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Equivalence of critical and subcritical sharp Trudinger–Moser inequalities in fractional dimensions and extremal functions

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Abstract. We establish critical and subcritical sharp Trudinger–Moser inequalities for fractional dimensions on the whole space. Moreover, we obtain asymptotic lower and upper bounds for the fractional subcritical Trudinger–Moser supremum from which we can prove the equivalence between critical and subcritical inequalities. Using this equivalence, we prove the existence of maximizers for both the subcritical and critical associated extremal problems. As a by-product of this development, we can explicitly calculate the value of the critical supremum in some special situations.

1. Introduction

Let $0 < R \leq \infty, \alpha, \theta \geq 0$ and $q \geq 1$ be real numbers. Let $L^q_\theta = L^q_\theta(0, R)$ be the weighted Lebesgue space defined as the set of all measurable functions u on $(0, R)$ such that

$$\|u\|_{L^q_\theta} = \begin{cases} \left(\int_0^R |u(r)|^q \,\mathrm{d}\lambda_\theta\right)^{1/q} < \infty & \text{if } 1 \leq q < \infty, \\ \operatorname*{ess\,sup}_{0<r<R} |u(r)| < \infty & \text{if } q = \infty, \end{cases}$$

where we are denoting

$$\int_0^R f(r)\,\mathrm{d}\lambda_\theta = \omega_\theta \int_0^R f(r) r^\theta \mathrm{d}r, \quad 0 < R \leq \infty, \tag{1.1}$$

with ω_θ defined by

$$\omega_\theta = \frac{2\pi^{(\theta+1)/2}}{\Gamma((\theta+1)/2)}, \quad \text{with} \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t.$$

In the case that θ is a positive integer number, ω_θ agrees precisely with the known spherical volume element for Euclidean space $\mathbb{R}^{\theta+1}$. In fact, according to the formalism

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in [33], the integration of a radially symmetric function $f(r)$ in a $(\theta+1)$-dimensional fractional space is given by (1.1), when $R=\infty$. Integration over non-integer dimensional spaces is often used in the dimensional regularization method as a powerful tool to obtain results in statistical mechanics and quantum field theory [7,30,35]. For a deeper discussion on this subject, we suggest [36] and the references therein.

We emphasize that the Lebesgue spaces L^q_θ is also related with the classical Hardy inequality [20], see [12, 23] for more details. In addition, we can use L^q_θ-spaces to define Sobolev type spaces that are suitable to investigate a general class of differential operators which includes the p-Laplacian, $p\geq 2$, and k-Hessian operators in the radial form, see for instance [6, 17, 23] and references therein. Indeed, as observed by P. Clément et al. [6], if we consider $X_R = X^{1,p}_R(\alpha,\theta)$, $\alpha,\theta\geq 0$, $p>1$ and $0<R\leq\infty$, as the set of all locally absolutely continuous functions on the interval $(0,R)$ such that $\lim_{r\to R} u(r)=0$, $u\in L^p_\theta$ and $u'\in L^p_\alpha$, then X_R becomes a Banach space endowed with the norm

$$\|u\| = \left(\|u\|^p_{L^p_\theta} + \|u'\|^p_{L^p_\alpha}\right)^{1/p}.$$

Further, we can distinguish two special behaviors for the weighted Sobolev spaces X_R. Namely, *the Sobolev case*, when the condition

$$\alpha - p + 1 > 0 \tag{1.2}$$

holds, and the *Trudinger–Moser case* if

$$\alpha - p + 1 = 0. \tag{1.3}$$

In the Sobolev case (1.2), the value

$$p^* := p^*(\alpha, p, \nu) = \frac{(\nu+1)p}{\alpha - p + 1}$$

is the *critical exponent* for the embedding

$$X^{1,p}_R(\alpha,\theta) \hookrightarrow L^q_\nu.$$

Indeed, for the bounded situation $0<R<\infty$, one has the following continuous embedding:

$$X^{1,p}_R(\alpha,\theta) \hookrightarrow L^q_\nu, \quad \text{if } q\in(1,p^*] \text{ and } \min\{\theta,\nu\}\geq \alpha - p.$$

Moreover, in the strict case $q<p^*$, the embedding is also compact. In contrast, for the *Trudinger–Moser case* one has the compact embedding

$$X^{1,p}_R(\alpha,\theta) \hookrightarrow L^q_\nu, \quad \text{if } q\in(1,\infty) \text{ and } \nu\geq 0. \tag{1.4}$$

However, $X_R \hookrightarrow L^\infty_\nu$ does not hold, as one can see taking $u(r)=\ln(\ln(eR/r))$.

It is worth pointing out that the weighted Sobolev spaces X_R are employed by several authors to investigate existence of solutions for a large class of differential equations. We recommend [6, 8, 9, 12, 18, 19] for a general class of radial operators, and for the k-Hessian equation [11, 13, 15], and recently [14]. This paper deals with intrinsic properties of X_R, which are related with sharp variational inequalities. In this direction, let us first recall some previous results. Firstly, the embedding in (1.4) does not find its threshold in the weighted Lebesgue spaces L^q_ν, instead, in [10] it was proved a sharp inequality of the Trudinger–Moser type (see [29,34]) for X_R which gets embedded into an weighted Orlicz space determined by exponential growth. In fact, let us denote

$$\mu_{\alpha,\theta} = (\theta+1)\omega_\alpha^{1/\alpha} \quad \text{and} \quad |B_R|_\theta = \int_0^R \mathrm{d}\lambda_\theta.$$

Then, in [10] the authors proved the following.

Theorem A. *Let* $0 < R < \infty$*,* $\alpha \geq 1$*,* $\theta \geq 0$ *and* $p = \alpha + 1$ *be real numbers. Then,*

(i) *we have* $\exp(\mu|u|^{p/(p-1)}) \in L^1_\theta$*, for any* $\mu > 0$ *and* $u \in X^{1,p}_R(\alpha,\theta)$*;*

(ii) *there exists* $c > 0$ *depending only on* α*,* p *and* θ *such that*

$$\sup_{\|u'\|_{L^p_\alpha} \leq 1} \frac{1}{|B_R|_\theta} \int_0^R e^{\mu|u|^{p/(p-1)}} \,\mathrm{d}\lambda_\theta \quad \begin{cases} \leq c & \text{if } \mu \leq \mu_{\alpha,\theta}, \\ = \infty & \text{if } \mu > \mu_{\alpha,\theta}; \end{cases} \tag{1.5}$$

(iii) *the supremum in* (1.5) *is attained for all* $0 < \mu \leq \mu_{\alpha,\theta}$*.*

In this paper we are mainly interested in the unbounded case when $R = \infty$. Here, according to [9], for the *Sobolev case*, we also have the following continuous embedding:

$$X^{1,p}_\infty(\alpha,\theta) \hookrightarrow L^q_\theta \quad \text{if } q \in [p, p^*] \text{ and } \theta \geq \alpha - p. \tag{1.6}$$

Also, the embeddings (1.6) are compact under the strict conditions $\theta > \alpha - p$ and $p < q < p^*$. In the *Trudinger–Moser case*, it holds the continuous embeddings

$$X^{1,p}_\infty(\alpha,\theta) \hookrightarrow L^q_\theta \quad \text{for all } q \in [p,\infty), \tag{1.7}$$

which are compact in the strict case $q > p$.

We recall the following Trudinger–Moser type inequality of the scaling invariant form obtained in [10].

Theorem B. *Assume* $p \geq 2$*,* $\alpha = p - 1$ *and* $\theta \geq 0$*. For any* $\mu < \mu_{\alpha,\theta}$*, there exists a positive constant* $C_{p,\mu,\theta}$ *such that, for all* $u \in X^{1,p}_\infty(\alpha,\theta)$*,* $\|u'\|_{L^p_\alpha} \leq 1$*,*

$$\int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \leq C_{p,\mu,\theta}\|u\|^p_{L^p_\theta},$$

where

$$\varphi_p(t) = e^t - \sum_{k=0}^{k_0-1}\frac{t^k}{k!} = \sum_{j\in\mathbb{N}:\, j\geq p-1}\frac{t^j}{j!}, \quad t \geq 0, \tag{1.8}$$

with $k_0 = \min\{j \in \mathbb{N} : j \geq p-1\}$*. The constant* $\mu_{\alpha,\theta}$ *is sharp in the sense that the supremum is infinity when* $\mu \geq \mu_{\alpha,\theta}$*.*

Theorem B is the fractional dimension counterpart of the result in S. Adachi and K. Tanaka [2]. We also refer to [5, 16, 31] concerning related work for the classical Sobolev spaces. Our first result in this paper yields a precise asymptotics result on the above inequality.

Theorem 1.1. *Assume $p \geq 2$, $\alpha = p-1$ and $\theta \geq 0$. For any $0 \leq \mu < \mu_{\alpha,\theta}$, we denote*

$$\mathrm{TMSC}(\mu,\alpha,\theta) = \sup_{\|u'\|_{L^p_\alpha}\leq 1} \frac{1}{\|u\|^p_{L^p_\theta}} \int_0^\infty \varphi_p\big(\mu |u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

Then there exist positive constants $c(\alpha,\theta)$ and $C(\alpha,\theta)$ such that, when μ is close enough to $\mu_{\alpha,\theta}$,

$$\frac{c(\alpha,\theta)}{1-(\mu/\mu_{\alpha,\theta})^{p-1}} \leq \mathrm{TMSC}(\mu,\alpha,\theta) \leq \frac{C(\alpha,\theta)}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}.$$

Moreover, the constant $\mu_{\alpha,\theta}$ is sharp in the sense that $\mathrm{TMSC}(\mu_{\alpha,\theta},\alpha,\theta) = \infty$.

One of the goals of this paper is to investigate the critical regime $\mu = \mu_{\alpha,\theta}$. In this case, we will firstly prove the following.

Theorem 1.2. *Assume $p \geq 2$, $\alpha = p-1$ and $\theta \geq 0$. For any $0 \leq \sigma \leq \mu_{\alpha,\theta}$, we denote*

$$\mathrm{TMC}(\sigma,\alpha,\theta) = \sup_{\|u\|\leq 1} \int_0^\infty \varphi_p\big(\sigma |u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

Then $\mathrm{TMC}(\sigma,\alpha,\theta)$ *is finite. The constant $\mu_{\alpha,\theta}$ is sharp. In addition, we have the following identity: for all $\sigma \leq \mu_{\alpha,\theta}$,*

$$\mathrm{TMC}(\sigma,\alpha,\theta) = \sup_{\mu\in(0,\sigma)} \left(\frac{1-(\mu/\sigma)^{p-1}}{(\mu/\sigma)^{p-1}}\right)\mathrm{TMSC}(\mu,\alpha,\theta). \tag{1.9}$$

For classical Sobolev spaces, the critical supremum $\mathrm{TMC}(\sigma,\alpha,\theta)$ was first investigated by B. Ruf in [32] and Y. Li and B. Ruf in [28]. There has been a growing interest in this kind of inequalities during the last decades, and a wide literature is available, see for instance [4, 21, 22, 24–26] and the references therein. We note that the boundedness of $\mathrm{TMC}(\sigma,\alpha,\theta)$ has already been investigated in [1]. In this work we give a new proof for the boundedness which enables in particular to get a useful relation between $\mathrm{TMSC}(\sigma,\alpha,\theta)$ and $\mathrm{TMC}(\sigma,\alpha,\theta)$ given by (1.9).

Another interesting question about the suprema $\mathrm{TMSC}(\mu,\alpha,\theta)$ and $\mathrm{TMC}(\sigma,\alpha,\theta)$, and for Trudinger–Moser inequalities in general, is whether extremal functions exist or not. Inspired by recent approaches in [4, 25–27], we will employ the identity (1.9) to investigate this question. Firstly, on the subcritical supremum $\mathrm{TMSC}(\mu,\alpha,\theta)$ we are able to prove the following.

Theorem 1.3. *Let α, p and θ satisfy the assumptions of Theorem 1.1. Then the fractional subcritical supremum* $\mathrm{TMSC}(\mu,\alpha,\theta)$ *is attained.*

By using Theorem 1.3 and the identity (1.9), we will first prove the following attainability result for the fractional critical supremum $\mathrm{TMC}(\sigma,\alpha,\theta)$.

Theorem 1.4. *Let α, p and θ be under the assumptions of Theorem 1.2.*

(i) *If $k_0 > p-1$ and $0 < \sigma < \mu_{\alpha,\theta}$, then* $\mathrm{TMC}(\sigma,\alpha,\theta)$ *is attained.*

(ii) *If $k_0 = p-1$ and $0 < \sigma < \mu_{\alpha,\theta}$, then* $\mathrm{TMC}(\sigma,\alpha,\theta)$ *is attained whenever*

$$\mathrm{TMC}(\sigma,\alpha,\theta) > \frac{\sigma^{p-1}}{(p-1)!}.$$

Theorem 1.4 has already been obtained in [1]; however, our proof here is new and relies on the critical and subcritical equivalence given in Theorem 1.2. In addition, following [22] we also are able to characterize precisely the attainability of $\mathrm{TMC}(\sigma,\alpha,\theta)$ for the case (ii) above. In order to get this, we define the value $\sigma_* = \sigma_*(\alpha,\theta) \in [0,\mu_{\alpha,\theta})$ by

$$\sigma_* = \inf\big\{\sigma \in (0,\mu_{\alpha,\theta}) : \mathrm{TMC}(\sigma,\alpha,\theta) \ \text{ is attained}\big\},$$

when $\mathrm{TMC}(\sigma,\alpha,\theta)$ is attained for some $\sigma \in (0,\mu_{\alpha,\theta})$. If $\mathrm{TMC}(\sigma,\alpha,\theta)$ is not attained for any $\sigma \in (0,\mu_{\alpha,\theta})$, then we set $\sigma_* = \infty$.

Theorem 1.5. *Let $k_0 = p-1$ and α,θ be as in Theorem 1.2. Suppose $\sigma_* < \mu_{\alpha,\theta}$. Then*

(i) $\mathrm{TMC}(\sigma,\alpha,\theta)$ *is attained for* $\sigma_* < \sigma < \mu_{\alpha,\theta}$.

(ii) *The function* $\nu\colon (\sigma_*,\mu_{\alpha,\theta}) \to \mathbb{R}$ *given by* $\nu(\sigma) = \frac{(p-1)!}{\sigma^{p-1}}\mathrm{TMC}(\sigma,\alpha,\theta)$ *is strictly increasing. Moreover, by setting* $\mathrm{TMC}(0,\alpha,\theta)=0$*, there holds*

$$\mathrm{TMC}(\sigma,\alpha,\theta) \begin{cases} = \dfrac{\sigma^{p-1}}{(p-1)!}, & \text{for } \sigma \in [0,\sigma_*], \\ > \dfrac{\sigma^{p-1}}{(p-1)!}, & \text{for } \sigma \in (\sigma_*,\mu_{\alpha,\theta}), \end{cases} \tag{1.10}$$

and in particular,

$$\sigma_* = \inf\big\{\sigma \in (0,\mu_{\alpha,\theta}) : \mathrm{TMC}(\sigma,\alpha,\theta) > \sigma^{p-1}/(p-1)!\big\}. \tag{1.11}$$

(iii) *If* $p>2$*, we have* $\sigma^* = 0$ *and thus* $\mathrm{TMC}(\sigma,\alpha,\theta)$ *is attained for any* $(0,\mu_{\alpha,\theta})$.

As a consequence of Theorem 1.5, since $\mathrm{TMC}(\sigma,1,\theta)$ is not attained for σ small enough (cf. Theorem 1.3 in [1]), Theorem 1.5 provides

$$\mathrm{TMC}(\sigma,1,\theta) = \sup_{\|u\|\le 1}\int_0^\infty \varphi_2(\sigma|u|^2)\,\mathrm{d}\lambda_\theta = \sigma, \quad \forall\, \sigma \in [0,\sigma_*].$$

The rest of this paper is organized as follows. In Section 2, we show Theorem 1.1. Section 3 is devoted to the subcritical and critical equivalence stated in Theorem 1.2. In Section 4 we will prove the existence of extremal functions for both subcritical TMSC and critical TMC fractional Trudinger–Moser suprema in Theorem 1.3 and Theorem 1.4. The proof of Theorem 1.5 is given in Section 5.

2. Sharp subcritical Trudinger–Moser inequality: Proof of Theorem 1.1

In this section, we will prove the asymptotic behavior for the supremum $\mathrm{TMSC}(\mu,\alpha,\theta)$ for the subcritical Trudinger–Moser inequality in Theorem 1.1.

2.1. Some elementary properties

Note that from the definition (1.1) and the change of variables $s=\tau r$, we have

$$\int_0^\infty f(\tau r)\,\mathrm{d}\lambda_\theta = \frac{1}{\tau^{\theta+1}}\int_0^\infty f(s)\,\mathrm{d}\lambda_\theta, \quad \tau>0. \tag{2.1}$$

Thus, by setting $u_\tau(r) = \zeta u(\tau r)$, with $\zeta, \tau > 0$ and $u \in X^{1,p}_\infty(\alpha, \theta)$, we can write

$$\|u_\tau'\|^p_{L^p_\alpha} = \frac{(\zeta\tau)^p}{\tau^{\alpha+1}}\|u'\|^p_{L^p_\alpha} \quad \text{and} \quad \|u_\tau\|^q_{L^q_\theta} = \frac{\zeta^q}{\tau^{\theta+1}}\|u\|^q_{L^q_\theta}, \quad q \geq p. \tag{2.2}$$

Also, we observe that

$$\begin{aligned} &\varphi_p(\rho t) \leq \rho^{p-1}\varphi_p(t), \quad \text{if } 0 \leq \rho \leq 1,\\ &\varphi_p(\rho t) \geq \rho^{p-1}\varphi_p(t), \quad \text{if } \rho \geq 1, \end{aligned} \tag{2.3}$$

where $\varphi_p(t)$ is given by (1.8).

Lemma 2.1. *For all $q \geq 1$ and $\varepsilon > 0$, it holds*

$$(x+y)^q \leq (1+\varepsilon)^{(q-1)/q} x^q + (1-(1+\varepsilon)^{-1/q})^{1-q} y^q, \quad x, y \geq 0.$$

Proof. Since $x \mapsto x^q$, $x \geq 0$, is a convex function, we have

$$\begin{aligned} (x+y)^q &= \Big(\frac{1}{(1+\varepsilon)^{1/q}}(1+\varepsilon)^{1/q}x + \Big(1-\frac{1}{(1+\varepsilon)^{1/q}}\Big)\Big(1-\frac{1}{(1+\varepsilon)^{1/q}}\Big)^{-1}y\Big)^q \\ &\leq \frac{1}{(1+\varepsilon)^{1/q}}(1+\varepsilon)x^q + \Big(1-\frac{1}{(1+\varepsilon)^{1/q}}\Big)^{1-q} y^q. \end{aligned}$$

■

Henceforth suppose that the condition $\alpha - p + 1 = 0$ holds. The next result ensures that the subcritical supremum $\mathrm{TMSC}(\mu, \alpha, \theta)$ can be normalized.

Lemma 2.2.

$$\mathrm{TMSC}(\mu,\alpha,\theta) = \sup_{\|u'\|_{L^p_\alpha} = \|u\|_{L^p_\theta} = 1} \int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

Proof. It is sufficient to show that

$$\mathrm{TMSC}(\mu,\alpha,\theta) \leq \sup_{\|u'\|_{L^p_\alpha} = \|u\|_{L^p_\theta} = 1} \int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

In order to get this, for each $u \in X^{1,p}_\infty \setminus \{0\}$, with $\|u'\|_{L^p_\alpha} \leq 1$, we set

$$v(r) = \frac{u(\tau r)}{\|u'\|_{L^p_\alpha}}, \quad \text{with} \quad \tau = \Bigg(\frac{\|u\|^p_{L^p_\theta}}{\|u'\|^p_{L^p_\alpha}}\Bigg)^{1/(\theta+1)}.$$

Since we are supposing $\alpha - p + 1 = 0$, (2.2) yields

$$\|v'\|_{L^p_\alpha} = \|v\|_{L^p_\theta} = 1.$$

Then, from (2.1) and (2.3) it follows that

$$\begin{aligned} \int_0^\infty \varphi_p\big(\mu|v|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta &= \frac{1}{\tau^{\theta+1}}\int_0^\infty \varphi_p\Bigg(\frac{1}{\|u'\|^{p/(p-1)}_{L^p_\alpha}}\mu|u|^{\frac{p}{p-1}}\Bigg)\,\mathrm{d}\lambda_\theta \\ &\geq \Bigg(\frac{\|u'\|^p_{L^p_\alpha}}{\|u\|^p_{L^p_\theta}}\Bigg)\frac{1}{\|u'\|^p_{L^p_\alpha}}\int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta = \frac{1}{\|u\|^p_{L^p_\theta}}\int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta, \end{aligned}$$

which completes the proof. ■

2.2. Proof of Theorem 1.1

Let $u \in X^{1,p}_\infty$, with $\|u'\|_{L^p_\alpha} \le 1$. From the Pólya–Szegö inequality obtained in [1, 3], we can assume that u is a non-increasing function. Also, by Lemma 2.2 it is sufficient to analyze the case $\|u\|_{L^p_\theta} = 1$.

Initially, we will prove that

$$\text{TMSC}(\mu,\alpha,\theta) \le \frac{C(\alpha,\theta)}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}. \tag{2.4}$$

Let us denote

$$A_u = \{r > 0 : |u(r)|^p > 1-(\mu/\mu_{\alpha,\theta})^{p-1}\}.$$

We observe that for all $|t| \le 1$ it holds

(2.5)

$$\varphi_p\big(\mu|t|^{\frac{p}{p-1}}\big) = \sum_{j\in\mathbb{N}\,:\,j\ge p-1} \frac{\mu^j}{j!}|t|^{\frac{jp}{p-1}} \le \sum_{j\in\mathbb{N}\,:\,j\ge p-1} \frac{\mu^j}{j!}|t|^p \le |t|^p \sum_{j=0}^{\infty}\frac{\mu^j}{j!} = e^{\mu}|t|^p.$$

Hence, if $A_u = \emptyset$ and consequently $u \le 1$ in $(0,\infty)$, the inequality (2.5) yields

$$\int_0^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \le e^{\mu}\int_0^\infty |u|^p\,\mathrm{d}\lambda_\theta \le \frac{e^{\mu_{\alpha,\theta}}}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}. \tag{2.6}$$

So we can assume $A_u \ne \emptyset$. Thus, there exists $R_u > 0$ such that $A_u = (0, R_u)$, because we are assuming u is a non-increasing function. Analogously to (2.6), we obtain

$$\begin{aligned}\int_{R_u}^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta &\le \int_{\{u\le 1\}} \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \\ &\le e^{\mu}\int_{\{u\le 1\}} |u|^p\,\mathrm{d}\lambda_\theta \le \frac{e^{\mu_{\alpha,\theta}}}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}.\end{aligned}$$

Now observe that

$$|B_{R_u}|_\theta = \int_0^{R_u}\mathrm{d}\lambda_\theta \le \frac{1}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}\int_0^\infty |u|^p\,\mathrm{d}\lambda_\theta \le \frac{1}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}. \tag{2.7}$$

For $r \in (0, R_u)$, we set

$$v(r) = u(r) - \Big(1-\Big(\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{p-1}\Big)^{1/p}.$$

It is clear that $v \in X^{1,p}_{R_u}(\alpha,\theta)$ and $\|v'\|_{L^p_\alpha(0,R_u)} \le 1$. Also, by choosing $\varepsilon = (\mu_{\alpha,\theta}/\mu)^p - 1$ and $q = p/(p-1)$ in Lemma 2.1, we have

$$\begin{aligned}|u|^{\frac{p}{p-1}} &\le (1+\varepsilon)^{\frac{1}{p}}|v|^{\frac{p}{p-1}} + \Big(1-\frac{1}{(1+\varepsilon)^{(p-1)/p}}\Big)^{-\frac{1}{p-1}}\Big(1-\Big(\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{p-1}\Big)^{\frac{1}{p-1}} \\ &= \frac{\mu_{\alpha,\theta}}{\mu}|v|^{\frac{p}{p-1}} + 1.\end{aligned}$$

Hence, the Trudinger–Moser type inequality (1.5) and (2.7) imply

$$\int_0^{R_u} \varphi_p\left(\mu |u|^{\frac{p}{p-1}}\right) \mathrm{d}\lambda_\theta \le \int_0^{R_u} e^{\mu |u|^{\frac{p}{p-1}}} \mathrm{d}\lambda_\theta \le e^{\mu} \int_0^{R_u} e^{\mu_{\alpha,\theta}|v|^{\frac{p}{p-1}}} \mathrm{d}\lambda_\theta$$
$$\le c_{\alpha,\theta}\, e^{\mu}\, |B_{R_u}|_\theta \le \frac{c_{\alpha,\theta}\, e^{\mu_{\alpha,\theta}}}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}.$$

Remark 2.3. At this point, we note that we have proved that

$$\mathrm{TMSC}(\mu,\alpha,\theta) \le \frac{C(\alpha,\theta)}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}$$

for any $\mu < \mu_{\alpha,\theta}$ not necessarily close to $\mu_{\alpha,\theta}$.

This proves (2.4). Next, we will prove the opposite inequality

$$\mathrm{TMSC}(\mu,\alpha,\theta) \ge \frac{c(\alpha,\theta)}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}.$$

To see this, let us consider the sequence

$$u_n(r) = \frac{1}{\omega_\alpha^{1/p}} \begin{cases} \left(\dfrac{n}{\theta+1}\right)^{(p-1)/p}, & \text{if } 0 \le r \le e^{-n/(\theta+1)},\\ \left(\dfrac{\theta+1}{n}\right)^{1/p} \ln \dfrac{1}{r}, & \text{if } e^{-n/(\theta+1)} < r < 1,\\ 0, & \text{if } r \ge 1. \end{cases} \tag{2.8}$$

Since that $\alpha = p-1$, it follows that

$$\|u_n'\|^p_{L^p_\alpha} = 1 \quad \text{and} \quad \|u_n\|^p_{L^p_\theta} = \frac{c}{n}\left[n^p e^{-n} + \int_0^n s^p\, e^{-s}\, \mathrm{d}s\right]$$

for some $c = c(\alpha,\theta) > 0$. Thus, since $\int_0^\infty s^p e^{-s} \mathrm{d}s = \Gamma(p+1) > 0$, there are $c_1 = c_1(\alpha,\theta) > 0$ and $n_1 \in \mathbb{N}$ such that

$$\|u_n\|^p_{L^p_\theta} \le \frac{c_1}{n}, \quad \forall n \ge n_1.$$

On the other hand,

$$\int_0^\infty \varphi_p\big(\mu |u_n|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta \ge \int_0^{e^{-n/(\theta+1)}} \varphi_p\Big(\frac{\mu}{\mu_{\alpha,\theta}} n\Big)\, \mathrm{d}\lambda_\theta = \frac{\omega_\theta}{\theta+1}\, \varphi_p\Big(\frac{\mu}{\mu_{\alpha,\theta}} n\Big) e^{-n}$$
$$= \frac{\omega_\theta}{\theta+1}\Big[e^{(\mu/\mu_{\alpha,\theta}-1)n} - \Big(\sum_{j=0}^{k_0-1} \Big(\frac{\mu}{\mu_{\alpha,\theta}}\Big)^j \frac{n^j}{j!}\Big) e^{-n}\Big]$$
$$\ge \frac{\omega_\theta}{\theta+1}\Big[e^{(\mu/\mu_{\alpha,\theta}-1)n} - \Big(\sum_{j=0}^{k_0-1} \frac{n^j}{j!}\Big) e^{-n}\Big].$$

Thus, for all $n \geq n_1$,

$$\begin{aligned}
\mathrm{TMSC}(\mu,\alpha,\theta) &\geq \frac{1}{\|u_n\|^p_{L^p_\theta}} \int_0^\infty \varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \\
&\geq c_2\Big[ne^{(\mu/\mu_{\alpha,\theta}-1)n} - \Big(\sum_{j=0}^{k_0-1}\frac{n^j}{j!}\Big)ne^{-n}\Big] \\
&= \frac{c_2}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}\Big(1-\Big(\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{p-1}\Big)\Big[ne^{-(1-\mu/\mu_{\alpha,\theta})n} - \Big(\sum_{j=0}^{k_0-1}\frac{n^j}{j!}\Big)ne^{-n}\Big], \qquad (2.9)
\end{aligned}$$

for some $c_2 = c_2(\alpha,\theta) > 0$. Now, we can choose $n_2 \geq n_1$ such that

$$\Big(1-\Big(\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{p-1}\Big)\Big(\sum_{j=0}^{k_0-1}\frac{n^j}{j!}\Big)ne^{-n} \leq \frac{1}{e^5}, \quad \forall n \geq n_2 \text{ and } 0 \leq \mu < \mu_{\alpha,\theta}.$$

Hence, for all $n \geq n_2$,

$$\mathrm{TMSC}(\mu,\alpha,\theta) \geq \frac{c_2}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}\Big[\Big(1-\frac{\mu}{\mu_{\alpha,\theta}}\Big)ne^{-(1-\mu/\mu_{\alpha,\theta})n} - e^{-5}\Big].$$

Now, if α is close enough to $\mu_{\alpha,\theta}$ such that $(1-\mu/\mu_{\alpha,\theta})^{-1} \geq n_2$, by picking $n \in \mathbb{N}$ such that

$$\Big(1-\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{-1} \leq n \leq 4\Big(1-\frac{\mu}{\mu_{\alpha,\theta}}\Big)^{-1},$$

we obtain

$$\mathrm{TMSC}(\mu,\alpha,\theta) \geq \frac{c_2}{1-(\mu/\mu_{\alpha,\theta})^{p-1}}\big[e^{-4}-e^{-5}\big].$$

Finally, from (2.9), for $\mu = \mu_{\alpha,\theta}$ we have

$$\mathrm{TMSC}(\mu_{\alpha,\theta},\alpha,\theta) \geq c_2\Big[n - \Big(\sum_{j=0}^{k_0-1}\frac{n^j}{j!}\Big)ne^{-n}\Big] \to \infty, \quad \text{as } n \to \infty.$$

3. Equivalence of critical and subcritical Trudinger–Moser inequalities

The aim of this section is to prove the critical and subcritical equivalence given in Theorem 1.2. We observe that we are not assuming that $\mathrm{TMC}(\mu_{\alpha,\theta},\alpha,\theta)$ is finite in our argument.

Lemma 3.1. *For any* $0 < \sigma \leq \mu_{\alpha,\theta}$ *and* $0 < \mu < \sigma$,

$$\mathrm{TMSC}(\mu,\alpha,\theta) \leq \Big(\frac{(\mu/\sigma)^{p-1}}{1-(\mu/\sigma)^{p-1}}\Big)\mathrm{TMC}(\sigma,\alpha,\theta).$$

In particular, if $\mathrm{TMC}(\mu_{\alpha,\theta},\alpha,\theta)$ *is finite, then* $\mathrm{TMSC}(\mu,\alpha,\theta)$ *is finite.*

Proof. Let $u \in X^{1,p}_{\infty}$, with $\|u'\|_{L^p_\alpha} = 1$ and $\|u\|_{L^p_\theta} = 1$. Set

$$u_t(r) = \left(\frac{\mu}{\sigma}\right)^{(p-1)/p} u(tr), \quad \text{with} \quad t = \left(\frac{(\mu/\sigma)^{p-1}}{1-(\mu/\sigma)^{p-1}}\right)^{1/(\theta+1)}.$$

By (2.2) we get

$$\|u_t'\|^p_{L^p_\alpha} = \left(\frac{\mu}{\sigma}\right)^{p-1} \|u'\|^p_{L^p_\alpha} = \left(\frac{\mu}{\sigma}\right)^{p-1},$$
$$\|u_t\|^p_{L^p_\theta} = \left(\frac{\mu}{\sigma}\right)^{p-1} \frac{\|u\|^p_{L^p_\theta}}{t^{\theta+1}} = 1 - \left(\frac{\mu}{\sigma}\right)^{p-1}.$$

Hence $\|u_t'\|^p_{L^p_\alpha} + \|u_t\|^p_{L^p_\theta} = 1$ and we have

$$\int_0^\infty \varphi_p\big(\mu|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta = t^{\theta+1}\int_0^\infty \varphi_p\big(\sigma|u_t|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \le \left(\frac{(\mu/\sigma)^{p-1}}{1-(\mu/\sigma)^{p-1}}\right)\mathrm{TMC}(\sigma,\alpha,\theta).$$

Since $u \in X^{1,p}_{\infty}$, with $\|u'\|_{L^p_\alpha} = 1$ and $\|u\|_{L^p_\theta} = 1$, is arbitrary, in view of Lemma 2.2, we conclude the proof. ∎

3.1. Proof of Theorem 1.2

Let $u \in X^{1,p}_{\infty}$ such that $0 < \|u'\|^p_{L^p_\alpha} + \|u\|^p_{L^p_\theta} \le 1$. Assume that

$$\|u'\|_{L^p_\alpha} = \vartheta \in (0,1) \quad \text{and} \quad \|u\|^p_{L^p_\theta} \le 1 - \vartheta^p.$$

If $1/2 < \vartheta < 1$, we set

$$u_t(r) = \frac{u(tr)}{\vartheta}, \quad \text{with} \quad t = \left(\frac{1-\vartheta^p}{\vartheta^p}\right)^{1/(\theta+1)} > 0.$$

From (2.2), we can write

$$\|u_t'\|_{L^p_\alpha} = \frac{\|u'\|_{L^p_\alpha}}{\vartheta} = 1,$$
$$\|u_t\|^p_{L^p_\theta} = \frac{1}{\vartheta^p}\frac{1}{t^{\theta+1}}\|u\|^p_{L^p_\theta} \le \frac{1-\vartheta^p}{\vartheta^p t^{\theta+1}} = 1.$$

Hence, for any $\sigma \le \mu_{\alpha,\theta}$, Theorem 1.1 (cf. Remark 2.3) yields

$$\int_0^\infty \varphi_p\big(\sigma|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \le t^{\theta+1}\int_0^\infty \varphi_p\big(\vartheta^{\frac{p}{p-1}}\mu_{\alpha,\theta}|u_t|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta$$
$$\le t^{\theta+1}\,\mathrm{TMSC}\big(\vartheta^{\frac{p}{p-1}}\mu_{\alpha,\theta},\alpha,\theta\big) \le \left(\frac{1-\vartheta^p}{\vartheta^p}\right)\frac{C(\alpha,\theta)}{1-\left(\frac{\vartheta^{p/(p-1)}\mu_{\alpha,\theta}}{\mu_{\alpha,\theta}}\right)^{p-1}}$$
$$= \left(\frac{1-\vartheta^p}{\vartheta^p}\right)\frac{C(\alpha,\theta)}{1-\vartheta^p} = \frac{C(\alpha,\theta)}{\vartheta^p} \le 2^p C(\alpha,\theta).$$

If $0 < \vartheta \le 1/2$, setting

$$v(r) = 2u(r/\vartheta)$$

we have

$$\|v'\|_{L^p_\alpha} = 2\|u'\|_{L^p_\alpha} \le 1,$$
$$\|v\|^p_{L^p_\theta} = 2^p\,\vartheta^{\theta+1}\,\|u\|^p_{L^p_\theta} \le 2^p\,\vartheta^{\theta+1}\,(1-\vartheta^p) \le 2^p\,\vartheta^{\theta+1}.$$

Consequently, Theorem 1.1 provides

$$\int_0^\infty \varphi_p\big(\sigma|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \le \frac{1}{\vartheta^{\theta+1}}\int_0^\infty \varphi_p\big(2^{-\frac{p}{p-1}}\mu_{\alpha,\theta}|v|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta$$
$$\le 2^p\,\mathrm{TMSC}\big(2^{-p/(p-1)}\,\mu_{\alpha,\theta},\alpha,\theta\big) \le C(\alpha,\theta)\Big(\frac{2^p}{1-2^{-p}}\Big).$$

Since $u \in X^{1,p}_\infty$, with $\|u\| \le 1$, is arbitrary, we obtain $\mathrm{TMC}(\sigma,\alpha,\theta) < \infty$, for any $\sigma \le \mu_{\alpha,\theta}$.

Next we will show that the constant $\mu_{\alpha,\theta}$ is sharp. To see this, we use the sequence (u_n) in (2.8) again. Indeed, we have

$$\|u_n'\|^p_{L^p_\alpha} = 1 \quad \text{and} \quad \|u_n\|^p_{L^p_\theta} = O\Big(\frac{1}{n}\Big), \quad \text{as } n \to \infty.$$

Now, for $\tau_n \in (0,1)$ such that

$$\tau_n^p(1+\|u_n\|^p_{L^p_\theta}) = 1, \quad \text{with} \quad \tau_n = 1 - O\Big(\frac{1}{n^{1/p}}\Big) \to 1, \quad \text{as } n \to \infty,$$

we set

$$v_n(r) = \tau_n u_n(r).$$

Then

$$\|v_n'\|^p_{L^p_\alpha} + \|v_n\|^p_{L^p_\theta} = \tau_n^p\,\|u_n'\|^p_{L^p_\alpha} + \tau_n^p\,\|u_n\|^p_{L^p_\theta} = \tau_n^p + \tau_n^p\,\|u_n\|^p_{L^p_\theta} = 1.$$

In addition, for any $\sigma > \mu_{\alpha,\theta}$,

$$\int_0^\infty \varphi_p\big(\sigma|v_n|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \ge \int_0^{e^{-n/(\theta+1)}} \Big(e^{\frac{n\sigma}{\mu_{\alpha,\theta}}\tau_n^{p/(p-1)}} - \sum_{k=0}^{k_0-1}\frac{1}{k!}\Big(\frac{n\sigma}{\mu_{\alpha,\theta}}\Big)^k \tau_n^{\frac{kp}{p-1}}\Big)\,\mathrm{d}\lambda_\theta$$
$$= \frac{\omega_\theta}{\theta+1}\Big[e^{(n\sigma/\mu_{\alpha,\theta})\,\tau_n^{p/(p-1)}-n} - O\Big(\frac{(n\tau_n)^{k_0-1}}{e^n}\Big)\Big] \to +\infty, \quad \text{as } n \to \infty.$$

Now we are going to show that

$$\mathrm{TMC}(\sigma,\alpha,\theta) = \sup_{\mu\in(0,\sigma)}\Big(\frac{1-(\mu/\sigma)^{p-1}}{(\mu/\sigma)^{p-1}}\Big)\mathrm{TMSC}(\mu,\alpha,\theta). \tag{3.1}$$

By Lemma 3.1, we obtain

$$\sup_{\mu\in(0,\sigma)}\Big(\frac{1-(\mu/\sigma)^{p-1}}{(\mu/\sigma)^{p-1}}\Big)\mathrm{TMSC}(\mu,\alpha,\theta) \le \mathrm{TMC}(\sigma,\alpha,\theta). \tag{3.2}$$

To obtain the reverse inequality, let (u_n) be a maximizing sequence of $\mathrm{TMC}(\sigma,\alpha,\theta)$, that is, $u_n \in X^{1,p}_\infty$, $0 < \|u_n'\|^p_{L^p_\alpha} + \|u_n\|^p_{L^p_\theta} \le 1$, such that

$$\mathrm{TMC}(\sigma,\alpha,\theta) = \lim_n \int_0^\infty \varphi_p\big(\sigma|u_n|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta.$$

We set

$$u_{\tau_n}(r) = \frac{u(\tau_n r)}{\|u_n'\|_{L^p_\alpha}}, \quad \text{with} \quad \tau_n = \left(\frac{1-\|u_n'\|^p_{L^p_\alpha}}{\|u_n'\|^p_{L^p_\alpha}}\right)^{1/(\theta+1)} > 0.$$

Then

$$\|u_{\tau_n}'\|_{L^p_\alpha} = 1 \quad \text{and} \quad \|u_{\tau_n}\|^p_{L^p_\theta} = \frac{1}{\|u_n'\|^p_{L^p_\alpha}}\frac{1}{\tau_n^{\theta+1}}\|u_n\|^p_{L^p_\theta} = \frac{\|u_n\|^p_{L^p_\theta}}{1-\|u_n'\|^p_{L^p_\alpha}} \le 1.$$

Consequently,

$$\begin{aligned}\int_0^\infty \varphi_p\big(\sigma|u_n|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta &= \tau_n^{\theta+1}\int_0^\infty \varphi_p\big(\sigma\|u_n'\|^{\frac{p}{p-1}}_{L^p_\alpha}|u_{\tau_n}|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta \\ &\le \tau_n^{\theta+1}\,\mathrm{TMSC}\big(\sigma\|u_n'\|^{\frac{p}{p-1}}_{L^p_\alpha},\alpha,\theta\big) = \left(\frac{1-\|u_n'\|^p_{L^p_\alpha}}{\|u_n'\|^p_{L^p_\alpha}}\right)\mathrm{TMSC}\big(\sigma\|u_n'\|^{\frac{p}{p-1}}_{L^p_\alpha},\alpha,\theta\big) \\ &\le \sup_{\mu\in(0,\sigma)}\left(\frac{1-(\mu/\sigma)^{p-1}}{(\mu/\sigma)^{p-1}}\right)\mathrm{TMSC}(\mu,\alpha,\theta).\end{aligned}$$

Hence, we obtain

$$\mathrm{TMC}(\sigma,\alpha,\theta) \le \sup_{\mu\in(0,\sigma)}\left(\frac{1-(\mu/\sigma)^{p-1}}{(\mu/\sigma)^{p-1}}\right)\mathrm{TMSC}(\mu,\alpha,\theta). \tag{3.3}$$

Now, (3.1) follows from (3.2) and (3.3).

4. Existence of extremal functions

In this section we will prove the existence of extremal functions for both subcritical and critical Trudinger–Moser inequalities Theorem 1.3 and Theorem 1.4. First of all, we present the following radial type lemma.

Lemma 4.1. *For each $u \in X^{1,p}_\infty(\alpha,\theta)$, $p \ge 2$, we have the inequality*

$$|u(r)|^p \le \frac{C}{r^{\frac{\alpha+\theta(p-1)}{p}}}\|u\|^p, \quad \forall r > 0,$$

where $C > 0$ depends only on α, p and θ. In addition,

$$\lim_{r\to\infty} r^{\frac{\alpha+\theta(p-1)}{p}}|u(r)|^p = 0.$$

Proof. Let $u \in X^{1,p}_\infty(\alpha,\theta)$ be arbitrary. For any $r > 0$, we have

$$|u(r)|^p = -\int_r^\infty \frac{d}{ds}\left(|u(s)|^p\right) \mathrm{d}s \le p\int_r^\infty |u(s)|^{p-1}|u'(s)|\,\mathrm{d}s.$$

Hence

$$r^{\frac{\alpha+\theta(p-1)}{p}}\,|u(r)|^p \le p\int_r^\infty |u(s)|^{p-1}\,s^{\theta(p-1)/p}\,|u'(s)|\,s^{\alpha/p}\,\mathrm{d}s$$

and the Young inequality yields

$$r^{\frac{\alpha+\theta(p-1)}{p}}\,|u(r)|^p \le C\Big[\int_r^\infty |u(s)|^p\,\mathrm{d}\lambda_\theta + \int_r^\infty |u'(s)|^p\,\mathrm{d}\lambda_\alpha\Big],$$

for some $C > 0$ depending only on α, p and θ. This proves the result. ∎

4.1. Maximizers for the subcritical Trudinger–Moser inequality

Let $(u_n) \subset X^{1,p}_\infty$ be a maximizing sequence to the subcritical Trudinger–Moser supremum $\mathrm{TMSC}(\mu,\alpha,\theta)$. From Lemma 2.2, we may suppose that

$$\begin{aligned}
&\mathrm{TMSC}(\mu,\alpha,\theta) = \lim_n \int_0^\infty \varphi_p\big(\mu\,|u_n|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta,\\
&\|u_n'\|_{L^p_\alpha} = \|u_n\|_{L^p_\theta} = 1,\\
&u_n \rightharpoonup u \quad \text{weakly in } X^{1,p}_\infty.
\end{aligned}$$

From the compact embedding (1.7), we also may assume that

$$u_n \to u \quad \text{in } L^q_\theta,\ q > p, \quad \text{and} \quad u_n(r) \to u(r) \quad \text{a.e in } (0,\infty). \tag{4.1}$$

Of course, we also have

$$\|u'\|_{L^p_\alpha} \le 1 \quad \text{and} \quad \|u\|_{L^p_\theta} \le 1.$$

At this point we observe that there exist $C = C(p,\mu) > 0$ such that

$$\varphi_p\big(\mu t^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,t^{\frac{k_0 p}{p-1}} \le C\,\varphi_p\big(\mu t^{\frac{p}{p-1}}\big)\,t^{\frac{p}{p-1}}, \quad t \ge 0. \tag{4.2}$$

Let $\varepsilon > 0$ be arbitrary. From Lemma 4.1, there exists $R > 0$ such that $|u_n(r)| \le \varepsilon$, for all $r \ge R$. Hence, from (4.2) and Theorem B we obtain

$$\begin{aligned}
\int_R^\infty \Big[\varphi_p\big(\mu\,|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u_n|^{\frac{k_0 p}{p-1}}\Big]\,\mathrm{d}\lambda_\theta &\le C(p,\mu)\int_R^\infty \varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big)\,|u_n|^{\frac{p}{p-1}}\,\mathrm{d}\lambda_\theta\\
&\le C(p,\mu)\,\varepsilon^{\frac{p}{p-1}}\int_R^\infty \varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta\\
&\le C(p,\mu,\theta)\,\varepsilon^{\frac{p}{p-1}}.
\end{aligned}$$

Also, we have (cf. (4.1))

$$\varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u_n|^{\frac{k_0 p}{p-1}} \to \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u|^{\frac{k_0 p}{p-1}} \quad \text{a.e in } (0,R), \text{ as } n\to\infty.$$

In addition, by setting $v_n(r) = u_n(r) - u_n(R)$ for all $r \in (0,R)$, we have $v_n \in X^{1,p}_R(\alpha,\theta)$ with $\|v_n'\|_{L^p_\alpha} \le 1$. Moreover, from Lemma 2.1, for any $q > 1$,

$$|u_n|^{p/(p-1)} \le q^{1/p}\,|v_n|^{p/(p-1)} + \big(1 - q^{-(p-1)/p}\big)^{-1/(p-1)}\,\varepsilon^{p/(p-1)}.$$

By choosing $q > 1$ close to 1 such that $q^{(p+1)/p}\mu < \mu_{\alpha,\theta}$, Theorem A yields

$$\begin{aligned}\int_0^R \Big[\varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u_n|^{\frac{k_0 p}{p-1}}\Big]^q \,\mathrm{d}\lambda_\theta &\le \int_0^R \Big[\varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big)\Big]^q \,\mathrm{d}\lambda_\theta \\ &\le \int_0^R e^{q\mu|u_n|^{\frac{p}{p-1}}}\,\mathrm{d}\lambda_\theta \le C(p,q,\alpha,\theta)\int_0^R e^{\mu_{\alpha,\theta}|v_n|^{\frac{p}{p-1}}}\,\mathrm{d}\lambda_\theta \\ &\le C(p,q,\mu,\theta,R).\end{aligned} \tag{4.3}$$

Thus, we may use Vitali's convergence theorem to obtain

$$\lim_{n\to\infty}\int_0^R \Big[\varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u_n|^{\frac{k_0 p}{p-1}}\Big]\mathrm{d}\lambda_\theta = \int_0^R \Big[\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u|^{\frac{k_0 p}{p-1}}\Big]\,\mathrm{d}\lambda_\theta.$$

Now, using the Brezis–Lieb lemma together with (4.1), we have

$$\lim_{n\to\infty}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta = \begin{cases}\displaystyle\int_0^\infty |u|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta, & \text{if } k_0 > p-1,\\ 1, & \text{if } k_0 = p-1.\end{cases}$$

Hence, if $k_0 > p-1$,

$$\begin{aligned}\mathrm{TMSC}(\mu,\alpha,\theta) &= \lim_n \int_0^\infty \varphi_p\big(\mu\,|u_n|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \\ &= \lim_n\Big[\int_0^\infty \Big(\varphi_p\big(\mu\,|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u_n|^{\frac{k_0 p}{p-1}}\Big)\,\mathrm{d}\lambda_\theta + \frac{\mu^{k_0}}{k_0!}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta\Big] \\ &\le \int_0^R \Big(\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u|^{\frac{k_0 p}{p-1}}\Big)\,\mathrm{d}\lambda_\theta + C(p,\mu,\theta)\,\varepsilon^{\frac{p}{p-1}} + \frac{\mu^{k_0}}{k_0!}\int_0^\infty |u|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta \\ &\le \int_0^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta + C(p,\mu,\theta)\,\varepsilon^{\frac{p}{p-1}}.\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\mathrm{TMSC}(\mu,\alpha,\theta) \le \int_0^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

It follows that $0 < \|u\|_{L^p_\theta} \le 1$ and thus

$$\mathrm{TMSC}(\mu,\alpha,\theta) \le \frac{1}{\|u\|^p_{L^p_\theta}}\int_0^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta$$

which completes the proof in the case $k_0 > p-1$. If $k_0 = p-1$, we can write

$$\begin{aligned}\mathrm{TMSC}(\mu,\alpha,\theta) &= \lim_n \int_0^\infty \Big(\varphi_p\big(\mu|u_n|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u_n|^p\Big)\, \mathrm{d}\lambda_\theta + \frac{\mu^{k_0}}{k_0!} \\ &\le \int_0^R \Big(\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u|^p\Big)\, \mathrm{d}\lambda_\theta + C(p,\mu,\theta)\,\varepsilon^{\frac{p}{p-1}} + \frac{\mu^{k_0}}{k_0!} \\ &\le \int_0^\infty \Big(\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}|u|^p\Big)\, \mathrm{d}\lambda_\theta + C(p,\mu,\theta)\,\varepsilon^{\frac{p}{p-1}} + \frac{\mu^{k_0}}{k_0!}.\end{aligned}$$

Letting $\varepsilon \to 0$, it follows that

$$\mathrm{TMSC}(\mu,\alpha,\theta) \le \int_0^\infty \Big(\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u|^p\Big)\, \mathrm{d}\lambda_\theta + \frac{\mu^{k_0}}{k_0!}. \tag{4.4}$$

Moreover, for any $w \in X^{1,p}_\infty(\alpha,\theta)$ with $\|w'\|_{L^p_\alpha} = \|w\|_{L^p_\theta} = 1$ we have

$$\int_0^\infty \varphi_p\big(\mu|w|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta \ge \frac{\mu^{k_0}}{k_0!}\int_0^\infty |w|^p\, \mathrm{d}\lambda_\theta + \frac{\mu^{k_0+1}}{(k_0+1)!}\int_0^\infty |w|^{\frac{p(k_0+1)}{p-1}}\, \mathrm{d}\lambda_\theta.$$

This implies that $\mathrm{TMSC}(\mu,\alpha,\theta) > \mu^{k_0}/k_0!$. Thus, from (4.4), we get $0 < \|u\|_{L^p_\theta} \le 1$ and

$$\begin{aligned}\mathrm{TMSC}(\mu,\alpha,\theta) &\le \frac{1}{\|u\|^p_{L^p_\theta}}\int_0^\infty \Big[\varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big) - \frac{\mu^{k_0}}{k_0!}\,|u|^p\Big]\, \mathrm{d}\lambda_\theta + \frac{\mu^{k_0}}{k_0!} \\ &= \frac{1}{\|u\|^p_{L^p_\theta}}\int_0^\infty \varphi_p\big(\mu\,|u|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta,\end{aligned}$$

and the result is proved.

4.2. Maximizers for the critical Trudinger–Moser inequality

Next we combine Theorems 1.2 and 1.3 to demonstrate Theorem 1.4. Firstly, for $0 < s < \mu_{\alpha,\theta}$, we set

$$f(s) = \mathrm{TMSC}(s,\alpha,\theta) \quad \text{and} \quad g(s) = \mathrm{TMC}(s,\alpha,\theta).$$

Hence, Theorem 1.2 yields

$$g(\sigma) = \sup_{s\in(0,\sigma)} \Big(\frac{1-(s/\sigma)^{p-1}}{(s/\sigma)^{p-1}}\Big) f(s). \tag{4.5}$$

Lemma 4.2. *f is a continuous function on $(0,\mu_{\alpha,\theta})$.*

Proof. By using Theorem 1.3, we can pick $\varepsilon_n \downarrow 0$ and $u_n \in X^{1,p}_\infty$, with $\|u_n'\|_{L^p_\alpha} \le 1$ and $\|u_n\|_{L^p_\theta} = 1$, such that

$$f(s+\varepsilon_n) = \int_0^\infty \varphi_p\big((s+\varepsilon_n)\,|u_n|^{\frac{p}{p-1}}\big)\, \mathrm{d}\lambda_\theta.$$

Then

$$0 \le f(s+\varepsilon_n) - f(s) \le \int_0^\infty \left[\varphi_p\big((s+\varepsilon_n)|u_n|^{\frac{p}{p-1}}\big) - \varphi_p\big(s|u_n|^{\frac{p}{p-1}}\big)\right] \mathrm{d}\lambda_\theta. \tag{4.6}$$

Without loss of generality, we also may assume that (cf. (1.7))

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } X^{1,p}_\infty, \\ &u_n \to u \quad \text{in } L^q_\theta,\ q > p, \quad \text{and} \quad u_n(r) \to u(r) \quad \text{a.e in } (0,\infty). \end{aligned}$$

In particular,

$$\varphi_p\big((s+\varepsilon_n)|u_n(r)|^{\frac{p}{p-1}}\big) - \varphi_p\big(s|u_n(r)|^{\frac{p}{p-1}}\big) \to 0 \quad \text{a.e in } (0,\infty).$$

In the same way as in (4.3), we can use Lemma 4.1 and Theorem A to obtain a positive constant $C(p,q,s,\theta,R)$ such that

$$\int_0^R \left[\varphi_p\big((s+\varepsilon_n)|u_n|^{\frac{p}{p-1}}\big) - \varphi_p\big(s|u_n|^{\frac{p}{p-1}}\big)\right]^q \mathrm{d}\lambda_\theta \le C(p,q,s,\theta,R),$$

for some $q > 1$ and for all $R > 0$. It follows that

$$\int_0^R \left[\varphi_p\big((s+\varepsilon_n)|u_n|^{\frac{p}{p-1}}\big) - \varphi_p\big(s|u_n|^{\frac{p}{p-1}}\big)\right] \mathrm{d}\lambda_\theta \to 0.$$

On the other hand, for R large enough, Lemma 4.1 yields

$$|u_n(r)| \le 1, \quad \text{for every } n \in \mathbb{N},\ r \ge R.$$

Then

$$\begin{aligned} &\int_R^\infty \left[\varphi_p\big((s+\varepsilon_n)|u_n|^{\frac{p}{p-1}}\big) - \varphi_p\big(s|u_n|^{\frac{p}{p-1}}\big)\right] \mathrm{d}\lambda_\theta \\ &\quad = \int_R^\infty \sum_{j\in\mathbb{N}\,:\, j\ge p-1} \left[\frac{(s+\varepsilon_n)^j}{j!} - \frac{s^j}{j!}\right] |u_n|^{\frac{jp}{p-1}} \mathrm{d}\lambda_\theta \\ &\quad \le \sum_{j\in\mathbb{N}\,:\, j\ge p-1} \left[\frac{(s+\varepsilon_n)^j}{j!} - \frac{s^j}{j!}\right] \int_R^\infty |u_n|^p \mathrm{d}\lambda_\theta \le [\varphi_p(s+\varepsilon_n) - \varphi_p(s)] \to 0. \end{aligned}$$

From (4.6), we obtain

$$0 \le f(s+\varepsilon_n) - f(s) \to 0, \quad \text{as } n \to \infty.$$

Similarly, we also have that

$$0 \le f(s) - f(s-\varepsilon_n) \to 0, \quad \text{as } n \to \infty. \qquad \blacksquare$$

Now, in order to ensure the existence of an extremal function for $\mathrm{TMC}(\sigma,\alpha,\theta)$ when $0 < \sigma < \mu_{\alpha,\theta}$, it is sufficient to show that

$$\limsup_{s\to 0^+} \left(\frac{1-(s/\sigma)^{p-1}}{(s/\sigma)^{p-1}}\right) f(s) < g(\sigma) \tag{4.7}$$

and

$$\limsup_{s\to\sigma^-}\Big(\frac{1-(s/\sigma)^{p-1}}{(s/\sigma)^{p-1}}\Big)f(s) < g(\sigma). \tag{4.8}$$

Indeed, (4.7), (4.8) together with (4.5) and Lemma 4.2 ensure the existence of $s_\sigma \in (0,\sigma)$ such that

$$g(\sigma) = \Big(\frac{1-(s/\sigma)^{p-1}}{(s_\sigma/\sigma)^{p-1}}\Big)f(s_\sigma). \tag{4.9}$$

Let u_σ be an extremal function for $\mathrm{TMSC}(s_\sigma,\alpha,\theta)$ ensured by Theorem 1.3. Set

$$v_\sigma(r) = \Big(\frac{s_\sigma}{\sigma}\Big)^{\frac{p-1}{p}} u_\sigma(\tau r),$$

where

$$\tau = \left(\frac{(s_\sigma/\sigma)^{p-1}\|u_\sigma\|^p_{L^p_\theta}}{1-(s_\sigma/\sigma)^{p-1}}\right)^{\frac{1}{\theta+1}}.$$

From (2.2), it follows that

$$\|v_\sigma\|^p = \|v_\sigma'\|^p_{L^p_\alpha} + \|v_\sigma\|^p_{L^p_\theta} = \Big(\frac{s_\sigma}{\sigma}\Big)^{p-1}\big[\|u_\sigma'\|^p_{L^p_\alpha} + \tau^{-(\theta+1)}\|u_\sigma\|^p_{L^p_\theta}\big] \le 1.$$

We also have (cf. (4.9))

$$\begin{aligned}\mathrm{TMC}(\sigma,\alpha,\theta) &= \frac{1-(s/\sigma)^{p-1}}{(s_\sigma/\sigma)^{p-1}}\,\frac{\tau^{\theta+1}}{\|u_\sigma\|^p_{L^p_\theta}}\int_0^\infty \varphi_p\big(\sigma|v_\sigma|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta\\ &= \int_0^\infty \varphi_p\big(\sigma|v_\sigma|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.\end{aligned}$$

Hence, v_σ is an extremal function of $\mathrm{TMC}(\sigma,\alpha,\theta)$. Now, since

$$\limsup_{s\to\sigma^-}\Big(\frac{1-(s/\sigma)^{p-1}}{(s/\sigma)^{p-1}}\Big)f(s) = 0 < g(\sigma),$$

it is clear that (4.8) holds.

Next, we will prove that (4.7) holds. Firstly, we provide the following useful estimate.

Lemma 4.3. *For all $q \ge p$ and $0<\mu<\mu_{\alpha,\theta}$, we have*

$$\sup_{\|u'\|_{L^p_\alpha}\le 1,\ \|u\|^p_{L^p_\theta}=1}\int_0^\infty e^{\mu|u|^{\frac{p}{p-1}}}|u|^q\,\mathrm{d}\lambda_\theta \le c$$

for some $c = c(\mu,\alpha,\theta,q) > 0$.

Proof. We proceed as in Theorem 1.1. Indeed, let $u \in X^{1,p}_{\infty} \setminus \{0\}$, with $\|u'\|_{L^p_\alpha} \le 1$ and $\|u\|^p_{L^p_\theta} = 1$. From the Pólya–Szegö inequality obtained in [1], we can assume that u is a non-increasing function. We write

$$\int_0^\infty e^{\mu |u|^{\frac{p}{p-1}}} |u|^q \, \mathrm{d}\lambda_\theta = \int_{\{u<1\}} e^{\mu |u|^{\frac{p}{p-1}}} |u|^q \, \mathrm{d}\lambda_\theta + \int_{\{u\ge 1\}} e^{\mu |u|^{\frac{p}{p-1}}} |u|^q \, \mathrm{d}\lambda_\theta.$$

Of course we have

$$\int_{\{u<1\}} e^{\mu |u|^{\frac{p}{p-1}}} |u|^q \, \mathrm{d}\lambda_\theta \le e^\mu \int_{\{u<1\}} |u|^q \, \mathrm{d}\lambda_\theta \le e^\mu \int_{\{u<1\}} |u|^p \, \mathrm{d}\lambda_\theta \le e^\mu.$$

Set

$$I_u = \{r > 0 : u(r) \ge 1\}.$$

Without loss of generality, we can assume $I_u \neq \emptyset$. Thus, there is $R_u > 0$ such that $I_u = (0, R_u)$. Now, if

$$v(r) = u(r) - 1, \quad r \in (0, R_u),$$

we have $v \in X^{1,p}_{R_u}(\alpha, \theta)$ and $\|v'\|_{L^p_\alpha} \le 1$. Also, from Lemma 2.1 we have

$$|u|^{p/(p-1)} \le (1+\varepsilon)^{1/p} |v|^{p/(p-1)} + c_1(\varepsilon, \alpha, \theta),$$

for some $c_1 = c_1(\varepsilon, \alpha, \theta) > 0$. Hence, by choosing $\varepsilon > 0$ small enough and $\eta > 1$ such that $\mu(1+\varepsilon)^{1/p} \frac{\eta}{\eta-1} \le \mu_{\alpha,\theta}$, the Hölder inequality and Theorem A imply

$$\begin{aligned}
\int_0^{R_u} e^{\mu |u|^{\frac{p}{p-1}}} |u|^q \, \mathrm{d}\lambda_\theta &\le \Big(\int_0^{R_u} |u|^{\eta q} \, \mathrm{d}\lambda_\theta \Big)^{1/\eta} \Big(\int_0^{R_u} e^{\frac{\eta\mu}{\eta-1} |u|^{\frac{p}{p-1}}} \, \mathrm{d}\lambda_\theta \Big)^{(\eta-1)/\eta} \\
&\le C(\varepsilon, \alpha, \theta, \eta, \mu) \, \|u\|^q_{L^{\eta q}_\theta} \Big(\int_0^{R_u} e^{\mu_{\alpha,\theta} |v|^{\frac{p}{p-1}}} \, \mathrm{d}\lambda_\theta \Big)^{(\eta-1)/\eta} \\
&\le C(\varepsilon, \alpha, \theta, \eta, \mu) \, \|u\|^q_{L^{\eta q}_\theta} \big(|B_{R_u}|_\theta \big)^{(\eta-1)/\eta}.
\end{aligned} \tag{4.10}$$

Finally, since

$$|B_{R_u}|_\theta = \int_0^{R_u} \mathrm{d}\lambda_\theta \le \int_0^{R_u} |u|^p \, \mathrm{d}\lambda_\theta \le \|u\|_{L^p_\theta} = 1$$

and (cf. (1.7))

$$\|u\|^q_{L^{\eta q}_\theta} \le C \|u\|^q \le C(\alpha, \theta, q, \eta),$$

the inequality (4.10) gives the result. ∎

Since we are supposing $\mathrm{TMC}(\sigma, \alpha, \theta) > \sigma^{k_0}/k_0!$, when $k_0 = p-1$, to complete the proof of (4.7), and then the proof of Theorem 1.4, it is now enough to prove the following.

Lemma 4.4. *For each* $0 < \sigma < \mu_{\alpha,\theta}$*, we have*

$$\limsup_{s \to 0^+} \Big[\frac{1 - (s/\sigma)^{p-1}}{(s/\sigma)^{p-1}} \Big] f(s) \begin{cases} = 0, & \text{if } k_0 > p-1, \\ \le \dfrac{\sigma^{k_0}}{k_0!}, & \text{if } k_0 = p-1. \end{cases}$$

Proof. Let (s_n) be an arbitrary sequence such that $s_n \downarrow 0$. From Theorem 1.3, we can find a sequence $(u_n) \subset X^{1,p}_\infty$, with $\|u_n'\|_{L^p_\alpha} \le 1$ and $\|u_n\|_{L^p_\theta} = 1$, such that

$$\begin{aligned} f(s_n) &= \frac{s_n^{k_0}}{k_0!}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta + s_n^{k_0+1}\sum_{j\ge k_0+1}\int_0^\infty \frac{s_n^{j-(k_0+1)}}{j!}|u_n|^{\frac{jp}{p-1}}\,\mathrm{d}\lambda_\theta \\ &= \frac{s_n^{k_0}}{k_0!}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta + s_n^{k_0+1}\sum_{\ell=0}^{\infty}\int_0^\infty \frac{s_n^{\ell}}{(\ell+k_0+1)!}|u_n|^{\frac{\ell p}{p-1}+\frac{(k_0+1)p}{p-1}}\,\mathrm{d}\lambda_\theta. \end{aligned}$$

Since $(\ell+k_0+1)! \ge \ell!$ and in view of Lemma 4.3, we can write

$$\begin{aligned} f(s_n) &\le \frac{s_n^{k_0}}{k_0!}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta + s_n^{k_0+1}\int_0^\infty e^{\sigma|u_n|^{\frac{p}{p-1}}}|u_n|^{\frac{(k_0+1)p}{p-1}}\,\mathrm{d}\lambda_\theta \\ &\le \frac{s_n^{k_0}}{k_0!}\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta + s_n^{k_0+1}c(\sigma,\alpha,\theta). \end{aligned}$$

It follows that

$$\begin{aligned} &\Big(\frac{1-(s_n/\sigma)^{p-1}}{(s_n/\sigma)^{p-1}}\Big)f(s_n) \\ &\quad\le \Big(\frac{1-(s_n/\sigma)^{p-1}}{(1/\sigma)^{p-1}}\Big)\frac{s_n^{k_0-(p-1)}}{k_0!}\Big[\int_0^\infty |u_n|^{\frac{k_0 p}{p-1}}\,\mathrm{d}\lambda_\theta + c(\sigma,\alpha,\theta)k_0!s_n\Big]. \end{aligned} \tag{4.11}$$

Case 1: $k_0 > p-1$.

From (1.7) and (4.11), we obtain

$$\Big(\frac{1-(s_n/\sigma)^{p-1}}{(s_n/\sigma)^{p-1}}\Big)f(s_n) \to 0, \quad \text{as } n\to\infty.$$

Case 2: $k_0 = p-1$.

We have

$$\Big(\frac{1-(s_n/\sigma)^{p-1}}{(s_n/\sigma)^{p-1}}\Big)f(s_n) \le \Big(\frac{1-(s_n/\sigma)^{p-1}}{(1/\sigma)^{p-1}}\Big)\frac{1}{k_0!}\big[1+c(\sigma,\alpha,\theta)s_n\big] \to \frac{\sigma^{k_0}}{k_0!}, \quad \text{as } n\to\infty.$$

The proof is completed. ∎

5. Proof of Theorem 1.5

In this section, we will analyze the attainability of $\mathrm{TMC}(\sigma,\alpha,\theta)$ when the condition $k_0 = p-1$ holds.

Lemma 5.1. *For any $\sigma \in (0,\mu_{\alpha,\theta})$, we have*

$$\mathrm{TMC}(\sigma,\alpha,\theta) \ge \frac{\sigma^{k_0}}{k_0!}. \tag{5.1}$$

In addition, if $p>2$, the above inequality becomes strict.

Proof. We follow the argument of Ishiwata [21]. Let $u \in X^{1,p}_{\infty}(\alpha,\theta)$ be such that $\|u\| = 1$, and set

$$u_t(r) = t^{1/p}\, u(t^{1/(\theta+1)} r).$$

We can easily show that

$$\|u_t'\|^p_{L^p_\alpha} = t\|u'\|^p_{L^p_\alpha} \quad \text{and} \quad \|u_t\|^q_{L^q_\theta} = t^{(q-p)/p}\|u\|^q_{L^q_\theta}, \quad \forall\, q \geq p.$$

In particular, for each $t > 0$ small enough, if $\xi_t = (t + (1-t)\|u\|^p_{L^p_\theta})^{-1/p}$ we have

$$\|\xi_t u_t\|^p = t\,\xi_t^p\,\|u'\|^p_{L^p_\alpha} + \xi_t^p\,\|u\|^p_{L^p_\theta} = 1.$$

Noticing that $\xi_t^p \to 1/\|u\|^p_{L^p_\theta}$ as $t \to 0^+$, then for $v_t = \xi_t u_t$ we have

$$\begin{aligned}
\mathrm{TMC}(\sigma,\alpha,\theta) &\geq \int_0^\infty \varphi_p\big(\sigma\,|v_t|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta \\
&\geq \frac{\sigma^{k_0}}{k_0!}\int_0^\infty |v_t|^p\,\mathrm{d}\lambda_\theta + \frac{\sigma^{k_0+1}}{(k_0+1)!}\int_0^\infty |v_t|^{\frac{p(k_0+1)}{p-1}}\,\mathrm{d}\lambda_\theta \\
&= \frac{\sigma^{p-1}}{(p-1)!}\Big[\xi_t^p\,\|u\|^p_{L^p_\theta} + \frac{\sigma}{p}\,\xi_t^{\frac{p}{p-1}+p}\,\|u\|^{\frac{p^2}{p-1}}_{L^{p^2/(p-1)}_\theta}\; t^{\frac{1}{p-1}}\Big] \\
&\to \frac{\sigma^{p-1}}{(p-1)!}, \qquad \text{as } t \to 0.
\end{aligned} \tag{5.2}$$

This proves (5.1). Moreover, if $p > 2$, we observe that the function

$$h_{p,\theta,\sigma}(t) = \xi_t^p\,\|u\|^p_{L^p_\theta} + \frac{\sigma}{p}\,\xi_t^{p/(p-1)+p}\,\|u\|^{p^2/(p-1)}_{L^{p^2/(p-1)}_\theta}\,t^{1/(p-1)}$$

satisfies $h_{p,\theta,\sigma}(0) = 1$ and $h'_{p,\theta,\sigma}(t) > 0$ for $t > 0$ small enough. Hence, the result follows from (5.2). ∎

Lemma 5.2.

(i) *The function* $\sigma \mapsto \frac{(p-1)!}{\sigma^{p-1}}\mathrm{TMC}(\sigma,\alpha,\theta)$ *is non-decreasing for* $0 < \sigma \leq \mu_{\alpha,\theta}$.

(ii) *Let* $0 < \sigma_1 < \sigma_2 \leq \mu_{\alpha,\theta}$. *Suppose that* $\mathrm{TMC}(\sigma_1,\alpha,\theta)$ *is attained. Then*

$$\frac{(p-1)!}{\sigma_2^{p-1}}\mathrm{TMC}(\sigma_2,\alpha,\theta) > \frac{(p-1)!}{\sigma_1^{p-1}}\mathrm{TMC}(\sigma_1,\alpha,\theta),$$

and $\mathrm{TMC}(\sigma_2,\alpha,\theta)$ *is also attained.*

Proof. (i) Since

$$\frac{(p-1)!}{\sigma^{p-1}}\,\varphi_p\big(\sigma|t|^{\frac{p}{p-1}}\big) = (p-1)!\sum_{j=p-1}^{\infty}\frac{\sigma^{j-(p-1)}}{j!}\,t^{\frac{jp}{p-1}},$$

it is clear that for all $t \neq 0$,

$$\frac{(p-1)!}{\sigma_1^{p-1}}\varphi_p\big(\sigma_1|t|^{\frac{p}{p-1}}\big) < \frac{(p-1)!}{\sigma_2^{p-1}}\varphi_p\big(\sigma_2|t|^{\frac{p}{p-1}}\big), \quad 0<\sigma_1<\sigma_2\le\mu_{\alpha,\theta}. \tag{5.3}$$

Thus, (i) is proved.

(ii) Since $\mathrm{TMC}(\sigma_1,\alpha,\theta)$ is attained, we can pick $u \in X_\infty^{1,p}$ such that $\|u\| = 1$ and

$$\mathrm{TMC}(\sigma_1,\alpha,\theta) = \int_0^\infty \varphi_p\big(\sigma_1|u|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta.$$

Thus, Lemma 5.1 and (5.3) yield

$$\begin{aligned}\frac{(p-1)!}{\sigma_2^{p-1}}\mathrm{TMC}(\sigma_2,\alpha,\theta) &\ge \frac{(p-1)!}{\sigma_2^{p-1}}\int_0^\infty \varphi_p\big(\sigma_2\,|u|^{\frac{p}{p-1}}\,\big)\,\mathrm{d}\lambda_\theta\\ &> \frac{(p-1)!}{\sigma_1^{p-1}}\int_0^\infty \varphi_p\big(\sigma_1\,|u|^{\frac{p}{p-1}}\,\big)\,\mathrm{d}\lambda_\theta = \frac{(p-1)!}{\sigma_1^{p-1}}\mathrm{TMC}(\sigma_1,\alpha,\theta)\ge 1.\end{aligned}$$

Then we have that $\frac{(p-1)!}{\sigma_2^{p-1}}\mathrm{TMC}(\sigma_2,\alpha,\theta) > 1$, and thus part (ii) of Theorem 1.4 asserts that $\mathrm{TMC}(\sigma_2,\alpha,\theta)$ is attained. ■

Proof of Theorem 1.5. (i) It follows directly from Lemma 5.2 and the definition of σ_*.

(ii) From Lemma 5.2, the function $\sigma \mapsto \frac{(p-1)!}{\sigma^{p-1}}\mathrm{TMC}(\sigma,\alpha,\theta)$ is strictly increasing on $(\sigma_*,\mu_{\alpha,\theta})$. Next, we will show that

$$\mathrm{TMC}(\sigma_*,\alpha,\theta) = \frac{\sigma_*^{p-1}}{(p-1)!}. \tag{5.4}$$

For our convention $\mathrm{TMC}(0,\alpha,\theta) = 0$, we may assume $\sigma_* \in (0,\mu_{\alpha,\theta})$. From Lemma 5.1, if (5.4) is not true we must have

$$\mathrm{TMC}(\sigma_*,\alpha,\theta) > \frac{\sigma_*^{p-1}}{(p-1)!}.$$

Thus, since $\sigma_* < \mu_\alpha$, Theorem 1.4(ii) implies that $\mathrm{TMC}(\sigma_*,\alpha,\theta)$ is achieved for some $u_* \in X_\infty^{1,p}$. Also, we have

$$\lim_{\sigma\to\sigma_*}\int_0^\infty \varphi_p\big(\sigma|u_*|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta = \int_0^\infty \varphi_p\big(\sigma_*|u_*|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta = \mathrm{TMC}(\sigma_*,\alpha,\theta) > \frac{\sigma_*^{p-1}}{(p-1)!}.$$

Hence, if $\sigma \in (0,\sigma_*)$ is sufficiently close to σ_*, we must have

$$\mathrm{TMC}(\sigma,\alpha,\theta) \ge \int_0^\infty \varphi_p\big(\sigma|u_*|^{\frac{p}{p-1}}\big)\,\mathrm{d}\lambda_\theta > \frac{\sigma_*^{p-1}}{(p-1)!} > \frac{\sigma^{p-1}}{(p-1)!}.$$

Thus, for such a $\sigma \in (0,\sigma_*)$, Theorem 1.4(ii) implies that $\mathrm{TMC}(\sigma,\alpha,\theta)$ is achieved, which contradicts the definition of σ_*. This proves (5.4). Now, from (5.4) and Lemma 5.2(ii), for each $\sigma \in (\sigma_*,\mu_{\alpha,\theta})$, the supremum $\mathrm{TMC}(\sigma,\alpha,\theta)$ is attained and we also have

$$\frac{(p-1)!}{\sigma^{p-1}}\mathrm{TMC}(\sigma,\alpha,\theta) > \frac{(p-1)!}{\sigma_*^{p-1}}\mathrm{TMC}(\sigma_*,\alpha,\theta) = 1. \tag{5.5}$$

In addition, Lemma 5.1, Theorem 1.4 (ii) and the definition of σ_* yield

$$\mathrm{TMC}(\sigma,\alpha,\theta) = \frac{\sigma^{p-1}}{(p-1)!}, \quad \text{for each } \sigma \in [0,\sigma_*]. \tag{5.6}$$

Now, it is clear that (5.5) and (5.6) give (1.10). Finally, let us denote

$$\bar{\sigma}_* = \inf\left\{\sigma \in (0,\mu_{\alpha,\theta}) : \mathrm{TMC}(\sigma,\alpha,\theta) > \sigma^{p-1}/(p-1)!\right\}.$$

Then, Theorem 1.4 (ii) yields $\sigma_* \le \bar{\sigma}_*$. If $\sigma_* < \bar{\sigma}_*$, we can pick $\sigma_0 \in (\sigma_*, \bar{\sigma}_*)$ for which we must have

$$\frac{(p-1)!}{\sigma_0^{p-1}}\mathrm{TMC}(\sigma_0,\alpha,\theta) > \frac{(p-1)!}{\sigma_*^{p-1}}\mathrm{TMC}(\sigma_*,\alpha,\theta) = 1,$$

that is,

$$\mathrm{TMC}(\sigma_0,\alpha,\theta) > \frac{\sigma_0^{p-1}}{(p-1)!},$$

which contradicts the definition of $\bar{\sigma}_*$. Hence (1.11) holds. Finally, (iii) follows directly from Lemma 5.1. ∎

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