Rev. Mat. Iberoam. 39 (2023), no. 3, 1039–1058
DOI 10.4171/RMI/1348

Curvature estimates for p-convex hypersurfaces of prescribed curvature

Weisong Dong

Abstract. In this paper, we establish curvature estimates for p-convex hypersurfaces in $\mathbb{R}^{n+1}$ of prescribed curvature with $p \geq n/2$. The existence of a star-shaped hypersurface of prescribed curvature is obtained. We also prove a type of interior C^2 estimates for solutions to the Dirichlet problem of the corresponding equation.

1. Introduction

Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface and let $\kappa(X) = (\kappa_1, \ldots, \kappa_n)$ be the principal curvatures of M at X. Given $1 \leq p \leq n$, a C^2 regular hypersurface M is called *p-convex* if, at each $X \in M$, $\kappa(X)$ satisfies

$$\kappa_{i_1} + \cdots + \kappa_{i_p} \geq 0, \quad \forall 1 \leq i_1 < \cdots < i_p \leq n.$$

In other words, the sum of the p smallest principal curvatures is nonnegative at each point of M. The notion of p-convexity goes back to Wu [36], and has been studied extensively by Wu [36], Sha [28, 29] and Harvey–Lawson [19, 20].

In this paper, we are interested in finding a p-convex hypersurface $M \subset \mathbb{R}^{n+1}$ of prescribed curvature as below:

$$\prod_{1 \leq i_1 < \cdots < i_p \leq n} (\kappa_{i_1} + \cdots + \kappa_{i_p}) = f(X, \nu(X)), \quad \forall X \in M, \tag{1.1}$$

where $\nu(X)$ is the unit outer normal of M at X, the function $f(X,\nu) \in C^2(\Gamma)$ is positive and Γ is an open neighborhood of unit normal bundle of M in $\mathbb{R}^{n+1} \times \mathbb{S}^n$. The Gaussian curvature equation, that corresponds to $p = 1$ in (1.1), was studied by Oliker [24]. The mean curvature equation, corresponding to $p = n$ in (1.1), was studied by Bakelman–Kantor [1] and Treibergs–Wei [33]. For general curvature equations, see Caffarelli–Nirenberg–Spruck [4] and Gerhardt [11]. When $p = n - 1$, the equation was studied by Chu–Jiao [7] and, in complex settings, it is related to the Gauduchon conjecture, which was solved by Székelyhidi–Tosatti–Weinkove [30]. For some previous work on this topic, see Tosatti–Weinkove [31, 32] and Fu–Wang–Wu [9, 10].

2020 Mathematics Subject Classification: Primary 53C21; Secondary 53C42.
Keywords: Curvature estimates, p-convex hypersurfaces, interior estimates.

It is of great interest in geometry and PDEs to derive a C^2 estimate for equation (1.1) for general $f(X,\nu(X))$. We have the following main result.

Theorem 1.1. *Suppose $M \subset \mathbb{R}^{n+1}$ is a closed star-shaped p-convex hypersurface with $p \geq n/2$ satisfying the curvature equation (1.1). Then, there is a positive constant C such that*

$$\sup_{X\in M, i=1,\dots,n} |\kappa_i(X)| \leq C, \tag{1.2}$$

where C depends on n, p, $|M|_{C^1}$, $\inf f$ and $|f|_{C^2}$.

We remark that in this theorem only a few conditions on f are assumed. Usually, to derive C^2 estimates for elliptic equations which are not strictly elliptic, there should be some extra assumptions on f due to the dependency on $\nu(X)$. We refer the reader to Ivochkina [21, 22], Guan–Guan [13], Guan–Lin–Ma [16], Guan–Li–Li [15] and Guan–Jiao [14] for more details. Moreover, Guan–Ren–Wang in [17] showed that for the following curvature equation:

$$\frac{\sigma_k}{\sigma_l}(\kappa(X)) = f(X,\nu(X)), \quad \forall X \in M,$$

where $0 < l < k \leq n$ and σ_k is the k-th elementary symmetric function, estimate (1.2) fails generally, though it may hold for special f as in Guan–Guan [13]. When $l = 0$, some results are known for general f. For instance, estimate (1.2) was proved for $k = n$ by Caffarelli–Nirenberg–Spruck [3] and, for $2 \leq k \leq n$, Guan–Ren–Wang [17] obtained the estimate for convex solutions. For $k = n-1$ and $n-2$, estimate (1.2) was established by Ren–Wang [25, 34]. They also conjectured that the estimate still holds for $k > n/2$ in [26, 34]. When f is independent of ν, Caffarelli–Nirenberg–Spruck [4] proved the C^2 estimate for a general class of fully nonlinear curvature equations. For hypersurfaces of prescribed curvature in Riemannian manifolds and Minkowski space, see [5] and [27, 35]. We also refer the reader to Guan–Zhang [18] and references therein for a class of curvature equations arising from convex geometry.

To obtain the existence of a p-convex hypersurface satisfying the prescribed curvature equation (1.1), we assume the following two conditions on f. The first one is that there exists two positive constants $r_1 < 1 < r_2$ such that

$$\begin{aligned} f\Big(X, \frac{X}{|X|}\Big) &\geq \frac{p^{C_n^p}}{r_1^{C_n^p}}, \quad \text{for } |X| = r_1; \\ f\Big(X, \frac{X}{|X|}\Big) &\leq \frac{p^{C_n^p}}{r_2^{C_n^p}}, \quad \text{for } |X| = r_2. \end{aligned} \tag{1.3}$$

This condition is used to derive C^0 estimates. The second one is that for any fixed unit vector ν,

$$\frac{\partial}{\partial\rho}\big(\rho^{C_n^p} f(X,\nu)\big) \leq 0, \quad \text{where } \rho = |X|, \tag{1.4}$$

and will be used to derive C^1 estimates. Actually, with suitable assumptions of f, Li [23] proved that the interior gradient estimate holds.

By the continuity method argument as in [4], we can obtain the following result.

Theorem 1.2. *Let $f \in C^2((\overline{B_{r_2}} \backslash B_{r_1}) \times \mathbb{S}^n)$ be a positive function satisfying (1.3) and (1.4). Then equation (1.1) has a unique $C^{3,\alpha}$ star-shaped p-convex solution M in $\{X \in \mathbb{R}^{n+1} : r_1 \le |X| \le r_2\}$, for any $\alpha \in (0,1)$, as long as $p \ge n/2$.*

The method of proving Theorem 1.1 can be applied to obtain an interior C^2 estimate for the Dirichlet problem of the corresponding equation in the Euclidean space. Suppose that Ω is a bounded domain in $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$, denote by $\lambda(D^2u) = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of the Hessian D^2u. We say that $u \in C^2(\Omega)$ is *p-plurisubharmonic* if the eigenvalues of D^2u satisfy $\lambda_{i_1} + \cdots + \lambda_{i_p} \ge 0$, for all $1 \le i_1 < \cdots < i_p \le n$ (see [20]). Given a C^2 p-plurisubharmonic function v on $\bar{\Omega}$, consider the following Dirichlet problem:

$$\prod_{1 \le i_1 < \cdots < i_p \le n} (\lambda_{i_1} + \cdots + \lambda_{i_p}) = f(x, u, Du), \quad \text{in } \Omega, \tag{1.5}$$

with boundary data

$$u = v \quad \text{on } \partial\Omega,$$

where $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function. By the same argument as that of Theorem 1.1, we can prove the following interior estimate.

Theorem 1.3. *Suppose that a p-plurisubharmonic function $u \in C^4(\Omega) \cap C^{1,1}(\bar{\Omega})$ is a solution to the Dirichlet problem (1.5) and satisfies $u < v$ in Ω. Then, there exist constants C and β depending only on n, p, $|u|_{C^1}$, $|v|_{C^1}$, $\inf f$, $|f|_{C^2}$ and Ω such that*

$$\sup_{\Omega} (v - u)^{\beta} \Delta u \le C \tag{1.6}$$

as long as $p \ge n/2$.

Remark 1.4. As a byproduct of the proof of the above theorem, one can conclude the following global C^2 estimate for equation (1.5):

$$\sup_{\Omega} |D^2u| \le C\big(1 + \sup_{\partial\Omega} |D^2u|\big),$$

where C is a constant as in Theorem 1.3.

We shall only give an outline for the proof of Theorem 1.3, as it is almost the same as that of Theorem 1.1. The estimate (1.6) can also be seen in some sense as a generalization of Theorem 0.4 in [8], since there the right-hand side function f does not depend on Du. Such an estimate for the k-Hessian equation $\sigma_k(\lambda) = f(x,u)$ has been proved by Chou–Wang [6]. The function f depending on ν or Du creates substantial difficulties to derive a C^2 estimate, as the bad term $-Ch_{11}$ or $-Cu_{11}$ appears when one applies the maximum principle to the test function. One way to overcome this is, like in [17, 25, 34], to control the bad third order terms, which is very hard even for the k-Hessian equation. Another way is, as in [7], to control $-Ch_{11}$ firstly by good terms (see Lemma 3.2), and then the bad third order terms can be eliminated easily by Lemma 3.4. Thanks to Dinew [8], where many properties of the operator have been proved, we can follow the argument in [7] to prove the estimate.

The paper is organized as follows. In Section 2, we recall some properties of the operator from [8]. In Section 3, we prove the curvature estimate. In Section 4 we derive the gradient estimate, and in Section 5 we apply the continuity method to prove Theorem 1.2. Finally, we give an outline of the proof of Theorem 1.3 in Section 6.

2. Preliminaries

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and let ∇ be the connection on it. Assume that M is star-shaped with respect to the origin, i.e., the position vector X of M can be written as $X(x) = \rho(x)x$, where $x \in \mathbb{S}^n$. Then the unit outer normal of M is given by

$$\nu = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}.$$

Let $\{e_1, \dots, e_n\}$ be a smooth local orthonormal frame on $\mathbb{S}^n$. Then the metric of M is given by $g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j$, and the second fundamental form of M is

$$h_{ij} = \frac{\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{2.1}$$

The principal curvatures $\kappa = (\kappa_1, \dots, \kappa_n)$ are the eigenvalues of h_{ij} with respect to g_{ij}.

At a point X_0 in M, choose a local orthonormal frame $\{e_1, e_2 \dots, e_n\}$. The following geometric formulas are well known:

$$\begin{aligned} X_{ij} &= -h_{ij}\nu \quad \text{(Gauss formula)},\\ (\nu)_i &= h_{ij}e_j \quad \text{(Weingarten equation)},\\ h_{ijk} &= h_{ikj} \quad \text{(Codazzi formula)},\\ R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)}, \end{aligned} \tag{2.2}$$

where R_{ijkl} is the $(4,0)$-Riemannian curvature tensor and we have the formula

$$h_{ijkl} = h_{klij} + h_{mk}(h_{mj}h_{il} - h_{ml}h_{ij}) + h_{mi}(h_{mj}h_{kl} - h_{ml}h_{kj}). \tag{2.3}$$

We recall the p-convex cones introduced by Harvey and Lawson [20].

Definition 2.1. Let $p \in \{1, \dots, n\}$. The cone $\mathcal{P}_p$ is defined by

$$\mathcal{P}_p = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \forall 1 \le i_1 < i_2 < \cdots < i_p \le n,\ \lambda_{i_1} + \cdots + \lambda_{i_p} > 0\}.$$

Associated to $\mathcal{P}_p$ is the cone of symmetric $n \times n$ matrices defined by

$$P_p = \{A \mid \forall 1 \le i_1 < i_2 < \cdots < i_p \le n,\ \lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) > 0\}.$$

We call A is p-positive if $A \in P_p$.

For convenience, we introduce the following notations:

$$F(h_{ij}) := F(\kappa) = \prod_{1 \le i_1 < \cdots < i_p \le n} (\kappa_{i_1} + \cdots + \kappa_{i_p}) \quad \text{and} \quad \tilde{F} = F^{1/C_n^p},$$

where $C_n^p = \frac{n!}{p!(n-p)!}$. Equation (1.1) then can be written as

$$\tilde{F}(h_{ij}) := \tilde{F}(\kappa) = \tilde{f}(X, \nu(X)), \tag{2.4}$$

where $\kappa = (\kappa_1, \dots, \kappa_n)$ and $\tilde{f} = f^{1/C_n^p}$. Denote

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}, \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}} \quad \text{and} \quad \mathcal{F} = \sum F^{ii}.$$

Direct calculations show that

$$\tilde{F}^{ij} = \frac{1}{C_n^p} F^{1/C_n^p - 1} F^{ij}$$

and

$$\tilde{F}^{ij,kl} = \frac{1}{C_n^p} F^{1/C_n^p - 1} F^{ij,kl} + \frac{1}{C_n^p}\Big(\frac{1}{C_n^p} - 1\Big) F^{1/C_n^p - 2} F^{ij} F^{kl}.$$

We remark that $\tilde{F}$ is concave with respect to h_{ij} by Lemma 1.13 and Corollary 1.14 in [8]. And the equation is elliptic as the matrix $\{\partial \tilde{F}/\partial h_{ij}\}$ is positive definite for $\{h_{ij}\} \in P_p$.

Now we do some basic calculations which will be used in the next section. Our calculations are carried out at a point X_0 on the hypersurface M, and we use coordinates such that at this point $\{h_{ij}\}$ is diagonal and its eigenvalues with respect to g_{ij} are ordered as $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. Note that F^{ij} is also diagonal at X_0 and we have the following formulas:

$$F^{kk} = \frac{\partial F}{\partial \kappa_k} = \sum_{k \in \{i_1, \dots, i_p\}} \frac{F(\kappa)}{\kappa_{i_1} + \cdots + \kappa_{i_p}},$$

for which we refer to Lemma 1.10 in [8]. We also have formulas for the second order derivatives of F at X_0:

$$F^{kk,ll} = \frac{\partial^2 F}{\partial \kappa_k \partial \kappa_l} = \sum_{\substack{k \in \{i_1, \dots, i_p\} \\ l \in \{j_1, \dots, j_p\} \\ \{i_1, \dots, i_p\} \neq \{j_1, \dots, j_p\}}} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})},$$

and, for $k \neq r$,

$$F^{kr,rk} = \frac{F^{kk} - F^{rr}}{\kappa_k - \kappa_r} = - \sum_{\substack{k \notin \{i_1, \dots, i_p\} \ni r \\ r \notin \{j_1, \dots, j_p\} \ni k \\ \{i_1, \dots, i_p\} \setminus \{r\} = \{j_1, \dots, j_p\} \setminus \{k\}}} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})}.$$

Otherwise, we have $F^{ij,kl} = 0$. See Lemma 1.12 in [8] for the above formulas. These formulas can also be easily obtained from Theorem 5.5 in [2]. The following properties of the function F, which are very similar to the properties of σ_k, were proved by Dinew [8].

Lemma 2.2 ([8]). *Suppose that the diagonal matrix $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ belongs to P_p and that $\lambda_1 \geq \cdots \geq \lambda_n$. Then,*

(1) $\tilde{F}^{11}(A)\lambda_1 \geq \frac{1}{n}\tilde{F}(A)$;

(2) $\sum_{k=1}^n \tilde{F}^{kk}(A) \geq p$;

(3) $\sum_{k=1}^n F^{kk}(A)\lambda_k = C_n^p F(A)$;

(4) *there is a constant $\theta = \theta(n, p)$ such that $F^{jj}(A) \geq \theta \sum F^{ii}$ for all $j \geq n - p + 1$.*

For the reader's convenience, we provide a short proof of the above lemma in the appendix.

3. Curvature estimates

Set $u = \langle X, \nu\rangle$, which is the support function of the hypersurface M. Clearly, we have

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}.$$

There exists a positive constant C, depending on $\inf_M \rho$ and $|\rho|_{C^1}$, such that

$$\frac{1}{C} \le \inf_M u \le u \le \sup_M u \le C.$$

In order to prove Theorem 1.1, we consider the following auxiliary function:

$$G = \log \kappa_{\max} - \log(u-a) + \frac{A}{2}|X|^2$$

where $\kappa_{\max}$ is the largest principal curvature, $a = \frac{1}{2}\inf_M u > 0$, and $A \ge 1$ is a large constant to be determined. Suppose the maximum of G is achieved at a point $X_0 \in M$. Choose a local orthonormal frame $\{e_1, \dots, e_n\}$ around X_0 such that

$$h_{ij} = \delta_{ij} h_{ii} \quad \text{and} \quad h_{11} \ge h_{22} \ge \cdots \ge h_{nn} \quad \text{at } X_0.$$

Since $\kappa_{\max}$ may not be differentiable, we define a new function $\hat{G}$ near X_0 by

$$\hat{G} = \log h_{11} - \log(u-a) + \frac{A}{2}|X|^2.$$

It is easy to see $\hat{G}$ achieves a maximum at X_0. Now, differentiating $\hat{G}$ at X_0 twice yields that

$$0 = \frac{h_{11i}}{h_{11}} - \frac{u_i}{u-a} + A\langle X, e_i\rangle \tag{3.1}$$

and

$$0 \ge \frac{h_{11ii}}{h_{11}} - \left(\frac{h_{11i}}{h_{11}}\right)^2 - \frac{u_{ii}}{u-a} + \left(\frac{u_i}{u-a}\right)^2 + A(1 + \langle X, X_{ii}\rangle). \tag{3.2}$$

Contracting (3.2) with $\tilde{F}^{ii}$, we get

$$0 \ge \frac{\tilde{F}^{ii} h_{11ii}}{h_{11}} - \frac{\tilde{F}^{ii} h_{11i}^2}{h_{11}^2} - \frac{\tilde{F}^{ii} u_{ii}}{u-a} + \frac{\tilde{F}^{ii} u_i^2}{(u-a)^2} + A\tilde{F}^{ii}(1 + \langle X, X_{ii}\rangle). \tag{3.3}$$

Lemma 3.1. *We have*

$$\begin{aligned} 0 \ge & -\frac{2}{h_{11}} \sum_{i\ge 2} \tilde{F}^{1i,i1} h_{11i}^2 - \frac{\tilde{F}^{ii} h_{11i}^2}{h_{11}^2} - C h_{11} \\ & + \frac{a\tilde{F}^{ii} h_{ii}^2}{u-a} + \frac{\tilde{F}^{ii} u_i^2}{(u-a)^2} + A\sum \tilde{F}^{ii} - CA. \end{aligned} \tag{3.4}$$

Proof. From the formula (2.3), we have

$$\tilde{F}^{ii}h_{11ii} = \tilde{F}^{ii}h_{ii}h_{11}^2 - \tilde{F}^{ii}h_{ii}^2 h_{11} + \tilde{F}^{ii}h_{ii11}. \tag{3.5}$$

Differentiating equation (1.1) twice at X_0, we obtain

$$\tilde{F}^{ii}h_{iik} = (d_X \tilde{f})(e_k) + h_{kk}(d_\nu \tilde{f})(e_k) \tag{3.6}$$

and

$$\tilde{F}^{ii}h_{iikk} + \tilde{F}^{pq,rs}h_{pqk}h_{rsk} \geq -C - Ch_{11}^2 + \sum_l h_{lkk}(d_\nu \tilde{f})(e_l). \tag{3.7}$$

By the concavity of $\tilde{F}$ and the Codazzi formula, we have

$$-\tilde{F}^{pq,rs}h_{pq1}h_{rs1} \geq -2\sum_{i\geq 2}\tilde{F}^{1i,i1}h_{11i}^2. \tag{3.8}$$

Note that by Lemma 2.2 (3) we have $\tilde{F}^{ii}h_{ii} = \tilde{f}$. Hence, we see that

$$\tilde{F}^{ii}h_{11ii} \geq -2\sum_{i\geq 2}\tilde{F}^{1i,i1}h_{11i}^2 - \tilde{F}^{ii}h_{ii}^2 h_{11} + \sum_l h_{l11}(d_\nu \tilde{f})(e_l) - C - Ch_{11}^2. \tag{3.9}$$

We now compute the term $\tilde{F}^{ii}u_{ii}$. By (2.2), we have

$$u_i = h_{ii}\langle X, e_i\rangle \quad \text{and} \quad u_{ii} = \sum_k h_{iik}\langle X, e_k\rangle - uh_{ii}^2 + h_{ii}.$$

Hence, we obtain

$$\begin{aligned} \tilde{F}^{ii}u_{ii} &= \sum_k \tilde{F}^{ii}h_{iik}\langle X, e_k\rangle - u\tilde{F}^{ii}h_{ii}^2 + \tilde{f} \\ &\leq \sum_k h_{kk}(d_\nu \tilde{f})(e_k)\langle X, e_k\rangle - u\tilde{F}^{ii}h_{ii}^2 + C. \end{aligned} \tag{3.10}$$

By the Gauss formula, we have

$$\langle X, X_{ii}\rangle = -h_{ii}\langle X, \nu\rangle = -h_{ii}u. \tag{3.11}$$

Substituting (3.9), (3.10) and (3.11) in (3.3), we obtain that

$$\begin{aligned} 0 \geq &-\frac{2}{h_{11}}\sum_{i\geq 2}\tilde{F}^{1i,i1}h_{11i}^2 - \tilde{F}^{ii}h_{ii}^2 + \frac{1}{h_{11}}\sum_l h_{l11}(d_\nu \tilde{f})(e_l) - Ch_{11} \\ &- \frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} - \frac{\sum_k h_{kk}(d_\nu \tilde{f})(e_k)\langle X, e_k\rangle}{u-a} + \frac{u\tilde{F}^{ii}h_{ii}^2}{u-a} - \frac{C}{u-a} \\ &+ \frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} + A\sum \tilde{F}^{ii} - CA. \end{aligned} \tag{3.12}$$

By the Codazzi formula, $u_k = h_{kk}\langle X, e_k\rangle$ and (3.1), we have

$$\frac{1}{h_{11}}\sum_k h_{k11}(d_\nu \tilde{f})(e_k) - \frac{h_{kk}(d_\nu \tilde{f})(e_k)\langle X, e_k\rangle}{u-a} \geq -CA.$$

Therefore, we arrive at

$$\begin{aligned} 0 \geq & -\frac{2}{h_{11}}\sum_{i\geq 2}\tilde{F}^{1i,i1}h_{11i}^2 - \tilde{F}^{ii}h_{ii}^2 - Ch_{11} - \frac{C}{u-a} \\ & - \frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} + \frac{u\tilde{F}^{ii}h_{ii}^2}{u-a} + \frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} + A\sum \tilde{F}^{ii} - CA, \end{aligned} \tag{3.13}$$

which is just the inequality (3.4). ■

Next, we deal with the bad term $-Ch_{11}$.

Lemma 3.2. *Suppose $p \geq n/2$. If h_{11} is large enough, we have*

$$Ch_{11} \leq \frac{a\tilde{F}^{ii}h_{ii}^2}{2(u-a)} + \frac{A}{2}\sum \tilde{F}^{ii} \tag{3.14}$$

for sufficiently large A.

Proof. Note that

$$\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n > 0.$$

We divide the proof into two cases.

Case 1. Suppose $\kappa_n \leq -\delta\kappa_1$, where $\delta > 0$ is a small constant to be determined later. By Lemma 2.2 we see $\tilde{F}^{nn} \geq \theta \sum \tilde{F}^{ii} \geq \theta p$. We then obtain that

$$\tilde{F}^{nn}h_{nn}^2 \geq \delta^2\kappa_1^2\tilde{F}^{nn} \geq \theta p\delta^2\kappa_1^2.$$

Therefore, for sufficiently large κ_1, we have

$$Ch_{11} \leq \frac{a\tilde{F}^{nn}h_{nn}^2}{2(u-a)}.$$

Case 2. Now $\kappa_n \geq -\delta\kappa_1$. We further divide this case into two cases.

Subcase 2.1. Suppose $\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n < \delta/\kappa_1$. Since

$$F^{nn} \geq \frac{F(\kappa)}{\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n},$$

we see that

$$\tilde{F}^{nn} = \frac{1}{C_n^p}F^{1/C_n^p - 1}F^{nn} \geq \frac{F^{1/C_n^p}}{C_n^p}\frac{\kappa_1}{\delta} = \frac{\tilde{f}}{C_n^p}\frac{\kappa_1}{\delta}.$$

Choosing δ sufficiently small, we obtain that

$$C\kappa_1 \leq \tilde{F}^{nn}.$$

Subcase 2.2. Suppose $\kappa_{n-p+1}+\kappa_{n-p+2}+\cdots+\kappa_n \geq \delta/\kappa_1$. For a fixed $(p-1)$-tuple $2 \leq i_1 < \cdots < i_{p-1} \leq n$, we have

$$\kappa_1+\kappa_{i_1}+\cdots+\kappa_{i_{p-1}} \geq (1-(p-1)\delta)\kappa_1.$$

Hence, we have

$$\begin{aligned} F^{nn} &\geq \prod_{2\leq i_1<\cdots<i_{p-1}\leq n} (\kappa_1+\kappa_{i_1}+\cdots+\kappa_{i_{p-1}}) \\ &\quad \times \prod_{\substack{2\leq i_1<\cdots<i_p\leq n \\ (i_1,\dots,i_p)\neq(n-p+1,\dots,n)}} (\kappa_{i_1}+\kappa_{i_2}+\cdots+\kappa_{i_p}) \\ &\geq [(1-(p-1)\delta)\kappa_1]^{C_{n-1}^{p-1}} \left[\frac{\delta}{\kappa_1}\right]^{C_{n-1}^{p}-1}. \end{aligned}$$

For $p \geq n/2$, a direct calculation shows that

$$C_{n-1}^{p-1}-C_{n-1}^{p} = \frac{(n-1)\cdots(n-p+1)}{(p-1)!}\left(1-\frac{n-p}{p}\right) \geq 0.$$

Therefore, we obtain

$$F^{nn} \geq c_\delta \kappa_1,$$

where $c_\delta = [(1-(p-1)\delta)]^{C_{n-1}^{p-1}} \delta^{C_{n-1}^{p}-1}$. It then follows that, for sufficiently large A,

$$C\kappa_1 \leq \frac{A}{2}\tilde{F}^{nn}. \qquad \blacksquare$$

By the above lemma, (3.4) becomes

$$\begin{aligned} 0 \geq &-\frac{2}{h_{11}}\sum_{i\geq 2}\tilde{F}^{1i,i1}h_{11i}^2 - \frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} + \frac{a\tilde{F}^{ii}h_{ii}^2}{2(u-a)} \\ &+ \frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} + \frac{A}{2}\sum\tilde{F}^{ii} - CA. \end{aligned} \tag{3.15}$$

Lemma 3.3. *For κ_1 sufficiently large, we have*

$$|\kappa_{n-p+1}|,\dots,|\kappa_n| \leq CA.$$

Proof. By the critical equation (3.1) and the Cauchy–Schwarz inequality, we have

$$-\frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} \geq -(1+\varepsilon)\frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} - \left(1+\frac{1}{\varepsilon}\right)A^2\tilde{F}^{ii}\langle X, e_i\rangle^2. \tag{3.16}$$

From (3.15) and $-\tilde{F}^{1i,i1} \geq 0$, we see that

$$0 \geq \frac{a\tilde{F}^{ii}h_{ii}^2}{2(u-a)} - \frac{\varepsilon\tilde{F}^{ii}u_i^2}{(u-a)^2} - \frac{CA^2}{\varepsilon}\sum\tilde{F}^{ii} - CA. \tag{3.17}$$

Using $u_i = h_{ii}\langle X, e_i\rangle$ and choosing ε sufficiently small, we obtain from (3.17) that

$$0 \geq \frac{a\tilde{F}^{ii}h_{ii}^2}{4(u-a)} - \frac{CA^2}{\varepsilon}\sum \tilde{F}^{ii}, \tag{3.18}$$

where we also used $\sum \tilde{F}^{ii} \geq p$. By Lemma 2.2, we now arrive at

$$0 \geq \frac{a\theta}{4(u-a)}\Big(\sum \tilde{F}^{ii}\Big)\Big(\sum_{i\geq n-p+1} h_{ii}^2\Big) - \frac{CA^2}{\varepsilon}\sum \tilde{F}^{ii}, \tag{3.19}$$

which implies that

$$\sum_{i\geq n-p+1} h_{ii}^2 \leq CA^2.$$

■

Lemma 3.4. *Given $1 > \delta > 0$, there is an $\varepsilon = \varepsilon(p, \delta) > 0$ such that*

$$-2F^{1i,i1} + 2\frac{F^{11}}{\kappa_1} \geq (1+\delta)\frac{F^{ii}}{\kappa_1}, \quad i = 2, 3, \cdots n,$$

for κ_1 sufficiently large.

Proof. Recall that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. By the formula

$$F^{1i,i1} = \frac{F^{11} - F^{ii}}{\kappa_1 - \kappa_i},$$

we see that

$$\frac{F^{ii}}{\kappa_1} = \frac{\kappa_1 - \kappa_i}{\kappa_1}(-F^{1i,i1}) + \frac{F^{11}}{\kappa_1}.$$

Since $\kappa_i > 0$ for $i \leq n-p+1$, we obtain

$$\frac{F^{ii}}{\kappa_1} \leq -F^{1i,i1} + \frac{F^{11}}{\kappa_1}, \quad \text{for } i = 2, \ldots, n-p+1.$$

By Lemma 3.3, we can assume that $|\kappa_i| \leq \epsilon\kappa_1$ for $i \geq n-p+2$ for sufficiently small ϵ and large κ_1. Hence, we have

$$\frac{F^{ii}}{\kappa_1} \leq -(1+\epsilon)F^{1i,i1} + \frac{F^{11}}{\kappa_1}, \text{ for } i = n-p+2, \ldots, n.$$

By the above two inequalities, we get the desired inequality. ■

By Lemma 3.4, (3.15) becomes

$$\begin{aligned} 0 \geq &\sum_{i\geq 2}\frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} - \frac{\tilde{F}^{ii}h_{11i}^2}{h_{11}^2} - 2\sum_{i\geq 2}\frac{\tilde{F}^{11}h_{11i}^2}{h_{11}^2} + \frac{a\tilde{F}^{ii}h_{ii}^2}{2(u-a)} \\ &+ \frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} + \frac{A}{2}\sum \tilde{F}^{ii} - CA. \end{aligned} \tag{3.20}$$

By the critical equation (3.1), the Cauchy–Schwarz inequality and $u_i = h_{ii}\langle X, e_i\rangle$, we see that

$$-\frac{\tilde{F}^{11}h_{111}^2}{h_{11}^2} \geq -(1+\varepsilon)\frac{\tilde{F}^{11}u_1^2}{(u-a)^2} - \left(1+\frac{1}{\varepsilon}\right)A^2\tilde{F}^{11}\langle X, e_1\rangle^2 \tag{3.21}$$
$$\geq -\frac{\tilde{F}^{11}u_1^2}{(u-a)^2} - C\varepsilon\frac{\tilde{F}^{11}h_{11}^2}{(u-a)^2} - \frac{CA^2}{\varepsilon}\tilde{F}^{11}$$

and

$$2\sum_{i\geq 2}\frac{\tilde{F}^{11}h_{11i}^2}{h_{11}^2} \leq C\sum_{i\geq 2}\tilde{F}^{11}h_{ii}^2 + CA^2F^{11}.$$

Note that

$$F^{11} = \sum_{1\notin\{i_1,\ldots,i_{p-1}\}}\frac{F(\kappa)}{\kappa_1+\kappa_{i_1}+\cdots+\kappa_{i_{p-1}}} \leq \frac{C}{\kappa_1}.$$

Hence, by Lemma 3.2, we obtain

$$2\sum_{i\geq 2}\frac{\tilde{F}^{11}h_{11i}^2}{h_{11}^2} \leq \frac{a\tilde{F}^{ii}h_{ii}^2}{4(u-a)} + \frac{A}{4}\sum\tilde{F}^{ii} + CA^2\tilde{F}^{11}. \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20), we have

$$0 \geq \frac{a\tilde{F}^{ii}h_{ii}^2}{4(u-a)} - \frac{\tilde{F}^{11}u_1^2}{(u-a)^2} - C\varepsilon\frac{\tilde{F}^{11}h_{11}^2}{(u-a)^2} - \frac{CA^2}{\varepsilon}\tilde{F}^{11} \tag{3.23}$$
$$+\frac{\tilde{F}^{ii}u_i^2}{(u-a)^2} + \frac{A}{4}\sum\tilde{F}^{ii} - CA^2\tilde{F}^{11} - CA.$$

Choosing ε sufficiently small and assuming h_{11} sufficiently large, we derive that

$$0 \geq \frac{a\tilde{F}^{ii}h_{ii}^2}{8(u-a)} + \frac{A}{4}\sum\tilde{F}^{ii} - CA. \tag{3.24}$$

It then follows that

$$\sum\tilde{F}^{ii} \leq C.$$

Next we prove that under this condition, one have

$$\tilde{F}^{11} \geq \frac{1}{C}.$$

Since $\sum F^{ii} \leq C$, in particular we have

$$\frac{F(\kappa)}{\kappa_{n-p+1}+\cdots+\kappa_n} \leq C.$$

This implies that

$$\kappa_{n-p+1}+\cdots+\kappa_n \geq \frac{1}{C},$$

where C also depends on $\inf f$. This yields

$$\tilde{F}^{ii} \geq \frac{1}{C}, \quad \forall 1 \leq i \leq n.$$

Substituting the above inequality into (3.24), we obtain

$$0 \geq \frac{h_{11}^2}{C} - CA,$$

from which we can derive an upper bound for h_{11}. Theorem 1.1 is proved.

4. Gradient estimates

Before we apply the continuity method to obtain a solution to equation (1.1), we need to derive a C^1 estimate for the equation. We show that there exists a positive constant C depending on n, p, $\inf \rho$, $\sup \rho$, $\inf f$ and $|f|_{C^1}$ such that

$$|\nabla \rho| \leq C,$$

where ∇ denotes the connection on $\mathbb{S}^n$. Note that

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla \rho|^2}}.$$

We only need to derive a positive lower bound of u. As in [16], we consider the following quantity:

$$w = -\log u + \gamma(|X|^2),$$

where the function $\gamma(\cdot)$ will be determined later. Suppose the maximum of w is achieved at $X_0 \in M$. If at X_0, X is parallel to ν, we have

$$u = \langle X, \nu \rangle = \rho \geq \inf_M \rho,$$

which gives a lower bound since ρ is assumed to have a positive lower bound. If X is not parallel to ν at X_0, we can choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ such that

$$\langle X, e_1 \rangle \neq 0 \quad \text{and} \quad \langle X, e_i \rangle = 0 \quad \text{for } i \geq 2.$$

Differentiate w at X_0 to obtain that

$$0 = w_i = -\frac{u_i}{u} + 2\gamma' \langle X, e_i \rangle = -\frac{h_{i1} \langle X, e_1 \rangle}{u} + 2\gamma' \langle X, e_i \rangle, \tag{4.1}$$

where in the last equality we used the Weingarten equation. Hence, we have

$$h_{11} = 2\gamma' u \quad \text{and} \quad h_{1i} = 0 \quad \text{for } i \geq 2.$$

Without loss of generality, we can assume $\{h_{ij}\}$ is diagonal at X_0. Differentiating w at X_0 a second time and contracting with $\{\tilde{F}^{ij}\}$, we obtain that

$$0 \geq \tilde{F}^{ii}\Big(-\frac{u_{ii}}{u} + \frac{u_i^2}{u^2} + \gamma''(|X|^2)_i^2 + \gamma'(|X|^2)_{ii}\Big). \tag{4.2}$$

Combining (4.1) with the above inequality, we arrive at

$$0 \geq -\frac{\tilde{F}^{ii}u_{ii}}{u} + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + \gamma'\tilde{F}^{ii}(|X|^2)_{ii}. \tag{4.3}$$

By (3.10), we have

$$\tilde{F}^{ii}u_{ii} = \langle X, e_1\rangle((d_X\tilde{f})(e_1) + h_{11}(d_\nu\tilde{f})(e_1)) - u\tilde{F}^{ii}h_{ii}^2 + \tilde{f}. \tag{4.4}$$

Also, we have

$$\tilde{F}^{ii}(|X|^2)_{ii} = 2\sum\tilde{F}^{ii} - 2u\tilde{f},$$

where we used $\tilde{F}^{ij}h_{ij} = \tilde{f}$. Recall that $h_{11} = 2\gamma' u$. Substituting the above two equalities into (4.3) we get

$$\begin{aligned} 0 \geq\ & -\frac{1}{u}\big(\langle X, e_1\rangle(d_X\tilde{f})(e_1) + \tilde{f}\big) - 2\langle X, e_1\rangle\gamma'(d_\nu\tilde{f})(e_1) \\ & + \tilde{F}^{ii}h_{ii}^2 + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + 2\gamma'\Big(\sum\tilde{F}^{ii} - u\tilde{f}\Big). \end{aligned} \tag{4.5}$$

At X_0, we see that $X = \langle X, e_1\rangle e_1 + \langle X, \nu\rangle\nu$. It then follows that

$$(d_X\tilde{f})(X) = \langle X, e_1\rangle(d_X\tilde{f})(e_1) + \langle X, \nu\rangle(d_X\tilde{f})(\nu).$$

From (1.4), we see that

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial\rho}\big(\rho^{C_n^p} f(X, \nu)\big) = \frac{\partial}{\partial\rho}\big(\rho^{C_n^p}\tilde{f}^{C_n^p}(X, \nu)\big) \\ &= C_n^p(\rho\tilde{f})^{C_n^p - 1}\big(\tilde{f} + (d_X\tilde{f})(X)\big) \\ &= C_n^p(\rho\tilde{f})^{C_n^p - 1}\big(\tilde{f} + \langle X, e_1\rangle(d_X\tilde{f})(e_1) + \langle X, \nu\rangle(d_X\tilde{f})(\nu)\big). \end{aligned}$$

We therefore obtain

$$-(\tilde{f} + \langle X, e_1\rangle(d_X\tilde{f})(e_1)) \geq \langle X, \nu\rangle(d_X\tilde{f})(\nu) = u(d_X\tilde{f})(\nu).$$

Substituting this into (4.5) we obtain

$$\begin{aligned} 0 \geq\ & (d_X\tilde{f})(\nu) - 2\langle X, e_1\rangle\gamma'(d_\nu\tilde{f})(e_1) + \tilde{F}^{ii}h_{ii}^2 \\ & + 4(\gamma'^2 + \gamma'')F^{11}\langle X, e_1\rangle^2 + 2\gamma'\Big(\sum\tilde{F}^{ii} - u\tilde{f}\Big). \end{aligned} \tag{4.6}$$

Now we choose $\gamma(t) = \alpha/t$, where α is a large constant to be determined later. Recall that $h_{11} = 2\gamma' u$ at X_0, which implies that $h_{11}(X_0) < 0$. This means that $h_{11} \in \{\kappa_{n-p+2}, \kappa_{n-p+3}, \ldots, \kappa_n\}$ and therefore by Lemma 2.2,

$$F^{11} \geq \theta\sum F^{ii}.$$

Similar to [7], we can assume $\langle X, e_1\rangle^2 \geq \frac{1}{2}\inf_M \rho^2$. Now we arrive at

$$0 \geq \Big(\frac{\alpha^2}{C} - C\alpha\Big)\sum \tilde{F}^{ii} - C\alpha. \tag{4.7}$$

Choosing α sufficiently large, we obtain a contradiction. Therefore, X_0 is parallel to ν, and u has a positive lower bound.

5. Existence of a solution

We use the continuity method as in [4] to prove Theorem 1.2. Consider the following family of functions:

$$f^t(X,\nu) = tf(X,\nu) + (1-t)p^{C_n^p}\Big[\frac{1}{|X|^{C_n^p}} + \varepsilon\Big(\frac{1}{|X|^{C_n^p}} - 1\Big)\Big],$$

where ε is a small positive constant such that

$$\min_{r_1\leq\rho\leq r_2}\Big[\frac{1}{\rho^{C_n^p}} + \varepsilon\Big(\frac{1}{\rho^{C_n^p}} - 1\Big)\Big] \geq c_0 > 0,$$

for some positive constant c_0. It is easy to see that $f^t(X,\nu)$ satisfies (1.3) and (1.4) with strict inequalities for $0 \leq t < 1$.

Let M_t be the solution of the equation

$$F(\kappa) = f^t(X_t, \nu_t),$$

where X_t and ν_t are position vector and unit outer normal of M_t respectively. Clearly, when $t = 0$, we have $M_0 = \mathbb{S}^n$ and $X_0 = x$. For $t \in (0,1)$, suppose $\rho_t = |X_t|$ attains its maximum at the point x_0. At this point, by (2.1), we have

$$g_{ij} = \rho_t^2\delta_{ij} \quad \text{and} \quad h_{ij} = -(\rho_t)_{ij} + \rho_t\delta_{ij} \geq \rho_t\delta_{ij}$$

under a smooth local orthonormal frame on $\mathbb{S}^n$. Then we have

$$F(\kappa) \geq F\Big(\frac{1}{\rho_t}(1,\dots,1)\Big) = \frac{p^{C_n^p}}{\rho_t^{C_n^p}}.$$

On the other hand, at x_0, the unit outer normal ν_t is parallel to X_t. If $\rho_t(x_0) = r_2$, we obtain

$$\frac{p^{C_n^p}}{r_2{}^{C_n^p}} \leq F(\kappa) = f^t(X_t,\nu_t) < \frac{p^{C_n^p}}{r_2{}^{C_n^p}},$$

which is a contradiction. So we have $\sup_{M_t}\rho_t \leq r_2$. Similarly argument at the minimum point of ρ_t gives that $\inf_{M_t}\rho_t \geq r_1$ on M_t. Hence, C^0 estimate follows. Combining our C^1 estimate, our C^2 estimate, the Evans–Krylov theorem with the argument in [4], we get the existence and uniqueness of solution to equation (1.1). Theorem 1.2 is proved.

6. Proof of Theorem 1.3

By Lemmas 3.1, 3.2, 3.3 and 3.4, one can prove Theorem 1.3. For completeness, we include an outline here.

Proof. We consider the following function:

$$G(x,\xi) = \log u_{\xi\xi} + \frac{a}{2}|\nabla u|^2 + \frac{A}{2}|x|^2 + \beta \log(v-u),$$

where a, A and β are constants to be determined later. Suppose that G achieves its maximum at (x_0,ξ_0). Around x_0, we choose a coordinate system such that $\xi_0 = e_1$ and $u_{ij}(x_0)$ is diagonal such that

$$u_{11} \geq u_{22} \geq \cdots \geq u_{nn} \quad \text{at } x_0.$$

This can be done as in [12]. Thus, the new function defined by

$$\hat{G}(x) = \log u_{11} + \frac{a}{2}|\nabla u|^2 + \frac{A}{2}|x|^2 + \beta \log(v-u)$$

also attains its maximum at x_0. Differentiate it once to obtain

$$0 = \frac{u_{11i}}{u_{11}} + a u_i u_{ii} + A x_i + \frac{\beta (v-u)_i}{v-u}. \tag{6.1}$$

Differentiating it twice and by similar computations as Lemma 3.1 and Lemma 3.2, we arrive at

$$\begin{aligned} 0 \geq & -\frac{2}{u_{11}} \sum_{i \geq 2} \tilde{F}^{1i,i1} u_{11i}^2 - \frac{\tilde{F}^{ii} u_{11i}^2}{u_{11}^2} + \frac{a \tilde{F}^{ii} u_{ii}^2}{2} \\ & + \frac{A}{2} \sum \tilde{F}^{ii} - CA - \frac{\beta \tilde{F}^{ii} (v-u)_i^2}{(v-u)^2} - \frac{C\beta}{v-u}. \end{aligned} \tag{6.2}$$

We remark that in the above inequality we used

$$\begin{aligned} \sum_k F^{kk} v_{kk} &= \sum_k \sum_{k \in \{i_1,\ldots,i_p\}} \frac{F(\lambda)}{\lambda_{i_1} + \cdots + \lambda_{i_p}} v_{kk} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \frac{F(\lambda)}{\lambda_{i_1} + \cdots + \lambda_{i_p}} (v_{i_1 i_1} + \cdots + v_{i_p i_p}), \end{aligned}$$

which is nonnegative since v is p-plurisubharmonic. Using the same argument as in Lemma 3.3, we obtain

$$|u_{ii}| \leq \frac{C}{v-u} \quad \text{for } i \geq n-p+1,$$

where C depends on a, A and β.

By (6.1) and the Cauchy–Schwarz inequality, we have

$$-\frac{\tilde{F}^{11}u_{111}^2}{u_{11}^2} \geq -Ca^2\tilde{F}^{11}u_{11}^2 - CA^2\tilde{F}^{11} - \frac{C\beta^2\tilde{F}^{11}}{(v-u)^2}$$

and

$$-\sum_{i\geq 2}\frac{\beta\tilde{F}^{ii}(v-u)_i^2}{(v-u)^2} \geq -\frac{3}{\beta}\sum_{i\geq 2}\frac{\tilde{F}^{ii}u_{11i}^2}{u_{11}^2} - \frac{Ca^2}{\beta}\sum_{i\geq 2}\tilde{F}^{ii}u_{ii}^2 - \frac{CA^2}{\beta}\sum_{i\geq 2}\tilde{F}^{ii}.$$

Substituting the above two inequalities into (6.2), we have

$$\begin{aligned} 0 \geq & -\frac{2}{u_{11}}\sum_{i\geq 2}\tilde{F}^{1i,i1}u_{11i}^2 - \Big(1+\frac{3}{\beta}\Big)\sum_{i\geq 2}\frac{\tilde{F}^{ii}u_{11i}^2}{u_{11}^2} + \Big(\frac{a}{2}-\frac{Ca^2}{\beta}\Big)\tilde{F}^{ii}u_{ii}^2 \\ & + \Big(\frac{A}{2}-\frac{CA^2}{\beta}\Big)\sum\tilde{F}^{ii} - Ca^2\tilde{F}^{11}u_{11}^2 - CA^2\tilde{F}^{11} - \frac{C\beta^2\tilde{F}^{11}}{(v-u)^2} \\ & - CA - \frac{2\beta\tilde{F}^{11}u_1^2}{(v-u)^2} - \frac{C\beta}{v-u}. \end{aligned} \tag{6.3}$$

By Lemma 3.2, similar to (3.22), we can get

$$2\sum_{i\geq 2}\tilde{F}^{11}\frac{u_{11i}^2}{u_{11}^2} \leq \frac{a}{8}\tilde{F}^{ii}u_{ii}^2 + \frac{A}{8}\sum\tilde{F}^{ii} + CA^2\tilde{F}^{11} + \frac{C\beta^2}{(v-u)^2}\tilde{F}^{11} \tag{6.4}$$

for sufficiently large $(v-u)u_{11}$ and A. Combining Lemma 3.4 with (6.4) and choosing a sufficiently small and β sufficiently large such that $\delta \geq 3/\beta$, we get from (6.3) that

$$0 \geq \frac{a}{4}\tilde{F}^{11}u_{11}^2 - \frac{C}{(v-u)^2}\tilde{F}^{11} - CA - \frac{C}{v-u} \geq \frac{a}{8}\tilde{F}^{11}u_{11}^2 - \frac{C}{v-u}, \tag{6.5}$$

where in the second inequality we assumed $(v-u)u_{11}$ is large enough.

By Lemma 2.2 (1), we have that $\tilde{F}^{11}u_{11} \geq c_0$, where $c_0 > 0$ depends on $\inf\tilde{f}$. Then, from (6.5) we obtain

$$(v-u)u_{11} \leq C,$$

which implies the estimate (1.6). ∎

A. Appendix

In this appendix, we include a proof of Lemma 2.2. For $A \in P_p$, recall the notations

$$F(A) := F(\lambda(A)) = \prod_{1\leq i_1<\cdots<i_p\leq n}(\lambda_{i_1}+\cdots+\lambda_{i_p}) \quad \text{and} \quad \tilde{F} = F^{1/C_n^p},$$

where $\lambda(A) = (\lambda_1,\ldots,\lambda_n)$ are the eigenvalues of A and P_p is defined in Definition 2.1. Suppose that the diagonal matrix $A = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ belongs to P_p and $\lambda_1 \geq \cdots \geq \lambda_n$.

Lemma A.1. $\tilde{F}^{11}(A)\lambda_1 \geq \frac{1}{n}\tilde{F}(A)$.

Proof. We have that

$$\begin{aligned}\tilde{F}^{11}(A) &= \frac{1}{C_n^p}\,[F(A)]^{1/C_n^p-1}\sum_{1\in\{i_1,\ldots,i_p\}}\frac{F(A)}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\\ &\geq \frac{1}{C_n^p}\,[F(A)]^{1/C_n^p-1}\,C_{n-1}^{p-1}\,\frac{F(A)}{p\lambda_1} = \frac{1}{n\lambda_1}\,[F(A)]^{1/C_n^p},\end{aligned}$$

where in the inequality we used $\lambda_{i_1}+\cdots+\lambda_{i_p} \leq p\lambda_1$. ∎

Lemma A.2. $\sum_{k=1}^n \tilde{F}^{kk}(A) \geq p$.

Proof. We have that

$$\begin{aligned}\sum_{k=1}^n \tilde{F}^{kk}(A) &= \frac{1}{C_n^p}\,[F(A)]^{1/C_n^p-1}\sum_{k=1}^n\sum_{k\in\{i_1,\ldots,i_p\}}\frac{F(A)}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\\ &= \frac{nC_{n-1}^{p-1}}{[C_n^p]^2}\,[F(A)]^{1/C_n^p}\sum_{1\leq i_1<\cdots<i_p\leq n}\frac{1}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\\ &\geq \frac{nC_{n-1}^{p-1}}{[C_n^p]^2}\,[F(A)]^{1/C_n^p}\,\frac{C_n^p}{[F(A)]^{1/C_n^p}} = p,\end{aligned}$$

where the inequality of arithmetic and geometric means was used in the inequality. ∎

Lemma A.3. $\sum_{k=1}^n F^{kk}(A)\lambda_k = C_n^p F(A)$.

Proof. Observe that

$$\begin{aligned}\sum_{k=1}^n F^{kk}(A)\,\lambda_k &= F(A)\sum_{k=1}^n\sum_{k\in\{i_1,\ldots,i_p\}}\frac{\lambda_k}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\\ &= F(A)\,\frac{nC_{n-1}^{p-1}}{p} = C_n^p F(A).\end{aligned}$$ ∎

Lemma A.4. $\sum_{k=1}^n \tilde{F}^{kk}(A)\lambda_k = \tilde{F}(A)$.

Proof. Observe that

$$\sum_{k=1}^n \tilde{F}^{kk}(A)\,\lambda_k = \frac{1}{C_n^p}\,[F(A)]^{1/C_n^p-1}\sum_{k=1}^n F^{kk}(A)\lambda_k = \tilde{F}(A).$$ ∎

Lemma A.5. *There is a constant $\theta = \theta(n,p)$ such that, for all $j \geq n-p+1$,*

$$F^{jj}(A) \geq \theta\sum_{i=1}^n F^{ii}(A).$$

Proof. Note that, for $j \geq n-p+1$,

$$F^{jj}(A) \geq \frac{F(A)}{\lambda_{n-p+1}+\cdots+\lambda_n} \geq \frac{1}{C_n^p} \sum_{1\leq i_1<\cdots<i_p\leq n} \frac{F(A)}{\lambda_{i_1}+\cdots+\lambda_{i_p}},$$

and

$$\begin{aligned}\sum_{k=1}^n F^{kk}(A) &= \sum_{k=1}^n \sum_{k\in\{i_1,\ldots,i_p\}} \frac{F(A)}{\lambda_{i_1}+\cdots+\lambda_{i_p}} \\ &= \frac{nC_{n-1}^{p-1}}{C_n^p} \sum_{1\leq i_1<\cdots<i_p\leq n} \frac{F(A)}{\lambda_{i_1}+\cdots+\lambda_{i_p}}.\end{aligned}$$

Thus the desired inequality is proved. ■

Acknowledgements. We would like to thank the anonymous referees for helpful comments and especially for pointing out a mistake.

Funding. This work was partially supported by the National Natural Science Foundation of China, No. 11801405.

References

[1] Bakel'man, I. J. and Kantor, B. E.: Existence of a hypersurface homeomorphic to the sphere in Euclidean space with a given mean curvature. In *Geometry and topology, no. 1*, pp. 3–10 (Russian). Leningrad. Gos. Ped. Inst. im. Gercena, Leningrad, 1974.

[2] Ball, J. M.: Differentiability properties of symmetric and isotropic functions. *Duke Math. J.* **51** (1984), no. 3, 699–728.

[3] Caffarelli, L., Nirenberg, L. and Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge–Ampère equation. *Comm. Pure Appl. Math.* **37** (1984), no. 3, 369–402.

[4] Caffarelli, L., Nirenberg, L. and Spruck, J.: Nonlinear second order elliptic equations. IV. Starshaped compact Weingarten hypersurfaces. In *Current topics in partial differential equations*, pp. 1–26. Kinokuniya, Tokyo, 1986.

[5] Chen, D., Li, H. and Wang, Z.: Starshaped compact hypersurfaces with prescribed Weingarten curvature in warped product manifolds. *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, article no. 42, 26 pp.

[6] Chou, K.-S. and Wang, X.-J.: A variational theory of the Hessian equation. *Comm. Pure Appl. Math.* **54** (2001), no. 9, 1029–1064.

[7] Chu, J. and Jiao, H.: Curvature estimates for a class of Hessian type equations. *Calc. Var. Partial Differential Equations* **60** (2021), no. 3, article no. 90, 18 pp.

[8] Dinew, S.: Interior estimates for p-plurisubharmonic functions. Preprint 2020, arXiv: 2006.12979.

[9] Fu, J., Wang, Z. and Wu, D.: Form-type Calabi–Yau equations. *Math. Res. Lett.* **17** (2010), no. 5, 887–903.

[10] Fu, J., Wang, Z. and Wu, D.: Form-type equations on Kähler manifolds of nonnegative orthogonal bisectional curvature. *Calc. Var. Partial Differential Equations* **52** (2015), no. 1-2, 327–344.

[11] Gerhardt, C.: Hypersurfaces of prescribed Weingarten curvature. *Math. Z.* **224** (1997), no. 2, 167–194.

[12] Guan, B.: Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. *Duke Math. J.* **163** (2014), no. 8, 1491–1524.

[13] Guan, B. and Guan, P.: Convex hypersurfaces of prescribed curvatures. *Ann. of Math. (2)* **156** (2002), no. 2, 655–673.

[14] Guan, B. and Jiao, H.: Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds. *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 2693–2712.

[15] Guan, P., Li, J. and Li, Y.: Hypersurfaces of prescribed curvature measure. *Duke Math. J.* **161** (2012), no. 10, 1927–1942.

[16] Guan, P., Lin, C. and Ma, X.-N.: The existence of convex body with prescribed curvature measures. *Int. Math. Res. Not. IMRN* (2009), no. 11, 1947–1975.

[17] Guan, P., Ren, C. and Wang, Z.: Global C^2-estimates for convex solutions of curvature equations. *Comm. Pure Appl. Math.* **68** (2015), no. 8, 1287–1325.

[18] Guan, P. and Zhang, X.: A class of curvature type equations. *Pure Appl. Math. Q.* **17** (2021), no. 3, 865–907.

[19] Harvey, F. R. and Lawson, H. B., Jr.: Geometric plurisubharmonicity and convexity: an introduction. *Adv. Math.* **230** (2012), no. 4-6, 2428–2456.

[20] Harvey, F. R. and Lawson, H. B., Jr.: p-convexity, p-plurisubharmonicity and the Levi problem. *Indiana Univ. Math. J.* **62** (2013), no. 1, 149–169.

[21] Ivochkina, N. M.: Solution of the Dirichlet problem for equations of mth order curvature. *Mat. Sb.* **180** (1989), no. 7, 867–887, 991; translation in *Math. USSR-Sb.* **67** (1990), no. 2, 317–339.

[22] Ivochkina, N. M.: The Dirichlet problem for the curvature equation of order m. *Algebra i Analiz* **2** (1990), no. 3, 192–217; translation in *Leningrad Math. J.* **2** (1991), no. 3, 631–654..

[23] Li, Y. Y.: Interior gradient estimates for solutions of certain fully nonlinear elliptic equations. *J. Differential Equations* **90** (1991), no. 1, 172–185.

[24] Oliker, V. I.: Hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge–Ampère type. *Comm. Partial Differential Equations* **9** (1984), no. 8, 807–838.

[25] Ren, C. and Wang, Z.: On the curvature estimates for Hessian equations. *Amer. J. Math.* **141** (2019), no. 5, 1281–1315.

[26] Ren, C. and Wang, Z.: Notes on the curvature estimates for Hessian equations. Preprint 2020, arXiv: 2003.14234v1.

[27] Ren, C., Wang, Z. and Xiao, L.: The prescribed curvature problem for entire hypersurfaces in Minkowski space. Preprint 2020, arXiv: 2007.04493v1.

[28] Sha, J.-P.: p-convex Riemannian manifolds. *Invent. Math.* **83** (1986), no. 3, 437–447.

[29] Sha, J.-P.: Handlebodies and p-convexity. *J. Differential Geom.* **25** (1987), no. 3, 353–361.

[30] Székelyhidi, G., Tosatti, V. and Weinkove, B.: Gauduchon metrics with prescribed volume form. *Acta Math.* **219** (2017), no. 1, 181–211.

[31] Tosatti, V. and Weinkove, B.: The Monge–Ampère equation for $(n-1)$-plurisubharmonic functions on a compact Kähler manifold. *J. Amer. Math. Soc.* **30** (2017), no. 2, 311–346.

[32] Tosatti, V. and Weinkove, B.: Hermitian metrics, $(n-1, n-1)$ forms and Monge–Ampère equations. *J. Reine Angew. Math.* **755** (2019), 67–101.

[33] Treibergs, A. E. and Wei, S. W.: Embedded hyperspheres with prescribed mean curvature. *J. Differential Geom.* **18** (1983), no. 3, 513–521.

[34] Wang, Z.: The global curvature estimate for the $n-2$ Hessian equation. Preprint 2020, arXiv: 2002.08702.

[35] Wang, Z. and Xiao, L.: Entire spacelike hypersurfaces with constant σ_k curvature in Minkowski space. *Math. Ann.* **382** (2022), no. 3-4, 1279–1322.

[36] Wu, H.: Manifolds of partially positive curvature. *Indiana Univ. Math. J.* **36** (1987), no. 3, 525–548.

Received August 6, 2021; revised March 15, 2022. Published online April 7, 2022.

Weisong Dong
School of Mathematics, Tianjin University, 135 Yaguan Road, Jinnan, Tianjin, P. R. China, 300354; dr.dong@tju.edu.cn