boldsymbol{\theta}}}_{[s,t],[u,v]}(Y,A), \tag{3.11}$$

where we use the same notation as in Lemma 3.9 and Corollary 3.10.

Proof. We only prove the bound (3.10); the bound (3.11) follows similarly. First, from Lemma 3.9 and its proof, we know that

$$\left(\mathcal{I}^{(Y;j)}_{\tilde{X}}\binom{\cdot}{u,v}\right)_{j\in J} \quad\text{and}\quad \left(\mathcal{I}^{(A;j)}_{\tilde{Z}}\binom{\cdot}{u,v}\right)_{j\in J}$$

are X-controlled and Z-controlled paths, respectively, with remainders

$$R^{(\mathcal{I}^{Y}_{\tilde{X}};\,j)}\binom{s,t}{u,v} := \int_u^v R^{(Y;j,\cdot)}\binom{s,t}{\cdot}\,d\tilde{X} \quad\text{and}\quad R^{(\mathcal{I}^{A}_{\tilde{Z}};\,j)}\binom{s,t}{u,v} := \int_u^v R^{(A;j,\cdot)}\binom{s,t}{\cdot}\,d\tilde{Z},$$

which by (3.8) are such that there exists $C'(T,\overline{\boldsymbol{\theta}},M)$ with

$$\left\|R^{(\mathcal{I}^{Y}_{\tilde{X}};\,j)}\binom{\cdot,\cdot}{u,v}\right\|_{q_j} \le C'(T,\overline{\boldsymbol{\theta}},M), \quad \left\|R^{(\mathcal{I}^{A}_{\tilde{Z}};\,j)}\binom{\cdot,\cdot}{u,v}\right\|_{q_j} \le C'(T,\overline{\boldsymbol{\theta}},M). \tag{3.12}$$

We now proceed with multiple applications of Lemma 3.11, the first of which yields

$$\begin{aligned}\left|\mathcal{I}^{Y}_{\tilde{X},X}\binom{s,t}{u,v} - \mathcal{I}^{A}_{\tilde{Z},Z}\binom{s,t}{u,v}\right| \le C\sum_{j=0}^{N}\Bigg[&\|X^{j+1}-Z^{j+1}\|_{p_j}\left|\mathcal{I}^{(Y;j)}_{\tilde{X}}\binom{s}{u,v}\right| \\ &+ \|X^{j+1}-Z^{j+1}\|_{p_j}\left\|R^{(\mathcal{I}^{Y}_{\tilde{X}};j)}\binom{\cdot,\cdot}{u,v}\right\|_{q_j} \\ &+ \left|\mathcal{I}^{(Y;j)}_{\tilde{X}}\binom{s}{u,v} - \mathcal{I}^{(A;j)}_{\tilde{Z}}\binom{s}{u,v}\right|\|Z^{j+1}\|_{p_j} \\ &+ \left\|R^{(\mathcal{I}^{Y}_{\tilde{X}};j)}\binom{\cdot,\cdot}{u,v} - R^{(\mathcal{I}^{A}_{\tilde{Z}};j)}\binom{\cdot,\cdot}{u,v}\right\|_{q_j}\|Z^{j+1}\|_{p_j}\Bigg],\end{aligned} \tag{3.13}$$

where $C = C(\omega,T,\overline{\boldsymbol{\theta}})$ is as in Lemma 3.11.

A second application gives us

$$\begin{aligned}\left|R^{(\mathcal{J}^{Y}_{\tilde{X}};j)}\binom{s,t}{u,v} - R^{(\mathcal{J}^{A}_{\tilde{Z}};j)}\binom{s,t}{u,v}\right| &\leq C\sum_{k=0}^{\tilde{N}}\Big[\|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\left|R^{(Y;j,k)}\binom{s,t}{u}\right| \\ &\quad + \|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\left\|\mathbf{R}^{(Y;j,k)}\binom{s,t}{\cdot,\cdot}\right\|_{\tilde{q}_k} \\ &\quad + \left|R^{(Y;j,k)}\binom{s,t}{u} - R^{(A;j,k)}\binom{s,t}{u}\right|\|\tilde{Z}^{k+1}\|_{\tilde{p}_k} \\ &\quad + \left\|\mathbf{R}^{(Y;j,k)}\binom{s,t}{\cdot,\cdot} - \mathbf{R}^{(A;j,k)}\binom{s,t}{\cdot,\cdot}\right\|_{\tilde{q}_k}\|\tilde{Z}^{k+1}\|_{\tilde{p}_k}\Big] \\ &\leq C\sum_{k=0}^{\tilde{N}}\Big[\|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\left\|R^{(Y;j,k)}\binom{\cdot,\cdot}{u}\right\|_{q_j} \\ &\quad + \|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\|\mathbf{R}^{(Y;j,k)}\|_{q_j,\tilde{q}_k} \\ &\quad + M\left\|R^{(Y;j,k)}\binom{\cdot,\cdot}{u} - R^{(A;j,k)}\binom{\cdot,\cdot}{u}\right\|_{q_j} \\ &\quad + M\left\|\mathbf{R}^{(Y;j,k)} - \mathbf{R}^{(A;j,k)}\right\|_{q_j,\tilde{q}_k}\Big]\omega(s,t)^{1/q_j}, \end{aligned} \tag{3.14}$$

which gives us a bound on the q_j-norm of the difference of the two integral remainders. Applying Lemma 3.11 yet again,

$$\begin{aligned}\left|\mathcal{J}^{(Y;j)}_{\tilde{X}}\binom{s}{u,v} - \mathcal{J}^{(A;j)}_{\tilde{Z}}\binom{s}{u,v}\right| &\leq C\sum_{k=0}^{\tilde{N}}\|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\,|Y^{(j,k)}_{s,u}| \\ &\quad + \|\tilde{X}^{k+1}-\tilde{Z}^{k+1}\|_{\tilde{p}_k}\left\|R^{(Y;j,k)}\binom{s}{\cdot,\cdot}\right\|_{\tilde{q}_k} + |Y^{(j,k)}_{s,u} - A^{(j,k)}_{s,u}|\,\|\tilde{Z}^{k+1}\|_{\tilde{p}_k} \\ &\quad + \left\|R^{(Y;j,k)}\binom{s}{\cdot,\cdot} - R^{(A;j,k)}\binom{s}{\cdot,\cdot}\right\|_{\tilde{q}_k}\|\tilde{Z}^{k+1}\|_{\tilde{p}_k}\Big]. \end{aligned} \tag{3.15}$$

For the last bound that we need, we recall (2.13), which in this scenario gives us

$$\begin{aligned}\left|\mathcal{J}^{(Y;j)}_{\tilde{X}}\binom{s}{u,v}\right| &\leq C'(T,\bar{\boldsymbol{\theta}})\sum_{k=0}^{\tilde{N}}\Big[|Y^{(j,k)}_{s,u}|\|\tilde{X}^{k+1}\|_{\tilde{p}_k} + \|\tilde{X}^{k+1}\|_{\tilde{p}_k}\left\|R^{(Y;j,k)}\binom{s}{\cdot,\cdot}\right\|_{\tilde{q}_k}\Big] \\ &\leq C''(T,\bar{\boldsymbol{\theta}},M). \end{aligned} \tag{3.16}$$

Combining the bounds (3.12), (3.14), (3.15), (3.16) into the inequality (3.13), we are able to reduce to (3.10) as required. ■

4. A maximal inequality for discrete joint integrals

Having justified the existence of the double integrals, we now work towards a Fubini type theorem for rough integration of jointly controlled paths. Our approach is akin to the case of Young integration, whereby we first establish a maximal inequality over the discrete two-parameter integrals and then leverage this inequality to prove existence and uniqueness of the integral as the limit of the discrete two-parameter integrals over partitions with decreasing mesh size. We will refer to the iterated rough integrals $\mathcal{I}^{Y}_{\tilde{X},X}$ and $\mathcal{I}^{Y}_{X,\tilde{X}}$ as *double rough integrals* and refer to the third type of integral to be constructed as the *joint rough integral*.

Just as in [8], we will use *grid-like partitions* of $[s,t]\times[u,v]$ to refer to partitions of the form $\mathcal{D}\times\tilde{\mathcal{D}}$, where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are partitions of $[s,t]$ and $[u,v]$ respectively. For a function $\Omega:[0,T]^2\times[0,T]^2\to E$ on a Banach space E and partitions $\mathcal{D}=\{s_0<\cdots<s_{m_0}\}\subset[s,t]$ and $\tilde{\mathcal{D}}=\{u_0<\cdots<u_{n_0}\}\subset[u,v]$, we write

$$\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega:=\sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Omega\binom{s_m,\,s_{m+1}}{u_n,\,u_{n+1}},$$

which will be used to write the discrete two-parameter integral over $\mathcal{D}\times\tilde{\mathcal{D}}$ in the context of two parameter rough integration.

We follow the same type of argument as the two dimensional Young–Towghi maximal inequality of [19] as it is presented in the appendix of [8]. The main idea is as follows: given a discrete integral over a partition, we select a point to remove from the partition and then observe the change in the discrete integral. The particular point we remove is carefully chosen to keep this change small. We then repeat until the partition is reduced to the trivial partition. To keep notation more succinct, we introduce the following quantities.

Definition 4.1. Let $\{Y^{(j,k)}\}\in\mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]\times[0,T];E)$, where $E=\mathrm{Bi}(V\times\tilde{V}\to W)$ for some Banach space W. Denote by $\Omega^{Y}\colon\Delta_T\times\Delta_T\to W$ the local approximation of the joint integral,

$$\Omega^{Y}\binom{s,\,t}{u,\,v}=\sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}Y^{(j,k)}_{s;u}(X^{j+1}_{s,t},\,\tilde{X}^{k+1}_{u,v}).$$

Define $\Theta^{Y}\colon\Delta^3_T\times\Delta^3_T\to W$ by

$$\Theta^{Y}\binom{s,\,s',\,t}{u,\,u',\,v}=\delta_1\,\delta_2\,\Omega^{Y}\binom{s,\,s',\,t}{u,\,u',\,v}=\delta_2\,\delta_1\,\Omega^{Y}\binom{s,\,s',\,t}{u,\,u',\,v},$$

where we note that equality of the latter two terms follows from how δ_1 and δ_2 commute with each other.

For partitions $\mathcal{D}=\{s=s_0<\cdots<s_{m_0}=t\}$ and $\tilde{\mathcal{D}}=\{u=u_0<\cdots<u_{n_0}=v\}$, define the differences

$$\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}}:=\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^{Y}-\sum_{\mathcal{D}\setminus\{s_m\}\times\tilde{\mathcal{D}}}\Omega^{Y}=-\sum_{n=1}^{n_0}\delta_1\,\Omega^{Y}\binom{s_{m-1},\,s_m,\,s_{m+1}}{u_{n-1},\,u_n},\tag{4.1}$$

$$\Delta^{(\tilde{\mathcal{D}};u_n)}_{\mathcal{D}\times\tilde{\mathcal{D}}}:=\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^{Y}-\sum_{\mathcal{D}\times\tilde{\mathcal{D}}\setminus\{u_n\}}\Omega^{Y}=-\sum_{m=1}^{m_0}\delta_2\,\Omega^{Y}\binom{s_{m-1},\,s_m}{u_{n-1},\,u_n,\,u_{n+1}}.\tag{4.2}$$

The quantity $\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}}$ is the change to the discrete integral over $\mathcal{D}\times\tilde{\mathcal{D}}$ when removing a point s_m from the partition $\mathcal{D}$, and similarly, $\Delta^{(\tilde{\mathcal{D}};u_n)}_{\mathcal{D}\times\tilde{\mathcal{D}}}$ is the change from removing u_n

from the other partition $\tilde{\mathcal{D}}$. From Lemma 2.7 we can show that Θ^Y may be expressed in terms of second order remainders of Y.

Lemma 4.2. *Let* $\{Y^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]^2; \mathrm{Bi}(V \times \tilde{V} \to W))$. *Then the following identities hold*:

$$-\delta_1 \Omega^Y \begin{pmatrix} s, s', t \\ u, v \end{pmatrix} = \sum_{j=0}^{N} \sum_{k=0}^{\tilde{N}} R^{(j,k)} \begin{pmatrix} s, s' \\ u \end{pmatrix} (X^{j+1}_{s',t}, \tilde{X}^{k+1}_{u,v}), \tag{4.3}$$

$$-\delta_2 \Omega^Y \begin{pmatrix} s, t \\ u, u', v \end{pmatrix} = \sum_{j=0}^{N} \sum_{k=0}^{\tilde{N}} R^{(j,k)} \begin{pmatrix} s \\ u, u' \end{pmatrix} (X^{j+1}_{s,t}, \tilde{X}^{k+1}_{u',v}), \tag{4.4}$$

$$\Theta^Y \begin{pmatrix} s, s', t \\ u, u', v \end{pmatrix} = \sum_{j=0}^{N} \sum_{k=0}^{\tilde{N}} \mathbf{R}^{(j,k)} \begin{pmatrix} s, s' \\ u, u' \end{pmatrix} (X^{j+1}_{s',t}, \tilde{X}^{k+1}_{u',v}). \tag{4.5}$$

Proof. The operators δ, δ_1 and δ_2 are linear, so applying Lemma 2.7 we have

$$\begin{aligned} \delta_1 \Omega^Y \begin{pmatrix} s, s', t \\ u, v \end{pmatrix} &= \sum_{k=0}^{\tilde{N}} \delta_1 \Big(\sum_{j=0}^{N} Y^{(j,k)}_{s;u} (X^{j+1}_{s,t}, \tilde{X}^{k+1}_{u,v}) \Big) \begin{pmatrix} s, s', t \\ u, v \end{pmatrix} \\ &= - \sum_{k=0}^{\tilde{N}} \sum_{j=0}^{N} R^{(j,k)} \begin{pmatrix} s, s' \\ u \end{pmatrix} (X^{j+1}_{s',t}, \tilde{X}^{k+1}_{u,v}), \end{aligned}$$

and similarly for $\delta_2 \Omega^Y$. We now use the fact that the remainders themselves are controlled paths (Lemma 3.3) to similarly apply Lemma 2.7 on $\delta_1 \Omega^Y$:

$$\begin{aligned} \delta_2 \delta_1 \Omega^Y \begin{pmatrix} s, s', t \\ u, u', v \end{pmatrix} &= - \sum_{j=0}^{N} \delta_2 \Big(\sum_{k=0}^{\tilde{N}} R^{(j,k)} \begin{pmatrix} s, s' \\ u, v \end{pmatrix} (X^{j+1}_{s',t}, \tilde{X}^{k+1}_{u,v}) \Big) \begin{pmatrix} s, s', t \\ u, u', v \end{pmatrix} \\ &= \sum_{j=0}^{N} \sum_{k=0}^{\tilde{N}} \mathbf{R}^{(j,k)} \begin{pmatrix} s, s' \\ u, u' \end{pmatrix} (X^{j+1}_{s',t}, \tilde{X}^{k+1}_{u',v}). \end{aligned}$$

■

The following selection lemma will allow us to later find "good" points to remove from partitions, and is a key step towards the maximal inequality

Lemma 4.3. *Let* I *be a finite non-empty index set,* $d_0 \in \mathbb{N}$*, and for* $d = 1, 2, \ldots, d_0$*, suppose* $\mathcal{P}_d := \sum_{i \in I} c_i(d)$ *are such that all* $c_i(d) \geq 0$*. Then there exist* $B \subset \{1, \ldots, d_0\}$*,* $d^* \in B$*, and* $i_* \in I$ *such that* $|B| \geq d_0/|I|$ *and*

$$\frac{\mathcal{P}_{d^*}}{|I|} \leq \Big[\prod_{d \in B} c_{i_*}(d) \Big]^{1/|B|}. \tag{4.6}$$

Proof. Define c^* by $c^*(d) := \sup_{i \in I} c_i(d)$, and let

$$B_i := \big\{ d \in \{1, \ldots, d_0\} \mid c^*(d) = c_i(d) \big\}$$

which may be empty. Then $|B_i|$ counts the number of d such that $c_i(d)$ is maximal. Since each d must belong to at least one B_i,

$$\sum_{i\in I} |B_i| \geq n_0,$$

and for $i_* = \arg\max\{|B_i| \mid i \in I\}$ we must have $|B_{i_*}| \geq d_0/|I|$, as assuming otherwise leads to the contradiction $\sum_{i\in I} |B_i| < d_0$. Letting $d^* = \arg\min\{c_{i_*}(d) \mid d \in B_{i_*}\}$, we have

$$\mathcal{P}_{d^*} \leq |I| \cdot c^*(d^*) \leq |I| \Big[\prod_{d\in B_{i_*}} c^*(d) \Big]^{1/|B_{i_*}|} = |I| \Big[\prod_{d\in B_{i_*}} c_{i_*}(d) \Big]^{1/|B_{i_*}|},$$

and letting $B = B_{i_*}$, we prove the statement. ■

We now use the identities of Lemma 4.2 together with the above selection lemma to prove the following lemma, which gives us a bound from suitable choice of a point in the partitions and is critical towards deriving the desired maximal inequality.

Lemma 4.4. *Let* $\{Y^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]^2; E)$*, where* $E = \mathrm{Bi}(V \otimes \tilde{V} \to W)$ *for some Banach space* W*. Suppose that Condition 3.6 holds and let* $\alpha \in (1/\theta_*, 1)$*. Given two partitions* $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\}$ *and* $\tilde{\mathcal{D}} = \{u = u_0 < \cdots < u_{n_0} = v\}$ *on* $[0,T]$*, define* $\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}$ *by*

$$\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}}; u_n) := \sum_{m=1}^{m_0-1} \left|\Theta^Y \begin{pmatrix} s_{m-1}, s_m, s_{m+1} \\ u_{n-1}, u_n, u_{n+1} \end{pmatrix}\right|^{\alpha},$$

$$\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\mathcal{D}; s_m) := \sum_{n=1}^{n_0-1} \left|\Theta^Y \begin{pmatrix} s_{m-1}, s_m, s_{m+1} \\ u_{n-1}, u_n, u_{n+1} \end{pmatrix}\right|^{\alpha},$$

for $m \in \{1, \ldots, m_0 - 1\}$ *and* $n \in \{1, \ldots, n_0 - 1\}$*. There exist* $m^* \in \{1, \ldots m_0 - 1\}$ *and* $n^* \in \{1, \ldots, n_0 - 1\}$ *such that*

$$\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}}; u_{n^*}) \leq \frac{C_1(\bar{\boldsymbol{\theta}}, T, \alpha)}{(n_0 - 1)^{\alpha\theta_*}} \big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*}\big]^{\alpha}, \tag{4.7}$$

$$\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\mathcal{D}; s_{m^*}) \leq \frac{C_1(\bar{\boldsymbol{\theta}}, T, \alpha)}{(m_0 - 1)^{\alpha\theta_*}} \big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*}\big]^{\alpha}, \tag{4.8}$$

where $C_1(\bar{\boldsymbol{\theta}}, T, \alpha)$ *is a constant independent of* Y*, and* $A(\bar{\boldsymbol{\theta}}, \mathbf{R}) = A(\bar{\boldsymbol{\theta}}, X, \tilde{X}, \mathbf{R})$ *is the quantity*

$$A(\bar{\boldsymbol{\theta}}, \mathbf{R}) = \Big(\sup_{j\in J} \|X^{j+1}\|_{p_j}\Big)\Big(\sup_{k\in K} \|\tilde{X}^{k+1}\|_{\tilde{p}_k}\Big)\Big(\sup_{(j,k)\in J\times K} \|\mathbf{R}^{(j,k)}\|_{q_j,\tilde{q}_k}\Big).$$

Proof. We will only prove the inequality for $\mathcal{P}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}}; u_n)$; the other inequality follows similarly. Since $\alpha < 1$, the inequality $(\sum_{i\in I} |a_i|)^{\alpha} \leq \sum_{i\in I} |a_i|^{\alpha}$ holds for any $(a_i)_{i\in I}$

over a finite index set I. Using the identity (4.5), we can then write

$$
\begin{aligned}
\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}};u_n) &\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\sum_{m=1}^{m_0-1}\left|\mathbf{R}^{(j,k)}\binom{s_{m-1},s_m}{u_{n-1},u_n}\right|^{\alpha}|X^{j+1}_{s_m,s_{m+1}}|^{\alpha}|\tilde{X}^{k+1}_{u_n,u_{n+1}}|^{\alpha}\\
&\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\Bigg[\Big(\sum_{m=1}^{m_0-1}\left|\mathbf{R}^{(j,k)}\binom{s_{m-1},s_m}{u_{n-1},u_n}\right|^{\alpha\theta_j q_j}\Big)^{\frac{1}{\theta_j q_j}}\\
&\qquad\cdot\Big(\sum_{m=1}^{m_0-1}|X^{j+1}_{s_m,s_{m+1}}|^{\alpha\theta_j p_j}\Big)^{\frac{1}{\theta_j p_j}}|\tilde{X}^{k+1}_{u_n,u_{n+1}}|^{\alpha}\Bigg]\\
&\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\Bigg[\Big(\sum_{m=1}^{m_0-1}\left|\mathbf{R}^{(j,k)}\binom{s_{m-1},s_m}{u_{n-1},u_n}\right|^{q_j}\Big)^{\alpha/q_j}\\
&\qquad\cdot\Big(\sum_{m=1}^{m_0-1}|X^{j+1}_{s_m,s_{m+1}}|^{p_j}\Big)^{\alpha/p_j}|\tilde{X}^{k+1}_{u_n,u_{n+1}}|^{\alpha}\Bigg]\\
&\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\Bigg[\Big(\sum_{m=1}^{m_0-1}\|\mathbf{R}^{(j,k)}\|^{q_j}_{q_j,\tilde{q}_k}\,\omega(s_{m-1},s_m)\,\tilde{\omega}(u_{n-1},u_n)^{q_j/\tilde{q}_k}\Big)^{1/q_j}\\
&\qquad\cdot\Big(\sum_{m=1}^{m_0-1}\|X^{j+1}\|^{p_j}_{p_j}\,\omega(s_m,s_{m+1})\Big)^{1/p_j}|\tilde{X}^{k+1}_{u_n,u_{n+1}}|\Bigg]^{\alpha}\\
&\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\Big[\|\mathbf{R}^{(j,k)}\|_{q_j,\tilde{q}_k}\,\|X^{j+1}\|_{p_j}\\
&\qquad\cdot|\tilde{X}^{k+1}_{u_n,u_{n+1}}|\,\omega(s,t)^{1/\theta_j}\,\tilde{\omega}(u_{n-1},u_n)^{1/\tilde{q}_k}\Big]^{\alpha},
\end{aligned}
$$

where in the second line we use Hölder's inequality, and in the next line we use that $\alpha\theta_j > 1$, so the $\ell^{\alpha\theta_j q_j}$-norm is dominated by the ℓ^{q_j}-norm.

Applying Lemma 4.3 with $I = K$, $d_0 = n_0 - 1$, and

$$c_k(n) := \big(|\tilde{X}^{k+1}_{u_n,u_{n+1}}|\,\tilde{\omega}(u_{n-1},u_n)^{1/\tilde{q}_k}\big)^{\alpha},$$

there exist B such that $|B| \ge (n_0-1)/\lfloor\tilde{p}\rfloor$, $n^* \in B$ and $k_* \in K$ such that

$$
\begin{aligned}
\sum_{k=0}^{\tilde{N}} c_{j,k}(n^*) &\le \lfloor\tilde{p}\rfloor\Big[\prod_{n\in B} c_{j_*,k_*}(n)\Big]^{1/|B|}\\
&= \lfloor\tilde{p}\rfloor\Big[\prod_{n\in B}|\tilde{X}^{k_*+1}_{u_n,u_{n+1}}|^{\tilde{p}_{k_*}}\Big]^{\frac{\alpha}{|B|\tilde{p}_{k_*}}}\Big[\prod_{n\in B}\tilde{\omega}(u_{n-1},u_n)\Big]^{\frac{\alpha}{|B|\tilde{q}_{k_*}}}.
\end{aligned}
$$

By the inequality of arithmetic and geometric means on each product,

$$\begin{aligned}\sum_{k=0}^{\tilde{N}} c_{j,k}(n^*) &\leq \lfloor \tilde{p} \rfloor \Big(\frac{1}{|B|} \sum_{n \in B} |\tilde{X}^{k_*+1}_{u_n,u_{n+1}}|^{\tilde{p}_{k*}} \Big)^{\alpha/\tilde{p}_{k*}} \Big(\frac{1}{|B|} \sum_{n \in B} \tilde{\omega}(u_{n-1}, u_n) \Big)^{\alpha/\tilde{q}_{k*}} \\ &\leq \frac{\lfloor \tilde{p} \rfloor^{\alpha\theta_{k*}+1}}{(n_0-1)^{\alpha\theta_{k*}}} \Big(\sum_{n=0}^{n_0-1} \|\tilde{X}^{k_*+1}\|^{\tilde{p}_{k*}}_{\tilde{p}_{k*}} \tilde{\omega}(u_n, u_{n+1})^{1/\tilde{p}_{k*}} \Big)^{\alpha/\tilde{p}_{k*}} \\ &\qquad \cdot \Big(\sum_{n=0}^{n_0-1} \tilde{\omega}(u_{n-1}, u_n) \Big)^{\alpha/\tilde{q}_{k*}} \\ &\leq \frac{\lfloor \tilde{p} \rfloor^{\alpha\theta_{k*}+1}}{(n_0-1)^{\alpha\theta_{k*}}} \|\tilde{X}^{k_*+1}\|^{\alpha}_{\tilde{p}_{k*}} \tilde{\omega}(u,v)^{\alpha\tilde{\theta}_{k*}}.\end{aligned}$$

Returning to $\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}}; u_n)$ with $n = n^*$,

$$\begin{aligned}\mathcal{P}^{\alpha}_{\mathcal{D}\times\tilde{\mathcal{D}}}(\tilde{\mathcal{D}}; u_{n^*}) &\leq \Big(\sup_{j \in J} \|X^{j+1}\|^{\alpha}_{p_j} \Big) \Big(\sup_{(j,k)\in J\times K} \|\mathbf{R}^{(j,k)}\|^{\alpha}_{q_j,\tilde{q}_k} \Big) \\ &\qquad \cdot C(\bar{\boldsymbol{\theta}}, T, \alpha)\, \omega(s,t)^{\alpha\theta_*} \sum_{j=0}^{N} \sum_{k=0}^{\tilde{N}} c_{j,k}(n^*) \\ &\leq A(\bar{\boldsymbol{\theta}}, \mathbf{R})^{\alpha}\, C(\bar{\boldsymbol{\theta}}, T, \alpha)\, \omega(s,t)^{\alpha\theta_*} \sum_{j=0}^{N} \frac{\lfloor \tilde{p} \rfloor^{\alpha\theta_{k*}+1}}{(n_0-1)^{\alpha\theta_{k*}}} \tilde{\omega}(u,v)^{\alpha\tilde{\theta}_{k*}} \\ &\leq \frac{C_1(\bar{\boldsymbol{\theta}}, T, \alpha)}{(n_0-1)^{\alpha\theta_*}} \big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*} \big]^{\alpha},\end{aligned}$$

where we use that for any $\mu > \nu > 1$ there exists $C(\mu, \nu, T)$ such that for all $(s,t) \in \Delta_T$, the inequality $\omega(s,t)^{\mu} \leq C(\mu,\nu,T)\, \omega(s,t)^{\nu}$ holds. ■

The proof of this bound draws inspiration from the proof of Lemma 6.4 of [8] (the original result of this lemma is from [19]), with suitable modifications made to be used in the context of rough paths and jointly controlled paths. The following bounds on $\Delta_{\mathcal{D}\times\tilde{\mathcal{D}}}$, defined by (4.1) and (4.2), are a consequence of the above lemma.

Corollary 4.5. *Under the conditions of Lemma 4.4, there exists a constant $C_2(\bar{\boldsymbol{\theta}}, T, \alpha)$ such that*

$$\sum_{m=1}^{m_0-1} \big| \Delta^{(\mathcal{D}; s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}; s_m)}_{\mathcal{D}\times\{u,v\}} \big|^{\alpha} \leq C_2(\bar{\boldsymbol{\theta}}, T, \alpha) \big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*} \big]^{\alpha}, \tag{4.9}$$

$$\sum_{n=1}^{n_0-1} \big| \Delta^{(\tilde{\mathcal{D}}; u_n)}_{\mathcal{D}\times\tilde{\mathcal{D}}} - \Delta^{(\tilde{\mathcal{D}}; u_n)}_{\{s,t\}\times\tilde{\mathcal{D}}} \big|^{\alpha} \leq C_2(\bar{\boldsymbol{\theta}}, T, \alpha) \big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*} \big]^{\alpha}, \tag{4.10}$$

where we recall that the $\Delta_{\mathcal{D}\times\tilde{\mathcal{D}}}$ are defined by (4.1) and (4.2).

Proof. For proving (4.9), we first note that for $(m,n) \in \{1,\dots,m_0-1\} \times \{1,\dots,n_0-1\}$, we have

$$\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}} - \Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}\setminus\{u_n\}} = \Theta^Y\begin{pmatrix} s_{m-1}, s_m, s_{m+1} \\ u_{n-1}, u_n, u_{n+1}\end{pmatrix}.$$

By Lemma 4.4, we can choose n^* such that (4.7) holds. We now apply this iteratively to successively remove points from the partition $\tilde{\mathcal{D}}$. Letting $\tilde{\mathcal{D}}_0 := \tilde{\mathcal{D}}$, for $l = 1,\dots,n_0-1$, recursively define

$$\tilde{\mathcal{D}}_l := \tilde{\mathcal{D}}'_{l-1} \setminus \{u_{n^*_l}\},$$

where n^*_l is chosen as in Lemma 4.4 applied to $\mathcal{D} \times \tilde{\mathcal{D}}_{l-1}$.

Since $\alpha < 1$, for any sequence $(a_n)_{n\in\mathbb{N}} \in \ell^1$,

$$\Big(\sum_{n\in\mathbb{N}} |a_n|\Big)^\alpha = \Big(\sum_{n\in\mathbb{N}} |a_n|^{\alpha/\alpha}\Big)^\alpha = \big\|(|a_n|^\alpha)_{n\in\mathbb{N}}\big\|_{\ell^{1/\alpha}} \le \big\|(|a_n|^\alpha)_{n\in\mathbb{N}}\big\|_{\ell^1} = \sum_{n\in\mathbb{N}} |a_n|^\alpha.$$

This then gives us the inequality

$$\begin{aligned}
\sum_{m=1}^{m_0-1} \big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}} - \Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\{u,v\}}\big|^\alpha &\le \sum_{m=1}^{m_0-1}\Big(\sum_{l=1}^{n_0-1}\big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}_{l-1}} - \Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}_l}\big|\Big)^\alpha \\
&\le \sum_{m=1}^{m_0-1}\sum_{l=1}^{n_0-1}\big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}_{l-1}} - \Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}_l}\big|^\alpha \\
&= \sum_{l=1}^{n_0-1} \mathcal{P}^\alpha_{\mathcal{D}\times\tilde{\mathcal{D}}_{l-1}}(\tilde{\mathcal{D}}_{l-1}; u_{n^*_l}) \\
&\le \sum_{l=1}^{n_0-1} \frac{C_1(\bar{\boldsymbol{\theta}}, T, \alpha)}{(n_0-l)^{\alpha\theta_*}}\big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\,\omega(s,t)^{\theta_*}\,\tilde{\omega}(u,v)^{\theta_*}\big]^\alpha \\
&\le C_1(\bar{\boldsymbol{\theta}}, T, \alpha)\,\zeta(\alpha\theta_*)\big[A(\bar{\boldsymbol{\theta}}, \mathbf{R})\,\omega(s,t)^{\theta_*}\,\tilde{\omega}(u,v)^{\theta_*}\big]^\alpha,
\end{aligned}$$

and similarly for (4.10), taking $C_2(\bar{\boldsymbol{\theta}}, T, \alpha) = C_1(\bar{\boldsymbol{\theta}}, T, \alpha)\,\zeta(\alpha\theta_*)$. ■

We are now in a position to make the following bounds on the discrete integrals, which are independent from the choice of grid-like partition.

Lemma 4.6. *Suppose that Condition 3.6 holds, and let $E = \mathrm{Bi}(V \times \tilde{V} \to W)$, and $\alpha \in (1/\theta_*, 1)$. Define $B_X(u,v) = B_X(\bar{\boldsymbol{\theta}}, R, u, v)$ and $B_{\tilde{X}}(s,t) = B_{\tilde{X}}(\bar{\boldsymbol{\theta}}, R, s, t)$ by*

$$\begin{aligned}
B_X(u,v) &= \sup_{(j,k)\in J\times K}\Big[\|X^{j+1}\|_{p_j}\,\|\tilde{X}^{k+1}\|_{\tilde{p}_k}\,\Big\|R^{(j,k)}\begin{pmatrix}\cdot,\cdot\\ u\end{pmatrix}\Big\|_{q_j}\tilde{\omega}(u,v)^{1/\tilde{p}_k}\Big], \\
B_{\tilde{X}}(s,t) &= \sup_{(j,k)\in J\times K}\Big[\|X^{j+1}\|_{p_j}\,\|\tilde{X}^{k+1}\|_{\tilde{p}_k}\,\Big\|R^{(j,k)}\begin{pmatrix} s\\ \cdot,\cdot\end{pmatrix}\Big\|_{\tilde{q}_k}\omega(s,t)^{1/p_j}\Big].
\end{aligned}$$

Then the following holds:

(1) *There exists a constant* $C_4(\overline{\boldsymbol{\theta}}, T, \alpha)$ *such that, for any two partitions* $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\}$ *and* $\tilde{\mathcal{D}} = \{u = u_0 < \cdots < u_{n_0} = v\}$*, we have*

$$\Big| \sum_{\mathcal{D}\times\{u,v\}} \Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big| \le C_4(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_X(u,v)\,\omega(s,t)^{\theta_*}, \tag{4.11}$$

$$\Big| \sum_{\{s,t\}\times\tilde{\mathcal{D}}} \Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big| \le C_4(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_{\tilde{X}}(s,t)\,\tilde{\omega}(u,v)^{\theta_*}. \tag{4.12}$$

(2) *There exists a constant* $C_5(\overline{\boldsymbol{\theta}}, T, \alpha)$ *such that, for any two partitions* $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\}$ *and* $\tilde{\mathcal{D}} = \{u = u_0 < \cdots < u_{n_0} = v\}$ *we have*

$$\begin{aligned} \Big| \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y - \sum_{\mathcal{D}\times\{u,v\}} \Omega^Y - \sum_{\{s,t\}\times\tilde{\mathcal{D}}} \Omega^Y + \Omega^Y\binom{s,t}{u,v}\Big| \\ \le C_5(\overline{\boldsymbol{\theta}}, T, \alpha)\, A(\overline{\boldsymbol{\theta}}, \mathbf{R})\,\omega(s,t)^{\theta_*}\,\tilde{\omega}(u,v)^{\theta_*}. \end{aligned} \tag{4.13}$$

Proof. (1) We first bound the sums

$$\sum_{m=1}^{m_0-1} \big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\{u,v\}}\big|^\alpha \quad \text{and} \quad \sum_{n=1}^{n_0-1} \big|\Delta^{(\tilde{\mathcal{D}};u_n)}_{\{s,t\}\times\tilde{\mathcal{D}}}\big|^\alpha$$

using similar calculations to those in Lemma 4.4:

$$\begin{aligned} \sum_{m=1}^{m_0-1} \big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\{u,v\}}\big|^\alpha &= \sum_{m=1}^{m_0-1} \big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\tilde{\mathcal{D}}} - \Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\{u,v\}}\big|^\alpha \\ &\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}}\sum_{m=1}^{m_0-1} \Big|R^{(j,k)}\binom{s_{m-1}, s_m}{u}\Big|^\alpha |X^{j+1}_{s_m,s_{m+1}}|^\alpha\, |\tilde{X}^{k+1}_{u,v}|^\alpha \\ &\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}} \Big[\Big(\sum_{m=1}^{m_0-1} \Big|R^{(j,k)}\binom{s_{m-1}, s_m}{u}\Big|^{q_j}\Big)^{1/q_j} \Big(\sum_{m=1}^{m_0-1} |X^{j+1}_{s_m,s_{m+1}}|^{p_j}\Big)^{1/p_j} |\tilde{X}^{k+1}_{u,v}|\Big]^\alpha \\ &\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}} \Big[\|X^{j+1}\|_{p_j}\,\|\tilde{X}^{k+1}\|_{\tilde{p}_k} \Big\|R^{(j,k)}\binom{\cdot,\cdot}{u}\Big\|_{q_j}\Big]^\alpha \omega(s,t)^{\alpha\theta_j}\,\tilde{\omega}(u,v)^{\alpha/\tilde{p}_k} \\ &\le \sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}} B_X(u,v)^\alpha\,\omega(s,t)^{\alpha\theta_j}. \end{aligned}$$

Using that $\omega(s,t)^{\theta_j} \le C_j(\overline{\boldsymbol{\theta}}, T)\,\omega(s,t)^{\theta_*}$ and $\tilde{\omega}(s,t)^{\tilde{\theta}_k} \le \tilde{C}_k(\overline{\boldsymbol{\theta}}, T)\,\tilde{\omega}(s,t)^{\theta_*}$ for some constants C_j and $\tilde{C}_k$, there exists $C_3(\overline{\boldsymbol{\theta}}, T, \alpha)$ such that

$$\sum_{m=1}^{m_0-1} \big|\Delta^{(\mathcal{D};s_m)}_{\mathcal{D}\times\{u,v\}}\big|^\alpha \le C_3(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_X(u,v)^\alpha\,\omega(s,t)^{\alpha\theta_*}, \tag{4.14}$$

$$\sum_{n=1}^{n_0-1} \big|\Delta^{(\tilde{\mathcal{D}};u_n)}_{\{s,t\}\times\tilde{\mathcal{D}}}\big|^\alpha \le C_3(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_{\tilde{X}}(s,t)^\alpha\,\tilde{\omega}(u,v)^{\alpha\theta_*}. \tag{4.15}$$

Again we arrive at inequalities (4.11), (4.12), (4.13) from here by repeatedly selecting points to remove from partitions in a way that allows us to adequately bound the change in sums over the partitions. Let $\mathcal{D}_0 := \mathcal{D}$, and for $l = 1, \ldots, m_0 - 1$, let $\mathcal{D}_l = \mathcal{D}_{l-1} \setminus \{m_l\}$, where

$$m_l := \arg\min \left\{ \left| \Delta^{(\mathcal{D}_{l-1}; s_m)}_{\mathcal{D}_{l-1} \times \{u,v\}} \right| : s_m \in \mathcal{D}_{l-1} \setminus \{s,t\} \right\}.$$

With this choice of m_l, it follows that

$$\begin{aligned} \left| \Delta^{(\mathcal{D}_{l-1}; s_{m_l})}_{\mathcal{D}_{l-1} \times \{u,v\}} \right|^{\alpha} &\leq \frac{1}{m_0 - l} \sum_{s_m \in \mathcal{D}_{l-1}} \left| \Delta^{(\mathcal{D}_{l-1}; s_m)}_{\mathcal{D}_{l-1} \times \{u,v\}} \right|^{\alpha} \\ &\leq \frac{1}{m_0 - l} \, C_3(\bar{\boldsymbol{\theta}}, T, \alpha) \, B_X(u,v)^{\alpha} \, \omega(s,t)^{\alpha\theta_*}. \end{aligned}$$

This then leads to the desired inequality (4.11):

$$\begin{aligned} \Big| \sum_{\mathcal{D} \times \{u,v\}} \Omega^Y - \Omega^Y \binom{s,t}{u,v} \Big| &\leq \sum_{l=1}^{m_0-1} \left| \Delta^{(\mathcal{D}_{l-1}; s_{m_l})}_{\mathcal{D}_{l-1} \times \{u,v\}} \right| \\ &\leq \sum_{l=1}^{m_0-1} \left[\frac{1}{m_0 - l} \, C_3(\bar{\boldsymbol{\theta}}, T, \alpha) \, B_X(u,v)^{\alpha} \, \omega(s,t)^{\alpha\theta_*} \right]^{1/\alpha} \\ &\leq \zeta\left(\frac{1}{\alpha}\right) \left[C_3(\bar{\boldsymbol{\theta}}, T, \alpha) \, B_X(u,v)^{\alpha} \, \omega(s,t)^{\alpha\theta_*} \right]^{1/\alpha}, \end{aligned}$$

and similarly for (4.12), where we let $C_4(\bar{\boldsymbol{\theta}}, T, \alpha) = \zeta(1/\alpha) \, C_3(\bar{\boldsymbol{\theta}}, T, \alpha)^{1/\alpha}$.

(2) We now repeat this process for (4.13). Let $\mathcal{D}^*_0 := \mathcal{D}$ and for $l = 1, \ldots, m_0 - 1$, recursively define $\mathcal{D}^*_l = \mathcal{D}^*_{l-1} \setminus \{m^*_l\}$, with

$$m^*_l := \arg\min \left\{ \left| \Delta^{(\mathcal{D}^*_{l-1}; s_m)}_{\mathcal{D}^*_{l-1} \times \tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1}; s_m)}_{\mathcal{D}^*_{l-1} \times \{u,v\}} \right| : s_m \in \mathcal{D}^*_{l-1} \setminus \{s,t\} \right\}.$$

Using this choice of m^*_l in conjunction with Corollary 4.5,

$$\begin{aligned} \left| \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \{u,v\}} \right|^{\alpha} &\leq \frac{1}{m_0 - l} \sum_{s_m \in \mathcal{D}^*_{l-1}} \left| \Delta^{(\mathcal{D}^*_{l-1}; s_m)}_{\mathcal{D}^*_{l-1} \times \tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1}; s_m)}_{\mathcal{D}^*_{l-1} \times \{u,v\}} \right|^{\alpha} \\ &\leq \frac{1}{m_0 - l} \, C_2(\bar{\boldsymbol{\theta}}, T, \alpha) \left[A(\bar{\boldsymbol{\theta}}, \mathbf{R}) \, \omega(s,t)^{\theta_*} \, \tilde{\omega}(u,v)^{\theta_*} \right]^{\alpha}. \end{aligned}$$

This then gives us

$$\begin{aligned} \Big| \sum_{l=1}^{m_0-1} \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \{u,v\}} \Big| &\leq \sum_{l=1}^{m_0-1} \left| \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1}; s_{m^*_l})}_{\mathcal{D}^*_{l-1} \times \{u,v\}} \right| \\ &\leq A(\bar{\boldsymbol{\theta}}, \mathbf{R}) \, \omega(s,t)^{\theta_*} \, \tilde{\omega}(u,v)^{\theta_*} \sum_{l=1}^{m_0-1} \left[\frac{1}{m_0 - l} \, C_2(\bar{\boldsymbol{\theta}}, T, \alpha) \right]^{1/\alpha} \\ &\leq \zeta\left(\frac{1}{\alpha}\right) C_2(\bar{\boldsymbol{\theta}}, T, \alpha)^{1/\alpha} \, A(\bar{\boldsymbol{\theta}}, \mathbf{R}) \, \omega(s,t)^{\theta_*} \, \tilde{\omega}(u,v)^{\theta_*}. \end{aligned} \tag{4.16}$$

To finish, we use the following identity:

$$\sum_{l=1}^{m_0-1} \Delta^{(\mathcal{D}^*_{l-1};s_{m^*_l})}_{\mathcal{D}^*_{l-1}\times\tilde{\mathcal{D}}} - \Delta^{(\mathcal{D}^*_{l-1};s_{m^*_l})}_{\mathcal{D}^*_{l-1}\times\{u,v\}}$$
$$= \sum_{l=1}^{m_0-1}\Big[\sum_{\mathcal{D}^*_{l-1}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}^*_{l}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}^*_{l-1}\times\{u,v\}}\Omega^Y + \sum_{\mathcal{D}^*_{l}\times\{u,v\}}\Omega^Y\Big]$$
$$= \sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}\times\{u,v\}}\Omega^Y - \sum_{\{s,t\}\times\tilde{\mathcal{D}}}\Omega^Y + \Omega^Y\binom{s,t}{u,v},$$

which combined with (4.16) yields (4.13) with $C_5(\overline{\boldsymbol{\theta}},T,\alpha) = \zeta(1/\alpha)\,C_2(\overline{\boldsymbol{\theta}},T,\alpha)^{1/\alpha}$. ■

From these bounds, the following maximal inequality is easily obtainable.

Theorem 4.7 (Two parameter maximal inequality). *Suppose that Condition 3.6 holds with $E = \mathrm{Bi}(V\times\tilde{V}\to W)$. Then, for any $\alpha\in(1/\theta_*,1)$, there exists a constant $C_6(\overline{\boldsymbol{\theta}},T,\alpha)$ such that, for any $(s,t),(u,v)\in\Delta_T$ and partitions $\mathcal{D}=\{s=s_0<\cdots<s_{m_0}=t\}$, $\tilde{\mathcal{D}}=\{u=u_0<\cdots<u_{n_0}=v\}$, we have the bound*

$$\Big|\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big| \le C_6(\overline{\boldsymbol{\theta}},T,\alpha)\,\omega(s,t)^{\theta_*}\,\tilde{\omega}(u,v)^{\theta_*} \cdot\Big[A(\overline{\boldsymbol{\theta}},T,\mathbf{R}) + \frac{B_X(u,v)}{\tilde{\omega}(u,v)^{\theta_*}} + \frac{B_{\tilde{X}}(s,t)}{\omega(s,t)^{\theta_*}}\Big]. \tag{4.17}$$

Proof. This follows immediately from writing

$$\Big|\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big| \le \Big|\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}\times\{u,v\}}\Omega^Y - \sum_{\{s,t\}\times\tilde{\mathcal{D}}}\Omega^Y + \Omega^Y\binom{s,t}{u,v}\Big|$$
$$+\Big|\sum_{\mathcal{D}\times\{u,v\}}\Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big| + \Big|\sum_{\{s,t\}\times\tilde{\mathcal{D}}}\Omega^Y - \Omega^Y\binom{s,t}{u,v}\Big|$$

and then applying Lemma 4.6, where $C_6(\overline{\boldsymbol{\theta}},T,\alpha) = \max\{C_4(\overline{\boldsymbol{\theta}},T,\alpha), C_5(\overline{\boldsymbol{\theta}},T,\alpha)\}$. ■

5. Joint rough integrals and a rough Fubini type theorem

We now arrive at the main results of this paper, where we show existence and uniqueness of the joint rough integral, as well as a rough Fubini type theorem over rectangles. In Section 5 of [9], two rough Fubini type theorems are proven (one on the simplex and one on the rectangle) in the $2\le p<3$ case under the additional assumption of admitting smooth approximations, which we are able to bypass at the cost of assumptions of controlled mixed variation.

Theorem 5.1 (Existence of rough joint integral). *Let*

$$\{Y^{(j,k)}\} \in \mathcal{D}^{p,\tilde{p}}_{X,\tilde{X}}([0,T]^2; \mathrm{Bi}(V \times \tilde{V} \to W))$$

for some Banach space W, and suppose that Condition 3.6 holds. Then the limit

$$\lim_{|\mathfrak{D}\times\tilde{\mathfrak{D}}|\to 0} \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}} \Omega^Y =: \int_{[s,t]\times[u,v]} Y\, d(X,\tilde{X}) \tag{5.1}$$

exists and is such that for all $\alpha \in (1/\theta_*, 1)$,

$$\begin{aligned} \left| \int_{[s,t]\times[u,v]} Y\, d\big(X,\tilde{X}\big) - \Omega^Y \begin{pmatrix} s,t \\ u,v \end{pmatrix} \right| &\le C_6 \big[A(\overline{\boldsymbol{\theta}}, T, \mathbf{R})\, \omega(s,t)^{\theta_*}\, \tilde{\omega}(u,v)^{\theta_*} \\ &\quad + B_X(u,v)\, \omega(s,t)^{\theta_*} + B_{\tilde{X}}(s,t)\, \tilde{\omega}(u,v)^{\theta_*} \big], \end{aligned} \tag{5.2}$$

where A is as defined in Lemma 4.4, B_X and $B_{\tilde{X}}$ are as in Lemma 4.6, and $C_6 = C_6(\overline{\boldsymbol{\theta}}, T, \alpha)$ is as in Theorem 4.7.

The proof of existence for this integral is inspired by the sewing lemma, and the presentation of the proof of Lemma 2.8 here intentionally serves as an analogue to the following proof.

Proof. By the maximal inequality of Theorem 4.7, the bound (5.2) follows immediately if the limit (5.1) is shown to exist. To do this, we show that for grid-like partitions $\mathfrak{D} \times \tilde{\mathfrak{D}}$ and $\mathfrak{D}' \times \tilde{\mathfrak{D}}'$, the difference between their discrete integrals disappears as the mesh size of these partitions goes to zero. We start by splitting the difference between the two discrete integrals into three parts, E_1, E_2 and E_3:

$$\begin{aligned} \left| \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}} \Omega^Y - \sum_{\mathfrak{D}'\times\tilde{\mathfrak{D}}'} \Omega^Y \right| \le & \underbrace{\left| \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}} \Omega^Y - \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}'} \Omega^Y - \sum_{\mathfrak{D}'\times\tilde{\mathfrak{D}}} \Omega^Y + \sum_{\mathfrak{D}'\times\tilde{\mathfrak{D}}'} \Omega^Y \right|}_{E_1} \\ & + \underbrace{\left| \sum_{\mathfrak{D}'\times\tilde{\mathfrak{D}}} \Omega^Y - \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}} \Omega^Y \right|}_{E_2} + \underbrace{\left| \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}'} \Omega^Y - \sum_{\mathfrak{D}\times\tilde{\mathfrak{D}}} \Omega^Y \right|}_{E_3}. \end{aligned}$$

From here on, the proof follows similarly to that of Lemma 2.8, again using the observation that controls are uniformly continuous on Δ_T. So for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\omega(r,r') < \varepsilon$ and $\tilde{\omega}(r,r') < \varepsilon$ for any $(r,r') \in \Delta_T$ such that $|r - r'| < \delta_\varepsilon$.

Let $\varepsilon > 0$, and suppose that $\max\{|\mathfrak{D}|, |\tilde{\mathfrak{D}}|, |\mathfrak{D}'|, |\tilde{\mathfrak{D}}'|\} < \delta_\varepsilon$. First we will assume that $\mathfrak{D}' \times \tilde{\mathfrak{D}}'$ refines $\mathfrak{D} \times \tilde{\mathfrak{D}}$, that is, $\mathfrak{D} \subset \mathfrak{D}'$ and $\tilde{\mathfrak{D}} \subset \tilde{\mathfrak{D}}'$. Writing $\mathfrak{D} = \{s = s_0 < \cdots < s_{m_0} = t\}$ and $\tilde{\mathfrak{D}} = \{u = u_0 < \cdots < u_{n_0} = v\}$, consider the following partitions on each of the intervals: for $m = 0, \ldots, m_0 - 1$ and $n = 0, \ldots, n_0 - 1$,

$$\mathfrak{D}'_m = \mathfrak{D}' \cap [s_m, s_{m+1}] \quad \text{and} \quad \tilde{\mathfrak{D}}'_n = \tilde{\mathfrak{D}}' \cap [u_n, u_{n+1}].$$

Let

$$\mathfrak{D}'_m = \{s_m = s^m_0 < \cdots < s^m_{a_m} = s_{m+1}\} \quad \text{and} \quad \tilde{\mathfrak{D}}'_n = \{u_n = u^n_0 < \cdots < u^n_{b_n} = u_{n+1}\}$$

We have the identities:

$$\sum_{\mathcal{D}'\times\tilde{\mathcal{D}}} \Omega^Y = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{a=0}^{a_m-1}\Omega^Y\Big(\begin{smallmatrix} s_a^m, s_{a+1}^m \\ u_n, u_{n+1}\end{smallmatrix}\Big)\Big) = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{\mathcal{D}'_m\times\{u_n,u_{n+1}\}}\Omega^Y\Big),$$

$$\sum_{\mathcal{D}\times\tilde{\mathcal{D}}'} \Omega^Y = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{b=0}^{b_n-1}\Omega^Y\Big(\begin{smallmatrix} s_m, s_{m+1} \\ u_b^n, u_{b+1}^n\end{smallmatrix}\Big)\Big) = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{\{s_m,s_{m+1}\}\times\tilde{\mathcal{D}}'_n}\Omega^Y\Big),$$

$$\sum_{\mathcal{D}'\times\tilde{\mathcal{D}}'} \Omega^Y = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{a=0}^{a_m-1}\sum_{b=0}^{b_n-1}\Omega^Y\Big(\begin{smallmatrix} s_a^m, s_{a+1}^m \\ u_b^n, u_{b+1}^n\end{smallmatrix}\Big)\Big) = \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\sum_{\mathcal{D}'_m\times\tilde{\mathcal{D}}'_n}\Omega^Y\Big).$$

We now rewrite E_1 and use (4.13) to obtain

$$\begin{aligned}
E_1 &= \Big|\sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big(\Omega^Y\Big(\begin{smallmatrix} s_m, s_{m+1} \\ u_n, u_{n+1}\end{smallmatrix}\Big) - \sum_{\{s_m,s_{m+1}\}\times\tilde{\mathcal{D}}'_n}\Omega^Y - \sum_{\mathcal{D}'_m\times\{u_n,u_{n+1}\}}\Omega^Y + \sum_{\mathcal{D}'_m\times\tilde{\mathcal{D}}'_n}\Omega^Y\Big)\Big| \\
&\le \sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\Big|\Omega^Y\Big(\begin{smallmatrix} s_m, s_{m+1} \\ u_n, u_{n+1}\end{smallmatrix}\Big) - \sum_{\{s_m,s_{m+1}\}\times\tilde{\mathcal{D}}'_n}\Omega^Y - \sum_{\mathcal{D}'_m\times\{u_n,u_{n+1}\}}\Omega^Y + \sum_{\mathcal{D}'_m\times\tilde{\mathcal{D}}'_n}\Omega^Y\Big| \\
&\le C_5(\overline{\boldsymbol{\theta}},T,\alpha)\,A(\overline{\boldsymbol{\theta}},\mathbf{R})\sum_{m=0}^{m_0-1}\sum_{n=0}^{n_0-1}\omega(s_m,s_{m+1})^{\theta_*}\,\tilde{\omega}(u_n,u_{n+1})^{\theta_*} \\
&\le C_5(\overline{\boldsymbol{\theta}},T,\alpha)\,A(\overline{\boldsymbol{\theta}},\mathbf{R})\,\omega(s,t)\,\tilde{\omega}(u,v)\,\varepsilon^{2(\theta_*-1)}.
\end{aligned}$$

For E_2, we first decompose it into the sum of two errors, and then use the inequalities (4.11) and (4.13),

$$\begin{aligned}
E_2 &\le \Big|\sum_{\mathcal{D}'\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}'\times\{u,v\}}\Omega^Y + \sum_{\mathcal{D}\times\{u,v\}}\Omega^Y\Big| + \Big|\sum_{\mathcal{D}'\times\{u,v\}}\Omega^Y - \sum_{\mathcal{D}\times\{u,v\}}\Omega^Y\Big| \\
&= \sum_{m=0}^{m_0-1}\Big|\sum_{\mathcal{D}'_m\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\{s_m,s_{m+1}\}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}'_m\times\{u,v\}}\Omega^Y + \Omega^Y\Big(\begin{smallmatrix} s_m, s_{m+1} \\ u, v\end{smallmatrix}\Big)\Big| \\
&\quad + \sum_{m=0}^{m_0-1}\Big|\sum_{\mathcal{D}'_m\times\{u,v\}}\Omega^Y - \Omega^Y\Big(\begin{smallmatrix} s_m, s_{m+1} \\ u, v\end{smallmatrix}\Big)\Big| \\
&\le \sum_{m=0}^{m_0-1}\big[C_5(\overline{\boldsymbol{\theta}},T,\alpha)\,A(\overline{\boldsymbol{\theta}},\mathbf{R})\,\tilde{\omega}(u,v)^{\theta_*} + C_4(\overline{\boldsymbol{\theta}},T,\alpha)\,B_X(u,v)\big]\,\omega(s_m,s_{m+1})^{\theta_*},
\end{aligned}$$

giving us the bound

$$E_2 \le \big[C_5(\overline{\boldsymbol{\theta}},T,\alpha)\,A(\overline{\boldsymbol{\theta}},\mathbf{R})\,\tilde{\omega}(u,v)^{\theta_*} + C_4(\overline{\boldsymbol{\theta}},T,\alpha)\,B_X(u,v)\big]\,\omega(s,t)\,\varepsilon^{1-\theta_*}. \tag{5.3}$$

Repeating the same process for E_3, we arrive at

$$E_3 \le \big[C_5(\overline{\boldsymbol{\theta}},T,\alpha)\,A(\overline{\boldsymbol{\theta}},\mathbf{R})\,\omega(s,t)^{\theta_*} + C_4(\overline{\boldsymbol{\theta}},T,\alpha)\,B_{\tilde{X}}(s,t)\big]\,\tilde{\omega}(u,v)\,\varepsilon^{1-\theta_*}, \tag{5.4}$$

using inequalities (4.12) and (4.13). Taking ε arbitrarily small, we deduce that the difference between discrete integrals disappears as the mesh sizes go to zero. For the general case, where $\mathcal{D}' \times \tilde{\mathcal{D}}'$ may not refine $\mathcal{D} \times \tilde{\mathcal{D}}$, we simply write

$$\Big|\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}'\times\tilde{\mathcal{D}}'}\Omega^Y\Big| \leq \Big|\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y - \sum_{\mathcal{D}\cup\mathcal{D}'\times\tilde{\mathcal{D}}\cup\tilde{\mathcal{D}}'}\Omega^Y\Big| + \Big|\sum_{\mathcal{D}\cup\mathcal{D}'\times\tilde{\mathcal{D}}\cup\tilde{\mathcal{D}}'}\Omega^Y - \sum_{\mathcal{D}'\times\tilde{\mathcal{D}}'}\Omega^Y\Big|,$$

which then allows us to use the bounds from the nested case, thus giving us existence of the limit (5.1). ∎

With the existence of the joint integral and the bound (5.2), we have the following Fubini type theorem for double rough integrals.

Theorem 5.2 (Rough Fubini type theorem on the rectangle). *Let everything be as in Theorem 5.1 and assume that the assumptions there hold. Then the double integrals of* (3.9) *are both equal to the joint integral* (5.1)*, that is,*

$$\int_{[s,t]\times[u,v]} Y\, d(X,\tilde{X}) = \mathcal{I}^Y_{\tilde{X},X}\begin{pmatrix} s,t \\ u,v \end{pmatrix} = \mathcal{I}^Y_{X,\tilde{X}}\begin{pmatrix} s,t \\ u,v \end{pmatrix}. \tag{5.5}$$

Proof. We prove this Fubini type theorem by showing that the double integrals $\mathcal{I}^Y_{\tilde{X},X}$ and $\mathcal{I}^Y_{X,\tilde{X}}$ can be arbitrarily well approximated by the discrete two parameter integrals $\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^Y$ with decreasing mesh sizes, and thus may also be written as the limit (5.1).

Recall the one-parameter controlled path integrals of Y, denoted $\mathcal{I}^{(Y;j)}_{\tilde{X}}$ and $\mathcal{I}^{(Y;k)}_{X}$, and defined by (3.6) and (3.7), respectively. For $(s,t),(u,v)\in\Delta_T$, we have

$$\begin{aligned}\sum_{j=0}^{N}\mathcal{I}^{(Y;j)}_{\tilde{X}}\begin{pmatrix} s \\ u,v \end{pmatrix}(X^{j+1}_{s,t}) &= \sum_{j=0}^{N}\Bigg(\lim_{|\tilde{\mathcal{D}}|\to 0}\sum_{[u_n,u_{n+1}]\subset\tilde{\mathcal{D}}}\Bigg(\sum_{k=0}^{\tilde{N}} Y^{(j,k)}_{s;u_n}(X^{j+1}_{s,t},\tilde{X}^{k+1}_{u_n,u_{n+1}})\Bigg)\Bigg)\\ &= \lim_{|\tilde{\mathcal{D}}|\to 0}\sum_{[u_n,u_{n+1}]\subset\tilde{\mathcal{D}}}\Bigg(\sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}} Y^{(j,k)}_{s;u_n}(X^{j+1}_{s,t},\tilde{X}^{k+1}_{u_n,u_{n+1}})\Bigg)\\ &= \lim_{|\tilde{\mathcal{D}}|\to 0}\sum_{\{s,t\}\times\tilde{\mathcal{D}}}\Omega^Y\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\tilde{N}}\mathcal{I}^{(Y;k)}_{X}\begin{pmatrix} s,t \\ u \end{pmatrix}(\tilde{X}^{k+1}_{u,v}) &= \sum_{k=0}^{\tilde{N}}\Bigg(\lim_{|\mathcal{D}|\to 0}\sum_{[s_m,s_{m+1}]\subset\mathcal{D}}\Bigg(\sum_{j=0}^{N} Y^{(j,k)}_{s_m;u}(X^{j+1}_{s_m,s_{m+1}},\tilde{X}^{k+1}_{u,v})\Bigg)\Bigg)\\ &= \lim_{|\mathcal{D}|\to 0}\sum_{[s_m,s_{m+1}]\subset\mathcal{D}}\Bigg(\sum_{j=0}^{N}\sum_{k=0}^{\tilde{N}} Y^{(j,k)}_{s_m;u}(X^{j+1}_{s_m,s_{m+1}},\tilde{X}^{k+1}_{u,v})\Bigg)\\ &= \lim_{|\mathcal{D}|\to 0}\sum_{\mathcal{D}\times\{u,v\}}\Omega^Y.\end{aligned}$$

Let $\varepsilon > 0$. By uniform continuity of controls on Δ_T, there exists $\delta^1_\varepsilon > 0$ such that for any $(r, r') \in \Delta_T$ with $|r - r'| < \delta^1_\varepsilon$, we have $\max\{\omega(r, r'), \tilde{\omega}(r, r')\} < \varepsilon$. Additionally, by the definition of controlled path integrals, there exists $\delta^2_\varepsilon > 0$ such that for any $\mathcal{D}$ and $\tilde{\mathcal{D}}$ with $|\mathcal{D} \times \mathcal{D}'| < \delta^2_\varepsilon$,

$$\left| \mathcal{I}^Y_{\tilde{X},X}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{m=0}^{m_0-1} \sum_{j=0}^{N} \mathcal{I}^{(Y;j)}_{\tilde{X}}\begin{pmatrix} s_m \\ u,v \end{pmatrix}(X^{j+1}_{s_m,s_{m+1}}) \right| < \varepsilon,$$

$$\left| \mathcal{I}^Y_{X,\tilde{X}}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{n=0}^{n_0-1} \sum_{k=0}^{\tilde{N}} \mathcal{I}^{(Y;k)}_{X}\begin{pmatrix} s,t \\ u_n \end{pmatrix}(\tilde{X}^{k+1}_{u_n,u_{n+1}}) \right| < \varepsilon,$$

letting $\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\}$ and $\tilde{\mathcal{D}} = \{u = u_0 < \cdots < u_{n_0} = v\}$ as usual. Let $\delta_\varepsilon = \max\{\delta^1_\varepsilon, \delta^2_\varepsilon\}$, and suppose that $\max\{|\mathcal{D}|, |\tilde{\mathcal{D}}|\} < \delta_\varepsilon$. Then

$$\begin{aligned} \left| \mathcal{I}^Y_{\tilde{X},X}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right| &\le \left| \mathcal{I}^Y_{\tilde{X},X}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{m=0}^{m_0-1} \sum_{j=0}^{N} \mathcal{I}^{(Y;j)}_{\tilde{X}}\begin{pmatrix} s_m \\ u,v \end{pmatrix}(X^{j+1}_{s_m,s_{m+1}}) \right| \\ &\quad + \underbrace{\left| \sum_{m=0}^{m_0-1} \sum_{j=0}^{N} \mathcal{I}^{(Y;j)}_{\tilde{X}}\begin{pmatrix} s_m \\ u,v \end{pmatrix}(X^{j+1}_{s_m,s_{m+1}}) - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right|}_{E_4}. \end{aligned}$$

The first term is bounded by ε. For the second term, we rewrite it and use the bound (5.4) on E_3 from the proof of Theorem 5.1 to get

$$\begin{aligned} E_4 &= \left| \sum_{m=0}^{m_0-1} \left(\lim_{|\tilde{\mathcal{D}}'|\to 0} \sum_{\{s_m,s_{m+1}\}\times\tilde{\mathcal{D}}'} \Omega^Y \right) - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right| = \lim_{|\tilde{\mathcal{D}}'|\to 0} \left| \sum_{\mathcal{D}\times\tilde{\mathcal{D}}'} \Omega^Y - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right| \\ &\le \left[C_5(\overline{\boldsymbol{\theta}}, T, \alpha)\, A(\overline{\boldsymbol{\theta}}, \mathbf{R})\, \omega(s,t)^{\theta_*} + C_4(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_{\tilde{X}}(s,t) \right] \tilde{\omega}(u,v)\, \varepsilon^{1-\theta_*}. \end{aligned}$$

Similarly, using the bound (5.3) on E_2,

$$\begin{aligned} \left| \mathcal{I}^Y_{X,\tilde{X}}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right| &\le \left| \mathcal{I}^Y_{X,\tilde{X}}\begin{pmatrix} s,t \\ u,v \end{pmatrix} - \sum_{n=0}^{n_0-1} \sum_{k=0}^{\tilde{N}} \mathcal{I}^{(Y;k)}_{X}\begin{pmatrix} u_n, u_{n+1} \\ s \end{pmatrix}(\tilde{X}^{k+1}_{u_n,u_{n+1}}) \right| \\ &\quad + \left| \sum_{n=0}^{n_0-1} \sum_{k=0}^{\tilde{N}} \mathcal{I}^{(Y;k)}_{X}\begin{pmatrix} u_n, u_{n+1} \\ s \end{pmatrix}(\tilde{X}^{k+1}_{u_n,u_{n+1}}) - \sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y \right| \\ &\le \varepsilon + \omega(s,t)\, \varepsilon^{1-\theta_*} \left[C_5(\overline{\boldsymbol{\theta}}, T, \alpha)\, A(\overline{\boldsymbol{\theta}}, \mathbf{R})\, \tilde{\omega}(u,v)^{\theta_*} \right. \\ &\quad \left. + C_4(\overline{\boldsymbol{\theta}}, T, \alpha)\, B_X(u,v) \right]. \end{aligned}$$

So for any $\varepsilon > 0$, the discrete integral $\sum_{\mathcal{D}\times\tilde{\mathcal{D}}} \Omega^Y$ is ε-close to $\mathcal{I}^Y_{\tilde{X},X}$ and $\mathcal{I}^Y_{X,\tilde{X}}$ for all grid-like partitions $\mathcal{D} \times \tilde{\mathcal{D}}$ such that $|\mathcal{D} \times \tilde{\mathcal{D}}| < \delta'_\varepsilon$ for some $\delta'_\varepsilon > 0$. Thus we arrive at the conclusion that both double integrals are equal to the joint integral (5.1). ■

The rough Fubini type theorems derived in [9] cover the case where the controlling rough paths have finite p-variation for $2 \leq p < 3$, and are used as tools in order to prove Hörmander's theorem for a class of SPDEs. In comparison with the Fubini type theorems of [9], we do not cover integrals over the simplex, but are able to generalise integrals over rectangles to the case where the controlling rough paths have arbitrary p-variation. With the introduction of this third type of double integral, the proof given here draws some parallels with the classical Fubini's theorem, and allows us to bypass the technical condition of smooth approximability.

An obvious corollary of the rough Fubini theorem is that the stability of the double integrals carries over to the joint integral:

Theorem 5.3 (Stability of joint integrals). *Suppose that all quantities are as in Theorem 3.12 and that the assumptions of the theorem hold. Then there exists $C_7(\bar{\boldsymbol{\theta}}, T, M)$ such that*

$$\Big| \int_{[s,t]\times[u,v]} Y \, d(X, \tilde{X}) - \int_{[s,t]\times[u,v]} A \, d(Z, \tilde{Z}) \Big| \leq C_7(T, \bar{\boldsymbol{\theta}}, M) \, d^{(X,\tilde{X}),(Z,\tilde{Z}),\bar{\boldsymbol{\theta}}}_{[s,t],[u,v]}(Y, A).$$

6. Signature kernels as jointly controlled paths

One notable example of a jointly controlled path satisfying Condition 3.6 is the signature kernel, a machine learning tool which has recently seen use in kernel methods in works such as [4,13,14,16–18]. An important result for the computation of the signature kernel is that it satisfies a second order Goursat PDE in the case where paths are differentiable [17]. An analogous result in the same paper tells us that, in the case of geometric rough paths, the signature kernel satisfies a two-parameter rough integral equation, and here we are able to give an alternative proof of this integral equation. First, we define the signature and signature kernel.

Definition 6.1. Let X be a p-rough path on V. The *signature* of X,

$$S(X) : \Delta_T \to T((V)),$$

is the unique extension of X from a multiplicative function on $T^{\lfloor p \rfloor}(V)$ with finite p-variation to a multiplicative functional on $T((V))$ with finite p-variation via the extension theorem (Theorem 2.2.1 of [15], Theorem 3.7 of [1]). That is,

$$S(X)_{s,t} = (1, X^1_{s,t}, \dots X^{\lfloor p \rfloor}_{s,t}, X^{\lfloor p \rfloor + 1}_{s,t}, \dots) \in T((V))$$

where for $l > \lfloor p \rfloor$, the $X^l \colon \Delta_T \to V^{\otimes l}$ are such that

$$\|X^l\|_{p/l} \leq \frac{1}{\beta_p \Gamma(l/p + 1)}, \tag{6.1}$$

and Chen's identity holds:

$$S(X)_{s,s'} \otimes S(X)_{s',t} = S(X)_{s,t}, \quad (s, s', t) \in \Delta^3_T.$$

For any two pairs of times $(s,t),(u,v) \in \Delta_T$ and any two rough paths X and $\tilde{X}$ on V, we define the *signature kernel* $K_{(s,t),(u,v)}(X,\tilde{X})$ by

$$K_{(s,t),(u,v)}(X,\tilde{X}) = \langle S(X)_{s,t}, S(\tilde{X})_{u,v}\rangle_{\overline{T(V)}} = \sum_{l=0}^{\infty} \langle X^l_{s,t}, \tilde{X}^l_{u,v}\rangle_{V^{\otimes l}},$$

where $\overline{T(V)}$ is the completion of $T(V)$ with respect to its inner product.

Proposition 6.2. *Let X and $\tilde{X}$ be two p-rough paths on V with control ω. Define the family of two parameter paths $Y^{\sigma;\tau}\colon [0,T]^2 \to \mathrm{Bi}(V \times V \to \mathbb{R})$ by*

$$Y^{\sigma;\tau}_{s;u}(a,b) := K_{(\sigma,s),(\tau,u)}(X,\tilde{X})\,\langle a,b\rangle_V = \sum_{l=0}^{\infty} \langle X^l_{\sigma,s} \otimes a, \tilde{X}^l_{\tau,u} \otimes b\rangle_{V^{\otimes l+1}},$$

where $a,b \in V$. Letting

$$Y^{(\sigma;\tau)(j,k)}\colon [0,T]^2 \to \mathrm{Bi}(V \otimes V \to \mathrm{Bi}(V \times V \to \mathbb{R}))$$

be defined for $(\xi,\eta) \in V^{\otimes j} \times V^{\otimes k}$ and by $a,b \in V$ by

$$Y^{(\sigma;\tau)(j,k)}_{s;u}(\xi,\eta)(a,b) := \sum_{l=\max\{j,k\}}^{\infty} \langle X^{l-j}_{\sigma,s} \otimes \xi \otimes a, \tilde{X}^{l-k}_{\tau,u} \otimes \eta \otimes b\rangle_{V^{\otimes l+1}},$$

then

$$\{Y^{(\sigma;\tau)(j,k)}\} \in \mathcal{D}^{p,p}_{X,X}([\sigma,T]\times[\tau,T]; \mathrm{Bi}(V^{\otimes j} \otimes V^{\otimes k} \to \mathbb{R})),$$

with remainders

(6.2)
$$R^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s \\ u,v\end{pmatrix}(\xi,\eta)(a,b) = \sum_{m=\lfloor p\rfloor}^{\infty}\sum_{l=m}^{\infty} \big\langle X^{l-j}_{\sigma,s} \otimes \xi \otimes a, \tilde{X}^{l-m}_{\tau,u} \otimes \tilde{X}^{m-k}_{u,v} \otimes \eta \otimes b\big\rangle_{V^{\otimes l+1}},$$

(6.3)
$$R^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s,t \\ u\end{pmatrix}(\xi,\eta)(a,b) = \sum_{m=\lfloor p\rfloor}^{\infty}\sum_{l=m}^{\infty} \big\langle X^{l-m}_{\sigma,s} \otimes X^{m-j}_{s,t} \otimes \xi \otimes a, \tilde{X}^{l-k}_{\tau,u} \otimes \eta \otimes b\big\rangle_{V^{\otimes l+1}},$$

(6.4)
$$\mathbf{R}^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s,t \\ u,v\end{pmatrix}(\xi,\eta)(a,b) = \sum_{l=\lfloor p\rfloor}^{\infty}\sum_{m=\lfloor p\rfloor}^{l}\sum_{n=\lfloor p\rfloor}^{l} \big\langle X^{l-n}_{\sigma,s} \otimes X^{n-j}_{s,t} \otimes \xi \otimes a,$$
$$\tilde{X}^{l-m}_{\tau,u} \otimes \tilde{X}^{m-k}_{u,v} \otimes \eta \otimes b\big\rangle_{V^{\otimes l+1}},$$

satisfying Condition 3.6 for $q_j = \tilde{q}_j = p/(\lfloor p\rfloor - j)$.

Proof. Here we will show that $Y^{(\sigma;\tau)(j,k)}$ satisfy the necessary conditions of a jointly controlled path, derivation of $Y^{(\sigma;\tau)(j,k)}$ is left as an exercise to the reader; this can be done by using that Y is a controlled path in both variables and that these controlled paths are also controlled paths in the other variable.

We first check that $Y^{(\sigma;\tau)(j,k)}$ and $R^{(\sigma;\tau;j,k)}$ properly satisfy the controlled path condition (2.7):

$$\begin{aligned}
Y^{(\sigma;\tau)(j,k)}_{t;u}(\xi,\eta)(a,b) &= \sum_{l=\max\{j,k\}}^{\infty} \langle X^{l-j}_{\sigma,t}\otimes\xi\otimes a, \tilde{X}^{l-k}_{\tau,u}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}} \\
&= \sum_{l=\max\{j,k\}}^{\infty} \Big\langle \Big(\sum_{m=0}^{l-j} X^{l-j-m}_{\sigma,s}\otimes X^{m}_{s,t}\Big)\otimes\xi\otimes a, \tilde{X}^{l-k}_{\tau,u}\otimes\eta\otimes b\Big\rangle_{V^{\otimes l+1}} \\
&= \sum_{m=0}^{\infty}\Big(\sum_{l=\max\{j+m,k\}}^{\infty} \langle X^{l-j-m}_{\sigma,s}\otimes X^{m}_{s,t}\otimes\xi\otimes a, \tilde{X}^{l-k}_{\tau,u}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}}\Big) \\
&= \sum_{m=0}^{N-j} Y^{(j+m,k)}_{s;u}(X^{m}_{s,t}\otimes\xi,\eta)(a,b) \\
&\qquad + \sum_{m=\lfloor p\rfloor-j}^{\infty}\Big(\sum_{l=j+m}^{\infty} \langle X^{l-j-m}_{\sigma,s}\otimes X^{m}_{s,t}\otimes\xi\otimes a, \tilde{X}^{l-k}_{\tau,u}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}}\Big) \\
&= \sum_{m=0}^{N-j} Y^{(\sigma;\tau)(j+m,k)}_{s;u}(X^{m}_{s,t}\otimes\xi,\eta)(a,b) + R^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s,t \\ u \end{pmatrix}(\xi,\eta)(a,b),
\end{aligned}$$

and, similarly, we have

$$\begin{aligned}
&Y^{(\sigma;\tau)(j,k)}_{s;v}(\xi,\eta)(a,b) \\
&\quad= \sum_{m=0}^{\infty}\Big(\sum_{l=\max\{j,k+m\}}^{\infty} \langle X^{l-j}_{\sigma,s}\otimes\xi\otimes a, \tilde{X}^{l-k-m}_{\tau,u}\otimes\tilde{X}^{m}_{u,v}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}}\Big) \\
&\quad= \sum_{m=0}^{N-j} Y^{(\sigma;\tau)(j,k+m)}_{s;u}(\xi,\tilde{X}^{m}_{u,v}\otimes\eta)(a,b) + R^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s \\ u,v \end{pmatrix}(\xi,\eta)(a,b).
\end{aligned}$$

Next, we check that $\mathbf{R}^{(j,k)}$ can be expressed in the form (6.4). For $\mathbf{R}^{(j,k)}$, we use the first identity of (3.4):

$$\begin{aligned}
&\mathbf{R}^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s,t \\ u,v \end{pmatrix}(\xi,\eta)(a,b) \\
&\quad= \sum_{m=\lfloor p\rfloor}^{\infty}\sum_{l=m}^{\infty} \langle X^{l-j}_{\sigma,t}\otimes\xi\otimes a, \tilde{X}^{l-m}_{\tau,u}\otimes\tilde{X}^{m-k}_{u,v}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}} \\
&\qquad - \sum_{n=0}^{N-j}\sum_{m=\lfloor p\rfloor}^{\infty}\sum_{l=m}^{\infty} \langle X^{l-j-n}_{\sigma,s}\otimes X^{n}_{s,t}\otimes\xi\otimes a, \tilde{X}^{l-m}_{\tau,u}\otimes\tilde{X}^{m-k}_{u,v}\otimes\eta\otimes b\rangle_{V^{\otimes l+1}}
\end{aligned}$$

$$= \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{m=\lfloor p \rfloor}^{l} \sum_{n=j}^{l} \langle X_{\sigma,s}^{l-n} \otimes X_{s,t}^{n-j} \otimes \xi \otimes a, \tilde{X}_{\tau,u}^{l-m} \otimes \tilde{X}_{u,v}^{m-k} \otimes \eta \otimes b \rangle_{V^{\otimes l+1}}$$
$$- \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{m=\lfloor p \rfloor}^{l} \sum_{n=j}^{N} \langle X_{\sigma,s}^{l-n} \otimes X_{s,t}^{n-j} \otimes \xi \otimes a, \tilde{X}_{\tau,u}^{l-m} \otimes \tilde{X}_{u,v}^{m-k} \otimes \eta \otimes b \rangle_{V^{\otimes l+1}}$$
$$= \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{m=\lfloor p \rfloor}^{l} \sum_{n=\lfloor p \rfloor}^{l} \langle X_{\sigma,s}^{l-n} \otimes X_{s,t}^{n-j} \otimes \xi \otimes a, \tilde{X}_{\tau,u}^{l-m} \otimes \tilde{X}_{u,v}^{m-k} \otimes \eta \otimes b \rangle_{V^{\otimes l+1}}.$$

For the regularity of remainders, it is easy to see that

- $R^{(j,k)}\binom{s}{u,v}$ contains $\tilde{X}_{u,v}^{m}$ terms only for $m \geq \lfloor p \rfloor - k$,
- $R^{(j,k)}\binom{s,t}{u}$ contains $X_{s,t}^{l}$ terms only for $l \geq \lfloor p \rfloor - j$,
- and $\mathbf{R}^{(j,k)}$ also contains these terms of these orders only.

We need now check that the sums do not diverge in the appropriate norms, which is a consequence of (6.1):

$$\left| R_{\sigma;\tau}^{(j,k)}\binom{s}{u,v} \right|$$
$$\leq \sum_{m=\lfloor p \rfloor}^{\infty} \sum_{l=m}^{\infty} \left(\|X^{l-j}\|_{\frac{p}{l-j}} \|\tilde{X}^{l-m}\|_{\frac{p}{l-m}} \|\tilde{X}^{m-k}\|_{\frac{p}{m-k}} \omega(0,T)^{(2l-j-k)/p} \omega(u,v)^{(m-k)/p} \right)$$
$$\leq \omega(u,v)^{(\lfloor p \rfloor - k)/p} \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{m=\lfloor p \rfloor}^{l} \frac{\omega(0,T)^{(2l-j-k)/p}}{(\beta_p)^3 \Gamma\left(\frac{l-j}{p}+1\right) \Gamma\left(\frac{l-m}{p}+1\right) \Gamma\left(\frac{m-k}{p}+1\right)}$$
$$\leq \omega(u,v)^{(\lfloor p \rfloor - k)/p} \sum_{l=\lfloor p \rfloor}^{\infty} \frac{(l-\lfloor p \rfloor)\, \omega(0,T)^{(2l-j-k)/p}}{(\beta_p)^3 \Gamma\left(\frac{l-j}{p}+1\right) \Gamma\left(\frac{l-k}{2p}+1\right)^2},$$

where we use the logarithmic convexity of the Gamma function in the last line, in particular, that for $\lambda \in [0,1]$,

$$\Gamma(x)^{\lambda}\, \Gamma(y)^{1-\lambda} \geq \Gamma(\lambda x + (1-\lambda) y).$$

This last sum is convergent, and for the other remainders we can similarly write

$$\left| R_{\sigma;\tau}^{(j,k)}\binom{s,t}{u} \right| \leq \omega(s,t)^{(\lfloor p \rfloor - j)/p} \sum_{l=\lfloor p \rfloor}^{\infty} \frac{(l-\lfloor p \rfloor)\, \omega(0,T)^{(2l-j-k)/p}}{(\beta_p)^3 \Gamma\left(\frac{l-k}{p}+1\right) \Gamma\left(\frac{l-j}{2p}+1\right)^2}$$

and

$$\left|\mathbf{R}^{(j,k)}_{\sigma;\tau}\begin{pmatrix} s,t \\ u,v \end{pmatrix}\right|$$
$$\leq \sum_{l=\lfloor p\rfloor}^{\infty}\sum_{m=\lfloor p\rfloor}^{l}\sum_{n=\lfloor p\rfloor}^{l}\Big(\|X^{l-n}\|_{\frac{p}{l-n}}\|X^{n-j}\|_{\frac{p}{n-j}}\|\tilde{X}^{l-m}\|_{\frac{p}{l-m}}\|\tilde{X}^{m-k}\|_{\frac{p}{m-k}}$$
$$\cdot\,\omega(0,T)^{(2l-n-m)/p}\,\omega(s,t)^{(n-j)/p}\,\omega(u,v)^{(m-k)/p}\Big)$$
$$\leq \sum_{l=\lfloor p\rfloor}^{\infty}\sum_{m=\lfloor p\rfloor}^{l}\sum_{n=\lfloor p\rfloor}^{l}\frac{\omega(0,T)^{(2l-n-m)/p}\,\omega(s,t)^{(n-j)/p}\,\omega(u,v)^{(m-k)/p}}{(\beta_p)^4\,\Gamma\big(\frac{l-n}{p}+1\big)\Gamma\big(\frac{n-j}{p}+1\big)\Gamma\big(\frac{l-m}{p}+1\big)\Gamma\big(\frac{m-k}{p}+1\big)}$$
$$\leq \sum_{l=\lfloor p\rfloor}^{\infty}\sum_{m=\lfloor p\rfloor}^{l}\sum_{n=\lfloor p\rfloor}^{l}\frac{\omega(0,T)^{(2l-j-k)/p}\,\omega(s,t)^{(\lfloor p\rfloor-j)/p}\,\omega(u,v)^{(\lfloor p\rfloor-k)/p}}{(\beta_p)^4\,\Gamma\big(\frac{l-j}{2p}+1\big)^2\,\Gamma\big(\frac{l-k}{2p}+1\big)^2}$$
$$= \omega(s,t)^{(\lfloor p\rfloor-j)/p}\,\omega(u,v)^{(\lfloor p\rfloor-k)/p}\sum_{l=\lfloor p\rfloor}^{\infty}\frac{(l-\lfloor p\rfloor)^2\,\omega(0,T)^{(2l-j-k)/p}}{(\beta_p)^4\,\Gamma\big(\frac{l-j}{2p}+1\big)^2\,\Gamma\big(\frac{l-k}{2p}+1\big)^2}.$$

Thus for any $\sigma,\tau\in[0,T]$, $\{Y^{(\sigma;\tau)(j,k)}\}$ is a jointly $(X,\tilde{X})$-controlled path (over the time intervals $[\sigma,T]\times[\tau,T]$) satisfying Condition 3.6 with $q_j=\tilde{q}_j=p/(\lfloor p\rfloor-j)$. ∎

We now show that the signature kernel satisfies a two-parameter rough integral equation. This integral equation matches that of Theorem 4.11 of [17], thereby showing that the two-parameter integral constructed here is consistent with the two-parameter integral constructed through one-form rough integration.

Proposition 6.3. *Let $\{Y^{(\sigma;\tau)(j,k)}\}$ be as defined in Proposition 6.2. Then for any $s,u\in[0,T]$, the jointly controlled path $Y^{s;u}$ satisfies the integral equation*

$$K_{(s,t),(u,v)}(X,\tilde{X}) = Y^{s;u}_{t;v}(\mathbb{1},\mathbb{1}) = 1+\int_{[s,t]\times[u,v]} Y^{s;u}\,d(X,\tilde{X}), \tag{6.5}$$

where $\mathbb{1}\in V$ is any element such that $\|\mathbb{1}\|_V=1$.

Proof. Let $\mathbb{1}\in V$ be such that $\|\mathbb{1}\|_V=1$. We rewrite $\Omega^{Y^{s;u}}$ in the following way:

$$\Omega^{Y^{s;u}}\begin{pmatrix} s',t \\ u',v \end{pmatrix} = \sum_{j=0}^{N}\sum_{k=0}^{N} Y^{(s;u)(j,k)}_{s';u'}\big(X^{j+1}_{s',t},\tilde{X}^{k+1}_{u',v}\big)$$
$$=\sum_{j=0}^{N}\sum_{k=0}^{N}\Big(\sum_{l=\max\{j,k\}}^{\infty}\big\langle X^{l-j}_{s,s'}\otimes X^{j+1}_{s',t},\tilde{X}^{l-k}_{u,u'}\otimes\tilde{X}^{k+1}_{u',v}\big\rangle_{V^{\otimes l+1}}\Big)$$
$$=\sum_{l=0}^{N}\sum_{j=0}^{l}\sum_{k=0}^{l}\big\langle X^{l-j}_{s,s'}\otimes X^{j+1}_{s',t},\tilde{X}^{l-k}_{u,u'}\otimes\tilde{X}^{k+1}_{u',v}\big\rangle_{V^{\otimes l+1}}$$
$$+\sum_{l=\lfloor p\rfloor}^{\infty}\sum_{j=0}^{N}\sum_{k=0}^{N}\big\langle X^{l-j}_{s,s'}\otimes X^{j+1}_{s',t},\tilde{X}^{l-k}_{u,u'}\otimes\tilde{X}^{k+1}_{u',v}\big\rangle_{V^{\otimes l+1}}$$

$$
\begin{aligned}
&= \sum_{l=0}^{N} \langle X_{s,t}^{l+1}, \tilde{X}_{u,v}^{l+1} \rangle_{V^{\otimes l+1}} + \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=0}^{N} \sum_{k=0}^{N} \langle X_{s,s'}^{l-j} \otimes X_{s',t}^{j+1}, \tilde{X}_{u,u'}^{l-k} \otimes \tilde{X}_{u',v}^{k+1} \rangle_{V^{\otimes l+1}} \\
&= Y_{t;v}^{s;u}(\mathbb{1}, \mathbb{1}) - 1 - \sum_{l=\lfloor p \rfloor}^{\infty} \langle X_{s,t}^{l+1}, \tilde{X}_{u,v}^{l+1} \rangle_{V^{\otimes l+1}} \\
&\quad + \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=0}^{N} \sum_{k=0}^{N} \langle X_{s,s'}^{l-j} \otimes X_{s',t}^{j+1}, \tilde{X}_{u,u'}^{l-k} \otimes \tilde{X}_{u',v}^{k+1} \rangle_{V^{\otimes l+1}} \\
&= Y_{t;v}^{s;u}(\mathbb{1}, \mathbb{1}) - 1 - \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=0}^{l+1} \sum_{k=0}^{l+1} \langle X_{s,s'}^{l+1-j} \otimes X_{s',t}^{j}, \tilde{X}_{u,u'}^{l+1-k} \otimes \tilde{X}_{u',v}^{k} \rangle_{V^{\otimes l+1}} \\
&\quad + \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=0}^{N} \sum_{k=0}^{N} \langle X_{s,s'}^{l-j} \otimes X_{s',t}^{j+1}, \tilde{X}_{u,u'}^{l-k} \otimes \tilde{X}_{u',v}^{k+1} \rangle_{V^{\otimes l+1}} \\
&= Y_{t;v}^{s;u}(\mathbb{1}, \mathbb{1}) - Y_{s';u'}^{s;u}(\mathbb{1}, \mathbb{1}) \\
&\quad - \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=\lfloor p \rfloor}^{l} \sum_{k=\lfloor p \rfloor}^{l} \langle X_{s,s'}^{l-j} \otimes X_{s',t}^{j+1}, \tilde{X}_{u,u'}^{l-k} \otimes \tilde{X}_{u',v}^{k+1} \rangle_{V^{\otimes l+1}}.
\end{aligned}
$$

For partitions

$$
\mathcal{D} = \{s = s_0 < \cdots < s_{m_0} = t\} \quad \text{and} \quad \tilde{\mathcal{D}} = \{u = u_0 < \cdots < u_{n_0} = t\},
$$

we have

$$
\begin{aligned}
\sum_{\mathcal{D} \times \tilde{\mathcal{D}}} \Omega^{Y^{s;u}} &= \sum_{n=0}^{n_0-1} \sum_{m=0}^{m_0-1} \Omega^{Y^{s;u}} \binom{s_m, s_{m+1}}{u_n, u_{n+1}} \\
&= \sum_{n=0}^{n_0-1} \sum_{m=0}^{m_0-1} \Big(Y_{s_{m+1};u_{n+1}}^{s;u}(\mathbb{1}, \mathbb{1}) - Y_{s_m;u_n}^{s;u}(\mathbb{1}, \mathbb{1}) \\
&\qquad - \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=\lfloor p \rfloor}^{l} \sum_{k=\lfloor p \rfloor}^{l} \langle X_{s,s_m}^{l-j} \otimes X_{s_m,s_{m+1}}^{j+1}, \tilde{X}_{u,u_n}^{l-k} \otimes \tilde{X}_{u_n,u_{n+1}}^{k+1} \rangle_{V^{\otimes l+1}} \Big) \\
&= Y_{t;v}^{s;u}(\mathbb{1}, \mathbb{1}) - 1 \\
&\qquad - \sum_{n=0}^{n_0-1} \sum_{m=0}^{m_0-1} \sum_{l=\lfloor p \rfloor}^{\infty} \sum_{j=\lfloor p \rfloor}^{l} \sum_{k=\lfloor p \rfloor}^{l} \langle X_{s,s_m}^{l-j} \otimes X_{s_m,s_{m+1}}^{j+1}, \tilde{X}_{u,u_n}^{l-k} \otimes \tilde{X}_{u_n,u_{n+1}}^{k+1} \rangle_{V^{\otimes l+1}}.
\end{aligned}
$$

With this identity, we can show that the sum above disappears as the mesh size of the partitions decreases:

$$\begin{aligned}
&\Big|1+\sum_{\mathcal{D}\times\tilde{\mathcal{D}}}\Omega^{Y^{s;u}}-Y^{s;u}_{t;v}(\mathbb{1},\mathbb{1})\Big| \\
&\le \sum_{n=0}^{n_0-1}\sum_{m=0}^{m_0-1}\sum_{l=\lfloor p\rfloor}^{\infty}\sum_{j=\lfloor p\rfloor}^{l}\sum_{k=\lfloor p\rfloor}^{l}\Big(\|X^{l-j}\|_{\frac{p}{l-j}}\|X^{j+1}\|_{\frac{p}{j+1}}\|\tilde{X}^{l-k}\|_{\frac{p}{l-k}}\|\tilde{X}^{k+1}\|_{\frac{p}{k+1}} \\
&\qquad\cdot\omega(0,T)^{\frac{2l-j-k}{p}}\,\omega(s_m,s_{m+1})^{\frac{j+1}{p}}\,\omega(u_n,u_{n+1})^{\frac{k+1}{p}}\Big) \\
&\le \sum_{n=0}^{n_0-1}\sum_{m=0}^{m_0-1}\sum_{l=\lfloor p\rfloor}^{\infty}\sum_{j=\lfloor p\rfloor}^{l}\sum_{k=\lfloor p\rfloor}^{l}\frac{\omega(0,T)^{\frac{2l-j-k}{p}}\,\omega(s_m,s_{m+1})^{\frac{j+1}{p}}\,\omega(u_n,u_{n+1})^{\frac{k+1}{p}}}{(\beta_p)^4\,\Gamma\big(\frac{l-j}{p}+1\big)\Gamma\big(\frac{j+1}{p}+1\big)\Gamma\big(\frac{l-k}{p}+1\big)\Gamma\big(\frac{k+1}{p}+1\big)} \\
&\le \sup_{|r-r'|\le|\mathcal{D}\times\tilde{\mathcal{D}}|}\omega(r,r')^{\frac{\lfloor p\rfloor+1-p}{p}}\sum_{l=\lfloor p\rfloor}^{\infty}\frac{(l-\lfloor p\rfloor)^2\,\omega(0,T)^{2+(2l+2)/p}}{(\beta_p)^4\,\Gamma\big(\frac{l+1}{2p}+1\big)^4},
\end{aligned}$$

and so we deduce (6.5) by taking the limit as the mesh size $|\mathcal{D}\times\tilde{\mathcal{D}}|$ goes to zero. ∎

Note that we do not make use of geometricity here, which is something that is necessary in [17] as it uses one-form integration of geometric rough paths.

7. Conclusion

In this paper we have expanded upon the idea of jointly controlled paths from [9], which serves as a framework for integration of two-parameter paths with respect to two rough paths, and established a rough Fubini theorem without the need for smooth approximability. A maximal inequality is established which provides a new bound for the double integrals, and some form of continuity is established for the double integrals of jointly controlled paths. Finally, we show that the signature kernel is a jointly controlled path and extend the integral equation (6.5) to non-geometric rough paths. Some potentially interesting questions remain to be answered, such as extension of a rough Fubini theorem to shapes other than rectangles, and whether in the geometric case the lifts of the double integrals to rough paths have any interesting relationships to each other.

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