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Extensions of a residually finite group by a weakly sofic group are weakly sofic

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Abstract. In this paper, we show that residually-finite-by-weakly-sofic extensions are weakly sofic. More precisely, we show that if in an exact sequence of groups $1 \to N \hookrightarrow K \twoheadrightarrow G \to 1$ the group G is residually finite and N is weakly sofic, then K is weakly sofic.

1. Introduction

In [8, 15], sofic groups have been defined in relation with the Gottschalk surjunctivity conjecture. It is an open question whether all groups are sofic. There is a hope that a non-sofic group may be constructed as an extension of a residually finite group by a finite one, [1, 10]. Notice, however, that an extension of an amenable group by a sofic group is sofic, [5]. (We say that a group K is an extension of G by N if there is an exact sequence $1 \to N \hookrightarrow K \twoheadrightarrow G \to 1$.)

The main result of [2] is an example of a non-approximable by $(U(n), \|\cdot\|_2)$ group. This example is a residually-finite-by-finite extension, it is not clear whether this example is a non-sofic group. Here, in contrast, we prove that every residually-finite-by-weakly-sofic extension is weakly sofic. In particular, residually-finite-by-residually-finite extensions are weakly sofic. It is an easy fact that all sofic groups are weakly sofic. The other inclusion is an open question.

The main result of the article is the following theorem.

Theorem 1.1. *Let H be a normal subgroup of a group K. If H is weakly sofic and $G = K/H$ is residually finite, then K is weakly sofic.*

Let us describe our approach to proving Theorem 1.1. Without loss of generality we may consider finitely generated K and G. By the Krasner–Kaloujnine embedding theorem, see [3, 11], any extension of G by H is embeddable in the (unrestricted) wreath product $H \wr G$. (Throughout this article, "wreath product" means "unrestricted wreath product", as well as "direct product" means "unrestricted direct product".) So, it suffices to show that $H \wr G$ is weakly sofic. Recall that the wreath product $H \wr G$ is a semidirect

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product $H^G \rtimes G$ with an action $(g.f)(x) = f(xg)$, for $g \in G$ and $f \in H^G$. In particular, $(f,g)(f',g') = (f(g.f'), gg')$. To show the weak soficity of $H \wr G$ we use the following characterization of weakly sofic groups.

Lemma 1.2. *Let K be a group. The following are equivalent*:

1) *K is weakly sofic*;
2) *every system of equations solvable in all finite groups is solvable over K*;
3) *K is a subgroup of a quotient of a direct product of finite groups.* (*The direct product may by uncountable.*)

This lemma is shown in [6] for a finitely generated group K. We prove it in full generality in Section 3. Let Sys(Fin) be the set of systems of equations solvable in all finite groups, see Definition 3.1 for details. Let $\hat{G}$ be the profinite completion of G. Now we are ready to formulate the main technical result that implies Theorem 1.1.

Theorem 1.3. *Let $\tilde{H}$ be a direct product of finite groups and G a finitely generated residually finite group. Let $\bar{w} \in$ Sys(Fyn). Then $\bar{w}$ is solvable over $\tilde{H} \wr G$ in $\tilde{H} \wr \hat{G}$.* (*$\tilde{H} \wr \hat{G}$ is an abstract wreath product, that is, we consider discontinuous functions $f \in \tilde{H}^{\hat{G}}$ as well.*)

Notice, that if X is a subgroup of Y, then, naturally, $H \wr X$ is a subgroup of $H \wr Y$. (We include $H^X \hookrightarrow H^Y$ as $f \to \tilde{f}$, where $\tilde{f}|_X = f$ and $\tilde{f}(y) = 1$ for $y \notin X$.) We will use such an inclusion in different places of the text as "obvious" without defining it explicitly. In particular, $\tilde{H} \wr G \hookrightarrow \tilde{H} \wr \hat{G}$, and Definition 3.1 makes sense in the context of Theorem 1.3.

We finish the introduction by describing the structure of the paper. In order to make the article more self-contained, we give definitions of sofic and weakly sofic groups in the next section. Section 3 recalls some definitions and results about group equations and establishes notations and terminology we are using. In the same section we show how Theorem 1.3 implies Theorem 1.1. In Section 4 we recall some definitions and notations of profinite completions of residually finite groups. In Section 5 we define an $\bar{a}$-universal solution and prove its existence. In Section 6 we finish the proof of Theorem 1.3.

2. Sofic and weakly sofic groups

The idea behind the concepts of sofic and weakly sofic groups is an approximation of a group by symmetric groups and by finite groups, respectively. For various equivalent definitions of sofic groups, see [7,12,14]. We will follow [7,14]. Let G be a group and let S be a group with a bi-invariant metric d. Bi-invariant means that $d(x,y) = d(sx,sy) = d(xs,ys)$ for every $x,y,s \in S$. For $X,Y \subset G$, let $XY = \{xy \mid x \in X,\ y \in Y\}$.

Definition 2.1. Let $\Phi \subseteq G$, $1 \in \Phi$. Let α and ε be non-negative real numbers. A map $\phi\colon (\Phi\Phi) \to S$ is called $(\Phi,\alpha,\varepsilon)$-homomorphism to a group S with a metric d if

- $\forall x,y \in \Phi,\ d(\phi(x)\phi(y), \phi(xy)) \le \varepsilon$,
- $\phi(1) = 1$,
- $d(\phi(x), 1) \ge \alpha$ for $x \in \Phi$, $x \ne 1$.

Remark. There is more general definition of $(\Phi, \alpha, \varepsilon)$-homomorphism where α is not a real number but a function $\alpha: G \to \mathbb{R}$ such that $\alpha(1) = 0$ and $\alpha(g) > 0$ for $g \neq 1$. Now, the third condition of Definition 2.1 is changed by $d(\phi(x), 1) \geq \alpha(x)$, see [14]. We do not need this more general definition to describe sofic and weakly sofic groups.

Let $\mathrm{Sym}(X)$ be a symmetric group on a finite set X. The Hamming distance d on $\mathrm{Sym}(X)$ is defined as $d(g, h) = \frac{1}{|X|}|\{x \in X \mid g(x) \neq h(x)\}|$.

Definition 2.2. A group G is sofic if for any finite $\Phi \subset G$, $1 \in \Phi$, and for any $\varepsilon > 0$, there exists a $(\Phi, 1/2, \varepsilon)$-homomorphism to a finite symmetric group with the Hamming distance.

Remark. There are several equivalent definitions. We can make an apparently more restrictive condition by changing $(\Phi, 1/2, \varepsilon)$-homomorphisms with $(\Phi, 1-\varepsilon, \varepsilon)$-homomorphisms. We can consider as well less restrictive conditions applying the definition of $(\Phi, \alpha, \varepsilon)$-homomorphism with a function $\alpha: G \to \mathbb{R}$ as described in the above Definition 2.1, see [14]. All these definitions are equivalent due to the so called amplification trick for the Hamming metric, see [4].

Definition 2.3. A group G is weakly sofic if for any finite $\Phi \subset G$, $1 \in \Phi$, and for any $\varepsilon > 0$, there exists a $(\Phi, 1/2, \varepsilon)$-homomorphism to a finite group with a bi-invariant metric.

Remark. One can see that conditions of Definition 2.3 are weaker than those of Definition 2.2. So sofic groups are weakly sofic. The other inclusion is an open question: although all finite groups are embeddable into finite symmetric groups, not all bi-invariant metric on a finite group is a restriction of a corresponding Hamming metric.

Immediately, by definition, it follows that soficity as well as weak soficity are local properties: K is sofic (weakly sofic) if any of its finitely generated subgroup is sofic (weakly sofic) respectively. Another useful simple fact: if G is sofic (weakly sofic), then any subgroup of G is sofic (weakly sofic).

3. Group equations

For a set X, we use the notation $X^* = \bigcup_{n \in \mathbb{N}} X^n$. Let $\bar{y} = (y_1, y_2, \ldots, y_j, \ldots)$ and $\bar{x} = (x_1, x_2, \ldots, x_j, \ldots)$ be countable sets of symbols for constants and variables, respectively. Let $F = F(\bar{y}, \bar{x})$ be the free group freely generated by $\bar{y}$ and $\bar{x}$. Let $\bar{w} \in F^*$. Notice that $\bar{w} \in F^r(y_1, \ldots, y_k, x_1, \ldots, x_n)$ for some $k, n, r \in \mathbb{N}$. By substitution, $\bar{w}$ defines a map $G^k \times G^n \to G^r$ for any group G. Consider the system of equations $\bar{w} = 1$.

Definition 3.1. We say that $\bar{w}$ is solvable in a group G if the sentence

$$\forall \bar{a} \in G^*,\ \exists \bar{x} \in G^* \text{ such that } \bar{w}(\bar{a}, \bar{x}) = 1$$

is valid. We say that a system $\bar{w}$ is solvable over a group G in H, $H > G$, if the sentence

$$\forall \bar{a} \in G^*,\ \exists \bar{x} \in H^* \text{ such that } \bar{w}(\bar{a}, \bar{x}) = 1$$

is valid. We say that $\bar{w}$ is solvable over a group G if it is solvable over G in some $H > G$.

Denote by $\mathrm{Sys}(G) \subseteq F^*$ the set of all finite systems of equations solvable in G. Let $\mathrm{Sys}(\mathrm{Fin}) = \bigcap_{|G|<\infty} \mathrm{Sys}(G)$. Specifying Corollary 19 of [6] for $\mathcal{K} = \mathrm{Fin}$ proves Lemma 1.2 for finitely generated K. In order to show Lemma 1.2 in full generality, we need the following two propositions about direct products of groups. (These propositions are probably well known, but we give a proof here for completeness.) In what follows, we say "X is a subquotient of Y" if X is a subgroup of a quotient of Y.

Proposition 3.2. *Let I be the set of all finitely generated subgroups of a group K. Let H_i, $i \in I$, be an indexed family of groups such that i is embeddable in H_i. Then K is a subquotient of the direct product $\prod_{i\in I} H_i$.*

Proof. Notice that I is partially ordered by inclusion. Let

$$N = \Big\{\alpha \in \prod_{i\in I} H_i \;\Big|\; \exists j \in I \text{ such that } \forall i \geq j, \alpha_i = 1\Big\}.$$

Using the fact that I is a lattice, we show that N is a normal subgroup of $\prod_{i\in I} H_i$. Indeed, let $\alpha, \beta \in N$ and let j_α (j_β) be such that $\alpha(i) = 1$ for $i \geq j_\alpha$ ($\beta(i) = 1$ for $i \geq j_\beta$). Then $\alpha_i\beta_i = 1$ for every $i > \langle j_\alpha, j_\beta\rangle$. This shows that N is a subgroup. Notice that α has the same support as its conjugate $\gamma\alpha\gamma^{-1}$. It follows that N is normal.

Now consider a map $K \to \prod_{i\in I} H_i$, $g \to \alpha(g)$, where

$$\alpha_i(g) = \begin{cases} 1 & \text{if } g \notin i, \\ g & \text{if } g \in i. \end{cases}$$

(Without loss of generality, we assume that $i < H_i$.) The composition of the map $g \to \alpha(g)$ with the natural quotient map $\prod_{i\in I} H_i \to \prod_{i\in I} H_i/N$ defines an inclusion $K \hookrightarrow \prod_{i\in I} H_i/N$. Indeed, $\alpha(g) \in N$ if and only if $g = 1$ by the definitions of α and N. On the other hand, $(\alpha(g)\alpha(h)\alpha^{-1}(gh))_i = 1$ for $i > \langle g, h\rangle$. So, $\alpha(g)\alpha(h)\alpha^{-1}(gh) \in N$. ∎

Proposition 3.3. *Let I and J be sets, and let H_{ij}, $i \in I$, $j \in J$, be an indexed family of groups. Let $N_i \triangleleft \prod_{j\in J} H_{ij}$. Then*

$$\prod_{i\in I}\Big(\prod_{j\in J} H_{ij}/N_i\Big) \cong \prod_{(i,j)\in I\times J} H_{ij}\Big/\prod_{i\in I} N_i.$$

Proof. It is almost a tautology; consider the natural inclusion $\prod_i N_i \hookrightarrow \prod_{(i,j)} H_{ij}$. ∎

Proof of Lemma 1.2. The implication 3) ⇒ 2) follows as "w is solvable in G_i" ⇒ "w is solvable in $\prod G_i$" ⇒ "w is solvable in a quotient of $\prod G_i$".

The implication 2) ⇒ 1). As every equation from $\mathrm{Sys}(\mathrm{Fin})$ is solvable over K, it is solvable over every finitely generated subgroup of K. So, by [6], every finitely generated subgroup of K is weakly sofic. Weak sofisity is a local property. It follows that K is weakly sofic.

The implication 1 ⇒ 3) is valid for every finitely generated subgroup of K. So, every finitely generated subgroup of K is a subquotient of a direct product of finite groups. So K is a subquotient of a direct product of finite groups by Propositions 3.2 and 3.3. ∎

3.1. Proof of Theorem 1.1

In this subsection we show how Theorem 1.1 follows from Theorem 1.3 and Lemma 1.2. Let H be weakly sofic and let G be a residually finite group. Let $\bar{w}(\bar{y}, \bar{x}) \in \mathrm{Sys}(\mathrm{Fin})$. By Lemma 1.2, it suffices to show that $\bar{w}$ is solvable over $H \wr G$. By Lemma 1.2 again, we know that $H < \tilde{H}/M$ for some direct product of finite groups $\tilde{H}$ and $M \lhd \tilde{H}$. There is a natural inclusion $H \wr G \hookrightarrow \tilde{H}/M \wr G$. Let $\bar{a} \in (H \wr G)^k < (\tilde{H}/M \wr G)^k$. Let $\tilde{a} \in (\tilde{H} \wr G)^k$ be a preimage of $\bar{a}$ under the natural homomorphism $\tilde{H} \wr G \to \tilde{H}/M \wr G$. By Theorem 1.3, there is a solution $\tilde{y} \in (\tilde{H} \wr \hat{G})^n$ of the equation $\bar{w}(\tilde{a}, \tilde{y}) = 1$. Obviously, an image of $\tilde{y}$ under the homomorphism $\tilde{H} \wr \hat{G} \to \tilde{H}/M \wr \hat{G}$ gives us a solution $\bar{y}$ of $\bar{w}(\bar{a}, \bar{y}) = 1$ in $\tilde{H}/M \wr \hat{G}$.

4. Profinite completion

Let G be a finitely generated residually finite group. Let $\mathfrak{M} = \{N \lhd G \mid G/N \text{ is finite}\}$, the set of co-finite normal subgroups of G. The order $N \preccurlyeq M \leftrightarrow N \supseteq M$ turns $\mathfrak{M}$ into a directed partially ordered set, see [13] for details. For $N \in \mathfrak{M}$, we denote $G_N = G/N$. For $N, M \in \mathfrak{M}$, $N \supseteq M$, let $\eta_{M,N} \colon G_M \to G_N$ be the natural homomorphisms. So, $I = (G_N, \eta_{M,N}, \mathfrak{M})$ is an inverse projective system of finite groups. Its inverse limit $\hat{G} = \lim_{\leftarrow I} G_N$ is the profinite completion of G, see [13]. A group $\hat{G}$ comes naturally with compatible epimorphisms $\eta_N \colon \hat{G} \to G_N$ and an inclusion $G \hookrightarrow \hat{G}$. The restriction of η_N on G is just the natural map $G \to G/N = G_N$; compatibility means that $\eta_N = \eta_{M,N} \circ \eta_M$ for every $N \supseteq M$. We will use the following notations. For $g \in \hat{G}$ ($g \in G_M$), let $g_N = \eta_N(g)$ ($g_N = \eta_{M,N}(g)$), respectively. If $\bar{g} = (g_1, g_2, \ldots, g_k) \in \hat{G}^k$, define $\bar{g}_N = ((g_1)_N, \ldots, (g_k)_N)$; if $\bar{f} = (f_1, \ldots, f_k)$, define $\bar{f}\bar{g} = (f_1 g_1, \ldots, f_k g_k)$. We will often use these notations in the situation when $\bar{g} \in G_N^k$ and $\bar{f} \in (H^{G_N})^k$, so, $\bar{f}\bar{g} \in (H \wr G_N)^k$.

Let $\bar{w} \in (F(\bar{y}, \bar{x}))^r$, $|\bar{y}| = k$ and $|\bar{x}| = n$.

Let $g, h \in \hat{G}$. By construction, $g \neq h$ if and only if $g_N \neq h_N$ for some $N \in \mathfrak{M}$. So the following lemma is valid.

Lemma 4.1. *Let $\bar{a} \in \hat{G}^k$ and $\bar{u} \in \hat{G}^n$ be such that $\bar{w}(\bar{a}_N, \bar{u}_N) = 1$ for every $N \in \mathfrak{M}$. Then $\bar{w}(\bar{a}, \bar{u}) = 1$.*

5. An $\bar{a}$-universal solution for $\bar{w}$

Let G be a finitely generated residually finite group and let $\hat{G}$ be its profinite completion. Fix $\bar{w} \in F^r(\bar{y}, \bar{x}) \cap \mathrm{Sys}(\mathrm{Fin})$ with $|\bar{y}| = k$, $|\bar{x}| = n$, and $|\bar{w}| = r$. Let $\bar{a} \in \hat{G}^k$.

Definition 5.1. $\bar{u} \in \hat{G}^n$ is called an $\bar{a}$-universal solution for $\bar{w}$ if the following statement is true:

$$\forall \text{ finite group } \Phi,\ \forall N \in \mathfrak{M},\ \forall \bar{f} \in (\Phi^{G_N})^k,\ \exists \bar{\phi} \in (\Phi^{G_N})^n \text{ such that } \bar{w}(\bar{f}\bar{a}_N, \bar{\phi}\bar{u}_N) = 1.$$

Lemma 5.2. *Each $\bar{w}$ has an $\bar{a}$-universal solution $\bar{u} \in \hat{G}^n$.*

Proof. For $N \in \mathfrak{M}$, we use the notation $\mathfrak{M}_N = \{M \in \mathfrak{M} \mid N \subseteq M\}$. In particular, $\mathfrak{M}_N$ is finite and $N \in \mathfrak{M}_N$. Let Φ be a finite group. Define

$$X_N^\Phi = \{\bar{u} \in (G_N)^n \mid \forall M \in \mathfrak{M}_N,\ \forall \bar{f} \in (\Phi^{G_M})^k,\ \exists \bar{\phi} \in (\Phi^{G_M})^n \text{ s.t. } \bar{w}(\bar{f}\bar{a}_M, \bar{\phi}\bar{u}_M) = 1\},$$

and

$$X_N = \bigcap_{|\Phi| \le \infty} X_N^\Phi$$

By definition, $\eta_{N,M}(X_N) \subseteq X_M$ for $N \subseteq M$. Notice that $\bar{u} \in X = \lim_{\leftarrow I} X_N$ would provide a proof of the lemma. So it suffices to show that $X \neq \emptyset$, or alternatively (by the properties of inverse limits of finite sets) that $X_N \neq \emptyset$ for all $N \in \mathfrak{M}$. Fix $N \in \mathfrak{M}$. Suppose, seeking for a contradiction, that $X_N = \emptyset$. As the sets X_N^Φ are finite, it follows that $X_N^{\Phi_1} \cap X_N^{\Phi_2} \cap \cdots \cap X_N^{\Phi_j} = \emptyset$ for some finite groups $\Phi_1, \Phi_2, \ldots, \Phi_j$. But then $X_N^\Phi = \emptyset$ for $\Phi = \Phi_1 \times \Phi_2 \cdots \times \Phi_j$.

So it suffices to show that $X_N^\Phi \neq \emptyset$ for every finite group Φ. Fix a finite group Φ. Let $\tilde{D}_M = (\Phi^{G_M})^m$, with $m = |\Phi|^{k|G_M|}$. Notice that $\tilde{D}_M$ has m different projections $\mathrm{Pr}_j \colon \tilde{D}_M \to \Phi^{G_M}$. Define now $\mathrm{Pr}_j \colon \tilde{D}_M^k \to (\Phi^{G_M})^k$ as follows: $\mathrm{Pr}_j(f_1, f_2, \ldots, f_k) = (\mathrm{Pr}_j(f_1), \mathrm{Pr}_j(f_2), \ldots, \mathrm{Pr}_j(f_k))$. By the choice of m, there exists a universal $\bar{f} \in D_M^k$ in the sense that among $\mathrm{Pr}_j(\bar{f})$, all elements of $(\Phi^{G_M})^k$ appear. Let $D_N = \prod_{M \in \mathfrak{M}_N} \tilde{D}_M$. A group G_N has a natural action on Φ^{G_M} for $M \in \mathfrak{M}_N$:

$$(g.f)(x) = f(xg_M), \quad \text{where } g \in G_N, f \in \Phi^{G_M}.$$

So G_N has an action on D_N defined componentwise as above. Consider the corresponding semidirect product $D_N \rtimes G_N$. As explained above, we may choose $\bar{f} \in D_N^k$ such that every its $\tilde{D}_M^k$ component is universal. Notice that the set

$$\tilde{X}_N^\Phi = \{(\bar{\phi}, \bar{u}) \in D_N \ltimes G_N \mid \bar{w}(\bar{f}\bar{a}_N, \bar{\phi}\bar{u}) = 1\}$$

is nonempty as $D_N \rtimes G_N$ is a finite group and $\bar{w} \in \mathrm{Sys}(\mathrm{Fin})$. On the other hand, X_N^Φ is the projection of $\tilde{X}_N^\Phi$ on $\bar{u}$ by the universality of $\bar{f}$. ∎

6. Proof of Theorem 1.3

Let $\bar{u} = (u_1, \ldots, u_n) \in \hat{G}^n$ be an $\bar{a}$-universal solution for $\bar{w} \in \mathrm{Sys}(\mathrm{Fin})$. Let Φ be a finite group and let $\Gamma = \langle G, \bar{u} \rangle \le \hat{G}$. Consider Φ^Γ as a compact topological group with respect to the Tichonov (direct product) topology. First of all we prove Theorem 1.3 for $\tilde{H} = \Phi$. The proof is topological and uses the profinite topology on Γ and the Tichonov topology on Φ^Γ. Denote by $\mathfrak{R}(\Phi) \subseteq \Phi^\Gamma$ the set of continuous function with respect to the profinite topology on Γ.

Lemma 6.1. *Let $\bar{f} \in \mathfrak{R}(\Phi)^k$. Then there exists $\bar{\phi} \in \mathfrak{R}(\Phi)^n$ that solves the equation $\bar{w}(\bar{f}\bar{a}, \bar{\phi}\bar{u}) = 1$.*

Proof. Let $\bar{f} \in \mathfrak{R}(\Phi)^k$. It follows that $\bar{f} = \tilde{f} \circ \gamma$ for some $\gamma \colon \Gamma \to \Gamma_N$, $N \in \mathfrak{M}$, and $\tilde{f} \in (\Phi^{\Gamma_N})^k$. By the universality of $\bar{u}$, it follows that there exists $\tilde{\phi} \in (\Phi^{\Gamma_N})^n$ such that $\bar{w}(\tilde{f}\bar{a}_N, \tilde{\phi}\bar{u}_N) = 1$. Then for $\bar{\phi} = \tilde{\phi} \circ \gamma$, we get $\bar{w}(\bar{f}\bar{a}, \bar{\phi}\bar{u}) = 1$. ∎

Lemma 6.2. *The set $\Re(\Phi)$ is dense in Φ^Γ (with respect to the Tichonov topology).*

Proof. Indeed, for any finite $A \subseteq \Gamma$ there is a homomorphism γ from Γ to a finite group such that $\gamma|_A$ is injective. The lemma follows by the definition of the Tichonov topology. ∎

For $\bar{f} \in \Phi^\Gamma$, we are going to find a solution of $\bar{w}(\bar{f}\bar{a}, \bar{x}) = 1$ in $\Phi \wr \Gamma$. The solution we are looking for is in the form $\bar{x} = \bar{\phi}\bar{u}$ for some $\bar{\phi} \in (\Phi^\Gamma)^n$. Its existence follows by Lemmas 6.1 and 6.2, and the compactness of Φ^Γ. Indeed, let $\bar{f}_j \in (\Re(\Phi))^k$ be a sequence converging to f: $\bar{f}_j \to \bar{f}$ as $j \to \infty$. By Lemma 6.1, there are $\bar{\phi}_j \in (\Phi^\Gamma)^n$ that solve the equation $\bar{w}(\bar{f}_j\bar{a}, \bar{\phi}_j\bar{u}) = 1$. By the compactness of Φ^Γ, we may assume, passing to a subsequence, that $\bar{\phi}_j \to \bar{\phi}$. Notice that the action of Γ is continuous on Φ^Γ and that the Γ part of $\bar{w}(\bar{f}\bar{a}, \bar{\phi}\bar{u})$ is independent of $\bar{f}$ and $\bar{\phi}$, and equals to 1. So, we conclude that $\bar{w}(\bar{f}\bar{a}, \bar{\phi}\bar{u}) = \lim_{j\to\infty} \bar{w}(\bar{f}_j\bar{a}, \bar{\phi}_j\bar{u}) = 1$. This proves Theorem 1.3 for $\tilde{H} = \Phi$. To finish the proof for direct products of finite groups, it suffices to notice that $(\prod \Phi_i)^G \cong \prod \Phi_i^G$, and one may find a solution componentwise.

7. Concluding remarks

It is also known that amenable-by-weakly-sofic extensions are weakly sofic, [9]. A class of sofic groups may be considered as a class of groups generalizing amenable and residually finite groups. Some proofs that works for amenable and residually finite groups may be generalized for sofic groups. So, is it possible to combine somehow the techniques of the present paper and those from [9] to prove that sofic-by-weakly-sofic extensions are weakly sofic?

The question whether residually-finite-by-residually finite extensions are sofic remains unanswered. Although there is a similar characterization of sofic groups: a group G is sofic if and only if every equation solvable in all permutation groups is solvable over G. The problem is that solvability in permutation groups is not enough to prove, say, the existence of universal solutions.

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