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On the Hölder regularity of signed solutions to a doubly nonlinear equation. Part II

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Abstract. We demonstrate two proofs for the local Hölder continuity of possibly sign-changing solutions to a class of doubly nonlinear parabolic equations whose prototype is

$$\partial_t(|u|^{q-1}u) - \Delta_p u = 0, \quad p > 2,\ 0 < q < p-1.$$

The first proof takes advantage of the expansion of positivity for the degenerate, parabolic p-Laplacian, thus simplifying the argument; the second proof relies solely on the energy estimates for doubly nonlinear parabolic equations. After proper adaptations of the interior arguments, we also obtain the boundary regularity for initial-boundary value problems of Dirichlet and Neumann type.

1. Introduction and main results

Initiated in [1], we continue our study on the Hölder regularity of weak solutions to a class of doubly nonlinear parabolic equations whose model case is

$$\partial_t(|u|^{q-1}u) - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{weakly in } E_T. \tag{1.1}$$

Here $E_T := E \times (0,T]$ for some $T > 0$ and some E open in $\mathbb{R}^N$. In [1] we have studied the borderline case, i.e., $p > 1$ and $q = p-1$, and in this note we will take on the doubly degenerate case, i.e., $p > 2$ and $0 < q < p-1$.

Our *main result* states that locally bounded, weak solutions to (1.1) are Hölder continuous in the interior, and up to the parabolic boundary of E_T, if proper assumptions on the boundary are imposed. Two proofs will be presented, both of which are entirely local and structural.

As a matter of fact, we shall consider parabolic partial differential equations of the general form

$$\partial_t(|u|^{q-1}u) - \operatorname{div}\mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T, \tag{1.2}$$

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where $\mathbf{A}(x,t,u,\zeta)\colon E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is a Carathéodory function. Namely, it is measurable with respect to $(x,t) \in E_T$ for all $(u,\zeta) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (u,ζ) for a.e. $(x,t) \in E_T$. Moreover, we assume the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,\zeta)\cdot\zeta \geq C_o|\zeta|^p, \\ |\mathbf{A}(x,t,u,\zeta)| \leq C_1|\zeta|^{p-1}, \end{cases} \quad \text{for a.e. } (x,t) \in E_T, \text{ all } u \in \mathbb{R} \text{ and all } \zeta \in \mathbb{R}^N, \tag{1.3}$$

where C_o and C_1 are given positive constants.

In the sequel, we will refer to the set of parameters $\{p,\, q,\, N,\, C_o,\, C_1\}$ as the *structural data*. We also write $\boldsymbol{\gamma}$ as a generic positive constant that can be quantitatively determined a priori only in terms of the data and that can change from line to line.

Postponing the formal definitions of weak solution, we will proceed to present the main results on the interior regularity in Section 1.1 and the boundary regularity in Section 1.2.

1.1. Interior regularity

Suppose that $u \in L^\infty(E_T)$ and set $M := \|u\|_{\infty,E_T}$. Let $\Gamma := \partial E_T - \bar{E} \times \{T\}$ be the parabolic boundary of E_T. For a compact set $\mathcal{K} \subset E_T$, we introduce the following intrinsic, parabolic distance from $\mathcal{K}$ to Γ by

$$\operatorname{dist}_p(\mathcal{K};\Gamma) := \inf_{\substack{(x,t)\in\mathcal{K}\\(y,s)\in\Gamma}} \{|x-y| + M^{(p-q-1)/p}|t-s|^{1/p}\}.$$

Now we state our main result concerning the interior Hölder continuity of weak solutions to (1.2), subject to the structure conditions (1.3). Throughout this note, we assume that $p > 2$ and $0 < q < p-1$ unless otherwise stated.

Theorem 1.1. *Let u be a bounded, local, weak solution to* (1.2)–(1.3) *in E_T. Then u is locally Hölder continuous in E_T. More precisely, there exist constants $\boldsymbol{\gamma} > 1$ and $\beta \in (0,1)$, that can be determined a priori only in terms of the data, such that for every compact set $\mathcal{K} \subset E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \boldsymbol{\gamma} M\left(\frac{|x_1-x_2| + M^{(p-q-1)/p}|t_1-t_2|^{1/p}}{\operatorname{dist}_p(\mathcal{K};\Gamma)}\right)^\beta,$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \mathcal{K}$.

Remark 1.2. Local boundedness is sufficient for Theorem 1.1 to hold. In fact, local boundedness is inherent in the notion of weak solutions, cf. Appendix A. Moreover, the method also applies to equations with lower order terms like in Chapters II–IV of [4] and in Appendix C of [6]. However, we will not pursue generality in this direction. Instead, concentration will be made on the actual novelty.

Remark 1.3. Theorem 1.1 implies a Liouville type theorem; the argument is the same as Corollary 1.1 of [1], which we refer to for details.

1.2. Boundary regularity

Results on the boundary regularity will be stated in this section. Let us first consider the following initial-boundary value problem of Dirichlet type:

$$\begin{cases} \partial_t(|u|^{q-1}u) - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T, \\ u(\cdot,t)|_{\partial E} = g(\cdot,t)|_{\partial E} \quad \text{for a.e. } t \in (0,T], \qquad u(\cdot,0) = u_o(\cdot), \end{cases} \tag{1.4}$$

where the structure conditions (1.3) are in force. Concerning the Dirichlet datum g at the lateral boundary $S_T := \partial E \times (0,T]$ and the initial datum u_o, we assume

(I) u_o is continuous in $\bar{E}$ with modulus of continuity $\omega_o(\cdot)$;

(D) $g \in L^p(0,T;W^{1,p}(E))$, and g is continuous on S_T with modulus of continuity $\omega_g(\cdot)$.

As for the geometry of the boundary ∂E, we introduce the property of *positive geometric density*:

(G) there exists $\alpha_* \in (0,1)$ and $\varrho_o > 0$ such that for all $x_o \in \partial E$, every cube $K_\varrho(x_o)$ and $0 < \varrho \le \varrho_o$, we have $|E \cap K_\varrho(x_o)| \le (1-\alpha_*)|K_\varrho|$.

Here, for $\varrho > 0$, we have set $K_\varrho(x_o)$ to be the cube with center at $x_o \in \mathbb{R}^N$ and edge 2ϱ, whose faces are parallel with the coordinate planes. When $x_o = 0$, we simply write K_ϱ. Intuitively, condition (G) means that there is an exterior cone whose vertex is attached to x_o and whose angle is quantified by α_*. This condition is also termed *Condition* (A), see (6.48) of [13].

Next, we consider the Neumann problem. The boundary ∂E is assumed to be of class C^1, such that the outward unit normal, which we denote by $\mathbf{n}$, is defined on ∂E. The initial-boundary value problem of Neumann type is formulated as

$$\begin{cases} \partial_t(|u|^{q-1}u) - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T, \\ \mathbf{A}(x,t,u,Du) \cdot \mathbf{n} = \psi(x,t,u) \quad \text{on } S_T, \qquad u(\cdot,0) = u_o(\cdot), \end{cases} \tag{1.5}$$

where the structure conditions (1.3) and assumption (I) for the initial data are still in force. For the Neumann datum ψ, we assume for simplicity, for some absolute constant C_2, that

$$|\psi(x,t,u)| \le C_2 \quad \text{for a.e. } (x,t,u) \in S_T \times \mathbb{R}. \tag{N}$$

Although more general conditions should also work (cf. Chapter II, Section 2 of [4]), we however will not pursue generality in this direction.

The formal definitions of weak solutions to (1.4) and (1.5) will be given in Section 1.4. Now we are ready to present the results concerning regularity of solutions to (1.4) or (1.5) up to the parabolic boundary Γ. Recall also that we have set $M := \|u\|_{\infty,E_T}$.

1.2.1. Near the initial time.

Theorem 1.4. *Let u be a bounded weak solution to the Dirichlet problem* (1.4) *under the assumption* (1.3)*. Assume* (I) *holds. Then u is continuous in $K \times [0,T]$ for any compact set $K \subset E$. More precisely, there is a modulus of continuity $\boldsymbol{\omega}(\cdot)$, determined by the data,* $\operatorname{dist}(K,\partial E)$*, M and $\boldsymbol{\omega}_o(\cdot)$, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \boldsymbol{\omega}(|x_1-x_2| + M^{(p-q-1)/p}|t_1-t_2|^{1/p}),$$

for every pair of points $(x_1,t_1),(x_2,t_2)\in K\times[0,T]$*. In particular, if* u_o *is Hölder continuous with exponent* β_o*, then* $\boldsymbol{\omega}(r)=\boldsymbol{\gamma} M r^{\beta}$ *for some* $\boldsymbol{\gamma}>0$ *and* $\beta\in(0,\beta_o]$ *depending on the data,* $\operatorname{dist}(K;\partial E)$ *and* β_o*.*

Remark 1.5. As we shall see in the proof of Theorem 1.4, the estimate on the modulus of continuity actually holds true for all $p>1$ and $q>0$, if $t_1=0$ or $t_2=0$.

1.2.2. Near S_T – Dirichlet type data.

Theorem 1.6. *Let u be a bounded weak solution to the Dirichlet problem* (1.4) *under the assumption* (1.3)*. Assume* (D) *and* (G) *hold. Then u is continuous in any compact set* $\mathcal{K}\subset\overline{E}_T$*. More precisely, there is a modulus of continuity* $\boldsymbol{\omega}(\cdot)$*, determined by the data,* α_**,* ϱ_o*,* $\operatorname{dist}(\mathcal{K};\{t=0\})$*, M and* $\boldsymbol{\omega}_g(\cdot)$*, such that*

$$|u(x_1,t_1)-u(x_2,t_2)|\le\boldsymbol{\omega}\big(|x_1-x_2|+M^{(p-q-1)/p}|t_1-t_2|^{1/p}\big),$$

for every pair of points $(x_1,t_1),(x_2,t_2)\in\mathcal{K}$*. In particular, if g is Hölder continuous with exponent* β_g*, then* $\boldsymbol{\omega}(r)=\boldsymbol{\gamma} M r^{\beta}$ *for some* $\boldsymbol{\gamma}>0$ *and* $\beta\in(0,\beta_g]$ *depending on the data,* α_**,* ϱ_o*,* $\operatorname{dist}(\mathcal{K};\{t=0\})$ *and* β_g*.*

1.2.3. Near S_T – Neumann type data.

Theorem 1.7. *Let u be a bounded weak solution to the Neumann problem* (1.5)*. Assume* ∂E *is of class* C^1 *and* (N) *holds. Then u is Hölder continuous in any compact set* $\mathcal{K}\subset\overline{E}_T$*. More precisely, there exist constants* $\boldsymbol{\gamma}>1$ *and* $\beta\in(0,1)$ *determined by the data,* C_2*,* $\operatorname{dist}(\mathcal{K};\{t=0\})$ *and the structure of* ∂E*, such that*

$$|u(x_1,t_1)-u(x_2,t_2)|\le\boldsymbol{\gamma} M\big(|x_1-x_2|+M^{(p-q-1)/p}|t_1-t_2|^{1/p}\big)^{\beta},$$

for every pair of points $(x_1,t_1),(x_2,t_2)\in\mathcal{K}$*.*

Remark 1.8. The proofs of Theorems 1.4–1.7 are local in nature. As a result, it suffices to require the boundary data in the Dirichlet problem (1.4) or the Neumann problem (1.5) to be taken just on a portion of the parabolic boundary.

1.3. Novelty and significance

The doubly nonlinear parabolic equation (1.2) accounts for many physical models, including dynamics of glaciers, shallow water flows and friction dominated flows in a gas network. We refer to [1] for a source of physical motivations. The mathematical interest of this equation lies in the degeneracy or the singularity or both it possesses, and a broader class of parabolic equations it generates, which include the parabolic p-Laplacian and the porous medium equation as particular instances.

The issue of local Hölder regularity for this equation has been investigated by a number of authors, in various forms and with different notions of solution, cf. [8, 9, 14, 16]. However, all of them assume that $p>2$ and $0<q<1$.

The main novelty of our results consists in extending the known range to a larger one, that is, $p>2$ and $0<q<p-1$, cf. Figure 1. On the other hand, even in the case $p>2$ and $0<q<1$, our results are not covered by the previous works, as they either use different notions of solution [9, 14, 16], or assume non-negativity of the solution [8].

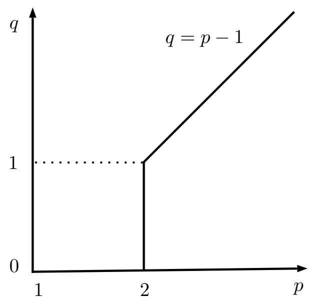

Figure 1. Range of p and q.

One of our main technical advances from the previous works lies in that we dispense with any kind of logarithmic type energy estimates. As such our arguments should have further implications in the context of the so-called Q-minima from the calculus of variations, cf. [16].

The expansion of positivity for the degenerate parabolic equations has been established in [5] as a key tool to study Harnack's inequalty. Roughly speaking, it asserts that the measure of the positivity set of a non-negative, super-solution translates into pointwise positivity at later times. Using it to handle the Hölder regularity seems new in the doubly degenerate setting. Similar ideas have appeared in [7, 11] in different forms, either for the parabolic p-Laplacian or for the porous medium equation. The real advantage of this important property lies in the simplification it brings and a geometric character it offers. On the other hand, the proof of this property is not easy, and meanwhile it is only known to hold in the context of partial differential equations. This latter point unfortunately results in certain restrictions for its application near the boundary. In particular, when we deal with the boundary regularity for Neumann problems, the original approach of DiBenedetto [3] has to be evoked and adapted.

Our arguments can be adapted to the borderline cases. In particular, when $q = 1$, the arguments deal with the degenerate, parabolic p-Laplacian; when $p = 2$, the porous medium equation can be treated; when $q = p - 1$, we are back to our first work [1]; see also [10] for non-negative solutions. Although a Harnack inequality has been established for all $0 < q < p - 1$ in [2], the Hölder regularity seems to present a different feature when $1 < p < 2$ and hence asks for a new argument. Beyond these borderline cases will be the subject of our next study.

1.4. Notations and definitions

1.4.1. Notion of local solution. A function

$$u \in C_{\text{loc}}(0, T; L^{q+1}_{\text{loc}}(E)) \cap L^{p}_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(E)) \tag{1.6}$$

is a local, weak sub(super)-solution to (1.2) with the structure conditions (1.3) if for every compact set $K \subset E$ and every sub-interval $[t_1, t_2] \subset (0, T]$,

$$\int_K |u|^{q-1} u \zeta \,\mathrm{d}x \Big|_{t_1}^{t_2} + \iint_{K \times (t_1, t_2)} [-|u|^{q-1} u \zeta_t + \mathbf{A}(x, t, u, Du) \cdot D\zeta] \,\mathrm{d}x \,\mathrm{d}t \le (\ge)\, 0 \tag{1.7}$$

for all non-negative test functions

$$\zeta \in W^{1,q+1}_{\mathrm{loc}}(0,T;L^{q+1}(K)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_o(K)).$$

This guarantees that all the integrals in (1.7) are convergent.

A function u that is both a local weak sub-solution and a local weak super-solution to (1.2)–(1.3) is a local weak solution.

1.4.2. Notion of solution to the Dirichlet problem. A function

$$u \in C(0,T;L^{q+1}(E)) \cap L^p(0,T;W^{1,p}(E))$$

is a weak sub(super)-solution to (1.4) if for every sub-interval $[t_1,t_2] \subset (0,T]$,

$$\int_E |u|^{q-1}u\zeta \,\mathrm{d}x \,\Big|_{t_1}^{t_2} + \iint_{E\times(t_1,t_2)} \big[-|u|^{q-1}u\zeta_t + \mathbf{A}(x,t,u,Du)\cdot D\zeta\big]\,\mathrm{d}x\,\mathrm{d}t \le (\ge)\, 0$$

for all non-negative test functions

$$\zeta \in W^{1,q+1}_{\mathrm{loc}}(0,T;L^{q+1}(E)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_o(E)).$$

Moreover, setting $\hat{q} := \min\{2, q+1\}$, the initial datum is taken in the sense that for any compact set $K \subset E$,

$$\int_{K\times\{t\}} (u-u_o)^{\hat{q}}_{\pm}\,\mathrm{d}x \to 0 \quad \text{as } t \downarrow 0.$$

The Dirichlet datum g is attained under $u \le (\ge)\, g$ on ∂E in the sense that the traces of $(u-g)_\pm$ vanish as functions in $W^{1,p}(E)$ for a.e. $t \in (0,T]$, i.e., we have $(u-g)_\pm \in L^p(0,T;W^{1,p}_o(E))$. Notice that no *a priori* information is assumed on the smoothness of ∂E.

A function u that is both a weak sub-solution and a weak super-solution to (1.4) is a weak solution.

1.4.3. Notion of solution to the Neumann problem. A function

$$u \in C(0,T;L^{q+1}(E)) \cap L^p(0,T;W^{1,p}(E))$$

is a weak sub(super)-solution to (1.5) if for every compact set $K \subset \mathbb{R}^N$ and every sub-interval $[t_1,t_2] \subset (0,T]$,

$$\begin{aligned}\int_{K\cap E} |u|^{q-1}u\zeta \,\mathrm{d}x \,\Big|_{t_1}^{t_2} &+ \iint_{\{K\cap E\}\times(t_1,t_2)} \big[-|u|^{q-1}u\zeta_t + \mathbf{A}(x,t,u,Du)\cdot D\zeta\big]\,\mathrm{d}x\,\mathrm{d}t \\ &\le (\ge) \iint_{\{K\cap\partial E\}\times(t_1,t_2)} \psi(x,t,u)\zeta\,\mathrm{d}\sigma\,\mathrm{d}t\end{aligned}$$

for all non-negative test functions

$$\zeta \in W^{1,q+1}_{\mathrm{loc}}(0,T;L^{q+1}(K)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_o(K)).$$

Here $\mathrm{d}\sigma$ denotes the surface measure on ∂E. The Neumann datum ψ is reflected in the boundary integral on the right-hand side. Moreover, the initial datum is taken as in the Dirichlet problem.

A function u that is both a weak sub-solution and a weak super-solution to (1.5) is a weak solution.

2. Energy estimates

In this section we present certain energy estimates for weak sub(super)-solutions to (1.2)–(1.3). They are analogues of the energy estimates derived in [1], which will be referred to for details. Moreover, it is noteworthy that they actually hold true for all $p > 1$ and $q > 0$.

The different roles played by sub-solutions and super-solutions are emphasized. When we state *"u is a sub(super)-solution..."* and use " $\pm$ " or " $\mp$ " in what follows, we mean the sub-solution corresponds to the upper sign and the super-solution corresponds to the lower sign in the statement.

For any $k \in \mathbb{R}$, we denote the truncated functions

$$(u-k)_+ \equiv \max\{u-k, 0\}, \quad (u-k)_- \equiv \max\{-(u-k), 0\}.$$

For $w, k \in \mathbb{R}$, we define two non-negative quantities

$$\mathfrak{g}_\pm(w,k) = \pm q \int_k^w |s|^{q-1}(s-k)_\pm \, \mathrm{d}s.$$

For $b \in \mathbb{R}$ and $\alpha > 0$, we will embolden $\boldsymbol{b}^\alpha$ to denote the signed α-power of b as

$$\boldsymbol{b}^\alpha = \begin{cases} |b|^{\alpha-1}b, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

Throughout the rest of this note, we will use $K_R(x_o)$ to denote the cube of side length $2R$ and center x_o, whose faces are parallel with the coordinate planes in $\mathbb{R}^N$, and the symbols

$$(x_o, t_o) + Q_\varrho(\theta) := K_\varrho(x_o) \times (t_o - \theta\varrho^p, t_o),$$
$$(x_o, t_o) + Q_{R,S} := K_R(x_o) \times (t_o - S, t_o)$$

to denote (backward) cylinders with the indicated positive parameters; when the context is unambiguous, we will omit the vertex (x_o, t_o) from the symbols for simplicity.

First of all, we present energy estimates for local weak sub(super)-solutions defined in Section 1.4.1. The proof is similar to Proposition 3.1 of [1], which we refer to for details. The only difference is that in the present situation, $\boldsymbol{u}^{p-1}$ must be replaced by $\boldsymbol{u}^q$ in terms related to the time derivative and $\mathfrak{g}_\pm$ has to be defined as above. Since the testing functions and the treatment of the term containing the vector-field $\mathbf{A}$ remain unchanged, the constant $\boldsymbol{\gamma}$ on the right-hand side of the estimates is independent of q.

Proposition 2.1. *Let u be a local weak sub(super)-solution to (1.2)–(1.3) in E_T. There exists a constant $\boldsymbol{\gamma}(C_o, C_1, p) > 0$ such that for all cylinders $Q_{R,S} \Subset E_T$, all $k \in \mathbb{R}$, and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(x_o) \times (t_o - S, t_o)$, we have*

$$\max\Big\{ \operatorname*{ess\,sup}_{t_o-S<t<t_o} \int_{K_R(x_o)\times\{t\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x, \iint_{Q_{R,S}} \zeta^p |D(u-k)_\pm|^p \,\mathrm{d}x\,\mathrm{d}t \Big\}$$
$$\le \boldsymbol{\gamma} \iint_{Q_{R,S}} \big[(u-k)_\pm^p |D\zeta|^p + \mathfrak{g}_\pm(u,k)|\partial_t \zeta^p|\big]\,\mathrm{d}x\,\mathrm{d}t + \int_{K_R(x_o)\times\{t_o-S\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x.$$

Next, we consider the situation near the initial level $t = 0$ when a continuous datum u_o is prescribed. Suppose the level k satisfies

$$\begin{cases} k \ge \sup_{K_R(x_o)} u_o & \text{for sub-solutions,} \\ k \le \inf_{K_R(x_o)} u_o & \text{for super-solutions.} \end{cases} \tag{2.1}$$

The following energy estimate can be obtained as in Proposition 3.2 of [1].

Proposition 2.2. *Let u be a local weak sub(super)-solution to (1.4) with (1.3) in E_T. There exists a constant $\boldsymbol{\gamma}(C_o, C_1, p) > 0$ such that for all cylinders $K_R(x_o) \times (0, S) \subset E_T$, every $k \in \mathbb{R}$ satisfying (2.1) and every non-negative, piecewise smooth cutoff function ζ independent of t and vanishing on $\partial K_R(x_o)$, we have*

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<t<S} \int_{K_R(x_o)\times\{t\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x + \iint_{K_R(x_o)\times(0,S)} \zeta^p |D(u-k)_\pm|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \le \boldsymbol{\gamma} \iint_{K_R(x_o)\times(0,S)} (u-k)_\pm^p |D\zeta|^p \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Next, we turn our attention to the energy estimates near S_T. When dealing with Dirichlet data, we need to assume the following restrictions on the level k:

$$\begin{cases} k \ge \sup_{Q_{R,S}\cap S_T} g & \text{for sub-solutions,} \\ k \le \inf_{Q_{R,S}\cap S_T} g & \text{for super-solutions.} \end{cases} \tag{2.2}$$

The following energy estimate can be obtained as in Proposition 3.3 of [1].

Proposition 2.3. *Let u be a local weak sub(super)-solution to (1.4) with (1.3) in E_T. There exists a constant $\boldsymbol{\gamma}(C_o, C_1, p) > 0$ such that for all cylinders $Q_{R,S}$ with the vertex $(x_o, t_o) \in S_T$, every $k \in \mathbb{R}$ satisfying (2.2), and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(x_o) \times (t_o - S, t_o)$, we have*

$$\begin{aligned} &\max\Big\{ \operatorname*{ess\,sup}_{t_o-S<t<t_o} \int_{\{K_R(x_o)\cap E\}\times\{t\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x, \iint_{Q_{R,S}\cap E_T} \zeta^p |D(u-k)_\pm|^p \,\mathrm{d}x\,\mathrm{d}t \Big\} \\ &\qquad \le \boldsymbol{\gamma} \iint_{Q_{R,S}\cap E_T} \big[(u-k)_\pm^p |D\zeta|^p + \mathfrak{g}_\pm(u,k)|\partial_t \zeta^p|\big] \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad + \int_{\{K_R(x_o)\cap E\}\times\{t_o-S\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x. \end{aligned}$$

Finally, we deal with the energy estimates for the Neumann problem (1.5). The following can be obtained as in Proposition 3.4 of [1].

Proposition 2.4. *Let u be a local weak sub(super)-solution to (1.5) with (1.3) in E_T. Assume ∂E is of class C^1 and that (N) holds. There exists a constant $\boldsymbol{\gamma} > 0$, depending on C_o, C_1, p and the structure of ∂E, such that for all cylinders $Q_{R,S}$ with the vertex $(x_o, t_o) \in S_T$, every $k \in \mathbb{R}$, and every non-negative, piecewise smooth cutoff function ζ*

vanishing on $\partial K_R(x_o) \times (t_o - S, t_o)$*, we have*

$$\max\Big\{ \operatorname*{ess\,sup}_{t_o-S<t<t_o} \int_{\{K_R(x_o)\cap E\}\times\{t\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x, \iint_{Q_{R,S}\cap E_T} \zeta^p |D(u-k)_\pm|^p \mathrm{d}x\,\mathrm{d}t \Big\}$$
$$\le \boldsymbol{\gamma} \iint_{Q_{R,S}\cap E_T} [(u-k)_\pm^p |D\zeta|^p + \mathfrak{g}_\pm(u,k)|\partial_t \zeta^p|]\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \boldsymbol{\gamma} C_2^{p/(p-1)} \iint_{Q_{R,S}\cap E_T} \zeta^p \chi_{\{(u-k)_\pm>0\}}\,\mathrm{d}x\,\mathrm{d}t + \int_{\{K_R(x_o)\cap E\}\times\{t_o-S\}} \zeta^p \mathfrak{g}_\pm(u,k)\,\mathrm{d}x.$$

3. Preliminary tools

For a compact set $K \subset \mathbb{R}^N$ and a cylinder $\mathcal{Q} := K \times (T_1, T_2] \subset E_T$, we introduce numbers $\boldsymbol{\mu}^\pm$ and $\boldsymbol{\omega}$ satisfying

$$\boldsymbol{\mu}^+ \ge \operatorname{ess\,sup}_{\mathcal{Q}} u, \quad \boldsymbol{\mu}^- \le \operatorname{ess\,inf}_{\mathcal{Q}} u, \quad \boldsymbol{\omega} \ge \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-.$$

In this section, we collect some lemmas, which will be the main ingredients in the proof of Theorem 1.1. The first one is a De Giorgi type lemma, which actually holds for all $p > 1$ and $q > 0$.

Lemma 3.1. *Let* u *be a locally bounded, local weak sub(super)-solution to* (1.2)–(1.3) *in* E_T. *Set* $\theta = (\xi\boldsymbol{\omega})^{q+1-p}$ *for some* $\xi \in (0,1)$ *and assume* $(x_o,t_o) + Q_\varrho(\theta) \subset \mathcal{Q}$. *There exists a constant* $\nu \in (0,1)$*, depending only on the data, such that if*

$$|\{\pm(\boldsymbol{\mu}^\pm - u) \le \xi\boldsymbol{\omega}\} \cap (x_o,t_o) + Q_\varrho(\theta)| \le \nu |Q_\varrho(\theta)|,$$

then either

$$|\boldsymbol{\mu}^\pm| > 8\xi\boldsymbol{\omega} \quad \textit{or} \quad \pm(\boldsymbol{\mu}^\pm - u) \ge \tfrac{1}{2}\xi\boldsymbol{\omega} \quad \textit{a.e. in } (x_o,t_o) + Q_{\varrho/2}(\theta).$$

Proof. The De Giorgi iteration has been performed in Lemma 2.2 of [12] for super-solutions, whereas the proof for sub-solutions is analogous. In order to obtain the present formulation, choose $a = 1/2$ and replace M by $\xi\boldsymbol{\omega}$. If $|\boldsymbol{\mu}^\pm| > 8\xi\omega$, there is nothing to prove. In the opposite case, the assumption $|\boldsymbol{\mu}^\pm| \le 8\xi\omega$ allows us to estimate $L \approx M$. Therefore, the critical number ν depends only on the data. ■

The next lemma is a variant of the previous one, involving quantitative initial data. Again, it actually holds for all $p > 1$ and $q > 0$.

Lemma 3.2. *Let* u *be a locally bounded, local weak sub(super)-solution to* (1.2)–(1.3) *in* E_T. *Set* $\theta = (\xi\boldsymbol{\omega})^{q+1-p}$ *for some* $\xi \in (0,1)$. *There exists a positive constant* ν_o *depending only on the data such that if*

$$\pm(\boldsymbol{\mu}^\pm - u(\cdot,t_o)) \ge \xi\boldsymbol{\omega} \quad \textit{a.e. in } K_\varrho(x_o),$$

then either

$$|\boldsymbol{\mu}^\pm| > 8\xi\boldsymbol{\omega} \quad \textit{or} \quad \pm(\boldsymbol{\mu}^\pm - u) \ge \tfrac{1}{2}\xi\boldsymbol{\omega} \quad \textit{a.e. in } K_{\varrho/2}(x_o) \times (t_o, t_o + \nu_o\theta\varrho^p],$$

provided the cylinders are included in $\mathcal{Q}$.

Proof. After enforcing that $|\boldsymbol{\mu}^-| \le 8\xi\boldsymbol{\omega}$, this is essentially the content of Lemma 3.1 in [12] for super-solutions, the case of sub-solutions being similar. More precisely, one has to choose $a = 1/2$, replace M by $\xi\boldsymbol{\omega}$ and note that $L \approx M$ whenever $|\boldsymbol{\mu}^-| \le 8\xi\boldsymbol{\omega}$. This allows to choose the parameter θ in Lemma 3.1 of [12] in the form $\nu_o(\xi\boldsymbol{\omega})^{q+1-p}$ for some ν_o depending only on the data. ∎

The previous lemma propagates pointwise information in a smaller cube, without a time lag. The next lemma translates measure theoretical information into a pointwise estimate over an expanded cube of later times. This is essentially the expansion of positivity for the degenerate, parabolic p-Laplacian established in [5]; see also Proposition 4.1 in Chapter 4 of [6]. As such, it actually holds for $p > 2$ and $q > 0$.

Lemma 3.3. *Let u be a locally bounded, local, weak sub(super)-solution to (1.2)–(1.3) in E_T. Let the parameters Λ, $c > 0$ and $\alpha \in (0, 1)$ be such that*

$$c\boldsymbol{\omega} \le \pm\boldsymbol{\mu}^\pm \le \Lambda\boldsymbol{\omega},$$

and for some $0 < a \le \frac{1}{2}c$,

$$|\{\pm(\boldsymbol{\mu}^\pm - u(\cdot, t_o)) \ge a\boldsymbol{\omega}\} \cap K_\varrho(x_o)| \ge \alpha|K_\varrho|.$$

There exist constants $b > 0$ and $\eta \in (0, 1)$, depending only on the data, Λ, c, a and α, such that

$$\pm(\boldsymbol{\mu}^\pm - u) \ge \eta\boldsymbol{\omega} \quad \text{a.e. in } K_{2\varrho}(x_o)\times(t_o+\tfrac{1}{2}b\boldsymbol{\omega}^{q-1}(\eta\boldsymbol{\omega})^{2-p}\varrho^p, t_o+b\boldsymbol{\omega}^{q-1}(\eta\boldsymbol{\omega})^{2-p}\varrho^p],$$

provided this cylinder is included in $\mathcal{Q}$.

Proof. We may assume $(x_o, t_o) = (0, 0)$ and prove the case of super-solutions only as the other case is similar. Let $k = \boldsymbol{\mu}^- + \frac{1}{2}c\boldsymbol{\omega}$. By Lemma A.1, $u_k := \min\{u, k\} = k - (u-k)_-$ is a local, weak super-solution to (1.2), i.e.,

$$\partial_t \boldsymbol{u}_k^q - \operatorname{div} \mathbf{A}(x, t, u_k, Du_k) \ge 0 \quad \text{weakly in } \mathcal{Q}.$$

Here the symbol $\boldsymbol{u}_k^q$ has been emboldened to denote the signed power of u_k defined in Section 2. To proceed, we define

$$v := \boldsymbol{u}_k^q - (\boldsymbol{\mu}^-)^q,$$

which is non-negative in $\mathcal{Q}$. Thanks to the restriction on $\boldsymbol{\mu}^-$, it is not hard to show that v belongs to the function space (1.6), with $q = 1$, defined on $\mathcal{Q}$ and satisfies

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x, t, v, Dv) \ge 0 \quad \text{weakly in } \mathcal{Q}.$$

Here $\bar{\mathbf{A}}$ is defined by

$$\bar{\mathbf{A}}(x, t, y, \zeta) := \mathbf{A}\big(x,\ t,\ |\tilde{y} + (\boldsymbol{\mu}^-)^q|^{(1-q)/q}(\tilde{y} + (\boldsymbol{\mu}^-)^q),\ \tfrac{1}{q}|\tilde{y} + (\boldsymbol{\mu}^-)^q|^{(1-q)/q}\zeta\big),$$

where $\tilde{y}$ denotes the truncation

$$\tilde{y} := \min\big\{\max\{y, 0\}, (1 - \tfrac{1}{2^q})(c\boldsymbol{\omega})^q\big\}.$$

Meanwhile, one verifies the structure conditions

$$\bar{\mathbf{A}}(x,t,y,\zeta)\cdot\zeta \geq \bar{C}_o\,\boldsymbol{\omega}^{(q-1)(1-p)}|\zeta|^p, \quad |\bar{\mathbf{A}}(x,t,y,\zeta)| \leq \bar{C}_1\,\boldsymbol{\omega}^{(q-1)(1-p)}|\zeta|^{p-1},$$

with positive constants $\bar{C}_i = \bar{C}_i(C_i, p, q, c, \Lambda)$, $i = 0, 1$.

In order to eliminate the dependence on $\boldsymbol{\omega}$ in the structure conditions of $\bar{\mathbf{A}}$, we consider the transformed function

$$\widetilde{v}(x,t) := v(x, \boldsymbol{\omega}^{(q-1)(p-1)}t),$$

which satisfies

$$\widetilde{v}_t - \operatorname{div}\widetilde{\mathbf{A}}(x,t,\widetilde{v}, D\widetilde{v}) \geq 0 \quad \text{weakly in } K \times (\boldsymbol{\omega}^{(q-1)(1-p)}T_1, \boldsymbol{\omega}^{(q-1)(1-p)}T_2]. \tag{3.1}$$

Here $\widetilde{\mathbf{A}}$ is defined by

$$\widetilde{\mathbf{A}}(x,t,y,\zeta) := \boldsymbol{\omega}^{(q-1)(p-1)}\bar{\mathbf{A}}(x, \boldsymbol{\omega}^{(q-1)(p-1)}t, y, \zeta)$$

for $(x,t) \in \widetilde{\mathcal{Q}} := K \times (\boldsymbol{\omega}^{(q-1)(1-p)}T_1, \boldsymbol{\omega}^{(q-1)(1-p)}T_2]$ and all $(y,\zeta) \in \mathbb{R}\times\mathbb{R}^N$. Thus, an easy calculation shows that $\widetilde{\mathbf{A}}$ satisfies the conditions

$$\widetilde{\mathbf{A}}(x,t,y,\zeta)\cdot\zeta \geq \bar{C}_o|\zeta|^p, \quad |\widetilde{\mathbf{A}}(x,t,y,\zeta)| \leq \bar{C}_1|\zeta|^{p-1}.$$

In other words, the function $\widetilde{v}$ is a non-negative, local, weak super-solution to the parabolic p-Laplacian type equation (3.1) in $\widetilde{\mathcal{Q}}$. This allows us to apply the expansion of positivity, see Proposition 4.1 in Chapter 4 of [6]. The measure theoretical information for u implies a similar inequality for u_k; in fact, we have

$$|\{u_k(\cdot,0) \geq \boldsymbol{\mu}^- + a\boldsymbol{\omega}\} \cap K_\varrho| \geq \alpha|K_\varrho|.$$

Taking into account $-\Lambda\boldsymbol{\omega} \leq u_k \leq -\frac{1}{2}c\boldsymbol{\omega}$, the information that $u_k(\cdot,0) \geq \boldsymbol{\mu}^- + a\boldsymbol{\omega}$ can be converted into an estimate from below for v. Indeed, by the mean value theorem, we estimate

$$\begin{aligned} v(\cdot,0) = \boldsymbol{u}_k^q(\cdot,0) - (\boldsymbol{\mu}^-)^q &\geq q\min\{\Lambda^{q-1}, (\tfrac{1}{2}c)^{q-1}\}\boldsymbol{\omega}^{q-1}(u_k(\cdot,0) - \boldsymbol{\mu}^-) \\ &\geq aq\min\{\Lambda^{q-1}, (\tfrac{1}{2}c)^{q-1}\}\boldsymbol{\omega}^q =: \widetilde{a}\boldsymbol{\omega}^q. \end{aligned}$$

In terms of $\widetilde{v}$, this becomes

$$|\{\widetilde{v}(\cdot,0) \geq \widetilde{a}\boldsymbol{\omega}^q\} \cap K_\varrho| \geq \alpha|K_\varrho|.$$

An application of Proposition 4.1 in Chapter 4 of [6] to $\widetilde{v}$ (with $C \equiv 0$ and $M = \widetilde{a}\boldsymbol{\omega}^q$) yields that for some positive constants $\eta, \delta \in (0,1)$ and $b > 1$, depending only on the data $\bar{C}_o, \bar{C}_1, p, N$ and on α, we have

$$\widetilde{v}(\cdot,t) \geq \eta\widetilde{a}\boldsymbol{\omega}^q \quad \text{a.e. in } K_{2\varrho},$$

for all

$$\frac{b^{p-2}}{2(\eta\widetilde{a}\boldsymbol{\omega}^q)^{p-2}}\,\delta\varrho^p < t \leq \frac{b^{p-2}}{(\eta\widetilde{a}\boldsymbol{\omega}^q)^{p-2}}\,\delta\varrho^p.$$

For v, this means that

$$v(\cdot,t) \geq \eta\tilde{a}\boldsymbol{\omega}^q \quad \text{a.e. in } K_{2\varrho},$$

for all t in the interval

$$\tfrac{1}{2} b^{p-2}\delta(\eta\tilde{a}\boldsymbol{\omega})^{2-p}\boldsymbol{\omega}^{q-1}\varrho^p < t \leq b^{p-2}\delta(\eta\tilde{a}\boldsymbol{\omega})^{2-p}\boldsymbol{\omega}^{q-1}\varrho^p.$$

We revert to the original function u with the aid of the mean value theorem. More precisely, we estimate

$$\begin{aligned}\eta\tilde{a}\boldsymbol{\omega}^q \leq v = \boldsymbol{u}_k^q - (\boldsymbol{\mu}^-)^q &\leq q\max\{\Lambda^{q-1}, (\tfrac{1}{2}c)^{q-1}\}\boldsymbol{\omega}^{q-1}(u_k - \boldsymbol{\mu}^-)\\ &\leq \tilde{\boldsymbol{\gamma}}\boldsymbol{\omega}^{q-1}(u - \boldsymbol{\mu}^-)\end{aligned}$$

for some positive $\tilde{\boldsymbol{\gamma}} = \tilde{\boldsymbol{\gamma}}(q,c,\Lambda)$. This, however, is equivalent to

$$u(\cdot,t) \geq \boldsymbol{\mu}^- + \frac{\eta\tilde{a}}{\tilde{\boldsymbol{\gamma}}}\boldsymbol{\omega} \quad \text{a.e. in } K_{2\varrho},$$

for all t in the above interval. Redefining $\eta\tilde{a}/\tilde{\boldsymbol{\gamma}}$ as η and $\tilde{\boldsymbol{\gamma}}^{2-p}b^{p-2}\delta$ as b, the claim follows. ■

Remark 3.4. An inspection of the above proof shows that $\eta = \boldsymbol{\gamma}a$ for some positive $\boldsymbol{\gamma}$ depending only on the data, α, c and Λ. The conclusion of Lemma 3.3 holds true for a smaller η by properly making a smaller.

The following lemma examines the situation when pointwise information is given at the initial level. It actually holds for $p > 2$ and $q > 0$.

Lemma 3.5. *Let u be a locally bounded, local weak sub(super)-solution to (1.2)–(1.3) in E_T. Introduce the parameters Λ, $c > 0$ and $\eta \in (0,1)$, and set $\theta = \boldsymbol{\omega}^{q-1}(\eta\boldsymbol{\omega})^{2-p}$. Suppose that*

$$c\boldsymbol{\omega} \leq \pm\boldsymbol{\mu}^{\pm} \leq \Lambda\boldsymbol{\omega}.$$

There exists a positive constant ν_1, depending only on the data, c and Λ, such that if

$$\pm(\boldsymbol{\mu}^{\pm} - u(\cdot,t_o)) \geq \eta\boldsymbol{\omega} \quad \text{a.e. in } K_\varrho(x_o),$$

then

$$\pm(\boldsymbol{\mu}^{\pm} - u) \geq \tfrac{1}{2}\eta\boldsymbol{\omega} \quad \text{a.e. in } K_{\varrho/2}(x_o) \times (t_o, t_o + \nu_1\theta\varrho^p],$$

provided the cylinders are included in $\mathcal{Q}$.

Proof. Suppose u is a local, weak super-solution as the other case is similar. Moreover, we may assume $(x_o,t_o) = (0,0)$. Introduce $\tilde{v}$ like in the proof of Lemma 3.3, which turns out to be a non-negative, local, weak super-solution to the parabolic p-Laplacian type equation (3.1) in $\tilde{\mathcal{Q}}$. Using the mean value theorem, the information that $u(\cdot,0) \geq \boldsymbol{\mu}^- + \eta\boldsymbol{\omega}$ in K_ϱ yields that $\tilde{v}(\cdot,0) \geq \boldsymbol{\gamma}\eta\boldsymbol{\omega}^q$ in K_ϱ for some positive $\boldsymbol{\gamma} = \boldsymbol{\gamma}(q,c,\Lambda)$. Consequently, we may apply Lemma 4.1 in Chapter 3 of [6] or Lemma 3.2 of [12] to $\tilde{v}$. For $a \in (0,1)$ at our disposal, we have

$$\tilde{v} \geq a\boldsymbol{\gamma}\eta\boldsymbol{\omega}^q \quad \text{a.e. on } K_{\varrho/2} \times (0, \vartheta(\tfrac{1}{2}\varrho)^p],$$

where

$$\vartheta = \bar{c}(1-a)^{N+3}(\boldsymbol{\gamma}\eta\boldsymbol{\omega}^q)^{2-p},$$

for some constant $\bar{c} \in (0,1)$ depending on $\overline{C}_o, \overline{C}_1, p, N$. As in the proof of Lemma 3.3, we convert this into an estimate for u. First, the scaling in time gives

$$\begin{aligned}\frac{\vartheta}{2^p \boldsymbol{\omega}^{(q-1)(1-p)}} &= 2^{-p}\bar{c}(1-a)^{N+3}\boldsymbol{\gamma}^{2-p}\boldsymbol{\omega}^{q-1}(\eta\boldsymbol{\omega})^{2-p}\\ &= 2^{-p}\bar{c}(1-a)^{N+3}\boldsymbol{\gamma}^{2-p}\theta =: \nu_1\theta,\end{aligned}$$

so that $v \geq a\boldsymbol{\gamma}\eta\boldsymbol{\omega}^q$ on $K_{\varrho/2} \times (0, \nu_1\theta\varrho^p]$. Note that ν_1 depends on $\overline{C}_o, \overline{C}_1, p, q, N, c, \Lambda$ and a. As in the proof of Lemma 3.3, we may apply the mean value theorem to estimate

$$a\boldsymbol{\gamma}\eta\boldsymbol{\omega}^q \leq v \leq \widetilde{\boldsymbol{\gamma}}\boldsymbol{\omega}^{q-1}(u - \boldsymbol{\mu}^-)$$

for some positive $\widetilde{\boldsymbol{\gamma}} = \widetilde{\boldsymbol{\gamma}}(q, c, \Lambda)$, and therefore on $K_{\varrho/2} \times (0, \nu_1\theta\varrho^p]$, we have

$$u \geq \boldsymbol{\mu}^- + \frac{a\boldsymbol{\gamma}}{\widetilde{\boldsymbol{\gamma}}}\eta\boldsymbol{\omega}.$$

Finally, choosing the free parameter a so that $a\boldsymbol{\gamma}/\widetilde{\boldsymbol{\gamma}} = 1/2$, on the one hand determines the value of ν_1 in dependence on the data, c and Λ, and on the other hand implies the desired bound from below. ■

4. The first proof of Theorem 1.1

The proof of Theorem 1.1 in this section relies on the expansion of positivity from Lemma 3.3. This important tool simplifies our arguments, though the attainment of it is difficult and turned out to be a major achievement in the recent theory, cf. [5,6]. As such, the same simplification can be carried out in [1]. On the other hand, the argument of this section does not seem applicable directly to the boundary regularity for the Neumann problem. For this reason, we will give a second proof of Theorem 1.1 in Section 5, referring back to our previous arguments in [1] that are modeled on [3].

4.1. The proof begins

Assume $(x_o, t_o) = (0,0)$, let $Q_o = K_\varrho \times (-\varrho^{p-1}, 0] \Subset E_T$ with radius $\varrho \leq 1$ and set

$$\boldsymbol{\mu}^+ = \operatorname{ess\,sup}_{Q_o} u, \quad \boldsymbol{\mu}^- = \operatorname{ess\,inf}_{Q_o} u, \quad \boldsymbol{\omega} \geq \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-.$$

Let $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$. For some $A > 1$ to be determined in terms of the data, we may assume that

$$Q_\varrho(A\theta) \subset Q_o, \quad \text{such that } \operatorname{ess\,osc}_{Q_\varrho(A\theta)} u \leq \boldsymbol{\omega}; \tag{4.1}$$

otherwise, we would have

$$\boldsymbol{\omega} \leq L\varrho^{1/(p-q-1)}, \quad \text{where } L = 4A^{1/(p-q-1)}. \tag{4.2}$$

Our proof unfolds along two main cases, namely, for some $\xi\in(0,1)$ to be determined,

$$\text{(4.3a)}\qquad \text{when } u \text{ is near zero:}\qquad \boldsymbol{\mu}^- \le \xi\boldsymbol{\omega} \text{ and } \boldsymbol{\mu}^+ \ge -\xi\boldsymbol{\omega};$$

$$\text{(4.3b)}\qquad \text{when } u \text{ is away from zero:}\quad \boldsymbol{\mu}^- > \xi\boldsymbol{\omega} \text{ or } \boldsymbol{\mu}^+ < -\xi\boldsymbol{\omega}.$$

Note that (4.3a) implies that $|\boldsymbol{\mu}^\pm|\le 2\boldsymbol{\omega}$. We deal with this case in Sections 4.2–4.4; the other case will be treated in Section 4.5.

4.2. Reduction of oscillation near zero – Part I

In this section, we will assume that (4.3a) holds true. We work with u as a super-solution near its infimum. To proceed further, we assume

$$\boldsymbol{\mu}^+ - \boldsymbol{\mu}^- > \tfrac12\boldsymbol{\omega}. \tag{4.4}$$

The other case $\boldsymbol{\mu}^+ - \boldsymbol{\mu}^- \le \frac12\boldsymbol{\omega}$ will be considered later. Observe that (4.4) implies

$$\boldsymbol{\mu}^+ - \tfrac18\boldsymbol{\omega} \ge \tfrac18\boldsymbol{\omega} \tag{4.5a}$$

$$\text{or}\quad \boldsymbol{\mu}^- + \tfrac18\boldsymbol{\omega} \le -\tfrac18\boldsymbol{\omega}. \tag{4.5b}$$

Let us consider for instance the first case, i.e., (4.5a), as the other one can be treated analogously. Hence we have $\frac14\boldsymbol{\omega}\le\boldsymbol{\mu}^+\le 2\boldsymbol{\omega}$ and Lemma 3.3 is at our disposal with $c=1/4$ and $\Lambda=2$.

Suppose A is a large number, and consider the "bottom" sub-cylinder of $Q_\varrho(A\theta)$, that is,

$$\widetilde{Q} := K_\varrho \times (-A\theta\varrho^p, -(A-1)\theta\varrho^p].$$

One of the following two alternatives must hold true:

$$\big|\{u\le\boldsymbol{\mu}^- + \tfrac14\boldsymbol{\omega}\}\cap\widetilde{Q}\big| \le \nu|\widetilde{Q}|, \tag{4.6a}$$

$$\big|\{u\le\boldsymbol{\mu}^- + \tfrac14\boldsymbol{\omega}\}\cap\widetilde{Q}\big| > \nu|\widetilde{Q}|. \tag{4.6b}$$

Here the number $\nu\in(0,1)$ is determined in Lemma 3.1 in terms of the data.

First suppose (4.6a) holds true. An application of Lemma 3.1 (with $\xi=1/4$) gives us that, recalling $|\boldsymbol{\mu}^-|\le 2\boldsymbol{\omega}$ due to (4.3a),

$$u \ge \boldsymbol{\mu}^- + \tfrac18\boldsymbol{\omega}\quad \text{a.e. in } \tfrac12\widetilde{Q}. \tag{4.7}$$

Here the notation $\frac12\widetilde{Q}$ should be self-explanatory in view of Lemma 3.1. In particular, the above pointwise lower bound of u holds at the time level $t_o=-(A-1)\theta\varrho^p$ for a.e. $x\in K_{\varrho/2}$, which serves as the initial datum for an application of Lemma 3.2. Indeed, we fix ν_o in Lemma 3.2 depending on the data and choose $\xi\in(0,1/8)$ so small that

$$0\le t_o + \nu_o(\xi\boldsymbol{\omega})^{q+1-p}(\tfrac12\varrho)^p = -(A-1)(\tfrac14\boldsymbol{\omega})^{q+1-p}\varrho^p + \nu_o(\xi\boldsymbol{\omega})^{q+1-p}(\tfrac12\varrho)^p,$$

i.e., we choose

$$\xi = \min\Big\{\frac18,\ \frac14\Big(\frac{\nu_o}{2^pA}\Big)^{1/(p-q-1)}\Big\}. \tag{4.8}$$

Thus, enforcing $|\boldsymbol{\mu}^-| \le \xi\boldsymbol{\omega}$, we obtain that

$$u \ge \boldsymbol{\mu}^- + \tfrac{1}{2}\xi\boldsymbol{\omega} \quad \text{a.e. in } K_{\varrho/4} \times (t_o, 0],$$

which in turn yields a reduction of oscillation

$$\operatorname{ess\,osc}_{Q_{\varrho/4}(\theta)} u \le (1 - \tfrac{1}{2}\xi)\boldsymbol{\omega}. \tag{4.9}$$

Here in (4.8) we have tacitly used the fact that $q < p - 1$ in the determination of ξ. Keep also in mind that $A > 1$ is yet to be determined in terms of the data.

The case $\boldsymbol{\mu}^- > \xi\boldsymbol{\omega}$ will be treated in Section 4.5; whereas if $-2\boldsymbol{\omega} < \boldsymbol{\mu}^- < -\xi\boldsymbol{\omega}$, we may apply Lemma 3.5 with $c = \xi$, $\Lambda = 2$ and $\eta = \eta_o \in (0, 1/8)$. Indeed, fixing ν_1 in Lemma 3.5 depending on the data and ξ, we choose η_o to satisfy

$$\nu_1 \boldsymbol{\omega}^{q-1}(\eta_o\boldsymbol{\omega})^{2-p}(\tfrac{1}{2}\varrho)^p \ge A(\tfrac{1}{4}\boldsymbol{\omega})^{q+1-p}\varrho^p,$$

i.e.,

$$\eta_o = \min\left\{\frac{1}{8},\, 4^{\frac{p-q-1}{p-2}}\left(\frac{\nu_1}{2^p A}\right)^{\frac{1}{p-2}}\right\}.$$

Here we have tacitly used the fact that $p > 2$ in the determination of η_o. In this way, Lemma 3.5 asserts that

$$u \ge \boldsymbol{\mu}^- + \tfrac{1}{2}\eta_o\boldsymbol{\omega} \quad \text{a.e. in } K_{\varrho/4} \times (t_o, 0],$$

which yields the reduction of oscillation

$$\operatorname{ess\,osc}_{Q_{\varrho/4}(\theta)} u \le (1 - \tfrac{1}{2}\eta_o)\boldsymbol{\omega}. \tag{4.10}$$

4.3. Reduction of oscillation near zero – Part II

In this section, we still assume that (4.3a) and (4.4) hold true. However, we turn our attention to the second alternative (4.6b). We work with u as a sub-solution near its supremum. Since under our assumptions we have $\boldsymbol{\mu}^+ - \frac{1}{4}\boldsymbol{\omega} \ge \boldsymbol{\mu}^- + \frac{1}{4}\boldsymbol{\omega}$, we may rephrase (4.6b) as

$$|\{\boldsymbol{\mu}^+ - u \ge \tfrac{1}{4}\boldsymbol{\omega}\} \cap \widetilde{Q}| > \nu|\widetilde{Q}|.$$

Then it is not hard to see that there exists

$$t_* \in [-A\theta\varrho^p, -(A-1)\theta\varrho^p - \tfrac{1}{2}\nu\theta\varrho^p]$$

such that

$$|\{\boldsymbol{\mu}^+ - u(\cdot, t_*) \ge \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho| > \tfrac{1}{2}\nu|K_\varrho|.$$

Indeed, if the above inequality does not hold for any s in the given interval, then

$$\begin{aligned}
|\{\boldsymbol{\mu}^+ - u \ge \tfrac{1}{4}\boldsymbol{\omega}\} \cap \widetilde{Q}| &= \int_{-A\theta\varrho^p}^{-(A-1)\theta\varrho^p - \frac{1}{2}\nu\theta\varrho^p} |\{\boldsymbol{\mu}^+ - u(\cdot, s) \ge \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho|\, \mathrm{d}s \\
&\quad + \int_{-(A-1)\theta\varrho^p - \frac{1}{2}\nu\theta\varrho^p}^{-(A-1)\theta\varrho^p} |\{\boldsymbol{\mu}^+ - u(\cdot, s) \ge \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho|\, \mathrm{d}s \\
&< \tfrac{1}{2}\nu|K_\varrho|\theta\varrho^p(1 - \tfrac{1}{2}\nu) + \tfrac{1}{2}\nu\theta\varrho^p|K_\varrho| < \nu|\widetilde{Q}|,
\end{aligned}$$

implying a contradiction to the above measure theoretical information. Recall that due to (4.5a), we actually have $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$. Then, based on the above measure theoretical information at t_*, an application of Lemma 3.3 with $c = 1/4$, $\Lambda = 2$ and a fixed constant $a = 1/8 = \frac{1}{2}c$ yields constants $b > 0$ and $\eta_1 \in (0, 1)$ depending only on the data such that

$$\boldsymbol{\mu}^+ - u(\cdot, t) \ge \eta_1 \boldsymbol{\omega} \quad \text{a.e. in } K_{\varrho/2},$$

for all times

$$t_* + \tfrac{1}{2} b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p \le t \le t_* + b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p.$$

Now, we determine A so that the set inclusion

$$(-\theta(\tfrac{1}{4}\varrho)^p, 0] \subset \big[t_* + \tfrac{1}{2} b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p,\, t_* + b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p\big]$$

is satisfied. To this end, we first consider the requirement $0 \le t_* + b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p$, which follows if the stronger condition

$$0 \le -A(\tfrac{1}{4}\boldsymbol{\omega})^{q+1-p} \varrho^p + b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p$$

is fulfilled. This leads to the choice

$$A = b\, 4^{q+1-p}\, \eta_1^{2-p}.$$

Note that we may assume $A > 1$, since we could choose a smaller constant η_1 in the definition of A by Remark 3.4 and use the fact that $p > 2$. The second requirement $-\theta(\frac{1}{4}\varrho)^p \ge t_* + \frac{1}{2} b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p$ is satisfied if we are able to verify the stronger condition

$$\begin{aligned} -(A-1)\theta\varrho^p - \tfrac{1}{2}\nu\theta\varrho^p + \tfrac{1}{2} b\, \boldsymbol{\omega}^{q-1} (\eta_1 \boldsymbol{\omega})^{2-p} \varrho^p &= -(A-1)\theta\varrho^p - \tfrac{1}{2}\nu\theta\varrho^p + \tfrac{1}{2} A\theta\varrho^p \\ &\le -\theta(\tfrac{1}{4}\varrho)^p, \end{aligned}$$

which is equivalent to

$$1 - \tfrac{1}{2}\nu + \tfrac{1}{4^p} \le \tfrac{1}{2} A.$$

Since $\nu \in (0, 1)$, the last inequality holds true if $A \ge 4$. However, as mentioned above, we may assume it by making η_1 smaller. Altogether, the above analysis determines A through η_1 and yields a reduction of oscillation

$$\operatorname{ess\,osc}_{Q_{\varrho/4}(\theta)} u \le (1 - \eta_1)\boldsymbol{\omega}. \tag{4.11}$$

To summarize, let us define

$$\eta = \min\{\tfrac{1}{2}\xi,\, \tfrac{1}{2}\eta_o,\, \eta_1\} \in (0, \tfrac{1}{2}),$$

where $\frac{1}{2}\xi$ is as in (4.9), $\frac{1}{2}\eta_o$ is as in (4.10) and η_1 is as in (4.11). Combining (4.9)–(4.11) gives the reduction of oscillation

$$\operatorname{ess\,osc}_{Q_{\varrho/4}(\theta)} u \le (1 - \eta)\boldsymbol{\omega}, \tag{4.12}$$

provided the intrinsic relation (4.1) is verified and under (4.3a) and (4.4).

In order to iterate the above argument, we introduce

$$\omega_1 = \max\{(1-\eta)\omega, L\varrho^{1/(p-q-1)}\};$$

we need to choose $\varrho_1 = \lambda\varrho$, for some $\lambda \in (0,1)$, such that

$$Q_{\varrho_1}(A\theta_1) \subset Q_{\varrho/4}(\theta) \cap Q_o, \quad \text{where } \theta_1 = (\tfrac{1}{4}\omega_1)^{q+1-p}.$$

To this end, we first let

$$\lambda = \tfrac{1}{4}A^{-1/p}(1-\eta)^{(p-q-1)/p}$$

and estimate

$$A\theta_1\varrho_1^p = A(\tfrac{1}{4}\omega_1)^{q+1-p}(\lambda\varrho)^p \le (\tfrac{1}{4}\omega)^{q+1-p}(\tfrac{1}{4}\varrho)^p = \theta(\tfrac{1}{4}\varrho)^p.$$

Consequently, the first set inclusion $Q_{\varrho_1}(A\theta_1) \subset Q_{\varrho/4}(\theta)$ holds. Note that $\lambda < 1/4$. The second set inclusion $Q_{\varrho_1}(A\theta_1) \subset Q_o$ is verified similarly with the same choice of λ. Therefore, taking into account (4.2), (4.12) and the violation of (4.4), i.e., the case where $\operatorname{ess\,osc}_{Q_o} u = \mu^+ - \mu^- \le \frac{1}{2}\omega$, we arrive at the intrinsic relation

$$\operatorname{ess\,osc}_{Q_{\varrho_1}(A\theta_1)} u \le \omega_1,$$

which takes the place of (4.1) in the next stage.

4.4. Reduction of oscillation near zero concluded

Now we may proceed by induction. Suppose that, up to $i = 1, 2, \dots, j-1$, we have built

$$\begin{gathered}\varrho_o = \varrho, \quad \varrho_i = \lambda\varrho_{i-1}, \quad \theta_i = (\tfrac{1}{4}\omega_i)^{q+1-p}, \\ \omega_o = \omega, \quad \omega_i = \max\bigl\{(1-\eta)\omega_{i-1}, L\varrho_{i-1}^{1/(p-q-1)}\bigr\}, \\ Q_i = Q_{\varrho_i}(\theta_i), \quad Q_i' = Q_{\varrho_i/4}(\theta_i), \\ \mu_i^+ = \operatorname{ess\,sup}_{Q_i} u, \quad \mu_i^- = \operatorname{ess\,inf}_{Q_i} u, \quad \operatorname{ess\,osc}_{Q_i} u \le \omega_i.\end{gathered}$$

For all the indices $i = 1, 2, \dots, j-1$, we always assume that (4.3a) holds true, i.e.,

$$\mu_i^- \le \xi\omega_i \quad \text{and} \quad \mu_i^+ \ge -\xi\omega_i.$$

This means that the previous arguments can be repeated and we have, for all $i = 1, 2, \dots, j$,

$$Q_{\varrho_i}(A\theta_i) \subset Q_{i-1}', \quad \operatorname{ess\,osc}_{Q_i} u \le (1-\eta)\omega_{i-1} \le \omega_i.$$

Consequently, iterating the above recursive inequality, we obtain for all $i = 1, 2, \dots, j$,

$$\begin{aligned}\operatorname{ess\,osc}_{Q_i} u \le \omega_i &\le \max\{(1-\eta)^i\omega, L\varrho^{1/(p-q-1)}\} \\ &= \max\Bigl\{\omega\Bigl(\frac{\varrho_i}{\varrho}\Bigr)^{\beta_o}, L\varrho^{1/(p-q-1)}\Bigr\},\end{aligned} \tag{4.13}$$

where

$$\beta_o = \frac{\ln(1-\eta)}{\ln\lambda}.$$

4.5. Reduction of oscillation away from zero

In this section, let us suppose j is the first index satisfying the second case in (4.3), i.e., either

$$\boldsymbol{\mu}_j^- > \xi\boldsymbol{\omega}_j \quad \text{or} \quad \boldsymbol{\mu}_j^+ < -\xi\boldsymbol{\omega}_j.$$

Let us treat, for instance, $\boldsymbol{\mu}_j^- > \xi\boldsymbol{\omega}_j$; the other case is analogous. We observe that since j is the first index for this to happen, one should have $\boldsymbol{\mu}_{j-1}^+ \le \boldsymbol{\mu}_{j-1}^- + \boldsymbol{\omega}_{j-1} \le (1+\xi)\boldsymbol{\omega}_{j-1}$. Here, we assume that there exists an index $j-1$ such that the first case in (4.3) is fulfilled. This can be justified by choosing $\boldsymbol{\omega} = \frac{1}{\xi}\|u\|_{L^\infty(E_T)}$ in Section 4.1. In addition, since $Q_j \subset Q_{j-1}$, by the definition of the essential supremum, one estimates

$$\boldsymbol{\mu}_j^- \le \boldsymbol{\mu}_j^+ \le \boldsymbol{\mu}_{j-1}^+ \le (1+\xi)\boldsymbol{\omega}_{j-1} \le \frac{1+\xi}{1-\eta}\boldsymbol{\omega}_j.$$

As a result, we have

$$\xi\boldsymbol{\omega}_j \le \boldsymbol{\mu}_j^- \le \frac{1+\xi}{1-\eta}\boldsymbol{\omega}_j. \tag{4.14}$$

The bound (4.14) indicates that starting from j, the equation (1.2) resembles the parabolic p-Laplacian type equation in Q_j. We drop the suffix j from our notation for simplicity, and introduce $v := u/\boldsymbol{\mu}^-$ in $Q = K_\varrho \times (-\theta\varrho^p, 0]$, where $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$. It is straightforward to verify that v belongs to the function space (1.6) defined on Q and satisfies

$$\partial_t v^q - \operatorname{div} \bar{\mathbf{A}}(x,t,v,Dv) = 0 \quad \text{weakly in } Q,$$

where, for $(x,t) \in Q$, $v \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$, we have defined

$$\bar{\mathbf{A}}(x,t,v,\zeta) = \mathbf{A}(x,t,\boldsymbol{\mu}^- v, \boldsymbol{\mu}^-\zeta)/(\boldsymbol{\mu}^-)^q,$$

which is subject to the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x,t,v,\zeta)\cdot\zeta \ge C_o(\boldsymbol{\mu}^-)^{p-q-1}|\zeta|^p, \\ |\bar{\mathbf{A}}(x,t,v,\zeta)| \le C_1(\boldsymbol{\mu}^-)^{p-q-1}|\zeta|^{p-1}, \end{cases} \quad \text{for a.e. } (x,t) \in Q, \text{ all } v \in \mathbb{R} \text{ and } \zeta \in \mathbb{R}^N.$$

Moreover, since $\boldsymbol{\omega}/\boldsymbol{\mu}^- \le 1/\xi$, we have that

$$1 \le v \le \frac{\boldsymbol{\mu}^+}{\boldsymbol{\mu}^-} \le \frac{\boldsymbol{\mu}^- + \boldsymbol{\omega}}{\boldsymbol{\mu}^-} \le \frac{1+\xi}{\xi} \quad \text{a.e. in } Q. \tag{4.15}$$

To proceed, it turns out to be more convenient to consider $w := v^q$, which because of (4.15) belongs to the function space (1.6), with $q = 1$, defined on Q and satisfies

$$\partial_t w - \operatorname{div} \widetilde{\mathbf{A}}(x,t,w,Dw) = 0 \quad \text{weakly in } Q,$$

where we have defined the vector-field $\widetilde{\mathbf{A}}$ by

$$\widetilde{\mathbf{A}}(x,t,y,\zeta) = \bar{\mathbf{A}}\big(x,t,\tilde{y}^{1/q}, \tfrac{1}{q}\tilde{y}^{(1-q)/q}\zeta\big),$$

for a.e. $(x,t) \in Q$, any $y \in \mathbb{R}$ and any $\zeta \in \mathbb{R}^N$. This time $\widetilde{y}$ is defined by

$$\widetilde{y} := \min\Big\{ \max\Big\{y, \frac{1}{2}\Big\}, 2\Big(\frac{1+\xi}{\xi}\Big)^q \Big\}.$$

Using (4.15) again, we verify that there exist positive constants $\widetilde{C}_o = \boldsymbol{\gamma}_o(p,q,\xi)C_o$ and $\widetilde{C}_1 = \boldsymbol{\gamma}_1(p,q,\xi)C_1$ such that

$$\begin{cases} \widetilde{\mathbf{A}}(x,t,y,\zeta)\cdot\zeta \ge \widetilde{C}_o(\boldsymbol{\mu}^-)^{p-q-1}|\zeta|^p, \\ |\widetilde{\mathbf{A}}(x,t,y,\zeta)| \le \widetilde{C}_1(\boldsymbol{\mu}^-)^{p-q-1}|\zeta|^{p-1}, \end{cases} \quad \text{for a.e. } (x,t) \in Q, \text{ all } y \in \mathbb{R} \text{ and all } \zeta \in \mathbb{R}^N.$$

Note that ξ is already fixed in (4.8) in terms of the data. To proceed, we introduce the function

$$\widehat{w}(x,t) := w(x, (\boldsymbol{\mu}^-)^{q+1-p}t),$$

which satisfies

$$\partial_t \widehat{w} - \operatorname{div} \widehat{\mathbf{A}}(x,t,\widehat{w}, D\widehat{w}) = 0 \quad \text{weakly in } \widehat{Q} := K_\varrho \times (-(\boldsymbol{\mu}^-)^{p-q-1}\theta\varrho^p, 0] \tag{4.16}$$

and belongs to the function space (1.6), with $q = 1$, defined on $\widehat{Q}$. Here the function $\widehat{\mathbf{A}}$ is defined by

$$\widehat{\mathbf{A}}(x,t,y,\zeta) := (\boldsymbol{\mu}^-)^{q+1-p}\widetilde{\mathbf{A}}(x,(\boldsymbol{\mu}^-)^{q+1-p}t, y, \zeta),$$

and it is subject to the structure conditions

$$\begin{cases} \widehat{\mathbf{A}}(x,t,y,\zeta)\cdot\zeta \ge \widetilde{C}_o|\zeta|^p, \\ |\widehat{\mathbf{A}}(x,t,y,\zeta)| \le \widetilde{C}_1|\zeta|^{p-1}, \end{cases} \quad \text{for a.e. } (x,t) \in \widehat{Q}, \text{ all } y \in \mathbb{R} \text{ and all } \zeta \in \mathbb{R}^N. \tag{4.17}$$

This shows that $\widehat{w}$ is a local weak solution to the parabolic p-Laplacian type equation in $\widehat{Q}$.

First proved in [3], the power-like oscillation decay for solutions to this kind of degenerate parabolic equation is well known by now. We state the conclusion in the following proposition in a form that favors our application, and refer to the monographs [4, 15] for a comprehensive treatment of this issue.

Proposition 4.1. *Let $p > 2$, $\sigma \in (0,1)$ and $\widehat{\boldsymbol{\omega}} > 0$. Then there exist constants $\beta_1 \in (0,1)$ and $\boldsymbol{\gamma} > 1$, depending only on the data $N, p, \widetilde{C}_o, \widetilde{C}_1$ and σ, such that if $\widehat{w}$ is a bounded, local, weak solution to (4.16)–(4.17) in $\widehat{Q}$ and, for $\widehat{\theta} = \widehat{\boldsymbol{\omega}}^{2-p}$, the assumptions*

$$\operatorname*{ess\,osc}_{Q_{\sigma\varrho}(\widehat{\theta})} \widehat{w} \le \widehat{\boldsymbol{\omega}} \quad \textit{and} \quad Q_{\sigma\varrho}(\widehat{\theta}) \subset \widehat{Q} \tag{4.18}$$

hold true, then for all $0 < r \le \varrho$, we have

$$\operatorname*{ess\,osc}_{Q_r(\widehat{\theta})} \widehat{w} \le \boldsymbol{\gamma}\widehat{\boldsymbol{\omega}}\Big(\frac{r}{\varrho}\Big)^{\beta_1}.$$

We tend to use Proposition 4.1. First we check that condition (4.18) is satisfied. Indeed, by the mean value theorem and (4.15), there exists some positive $\widetilde{\boldsymbol{\gamma}} = \widetilde{\boldsymbol{\gamma}}(q,\xi)$ such that

$$\operatorname*{ess\,osc}_{\widehat{Q}} \widehat{w} = \operatorname*{ess\,osc}_{Q} w \le \widetilde{\boldsymbol{\gamma}} \operatorname*{ess\,osc}_{Q} v \le \widetilde{\boldsymbol{\gamma}}\,\frac{\boldsymbol{\omega}}{\boldsymbol{\mu}^-} =: \widehat{\boldsymbol{\omega}}.$$

According to (4.14), we find that

$$\frac{1-\eta}{1+\xi} \le \frac{\boldsymbol{\omega}}{\boldsymbol{\mu}^-} \le \frac{1}{\xi}.$$

Further, by the definition of the corresponding cylinders, we obtain that $Q_{\sigma\varrho}(\widehat{\theta}) \subset \widehat{Q}$, provided

$$\Big(\widetilde{\boldsymbol{\gamma}}\,\frac{\boldsymbol{\omega}}{\boldsymbol{\mu}^-}\Big)^{2-p}(\sigma\varrho)^p \le (\boldsymbol{\mu}^-)^{p-q-1}\Big(\frac{1}{4}\,\boldsymbol{\omega}\Big)^{q+1-p}\varrho^p$$

holds true. This can be achieved by choosing σ small enough, i.e.,

$$\sigma \le \Big(\frac{1}{4}\Big)^{q+1-p}\widetilde{\boldsymbol{\gamma}}^{\,p-2}\Big(\frac{\boldsymbol{\omega}}{\boldsymbol{\mu}^-}\Big)^{q-1}.$$

In view of the lower and upper bound on the ratio $\boldsymbol{\omega}/\boldsymbol{\mu}^-$, the number σ can be chosen only in terms of the data, so that

$$\operatorname{ess\,osc}_{Q_{\sigma\varrho}(\widehat{\theta})} \widehat{w} \le \widehat{\boldsymbol{\omega}},$$

i.e., the condition (4.18) is fulfilled. Consequently, by Proposition 4.1, we have

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta})} \widehat{w} \le \boldsymbol{\gamma}\widehat{\boldsymbol{\omega}}\Big(\frac{r}{\varrho}\Big)^{\beta_1} \le \bar{\boldsymbol{\gamma}}\Big(\frac{r}{\varrho}\Big)^{\beta_1}$$

for $\bar{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\widetilde{\boldsymbol{\gamma}}/\xi$ and for any $0 < r \le \varrho$, with some $\beta_1 \in (0,1)$ depending only on the data. Since $p > 2$, we may estimate

$$\widehat{\theta} > \widehat{\theta}_o := \Big(\frac{\widetilde{\boldsymbol{\gamma}}}{\xi}\Big)^{2-p}$$

and conclude that

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_o)} \widehat{w} \le \bar{\boldsymbol{\gamma}}\Big(\frac{r}{\varrho}\Big)^{\beta_1} \quad \text{for all } 0 < r \le \varrho.$$

Reverting to w and using the fact that $q+1 < p$ and (4.14) in order to estimate

$$(\boldsymbol{\mu}^-)^{q+1-p} \ge \Big(\frac{1+\xi}{1-\eta}\,\boldsymbol{\omega}\Big)^{q+1-p},$$

we obtain that

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}^{q+1-p})} w \le \bar{\boldsymbol{\gamma}}\Big(\frac{r}{\varrho}\Big)^{\beta_1},$$

where

$$\widehat{\theta}_1 := \Big(\frac{1+\xi}{1-\eta}\Big)^{q+1-p}\widehat{\theta}_o$$

depends only on the data. Recalling the definition of w, by the mean value theorem and (4.15), one easily estimates that for some positive $\widetilde{\boldsymbol{\gamma}} = \widetilde{\boldsymbol{\gamma}}(q,\xi)$,

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}^{q+1-p})} v \le \widetilde{\boldsymbol{\gamma}}(q,\xi)\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}^{q+1-p})} w.$$

Finally, we revert to u and the suffix j, and use (4.14) to estimate $\boldsymbol{\mu}_j^- \le \frac{1+\xi}{1-\eta}\boldsymbol{\omega}_j$, which leads to

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}_j^{q+1-p})} u \le \boldsymbol{\mu}_j^- \operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}_j^{q+1-p})} v \le \boldsymbol{\gamma}\boldsymbol{\omega}_j\Big(\frac{r}{\varrho_j}\Big)^{\beta_1}, \tag{4.19}$$

whenever $0 < r < \varrho_j$. Since $\varrho \le 1$, we have that $\boldsymbol{\omega}_j \le \boldsymbol{\omega}_1 \le \max\{\boldsymbol{\omega}, L\} =: \boldsymbol{\omega}_L$, and therefore we obtain that $Q_r(\widehat{\theta}_1\boldsymbol{\omega}_L^{q+1-p}) \subset Q_r(\widehat{\theta}_1\boldsymbol{\omega}_j^{q+1-p})$. Combining this with (4.13) and (4.19), we arrive at the following, for all $0 < r < \varrho$:

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}_L^{q+1-p})} u \le \boldsymbol{\gamma}\boldsymbol{\omega}\Big(\frac{r}{\varrho}\Big)^{\beta_2} + \boldsymbol{\gamma} L\varrho^{1/(p-q-1)}, \quad \text{where } \beta_2 = \min\{\beta_o, \beta_1\}.$$

Without loss of generality, we may assume the above oscillation estimate holds with ϱ replaced by some $\widetilde{\varrho} \in (r, \varrho)$. Then taking $\widetilde{\varrho} = (r\varrho)^{1/2}$ and properly adjusting the Hölder exponent, we obtain the power-like decay of oscillation

$$\operatorname{ess\,osc}_{Q_r(\widehat{\theta}_1\boldsymbol{\omega}_L^{q+1-p})} u \le \boldsymbol{\gamma}\boldsymbol{\omega}\Big(\frac{r}{\varrho}\Big)^{\beta_2/2} + \boldsymbol{\gamma} L\varrho^{1/(p-q-1)}\Big(\frac{r}{\varrho}\Big)^{1/[2(p-q-1)]} \le \boldsymbol{\gamma}\boldsymbol{\omega}_L\Big(\frac{r}{\varrho}\Big)^{\beta},$$

where

$$\beta = \min\Big\{\frac{\beta_2}{2}, \frac{1}{2(p-q-1)}\Big\}.$$

At this stage, the proof of Theorem 1.1 can be completed by a standard covering argument.

5. The second proof of Theorem 1.1

The purpose of this section is to present another proof of Theorem 1.1 without using the expansion of positivity (Lemma 3.3). As we shall see, the arguments in Section 5.2 are similar to that of Section 4.2. The main difference appears in Section 5.3. To avoid using Lemma 3.3 as done in Section 4.3, we perform an argument of DiBenedetto [3], adapted in [1]. The real advantage of this section is that the proof relies solely on the energy estimates in Proposition 2.1. As such, it offers an amenable adaptation near the boundary given Neumann data, cf. Section 6.3.

5.1. The proof begins

The set-up is the same as in Section 4.1. Namely, we introduce the quantities $\boldsymbol{\mu}^{\pm}$, $\boldsymbol{\omega}$, θ, L, A and the cylinders $Q_\varrho(A\theta) \subset Q_o$. Moreover, they are connected by the intrinsic relation (4.1). For a positive ξ to be determined, the proof unfolds along two main cases, as in (4.3).

5.2. Reduction of oscillation near zero – Part I

Like in Section 4.2, we assume that (4.3a) holds and work with u as a super-solution near its infimum. Then we proceed with the assumption (4.4), which implies one of (4.5) holds. We may take (4.5a) such that $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$.

The second proof begins here. Suppose that for some $\bar{t} \in (-(A-1)\theta\varrho^p, 0]$,

$$\left|\{u \le \boldsymbol{\mu}^- + \tfrac{1}{4}\boldsymbol{\omega}\} \cap (0,\bar{t}) + Q_\varrho(\theta)\right| \le \nu |Q_\varrho(\theta)|, \tag{5.1}$$

where ν is the constant determined in Lemma 3.1 in terms of the data. According to Lemma 3.1, applied with $\xi = 1/4$, we have

$$u \ge \boldsymbol{\mu}^- + \tfrac{1}{8}\boldsymbol{\omega} \quad \text{a.e. in } (0,\bar{t}) + Q_{\varrho/2}(\theta),$$

since the other alternative, i.e., $|\boldsymbol{\mu}^-| \ge 2\boldsymbol{\omega}$, does not hold due to (4.3a). This pointwise information parallels (4.7) in Section 4.2. Similar arguments can be reproduced as in Section 4.2 to obtain the reduction of oscillation as in (4.9)–(4.10). In particular, only Lemma 3.1, Lemma 3.2 and Lemma 3.5 are used. In this process we fix the constant ξ as in (4.8) depending on the data and A, which will be chosen next in terms of the data.

5.3. Reduction of oscillation near zero – Part II

In this section we still assume that (4.3a) holds. However, now we work with u as a sub-solution near its supremum. Keep also in mind that (4.5a) is enforced so that $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$ may be assumed.

Suppose contrary to (5.1) that, recalling $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$,

$$\left|\{u \le \boldsymbol{\mu}^- + \tfrac{1}{4}\boldsymbol{\omega}\} \cap (0,\bar{t}) + Q_\varrho(\theta)\right| > \nu |Q_\varrho(\theta)| \quad \text{for all } \bar{t} \in (-(A-1)\theta\varrho^p, 0].$$

Then for any such $\bar{t}$, it is easy to see that there exists some $s \in [\bar{t} - \theta\varrho^p, \bar{t} - \frac{1}{2}\nu\theta\varrho^p]$ with

$$|\{u(\cdot,s) \le \boldsymbol{\mu}^- + \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho| > \tfrac{1}{2}\nu |K_\varrho|.$$

Since we assumed that $\boldsymbol{\mu}^+ - \boldsymbol{\mu}^- > \frac{1}{2}\boldsymbol{\omega}$, we have $\boldsymbol{\mu}^+ - \frac{1}{4}\boldsymbol{\omega} > \boldsymbol{\mu}^- + \frac{1}{4}\boldsymbol{\omega}$, which implies

$$|\{u(\cdot,s) \le \boldsymbol{\mu}^+ - \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho| \ge \tfrac{1}{2}\nu |K_\varrho|.$$

Recall that due to (4.5a), we have $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$. Thus our assumptions for the following Sections 5.3.1–5.3.3 are

(5.2) $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$,

(5.3) for any $\bar{t} \in (-(A-1)\theta\varrho^p, 0]$, there exists $s \in [\bar{t} - \theta\varrho^p, \bar{t} - \frac{1}{2}\nu\theta\varrho^p]$ such that

$$|\{u(\cdot,s) \le \boldsymbol{\mu}^+ - \tfrac{1}{4}\boldsymbol{\omega}\} \cap K_\varrho| \ge \tfrac{1}{2}\nu |K_\varrho|.$$

They would allow us to determine A and reduce the oscillation in this case. Similar arguments in Sections 5.3.1–5.3.3 have been carried out in [1]. However we think it is necessary to adapt them in the new setting because of the technical nature.

5.3.1. Propagation of measure theoretical information.

Lemma 5.1. *Suppose (5.2) and (5.3) are in force. There exists $\varepsilon \in (0,1)$, depending only on ν and the data, such that*

$$|\{u(\cdot,t) \le \boldsymbol{\mu}^+ - \varepsilon\boldsymbol{\omega}\} \cap K_\varrho| \ge \tfrac{1}{4}\nu |K_\varrho| \quad \textit{for all } t \in (s,\bar{t}].$$

Proof. For ease of notation, we set $s=0$. Further, for $\delta>0$ and $0<\varepsilon\le\frac{1}{8}$ to be determined by the data and ν, we consider $Q:=K_\varrho\times(0,\delta\varepsilon^{2-p}\theta\varrho^p]$ and $k=\boldsymbol{\mu}^+-\varepsilon\boldsymbol{\omega}\ge\frac{1}{8}\boldsymbol{\omega}$. Applying the energy estimate in Proposition 2.1 with a standard non-negative, time independent cutoff function $\zeta(x,t)\equiv\zeta(x)$ that equals 1 on $K_{(1-\sigma)\varrho}$ for some $\sigma\in(0,1)$ to be fixed later, vanishes on ∂K_ϱ and satisfies $|D\zeta|\le(\sigma\varrho)^{-1}$, we obtain, for all $0<t<\delta\varepsilon^{2-p}\theta\varrho^p$, that

$$\begin{aligned}&\int_{K_\varrho\times\{t\}}\int_k^u \tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau\,\zeta^p\,\mathrm{d}x\\&\qquad\le\int_{K_\varrho\times\{0\}}\int_k^u \tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau\,\zeta^p\,\mathrm{d}x+\boldsymbol{\gamma}\iint_Q(u-k)_+^p|D\zeta|^p\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Defining $k_{\tilde\varepsilon}=\boldsymbol{\mu}^+-\tilde\varepsilon\varepsilon\boldsymbol{\omega}$ for some $\tilde\varepsilon\in(0,\frac{1}{2})$, we estimate the term on the left-hand side by

$$\int_{K_\varrho\times\{t\}}\int_k^u \tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau\,\zeta^p\,\mathrm{d}x\ge|\{u(\cdot,t)>k_{\tilde\varepsilon}\}\cap K_{(1-\sigma)\varrho}|\int_k^{k_{\tilde\varepsilon}}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau.$$

Further, note that by the mean value theorem and the restriction $\frac{1}{4}\boldsymbol{\omega}\le\boldsymbol{\mu}^+\le2\boldsymbol{\omega}$, there exists a constant $\boldsymbol{\gamma}=\boldsymbol{\gamma}(q)$ such that

$$\int_k^{k_{\tilde\varepsilon}}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau\ge\boldsymbol{\gamma}\boldsymbol{\omega}^{q-1}(\varepsilon\boldsymbol{\omega})^2=\boldsymbol{\gamma}\varepsilon^2\boldsymbol{\omega}^{q+1}.$$

Next, by (5.3), we obtain for the first term on the right-hand side of the energy estimate that

$$\int_{K_\varrho\times\{0\}}\int_k^u \tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau\,\zeta^p\,\mathrm{d}x\le(1-\tfrac{1}{2}\nu)|K_\varrho|\int_k^{\boldsymbol{\mu}^+}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau$$

and, by the choice of ζ and $u\le\boldsymbol{\mu}^+$ for the second term on the right-hand side, that

$$\iint_Q(u-k)_+^p|D\zeta|^p\,\mathrm{d}x\,\mathrm{d}t\le\frac{\boldsymbol{\gamma}\delta}{\sigma^p}\,\varepsilon^{2-p}\theta(\varepsilon\boldsymbol{\omega})^p|K_\varrho|\le\frac{\boldsymbol{\gamma}\delta}{\sigma^p}\,\varepsilon^2\boldsymbol{\omega}^{q+1}|K_\varrho|.$$

Combining the preceding estimates leads to

$$|\{u(\cdot,t)>k_{\tilde\varepsilon}\}\cap K_{(1-\sigma)\varrho}|\le\frac{\int_k^{\boldsymbol{\mu}^+}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}{\int_k^{k_{\tilde\varepsilon}}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}(1-\tfrac{1}{2}\nu)|K_\varrho|+\frac{\boldsymbol{\gamma}\delta}{\sigma^p}|K_\varrho|.$$

Rewriting the fractional number of integrals on the right-hand side and using the mean value theorem as well as the restrictions $\frac{1}{4}\boldsymbol{\omega}\le\boldsymbol{\mu}^+\le2\boldsymbol{\omega}$ and $k\ge\frac{1}{8}\boldsymbol{\omega}$ yields the bound

$$\frac{\int_k^{\boldsymbol{\mu}^+}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}{\int_k^{k_{\tilde\varepsilon}}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}=1+\frac{\int_{k_{\tilde\varepsilon}}^{\boldsymbol{\mu}^+}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}{\int_k^{k_{\tilde\varepsilon}}\tau^{q-1}(\tau-k)_+\,\mathrm{d}\tau}\le1+\boldsymbol{\gamma}\tilde\varepsilon,$$

where $\boldsymbol{\gamma}$ depends only on q. Inserting this into the previous inequality, we conclude that

$$|\{u(\cdot,t)>k_{\tilde\varepsilon}\}\cap K_\varrho|\le(1-\tfrac{1}{2}\nu)(1+\boldsymbol{\gamma}\tilde\varepsilon)|K_\varrho|+\frac{\boldsymbol{\gamma}\delta}{\sigma^p}|K_\varrho|+N\sigma|K_\varrho|.$$

Now, we first fix $\tilde{\varepsilon} = \tilde{\varepsilon}(q, \nu)$ small enough that

$$(1 - \tfrac{1}{2}\nu)(1 + \boldsymbol{\gamma}\tilde{\varepsilon}) \le 1 - \tfrac{3}{8}\nu$$

and define $\sigma := \frac{\nu}{16N}$. Then, we choose δ small enough that $\frac{\boldsymbol{\gamma}\delta}{\sigma^p} \le \frac{1}{16}\nu$ and ε small enough that $\delta\varepsilon^{2-p} \ge 1$, where we take into account that $p > 2$. Redefining $\tilde{\varepsilon}\varepsilon$ as ε, we finish the proof of the lemma. ■

Since $\bar{t}$ is arbitrary in $(-(A-1)\theta\varrho^p, 0]$, the previous lemma actually yields the measure theoretical information

$$|\{u(\cdot, t) \le \boldsymbol{\mu}^+ - \varepsilon\boldsymbol{\omega}\} \cap K_\varrho| \ge \tfrac{1}{4}\nu|K_\varrho| \quad \text{for all } t \in (-(A-1)\theta\varrho^p, 0]. \tag{5.4}$$

5.3.2. Shrinking the measure near the supremum. Let $\varepsilon \in (0, 1)$ denote the constant from Lemma 5.1 depending only on the data. Further, we choose the number A in the form

$$A = 2^{j_*(p-2)} + 1,$$

with some j_* to be fixed later and consider the cylinder $Q_\varrho((A-1)\theta) = Q_\varrho(2^{j_*(p-2)}\theta)$, where $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$.

Lemma 5.2. *Suppose* (5.2) *and* (5.4) *hold. Then there exists a constant* $\boldsymbol{\gamma} > 0$, *depending only on the data, such that for any positive integer* j_*, *we have*

$$\left|\left\{u \ge \boldsymbol{\mu}^+ - \frac{\varepsilon\boldsymbol{\omega}}{2^{j_*}}\right\} \cap Q_\varrho((A-1)\theta)\right| \le \frac{\boldsymbol{\gamma}}{j_*^{(p-1)/p}}|Q_\varrho((A-1)\theta)|.$$

Proof. Consider the cylinder $K_{2\varrho} \times (-(A-1)\theta\varrho^p, 0]$ and a time independent cutoff function $\zeta(x,t) \equiv \zeta(x)$ vanishing on $\partial K_{2\varrho}$ and equal to 1 in K_ϱ such that $|D\zeta| \le 2\varrho^{-1}$. Applying the energy estimate from Proposition 2.1 with levels $k_j = \boldsymbol{\mu}^+ - 2^{-j-1}\varepsilon\boldsymbol{\omega}$ for $j = 0, \dots, j_* - 1$, we obtain that

$$\begin{aligned}
&\iint_{Q_\varrho((A-1)\theta)} |D(u-k_j)_+|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{K_{2\varrho}\times\{-(A-1)\theta\varrho^p\}} \zeta^p \mathfrak{g}_+(u, k_j)\,\mathrm{d}x + \boldsymbol{\gamma} \iint_{K_{2\varrho}\times(-(A-1)\theta\varrho^p, 0]} (u-k_j)_+^p |D\zeta|^p \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By the mean value theorem, the restriction $\frac{1}{4}\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$ and the fact that the parameter ε is already fixed in Lemma 5.1 in dependence on the data, the first term on the right-hand side of the preceding inequality is estimated by

$$\begin{aligned}
\int_{K_{2\varrho}\times\{-(A-1)\theta\varrho^p\}} \zeta^p \mathfrak{g}_+(u, k_j)\,\mathrm{d}x &\le \boldsymbol{\gamma}\boldsymbol{\omega}^{q-1}\left(\frac{\varepsilon\boldsymbol{\omega}}{2^j}\right)^2 |K_{2\varrho}| \\
&\le \frac{\boldsymbol{\gamma}}{\varrho^p\varepsilon^{p-2}}\left(\frac{\varepsilon\boldsymbol{\omega}}{2^j}\right)^p |Q_\varrho((A-1)\theta)| \\
&\le \frac{\boldsymbol{\gamma}}{\varrho^p}\left(\frac{\varepsilon\boldsymbol{\omega}}{2^j}\right)^p |Q_\varrho((A-1)\theta)|.
\end{aligned}$$

For the second term on the right, we use $u \le \boldsymbol{\mu}^+$ and the bound for $|D\zeta|$. Thus, we arrive at

$$\iint_{Q_\varrho((A-1)\theta)} |D(u-k_j)_+|^p \,\mathrm{d}x\,\mathrm{d}t \le \frac{\boldsymbol{\gamma}}{\varrho^p}\Big(\frac{\varepsilon\boldsymbol{\omega}}{2^j}\Big)^p |Q_\varrho((A-1)\theta)|.$$

Next, we apply Lemma 2.2 in Chapter I of [4] with levels $k_{j+1} > k_j$ slicewise to $u(\cdot,t)$ for fixed $t \in (-(A-1)\theta\varrho^p, 0]$. Taking into account the measure theoretical information from (5.4), which implies

$$|\{u(\cdot,t) < k_j\} \cap K_\varrho| \ge \tfrac{1}{4}\nu|K_\varrho| \quad \text{for all } t \in (-(A-1)\theta\varrho^p, 0],$$

and using Hölder's inequality, we conclude that

$$\begin{aligned}
&(k_{j+1}-k_j)|\{u(\cdot,t) > k_{j+1}\} \cap K_\varrho| \\
&\quad\le \frac{\boldsymbol{\gamma}\varrho^{N+1}}{|\{u(\cdot,t) < k_j\} \cap K_\varrho|} \int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho} |Du(\cdot,t)|\,\mathrm{d}x \\
&\quad\le \frac{\boldsymbol{\gamma}\varrho}{\nu}\Big[\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho} |Du(\cdot,t)|^p\,\mathrm{d}x\Big]^{1/p} |\{k_j < u(\cdot,t) < k_{j+1}\} \cap K_\varrho|^{1-1/p} \\
&\quad= \frac{\boldsymbol{\gamma}\varrho}{\nu}\Big[\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho} |Du(\cdot,t)|^p\,\mathrm{d}x\Big]^{1/p} [|A_j(t)| - |A_{j+1}(t)|]^{1-1/p}.
\end{aligned}$$

Here, we abbreviated $A_j(t) := \{u(\cdot,t) > k_j\} \cap K_\varrho$. Further, we define $A_j = \{u > k_j\} \cap Q_\varrho((A-1)\theta)$. Integrating the preceding inequality with respect to t over $(-(A-1)\theta\varrho^p, 0]$ and applying Hölder's inequality slicewise leads to the measure estimate

$$\begin{aligned}
\frac{\varepsilon\boldsymbol{\omega}}{2^{j+1}}|A_{j+1}| &\le \frac{\boldsymbol{\gamma}\varrho}{\nu}\Big[\iint_{Q_\varrho((A-1)\theta)} |D(u-k_j)_+|^p\,\mathrm{d}x\,\mathrm{d}t\Big]^{1/p} [|A_j| - |A_{j+1}|]^{1-1/p} \\
&\le \boldsymbol{\gamma}\,\frac{\varepsilon\boldsymbol{\omega}}{2^j}|Q_\varrho((A-1)\theta)|^{1/p}\,[|A_j| - |A_{j+1}|]^{1-1/p}.
\end{aligned}$$

Taking the power $\frac{p}{p-1}$ on both sides, we find that

$$|A_{j+1}|^{p/(p-1)} \le \boldsymbol{\gamma}|Q_\varrho((A-1)\theta)|^{1/(p-1)}\,[|A_j| - |A_{j+1}|].$$

Finally, adding the inequalities with respect to j from 0 to $j_* - 1$, we obtain that

$$j_*|A_{j_*}|^{p/(p-1)} \le \boldsymbol{\gamma}|Q_\varrho((A-1)\theta)|^{p/(p-1)},$$

which is equivalent to

$$|A_{j_*}| \le \frac{\boldsymbol{\gamma}}{j_*^{(p-1)/p}}|Q_\varrho((A-1)\theta)|.$$

To conclude, it suffices to replace j_* by $j_* - 1$ in the above line and adjust $\boldsymbol{\gamma}$. ∎

5.3.3. A De Giorgi-type lemma. As in the preceding section, let $\varepsilon \in (0,1)$ denote the constant from Lemma 5.1 depending only on the data.

Lemma 5.3. *Suppose that the assumptions* (5.2) *and* (5.3) *hold true. Then there exists a constant* $\nu_1 \in (0,1)$ *depending only on the data such that if for some* $j_* > 1$*, the measure bound*

$$\Big|\Big\{\boldsymbol{\mu}^+ - u \le \frac{\varepsilon\boldsymbol{\omega}}{2^{j_*}}\Big\} \cap Q_\varrho((A-1)\theta)\Big| \le \nu_1 |Q_\varrho((A-1)\theta)|$$

holds true, where $A = 2^{j_*(p-2)} + 1$ *and* $\theta = (\frac14\boldsymbol{\omega})^{q+1-p}$*, then*

$$\boldsymbol{\mu}^+ - u \ge \frac{\varepsilon\boldsymbol{\omega}}{2^{j_*+1}} \quad \textit{a.e. in } Q_{\varrho/2}((A-1)\theta).$$

Proof. Let $M := 2^{-j_*}\varepsilon\boldsymbol{\omega}$ and define

$$\begin{aligned} k_n &= \boldsymbol{\mu}^+ - \frac{M}{2} - \frac{M}{2^{n+1}}, \quad & \tilde{k}_n &= \frac{k_n + k_{n+1}}{2},\\ \varrho_n &= \frac{\varrho}{2} + \frac{\varrho}{2^{n+1}}, \quad & \tilde{\varrho}_n &= \frac{\varrho_n + \varrho_{n+1}}{2},\\ K_n &= K_{\varrho_n}, \quad & \widetilde{K}_n &= K_{\tilde{\varrho}_n},\\ Q_n &= Q_{\varrho_n}((A-1)\theta), \quad & \widetilde{Q}_n &= Q_{\tilde{\varrho}_n}((A-1)\theta). \end{aligned}$$

We employ the energy estimate from Proposition 2.1 with cutoff functions ζ that vanish on the parabolic boundary of Q_n, equal the identity in $\widetilde{Q}_n$ and fulfill

$$|D\zeta| \le \boldsymbol{\gamma}\frac{2^n}{\varrho} \quad \text{and} \quad |\zeta_t| \le \boldsymbol{\gamma}\frac{2^{pn}}{(A-1)\theta\varrho^p}.$$

Using the condition $\frac14\boldsymbol{\omega} \le \boldsymbol{\mu}^+ \le 2\boldsymbol{\omega}$ to estimate the terms on the right-hand side, we find

$$\begin{aligned} &\boldsymbol{\omega}^{q-1} \operatorname*{ess\,sup}_{-(A-1)\theta\tilde{\varrho}_n^p<t<0} \int_{\widetilde{K}_n} (u-\tilde{k}_n)_+^2\,\mathrm{d}x + \iint_{\widetilde{Q}_n} |D(u-\tilde{k}_n)_+|^p\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\le \boldsymbol{\gamma}\frac{2^{pn}}{\varrho^p}M^p\Big(1+\frac{\boldsymbol{\omega}^{q-1}}{(A-1)\theta M^{p-2}}\Big)|A_n| = \boldsymbol{\gamma}\frac{2^{pn}}{\varrho^p}M^p(1+\varepsilon^{2-p})|A_n|, \end{aligned}$$

where we abbreviated

$$A_n = \{u > k_n\} \cap Q_n.$$

Taking into account the choice of ζ, by an application of the Sobolev imbedding (see Proposition 3.1 in Chapter I of [4]) and the preceding estimate, we conclude that

$$\begin{aligned} \Big(\frac{M}{2^{n+3}}\Big)^p |A_{n+1}| &\le \iint_{\widetilde{Q}_n} (u-\tilde{k}_n)_+^p\zeta^p\,\mathrm{d}x\,\mathrm{d}t\\ &\le \Big[\iint_{\widetilde{Q}_n} [(u-\tilde{k}_n)_+\zeta]^{p\frac{N+2}{N}}\,\mathrm{d}x\,\mathrm{d}t\Big]^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}}\\ &\le \boldsymbol{\gamma}\Big[\iint_{\widetilde{Q}_n} |D[(u-\tilde{k}_n)_+\zeta]|^p\,\mathrm{d}x\,\mathrm{d}t\Big]^{\frac{N}{N+2}}\\ &\quad\times\Big[\operatorname*{ess\,sup}_{-(A-1)\theta\tilde{\varrho}_n^p<t<0} \int_{\widetilde{K}_n} (u-\tilde{k}_n)_-^2\,\mathrm{d}x\Big]^{\frac{p}{N+2}} |A_n|^{\frac{2}{N+2}}\\ &\le \boldsymbol{\gamma}\boldsymbol{\omega}^{\frac{p(1-q)}{N+2}}\Big(\frac{2^{pn}}{\varrho^p}M^p\Big)^{\frac{N+p}{N+2}}(1+\varepsilon^{2-p})^{\frac{N+p}{N+2}}|A_n|^{1+\frac{p}{N+2}}. \end{aligned}$$

Hence, for the quantity $\boldsymbol{Y}_n = |A_n|/|Q_n|$, we deduce the recursive inequality

$$\begin{aligned}\boldsymbol{Y}_{n+1} &\leq \boldsymbol{\gamma} \boldsymbol{b}^n \Big(\frac{(A-1)\theta M^{p-2}}{\boldsymbol{\omega}^{q-1}}\Big)^{\frac{p}{N+2}} (1+\varepsilon^{2-p})^{\frac{N+p}{N+2}} \boldsymbol{Y}_n^{1+\frac{p}{N+2}} \\ &= \boldsymbol{\gamma} \boldsymbol{b}^n \varepsilon^{\frac{p(p-2)}{N+2}} (1+\varepsilon^{2-p})^{\frac{N+p}{N+2}} \boldsymbol{Y}_n^{1+\frac{p}{N+2}},\end{aligned}$$

where $\boldsymbol{b} = 2^{\frac{p(2N+p+2)}{N+2}}$, and $\boldsymbol{\gamma}$ only depends on the data. Thus, the lemma on fast geometric convergence, i.e., Lemma 4.1 in Chapter I of [4], ensures the existence of a constant $\nu_1 \in (0,1)$ depending only on the data such that $\boldsymbol{Y}_n \to 0$ if we assume that the smallness condition $\boldsymbol{Y}_o \leq \nu_1$ holds true. ■

At this stage, we conclude the reduction of oscillation in the remaining case where (5.2) and (5.3) hold. To this end, denote by $\varepsilon \in (0,1)$, $\boldsymbol{\gamma} > 0$ and $\nu_1 \in (0,1)$ the corresponding constants from Lemmas 5.1, 5.2 and 5.3. Choose a positive integer j_* large enough so that

$$\frac{\boldsymbol{\gamma}}{j_*^{(p-1)/p}} \leq \nu_1$$

and notice $Q_{\varrho/2}((A-1)\theta) \supset Q_{\varrho/4}(\theta)$, where $A = 2^{j_*(p-2)}+1$. Hence, applying in turn Lemmas 5.2 and 5.3, we arrive at

$$\boldsymbol{\mu}^+ - u \geq \frac{\varepsilon\boldsymbol{\omega}}{2^{j_*+1}} \quad \text{a.e. in } Q_{\varrho/4}(\theta).$$

This gives the reduction of oscillation

$$\operatorname{ess\,osc}_{Q_{\varrho/4}(\theta)} u \leq \Big(1 - \frac{\varepsilon}{2^{j_*+1}}\Big)\boldsymbol{\omega}.$$

Recall the reduction of oscillation achieved in Section 5.2 via arguments of Section 4.2. Namely, $\frac{1}{2}\xi$ is chosen in the reduction of oscillation (4.9) and $\frac{1}{2}\eta_o$ is chosen in the reduction of oscillation (4.10). Combining all cases gives the reduction of oscillation exactly as in (4.12) with the choice

$$\eta = \min\Big\{\frac{\xi}{2}, \frac{\eta_o}{2}, \frac{\varepsilon}{2^{j_*+1}}\Big\},$$

from which the rest of the proof can be reproduced just like in Section 4.

6. Proof of boundary regularity

Since Theorems 1.4–1.7 can be proved in a similar way as the interior Hölder continuity, we will only give sketchy proofs, where we keep reference to the tools and strategies used in the interior case and highlight the main differences.

6.1. Proof of Theorem 1.4

Consider the cylinder $Q_o = K_\varrho(x_o) \times (0, \varrho^{p-1}] \subset E_T$ whose vertex $(x_o, 0)$ is attached to the initial boundary $E \times \{0\}$. For ease of notation, assume $x_o = 0$ and set

$$\boldsymbol{\mu}^+ = \operatorname{ess\,sup}_{Q_o} u, \quad \boldsymbol{\mu}^- = \operatorname{ess\,inf}_{Q_o} u, \quad \boldsymbol{\omega} \geq \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-.$$

Let $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$. We may assume that

$$Q_\varrho(\theta) \subset Q_o = K_\varrho \times (-\varrho^{p-1}, 0], \quad \text{such that } \operatorname{ess\,osc}_{Q_\varrho(\theta)} u \le \boldsymbol{\omega};$$

otherwise we would have

$$\boldsymbol{\omega} \le 4\varrho^{1/(p-q-1)}.$$

Like in the proof of interior regularity, we start by distinguishing between the main cases

- when u is near zero: $\boldsymbol{\mu}^- \le \boldsymbol{\omega}$ and $\boldsymbol{\mu}^+ \ge -\boldsymbol{\omega}$,
- when u is away from zero: $\boldsymbol{\mu}^- > \boldsymbol{\omega}$ or $\boldsymbol{\mu}^+ < -\boldsymbol{\omega}$.

The second case reduces to the corresponding estimate for weak solutions to parabolic p-Laplacian equations, see Lemma 11.1 in Chapter III of [4]. In the first case, which implies $|\boldsymbol{\mu}^\pm| \le 2\boldsymbol{\omega}$, we proceed by a comparison to the initial datum u_o. More precisely, we assume that either

$$\boldsymbol{\mu}^+ - \tfrac{1}{4}\boldsymbol{\omega} > \sup_{K_\varrho} u_o \quad \text{or} \quad \boldsymbol{\mu}^- + \tfrac{1}{4}\boldsymbol{\omega} < \inf_{K_\varrho} u_o,$$

since otherwise, we would obtain the bound

$$\operatorname{ess\,osc}_{Q_o} u \le 2 \operatorname{ess\,osc}_{K_\varrho} u_o.$$

As both cases can be treated analogously, we consider only the second inequality with $\boldsymbol{\mu}^-$ and work with u as a super-solution. Using $|\boldsymbol{\mu}^-| \le 2\boldsymbol{\omega}$, Lemma 3.2 (with $\xi = 1/4$) yields a constant $\nu_o \in (0,1)$, depending only on the data, such that

$$u \ge \boldsymbol{\mu}^- + \tfrac{1}{8}\boldsymbol{\omega} \quad \text{a.e. in } \widehat{Q}_1 := K_{\varrho/2} \times (0, \nu_o \theta \varrho^p].$$

Thus, we arrive at the reduction of oscillation

$$\operatorname{ess\,osc}_{\widehat{Q}_1} u \le \tfrac{7}{8}\boldsymbol{\omega}.$$

Finally, taking the initial datum into account, we conclude that

$$\operatorname{ess\,osc}_{\widehat{Q}_1} u \le \max\{\tfrac{7}{8}\boldsymbol{\omega}, 2\boldsymbol{\omega}_{u_o}(\varrho)\}.$$

Now we may proceed by an iteration argument as in Section 7.1 of [1] to conclude the proof.

6.2. Proof of Theorem 1.6

Consider the cylinder $Q_o = K_\varrho(x_o) \times (t_o - \varrho^{p-1}, t_0]$ whose vertex (x_o, t_o) is attached to S_T. Suppose that ϱ is so small that $t_o - \varrho^{p-1} > 0$ and $\varrho < \varrho_o$, where ϱ_o is the constant from the geometric condition (G). Further, we assume that $(x_o, t_o) = (0,0)$ for ease of notation and define

$$\boldsymbol{\mu}^+ = \operatorname{ess\,sup}_{Q_o \cap E_T} u, \quad \boldsymbol{\mu}^- = \operatorname{ess\,inf}_{Q_o \cap E_T} u, \quad \boldsymbol{\omega} \ge \boldsymbol{\mu}^+ - \boldsymbol{\mu}^-.$$

Let $\theta = (\frac{1}{4}\boldsymbol{\omega})^{q+1-p}$. For some $A > 1$ to be determined in terms of the data, we may assume that

$$Q_\varrho(A\theta) \subset Q_o, \quad \text{such that } \operatorname{ess\,osc}_{Q_\varrho(A\theta) \cap E_T} u \le \boldsymbol{\omega};$$

otherwise we would have

$$\boldsymbol{\omega} \le L\varrho^{1/(p-q-1)}, \quad \text{where } L = 4A^{1/(p-q-1)}.$$

As in the proof of interior Hölder continuity, we consider the main cases

(6.1a) when u is near zero: $\boldsymbol{\mu}^- \le \xi\boldsymbol{\omega}$ and $\boldsymbol{\mu}^+ \ge -\xi\boldsymbol{\omega}$;

(6.1b) when u is away from zero: $\boldsymbol{\mu}^- > \xi\boldsymbol{\omega}$ or $\boldsymbol{\mu}^+ < -\xi\boldsymbol{\omega}$.

Here $\xi \in (0,1)$ will be fixed in terms of the data and α_*, where α_* comes from the geometric condition (G) of ∂E.

When (6.1a) holds true, we either arrive at the bound

$$\operatorname{ess\,osc}_{Q_o\cap E_T} u \le 2\operatorname{ess\,osc}_{Q_o\cap S_T} g$$

or we continue with a comparison to the boundary datum g, i.e., we are concerned with the cases either

$$\boldsymbol{\mu}^+ - \tfrac14\boldsymbol{\omega} > \sup_{Q_o\cap S_T} g \quad \text{or} \quad \boldsymbol{\mu}^- + \tfrac14\boldsymbol{\omega} < \inf_{Q_o\cap S_T} g.$$

Since the inequalities can be treated analogously, let us consider only the second one. Observe that k satisfies the second inequality in (2.2) with $Q_{R,S}$ replaced by Q_o, since $(u-k)_-$ vanishes on $Q_o \cap S_T$ for all $k \le \boldsymbol{\mu}^- + \frac14\boldsymbol{\omega}$. Therefore, we may employ the energy estimate in Proposition 2.2 for super-solutions if we extend all integrals in the energy estimates to zero outside of E_T. The extended function $(u-k)_-$, which will be denoted by the same symbol, is still contained in the functional space in (1.6) within Q_o.

The proof of Lemma 4.2 of [1] can be adapted to the current situation, bearing in mind that we have assumed that ∂E fulfills the property of positive geometric density (G), and therefore for $k = \boldsymbol{\mu}^- + \frac14\boldsymbol{\omega}$, we have

$$|\{u_k^-(\cdot,t) - \boldsymbol{\mu}^- \ge \tfrac14\boldsymbol{\omega}\} \cap K_\varrho(x_o)| \ge \alpha_*|K_\varrho| \quad \text{for all } t \in (-A\theta\varrho^p, 0]. \tag{6.2}$$

Here we have used u_k^- as the extension of u to the whole Q_o, defined by

$$u_k^- := \begin{cases} k-(u-k)_- & \text{in } Q_o\cap E_T,\\ k & \text{in } Q_o\setminus E_T.\end{cases}$$

By Lemma A.2, the extension u_k^- turns out to be a local, weak super-solution to (1.2) in Q_o, with a properly extended principle part $\widetilde{\mathbf{A}}$, cf. Appendix A. The extended $\widetilde{\mathbf{A}}$ enjoys the same type of structural conditions as in (1.3). For simplicity, we still use u to denote the extended function in what follows.

Consequently, like in Lemma 4.2 of [1], there exists $\boldsymbol{\gamma}$, depending only on the data and α_*, such that for any positive integer j_*, we have

$$\left|\left\{u-\boldsymbol{\mu}^- \le \frac{\boldsymbol{\omega}}{2^{j_*+2}}\right\} \cap \widehat{Q}_\varrho\right| \le \frac{\boldsymbol{\gamma}}{j_*^{(p-1)/p}}|\widehat{Q}_\varrho|,$$

where

$$\widehat{Q}_\varrho = K_\varrho \times (-(2^{-j_*-2}\boldsymbol{\omega})^{q+1-p}\varrho^p, 0),$$

provided $|\boldsymbol{\mu}^-| \le 2^{-j_*-2}\boldsymbol{\omega}$. Assuming this condition on $\boldsymbol{\mu}^-$ is fulfilled and letting ν be the number determined in Lemma 3.1, we may choose j_* to satisfy that $\boldsymbol{\gamma} j_*^{-(p-1)/p} \le \nu$. Then setting

$$\xi = 2^{-j_*-2}, \quad A_1 = 2^{j_*(p-q-1)},$$

Lemma 3.1 implies that

$$u - \boldsymbol{\mu}^- \ge \tfrac{1}{2}\xi\boldsymbol{\omega} \quad \text{a.e. in } Q_{\varrho/2}(A_1\theta),$$

which in turn yields

$$\operatorname{ess\,osc}_{\widehat{Q}_{\varrho/2}\cap E_T} u \le (1 - \tfrac{1}{2}\xi)\boldsymbol{\omega}.$$

Hence, the oscillation is reduced when $|\boldsymbol{\mu}^-| < \xi\boldsymbol{\omega}$ for some $\xi \in (0,1)$ determined by the data and α_*. To proceed, one still needs to handle the situation when $\boldsymbol{\mu}^- < -\xi\boldsymbol{\omega}$, since this is not excluded in (6.1a).

Our current hypothesis to proceed consists of the measure information (6.2) and that $-2\boldsymbol{\omega} < \boldsymbol{\mu}^- < -\xi\boldsymbol{\omega}$, as we have assumed $\boldsymbol{\mu}^+ \ge -\xi\boldsymbol{\omega}$ in (6.1a). From this, we have two ways to proceed: one is to use the expansion of positivity (Lemma 3.3); the other is to follow the arguments in Sections 5.3.2–5.3.3. We only describe the first option.

In fact, by Lemma 3.3, the measure information (6.2) translates into the pointwise estimate

$$u \ge \boldsymbol{\mu}^- + \eta\boldsymbol{\omega} \quad \text{a.e. in } Q_{\varrho/2}(A_2\theta),$$

for some $\eta \in (0,1)$ depending on the data and ξ. This gives us a reduction of oscillation as usual, and hence finishes the reduction of oscillation under the condition (6.1a). The constant A_2 is determined by the data in this step, through b and η of Lemma 3.3. The final choice of A is given by the larger one of A_1 and A_2.

As in the interior case, we repeat the arguments inductively until the second case of (6.1) is satisfied for some index j for the first time. Starting from j, the equation behaves like the parabolic p-Laplacian type equation within $Q_j \cap E_T$. In order to render this point technically, we adapt the proof for interior regularity, where we use in particular the boundary regularity result Proposition 7.2 of [1] for the parabolic p-Laplacian near the lateral boundary.

6.3. Proof of Theorem 1.7

First of all, we observe that the second proof of interior regularity (Theorem 1.1) in Section 5 is based solely on the energy estimates in Proposition 2.1 and a corresponding Hölder estimate for solutions to the parabolic p-Laplacian.

A key ingredient – the Sobolev imbedding (cf. Proposition 3.1 in Chapter I of [4]) – was used in order to establish Lemma 3.1, Lemma 3.2, Lemma 3.5, Lemma 5.1 and Lemma 5.3, assuming the functions $(u-k)_\pm\zeta^p$ vanish on the lateral boundary of the domain of integration. This assumption in turn is fulfilled by choosing a proper cutoff function ζ. In the boundary situation, similar arguments have been employed in Section 6.2 and Section 6.1 by restricting the value of the level k according to the Dirichlet data as in (2.2) and the initial data as in (2.1), respectively.

However, in the current situation of Neumann data, the functions $(u-k)_\pm\zeta^p$ under conditions of Proposition 2.4 do not vanish on S_T and therefore such a Sobolev imbedding cannot be used in general. On the other hand, a similar Sobolev imbedding (cf. Proposition 3.2 in Chapter I of [4]) that does not require functions to vanish on the boundary still holds for the functional space

$$u \in C(0,T; L^p(E)) \cap L^p(0,T; W^{1,p}(E)).$$

The appearing constant now depends on N, the structure of ∂E and the ratio $T/|E|^{p/N}$, which is invariant for cylinders of the type $Q_\varrho = K_\varrho \times (-\varrho^p, 0]$ and $Q_\varrho \cap E_T$ as well, provided ∂E is smooth enough. In particular, Lemmas 3.1, 3.2, 3.5, 5.1 and 5.3 can be proved in this boundary setting.

Finally, we remark that the use of De Giorgi's isoperimetric inequality (cf. Lemma 2.2 in Chapter I of [4] and Theorem 4.2.1 of [17]) is permitted for extension domains, and thus in particular for C^1-domains. Thus, the machinery used in Lemma 5.2 can be reproduced.

For the proof of Theorem 1.7, we now consider a cylinder $Q_o = K_\varrho(x_o) \times (t_o - \varrho^{p-1}, t_o]$ whose vertex (x_o, t_o) is attached to S_T and ϱ is so small that $t_o - \varrho^{p-1} > 0$. According to the preceding considerations, we proceed exactly as in the second proof of interior regularity in Section 5. Obviously, in the present situation all cylinders have to be intersected with E_T. In this way, we conclude a reduction of oscillation for the lateral boundary point (x_o, t_o).

A. On the notion of parabolicity

We collect some useful lemmas regarding the notion of parabolicity for (1.2)–(1.3).

Lemma A.1. *Let u be a local weak sub(super)-solution to* (1.2)–(1.3)*. Then, for any $k \in \mathbb{R}$, the truncation $k \pm (u-k)_\pm$ is a local weak sub(super)-solution to* (1.2)–(1.3)*.*

The analysis has been carried out in Appendix A of [1] for $q = p-1$. However, the same proof actually works for all $p > 1$ and $q > 0$ after minor changes.

In particular, when u is a local weak solution, u_+ and u_- are non-negative, local weak sub-solutions to (1.2)–(1.3). By Theorem 4.1 of [2], they are locally bounded and hence u is also.

In order to formulate an analogue of Lemma A.1 near the lateral boundary S_T for a sub(super)-solution u to (1.4), consider the cylinder $Q_{R,S} = K_R(x_o) \times (t_o - S, t_o)$ whose vertex (x_o, t_o) is attached to S_T. Further, for a level k satisfying (2.2), we are concerned with the following truncated extension of u in $Q_{R,S}$:

$$u_k^\pm := \begin{cases} k \pm (u-k)_\pm & \text{in } Q_{R,S} \cap E_T, \\ k & \text{in } Q_{R,S} \setminus E_T. \end{cases}$$

Moreover, the extension of $\mathbf{A}$, defined by

$$\widetilde{\mathbf{A}}(x,t,u,\zeta) := \begin{cases} \mathbf{A}(x,t,u,\zeta) & \text{in } Q_{R,S} \cap E_T, \\ |\zeta|^{p-2}\zeta & \text{in } Q_{R,S} \setminus E_T, \end{cases}$$

is a Carathéodory function satisfying (1.3) with structure constants C_o and C_1 replaced by $\min\{1, C_o\}$ and $\max\{1, C_1\}$, respectively. In this situation, the following lemma holds.

Lemma A.2. *Suppose u is a sub(super)-solution to (1.4) with (1.3) and the level k satisfies (2.2). Let $u_k^{\pm}$ be defined as above. Then $u_k^{\pm}$ is a local weak sub(super)-solution to (1.2) with $\widetilde{\mathbf{A}}$ in $Q_{R,S}$.*

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