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Geometry and holonomy of indecomposable cones

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Abstract. We study the geometry and holonomy of semi-Riemannian, time-like metric cones that are indecomposable, i.e., which do not admit a local decomposition into a semi-Riemannian product. This includes irreducible cones, for which the holonomy can be classified, as well as non-irreducible cones. The latter admit a parallel distribution of null k-planes, and we study the cases $k = 1$ in detail. We give structure theorems about the base manifold and in the case when the base manifold is Lorentzian, we derive a description of the cone holonomy. This result is obtained by a computation of certain cocycles of indecomposable subalgebras in $\mathfrak{so}(1, n-1)$.

1. Introduction

1.1. Background

Cone constructions are a valuable tool in differential geometry to study overdetermined PDEs on manifolds. They are applied in conformal [15,16] and projective geometry [4,32] but the most striking example is Bär's classification of Riemannian manifolds with real Killing spinors [5]. Bär's observation that real Killing spinors on a Riemannian manifold (M, g) correspond to parallel spinors on the cone

$$(\check{M} = \mathbb{R}^{>0} \times M, \ \check{g} = dr^2 + r^2 g),$$

allows to apply several fundamental results in differential geometry: Berger's list of irreducible Riemannian holonomy groups [10] and the classification of those that belong to manifolds with parallel spinors by Wang [33], the understanding of the geometric structures that correspond to these holonomy groups, and finally Gallot's theorem [20] that the cone $(\check{M}, \check{g})$ over a complete manifold (M, g) is either flat or irreducible. This result allows to determine the geometry of (M, g): If the cone $(\check{M}, \check{g})$ is flat, then (M, g) has constant sectional curvature 1, and if the cone is irreducible, the geometry of (M, g) is determined by the special holonomy of the cone (Ricci-flat Kähler, hyper-Kähler, or exceptional).

One of the motivations to study semi-Riemannian cones is the Killing spinor equation on semi-Riemannian manifolds, but indefinite cones already become relevant in the

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Riemannian context. Indeed, *imaginary* Killing spinors on a Riemannian manifold (M, g) correspond to parallel spinors on the time-like cone

$$(\hat{M} = \mathbb{R}^{>0} \times M,\ \hat{g} = -\mathrm{d}r^2 + r^2 g). \tag{1.1}$$

Riemannian manifolds with imaginary Killing spinors were classified by Baum in [6, 7] without using the cone construction, but our results about Lorentzian cones in [2] allow to reprove Baum's classification.

Another motivation stems from supergravity (and string theory), where semi-Riemannian cones play a two-fold role. One the one hand, they appear as scalar geometries (of arbitrary dimension) in the superconformal formulation of supergravity theories, on the other hand, they can be used to study space-times which are part of supersymmetric solutions of the equations of motion of theories of (Poincaré) supergravity or of string theories. In the latter case, the supersymmetry equations can be analysed by passing to the time-like cone over the Lorentzian space-time manifold, which is a semi-Riemannian cone of index 2.

A generalisation of Bär's method to *indefinite* semi-Riemannian manifolds has two aspects: a holonomy classification of indefinite semi-Riemannian cones and the description of the corresponding geometry of the base. Both tasks face several difficulties in the semi-Riemannian context. The fundamental difficulty is that for metrics of arbitrary signature the holonomy group may not act completely reducibly: there are semi-Riemannian manifolds whose holonomy group admits an invariant subspace that is degenerate for the metric. As a consequence, those manifolds cannot be decomposed into a product of manifolds with irreducible holonomy, as it is the case for Riemannian manifolds. Hence, in an indefinite semi-Riemannian context, irreducibility has to be replaced by indecomposability. A semi-Riemannian manifold is *indecomposable* if its holonomy representation (i.e., the representation of the holonomy algebra on the tangent space) does not admit an invariant subspace that is non-degenerate for the metric. By the splitting theorems of de Rham [12] and Wu [34], such metrics do not have a local decomposition into product metrics, hence the term *indecomposable*. Therefore, a generalisation of Bär's method to semi-Riemannian geometry requires two steps:

(A) Generalise Gallot's theorem to the case of semi-Riemannian cones.

(B) For indecomposable semi-Riemannian cones, describe the holonomy of the cone and the local geometry of the base.

The problem in (A) was solved in [2], where we studied decomposable indefinite semi-Riemannian cones and obtained a generalisation of Gallot's result. In fact, we showed that a cone over a complete and compact semi-Riemannian manifold is either flat or indecomposable. The results in [2] have been generalised in the compact and in the complete case in [26–28]. Further results about decomposable cones have been obtained in Theorems 5 and 6 of [14]. Cones over Lorentzian Sasaki manifolds and their holonomy were studied in the decomposable and indecomposable case in [18, 19].

1.2. Results

In this article we deal with problem (B), i.e., we study the local geometry of the base and the holonomy of the cone in the case when the cone is *indecomposable*. This setting naturally splits into two different scenarios: the holonomy of the cone is *irreducible*, or it admits an *invariant subspace that is totally null but no non-degenerate invariant subspace*. The irreducible case is well understood as there is Berger's classification of irreducible holonomy groups [10], which we describe in Section 2.2 with the following result.

Theorem 1.1. *If* $(\widehat{M}, \widehat{g})$ *is a time-like cone with irreducible holonomy algebra* $\mathfrak{g}$*, then* $\mathfrak{g}$ *is isomorphic to one of the following Lie algebras*:

$$\begin{array}{lll} \mathfrak{so}(t,s), & \mathfrak{u}(p,q),\ \mathfrak{su}(p,q) \subset \mathfrak{so}(2p,2q), & \mathfrak{sp}(p,q) \subset \mathfrak{so}(4p,4q), \\ \mathfrak{so}(n,\mathbb{C}) \subset \mathfrak{so}(n,n), & \mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7,7), & \mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,8), \\ & \mathfrak{g}_2 \subset \mathfrak{so}(7), & \mathfrak{spin}(7) \subset \mathfrak{so}(8), \\ & \mathfrak{g}_{2(2)} \subset \mathfrak{so}(3,4), & \mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4). \end{array}$$

More interesting is the non-irreducible indecomposable case, where a holonomy classification is only available in specific situations [8, 9, 17–19, 25]. Here the cone admits a totally null vector distribution of rank $k > 0$ that is invariant under parallel transport or, equivalently, its space of sections is invariant under differentiation with respect to the Levi-Civita connection. In general, this case is rather difficult and no general holonomy classification is known. However, the parallel vector distribution determines the local structure of the base. This became obvious in [2] where we studied the case of *Lorentzian* indecomposable cones. As mentioned, some of our motivation comes from the equations of motion of supersymmetric theories of gravity, where the space-time metric is Lorentzian (that is of index 1). Hence, we will focus on cones that have index 2, that is, signature $(2, n-2)$. For these the totally null parallel vector distribution is of rank 1, i.e., a *null line*, but many of our results will hold for cones in arbitrary signature. We deal with the case of of rank 2, i.e., with parallel null planes, in Section 7 of [3]. In Section 3 we describe the local structure of the base as well as of the cone:

Theorem 1.2. *Let* $(\widehat{M}, \widehat{g})$ *be the time-like cone over a semi-Riemannian manifold* (M, g)*. If the cone admits a parallel null line field* $\mathbf{L}$*, then locally there is a parallel trivializing section of* $\mathbf{L}$*. Moreover, on a dense open subset* $\widehat{M}_{\mathrm{reg}} \subset \widehat{M}$*, the metric* $\widehat{g}$ *is locally isometric to a warped product of the form*

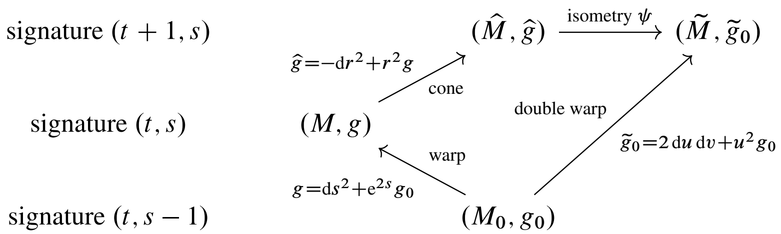
$$\widetilde{g}_0 = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g_0, \tag{1.2}$$

with a semi-Riemannian metric g_0*, and the metric* g *is locally of the form*

$$g = \mathrm{d}s^2 + \mathrm{e}^{2s} g_0.$$

In the case when the above decompositions hold globally (see Theorem 3.5), the situation can be summarised in the commutative diagram

(1.3)



Here $\widetilde{M} = \mathbb{R}^+ \times \mathbb{R}^- \times M_0$, see (3.1) for the definition of ψ. This result motivates the study of metrics of the form (1.2) in Section 4. Such metrics have a parallel null vector field ∂_v, and it was shown in [24] that their holonomy algebra $\widetilde{\mathfrak{g}} = \mathfrak{hol}(\widetilde{g}_0)$ is contained in $\mathfrak{hol}(g_0) \ltimes \mathbb{R}^{t,s}$, where (t,s) is the signature of the metric g_0, and moreover that $\mathrm{pr}_{\mathfrak{so}(n)}(\widetilde{\mathfrak{g}}) = \mathfrak{hol}(g_0)$. For a Lorentzian metric $\widetilde{g}$, i.e., when g_0 is Riemannian, it was shown in [2, 24] that we have, in fact,

$$\widetilde{\mathfrak{g}} = \mathfrak{hol}(g_0) \ltimes \mathbb{R}^n,$$

which means that the holonomy of the cone is determined solely by the holonomy of the metric g_0. In higher signatures, i.e., when g_0 is not Riemannian, this is no longer true, as examples will show. Our approach is to consider the ideal of translations in $\mathfrak{hol}(\widetilde{g}_0)$,

$$T := \mathfrak{hol}(\widetilde{g}_0) \cap \mathbb{R}^{t,s},$$

and use this for a first, purely algebraic study of indecomposable subalgebras in the stabiliser of a null vector. This will be carried out in Section 5.1, which is the most technical section of the paper. The key observation is that

$$\widetilde{\mathfrak{g}}/T = \{(X, \varphi(X)) \mid X \in \mathfrak{hol}(g_0)\}, \quad \text{with } \varphi \in Z^1(\mathfrak{hol}(g_0), \mathbb{R}^{t,s}/T),$$

where $Z^1(\mathfrak{hol}(g_0), \mathbb{R}^{t,s}/T)$ denotes the cocycles of $\mathfrak{hol}(g_0)$ with values in $\mathbb{R}^{t,s}/T$. For example, in order to obtain results for time-like cones over Lorentzian manifolds, we will compute $Z^1(\mathfrak{g}, \mathbb{R}^{1,n-1}/T)$, for indecomposable subalgebras $\mathfrak{g}$ of $\mathfrak{so}(1, n-1)$ (these belong to one of four types according to [8]). In Section 6 we apply these algebraic results to obtain the following theorem.

Theorem 1.3. *Let g_0 be a Lorentzian metric on an n-dimensional simply connected manifold M and $\widetilde{g}_0$ the metric of signature $(2,n)$ on $\mathbb{R}^+ \times \mathbb{R} \times M$ defined in (1.2). If the holonomy of $\widetilde{g}_0$ acts indecomposably and with invariant null line, then*

$$\mathfrak{hol}(\widetilde{g}_0) = \mathfrak{hol}(g_0) \ltimes \mathbb{R}^{1,n-1},$$

or g_0 admits a parallel null vector field and $\widetilde{g}_0$ admits two linearly independent parallel null vector fields that are orthogonal to each other.

This theorem shows that if the holonomy of $\widetilde{g}_0$ is not equal to the semi-direct product $\mathfrak{hol}(g_0) \ltimes \mathbb{R}^{1,n-1}$, then $\widetilde{g}_0$, and hence the cone admits a parallel null plane (which in addition is spanned by two parallel null vector fields). We study the case of cones admitting a totally null parallel 2-plane in Section 7 of [3].

2. Preliminaries

2.1. Fundamental properties of time-like cones

Let (M, g) be a semi-Riemannian manifold and let $\widehat{M} := \mathbb{R}^+ \times M$, with the metric

$$\widehat{g} := -\mathrm{d}r^2 + r^2 g, \tag{2.1}$$

be the *time-like cone* or just the *cone over* (M, g). We denote by

$$\xi = r\frac{\partial}{\partial r}$$

the *Euler vector field*. The Levi-Civita connection $\widehat{\nabla}$ of $\widehat{g}$ reduces to the Levi-Civita connection ∇ of g in the following way:

$$\widehat{\nabla}\xi = \mathrm{Id}_{TM}, \quad \widehat{\nabla}_X Y = \nabla_X Y + g(X, Y)\xi, \tag{2.2}$$

where here and in the following formulae $X, Y, Z \in \mathfrak{X}(M)$, and the curvature is given as

$$\xi \lrcorner \widehat{R} = 0, \quad \widehat{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y. \tag{2.3}$$

Hence, for the Ricci tensor, we obtain that

$$\xi \lrcorner \widehat{\mathrm{Ric}} = 0, \quad \widehat{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) + (n-1)g(X, Y). \tag{2.4}$$

This leads to the following observations.

Proposition 2.1. *Let* $(\widehat{M}, \widehat{g})$ *be the cone over* (M, g).

(1) (M, g) *has constant curvature* -1 *if and only if the cone* $(\widehat{M}, \widehat{g})$ *is flat.*
(2) *If* $(\widehat{M}, \widehat{g})$ *is Einstein, then it is Ricci-flat.*
(3) *If* (M, g) *is Einstein with* $\mathrm{Ric} = (1-n)g$, *then* $(\widehat{M}, \widehat{g})$ *is Ricci-flat.*

Finally, we recall the important known fact that the existence of a time-like vector field ξ, with $\nabla\xi = \mathrm{Id}$, characterises cones locally, see, for example, [21] or Lemma 1 of [14]. We include the proof here for expository reasons.

Proposition 2.2. *Let* $(\widehat{M}, \widehat{g})$ *be a semi-Riemannian manifold of dimension* $n+1$ *that admits a time-like vector field* ξ *such that* $\widehat{\nabla}\xi = \mathrm{Id}$. *Then there are local coordinates* $(r, x^1, \ldots x^n)$ *such that* $\widehat{g}$ *is of the form*

$$\widehat{g} = -\mathrm{d}r^2 + r^2 g_{ij}(x^1, \ldots, x^n)\, \mathrm{d}x^i\, \mathrm{d}x^j,$$

where i, j *run from* 1 *to* n, $g_{ij} = g_{ij}(x^1, \ldots, x^n)$ *are functions of the* x^k *coordinates only and we use the Einstein summation convention.*

Proof. The vector field ξ defines a positive function r via

$$\widehat{g}(\xi, \xi) = -r^2.$$

Differentiating this relation gives

$$2r\mathrm{d}r = \mathrm{d}(r^2) = -\mathrm{d}(\widehat{g}(\xi, \xi)) = -2g(\xi, \cdot) = -2\xi^\flat,$$

where the musical isomorphism $\flat$ denotes the metric dual with respect to $\widehat{g}$. Hence,

$$\xi^\flat = -\mathrm{d}\Big(\frac{r^2}{2}\Big)$$

is exact and therefore $\xi = -\widehat{\nabla} r^2/2$ is a gradient vector field. The level sets of the function r are orthogonal to ξ and we can fix coordinates $(x^1, \dots, x^n)$ on the level sets such that $(r, x^1, \dots, x^n)$ are local coordinates on $\widehat{M}$. In these coordinates the metric has the form

$$g = -\mathrm{d}r^2 + \widehat{g}_{ij}(r, x^1, \dots, x^n)\, \mathrm{d}x^i\, \mathrm{d}x^j,$$

and $\xi = r\partial_r$ holds. Since $\widehat{\nabla}\xi = \mathrm{Id}$, the vector field ξ is homothetic, $\mathcal{L}_\xi \widehat{g} = 2\widehat{g}$, which implies that

$$\widehat{g}_{ij}(r, x^1, \dots, x^n) = r^2 g_{ij}(x^1, \dots, x^n)$$

for some functions $g_{ij}(x^1, \dots, x^n)$ of the x^i coordinates. ■

2.2. The holonomy of irreducible cones

For irreducible cones, the possible holonomy groups are known from the Berger list [10], which comprises the orthogonal algebra and the three lists (2.5)–(2.7) below. In the following, let $\mathfrak{h} \subset \mathfrak{so}(t+1, q)$ be the irreducible holonomy algebra of a semi-Riemannian manifold $(\widehat{M}, \widehat{g})$, i.e., one of the entries in Berger's list. For each possible $\mathfrak{h}$, we will now determine if it can be the holonomy algebra of a cone.

Case 1: $\mathfrak{h} = \mathfrak{so}(t+1, s)$. This is the holonomy algebra of a generic semi-Riemannian manifold of signature $(t+1, s)$.

Proposition 2.3. *Let (M, g) be a semi-Riemannian manifold of signature (t, s) and of constant curvature $\kappa \neq -1$ and let $(\widehat{M}, \widehat{g})$ be the time-like cone over (M, g). Then we have $\mathfrak{hol}(\widehat{M}, \widehat{g}) = \mathfrak{so}(t+1, s)$.*

Proof. The curvature endomorphisms of (M, g) are of the form

$$R(X, Y) = \kappa(g(Y, \cdot)X - g(X, \cdot)Y).$$

Since the holonomy algebra contains all curvature endomorphisms, equation (2.3) shows that

$$\mathfrak{so}(t, s) \subset \mathfrak{hol}(\widehat{M}, \widehat{g}),$$

where $\mathfrak{so}(t, s)$ is included as the stabiliser of the vector ξ. Moreover, equations (2.2)–(2.3) show that

$$(\widehat{\nabla}_X \widehat{R})(Y, Z)\xi = -\widehat{R}(X, Y)Z = -2(g(Y, Z)X - g(X, Z)Y).$$

This establishes $\mathfrak{hol}(\widehat{M}, \widehat{g}) = \mathfrak{so}(t+1, s)$. ■

Case 2: $\mathfrak{h}$ is the holonomy of an irreducible symmetric space or one of the following algebras:

$$\begin{aligned} &\mathfrak{sp}(1) \oplus \mathfrak{sp}(p, q) \subset \mathfrak{so}(2p, 2q), \\ &\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(m, \mathbb{R}) \subset \mathfrak{so}(2m, 2m), \\ &\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(m, \mathbb{C}) \subset \mathfrak{so}(4m, 4m), \end{aligned} \tag{2.5}$$

where $p+q$ and m are greater than 1. In the first case the metric is quaternionic Kähler of signature $(4p, 4q)$ and in the second it is quaternionic para-Kähler. Examples of the third type are obtained by complexifying manifolds with holonomy of the first two types, as discussed below. In these examples, $(\widehat{M}, \widehat{g})$ is Einstein with *nonzero* Einstein constant, see Theorem 3 of [1]. Hence, these cases can be excluded as holonomy of cones, by Proposition 2.1.

Case 3: $\mathfrak{h}$ is one of the following:

$$\begin{array}{rr} \mathfrak{u}(p,q),\ \mathfrak{su}(p,q) \subset \mathfrak{so}(2p,2q), & \mathfrak{sp}(p,q) \subset \mathfrak{so}(4p,4q), \\ \mathfrak{g}_2 \subset \mathfrak{so}(7), & \mathfrak{spin}(7) \subset \mathfrak{so}(8), \\ \mathfrak{g}_{2(2)} \subset \mathfrak{so}(3,4), & \mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4). \end{array} \tag{2.6}$$

The geometric structures corresponding to these algebras do exist on cones over semi-Riemannian manifolds with certain structures. In fact, the following relations between structure on the base (M, g) and on the cone are well known (see, for example, [5] for the Riemannian case and [23] for the indefinite cases, and references therein):

(i) The cone over a (semi-Riemannian) Sasaki, Einstein–Sasaki or 3-Sasaki manifold is, respectively, a Kähler, Ricci-flat Kähler or hyper-Kähler manifold and hence has holonomy contained in $\mathfrak{u}(p,q)$, $\mathfrak{su}(p,q)$ or $\mathfrak{sp}(p,q)$.

(ii) The cone over a strict nearly-Kähler manifold of dimension 6, Riemannian or of signature $(2,4)$, has a parallel $\mathbf{G}_2$- or $\mathbf{G}_{2(2)}$-structure and hence has holonomy contained in $\mathfrak{g}_2$ or $\mathfrak{g}_{2(2)}$. Similarly, the cone over a nearly para-Kähler manifold, with $|\nabla J|^2 \neq 0$, has holonomy contained in $\mathfrak{g}_{2(2)}$, see Proposition 3.1 of [11].

(iii) The cone over a 7-manifold with a nearly-parallel $\mathbf{G}_2$-structure, Riemannian or of signature $(3,4)$, has a parallel $\mathbf{Spin}(7)$- or $\mathbf{Spin}(3,4)$-structure and hence has holonomy contained in $\mathfrak{spin}(7)$ or $\mathfrak{spin}(3,4)$.

The question remains, whether the holonomy of the cone is not only contained but actually *equal* to one of the algebras in the list (2.6). In the Riemannian setting (which corresponds to the case where the base of the time-like cone is negative definite), this can be established by using Gallot's theorem that the (space-like) cone over a *complete* Riemannian manifold (M, g) is either flat or irreducible, and then by constructing a complete (M, g) with the corresponding structure. For indefinite metrics, several gaps open up in this argument: our generalisation of Gallot's theorem in [2] assumes that (M, g) is compact and complete and implies that the cone is flat or *indecomposable*, but not necessarily irreducible. Hence, even if one constructed compact and complete indefinite semi-Riemannian manifolds with the above structures, the cone would not have to be irreducible, and hence its holonomy could be an indecomposable, non-irreducible subalgebra of the algebras in (2.6). We suspect, however, that for a "generic" semi-Riemannian manifold with one of the above structures, the cone has holonomy equal to the algebras in (2.6). An explicit way of constructing examples of cones with special holonomy is given below in Remark 2.7.

Case 4: $\mathfrak{h}$ is one of the following algebras:

$$\begin{array}{rr} \mathfrak{so}(n,\mathbb{C}) \subset \mathfrak{so}(n,n), & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(m,\mathbb{C}) \subset \mathfrak{so}(4m,4m), \\ \mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7,7), & \mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,8). \end{array} \tag{2.7}$$

Examples can be obtained by complexification as we will explain now in detail. In the case of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sp}(m,\mathbb{C})$, the metric is Einstein of nonzero scalar curvature (incompatible with a cone), whereas in the two exceptional cases, it is Ricci-flat.

Realisation of complex holonomy algebras. Let (M, g) be a connected real analytic manifold endowed with a real analytic semi-Riemannian metric. Then it is easy to see that M can be embedded into a connected complex manifold $M^{\mathbb{C}}$ with the following properties.

(1) There exists an atlas of $M^{\mathbb{C}}$ such that each of its charts $\varphi\colon U \to \mathbb{C}^n$ is real-valued on $U \cap M$ and the restrictions $\varphi|_{U\cap M}\colon U \cap M \to \mathbb{R}^n$, $U \cap M \neq \emptyset$, form an atlas of M.

(2) The metric coefficients $g_{ij}(x)$ with respect to the real coordinates $x = (x^1, \dots, x^n) = \varphi|_{U\cap M}$ are given by real power series converging in $U \cap M$.

(3) The power series $g_{ij}(z)$ in the holomorphic coordinates $z = (z^1, \dots, z^n) = \varphi$ converges in U for all i, j.

It follows that we can define a holomorphic symmetric tensor field $g^{\mathbb{C}}$ on $M^{\mathbb{C}}$ by

$$g^{\mathbb{C}}|_U = \sum g_{ij}(z)\, \mathrm{d}z^i\, \mathrm{d}z^j.$$

The tensor field is non-degenerate on a neighbourhood of M, and by restriction, we can always assume that it is non-degenerate on $M^{\mathbb{C}}$. Then it defines what is called a *holomorphic Riemannian metric* on $M^{\mathbb{C}}$. We will call $(M^{\mathbb{C}}, g^{\mathbb{C}})$ a *complexification* of (M, g). Recall that a pair consisting of a complex manifold and a holomorphic Riemannian metric on that manifold is called a *holomorphic Riemannian manifold*. Note that $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is unique as a germ of holomorphic Riemannian manifold along M.

We define the *holonomy algebra* of a holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ at $p \in M^{\mathbb{C}}$ as the Lie algebra spanned by all the skew-symmetric endomorphisms

$$((\nabla^{\mathbb{C}})^k_{v_1,\dots,v_k} R^{\mathbb{C}})(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T_p^{1,0} M^{\mathbb{C}}) \cong \mathfrak{so}(T_p M)^{\mathbb{C}},$$

where $v_1, \dots, v_{k+2} \in T_p^{1,0} M^{\mathbb{C}}$ and $k \geq 0$. Here $\nabla^{\mathbb{C}}$ denotes the (holomorphic) Levi-Civita connection of $g^{\mathbb{C}}$ and $R^{\mathbb{C}}$ its curvature tensor.

Proposition 2.4. *Let $(M^{\mathbb{C}}, g^{\mathbb{C}})$ be a complexification of a connected semi-Riemannian manifold (M, g). Then the holonomy algebra of $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is given by the complexification $\mathfrak{h}^{\mathbb{C}}$ of the holonomy algebra $\mathfrak{h}$ of (M, g).*

Proof. By the Ambrose–Singer theorem, for real analytic semi-Riemannian manifolds, we know that $\mathfrak{h}$ is spanned by all the endomorphisms $(\nabla^k_{v_1,\dots,v_k} R)(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T_p M)$, where $v_1, \dots, v_{k+2} \in T_p M$ and $k \geq 0$. From the definition of $g^{\mathbb{C}}$ as complex-analytic extension of g, it is clear that the Levi-Civita connection $\nabla^{\mathbb{C}}$ of $g^{\mathbb{C}}$ coincides with the complex-analytic extension of the Levi-Civita connection ∇ of g. The same relation holds for the curvature tensors and their covariant derivatives. This implies the proposition. ∎

Next we consider the real analytic manifold N of dimension $2n$ underlying the complex manifold $M^{\mathbb{C}}$. It carries a corresponding integrable complex structure J and we can identify (N, J) with $M^{\mathbb{C}}$. We endow N with the real analytic semi-Riemannian metric

$$g_N := 2\,\mathrm{Re}\, g^{\mathbb{C}}. \tag{2.8}$$

Note that g_N can be considered as a (fibrewise) real bilinear form on TN by means of the canonical identification

$$TN \cong T^{1,0}N, \quad X \mapsto X^{1,0} = \frac{1}{2}(X - iJX).$$

The factor 2 in (2.8) is chosen so that $g^{\mathbb{C}}$ is obtained by restricting (the complex bilinear extension of) g_N to $T^{1,0}N$.

We observe that the metric g_N can be defined on the real analytic manifold N underlying any holomorphic Riemannian manifold $(M^{\mathbb{C}}, g^{\mathbb{C}})$ irrespective of whether $(M^{\mathbb{C}}, g^{\mathbb{C}})$ is a complexification of a semi-Riemannian manifold (M, g).

Theorem 2.5. *Let $(M^{\mathbb{C}}, g^{\mathbb{C}})$ be a connected holomorphic Riemannian manifold and (N, g_N) the corresponding semi-Riemannian manifold. Then (N, g_N) has neutral signature and its holonomy algebra is isomorphic to the holonomy algebra of $(M^{\mathbb{C}}, g^{\mathbb{C}})$.*

Proof. Note first that $g_N(J\cdot, J\cdot) = -g_N$, since g_N is of type $(2,0) + (0,2)$ with respect to J. This implies that g_N has neutral signature, since J it maps a maximal definite subspace of T_pN to a maximal definite subspace of the same dimension and of opposite signature.

We consider first the Lie algebra $\mathfrak{so}(T_pN)$, $p \in N$, with respect to g_N and its subalgebra

$$\mathfrak{so}(T_pN)^J := \{A \in \mathfrak{so}(T_pN) \mid [A, J] = 0\}.$$

The latter can be considered as a complex Lie algebra with the complex structure $A \mapsto JA$. Indeed, JA is g_N-skew-symmetric as the product of a symmetric with a commuting skew-symmetric endomorphism. The symmetry of J follows from the fact that J is an anti-isometry squaring to minus one.

Now we claim that $\mathfrak{so}(T_pN)^J$ is canonically isomorphic to the complex Lie algebra $\mathfrak{so}(T_p^{1,0}N)$ with respect to $g^{\mathbb{C}}$. Using the metric g_N, we can identify $\mathfrak{so}(T_pN)^J$ with the set of real points in $\bigwedge^{2,0} T_pN \oplus \bigwedge^{0,2} T_pN$ and the latter can be identified with $\bigwedge^{2,0} T_pN \cong \bigwedge^2 T^{1,0}M$ by projecting to the $(2,0)$-component. Finally, using the metric $g^{\mathbb{C}}$, we can identify $\bigwedge^2 T^{1,0}M$ with $\mathfrak{so}(T_p^{1,0}N)$. This yields a canonical isomorphism

$$\Phi\colon \mathfrak{so}(T_pN)^J \to \mathfrak{so}(T_p^{1,0}N) \tag{2.9}$$

of complex vector spaces. It simply maps $A \in \mathfrak{so}(T_pN)^J$ to its restriction to $T^{1,0}N$. Therefore, it is even an isomorphism of Lie algebras.

Next we show, for all $v_1, \dots v_{k+2} \in T_pN$, that under the canonical isomorphism (2.9), the tensor $(\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2})$ is mapped to $(\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2})$, where $w_j = v_j^{1,0}$, ∇^N denotes the Levi-Civita connection of g_N and R^N its curvature. This implies the theorem, in virtue of the Ambrose–Singer theorem. First we show that ∇^N can be constructed from the holomorphic connection $\nabla^{\mathbb{C}}$. Let ∇' be the unique connection in $(TN)^{\mathbb{C}}$ with the following properties:

(1) $\nabla'_Z W = \nabla^{\mathbb{C}}_Z W$ for all holomorphic vector fields Z, W on $M^{\mathbb{C}}$.

(2) $\nabla'_{\bar{Z}} W = 0$ for all holomorphic vector fields Z, W on $M^{\mathbb{C}}$.

(3) ∇' is real, that is, it restricts to a connection in TN.

Notice that the above properties imply that the subbundles $T^{1,0}N$ and $T^{0,1}N$ are ∇'-parallel and, hence, that $\nabla' J = 0$. Moreover, using these properties, it is straightforward to check that ∇' is metric torsion-free, since $\nabla^{\mathbb{C}}$ is. This implies that ∇' (when considered as a connection in TN) coincides with the Levi-Civita connection ∇^N. As a consequence, we see that $\nabla^N J = 0$, and thus $(\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2}) \in \mathfrak{so}(T_pN)^J$. Now let X, Y be real vector fields on an open set $U \subset N$ which are infinitesimal automorphisms of J. Then we have the formula

$$(\nabla^N_X Y)^{1,0} = \nabla^{\mathbb{C}}_{X^{1,0}} Y^{1,0}. \tag{2.10}$$

This follows immediately from the defining properties of $\nabla' = \nabla^N$, by decomposing $X = Z + \bar{Z}$ and $Y = W + \bar{W}$, where $Z = X^{1,0}$, $W = Y^{1,0}$ are holomorphic. From (2.10), we deduce that

$$\Big(\big((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2})\big) v_{k+3}\Big)^{1,0} = \big((\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2})\big) w_{k+3},$$

for all $v_1, \dots, v_{k+3} \in T_pN$, where we recall that $w_j = v_j^{1,0}$. Since the left-hand side is precisely

$$\Phi\big((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2})\big) w_{k+3},$$

we can conclude that

$$\Phi\big((\nabla^N)^k_{v_1,\dots,v_k} R^N(v_{k+1}, v_{k+2})\big) = (\nabla^{\mathbb{C}})^k_{w_1,\dots,w_k} R^{\mathbb{C}}(w_{k+1}, w_{k+2}),$$

finishing the proof. ∎

This leads to the following consequence.

Corollary 2.6. *The complex holonomies*

$$\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n, n), \quad \mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7, 7), \quad \mathfrak{spin}(7, \mathbb{C}) \subset \mathfrak{so}(8, 8) \tag{2.11}$$

are holonomy algebras of time-like cones.

Proof. This follows from the above considerations and from the fact that the compact real forms of the complex holonomy algebras in (2.11) can be realised by timelike cones over negative definite manifolds. Indeed, if $(\widehat{M}, \widehat{g})$ is a time-like cone with holonomy $\mathfrak{so}(n)$, $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, or $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, then there is the Euler vector field $\xi \in \Gamma(T\widehat{M})$. Hence, the real analytic metric $\widehat{g}^{\mathbb{C}}$ on $\widehat{M}^{\mathbb{C}}$ has the holomorphic Euler vector field $\xi^{\mathbb{C}}$, with $\widehat{\nabla}^{\mathbb{C}} \xi^{\mathbb{C}} = \mathrm{Id}$. On the real manifold $N = \widehat{M}^{\mathbb{C}}$, we then have that $\eta = 2\,\mathrm{Re}\,\xi^{\mathbb{C}}$ satisfies $\nabla^N \eta = \mathrm{Id}$, as a consequence of equation (2.10) applied here to $N = \widehat{M}^{\mathbb{C}}$ instead of $M^{\mathbb{C}}$. By Proposition 2.2, we then get that N is locally a cone, which, by Theorem 2.5, has one of the complex holonomies in (2.11) as holonomy algebra. ∎

This proof can be made more explicit in local coordinates. Locally the metric $\widehat{g}$ is of the form

$$\widehat{g} = -\mathrm{d}r^2 + r^2 g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

with Euler vector field $\xi = r\partial_r \in \Gamma(T\widehat{M})$. The analytic metric $\widehat{g}^{\mathbb{C}}$ on $\widehat{M}^{\mathbb{C}}$ then is of the form

$$\widehat{g}^{\mathbb{C}} = -\mathrm{d}u^2 + u^2 g_{ij}(z^k)\,\mathrm{d}z^i\,\mathrm{d}z^j$$

with coordinates $(u = r + is, z^1, \dots, z^n)$, where $z^k = x^k + iy^k$, and holomorphic Euler vector field $\xi^{\mathbb{C}} = u\partial_u$, with $\widehat{\nabla}^{\mathbb{C}}\xi^{\mathbb{C}} = \mathrm{Id}$. Then the metric $\widehat{h} = \frac{1}{2}g_N$ on $N = \widehat{M}^{\mathbb{C}}$ is given by

$$\widehat{h} = -\mathrm{d}r^2 + \mathrm{d}s^2 + (r^2 - s^2)\,\mathrm{Re}(g_{ij}(z^k)\,\mathrm{d}z^i\,\mathrm{d}z^j) - 2rs\,\mathrm{Im}(g_{ij}(z^k)\,\mathrm{d}z^i\,\mathrm{d}z^j).$$

One can directly check that $\eta = r\partial_r + s\partial_s$ satisfies $\nabla^N\eta = \mathrm{Id}$. Moreover, the cone coordinate with respect to $\widehat{h}$ is given by $\rho = \sqrt{r^2 - s^2}$, which satisfies $\widehat{h}(\eta, \cdot) = -\rho\,\mathrm{d}\rho$.

Remark 2.7. Finally, we note that it is possible to construct examples of pseudo-Riemannian cones with these holonomies using different real forms of the complexified metrics and Proposition 2.4 and Corollary 2.6. For example, $\mathbf{SL}(2,\mathbb{R}) \times \mathbf{SL}(2,\mathbb{R})$ admits a unique left-invariant nearly pseudo-Kähler structure, which is a different real form of the complexification of the Riemannian nearly Kähler structure on $\mathbf{SU}(2) \times \mathbf{SU}(2)$, see [30]. Hence, the cone metrics are different real forms of the holomorphic cone metric. Since the cone over $\mathbf{SU}(2) \times \mathbf{SU}(2)$ has holonomy $\mathfrak{g}_2$, the cone over $\mathbf{SL}(2,\mathbb{R}) \times \mathbf{SL}(2,\mathbb{R})$ must have holonomy equal to $\mathfrak{g}_{2(2)}$.

2.3. Manifolds with parallel null line bundle

In the following manifolds with a parallel null line bundle will be crucial. In this section we will collect some facts about them.

Let (M, g) be a semi-Riemannian manifold with a *parallel null line bundle* $\mathbf{L}$, i.e., $\mathbf{L}$ is a rank 1 subbundle of TM, the fibres of which are null with respect to the metric g and invariant under parallel transport with respect to the Levi-Civita connection ∇ of g. This implies that every non-vanishing section $\chi \in \Gamma(\mathbf{L})$ satisfies

$$\nabla\chi = \alpha \otimes \chi, \tag{2.12}$$

for a uniquely determined 1-form α. Any vector field that satisfies equation (2.12) for some 1-form α is called a *recurrent vector field*.

Proposition 2.8. *Let χ be a recurrent vector field on a connected semi-Riemannian manifold (M, g). Then the function $f = g(\chi, \chi)$ is either everywhere positive, negative or zero. In particular, χ can only have zeros if $f \equiv 0$.*

Proof. The equation (2.12) yields the ODE $X(f) = 2\alpha(X)f$ for every vector field X. These ODEs imply that if f vanishes at a point, then f vanishes in a neighbourhood of this point. Due to the continuity of f, this shows that M is a disjoint union of the three open sets $\{f > 0\}$, $\{f < 0\}$ and $\{f = 0\}$. Now, since M is connected, the proposition follows. ■

Hence, locally, the existence of a parallel null line bundle is equivalent to the existence of a *recurrent null vector field*, where we recall that a vector field χ is null if $g(\chi, \chi) = 0$ and χ does not vanish, Definition 3 in Chapter 3 of [29]. Moreover, a nowhere vanishing recurrent vector field χ can be rescaled to parallel vector field $\lambda \cdot \chi$, for a non-vanishing function λ, if and only if the 1-form α is exact. Indeed, if $\alpha = dh$, then

$$\nabla(\mathrm{e}^{-h}\chi) = 0,$$

so that $\lambda = \mathrm{e}^{-h}$ and $\lambda \cdot \chi$ is parallel. Conversely, if $\lambda \cdot \chi$ is parallel, then

$$0 = R(X,Y)\chi = \mathrm{d}\alpha(X,Y)\chi$$

for all $X, Y \in TM$.

Hence, on simply connected manifolds (M,g), nowhere vanishing recurrent vector fields can be rescaled to parallel ones if and only if α is closed. The choice we have when locally choosing a recurrent vector field that spans a null line bundle $\mathbf{L}$ can be used to find special recurrent sections of $\mathbf{L}$.

Lemma 2.9. *Let* $\mathbf{L}$ *be a parallel null line bundle. Then locally there is a recurrent gradient vector field* χ *which spans* $\mathbf{L}$*. This vector field satisfies* $\nabla\chi = h\,\chi^\flat \otimes \chi$ *for a function* h*.*

Proof. Since $\mathbf{L}$ is parallel, the hyperplane distribution $\mathbf{L}^\perp = \{X \in TM \mid g(X,\mathbf{L}) = 0$ is parallel and hence involutive. Hence, by Frobenius' theorem, $\mathbf{L}^\perp$ is integrable and the integral manifolds are given as $f \equiv$ constant for some local function f. Hence, $\mathbf{L}^\perp = \ker(df)$, and the gradient $\chi := \mathrm{grad}(f)$ of f spans $\mathbf{L}$. Then χ is recurrent, i.e., we have $\nabla\chi = \alpha \otimes \chi$. But then $\chi = \mathrm{grad}(f)$ implies that

$$0 = \mathrm{d}\chi^\flat = \alpha \wedge \chi^\flat,$$

which shows that $\alpha = h\,\chi^\flat$ for a local function h. ∎

3. Cones with parallel null lines

In this section we assume that the cone (2.1) over a semi-Riemannian manifold (M,g) admits a null line that is invariant under parallel transport. We will show that locally this implies that the cone admits a parallel null vector field and that the base (M,g) is locally an exponential extension of a semi-Riemannian manifold (M_0, g_0), see Definition 3.2. The total space of the cone will then be shown to be locally isometric to a double warped extension $(\widetilde{M}, \widetilde{g})$ of (M_0, g_0), see Definition 3.2. This will generalise our results for Lorentzian cones in Section 9 of [2].

Proposition 3.1. *Let* $(\widehat{M}, \widehat{g})$ *be a timelike cone and assume that* $(\widehat{M}, \widehat{g})$ *admits a parallel null line* $\mathbf{L}$*. Then the following hold*:

(i) *The set* $\widehat{M}_{\mathrm{reg}}$*, where* $\mathbf{L}$ *is not perpendicular to the Euler vector field* ξ*, is open and dense and invariant under the flow of* ξ*. So, in particular,* $\widehat{M}_{\mathrm{reg}} = \widehat{M}_{\mathrm{reg}}$*, where* $M_{\mathrm{reg}} := \widehat{M}_{\mathrm{reg}} \cap M$*.*

(ii) $\mathbf{L}$ *is flat and, hence, locally (and globally if* M *is simply connected) there is a parallel null vector field that spans* $\mathbf{L}$*.*

Proof. By passing to the universal cover of $(\widehat{M}, \widehat{g})$, that is, to the cone over the universal cover of M, we can assume that M and $\widehat{M}$ are simply connected. Hence, we can assume that the parallel null line $\mathbf{L}$ is spanned by a nowhere vanishing recurrent vector field χ on $(\widehat{M}, \widehat{g})$. Then we decompose

$$\chi = f\,\xi + Z,$$

where Z is tangent to M and nowhere vanishing. We claim that the function f cannot vanish on a nonempty open set. If it did, formulae (2.2) would give

$$\alpha(X)Z = \widehat{\nabla}_X \chi = \nabla_X Z + g(X, Z)\xi$$

on the open set, and hence $g(X, Z) = 0$ for all $X \in TM$, which is a contradiction. This proves that the open set $\widehat{M}_{\mathrm{reg}} = \{p \in \widehat{M} \mid f(p) \neq 0\}$ is dense. The invariance of $\widehat{M}_{\mathrm{reg}}$ under the homothetic flow of ξ follows from the invariance of $\mathbf{L}$ under the flow. The latter is obtained by writing the Lie derivative as $\mathcal{L}_\xi = \widehat{\nabla}_\xi - \mathrm{Id}$ and using that $\mathbf{L}$ is parallel.

On $\widehat{M}_{\mathrm{reg}}$, we have

$$\mathrm{d}\alpha(X, \xi)\chi = \widehat{R}(X, \xi)\chi = 0$$

and

$$\mathrm{d}\alpha(X, Y)\chi = \widehat{R}(X, Y)\chi = \widehat{R}(X, Y)Z \in TM,$$

for $X, Y \in TM$. This implies $\mathrm{d}\alpha = 0$, since $\widehat{M}_{\mathrm{reg}}$ is dense, proving that $\mathbf{L}$ is flat. ∎

In the next proposition we describe an example of a cone with a parallel null line before showing that every example is locally of this form.

Definition 3.2. Let (M_0, g_0) be a semi-Riemannian manifold. Then the warped products $(M = \mathbb{R} \times M_0, g = ds^2 + e^{2s} g_0)$ and $(\widetilde{M} = \mathbb{R}^+ \times \mathbb{R}^- \times M_0, \widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g_0)$ will be called the *exponential extension* and the *double warped extension* of (M_0, g_0), respectively.

Proposition 3.3. *Let (M_0, g_0) be a semi-Riemannian manifold. The time-like cone $(\widehat{M}, \widehat{g})$ over the exponential extension (M, g) of (M_0, g_0) is globally isometric to the double warped extension $(\widetilde{M}, \widetilde{g})$ of (M_0, g_0). In particular, the cone admits the parallel null vector field ∂_v.*

Proof. The cone metric over (M, g) is given by

$$\widehat{g} = -\mathrm{d}r^2 + r^2 \mathrm{d}s^2 + r^2 e^{2s} g_0,$$

with $r \in \mathbb{R}^+$ and $s \in \mathbb{R}$. For the diffeomorphism

$$\begin{aligned} \psi\colon \widehat{M} &= \mathbb{R}^+ \times \mathbb{R} \times M_0 \ni (r, s, p) \\ &\mapsto \left(u = re^s, v = -\tfrac{1}{2} re^{-s}, p\right) \in \widetilde{M} = \mathbb{R}^+ \times \mathbb{R}^- \times M_0, \end{aligned} \tag{3.1}$$

one checks that

$$(\psi^{-1})^* \widehat{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g_0.$$

This proves the statement. ∎

Theorem 3.4. *Let $(\widehat{M}, \widehat{g})$ be a time-like cone over a semi-Riemannian manifold (M, g). Assume that $(\widehat{M}, \widehat{g})$ admits a parallel null line $\mathbf{L}$. Then the open dense subset $\widehat{M}_{\mathrm{reg}} \subset (\widehat{M}, \widehat{g})$, cf. Proposition 3.1, is locally isometric to the double warped extension $(\widetilde{M}, \widetilde{g})$ of a semi-Riemannian manifold (M_0, g_0), and the open dense subset $M_{\mathrm{reg}} \subset (M, g)$ is locally isometric to the exponential extension of (M_0, g_0).*

Proof. Since we have to show the existence of a *local* isometry, by Proposition 3.1, we can assume that $\mathbf{L}$ admits a parallel trivializing section χ. We write the parallel null vector field χ on $\widehat{M}$ as

$$\chi = \hat{f}\,\xi + \hat{Z},$$

with $\hat{Z}$ a nowhere vanishing vector field tangent to M and $\hat{f}$ a smooth function on $\widehat{M}$. We will show that $\hat{Z}$ defines a vector field Z on M. From

$$[\xi,\hat{Z}] = -d\hat{f}(\xi)\xi + [\xi,\chi] = -d\hat{f}(\xi)\xi - \widehat{\nabla}_\chi \xi = -d\hat{f}(\xi)\xi - \chi = -(d\hat{f}(\xi)+\hat{f})\xi - \hat{Z},$$

with $[\xi, \hat{Z}]$ being tangent to M, we get, on the one hand, that

$$\mathrm{d}\hat{f}(\xi) + \hat{f} = r\partial_r(\hat{f}) + \hat{f} = 0$$

and, on the other, that

$$[\xi,\hat{Z}] + \hat{Z} = 0.$$

The first equation shows that $\hat{f} = \frac{1}{r}f$, with f a function on M, and the second that $\hat{Z} = \frac{1}{r}Z$, with $Z = r\hat{Z}$ a vector field on M, i.e., $[\xi, Z] = 0$. Hence, we have

$$\chi = \frac{1}{r}(f\,\xi + Z).$$

Differentiating in direction of $X \in TM$, by (2.2), we get

$$0 = r\widehat{\nabla}_X\chi = (df(X) + g(X,Z))\xi + f\,X + \nabla_X Z,$$

which shows that $Z = -\mathrm{grad}(f)$, where grad denotes the gradient with respect to g, and

$$\nabla Z = -f\,\mathrm{Id}. \tag{3.2}$$

Hence, the distribution $Z^\perp$ on M is integrable and its leafs are given by the level sets of f. The vector field Z is not only a gradient but also a conformal vector field, since from (3.2), we compute

$$\mathcal{L}_Z g = -2fg.$$

Note also that on $M_{\mathrm{reg}} = \widehat{M}_{\mathrm{reg}} \cap M$, the vector field Z is transversal to the level sets of f. This follows from $df(Z) = g(\mathrm{grad}(f), Z) = -g(Z,Z) = -f^2$. Hence, locally on M_{reg}, the metric g is given as

$$g = \frac{(\mathrm{d}f)^2}{f^2} + f^2 g_0,$$

where $c^2 g_0$ is the metric g restricted to a level set $\{f = c\}$. Setting $s = \log|f|$ and using Proposition 3.3, this equation proves the statement in the theorem. ∎

The local geometry described in this theorem is summarised in diagram (1.3) in the introduction. We also have the following global result.

Theorem 3.5. *Let $(\widehat{M}, \widehat{g})$ be a time-like cone over a simply connected and space-like complete semi-Riemannian manifold (M, g). Assume that $(\widehat{M}, \widehat{g})$ admits a parallel null line $\mathbf{L}$ which is nowhere perpendicular to ξ. Then $(\widehat{M}, \widehat{g})$ is isometric to the double warped extension $(\widetilde{M}, \widetilde{g})$ of a semi-Riemannian manifold (M_0, g_0) and (M, g) is isometric to the exponential extension of (M_0, g_0), cf. Definition 3.2.*

Proof. Since M (and thus $\widehat{M}$) is simply connected, the flat line bundle **L** (see Proposition 3.1) admits a parallel section $\chi \neq 0$. By assumption, the function $\widehat{g}(\chi, \xi)$ has no zeros. As in the proof of Theorem 3.4, we can thus write

$$\chi = \frac{1}{r}(f\,\xi + Z)$$

for a nowhere vanishing function f and $Z = -\mathrm{grad}(f)$ on $M = M_{\mathrm{reg}}$. Then $Z' = \frac{1}{f} Z$ is a space-like geodesic unit vector field, as follows from $g(Z, Z) = f^2$ and equation (3.2):

$$\nabla_Z Z' = -\frac{df(Z)}{f^2}\, Z - Z = 0.$$

From the space-like completeness assumption, we conclude that Z' is complete, giving rise to a global diffeomorphism $M \simeq \mathbb{R} \times M_0$ under which the metric takes the form $g = ds^2 + e^{2s} g_0$. ■

Remark 3.6. The assumption that **L** is nowhere perpendicular to ξ in Theorem 3.5 cannot be dropped. In fact, the universal covering of anti-de Sitter space is simply connected and complete but any parallel line distribution over the time-like cone is somewhere perpendicular to ξ. In fact, a complete Lorentzian metric of constant negative curvature cannot be globally written in the form $ds^2 + e^{2s} g_0$, since the latter metric is incomplete, see Proposition 2.5 of [2]. Locally, it admits such description, where the Lorentzian metric g_0 is moreover flat.

In the following we will study metrics of the form $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g_0$. For brevity, we will drop the index 0 at g_0.

4. Metrics of the form $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$

4.1. Levi-Civita connection, curvature and holonomy

Let g be a semi-Riemannian metric (of signature (t, s)) on a manifold M of dimension n. We want to study the geometry and the holonomy of metrics of signature $(t+1, s+1)$ of the form

$$\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g, \tag{4.1}$$

from now on to be considered on the maximal domain $\widetilde{M} := \mathbb{R}^+ \times \mathbb{R} \times M \supset \mathbb{R}^+ \times \mathbb{R}^- \times M$. Such metrics admit a 2-dimensional solvable group of homotheties given by $(u, v, p) \mapsto (\mathrm{e}^r u, \mathrm{e}^r v + s, p)$. Its infinitesimal generators are the parallel null vector field ∂_v and the homothetic vector field $U = u\partial_u + v\partial_v$.

There are obvious inclusions of $M = \{1\} \times \{0\} \times M \subset \widetilde{M}$, $TM \subset T\widetilde{M}$ and $\Gamma(TM) \subset \Gamma(T\widetilde{M})$. Using these identifications, the Levi-Civita connection of $\widetilde{g}$ can be expressed by

$$\widetilde{\nabla}_X Y = \nabla_X Y - u\,g(X, Y)\,\partial_v \quad \text{and} \quad \widetilde{\nabla}_X \partial_u = \frac{1}{u}\, X, \tag{4.2}$$

with $X \in TM$, $Y \in \Gamma(TM)$, ∇ the Levi-Civita connection of g, and all other derivatives either vanish or they are determined by the vanishing of the torsion of ∇. Note that the homothetic vector field $U = u\partial_u + v\partial_v$ satisfies $\widetilde{\nabla} U = \mathrm{Id}$. Moreover, for the curvature of $\widetilde{g}$, one computes that

$$\partial_v \lrcorner \widetilde{R} = \partial_u \lrcorner \widetilde{R} = 0, \qquad \widetilde{R}(X,Y)Z = R(X,Y)Z \quad \text{for all } X, Y, Z \in TM, \tag{4.3}$$

where R is the curvature tensor of (M, g). Note that this implies, for an arbitrary tensor field Q, that

$$\widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_X Q = \widetilde{\nabla}_X \widetilde{\nabla}_{\partial_u} Q \quad \text{for all } X \in \Gamma(TM). \tag{4.4}$$

For the derivatives of $\widetilde{R}$, we get the following formulae, which determine all possible derivatives. First we observe that

$$(\widetilde{\nabla}_{\partial_u} \widetilde{R})(\partial_u, X) = 0, \tag{4.5}$$

for all $X \in TM$. For the q-th derivative in ∂_u-direction, we compute

$$(\widetilde{\nabla}_{\partial_u} \cdots \widetilde{\nabla}_{\partial_u} \widetilde{R})(X,Y)Z = \frac{(-1)^q (q+1)!}{u^q} R(X,Y)Z. \tag{4.6}$$

Moreover, a simple induction shows

$$\begin{aligned}(\widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(X,Y)Z = {} & (\nabla_{X_1} \cdots \nabla_{X_p} R)(X,Y)Z \\ & - u \sum_{i=1}^{p} (\nabla_{X_1} \overset{i}{\cdot\uparrow\cdot} \nabla_{X_p} R)(X,Y,Z,X_i)\partial_v,\end{aligned} \tag{4.7}$$

for all $X_i, X, Y, Z, W \in TM$, where the symbol $\overset{i}{.\uparrow.}$ indicates the omission of the ith term. In general, a straightforward computations shows the following.

Proposition 4.1. *The $(p+q)$th derivative of $\widetilde{R}$ is determined by the relations*

$$\partial_v \lrcorner \widetilde{\nabla}^k \widetilde{R} = 0, \quad (\widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z)(W) = (\widetilde{\nabla}_{X_1} \widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_2} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(Y,Z)(W),$$

and the formula

$$\begin{aligned}&(\widetilde{\nabla}_{\partial_u} \cdots \widetilde{\nabla}_{\partial_u} \widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_p} \widetilde{R})(X,Y)Z \\ &\quad = \frac{c(p,q)}{u^q}\Big((\nabla_{X_1} \cdots \nabla_{X_p} R)(X,Y)Z - u \sum_{i=1}^{p} (\nabla_{X_1} \overset{i}{\cdot\uparrow\cdot} \nabla_{X_p} R)(X,Y,Z,X_i)\,\partial_v \Big),\end{aligned}$$

where $c(p,0) = 1$ and $c(p,q) = (-1)^q (p+2)\cdots(p+q+1)$ when $q \geq 1$.

Our aim is to study the holonomy of metrics $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$. Since ∂_v is a parallel vector field on $(\widetilde{M}, \widetilde{g})$, the holonomy of $(\widetilde{M}, \widetilde{g})$ is contained in the stabiliser of the vector ∂_v at a point. By splitting $T\widetilde{M} = \mathbb{R}\partial_v \oplus TM \oplus \mathbb{R}\partial_u$, where $\mathrm{span}\{\partial_v, \partial_u\} = TM^{\perp}$, and fixing an orthonormal basis in T_pM, we can identify $\mathfrak{so}(T_pM, g) \simeq \mathfrak{so}(t,s)$ and have $\mathfrak{hol}(M,g) \subset \mathfrak{so}(t,s)$. Hence, we can identify the stabiliser of ∂_v in $\mathfrak{so}(t+1, s+1)$ with $\mathfrak{so}(t+1,s+1)_{\partial_v} = \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s}$, and we get that

$$\mathfrak{hol}(\widetilde{M}, \widetilde{g}) \subset \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s} = \left\{ \begin{pmatrix} 0 & g(w,\cdot) & 0 \\ 0 & A & -w \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{so}(t,s), w \in \mathbb{R}^{t,s} \right\}, \tag{4.8}$$

where the matrices are with respect to the splitting $T\widetilde{M} = \mathbb{R}\partial_v \oplus TM \oplus \mathbb{R}\partial_u$ and the identification $T_pM = \mathbb{R}^{t,s}$. With these identifications, there are two projections

$$\mathrm{pr}_{\mathfrak{so}(t,s)}\colon \mathfrak{hol}(\widetilde{M}, \widetilde{g}) \to \mathfrak{so}(t,s), \quad \mathrm{pr}_{\mathbb{R}^{t,s}}\colon \mathfrak{hol}(\widetilde{M}, \widetilde{g}) \to \mathbb{R}^{t,s},$$

to the linear part A and the translational part w in (4.8) of $\mathfrak{so}(t+1,s+1)_{\partial_v} = \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s}$. Since derivatives of the curvature are contained in the holonomy algebra, Proposition 4.1 implies that

$$\begin{aligned} \mathrm{pr}_{\mathfrak{so}(t,s)}\big(\widetilde{\nabla}^q_{\partial_u}\widetilde{\nabla}_{X_1}\cdots\widetilde{\nabla}_{X_p}\widetilde{R})(Y,Z)\big) &= \frac{c}{u^q}(\nabla_{X_1}\cdots\nabla_{X_p}R)(Y,Z), \\ \mathrm{pr}_{\mathbb{R}^{t,s}}\big(\widetilde{\nabla}^q_{\partial_u}\widetilde{\nabla}_{X_1}\cdots\widetilde{\nabla}_{X_p}\widetilde{R})(Y,Z)\big) &= \frac{c}{u^{q-1}}\sum_{i=1}^{p}(\nabla_{X_1}\overset{i}{\overset{\uparrow}{\cdots}}\nabla_{X_p}R)(Y,Z)X_i, \end{aligned} \tag{4.9}$$

where $X_i, Y, Z \in T_pM$ and c is a nonzero constant.

A first description of the holonomy of $(\widetilde{M}, \widetilde{g})$ was obtained in [24]. This description is the first part of the following proposition.

Proposition 4.2 (Theorem 4.2 of [24]). *Let g be a semi-Riemannian metric on M with holonomy algebra $\mathfrak{hol}(g)$ and $\widetilde{g}$ the metric $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ on $\mathbb{R}^+ \times \mathbb{R} \times M$. Then*

$$\mathfrak{hol}(\widetilde{g}) \subset \mathfrak{hol}(g) \ltimes \mathbb{R}^{t,s} \subset \mathfrak{so}(t,s) \ltimes \mathbb{R}^{t,s} = \mathfrak{so}(t+1,s+1)_{\partial_v}$$

and

$$\mathrm{pr}_{\mathfrak{so}(t,s)}(\mathfrak{hol}(\widetilde{g})) = \mathfrak{hol}(g).$$

Moreover, if (M,g) admits a nonzero parallel vector field X, then

$$\mathfrak{hol}(\widetilde{g}) \;\subset\; \mathfrak{hol}(g) \ltimes X^{\perp},$$

where $X^{\perp} \subset T_pM$ denotes the subspace orthogonal to X_p with respect to g.

Proof. The proof of the first part of the proposition was given in [24] and uses equations (4.2) to compute explicitly the parallel transport in $(\widetilde{M}, \widetilde{g})$. Indeed, let $\widetilde{\gamma}\colon [t_0,t_1] \to \widetilde{M}$ be a piecewise smooth curve given by $\widetilde{\gamma}(t) = (u(t), v(t), \gamma(t))$ with a curve γ in M. Let $Y(t)$ be a parallel vector field along γ with respect to ∇ and tangential to M. Then one checks that the vector field

$$\widetilde{Y}(t) = \frac{1}{u(t)}\,Y(t) + f(t)\cdot\partial_v$$

is parallel with respect to $\widetilde{\nabla}$ along $\widetilde{\gamma}$, where $f(t) = \int_{t_0}^{t} g_{\gamma(s)}(\dot{\gamma}(s), Y(s))\,\mathrm{d}s$. In particular, the parallel transport of $Z \in T_{(u(t_0),v(t_0),\gamma(t_0))}M$ along $\widetilde{\gamma}$ is given by

$$\widetilde{\mathcal{P}}_{\widetilde{\gamma}}(Z) = \frac{1}{u(t_1)}\,\mathcal{P}_{\gamma}(Z) + \Big(\int_{t_0}^{t_1} g_{\gamma(t)}(\dot{\gamma}(t), \mathcal{P}_{\gamma}|_{[t_0,t]}(Z))\,\mathrm{d}t\Big)\partial_v|_{\widetilde{\gamma}(t_1)}.$$

This implies that for a loop $\widetilde{\gamma}$ starting and ending at $(u,v,p) \in \widetilde{M}$, we have that

$$\mathrm{pr}_{\mathfrak{so}(t,s)}(\widetilde{\mathcal{P}}_{\widetilde{\gamma}}) = \frac{1}{u}\,\mathcal{P}_{\gamma},$$

which shows that $\mathrm{pr}_{\mathfrak{so}(t,s)}(\mathfrak{hol}(\widetilde{g})) = \mathfrak{hol}(g)$.

For the second part, in the case where (M, g) admits a parallel vector field X, the statement follows from the Ambrose–Singer holonomy theorem and the second equation in (4.9) as $(\nabla_{X_1} \cdots \nabla_{X_p} R)(Y, Z, X_i, X) = 0$ for all $X_i \in TM$ if X is parallel. ∎

Note that this does *not* establish the inclusion $\mathfrak{hol}(g) \subset \mathfrak{hol}(\tilde{g})$. Hence, for a metric of the form $\tilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$, this result allows for the possibility that $\mathfrak{hol}(\tilde{g})$ is not completely determined by $\mathfrak{hol}(g)$. Indeed, for the space of translations in $\mathfrak{hol}(\tilde{g})$

$$T := \mathfrak{hol}(\tilde{g}) \cap \mathbb{R}^{t,s},$$

we have the following possibilities:

(1) $T = \mathbb{R}^{t,s}$: In this case the holonomy of $\tilde{g}$ is completely determined by the holonomy of g and we have $\mathfrak{hol}(\tilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^{t,s}$.

(2) $T \neq \mathbb{R}^{t,s}$: In this case we can distinguish two situations:

(a) $\mathfrak{hol}(g) \subset \mathfrak{hol}(\tilde{g})$, in which case there is a subspace of translations $T \subsetneq \mathbb{R}^{t,s}$ such that $\mathfrak{hol}(\tilde{g}) = \mathfrak{hol}(g) \ltimes T$,

(b) $\mathfrak{hol}(g) \not\subset \mathfrak{hol}(\tilde{g})$.

In both cases in (2), it seems as if $\mathfrak{hol}(g)$ does not determine $\mathfrak{hol}(\tilde{g})$ completely and that further knowledge about the geometry of g is needed in order to decide whether (a) or (b) occur, to determine T, etc. In Sections 5 and 6 we will study these questions further, first purely algebraically and then using geometric properties of $\tilde{g}$. But first we will give some examples.

4.2. Locally symmetric spaces and other examples

4.2.1. Locally symmetric spaces. Here we consider manifolds $(\tilde{M}, \tilde{g})$ that arise via the construction (4.1) from semi-Riemannian locally symmetric spaces (M, g).

Theorem 4.3. *Let (M, g) be a semi-Riemannian locally symmetric space, i.e., a semi-Riemannian manifold with $\nabla R = 0$. For (M, g), we consider the metric $\tilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$ on $\tilde{M} = \mathbb{R}^+ \times \mathbb{R} \times M$. Then*

$$\mathfrak{hol}_{\tilde{p}}(\tilde{M}, \tilde{g}) = \mathfrak{hol}_p(M, g) \ltimes T,$$

where $T = \mathfrak{hol}_p(M, g)V$, with $V = T_pM$ and $\tilde{p} = (1, 0, p) \in \tilde{M}$.

Proof. As a consequence of the Ambrose–Singer theorem and $\nabla R = 0$, we have that

$$\mathfrak{hol}_p(M, g) = \operatorname{span}\{R(X, Y)|_p : X, Y \in T_pM\}. \tag{4.10}$$

The curvature $\tilde{R}$ of $(\tilde{M}, \tilde{g})$ satisfies equation (4.3), which, together with equation (4.10), shows that $\mathfrak{g} = \mathfrak{hol}(M, g)$ is contained in $\tilde{\mathfrak{g}} = \mathfrak{hol}_p(\tilde{M}, \tilde{g})$. Moreover, by Proposition 4.1, we have that

$$(\tilde{\nabla}_{\partial_u} \cdots \tilde{\nabla}_{\partial_u} \tilde{\nabla}_{X_1} \tilde{R})(X, Y)Z = \frac{c}{u^{q-1}} R(X, Y, Z, X_1)\partial_v,$$

for a nonzero constant c and $X, Y, Z, X_1 \in TM$, and all other derivatives of $\tilde{R}$ are zero. This implies the claim. ∎

Corollary 4.4. *Let (M, g) be a semi-Riemannian locally symmetric space, which is locally the product of (non-flat) irreducible symmetric spaces. Then*

$$\mathfrak{hol}_{\tilde{p}}(\widetilde{M}, \widetilde{g}) = \mathfrak{hol}_p(M, g) \ltimes T_p M.$$

Example 4.5. The following example shows that Corollary 4.4 does not extend to indecomposable symmetric spaces such as the Cahen–Wallach space of dimension $n = m + 2$,

$$(M, g) = \Big(\mathbb{R}^n, g_{\mathrm{CW}} = 2\,\mathrm{d}x\,\mathrm{d}z + \sum_{i,j=1}^{m} \lambda_{ij} y^i y^j \,\mathrm{d}z^2 + \sum_{i=1}^{m} (\mathrm{d}y^i)^2\Big),$$

where $(x, y^1, \ldots, y^m, z)$ are global coordinates on $\mathbb{R}^{m+2}$ and where $S = (\lambda_{ij})$ is a constant symmetric matrix with $\det(S) \neq 0$. In this case, we have $\mathfrak{hol}(M, g) = \mathbb{R}^m \subset \mathfrak{so}(1, m+1)_{\partial_x} = \mathfrak{so}(m) \ltimes \mathbb{R}^m$ and $T = \mathrm{span}(\partial_x, \partial_1, \ldots, \partial_m)$, where $\partial_i = \partial/\partial y^i$. We will explain these Lie algebras in more detail later on.

4.2.2. pp-waves and plane waves. The pp-waves are Lorentzian manifolds that are generalisations of Cahen–Wallach spaces. Again we consider $M = \mathbb{R}^n = \mathbb{R}^{m+2}$ with global coordinates $(x, y^1, \ldots, y^m, z)$ and f a function $f = f(y^1, \ldots, y^m, z)$ of $y^1, \ldots, y^m$ and z but not of x. Then a general pp-wave metric on $\mathbb{R}^{m+2}$ is given by

$$g = 2\,\mathrm{d}x\,\mathrm{d}z + 2f(y^1, \ldots, y^m, z)\,\mathrm{d}z^2 + \sum_{i=1}^{m} (\mathrm{d}y^i)^2. \tag{4.11}$$

The Levi-Civita connection and the curvature are determined by

$$\nabla \partial_x = 0, \quad \nabla_{\partial_i} \partial_j = 0, \quad \nabla_{\partial_z} \partial_i = \partial_i f\, \partial_x, \quad \nabla_{\partial_z} \partial_z = \partial_z f\, \partial_x - \sum_{i=1}^{m} \partial_i f\, \partial_i,$$

and

$$\partial_x \lrcorner R = 0, \quad R(\partial_i, \partial_j) = 0, \quad R(\partial_i, \partial_z, \partial_z, \partial_j) = -\partial_i \partial_j f.$$

In the basis $(\partial_x, \partial_1, \ldots, \partial_m, \partial_z - f\,\partial_x)$ the metric is given by

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_m & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and we can write the curvature and its derivatives as endomorphisms in $\mathfrak{so}(\eta)$ as

$$\begin{aligned} &(\nabla_{X_1} \cdots \nabla_{X_p} R)(\partial_i, \partial_z) \\ &\quad = \begin{pmatrix} 0 & (X_1 \cdots X_p \partial_i \partial_j (f))_{j=1}^m & 0 \\ 0 & 0 & -(X_1 \cdots X_p \partial_i \partial_j (f))_{j=1}^m \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.12}$$

where the X_i are constant vector fields on $M = \mathbb{R}^n$. As for Cahen–Wallach spaces, their holonomy algebra is contained in (and equal to, if the Hessian $\partial_i \partial_j f$ of f is invertible) $\mathbb{R}^m \subset \mathfrak{so}(1, m+1)_{\partial_x}$ and hence abelian.

Now we consider the semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of signature $(2, m+2)$ for a given pp-wave (M, g) of dimension $n = m + 2$. Then, by setting

$$A_{qrk_1\cdots k_s i} := (\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{\nabla}_{\partial_{k_1}} \cdots \widetilde{\nabla}_{\partial_{k_s}} \widetilde{R})(\partial_i, \partial_z),$$

equations (4.9) in this case are

$$\mathrm{pr}_{\mathfrak{so}(1,n-1)}(A_{qrk_1\cdots k_s i}) = \frac{c}{u^q} \begin{pmatrix} 0 & (\partial_{k_1} \cdots \partial_{k_s} \partial_i \partial_j f^{(r)})_{j=1}^m & 0 \\ 0 & 0 & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{pr}_{\mathbb{R}^{1,n-1}}(A_{qrk_1\cdots k_s i}) = \frac{c}{u^{q-1}} \Big(s \partial_{k_1} \cdots \partial_{k_s} \partial_i f^{(r)} \partial_x + \sum_{j=1}^m \partial_{k_1} \cdots \partial_{k_s} \partial_i \partial_j f^{(r-1)} \partial_j \Big). \tag{4.13}$$

where $f^{(r)}$ denotes the r-th derivative of f with respect to the coordinate z. This shows that $\mathfrak{hol}(\widetilde{M}, \widetilde{g}) \subset \mathfrak{hol}(M, g) \ltimes \partial_x^\perp$, with $\partial^\perp = \mathrm{span}(\partial_x, \partial_1, \ldots, \partial_m)$, as claimed in Proposition 4.2. In general, these projections could be coupled to each other, but for a special case, we can say more as follows.

Proposition 4.6. *Let (M, g) be a pp-wave as in (4.11) but with the condition that f does not depend on z, i.e., $f = f(y^1, \ldots, y^n)$ and such that $\det(\partial_i \partial_j f) \neq 0$ at one point or, more generally, such that at one point,*

$$\mathrm{span}\big\{ \mathrm{d}(\partial_{k_1} \cdots \partial_{k_p} f) \mid p \geq 1,\ k_1, \ldots, k_p \in \underline{m} \big\} = (\mathbb{R}^m)^*, \tag{4.14}$$

where $\underline{m} = \{1, \ldots, m\}$. Then

$$\mathfrak{hol}(\widetilde{M}, \widetilde{g}) = \mathfrak{hol}(M, g) \ltimes \partial_x^\perp.$$

Proof. We evaluate formulae (4.9) for $r = 1$. Since f is independent of z, we have $f' = 0$, and hence

$$\mathrm{pr}_{\mathfrak{so}(1,m+1)}\big((\widetilde{\nabla}_{\partial_z} \widetilde{\nabla}_{\partial_{k_1}} \cdots \widetilde{\nabla}_{\partial_{k_p}} \widetilde{R})(\partial_i, \partial_z) \big) = 0$$

and

$$\mathrm{pr}_{\mathbb{R}^{1,m+1}}\big((\widetilde{\nabla}_{\partial_z} \widetilde{\nabla}_{\partial_{k_1}} \cdots \widetilde{\nabla}_{\partial_{k_p}} \widetilde{R})(\partial_i, \partial_z) \big) = \sum_{j=1}^m \partial_{k_1}! \cdots \partial_{k_p} \partial_i \partial_j f\, \partial_j.$$

If $\det(\partial_i \partial_j f) \neq 0$ (or if (4.14) holds at one point), this shows that $\mathrm{span}(\partial_1, \ldots, \partial_m) \subset \mathfrak{hol}(\widetilde{M}, \widetilde{g}) \cap \mathbb{R}^{1,m+1}$. This space, however, is not invariant under $\mathfrak{hol}(M, g)$ and is mapped under the adjoint representation in $\mathfrak{hol}(\widetilde{M}, \widetilde{g})$ to $\mathbb{R}\partial_x$, so that we have $\mathfrak{hol}(\widetilde{M}, \widetilde{g}) = \mathfrak{hol}(M, g) \ltimes \partial_x^\perp$. ■

This proposition can be clearly generalised to functions f that are polynomial, say of degree d, in z (and have arbitrary dependence on the y^i). It suffices to replace $r = 1$ in the proof with $r = d + 1$ and the condition on f by the corresponding condition on $f^{(d)}$. It does not hold, however, for general f as the following example shows.

Example 4.7. Let $f(y, z) = \mathrm{e}^z y^2$ and g a plane wave metric[1] on $\mathbb{R}^3$ defined by f,

$$g = 2\,\mathrm{d}x\,\mathrm{d}z + \mathrm{e}^z y^2 \mathrm{d}z^2 + \mathrm{d}y^2.$$

[1] Plane waves are pp-waves for which the function f is a quadratic polynomial in the y^i's with z-dependent coefficients, i.e., $f(y^1, \ldots, y^m, z) = \sum_{i,j=1}^m f_{ij}(z) y^i y^j$, with $(f_{ij}(z))$ a symmetric matrix of functions of $z \in \mathbb{R}$.

Its curvature and derivatives thereof are given by equation (4.12) as follows:

$$\nabla_{\partial_y} R = 0, \quad (\nabla^{(r)}_{\partial_z} R)(\partial_y, \partial_z) = 2 \begin{pmatrix} 0 & e^z & 0 \\ 0 & 0 & -e^z \\ 0 & 0 & 0 \end{pmatrix} =: A(z),$$

for all $r \geq 0$. Its holonomy algebra is 1-dimensional. When we now consider the metric $\widetilde{g}$, formula (4.13) shows that

$$(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{R})(\partial_y, \partial_z) = \frac{c}{u^q} \begin{pmatrix} 0 & 2u\, e^z dy & 0 \\ 0 & A(z) & -2u\, e^z \partial_y \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{\nabla}_{\partial_y} \widetilde{R})(\partial_y, \partial_z) = \frac{2c}{u^{q-1}} e^z \partial_x,$$

with all other derivatives of the curvature being zero. Since $\widetilde{g}$ is analytic, its holonomy is determined by the derivatives of the curvature at a point, say at $v = x = y = z = 0$ and $u = 1$, and it is spanned by the two matrices arising from $(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{R})(\partial_y, \partial_z)$ and $(\widetilde{\nabla}^q_{\partial_u} \widetilde{\nabla}^r_{\partial_z} \widetilde{\nabla}_{\partial_y} \widetilde{R})(\partial_y, \partial_z)$,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that the holonomy of $\widetilde{g}$ is abelian and is neither purely translational nor a semi-direct sum of $\mathfrak{hol}(g)$ with a Lie algebra of translations.

4.3. Lift of parallel objects

In this section we analyse how parallel objects on (M, g), such as vector fields and vector distributions, lift to $(\widetilde{M}, \widetilde{g})$. First we analyse how certain vector fields on M lift to $\widetilde{M}$.

Lemma 4.8. *Let ξ be a homothetic gradient vector field on (M, g), i.e., a vector field with*

$$\nabla \xi = a\,\mathrm{Id}, \tag{4.15}$$

for a constant $a \in \mathbb{R}$ and such that $\xi^\flat$ is not only closed but exact, $\xi^\flat = df$ for a smooth function f. Then the vector field $\widetilde{\xi}$, defined by

$$\widetilde{\xi} = f\,\partial_v + \frac{1}{u}\xi - a\partial_u,$$

is parallel for $\widetilde{\nabla}$. In particular, if ξ is parallel for (M, g), then $\widetilde{\xi} = f\,\partial_v + \frac{1}{u}\xi$ is parallel for $(\widetilde{M}, \widetilde{g})$.

Proof. First note that condition (4.15) implies that $\xi^\flat$ is closed, i.e., locally we can always find a function f such that $\xi^\flat = df$. Then we compute

$$\widetilde{\nabla}_{\partial_u} \widetilde{\xi} = -\frac{1}{u^2}\xi + \frac{1}{u}\widetilde{\nabla}_{\partial_u}\xi = 0,$$

because of (4.2). Moreover, for every $X \in TM$, we have that

$$\widetilde{\nabla}_X \widetilde{\xi} = df(X)\partial_v + \frac{a}{u}X - g(\xi, X)\partial_v - a\widetilde{\nabla}_X \partial_u = 0,$$

again by (4.2) and $\mathrm{d}f = \xi^\flat$. ∎

In a similar way we can prove:

Lemma 4.9. *Let* $\mathbf{L}$ *be a parallel null line bundle on* (M, g)*. Then the totally null* 2*-plane bundle* $\mathbf{P}$ *on* $(\widetilde{M}, \widetilde{g})$ *spanned by* ∂_v *and* $\mathbf{L}$ *is parallel for* $\widetilde{\nabla}$.

Proof. This follows from applying equation (4.2) to a recurrent null vector field ξ spanning $\mathbf{L}$ and ∂_v being parallel for $\widetilde{\nabla}$. ∎

The following proposition will be used in Section 6 for the proof of Theorem 1.3.

Proposition 4.10. *Let* (M, g) *be a manifold with parallel null line bundle* $\mathbf{L}$*. Assume that the metric* $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$ *admits a recurrent vector field in the span of* ∂_v *and* $\mathbf{L}$ *that is not a multiple of* ∂_v*. Then locally* g *admits a parallel null vector field in* $\mathbf{L}$.

Proof. By Lemma 2.9, we can assume that $\mathbf{L}$ is spanned by a recurrent *gradient* vector field $\xi = \operatorname{grad}(f)$, i.e., with $\xi^\flat = df$ and $\nabla\xi = \theta \otimes \xi$, where θ is a multiple of $\xi^\flat$. Then the vector field

$$\widetilde{\xi} = f\,\partial_v + \frac{1}{u}\xi$$

satisfies

$$\widetilde{\nabla}_{\partial_u} \widetilde{\xi} = 0, \tag{4.16}$$

$$\widetilde{\nabla}_X \widetilde{\xi} = \frac{1}{u}\theta(X)\xi = \theta(X)(\widetilde{\xi} - f\,\partial_v) \quad \text{for } X \in TM. \tag{4.17}$$

Without loss of generality, the assumption implies that $\widetilde{g}$ admits a recurrent vector field of the form $\zeta = \widetilde{\xi} + h\partial_v$ for a function h. It defines a one-form α by $\widetilde{\nabla}\zeta = \alpha \otimes \zeta$. Then the fact that ∂_v is parallel and equation (4.16) immediately show that

$$\partial_u h = \alpha(\partial_u) = \partial_v h = \alpha(\partial_v) = 0.$$

Equation (4.17) implies that

$$\widetilde{\nabla}_X \zeta = \theta(X)\widetilde{\xi} + (dh(X) - f\,\theta(X))\partial_v.$$

Hence, the equation $\widetilde{\nabla}\zeta = \alpha \otimes \zeta$ implies that $\alpha = \theta$ and

$$\mathrm{d}h = (f + h)\theta.$$

Differentiating this and taking into account that $\mathrm{d}f \wedge \theta = \mathrm{d}h \wedge \theta = 0$ gives

$$0 = (f + h)\,\mathrm{d}\theta.$$

If $\mathrm{d}\theta \neq 0$ this implies $h = -f$. This contradicts the above $\mathrm{d}h = (f + h)\theta$, as it would imply that h and hence f are constant. So we must have $\mathrm{d}\theta = 0$. This however implies that one can rescale ξ to a parallel null vector field. ∎

Finally, for parallel distributions of (M, g), we get the following.

Lemma 4.11. *Let $W \subset TM$ be a parallel distribution on (M, g). Then the distribution $\mathbb{R}\,\partial_v \oplus W \subset T\widetilde{M}$ is parallel.*

Proof. The distribution W is locally spanned by vector fields $W_1, \dots, W_k$. Then one checks that, for $\widetilde{W}_i := \partial_v + \frac{1}{u} W_i$, we have $\widetilde{\nabla}_{\partial_u} \widetilde{W}_i = 0$ and

$$\widetilde{\nabla}_X \widetilde{W}_i = -g(X, W_i)\partial_v + \frac{1}{u} \nabla_X W_i \in \mathbb{R}\,\partial_v \oplus W \quad \text{for all } X \in TM. \qquad \blacksquare$$

5. Results about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$

In this section we will prove several algebraic results about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$ stabilising a null line or a null vector. We will use these results in the next section when studying further the holonomy of metrics of the form $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$.

5.1. Indecomposable subalgebras stabilising a null vector

In this section we will fix some notations and observe some fundamental facts about indecomposable subalgebras of $\mathfrak{so}(t+1, s+1)$ stabilising a null vector. In particular, in this section we will see why the vector space $Z^1(\mathfrak{g}, V)$ of 1-cocycles of a Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$-module V comes into play. Recall that

$$(5.1) \qquad Z^1(\mathfrak{g}, V) := \{\varphi\colon \mathfrak{g}^* \otimes V \mid \varphi([X,Y]) = X\,\varphi(Y) - Y\,\varphi(X) \text{ for all } X, Y \in \mathfrak{g}\}$$

and

$$H^1(\mathfrak{g}, V) := \frac{Z^1(\mathfrak{g}, V)}{\mathrm{d}V},$$

where

$$\mathrm{d}\colon V \to Z^1(\mathfrak{g}, V), \quad \mathrm{d}v(X) := Xv, \quad v \in V, \quad X \in \mathfrak{g}.$$

Let $\widetilde{V}$ be a semi-Euclidean vector space of signature $(t+1, s+1)$ with metric $\widetilde{g}$ and let $\mathbf{e}_\pm$ be two null vectors such that $\widetilde{g}(\mathbf{e}_-, \mathbf{e}_+) = 1$. We split $\widetilde{V} = L_- \oplus V \oplus L_+$, with $L_\pm = \mathbb{R} \cdot \mathbf{e}_\pm$ and $V = (L_- \oplus L_+)^\perp$, which is equipped with the metric $g = \widetilde{g}|_{V \times V}$. With respect to this splitting, the stabiliser of L_- in $\mathfrak{so}(\widetilde{V})$, denoted by $\mathfrak{so}(\widetilde{V})_{L_-}$, is given as

$$\begin{aligned}\mathfrak{so}(\widetilde{V})_{L_-} &= (\mathbb{R} \oplus \mathfrak{so}(V)) \ltimes V \\ &= \left\{ (a, X, v) := \begin{pmatrix} a & -v^\flat & 0 \\ 0 & X & v \\ 0 & 0 & -a \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ X \in \mathfrak{so}(V),\ v \in V \right\}.\end{aligned}$$

The action of $\mathfrak{so}(\widetilde{V})_{L_-}$ on $\widetilde{V} = L_- \oplus V \oplus L_+ \cong \mathbb{R} \oplus V \oplus \mathbb{R}$ is given by

$$(5.2) \qquad (a, X, v) \cdot \begin{pmatrix} r \\ u \\ s \end{pmatrix} = \begin{pmatrix} ar - g(v, u) \\ Xu + sv \\ -as \end{pmatrix}.$$

Furthermore, we record the formula for the Lie bracket in $\mathfrak{so}(\widetilde{V})_{L_-}$:

$$(5.3) \qquad [(a, X, v), (b, Y, w)] = (0, [X, Y], (X + a)w - (Y + b)v).$$

The stabiliser of the *vector* $\mathbf{e}_-$ is given as $\mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$, i.e., it is obtained by requiring a to be zero in the above formulae. Note that, the adjoint action of $\mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ preserves the ideal V, whereas the linear action on $\widetilde{V}$ does not preserve the subspace $V \subset \widetilde{V}$.

Furthermore, note that there are natural projections pr_V and $\mathrm{pr}_{\mathfrak{so}(V)}$ on V and $\mathfrak{so}(V)$. For a subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(V) \ltimes V$, we call $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(V)}(\widetilde{\mathfrak{g}})$ the *linear part of* $\widetilde{\mathfrak{g}}$ and $T := \widetilde{\mathfrak{g}} \cap V$ the *translations in* $\widetilde{\mathfrak{g}}$. Note that $\widetilde{\mathfrak{g}} \subset \mathfrak{g} \ltimes V$ but, in general, $\mathfrak{g} \not\subset \widetilde{\mathfrak{g}}$.

Proposition 5.1. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ be a subalgebra, $\mathfrak{g}$ its linear part and T the translations in $\widetilde{\mathfrak{g}}$. Then*:

(1) *T is an ideal in $\widetilde{\mathfrak{g}}$.*

(2) *$T \subset V$ is invariant under $\mathfrak{g}$, and consequently $\mathfrak{g}$ acts on V/T.*

(3) *We have an inclusion of Lie algebras $\widetilde{\mathfrak{g}}/T \subset \mathfrak{g} \ltimes V/T$.*

(4) *There is a $\varphi \in Z^1(\mathfrak{g}, V/T)$ such that $\widetilde{\mathfrak{g}}/T = \{(X, \varphi(X)) \mid X \in \mathfrak{g}\}$.*

(5) *If T has a $\mathfrak{g}$-invariant complement T', then there is a $\varphi \in Z^1(\mathfrak{g}, T')$ such that*

$$\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T, \quad \text{where } \mathfrak{h}_\varphi = \{(X, \varphi(X)) \in \widetilde{\mathfrak{g}} \mid X \in \mathfrak{g}\}.$$

Proof. Items (1), (2) and (3) are obvious from the definitions. For item (4), we define $\varphi(X) = v \bmod T$ if $(X, v) \in \widetilde{\mathfrak{g}}$. Since $(X, v) \in \widetilde{\mathfrak{g}}$ and $(X, w) \in \widetilde{\mathfrak{g}}$ implies that $v - w \in T$, this map is well defined. From equation (5.3), we see that φ is an element in $Z^1(\mathfrak{g}, V/T)$. Finally, item (5) follows easily from item (4) using the identification $V/T = T'$ as $\mathfrak{g}$-modules. ∎

Theorem 5.2. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ be a subalgebra acting indecomposably on $\widetilde{V}$. Let $\mathfrak{g} \subset \mathfrak{so}(V)$ and $T \subset V$ be, respectively, the linear part and translational ideal of $\widetilde{\mathfrak{g}}$.*

(1) *If T has a $\mathfrak{g}$-invariant complement T' and $H^1(\mathfrak{g}, V) = 0$, then, up to conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$ and $T^\perp$ is degenerate or zero. In particular, if T is nondegenerate and $H^1(\mathfrak{g}, V) = 0$, then $T = V$.*

(2) *If T is degenerate such that $L = T \cap T^\perp$ is a null line (this is the case, for example, when T is degenerate and g Lorentzian) and if the representation of $\mathfrak{g}$ on $V/L^\perp$ satisfies that $H^1(\mathfrak{g}, V/L^\perp) = 0$, then, up to conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}}$ preserves L.*

Proof. (1) First assume $V = T \oplus T'$ is a $\mathfrak{g}$-invariant decomposition. In virtue of Proposition 5.1, $\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, for some $\varphi \in Z^1(\mathfrak{g}, T')$. Since $Z^1(\mathfrak{g}, V) = \mathrm{d}V$ and $Z^1(\mathfrak{g}, T') \subset Z^1(\mathfrak{g}, V)$, we find a $v \in V$ such that

$$\varphi(X) = Xv \quad \text{for all } X \in \mathfrak{g}.$$

Then every element $(X, \varphi(X)) = (X, Xv) \in \mathfrak{h}_\varphi$ can be conjugated to X by a conjugation with the translation given by v, i.e., with

$$A_v = \begin{pmatrix} 1 & -v^\flat & -\frac{1}{2}g(v,v) \\ 0 & \mathbf{1} & v \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

Indeed, for each $X \in \mathfrak{g}$, we get

$$A_v \begin{pmatrix} 0 & -(Xv)^\flat & 0 \\ 0 & X & Xv \\ 0 & 0 & 0 \end{pmatrix} A_v^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & Xv \\ 0 & 0 & 0 \end{pmatrix} A_{-v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

using that $X \in \mathfrak{so}(V)$. This shows that after conjugation with a translation, we have that $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. Hence, $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$, where $T = \widetilde{\mathfrak{g}} \cap V$. Note that this already implies that T is nonzero, because otherwise $\widetilde{\mathfrak{g}} = \mathfrak{g} \subset \mathfrak{so}(V)$, which contradicts indecomposability. Since T is $\mathfrak{g}$ invariant, also the orthogonal complement $T^\perp$ of T in V is $\mathfrak{g}$ invariant. Then equation (5.2) shows that $T^\perp \subset \widetilde{V}$ is also invariant under the action of $T \subset \mathfrak{so}(\widetilde{V})$ on $\widetilde{V}$, and therefore $T^\perp$ is $\widetilde{\mathfrak{g}}$-invariant. Hence, by indecomposability of $\widetilde{\mathfrak{g}}$, $T^\perp$ has to be degenerate or zero.

(2) Assume that T is degenerate such that $L := T \cap T^\perp$ is a null line. By item (2) of Proposition 5.1, L is invariant under $\mathfrak{g}$. Moreover, by item (5) of Proposition 5.1, there is a $\varphi \in Z^1(\mathfrak{g}, V/T)$ such that $\widetilde{\mathfrak{g}}/T = \{(X, \varphi(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \ltimes V/T$. Hence, if $\widetilde{\varphi}\colon \mathfrak{g} \to V$ is a lift of φ, we can write $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$, where $\mathfrak{h}_{\widetilde{\varphi}} = \{(X, \widetilde{\varphi}(X)) \in \mathfrak{g} \ltimes V \mid X \in \mathfrak{g}\}$. Note that, since T may not have an invariant complement, in general, we do not have that $\widetilde{\varphi} \in Z^1(\mathfrak{g}, V)$ and neither that $\mathfrak{h}_{\widetilde{\varphi}}$ is a subalgebra.

Let $L^\perp$ be the hyperplane in V that is orthogonal to L. We have $L \subset T \subset L^\perp$, and hence, by formula (5.2), L is annihilated by the translations T in $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$. It remains to show that L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$, unless $\mathfrak{g}$ acts trivially on L. For this we consider the projection $\pi\colon V/T \twoheadrightarrow V/L^\perp$ and distinguish two cases:

Case 1: $\pi \circ \varphi\colon \mathfrak{g} \to V/L^\perp$ is zero. This means that the image of the lift $\widetilde{\varphi}$ is contained in $L^\perp$. This, however, implies that L is not only invariant under $\mathfrak{g}$ but also under $\widetilde{\mathfrak{g}} = \mathfrak{h}_{\widetilde{\varphi}} + T$. Indeed, from formula (5.2), it follows, for an element $(X, \widetilde{\varphi}(X)) \in \mathfrak{h}_{\widetilde{\varphi}}$ and $\ell \in L$, that $(X, \widetilde{\varphi}(X)) \cdot \ell = X \cdot \ell - g(\widetilde{\varphi}(X), \ell)\mathrm{e}_- = X \cdot \ell \in L$, since $\widetilde{\varphi}(X) \in L^\perp$ and $\mathfrak{g}$ leaves L invariant. Hence, in this case L is $\widetilde{\mathfrak{g}}$-invariant.

Case 2: $\pi \circ \varphi\colon \mathfrak{g} \to V/L^\perp$ is not zero, i.e., the image of $\widetilde{\varphi}$ is not contained in $L^\perp$. In this case, similarly to (1), we try to find a conjugation with a translation that shows that L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$ (after conjugation). For $v \in V$ to be determined, we consider the associated translation A_v as in equation (5.4). Then, as in case 1, for an element

$$(X, \widetilde{\varphi}(X)) = \begin{pmatrix} 0 & -(\widetilde{\varphi}(X))^\flat & 0 \\ 0 & X & \widetilde{\varphi}(X) \\ 0 & 0 & 0 \end{pmatrix},$$

we get that

$$A_v (X, \widetilde{\varphi}(X)) A_v^{-1} = \begin{pmatrix} 0 & -(\widetilde{\varphi}(X) - Xv)^\flat & 0 \\ 0 & X & \widetilde{\varphi}(X) - Xv \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Fix $\ell \in L$ and $\hat{\ell} \in V$ such that $g(\ell, \hat{\ell}) = 1$. Then define $0 \neq \lambda \in \mathfrak{g}^*$ and $\rho \in \mathfrak{g}^*$ by $\widetilde{\varphi}(X) = \lambda(X)\hat{\ell} \bmod L^\perp$ and $X\ell = -\rho(X)\ell$, for $X \in \mathfrak{g}$. This is summarised in $(X, \widetilde{\varphi}(X)) \cdot \ell = -\lambda(X)\mathrm{e}_- - \rho(X)\ell$. It also implies that $X\hat{\ell} = \rho(X)\hat{\ell} \bmod L^\perp$, i.e., $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V/L^\perp)$

is the induced representation of $\mathfrak{g}$ on $V/L^\perp$. Note that $\varphi \in Z^1(\mathfrak{g}, V/T)$ induces the non-zero element $\overline{\varphi} := \pi \circ \varphi \in Z^1(\mathfrak{g}, V/L^\perp)$. So $H^1(\mathfrak{g}, V/L^\perp) = 0$ implies that

$$\overline{\varphi}(X) = X(c\hat{\ell} \bmod L^\perp) = cX\hat{\ell} \bmod L^\perp = c\rho(X)\hat{\ell} \bmod L^\perp,$$

and thus $\widetilde{\varphi}(X) = c\rho(X)\hat{\ell} \bmod L^\perp$ for some $c \neq 0$.

Now, in equation (5.4), we set $v := c\hat{\ell}$. Taking into account that $g(\hat{\ell}, \ell) = 1$, formula (5.5) shows that

$$A_v(X, \widetilde{\varphi}(X))A_v^{-1} \cdot \ell = -(\lambda(X) - c\rho(X))\mathbf{e}_- - \rho(X)\ell = -\rho(X)\ell.$$

This shows that after conjugation with a translation the null line L is invariant under $\mathfrak{h}_{\widetilde{\varphi}}$ and hence under $\widetilde{\mathfrak{g}}$. ■

Example 5.3. Consider $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n = \mathfrak{so}(1, n+1)_{\mathrm{e}_0}$, where $\mathrm{e}_0 \in \mathbb{R}^{1,n+1}$ is a null vector. Then, for $T = \mathbb{R} \cdot \mathrm{e}_0$, one can check that $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T \subset \mathfrak{so}(2, n+2)_{\mathrm{e}_-}$ is indecomposable. Similarly, for $T = \mathrm{span}(\mathrm{e}_0, \dots \mathrm{e}_n)$, $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes T$ is indecomposable. Note that the latter is the holonomy algebra of a $(\widetilde{M}, \widetilde{g})$ for a Cahen–Wallach space (M, g_{CW}) of dimension $n + 2$ presented in Example 4.5.

5.2. Indecomposable subalgebras with completely reducible linear part

The main result of this section is Theorem 5.5, which is a generalisation to arbitrary signature of a result in [8] for an indecomposable stabiliser in $\mathfrak{so}(1, n+1)$ of a null *vector*.[2] It gives a description of all indecomposable subalgebras $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ with completely reducible linear part and non-degenerate translational part.

The main results of this and the next section use a result about Lie algebra cohomology,[3] which we will present first. In the following, for a $\mathfrak{g}$-module V, we denote by $V^{\mathfrak{g}}$ the $\mathfrak{g}$ invariant vectors,

$$V^{\mathfrak{g}} = \{v \in V \mid Xv = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Theorem 5.4 (Theorem 13 of [22], Theorem 2.28 of [31]). *Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module, both finite-dimensional and over a field $\mathbb{F}$ of characteristic zero. Assume that there is an ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that*:

(1) *there is a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{b}$, and*

(2) *V and $\mathfrak{g}$ are completely reducible as $\mathfrak{h}$-modules.*

Then

$$H^p(\mathfrak{g}, V) \simeq \sum_{i+j=p} H^i(\mathfrak{h}, \mathbb{F}) \otimes H^j(\mathfrak{b}, V)^{\mathfrak{g}}.$$

In particular, when $p = 1$,

$$H^1(\mathfrak{g}, V) \simeq H^1(\mathfrak{b}, V)^{\mathfrak{g}} + (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}])^* \otimes V^{\mathfrak{g}}. \tag{5.6}$$

[2] We point out that in [8] a similar result for the stabiliser in $\mathfrak{so}(1, n+1)$ of a null *line* is given.

[3] We do have self-contained proofs of Theorems 5.5 and 5.7 that do not use Theorem 5.4, but for the sake of brevity we do not present them here as they are longer.

The original version of this theorem is due to Hochschild and Serre, Theorem 13 of [22], in which the existence of $\mathfrak{h}$ was not assumed but that $\mathfrak{g}/\mathfrak{b}$ is semisimple. Solleveld proved the generalisation that is given here in his Master's thesis, Theorem 2.28 of [31]. Equation (5.6) for $p = 1$ follows from the facts that $H^0(\mathfrak{h}, \mathbb{F}) = \mathbb{F}$, $H^0(\mathfrak{b}, V) = V^{\mathfrak{b}}$ and that $H^1(\mathfrak{h}, \mathbb{F})$ is isomorphic to $(\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}])^*$.

Now we turn to the main result of this section. We use the same conventions as in Section 5.1.

Theorem 5.5. *Let $\widetilde{\mathfrak{g}} \subset \mathfrak{so}(\widetilde{V})_{\mathbf{e}_-} = \mathfrak{so}(V) \ltimes V$ be an indecomposable subalgebra which satisfies the following properties*:

(1) $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(V)}(\widetilde{\mathfrak{g}})$ *acts completely reducibly on V, and*

(2) *the translational ideal $T = \widetilde{\mathfrak{g}} \cap V$ is non-degenerate.*

Under these assumptions, let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ be the decomposition of $\mathfrak{g}$ into its centre and the semisimple derived Lie algebra. Then $\mathfrak{g}$ acts trivially on $T^\perp$ and $T \neq 0$. Moreover, there is a linear map $\varphi\colon \mathfrak{g} \to T^\perp$ with $\varphi|_{\mathfrak{g}'} = 0$ such that after conjugation in $\mathfrak{so}(V) \ltimes V$, $\widetilde{\mathfrak{g}}$ is of the form $\widetilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, where

$$\mathfrak{h}_\varphi = \big\{(X, \varphi(X)) \in \mathfrak{so}(V) \ltimes V \mid X \in \mathfrak{g}\big\}, \tag{5.7}$$

and the image of φ is co-null in $T^\perp$, i.e., $(\mathrm{im}\,\varphi)^\perp \subset T^\perp$ is totally null.

The proof of this theorem is based on a lemma which will follow from Theorem 5.4. Since V is a completely reducible module, $\mathfrak{g}$ is reductive and hence $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple, $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and we denote the projection to $\mathfrak{z}$ by $\pi_{\mathfrak{z}}\colon \mathfrak{g} \to \mathfrak{z}$.

Lemma 5.6. *Let V be a semi-Euclidean vector space and let $\mathfrak{g} \subset \mathfrak{so}(V)$ be a Lie subalgebra which acts completely reducibly on V. Then*

$$Z^1(\mathfrak{g}, V) = \mathrm{d}V \oplus \iota(Z^1(\mathfrak{z}, V^{\mathfrak{g}})),$$

where $\mathfrak{z}$ the centre of $\mathfrak{g}$ and $\iota\colon Z^1(\mathfrak{z}, V^{\mathfrak{g}}) \to Z^1(\mathfrak{g}, V)$ is the inclusion $\iota(\varphi) = \varphi \circ \pi_{\mathfrak{z}}$ with $\pi_{\mathfrak{z}}\colon \mathfrak{g} \to \mathfrak{z}$. In particular,

$$H^1(\mathfrak{g}, V) \simeq H^1(\mathfrak{z}, V^{\mathfrak{g}}).$$

Proof. First note that, for $\varphi \in Z^1(\mathfrak{z}, V^{\mathfrak{g}}) = \mathrm{Hom}(\mathfrak{z}, V^{\mathfrak{g}})$, $\iota(\varphi)$ is indeed a cocycle in $Z^1(\mathfrak{g}, V)$. Moreover, with V completely reducible, we have $V^{\mathfrak{g}} \cap \mathfrak{g}V = \{0\}$ and hence that

$$\mathrm{d}V \cap \iota(Z^1(\mathfrak{z}, V^{\mathfrak{g}})) = \{0\}.$$

It remains to show that

$$H^1(\mathfrak{g}, V) \simeq Z^1(\mathfrak{z}, V^{\mathfrak{g}}).$$

But we can apply Theorem 5.4 to $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{z}$ and $\mathfrak{h} = \mathfrak{g}'$ to get, from equation (5.6),

$$H^1(\mathfrak{g}, V) \simeq H^1(\mathfrak{z}, V)^{\mathfrak{g}}.$$

Therefore, it remains to show that $H^1(\mathfrak{z}, V)^{\mathfrak{g}}$ is isomorphic to $Z^1(\mathfrak{z}, V^{\mathfrak{g}})$. We note that

$$H^1(\mathfrak{z}, V)^{\mathfrak{g}} = \big\{[\varphi] \in H^1(\mathfrak{z}, V) \mid \varphi \in Z^1(\mathfrak{z}, V) : \forall X \in \mathfrak{g}, \exists v \in V, X\varphi = \mathrm{d}_{\mathfrak{z}} v\big\},$$

where $d_{\mathfrak{z}}: V \to Z^1(\mathfrak{z}, V)$, $d_{\mathfrak{z}} v = dv|_{\mathfrak{z}}$ is the differential of $\mathfrak{z}$. Clearly, $Z^1(\mathfrak{z}, V^{\mathfrak{g}})$ injects into $H^1(\mathfrak{z}, V)^{\mathfrak{g}}$ by mapping a cocycle to its equivalence class in $H^1(\mathfrak{z}, V)^{\mathfrak{g}}$, but we have to show that this is surjective.

For this, note that if $[\varphi] \in H^1(\mathfrak{z}, V)^{\mathfrak{g}}$, then $\varphi \in Z^1(\mathfrak{z}, V)$ is such that for each $X \in \mathfrak{g}$, there is a $v_X \in V$ such that

$$X\varphi(Z) = Zv_X.$$

This defines a linear map $\hat{\varphi}: \mathfrak{g} \to V/V^{\mathfrak{z}}$ by the relation

$$\hat{\varphi}(X) = v_X + V^{\mathfrak{z}}.$$

Since $Z \in \mathfrak{z}$, we have

$$Zv_{[X,Y]} = [X, Y]\varphi(Z) = Z(Xv_Y - Yv_X),$$

and so $\hat{\varphi}$ is a cocycle, i.e., $\hat{\varphi} \in Z^1(\mathfrak{g}, V/V^{\mathfrak{z}})$. This induces a linear map

$$\Psi: H^1(\mathfrak{z}, V)^{\mathfrak{g}} \ni [\varphi] \mapsto [\hat{\varphi}] \in H^1(\mathfrak{g}, V/V^{\mathfrak{z}}),$$

which clearly has the kernel $Z^1(\mathfrak{z}, V^{\mathfrak{g}})$. Therefore,

$$H^1(\mathfrak{z}, V)^{\mathfrak{g}}/Z^1(\mathfrak{z}, V^{\mathfrak{g}}) \simeq \mathrm{im}(\Psi) \subset H^1(\mathfrak{g}, V/V^{\mathfrak{z}}).$$

Now we use again equation (5.6) in Theorem 5.4 to get that

$$H^1(\mathfrak{g}, V/V^{\mathfrak{z}}) \simeq H^1(\mathfrak{z}, V/V^{\mathfrak{z}})^{\mathfrak{g}}.$$

The last step in the proof is to show that $H^1(\mathfrak{z}, V/V^{\mathfrak{z}}) = \{0\}$. For this we set $W := V/V^{\mathfrak{z}}$ and we have to show that $Z^1(\mathfrak{z}, W) = d_{\mathfrak{z}} W$. The $\mathfrak{z}$-module W is an orthogonal sum of 2-dimensional indecomposable modules W_i and $Z^1(\mathfrak{z}, W) = \oplus_i Z^1(\mathfrak{z}, W_i)$. Therefore, we can assume without loss of generality that $W = W_1$ is 2-dimensional. Let us denote by I a generator of the 1-dimensional Lie algebra $\mathfrak{so}(W)$ such that $I^2 = \epsilon\,\mathrm{Id}$, $\epsilon = \pm 1$. Then there exists $0 \neq \lambda \in \mathfrak{z}^*$ such that $Xv = \lambda(X)Iv$ for all $X \in \mathfrak{z}$ and $v \in W$. Given $\varphi \in Z^1(\mathfrak{z}, W)$, we have

$$0 = X\varphi(Y) - Y\varphi(X) = \lambda(X)I\varphi(Y) - \lambda(Y)I\varphi(X),$$

for all $X, Y \in \mathfrak{z}$. The latter equation implies that there exists a vector $v \in W$ such that

$$I\varphi(X) = \lambda(X)v \quad \text{for all } X \in \mathfrak{z}.$$

This shows that $\varphi = \epsilon\lambda \otimes Iv = \epsilon d_{\mathfrak{z}} v \in dW$ and hence that $H^1(\mathfrak{z}, V/V^{\mathfrak{z}}) = \{0\}$.

This implies that $\mathrm{im}(\Psi) = \{0\}$, and hence $Z^1(\mathfrak{z}, V^{\mathfrak{g}}) = H^1(\mathfrak{z}, V)^{\mathfrak{g}} \simeq H^1(\mathfrak{g}, V)$. ∎

Now we are in a position to prove Theorem 5.5.

Proof of Theorem 5.5. From Proposition 5.1 we have that $\tilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$, where $\mathfrak{h}_\varphi$ is given by equation (5.7), with $\varphi \in Z^1(\mathfrak{g}, T^\perp)$. It remains to verify that $\varphi|_{\mathfrak{g}'} = 0$. Lemma 5.6 shows that, up to conjugation of $\tilde{\mathfrak{g}}$ in $\mathfrak{so}(V) \ltimes V$ by a translation in $T^\perp$, we have $\varphi \in \iota(Z^1(\mathfrak{z}, T^\perp \cap V^{\mathfrak{g}}))$. This shows that φ vanishes on $\mathfrak{g}'$ and takes values in $T^\perp \cap V^{\mathfrak{g}}$. The $\mathfrak{g}$-invariant decomposition

$$T^\perp = (T^\perp \cap V^{\mathfrak{g}}) \overset{\perp}{\oplus} \mathfrak{g}T^\perp$$

shows that the subspace $\mathfrak{g}T^{\perp} \subset V$ is non-degenerate. Let us check that it is not only invariant under $\mathfrak{g}$ but also under $\widetilde{\mathfrak{g}}$. For this, it suffices to observe that, by our description of $\widetilde{\mathfrak{g}}$ and the fact that $\operatorname{im}\varphi \subset T^{\perp} \cap V^{\mathfrak{g}}$, the translational part of any element of $\widetilde{\mathfrak{g}}$ is contained in $(T^{\perp} \cap V^{\mathfrak{g}}) \oplus T$. Therefore, it is perpendicular to $\mathfrak{g}T^{\perp}$, which shows that $\mathfrak{g}T^{\perp} \subset V \subset \widetilde{V}$ is $\widetilde{\mathfrak{g}}$-invariant. Since $\widetilde{\mathfrak{g}}$ is indecomposable, this proves that $\mathfrak{g}T^{\perp} = 0$.

Note that this implies that $T \neq 0$, because otherwise $T^{\perp} = V$, and hence $\mathfrak{g} = 0$ and $\widetilde{\mathfrak{g}} = T = 0$, which contradicts the indecomposability of $\widetilde{\mathfrak{g}}$.

Finally, let $(\operatorname{im}\varphi)^{\perp}$ be the orthogonal space of $\operatorname{im}\varphi$ in $T^{\perp}$ and let W be a $\mathfrak{g}$-invariant complement of $\operatorname{im}\varphi \cap (\operatorname{im}\varphi)^{\perp}$ in $(\operatorname{im}\varphi)^{\perp}$. Then W is non-degenerate. Again it is not only $\mathfrak{g}$-invariant but also $\widetilde{\mathfrak{g}}$-invariant because the translational part of any element in $\widetilde{\mathfrak{g}}$ is contained in $(\operatorname{im}\varphi) \oplus T$ and $W \subset (\operatorname{im}\varphi)^{\perp} \subset T^{\perp}$. Since $\widetilde{\mathfrak{g}}$ is indecomposable, this shows that $W = 0$, and hence that $(\operatorname{im}\varphi)^{\perp} \subset \operatorname{im}\varphi$. ∎

5.3. Cohomology of indecomposable subalgebras in $\mathfrak{so}(1, n+1)$

In this section we compute the 1-cocycles for subalgebras $\mathfrak{g}$ of $\mathfrak{so}(1, n+1)$ that act indecomposably on $V = \mathbb{R}^{1,n+1}$. Such a subalgebra is either irreducible, in which case it is equal to $\mathfrak{so}(1, n+1)$, see [13], and hence $H^1(\mathfrak{g}, V) = 0$, or admits a parallel null-line $L = L_- = \mathbb{R}\mathrm{e}_-$. That such a subalgebra belongs to one of the four types discussed in the proof of Lemma 5.7 below, was proven in [8].

In the following, we will use equations (5.2) and (5.3) and the identifications in Section 5.1 with $(\widetilde{V}, \widetilde{g}, V, g)$ replaced by (V, g, V_0, g_0). Note that $g_0 = g|_{V_0 \times V_0}$ is the standard Euclidean scalar product on $V_0 = \mathbb{R}^n$. We will use the standard decomposition $V = \mathbb{R} \cdot \mathrm{e}_- \oplus V_0 \oplus \mathbb{R} \cdot \mathrm{e}_+$ and the notation $\mathfrak{g}_0 = \mathrm{pr}_{\mathfrak{so}(V_0)}(\mathfrak{g})$, $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$, $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}_0)$ for a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)_L$.

Theorem 5.7. *Let $V = \mathbb{R} \cdot \mathrm{e}_- \oplus V_0 \oplus \mathbb{R} \cdot \mathrm{e}_+$ be the Minkowski space with null vectors $\mathrm{e}_\pm$ and Euclidean vector space V_0, and let $\mathfrak{g} \subset \mathfrak{so}(V)_L \subset \mathfrak{so}(V)$ be an indecomposable subalgebra. Then*

$$H^1(\mathfrak{g}, V) = 0,$$

or $\mathfrak{g}$ annihilates $\mathbf{e}_-$.

Proof. First note that if $\dim(V) = 2$, i.e., $V_0 = 0$, then

$$\mathfrak{g} = \mathfrak{so}(1,1) = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $H^1(\mathfrak{g}, V)$ is clearly trivial.

If $\dim(V) \geq 3$, then according to [8], any indecomposable subalgebra $\mathfrak{g}$ of $\mathfrak{so}(V)_L$, belongs to one of four different types. Two of them annihilate e_-, whereas the other two act non-trivially on $\mathbb{R} \cdot \mathrm{e}_-$. The latter are given as follows, where $\mathfrak{z}$ denotes the centre of $\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{g}_0'$ with $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$ semisimple:

(1) $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{g}_0) \ltimes V_0$. We can set

$$\mathfrak{b} := (\mathbb{R} \oplus \mathfrak{z}) \ltimes V_0,$$

and then $\mathfrak{g}/\mathfrak{b} = \mathfrak{g}_0'$ is semisimple and acts completely reducibly on $V = \mathbb{R}\mathrm{e}_- \oplus V_0 \oplus \mathbb{R}\mathrm{e}_+$.

(2) $\mathfrak{g} = (\mathfrak{h}_f \oplus \mathfrak{g}_0') \ltimes V_0$, with $0 \neq f \in \mathfrak{z}^*$ and $\mathfrak{h}_f = \{(f(Z), Z) \mid Z \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$. Here we set

$$\mathfrak{b} := \mathfrak{h}_f \ltimes V_0,$$

so that $\mathfrak{g}/\mathfrak{b} = \mathfrak{g}_0'$ acts again completely reducibly on V.

Now we apply Theorem 5.4 to $\mathfrak{g}$, the ideal $\mathfrak{b}$ as given in the above and $\mathfrak{g}_0' = \mathfrak{g}/\mathfrak{b}$. Since $\mathfrak{g}_0'$ is semisimple, the second summand in (5.6) vanishes and we get

$$H^1(\mathfrak{g}, V) \simeq H^1(\mathfrak{b}, V)^{\mathfrak{g}}.$$

In order to determine $H^1(\mathfrak{b}, V)$, we can apply Theorem 5.4 again, this time to $\mathfrak{b}$, the ideal $\mathfrak{a} = V_0$ and the subalgebra $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{z}$ in case (1) and $\mathfrak{h} = \mathfrak{h}_f$ in case (2). In both cases $\mathfrak{h}$ is abelian and acts completely reducibly on $\mathfrak{b}$ and on V, so the assumptions of Theorem 5.4 are satisfied and we get

$$H^1(\mathfrak{b}, V) \simeq H^1(\mathfrak{a}, V)^{\mathfrak{b}} + \mathfrak{h}^* \otimes V^{\mathfrak{b}}.$$

Since for both types of $\mathfrak{g}$, $\mathfrak{b}$ scales e_- and contains $\mathfrak{a} = V_0$, we have that $V^{\mathfrak{b}} = \{0\}$, cf. (5.2). So, it remains to show that $(H^1(\mathfrak{a}, V)^{\mathfrak{b}})^{\mathfrak{g}} = H^1(\mathfrak{a}, V)^{\mathfrak{g}}$ is trivial. Even though $\mathfrak{a} = V_0$ is abelian, we cannot apply Lemma 5.6 to find $H^1(\mathfrak{a}, V)$, because $\mathfrak{a}$ does not act completely reducibly on V. Instead, we first note that if $\dim(\mathfrak{a}) = \dim(V_0) = 1$, then $Z^1(\mathfrak{a}, V) = \mathfrak{a}^* \otimes V$, $\mathrm{d}V = \mathfrak{a}^* \otimes (\mathbb{R}\mathrm{e}_- \oplus V_0)$, and the line $\mathfrak{a}^* \otimes \mathrm{e}_+ \subset Z^1(\mathfrak{a}, V)$ projects isomorphically onto $H^1(\mathfrak{a}, V)$. From (5.2) we see that the action of an element $(a, X, v) \in \mathfrak{g}$ on $H^1(\mathfrak{a}, V)$ is given by multiplication with $-a$. Since for both types of $\mathfrak{g}$ there are elements with $a \neq 0$, we conclude that $H^1(\mathfrak{a}, V)^{\mathfrak{g}} = 0$. Thus, we can assume that $\dim(V_0) \geq 2$. Then $\varphi \in Z^1(\mathfrak{a}, V)$ splits into components $\varphi = (\varphi_-, \varphi_0, \varphi_+)$ with respect to $V = \mathbb{R} \cdot \mathrm{e}_- \oplus V_0 \oplus \mathbb{R} \cdot \mathrm{e}_+$, with $\varphi_\pm \in \mathfrak{a}^*$ and $\varphi_0 \in V_0^* \otimes V_0$. From (5.2) we see that $u \in V_0 = \mathfrak{a}$ acts on $(v_-, v, v_+) \in V$ as

$$u \cdot (v_-, v, v_+) = (-u^\top v, v_+ u, 0).$$

Since $\mathfrak{a}$ is abelian, the cocycle condition for φ yields

$$u^\top \varphi_0(v) - v^\top \varphi_0(u) = 0, \quad \varphi_+(u)v - \varphi_+(v)u = 0,$$

for all $u, v \in V_0$. Since $\dim(V_0) \geq 2$, the second equation implies that $\varphi_+ = 0$. The first equation implies that φ_0 is a symmetric endomorphism of V_0. This shows that $Z^1(\mathfrak{a}, V) = V_0^* \oplus S(V_0)$ and that

$$H^1(\mathfrak{a}, V) \simeq S_0(V_0),$$

where $S(V_0)$ and $S_0(V_0)$ denote the symmetric and the symmetric trace-free endomorphisms of V_0, respectively. Hence, every element $[\varphi] \in H^1(\mathfrak{a}, V)^{\mathfrak{g}}$ can be represented by a symmetric trace free-matrix S. Therefore, the equation that $[\varphi]$ is $\mathfrak{g}$-invariant, which means that for every $(a, X, v) \in \mathfrak{g}$, there is a $(w_-, w, w_+) \in V$ such that

$$(a, X, v) \cdot \varphi = \mathrm{d}(w_-, w, w_+),$$

becomes, by (5.2),

$$(a, X, v)(\varphi(u)) - \varphi([(a, X, v), (0, 0, u)]) = \begin{pmatrix} -v^\top S u \\ [X, S]u - aSu \\ 0 \end{pmatrix} = \begin{pmatrix} -w^\top u \\ w_+ u \\ 0 \end{pmatrix},$$

for all $u \in V_0$. This implies that

$$[X, S] = (w_+ \mathrm{Id} + aS).$$

Taking the trace yields $w_+ = 0$ and multiplying both sides by S and taking the trace gives

$$a\,\mathrm{tr}(S^2) = \mathrm{tr}([X, S]S) = 0.$$

Since we can chose $a \neq 0$ for both types, this implies that $\mathrm{tr}(S^2) = 0$. With S symmetric, we obtain that $S = 0$, hence $H^1(\mathfrak{a}, V)^{\mathfrak{g}} = \{0\}$ and consequently that $H^1(\mathfrak{g}, V) = 0$. ■

Remark 5.8. Similar arguments can be used to determine $H^1(\mathfrak{g}, V)$ for the other two types of indecomposable subalgebras of $\mathfrak{so}(V)_L$, those that leave invariant the null vector e_- (notations as in Lemma 5.7; for details about these subalgebras, see [8]). One of them is of the form $\mathfrak{g} = \mathfrak{g}_0 \ltimes V_0$ and by applying to above arguments to $\mathfrak{b} := \mathfrak{z} \ltimes V_0$, one can show that

$$H^1(\mathfrak{g}, V) = S_0(V_0)^{\mathfrak{g}_0} \oplus \mathfrak{z}^* \oplus (V_0^{\mathfrak{g}_0})^*,$$

where $S_0(V_0)^{\mathfrak{g}_0}$ denotes the trace-free, symmetric matrices that commute with $\mathfrak{g}_0$.

A similar statement holds for the remaining fourth type, where $\mathfrak{g} = (\mathfrak{h}_f \oplus \mathfrak{g}_0') \ltimes T_0$, with $0 \neq T_0 \subsetneq V_0$ invariant under $\mathfrak{g}_0$ such that $T_0^\perp \subset \ker(\mathfrak{g}_0)^\perp$ and

$$\mathfrak{h}_f = \{(0, Z, f(Z)) \mid Z \in \mathfrak{z}\}, \quad \text{with } f : \mathfrak{z} \to T_0^\perp \text{ surjective.}$$

Here one can apply the above strategy to $\mathfrak{b} := \mathfrak{h}_f \ltimes T_0$. However, since the result is somewhat technical and we do not need it for what follows, we will not give the details here.

Finally, we study the two types of indecomposable subalgebras of $\mathfrak{so}(1, n+1)$ that stabilise the null line L but act non-trivially on L, i.e., the types considered in the previous theorem.

Proposition 5.9. *Let $V = \mathbb{R} \cdot \mathrm{e}_- \oplus V_0 \oplus \mathbb{R} \cdot \mathrm{e}_+$ be the Minkowski space with null vectors $\mathrm{e}_\pm$, and let $\mathfrak{g} \subset \mathfrak{so}(V)_L \subset \mathfrak{so}(V)$ be an indecomposable subalgebra stabilising a null line $L = \mathbb{R}\mathrm{e}_-$ but acting non-trivially on L. Let $\rho \in \mathfrak{g}^*$ be defined by the representation of $\mathfrak{g}$ on $V/L^\perp$, i.e.,*

$$(a, X, v)[u] = \rho(a, X, v)[u], \quad \text{i.e., } \rho(a, X, v) = -a$$

(according to formula (5.2)). Then, every $\varphi \in Z^1(\mathfrak{g}, V/L^\perp) \subset \mathfrak{g}^*$ is a multiple of ρ or, equivalently, $Z^1(\mathfrak{g}, V/L^\perp) = \mathrm{d}(V/L^\perp)$, i.e., $H^1(\mathfrak{g}, V/L^\perp) = \{0\}$.

Proof. First we consider the type $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{g}_0) \ltimes V_0$. Note that we do not exclude the case $V_0 = 0$, for which $\mathfrak{g} = \mathfrak{so}(1, 1)$. For $a \neq 0$, every $\varphi \in Z^1(\mathfrak{g}, V/L^\perp)$ satisfies

$$0 = \varphi([(a, 0, 0), (0, X, 0)]) = -a\varphi(0, X, 0),$$

for all $X \in \mathfrak{g}_0$. Hence, $\varphi|_{\mathfrak{g}_0} = 0$. Similarly, we get

$$a\varphi(0, 0, v) = \varphi([(a, 0, 0), (0, 0, v)]) = -a\varphi(0, 0, v),$$

for all $v \in \mathbb{R}^n$. Hence, $\varphi|_{V_0} = 0$. This implies that φ is a multiple of ρ.

Now we assume that $\mathfrak{g} = (\mathbb{R}\zeta_0 \oplus \mathfrak{k}) \ltimes V_0$, where $\mathfrak{k} = \ker f \oplus \mathfrak{g}_0' \subset \mathfrak{z} \oplus \mathfrak{g}_0' = \mathfrak{g}_0 = \mathrm{pr}_{\mathfrak{so}(n)}\mathfrak{g}$, $f \in \mathfrak{z}^*$, $\zeta_0 = (1, Z_0)$ and $Z_0 \in \mathfrak{z}$ is a vector in the centre $\mathfrak{z}$ of $\mathfrak{g}_0$ such that $f(Z_0) = 1$. In particular, $\dim(V_0) \geq 2$. For $X \in \mathfrak{k}$, we obtain

$$0 = \varphi([\zeta_0, (0, X, 0)]) = -\varphi(0, X, 0),$$

i.e., $\varphi|_{\mathfrak{k}} = 0$. Moreover, for all $v \in \mathbb{R}^n$, from the cocycle condition, we get

$$-\varphi(0, 0, v) = \varphi([\zeta_0, (0, 0, v)]) = \varphi(0, 0, (1 + Z_0)v) = \varphi(0, 0, v) + \varphi(0, 0, Z_0 v),$$

i.e., that

$$\varphi(0, 0, Z_0 v) = -2\varphi(0, 0, v). \tag{5.8}$$

Applying equation (5.8) twice, one obtains

$$\varphi(0, 0, Z_0^2 v) = -2\varphi(0, 0, Z_0 v) = 4\varphi(0, 0, v).$$

Since $Z_0 \in \mathfrak{so}(n)$, its square Z_0^2 is diagonalisable with only nonpositive eigenvalues. Hence, we get that $\varphi|_{V_0} = 0$. This implies that φ is a multiple of ρ. ∎

6. Holonomy of metrics $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$

In this section we will use the geometric lifting properties of metrics of the form $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$ derived in Section 4 and the algebraic results of Section 5 in order study the holonomy of $\widetilde{g}$. For cones over manifolds (M, g) of arbitrary signature but with completely reducible holonomy, Theorem 5.5 has the following consequences.

Corollary 6.1. *Let g be a semi-Riemannian metric of signature (t, s) on a manifold M, the holonomy algebra $\mathfrak{hol}(g)$ of which acts completely reducibly. Consider the metric*

$$\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$$

on $\widetilde{M} = \mathbb{R}^+ \times \mathbb{R} \times M$ and assume that the holonomy $\widetilde{\mathfrak{g}} := \mathfrak{hol}(\widetilde{g})$ of $\widetilde{g}$ acts indecomposably, i.e., without a proper non-degenerate invariant subspace, and that the translational ideal $T := \widetilde{\mathfrak{g}} \cap V$ is non-degenerate. Then

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes V.$$

Proof. First Proposition 4.2 gives that $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(t,s)}(\widetilde{\mathfrak{g}}) = \mathfrak{hol}(g)$. Then Theorem 5.5 applied to $\widetilde{\mathfrak{g}}$ shows that $\mathfrak{g}T^{\perp} = 0$. If $T^{\perp} \neq \{0\}$, then g admits a non-degenerate parallel vector field which, according to Lemma 4.8, would lift to a non-degenerate parallel vector field for $\widetilde{g}$. This is excluded by the assumption of indecomposability of $\widetilde{g}$. ∎

As an aside, let us record the consequence of Theorem 5.5 for Lorentzian metrics of the form $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$. We have obtained this result in Section 9 of [2].

Corollary 6.2. *Let g be a Riemannian metric in dimension n and $\widetilde{g} = 2\mathrm{d}u\,\mathrm{d}v + u^2 g$ a Lorentzian metric. If the holonomy of $\widetilde{g}$ acts indecomposably, then*

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^n.$$

In the main result of this section we deal with metrics $\widetilde{g}$ over *Lorentzian* metrics g.

Theorem 6.3. *Let g be a Lorentzian metric on an n-dimensional simply connected manifold M and $\widetilde{g} = 2\,\mathrm{d}u\,\mathrm{d}v + u^2 g$ of signature $(2,n)$ on $\mathbb{R}^+ \times \mathbb{R} \times M$. If the holonomy of $\widetilde{g}$ acts indecomposably, then*

$$\mathfrak{hol}(\widetilde{g}) = \mathfrak{hol}(g) \ltimes \mathbb{R}^{1,n-1},$$

or g admits a parallel null vector field and $\widetilde{g}$ admits two linearly independent parallel null vector fields that are orthogonal to each other.

Proof. Set $\widetilde{\mathfrak{g}} := \mathfrak{hol}(\widetilde{g})$, $\mathfrak{g} := \mathfrak{hol}(g)$ and $V := \mathbb{R}^{1,n-1}$. Let $T = \widetilde{\mathfrak{g}} \cap V$ be the pure translations in $\widetilde{\mathfrak{g}}$. We have to show that $T = V$, in which case we have that $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes V$, or that $\mathfrak{g}$ admits an invariant null vector. Hence, we assume from now on that $T \neq V$. By Proposition 4.2, we have that $\widetilde{\mathfrak{g}} \subset \mathfrak{g} \ltimes V$ with $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(1,n+1)}(\widetilde{\mathfrak{g}})$ and T is $\mathfrak{g}$ invariant.

Since $\mathfrak{g}$ is a holonomy algebra, we can apply the Wu splitting theorem and obtain $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and

$$V = \mathbb{R}^{1,n-1} = V_0 \oplus^\perp V_1 \oplus^\perp V_2 \oplus^\perp \cdots \oplus^\perp V_k,$$

with $\mathfrak{g}_i$ acting trivially on V_j for $i \neq j$, and all the V_i's are non-degenerate, with V_0 a trivial representation and V_i indecomposable for $i = 1, \dots, k$. Since we assume that $\widetilde{\mathfrak{g}}$ acts indecomposably, $\widetilde{g}$ does not admit non-degenerate parallel vector fields. Therefore, Lemma 4.8 implies that $V_0 = \{0\}$. Hence, we can choose the V_i in a way that V_1 is the Minkowski space and indecomposable for $\mathfrak{g}_1$, and the remaining V_i are Euclidean and irreducible for $\mathfrak{g}_i$. Note that for $i = 2, \dots, k$, we have that $\mathfrak{g}_i \subset \mathfrak{so}(n_i)$, where $n_i = \dim(V_i)$. Moreover, we can write

$$\mathfrak{g} \ltimes V = (\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k) \ltimes (V_1 \oplus \cdots \oplus V_k) = (\mathfrak{g}_1 \ltimes V_1) \oplus \cdots \oplus (\mathfrak{g}_k \ltimes V_k).$$

Not only T but also $T_i = \widetilde{\mathfrak{g}} \cap V_i$ is $\mathfrak{g}$-invariant. Hence, we have, for $i = 2, \dots, k$, that $T_i = \{0\}$ or $T_i = V_i$, and that T_1 is degenerate, trivial or equal to V_1. The same holds for $P_i = \mathrm{pr}_{V_i} T$ containing T_i.

Since V_1 is indecomposable but not necessarily irreducible, we have to consider several cases for T.

Case 1: T is indefinite, i.e., of signature $(1, \dim(T) - 1)$. In this case we have that $T \cap V_1 = V_1$ and that $T^\perp$ is positive definite and hence a direct sum of irreducibles that can be arranged such that $T^\perp = V_{\ell+1} \oplus \cdots \oplus V_k$ with $1 \leq \ell \leq k-1$ (recall that $T \neq \{0\}$ and that we are working under the assumption $T \neq V$). We apply Theorem 5.5 to the following data.

We define $\widetilde{W} := \mathbb{R}\mathrm{e}_- \oplus T^\perp \oplus \mathbb{R}\mathrm{e}_+$ and a representation $\rho\colon \widetilde{\mathfrak{g}} \to \mathfrak{so}(\widetilde{W})_{\mathrm{e}_-}$ by $\rho(X, v) = (X|_{T^\perp}, \mathrm{pr}_{T^\perp}(v))$. Since $T^\perp$ is positive definite, we have $T \cap T^\perp = \{0\}$, so by its very definition, $\rho(\widetilde{\mathfrak{g}})$ satisfies that $\rho(\widetilde{\mathfrak{g}}) \cap T^\perp = \{0\}$. On the other hand, $\rho(\widetilde{\mathfrak{g}})$ satisfies the assumptions of Theorem 5.5. Hence, with $\rho(\widetilde{\mathfrak{g}}) \cap T^\perp = \{0\}$, the projection of $\rho(\widetilde{\mathfrak{g}})$ onto $\mathfrak{so}(T^\perp)$ acts trivially on $T^\perp$. But this contradicts the fact that $T^\perp = V_{\ell+1} \oplus \cdots \oplus V_k$, where the V_i's are irreducible for $\mathrm{pr}_{\mathfrak{so}(1,n-1)}(\widetilde{\mathfrak{g}})$ and hence for $\mathrm{pr}_{\mathfrak{so}(T^\perp)}(\rho(\widetilde{\mathfrak{g}}))$.

Case 2: T is positive definite (including the case $T = 0$), i.e., $T \cap V_1 = \{0\}$ in virtue of the indecomposability of the $\mathfrak{g}_1$-module V_1. In this case $T^\perp$ is non-degenerate and $V_1 \subset T^\perp$, i.e.,

$$T^\perp = V_1 \oplus \cdots \oplus V_\ell \quad \text{and} \quad T = V_{\ell+1} \oplus \cdots \oplus V_k.$$

Set

$$\mathfrak{g}_- = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell \quad \text{and} \quad \mathfrak{g}_+ = \mathfrak{g}_{\ell+1} \oplus \cdots \oplus \mathfrak{g}_k,$$

where $\mathfrak{g}_+ = \mathfrak{z}_+ + \mathfrak{g}'_+$ is reductive with centre $\mathfrak{z}_+$ and derived algebra $\mathfrak{g}'_+$, and $\mathfrak{g}_1$ is either irreducible or indecomposable but with an invariant null line L.

In the case when $\mathfrak{g}_1$ acts irreducibly on V_1, $\mathfrak{g}$ acts completely reducibly on V and, since T is positive definite, we can apply Corollary 6.1 to get a contradiction to $T \neq V$.

Hence, we can assume that $\mathfrak{g}_1$ is contained in the stabiliser of the null line L, i.e., $\mathfrak{g}_1 \subset \mathfrak{so}(V_1)_L$. Since $\mathfrak{g}_+$ acts trivially on $T^\perp$, the V_i's are irreducible for $i = \ell + 1, \ldots, k$, and $\mathfrak{g}_-$ acts trivially on T, we have that

$$V^{\mathfrak{g}_-} \cap T^\perp = V^{\mathfrak{g}}. \tag{6.1}$$

As in Proposition 5.1, there is a $\varphi \in Z^1(\mathfrak{g}, T^\perp)$ such that $\tilde{\mathfrak{g}} = \mathfrak{h}_\varphi \ltimes T$. Then for $X_\pm \in \mathfrak{g}_\pm$, we have

$$0 = \varphi([X_+, X_-]) = X_- \varphi(X_+).$$

Hence, using equality (6.1), we obtain $\varphi(\mathfrak{g}_+) \subset V^{\mathfrak{g}_-} \cap T^\perp = V^{\mathfrak{g}}$. If $\varphi|_{\mathfrak{g}_+} \neq 0$, we conclude that $V^{\mathfrak{g}}$ is a non-trivial subspace of $T^\perp$ and thus $V^{\mathfrak{g}} = L$. Hence, if $\varphi|_{\mathfrak{g}_+} \neq 0$, there is a non-zero vector in L that is annihilated by $\mathfrak{g}$ and therefore the metric g admits a parallel null vector field.

Hence, for case 2, we can assume that $\varphi|_{\mathfrak{g}_+} = 0$ and we are left with

$$\varphi: \mathfrak{g}_- \to T^\perp = V_1 \oplus \cdots \oplus V_\ell.$$

Then for $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$, with $i, j \in \{1, \ldots, \ell\}$, and $i \neq j$, we have

$$0 = X_i \varphi(X_j) - X_j \varphi(X_i),$$

and hence

$$X_i \varphi(X_j) = 0. \tag{6.2}$$

Since the $V_{j \geq 2}$ are irreducible, this relation for $j = 1$ implies that

$$\varphi|_{\mathfrak{g}_1} \in Z^1(\mathfrak{g}_1, V_1).$$

On the other hand, for $j \geq 2$, we have that

$$\varphi|_{\mathfrak{g}_j} \in Z^1(\mathfrak{g}_j, L \oplus V_j),$$

where L is the $\mathfrak{g}$-invariant null line. If we write $\varphi = \varphi_1 + \cdots + \varphi_\ell$ with $\varphi_i: \mathfrak{g}_- \to V_i$, then relation (6.2) implies that if there exists $X_j \in \mathfrak{g}_j$ for some $j \geq 2$ such that $\varphi_1(X_j) \neq 0$, and thus $\varphi_1(\mathfrak{g}_j) = L$, then $\mathfrak{g}_1$ and hence $\mathfrak{g}$ acts trivially on L. The latter case implies again that the metric g admits a parallel null vector field.

Hence, we have obtained that g admits a parallel null vector field or that $\varphi = \varphi_1 + \cdots + \varphi_\ell$, with $\varphi_i \in Z^1(\mathfrak{g}_i, V_i)$ for $i = 1, \ldots \ell$. Since the V_i for $i \geq 2$ are irreducible, we have that $Z^1(\mathfrak{g}_i, V_i) = \mathrm{d}V_i$, by Lemma 5.6. The case $i = 1$ is covered by Lemma 5.7, where we have shown that $H^1(\mathfrak{g}_1, V_1) = 0$ whenever g does not admit a parallel null vector field. Hence, if g does not admit a parallel null vector field, we obtain from (1) in Theorem 5.2 that $T^\perp$ is degenerate or zero. But this contradicts $T \neq V$ and that $T^\perp$ in case 2 is non-degenerate.

Case 3: T is degenerate, i.e., there is a $\mathfrak{g}$-invariant null line $L = T \cap T^\perp$. Our aim is to apply point (2) in Theorem 5.2 and Proposition 5.9. First note that $\mathfrak{g}$ and therefore the indecomposable subalgebra $\mathfrak{g}_1 \subset \mathfrak{so}(V_1)$ both leave T and hence the null line L invariant. If $\mathfrak{g}_1$ acts trivially on L, then $\mathfrak{g}$ acts trivially on L and the metric g admits a parallel null vector field. Therefore, we can assume that $\mathfrak{g}_1$ does not act trivially on L. This means that we can apply Proposition 5.9 to $\mathfrak{g}_1$ and $L^\perp \cap V_1$ to get that

$$Z^1(\mathfrak{g}_1, V_1/(L^\perp \cap V_1)) = \mathrm{d}(V_1/(L^\perp \cap V_1)).$$

On the other hand, we note that there is a canonical identification

$$V/L^\perp \simeq V_1/(L^\perp \cap V_1),$$

which shows that $\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ acts trivially on $V/L^\perp$. Hence,

$$Z^1(\mathfrak{g}, V/L^\perp) = Z^1(\mathfrak{g}_1, V_1/(L^\perp \cap V_1)) = \mathrm{d}(V/L^\perp).$$

Hence, $H^1(\mathfrak{g}, V/L^\perp) = 0$ and we can apply (2) in Theorem 5.2, which implies that, up to conjugation, $\widetilde{\mathfrak{g}}$ leaves invariant a null line L. This means that $(\widetilde{M}, \widetilde{g})$ admits a recurrent null vector field in the span of ∂_v and L (even a recurrent section in L). But in this situation, Proposition 4.10 ensures the existence of a parallel null vector field on (M, g). ■

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