



Removable sets for Newtonian Sobolev spaces and a characterization of p -path almost open sets

Anders Björn, Jana Björn and Panu Lahti

Abstract. We study removable sets for Newtonian Sobolev functions in metric measure spaces satisfying the usual (local) assumptions of a doubling measure and a Poincaré inequality. In particular, when restricted to Euclidean spaces, a closed set $E \subset \mathbb{R}^n$ with zero Lebesgue measure is shown to be removable for $W^{1,p}(\mathbb{R}^n \setminus E)$ if and only if $\mathbb{R}^n \setminus E$ supports a p -Poincaré inequality as a metric space. When $p > 1$, this recovers Koskela's result (*Ark. Mat.* **37** (1999), 291–304), but for $p = 1$, as well as for metric spaces, it seems to be new. We also obtain the corresponding characterization for the Dirichlet spaces $L^{1,p}$. To be able to include $p = 1$, we first study extensions of Newtonian Sobolev functions in the case $p = 1$ from a noncomplete space X to its completion \hat{X} . In these results, p -path almost open sets play an important role, and we provide a characterization of them by means of p -path open, p -quasiopen and p -finely open sets. We also show that there are nonmeasurable p -path almost open subsets of \mathbb{R}^n , $n \geq 2$, provided that the continuum hypothesis is assumed to be true. Furthermore, we extend earlier results about measurability of functions with L^p -integrable upper gradients, about p -quasiopen, p -path open and p -finely open sets, and about Lebesgue points for $N^{1,1}$ -functions, to spaces that only satisfy local assumptions.

1. Introduction

The first-order analysis on metric spaces makes it possible to define Sobolev-type spaces also on nonopen subsets of \mathbb{R}^n . This, in particular, leads to questions about extensions and restrictions of Sobolev functions, as well as about the gradients of such restrictions in arbitrary (possibly nonmeasurable) sets. In this paper, we address some of these questions in rather general metric spaces and sets.

Standard assumptions in the area are that the metric space is complete and equipped with a globally doubling measure supporting a global p -Poincaré inequality. The integrability exponent for Sobolev functions and their gradients is often assumed to be $p > 1$,

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since this gives reflexive spaces and provides useful tools. At the same time, in many concrete situations, it is desirable to consider noncomplete spaces and to relax the global assumptions to local ones. Last, but not least, the case $p = 1$ is also attracting a lot of interest.

It was shown by Koskela [44, Theorem C] that a closed set $E \subset \mathbb{R}^n$ of zero Lebesgue measure is removable for the Sobolev space $W^{1,p}(\mathbb{R}^n \setminus E)$, with $p > 1$, if and only if $\mathbb{R}^n \setminus E$ supports a p -Poincaré inequality. One of our results is that a similar equivalence holds also for $p = 1$ and for metric spaces, even noncomplete ones and with only local Poincaré inequalities. Moreover, we do not require E to be closed, only that its complement Ω is p -path almost open, i.e., for p -almost every curve γ , the preimage $\gamma^{-1}(\Omega)$ is a union of an open set and a set of zero 1-dimensional Lebesgue measure.

When specialized to weighted Euclidean spaces, as in Heinonen–Kilpeläinen–Martio [36], these results (obtained in Theorems 5.4 and 5.9) can be formulated as follows. Here we follow the notation of [36] and denote the weighted Sobolev and Dirichlet spaces by $H^{1,p}(\Omega, \mu)$ and $L^{1,p}(\Omega, \mu)$, respectively. These spaces coincide with $\hat{N}^{1,p}(\Omega)$ and $\hat{D}^p(\Omega)$, with respect to μ , as defined in Section 5, see the discussion after Theorem 5.4.

Theorem 1.1. *Let $1 \leq p < \infty$ and $d\mu = w dx$, where w is a p -admissible weight on \mathbb{R}^n in the sense of [36]. Let $\Omega = \mathbb{R}^n \setminus E$, where $\mu(E) = 0$. Assume that Ω is p -path almost open, which in particular holds if Ω is open. Then the following statements are equivalent:*

- (a) E is removable for the Sobolev space $H^{1,p}(\Omega, \mu)$.
- (b) E is removable for the Dirichlet space $L^{1,p}(\Omega, \mu)$.
- (c) (Ω, μ) supports a global p -Poincaré inequality.

Note that every set Ω with $\mu(\Omega \cap \partial\Omega) = 0$ is p -path almost open, by Theorem 1.3. When Ω is not open and in the metric setting, the Sobolev and Dirichlet spaces have to be interpreted by means of upper gradients as in Section 2.

Removable sets for Sobolev spaces is a classical topic, also related to sets of capacity zero and to singularities of quasiconformal mappings. We refer to Koskela [44] for further references and a much more extensive discussion. Among other results in [44], p -porous sets contained in a hyperplane were shown to be removable for $H^{1,p}$ (and equivalently for the p -Poincaré inequality).

Removable sets for Poincaré inequalities in metric spaces were studied in Koskela–Shanmugalingam–Tuominen [46]. Their results on porous sets, together with our Theorems 1.1, 5.4 and 5.9, therefore provide examples of removable sets for Sobolev and Dirichlet spaces, see [46, Theorems A, B and Proposition 3.3]. Removability for Dirichlet spaces was not discussed in [44] or [46].

As mentioned in [46, p. 335], Koskela’s proof can be generalized to metric spaces with global assumptions, provided that E is compact, its complement is connected and $p > 1$. We approach the problem from a different angle, though similar methods lie behind some of our arguments as well. Namely, we rely on extensions of Newtonian (Sobolev) functions from a noncomplete metric space X to its completion \hat{X} , recently considered in [8] for $p > 1$.

To be able to handle also $p = 1$, we therefore first prove the following extension result. In addition, as in [8], we replace the global assumptions of a doubling measure and

a 1-Poincaré inequality by weaker local conditions. These local assumptions, as well as the Newtonian and Dirichlet spaces $N^{1,p}(X)$ and $D^p(X)$, will be defined in Section 2.

Theorem 1.2. *Assume that the doubling property and the 1-Poincaré inequality hold within an open set $\Omega \subset X$ in the sense of Definition 2.5. Let $\Omega^\wedge = \widehat{X} \setminus \overline{X} \setminus \overline{\Omega}$, where the closure is taken in the completion \widehat{X} of X . Let $u \in D^1(\Omega)$. Then there is $\hat{u} \in D^1(\Omega^\wedge)$ such that $\hat{u} = u$ C_1^X -q.e. in Ω and the minimal 1-weak upper gradients $g_u := g_{u,\Omega}$ of u and $g_{\hat{u}} := g_{\hat{u},\Omega^\wedge}$ of \hat{u} , with respect to Ω and Ω^\wedge , respectively, satisfy*

$$g_{\hat{u}} \leq A_0 g_u \quad \text{a.e. in } \Omega,$$

where A_0 is a constant depending only on the doubling constant and both constants in the 1-Poincaré inequality within Ω . In particular, the function \hat{u} can be taken to be

$$(1.1) \quad \hat{u}(x) = \limsup_{r \rightarrow 0} \int_{\widehat{B}(x,r) \cap \Omega} u \, d\mu, \quad x \in \Omega^\wedge.$$

If Ω is also 1-path open in \widehat{X} , then we can, in the above conclusion (except for (1.1)), take $\hat{u} \equiv u$ and $g_{\hat{u}} \equiv g_u$ in Ω .

The idea of the proof is to approximate u by discrete convolutions that immediately extend to Ω^\wedge . This goes back to the aforementioned paper by Koskela [44, Theorem C] and is similar to [8] and Heikkinen–Koskela–Tuominen [34]. When $1 < p < \infty$, one can use the reflexivity of L^p to extract a weakly converging subsequence from the p -weak upper gradients of these discrete convolutions. In the case $p = 1$, we instead show that the sequence of 1-weak upper gradients is *equi-integrable*, and then apply the Dunford–Pettis theorem to obtain a weakly converging subsequence. In this way, at the limit we obtain the desired function $\hat{u} \in D^1(\Omega^\wedge)$. Just as in the case $p > 1$ considered in [8], we do not know whether it is ever necessary to have $A_0 > 1$.

To replace the usual global assumptions by similar local ones in our results, we apply a recent result of Rajala [51] about approximations by uniform domains. In particular, we extend results about measurability of functions with L^p -integrable upper gradients (from [41]), about p -quasiopen, p -path open and p -finely open sets (from [10], [11] and [48]), and about Lebesgue points for $N^{1,1}$ -functions (from [43]), to spaces that only satisfy local assumptions, see Section 3 and Proposition 4.11. These localized results are useful later on in the paper.

Observe that in Theorem 1.2 we do not require Ω to be measurable in \widehat{X} , see Section 4 for details. It is not known if Ω can satisfy the assumptions in Theorem 1.2 and at the same time be nonmeasurable in \widehat{X} . Nevertheless, in Section 6 we construct a measurable set in \mathbb{R}^2 , with full measure and satisfying the conclusions in Theorems 1.1 and 1.2 (except for the last part), but which is not even p -path almost open in \mathbb{R}^2 .

The role of p -path (almost) open sets in our arguments is that they preserve minimal p -weak upper gradients and sets with zero capacity, see Lemmas 4.1, 4.2 and Björn–Björn [5, Proposition 3.5]. In Section 7, we study these sets in more detail and prove the following characterization, which combines Theorems 3.7 and 7.3.

Theorem 1.3. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality, $1 \leq p < \infty$. Let $U \subset X$ be measurable. Then the following are equivalent:*

- (a) U is p -path almost open.
- (b) $U = V \cup N$, where V is p -path open and $\mu(N) = 0$.
- (c) $U = V \cup N$, where V is p -quasiopen and $\mu(N) = 0$.
- (d) $U = V \cup N$, where V is p -finely open and $\mu(N) = 0$.

The p -path almost open sets were introduced in [5] and the implication (b) \Rightarrow (a) was proved therein [5, Lemma 3.2]. Since p -quasiopen, p -path open and p -finely open sets are measurable (under the above assumptions), the characterization in Theorem 1.3 is not possible for nonmeasurable p -path almost open sets. At the same time, we show that there are nonmeasurable p -path almost open subsets of \mathbb{R}^n , $n \geq 2$, provided that the continuum hypothesis is assumed. Together with Theorem 1.3 and Example 7.7, this answers Open problem 3.4 in [5].

Quasiopen and finely open sets have earlier been used in various areas of mathematics. For example, quasiopen sets appear naturally as minimizing sets in shape optimization problems, see e.g. Buttazzo–Dal Maso [19], Buttazzo–Shrivastava [20, Examples 4.3 and 4.4] and Fusco–Mukherjee–Zhang [28]. They are also level sets of Sobolev functions and are thus (together with p -finely open sets) suitable for the theory of Sobolev spaces, see Kilpeläinen–Malý [42], Malý–Ziemer [49] and Fuglede [26, 27]. In this context, our Theorems 1.1, 5.4 and 5.9 fully characterize removable singularities with zero measure for Sobolev (and Dirichlet) spaces on p -quasiopen (and thus also p -finely open) sets. Finely open sets define the fine topology and are closely related to superharmonic functions. Fine potential theory on finely open sets has been studied since the 1940s, see Cartan [22] (the linear case, $p = 2$).

2. Upper gradients and Newtonian spaces

We assume throughout the paper, except for Section 5, that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$.

It follows that X is separable and Lindelöf. To avoid pathological situations we assume that X contains at least two points. Proofs of the results in this section can be found in the monographs Björn–Björn [4] and Heinonen–Koskela–Shanmugalingam–Tyson [38].

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. Unless said otherwise, we will only consider curves that are non-constant, compact and rectifiable, and thus each curve can be parameterized by its arc length ds . A property is said to hold for p -almost every curve if it fails only for a curve family Γ with zero p -modulus. Here the p -modulus of Γ is

$$\text{Mod}_{p,X}(\Gamma) := \inf_{\rho} \int_X \rho^p d\mu,$$

with the infimum taken over all nonnegative Borel functions ρ on X such that $\int_{\gamma} \rho ds \geq 1$ for each $\gamma \in \Gamma$.

Following Heinonen–Koskela [37], we next introduce upper gradients (called very weak gradients in [37]).

Definition 2.1. A Borel function $g: X \rightarrow [0, \infty]$ is an *upper gradient* of a function $u: X \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ if for all curves $\gamma: [0, l_\gamma] \rightarrow X$,

$$(2.1) \quad |u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite. If $g: X \rightarrow [0, \infty]$ is measurable and (2.1) holds for p -almost every curve, then g is a *p -weak upper gradient* of u .

The p -weak upper gradients were introduced in Koskela–MacManus [45]. It was also shown therein that if $g \in L^p_{\text{loc}}(X)$ is a p -weak upper gradient of u , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $\|g_j - g\|_{L^p(X)} \rightarrow 0$. If u has an upper gradient in $L^p_{\text{loc}}(X)$, then it has an a.e. unique *minimal p -weak upper gradient* $g_u \in L^p_{\text{loc}}(X)$ in the sense that $g_u \leq g$ a.e. for every p -weak upper gradient $g \in L^p_{\text{loc}}(X)$ of u , see Shanmugalingam [54] and Hajlasz [30]. Following Shanmugalingam [53], we define a version of Sobolev spaces on the metric space X .

Definition 2.2. For a measurable function $u: X \rightarrow \overline{\mathbb{R}}$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of u . The *Newtonian space* on X is

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [53]. We also define

$$D^p(X) = \{u : u \text{ is measurable, finite a.e. and has an upper gradient in } L^p(X)\}.$$

This definition deviates from the definition in [4, Definition 1.54] in that it requires the functions to be finite a.e., which will be useful in e.g. Theorem 5.7, see Remark 5.8. The two definitions coincide whenever X supports a local p -Poincaré inequality, since any measurable function with an upper gradient in $L^p(X)$ then belongs to $L^1_{\text{loc}}(X)$, see [4, Proposition 4.13] and [8, p. 50].

In this paper we assume that functions in $N^{1,p}(X)$ and $D^p(X)$ are defined everywhere (with values in $\overline{\mathbb{R}}$), not just up to an equivalence class in the corresponding function space. This is important for upper gradients to make sense.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $u \in N^{1,p}(E)$ if for every $x \in E$, there exists a ball $B_x \ni x$ such that $u \in N^{1,p}(B_x \cap E)$. The spaces $L^p(E)$, $L^p_{\text{loc}}(E)$, $D^p(E)$ and $D^p_{\text{loc}}(E)$ are defined similarly. If $u, v \in D^p_{\text{loc}}(X)$, then $g_u = g_v$ a.e. in the set $\{x \in X : u(x) = v(x)\}$. In particular for $c \in \mathbb{R}$, we have $g_{\min\{u,c\}} = g_u \chi_{\{u < c\}}$ a.e. Moreover, if $u, v \in D^p(X)$, then $|u|g_v + |v|g_u$ is a p -weak upper gradient of uv .

It is easily verified by gluing curves together that if g_1 and g_2 are upper gradients for a function u in the open sets G_1 and G_2 , respectively, then $g_1 \chi_{G_1} + g_2 \chi_{G_2}$ is an upper gradient for u in $G_1 \cup G_2$. From this, it follows that if $u \in N^{1,p}(G_j)$, $j = 1, 2$, then $u \in N^{1,p}(G_1 \cup G_2)$. A similar sheaf property holds for D^p .

Definition 2.3. The (Sobolev) *capacity* of a set $E \subset X$ is the number

$$C_p^X(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E .

We say that a property holds C_p^X -*quasieverywhere* (C_p^X -q.e.) if the set of points for which the property fails has zero C_p^X -capacity. The capacity is the correct gauge for distinguishing between two Newtonian functions. Namely, if $v \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ C_p^X -q.e. Moreover, if $u, v \in D_{loc}^p(X)$ and $u = v$ a.e., then $u = v$ C_p^X -q.e. In this paper we will use many different C_p^X -capacities with respect to different metric spaces X ; this will always be carefully denoted in the superscript.

Definition 2.4. An \mathbb{R} -valued function u , defined on a set $E \subset X$, is C_p^X -*quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p^X(G) < \varepsilon$ and $u|_{E \setminus G}$ is \mathbb{R} -valued and continuous.

For a ball $B = B(x, r)$ with centre x and radius r , we let $\lambda B = B(x, \lambda r)$. In metric spaces, it can happen that balls with different centres and/or radii denote the same set. We will, however, make the convention that a ball comes with a predetermined centre and radius. In this paper, balls are assumed to be open.

Definition 2.5. The measure μ is *doubling within an open set* $\Omega \subset X$ if there is $C > 0$ (depending on Ω) such that $\mu(2B) \leq C\mu(B)$ for all balls $B \subset \Omega$.

Similarly, the *p-Poincaré inequality holds within an open set* Ω if there are constants $C > 0$ and $\lambda \geq 1$ (depending on Ω) such that for all balls $B \subset \Omega$, all integrable functions u on λB , and all upper gradients g of u in λB ,

$$(2.2) \quad \int_B |u - u_B| \, d\mu \leq Cr_B \left(\int_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where $u_B := \int_B u \, d\mu / \mu(B)$ and r_B is the radius of B .

Each of these properties is called *local* if for every $x \in X$ there is $r > 0$ (depending on x) such that the property holds within $B(x, r)$. The property is called *semilocal* if it holds within every ball $B(x_0, r_0)$ in X . If moreover C and λ are independent of x_0 and r_0 , then it is called *global*.

Note that there is a difference between a property holding *within* $\Omega \subset X$ (i.e., for balls taken in the underlying space X) and *on* Ω , seen as a metric space in its own right, where balls are taken with respect to Ω .

The p -Poincaré inequality can equivalently be defined using p -weak upper gradients. We will need the following characterization of the p -Poincaré inequality showing that it is enough to test with bounded $u \in N^{1,p}(X)$.

Lemma 2.6. *Let $\Omega \subset X$ be open. Assume that there are constants $C > 0$ and $\lambda \geq 1$ such that (2.2) holds for all balls $B \subset \Omega$ and all bounded $u \in N^{1,p}(X)$. Then the p -Poincaré inequality holds within Ω with the same constants C and λ .*

Below and later, we write $u_+ = \max\{0, u\}$ and $u_- = \max\{0, -u\}$.

Proof. Let $B = B(x, r) \subset \Omega$ be a ball, let u be an integrable function on λB , and let g be an upper gradient of u in λB . We may assume that $g \in L^p(\lambda B)$, as otherwise there is nothing to prove. Thus, $u \in D^p(\lambda B)$.

For $j = 1, 2, \dots$, let $B_j = B(x, (1 - 2^{-j})r)$,

$$u_j = \max\{\min\{u, j\}, -j\} \quad \text{and} \quad v_j = (1 - 2^{j+1}r \operatorname{dist}(x, \lambda B_j))_+ u_j,$$

extended by zero outside λB . Then $v_j \in N^{1,p}(X)$ is bounded and g is an upper gradient of v_j in λB_j . Thus, (2.2) applied to v_j and B_j gives

$$\int_{B_j} |u_j - (u_j)_{B_j}| d\mu = \int_{B_j} |v_j - (v_j)_{B_j}| d\mu \leq Cr_{B_j} \left(\int_{\lambda B_j} g^p d\mu \right)^{1/p}.$$

The result now follows from the fact that $u_j \rightarrow u$ in $L^1(B)$ as $j \rightarrow \infty$. ■

3. From global to local assumptions

In this section we show how a recent result due to Rajala [51, Theorem 1.1] can be used to lift results, that have earlier been obtained under global assumptions, to spaces with only local assumptions. This will be useful later in our considerations. The main idea in this localization approach is to see suitable neighbourhoods of points in X as “good” metric spaces in their own right. Since balls may be disconnected and need not support a Poincaré inequality, they do not in general serve as such good neighbourhoods. Even when a ball, or its closure, is connected it can fail to support a Poincaré inequality and the measure may fail to be globally doubling on it. Instead, closures of the uniform domains constructed by Rajala [51] will do the job, since they are compact, support global Poincaré inequalities and the measure is globally doubling on them.

Recall that a space is *geodesic* if every pair of points can be connected by a curve whose length equals the distance between the points, and that a *domain* is an open connected set. A domain $G \subset X$ is *uniform* if there is a constant $A \geq 1$ such that for every pair $x, y \in G$ there is a curve $\gamma: [0, l_\gamma] \rightarrow G$ with $\gamma(0) = x$ and $\gamma(l_\gamma) = y$ such that $l_\gamma \leq Ad(x, y)$ and

$$\operatorname{dist}(\gamma(t), X \setminus G) \geq \frac{1}{A} \min\{t, l_\gamma - t\} \quad \text{for } 0 \leq t \leq l_\gamma.$$

As usual, $\operatorname{dist}(x, \emptyset) = \infty$. Moreover, X is *globally doubling* if there is a constant N such that every ball $B(x, r)$ can be covered by N balls with radii $\frac{1}{2}r$.

The following result was proved in [51] under the assumption that X is *quasiconvex*. In particular, it applies to geodesic spaces because their quasiconvexity constant is 1.

Theorem 3.1 (Rajala [51, Theorem 1.1]). *Let X be a geodesic metric space and let $U \subset X$ be a bounded domain. If U is globally doubling and $\varepsilon > 0$, then there is a uniform domain G such that*

$$\{x \in U : \operatorname{dist}(x, X \setminus U) \geq \varepsilon\} \subset G \subset U.$$

Note that if $U = X$, then X itself is a uniform domain and $G = U = X$ above. In the definition of uniform domains it is often assumed that $G \subsetneq X$. Allowing $G = X$, as in [51], is convenient when formulating Theorems 3.1 and 3.2.

In [51] it is assumed that X is globally doubling, and approximation from outside by uniform domains is also deduced. However, when approximating from inside as in Theorem 3.1, it is easy to see that in the proof in [51], it is enough to apply [51, Lemma 2.1] with respect to U . It is therefore enough to assume that U is globally doubling, which makes it possible to deduce the following result.

Theorem 3.2. *Assume that the p -Poincaré inequality and the doubling property for μ hold (with constants C_{PI} , λ and C_μ) within $B_1 = B(x_1, r_1)$. Also assume that $\overline{\Lambda B_1}$ is compact, where $\Lambda = 3C_\mu^3 C_{PI}$. Then there is a bounded uniform domain G such that*

$$\tau B_1 \Subset G \subset \frac{1}{6} B_1, \quad \text{where } \tau = \frac{1}{60\Lambda}.$$

Moreover, $\mu|_G$ and $\mu|_{\overline{G}}$ are globally doubling and support global p -Poincaré inequalities on the metric spaces G and \overline{G} , respectively.

Note that Λ is independent of λ , as in [6, Lemma 4.9] and [4, Theorem 4.32]. As usual, by $E \Subset G$ we mean that \overline{E} is a compact subset of G .

Remark 3.3. Note that G , being uniform, satisfies the so-called *corkscrew condition*. Applying Theorem 2.8 in Björn–Shanmugalingam [17] to $A = \overline{G}$ and letting $\rho \rightarrow 0$ in [17, (2.2)] shows that $\mu(\partial G) = 0$.

Proof of Theorem 3.2. Define the inner metric

$$d_{\text{in}, \overline{\Lambda B_1}}(x, y) = \inf \text{length}(\gamma),$$

where the infimum is taken over all curves $\gamma \subset \overline{\Lambda B_1}$ connecting x and y . Let

$$Y = \{x \in \overline{\Lambda B_1} : d_{\text{in}, \overline{\Lambda B_1}}(x, x_1) < \infty\}$$

be the rectifiably connected component of $\overline{\Lambda B_1}$ containing x_1 . As $\overline{\Lambda B_1}$ is compact, it follows from Ascoli’s theorem that $(Y, d_{\text{in}, \overline{\Lambda B_1}})$ is a geodesic metric space. By Björn–Björn [6, Lemma 4.9], every pair of points $x, y \in \frac{1}{5} B_1$ can be connected by a curve in $\overline{\Lambda B_1}$, of length at most $9\Lambda d(x, y)$. Hence, both $\frac{1}{6} B_1$ and

$$B_{\text{in}} := \{x \in Y : d_{\text{in}, \overline{\Lambda B_1}}(x, x_1) < \frac{1}{6} r_1\}$$

are open and $\tau B_1 \Subset B_{\text{in}} \subset \frac{1}{6} B_1 \subset Y$. The reason for using the inner metric is that inner balls are always connected, while standard balls, such as $\frac{1}{6} B_1$, need not be connected.

By [6, Proposition 3.4], the ball $\frac{1}{6} B_1$ is globally doubling. As d and $d_{\text{in}, \overline{\Lambda B_1}}$ are comparable within $\frac{1}{6} B_1$, also $(B_{\text{in}}, d_{\text{in}, \overline{\Lambda B_1}})$ is globally doubling. Since $(B_{\text{in}}, d_{\text{in}, \overline{\Lambda B_1}})$ is connected, we can therefore apply Theorem 3.1 with respect to $(Y, d_{\text{in}, \overline{\Lambda B_1}})$ and obtain a uniform domain G such that $\tau B_1 \Subset G \subset B_{\text{in}}$. Note that since d and $d_{\text{in}, \overline{\Lambda B_1}}$ are comparable within $\frac{1}{5} B_1$ and $\text{dist}(\frac{1}{6} B_1, X \setminus Y) \geq \frac{1}{30} r_1$, uniformity is the same with respect to $(Y, d_{\text{in}, \overline{\Lambda B_1}})$ and (X, d) (although the uniformity constants may be different).

By Björn–Shanmugalingam [17, Lemmas 2.5 and 4.2], $\mu|_G$ is globally doubling on G . Next, we use [17, Theorem 4.4], to see that $\mu|_G$ supports a global p -Poincaré inequality on G . Since the proof of [17, Theorem 4.4] only uses balls contained (together with their dilations) in G , the proof applies under our assumptions. As $\mu(\partial G) = 0$, by Remark 3.3, the same conclusions hold for $\mu|_{\bar{G}}$. (To see that the Poincaré inequality holds on \bar{G} , one only needs to observe that if g is an upper gradient of u on \bar{G} , then $g|_G$ is an upper gradient of $u|_G$ on G , see also [8, Proposition 3.6] for further details.) ■

One result that can be obtained using Theorem 3.2 is the following extension of Theorem 1.11 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [41]. Since local assumptions are inherited by open subsets, it directly applies also to open $\Omega \subset X$ (in place of X), cf. Remark 4.8.

Theorem 3.4. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality. Let $g \in L^p_{\text{loc}}(X)$ be an upper gradient of $u: X \rightarrow \mathbb{R}$. Then $u \in L^p_{\text{loc}}(X)$ and u is in particular measurable.*

Proof. Let $x \in X$. It follows from Theorem 3.2 that there is a bounded uniform domain $G_x \ni x$ such that \bar{G}_x is compact and $\mu|_{\bar{G}_x}$ is globally doubling and supports a global p -Poincaré inequality on \bar{G}_x . In particular, $g|_{\bar{G}_x} \in L^p(\bar{G}_x)$ is an upper gradient of $u|_{\bar{G}_x}$ in \bar{G}_x and Theorem 1.11 in [41] shows that $u|_{\bar{G}_x}$ is measurable and belongs to $L^p_{\text{loc}}(\bar{G}_x)$.

As X is Lindelöf it follows that u is measurable on X and $u \in L^p_{\text{loc}}(X)$. ■

Another consequence of Rajala’s theorem is a characterization of p -path open sets under local assumptions. These sets will play a prominent role in our studies, since they preserve minimal p -weak upper gradients and sets with zero capacity, see Lemmas 4.1 and 4.2 below and Björn–Björn [5, Proposition 3.5]. The relation between p -path open and p -path almost open sets will be studied in Section 7.

Definition 3.5. A set $G \subset A$ is p -path open in $A \subset X$ if for p -almost every curve $\gamma: [0, l_\gamma] \rightarrow A$, the set $\gamma^{-1}(G)$ is (relatively) open in $[0, l_\gamma]$. Further, $G \subset A$ is p -path almost open in $A \subset X$ if for p -almost every curve $\gamma: [0, l_\gamma] \rightarrow A$, the set $\gamma^{-1}(G)$ is the union of an open set and a set with zero 1-dimensional Lebesgue measure.

The p -modulus $\text{Mod}_p(\Gamma)$ of the exceptional curve family Γ can equivalently be measured within X or A , provided that A is equipped with the appropriate restriction $\tilde{\mu}$ of μ to A . Since A may be nonmeasurable, $\tilde{\mu}$ is defined by letting

$$\tilde{\mu}(\tilde{E}) = \inf\{\mu(E) : E \supset \tilde{E} \text{ and } E \text{ is a Borel set with respect to } X\}$$

for Borel sets \tilde{E} in A , and then completing $\tilde{\mu}$. This makes $\tilde{\mu}$ a complete Borel regular measure on A , which coincides with the restriction $\mu|_A$ when A is μ -measurable. It also follows that every Borel function $\tilde{\rho}$ on A has a Borel extension ρ to X such that

$$\int_A \tilde{\rho} d\tilde{\mu} = \int_X \rho d\mu.$$

Hence, $\text{Mod}_{p,A}(\Gamma) = \text{Mod}_{p,X}(\Gamma)$ as claimed. The relation between $\tilde{\mu}$ and μ is quite similar to the relation between μ and $\hat{\mu}$ as discussed in the beginning of Section 4 and in the corrigendum of Björn–Björn [8], and the relation between μ_X and μ_Y in Section 5.

The two properties in Definition 3.5 are transitive, as shown by the following result. Note also that it follows from [4, Proposition 2.45] that if $1 \leq p < q$ and G is q -path open (resp. q -path almost open) in X , then G is p -path open (resp. p -path almost open) in X .

Lemma 3.6. *Assume that $G_1 \subset G_2 \subset G_3$ and that G_2 is p -path almost open in G_3 . Then G_1 is p -path almost open in G_2 if and only if it is p -path almost open in G_3 . The corresponding result also holds if “ p -path almost open” is replaced by “ p -path open” throughout.*

Proof. If G_1 is p -path almost open in G_3 , then it is p -path almost open in G_2 (in view of the discussion above) simply because every curve in G_2 is a curve in G_3 .

Conversely, assume that G_1 is p -path almost open in G_2 . Let $\Gamma_j, j = 1, 2$, be the family of curves γ in G_{j+1} such that $\gamma^{-1}(G_j)$ is not a union of an open set and a set of measure zero. Let Γ' be the family of curves in G_3 which contain a subcurve in Γ_1 . Then by assumption, [4, Lemma 1.34 (c)] and the discussion above,

$$\text{Mod}_{p,G_3}(\Gamma') \leq \text{Mod}_{p,G_3}(\Gamma_1) = \text{Mod}_{p,G_3}(\Gamma_2) = 0.$$

Next take a curve $\gamma: [0, l_\gamma] \rightarrow G_3$ such that $\gamma \notin \Gamma_2 \cup \Gamma'$. Then $\gamma^{-1}(G_2)$ is a union of an open set A and a set of measure zero. Since $A \subset \mathbb{R}$, it can be written as a countable or finite union of pairwise disjoint open intervals A_j . Each A_j can be written as an increasing countable union of compact intervals, and since $\gamma \notin \Gamma'$, we see that $A_j \cap \gamma^{-1}(G_1)$ is a union of an open set and a set of measure zero. Hence, $\gamma^{-1}(G_1)$ is a union of an open set and a set of measure zero. As $\text{Mod}_{p,G_3}(\Gamma_2 \cup \Gamma') = 0$, we have shown that G_1 is p -path almost open in G_3 .

The p -path open case is similar. ■

Next, we shall characterize p -path open sets in terms of p -quasiopen and p -finely open sets, under local assumptions. Such characterizations have been done under global assumptions, and as earlier in this section we will show how to “lift” them to local assumptions.

A set $V \subset X$ is p -quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p^X(G) < \varepsilon$ and $G \cup V$ is open. Every p -quasiopen set is measurable by [5, Lemma 9.3]. The family of p -quasiopen sets does not form a topology (in general) but it is closed under countable unions.

If $E \subset A$ are bounded subsets of X , then the variational capacity of E with respect to A is

$$\text{cap}_p^X(E, A) = \inf_u \int_X g_u^p d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E and $u = 0$ on $X \setminus A$. (If no such function u exists, then $\text{cap}_p^X(E, A) = \infty$.)

A set $E \subset X$ is p -thin at $x \in X$ if

$$(3.1) \quad \int_0^1 \left(\frac{\text{cap}_p^X(E \cap B(x, r), B(x, 2r))}{\text{cap}_p^X(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty$$

when $p > 1$, and if

$$(3.2) \quad \lim_{r \rightarrow 0} \frac{\text{cap}_1^X(E \cap B(x, r), B(x, 2r))}{\text{cap}_1^X(B(x, r), B(x, 2r))} = 0$$

when $p = 1$. (The quotients in (3.1) and (3.2) are interpreted as 1 if the denominators therein are zero.) Note that, under the assumptions of Theorem 3.7 presented below, $\text{cap}_p^X(B(x, r), B(x, 2r))$ is comparable to $\mu(B(x, r))/r^p$ for sufficiently small r , by e.g. the proof of [4, Proposition 6.16], and so the latter quantity could also be used in (3.1) and (3.2), as was done in e.g. Lahti [48].

A set $V \subset X$ is p -finely open if $X \setminus V$ is p -thin at each point $x \in V$. The family of p -finely open sets forms the p -fine topology.

The following theorem gives the equivalence of (b)–(d) in Theorem 1.3, since we have $\mu(Z) = 0$ whenever $C_p^X(Z) = 0$ (this follows directly from Definition 2.3).

Theorem 3.7. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality: Let $U \subset X$. Then the following are equivalent:*

- (a) U is p -path open.
- (b) U is p -quasiopen.
- (c) $U = V \cup Z$, where V is p -finely open and $C_p^X(Z) = 0$.

When X is complete and μ is globally doubling and supports a global p -Poincaré inequality, these characterizations are due to Björn–Björn–Latvala [9, Theorem 4.9], [10, Theorem 1.4], Björn–Björn–Malý [11, Theorem 1.1], Shanmugalingam [54, Remark 3.5], and Lahti [48, Corollary 6.12] combined with Hakkarainen–Kinnunen [32, Theorems 4.3 and 5.1]. We will use these results, and the proof below just shows how to lift them to local assumptions, without repeating the arguments.

Proof. We start by some preliminary observations. By Theorem 3.2, for every $x \in X$ there is a bounded uniform domain $G_x \ni x$ such that \overline{G}_x is compact and $\mu|_{\overline{G}_x}$ is globally doubling and supports a global p -Poincaré inequality on \overline{G}_x . As X is Lindelöf, there is a countable cover $\{G_j\}_{j=1}^\infty$ of X , where $G_j = G_{x_j}$.

We also note for later use that Proposition 3.3 in [11] (applied both to \overline{G}_j and to X as the underlying space) implies that $U \cap G_j$ is p -quasiopen with respect to \overline{G}_j if and only if it is p -quasiopen with respect to G_j , which in turn is equivalent to it being p -quasiopen with respect to X .

(a) \Rightarrow (b) For each j , the set $U_j := U \cap G_j$ is p -path open in \overline{G}_j . By Theorem 1.1 in [11], we see that U_j is p -quasiopen with respect to \overline{G}_j , and by the above argument also with respect to X . Hence, $U = \bigcup_{j=1}^\infty U_j$ is p -quasiopen in X .

(b) \Rightarrow (a) This is proved in Shanmugalingam [54, Remark 3.5], without any assumptions on X .

To prove the equivalence with (c), note that in the case $p > 1$, a set $W \subset G_j$ is p -finely open with respect to \overline{G}_j if and only if for every $x \in W$,

$$\int_0^{r_x} \left(\frac{\text{cap}_p^{\overline{G}_j}(B(x, r) \setminus W, B(x, 2r))}{\text{cap}_p^{\overline{G}_j}(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty,$$

where $r_x > 0$ is such that $B(x, 2r_x) \subset G_j$. Clearly, for $0 < r < r_x$ and $A \subset B(x, r)$,

$$(3.3) \quad \text{cap}_p^{\bar{G}_j}(A, B(x, 2r)) = \text{cap}_p^X(A, B(x, 2r)),$$

and hence W is p -finely open with respect to \bar{G}_j if and only if it is p -finely open with respect to X . The equality (3.3) holds also in the case $p = 1$ and implies directly that $W \subset G_j$ is 1-finely open with respect to \bar{G}_j if and only if it is 1-finely open with respect to X .

(b) \Rightarrow (c) By the above argument, $U_j := U \cap G_j$ is p -quasiopen with respect to \bar{G}_j . Theorem 4.9 in [9] (for $p > 1$) and [48, Corollary 6.12] combined with [32, Theorems 4.3 and 5.1] (for $p = 1$) show that it can be written as $U_j = V_j \cup Z_j$, where V_j is p -finely open with respect to \bar{G}_j (and equivalently X) and $C_p^{\bar{G}_j}(Z_j) = 0$. Hence, $\bigcup_{j=1}^\infty V_j$ is p -finely open with respect to X . Moreover, it follows from e.g. [4, Lemma 2.24] that the capacities $C_p^{\bar{G}_j}$ and C_p^X have the same zero sets in G_j and so $C_p^X(\bigcup_{j=1}^\infty Z_j) = 0$. Since $U = \bigcup_{j=1}^\infty V_j \cup \bigcup_{j=1}^\infty Z_j$, (c) holds.

(c) \Rightarrow (b) For each j , the set $V \cap G_j$ is p -finely open in X and thus in \bar{G}_j , by the above observation. Also

$$C_p^{\bar{G}_j}(Z \cap G_j) \leq C_p^X(Z) = 0.$$

It then follows from [10, Theorem 1.4] (for $p > 1$) and [48, Corollary 6.12] combined with [32, Theorems 4.3 and 5.1] (for $p = 1$) that the set $U_j := (V \cap G_j) \cup (Z \cap G_j)$ is p -quasiopen with respect to \bar{G}_j , and thus also with respect to X , by the above argument. Hence, $U = \bigcup_{j=1}^\infty U_j$ is p -quasiopen in X . ■

Theorems 1.2 and 1.3 in [11] can be extended similarly. See also Corollary 4.10 below and Björn–Björn [6, Theorem 9.1].

4. Extending $N^{1,1}$ -functions to the completion \hat{X}

The main goal of this section is to prove Theorem 1.2. We let \hat{X} be the completion of X with respect to the metric d . The metric immediately extends to \hat{X} . We extend the measure to \hat{X} by defining

$$\hat{\mu}(E) = \mu(E \cap X) \quad \text{for every Borel set } E \subset \hat{X},$$

and then complete it to obtain a Borel regular measure $\hat{\mu}$. Saksman [52, Lemma 1] used a similar construction when studying globally doubling measures.

Now $\hat{X} \setminus X$ either has zero $\hat{\mu}$ -measure or is $\hat{\mu}$ -nonmeasurable. In both cases, we have $\hat{\mu}_{\text{in}}(\hat{X} \setminus X) = 0$, where the inner measure $\hat{\mu}_{\text{in}}$ is defined by

$$(4.1) \quad \begin{aligned} \hat{\mu}_{\text{in}}(E) &= \sup\{\hat{\mu}(A) : A \subset E \text{ is } \hat{\mu}\text{-measurable}\} \\ &= \sup\{\hat{\mu}(A) : A \subset E \text{ is a Borel set in } \hat{X}\}. \end{aligned}$$

The latter equality follows from the fact that $\hat{\mu}$ is a complete Borel regular measure. Moreover,

$$\hat{\mu}(E) = \mu(E \cap X) \quad \text{for every } \hat{\mu}\text{-measurable set } E \subset \hat{X},$$

and thus for $E \subset X$, we have

$$(4.2) \quad \mu(E) = 0 \quad \text{if and only if} \quad \hat{\mu}(E) = 0.$$

It also follows that every μ -measurable (respectively, Borel) function $u: X \rightarrow \mathbb{R}$ has a $\hat{\mu}$ -measurable (respectively, Borel) extension $\hat{u}: \hat{X} \rightarrow \mathbb{R}$ such that $\hat{u}|_X = u$ and

$$(4.3) \quad \int_{\hat{X}} \hat{u} \, d\hat{\mu} = \int_X u \, d\mu,$$

whenever at least one of the integrals exists. Conversely, it follows from the above definition of $\hat{\mu}$ that $v|_X$ is μ -measurable (respectively, Borel) and

$$(4.4) \quad \int_X v \, d\mu = \int_{\hat{X}} v \, d\hat{\mu},$$

whenever $v: \hat{X} \rightarrow \mathbb{R}$ is $\hat{\mu}$ -measurable (respectively, Borel) and one of the integrals exists. See the corrigendum of Björn–Björn [8] for further details; the $\hat{\mu}$ -nonmeasurable case was unfortunately overlooked in the original paper.

The following two auxiliary results relate notions on \hat{X} to the same notions on p -path (almost) open sets.

Lemma 4.1. *Assume that $\Omega \subset \hat{X}$ is $\hat{\mu}$ -measurable and p -path almost open in \hat{X} , $p \geq 1$, and that $u \in D_{\text{loc}}^p(\Omega)$. If $G \subset \Omega \cap X$ is μ -measurable and p -path almost open in Ω , then the minimal p -weak upper gradients $g_{u,G}$ and $\hat{g}_{u,\Omega}$ of u with respect to (G, μ) and $(\Omega, \hat{\mu})$, respectively, coincide a.e. in G .*

Note that by Lemma 3.6, G is p -path almost open in Ω if and only if it is p -path almost open in \hat{X} .

Proof. This is proved verbatim as in Proposition 3.5 in Björn–Björn [5], with the obvious interpretations of the integrals with respect to $\hat{\mu}$. The only additional observation needed is that if Γ is a family of curves in G , then by (4.3) and (4.4),

$$(4.5) \quad \text{Mod}_{p,G}(\Gamma) = \inf_{\rho} \int_G \rho^p \, d\mu = \inf_{\rho} \int_{\Omega \cap X} \rho^p \, d\mu = \inf_{\hat{\rho}} \int_{\Omega} \hat{\rho}^p \, d\hat{\mu} = \text{Mod}_{p,\Omega}(\Gamma),$$

where the infima are taken over all $\rho \in L^p(G, \mu)$, $\rho \in L^p(\Omega \cap X, \mu)$ and $\hat{\rho} \in L^p(\Omega, \hat{\mu})$ satisfying for all $\gamma \in \Gamma$, respectively,

$$\int_{\gamma} \rho \, ds \geq 1 \quad \text{and} \quad \int_{\gamma} \hat{\rho} \, ds \geq 1. \quad \blacksquare$$

Lemma 4.2. *Let $G \subset X$ be μ -measurable and p -path open in \hat{X} , $p \geq 1$, and $E \subset G$. Then $C_p^G(E) = 0$ if and only if $C_p^{\hat{X}}(E) = 0$.*

Proof. By [4, Proposition 1.48], we have that $C_p^G(E) = 0$ if and only if both $\mu(E) = 0$ and $\text{Mod}_{p,G}(\Gamma_E^G) = 0$, where Γ_E^G consists of all curves $\gamma \subset G$ which hit E , i.e., $\gamma^{-1}(E) \neq \emptyset$. A similar equivalence holds for $C_p^{\hat{X}}(E) = 0$ and

$$\Gamma_E^{\hat{X}} = \{\gamma \subset \hat{X} : \gamma^{-1}(E) \neq \emptyset\}.$$

Since G is p -path open in \widehat{X} , for $\text{Mod}_{p,\widehat{X}}$ -almost all curves $\gamma \in \Gamma_E^{\widehat{X}}$, the preimage $\gamma^{-1}(G)$ is relatively open in $[0, l_\gamma]$ and nonempty, and thus γ contains a nonconstant subcurve $\gamma' \in \Gamma_E^G$. Hence, by [4, Lemma 1.34 (c)] and (4.5),

$$\text{Mod}_{p,\widehat{X}}(\Gamma_E^{\widehat{X}}) \leq \text{Mod}_{p,G}(\Gamma_E^G).$$

The reverse inequality is trivial. Together with (4.2), this concludes the proof. ■

The following examples show that there is no hope to obtain Lemma 4.2 for μ -measurable sets that are only p -path almost open in \widehat{X} .

Example 4.3. Let $X = \mathbb{R}^n$ (unweighted), $p \geq 1$, $E = \{x \in \mathbb{R}^n : |x| = 1\}$ and

$$G = \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} \{x : |x| = 1 \pm 2^{-k}\}.$$

Then G is the union of an open set and a set of Lebesgue measure zero, and is thus p -path almost open for all $p \geq 1$, by Theorem 1.3. Moreover, $\widehat{X} = \mathbb{R}^n$ and $C_p^{\widehat{X}}(E) > 0 = C_p^G(E)$. Indeed, the characteristic function $\chi_E \in N^{1,p}(G)$ is admissible for $C_p^G(E)$ and has zero as a p -weak upper gradient with respect to G . At the same time, the $(n - 1)$ -dimensional Hausdorff measure of E is nonzero, and so by Adams [1, equation (12), p. 122] or Hakkarainen–Kinnunen [32, Theorems 4.3 and 5.1], $C_p^{\widehat{X}}(E) > 0$ holds for $p = 1$ and thus for all $p \geq 1$. When $p > n$, one can also choose $E = \{0\}$ and

$$G = \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} \{x : |x| = 2^{-k}\} \quad \text{or} \quad G = \{0\}.$$

Example 4.4. For $\alpha > 1$, let $G = \{x = (x', x_n) \in \mathbb{R}^n : |x'| \leq x_n^\alpha \leq 1\}$ be a closed cusp in $X = \widehat{X} = \mathbb{R}^n$, $n \geq 2$, equipped with the measure $d\mu(x) = |x|^\beta dx$, where $\beta > -n$. Note that μ is globally doubling and supports a global 1-Poincaré inequality on \mathbb{R}^n , by Corollary 15.35 in Heinonen–Kilpeläinen–Martio [36] and Theorem 1 in Björn [16]. Since G is the union of an open set and a set of Lebesgue measure zero, it is p -path almost open for all $p \geq 1$, by Theorem 1.3. Testing with $u_j(x) = \min\{1, -(\log x_n)/j\}$ shows that

$$C_p^G(\{0\}) = 0 \quad \text{if } 1 < p \leq \alpha(n - 1) + \beta + 1 \text{ or } 1 = p < \alpha(n - 1) + \beta + 1,$$

while $C_p^{\widehat{X}}(\{0\}) > 0$ for $p > \max\{n + \beta, 1\}$, by [36, Example 2.22], and for $p = 1 \geq n + \beta$, by Hakkarainen–Kinnunen [32, Theorems 4.3 and 5.1]. Note that for each $p \geq 1$ it is possible to find $\beta > -n$ so that $C_p^{\widehat{X}}(\{0\}) > 0 = C_p^G(\{0\})$.

Recall that for an open set Ω in X , we let

$$\Omega^\wedge = \widehat{X} \setminus \overline{X \setminus \Omega},$$

where the closure is taken in \widehat{X} . This makes Ω^\wedge into the largest open set in \widehat{X} such that $\Omega = \Omega^\wedge \cap X$. Note that $X^\wedge = \widehat{X}$. We denote balls with respect to \widehat{X} by \widehat{B} or $\widehat{B}(x, r) = \{y \in \widehat{X} : d(x, y) < r\}$, and balls with respect to X by B . The inclusion $\widehat{B}(x, r) \subset B(x, r)^\wedge$ can be strict.

If a function $u: \widehat{X} \rightarrow \overline{\mathbb{R}}$ has a (1-weak) upper gradient g on \widehat{X} , then clearly $g|_X$ is a (1-weak) upper gradient of $u|_X$. The converse is not true in general, as seen e.g. in $X = \mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R} = \widehat{X}$, but Theorem 1.2 provides a converse under suitable assumptions.

For $p > 1$ the result corresponding to Theorem 1.2 was obtained in Björn–Björn [8, Theorem 4.1], where the reflexivity of L^p was used through the application of [4, Lemma 6.2]. We shall now explain how Theorem 1.2 can be obtained for $p = 1$ using the Dunford–Pettis theorem (see e.g. Ambrosio–Fusco–Pallara [2, Theorem 1.38]) instead of reflexivity. In both cases, the proof is based on discrete convolutions and their gradients, as in Koskela [44, proof of Theorem C] and Heikkinen–Koskela–Tuominen [34].

Definition 4.5. Given a measurable set $H \subset X$, a sequence $\{g_i\}_{i=1}^\infty$ of functions in $L^1(H)$ is *equi-integrable* if the following two conditions are satisfied:

- (a) For any $\varepsilon > 0$, there is a measurable set $A \subset H$ with $\mu(A) < \infty$ such that

$$\int_{H \setminus A} |g_i| d\mu < \varepsilon \quad \text{for } i = 1, 2, \dots$$

- (b) For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $D \subset H$ is measurable and $\mu(D) < \delta$, then

$$\int_D |g_i| d\mu < \varepsilon \quad \text{for } i = 1, 2, \dots$$

Let $\lambda \geq 1$ and let $\Omega \subset X$ be an open set such that the doubling property holds within Ω . For each $k = 1, 2, \dots$, consider a Whitney-type covering of Ω by balls $\{B_{ik}\}_i$ with radii $r_{ik} \leq 1/k$ and a subordinate Lipschitz partition of unity $\{\varphi_{ik}\}_i$ so that

- (i) the balls $\frac{1}{5}B_{ik}$ are pairwise disjoint, and $80\lambda B_{ik} \subset \Omega$ for all i ;
- (ii) the balls $\{10\lambda B_{ik}\}_i$ have bounded overlap

$$(4.6) \quad \sum_i \chi_{10\lambda B_{ik}}(x) \leq m, \quad x \in \Omega;$$

- (iii) if $10\lambda B_{ik} \cap 10\lambda B_{jk} \neq \emptyset$, then $r_{jk} \leq 2r_{ik}$;
- (iv) each φ_{ik} is a nonnegative C/r_{ik} -Lipschitz function vanishing outside $2B_{ik}$;
- (v) $\sum_i \varphi_{ik} = 1$ in Ω .

Here m and C are constants depending only on λ and the doubling constant C_μ of μ within Ω .

For each fixed k , we can construct the covering as follows: For each $x \in \Omega$, let t_x be the smallest nonnegative integer such that

$$(4.7) \quad r_x := \frac{2^{-t_x}}{k} \leq \frac{\text{dist}(x, X \setminus \Omega)}{80\lambda}.$$

Since X is separable and $\{B(x, r_x)\}_{x \in \Omega}$ covers Ω , we can use the $5B$ -covering lemma (see e.g. Heinonen–Koskela–Shanmugalingam–Tyson [38, p. 60]) to find an at most countable cover of Ω by balls $B_{ik} := B(x_{ik}, r_{ik})$, $r_{ik} = r_{x_{ik}}$, such that the balls $\frac{1}{5}B_{ik}$ are pairwise disjoint. Property (i) is now easy to verify. For (iii), when $10\lambda B_{ik} \cap 10\lambda B_{jk} \neq \emptyset$, we have from (4.7),

$$80\lambda r_{jk} \leq \text{dist}(x_{jk}, X \setminus \Omega) \leq \text{dist}(x_{ik}, X \setminus \Omega) + 10\lambda(r_{jk} + r_{ik}),$$

so that $7r_{jk} \leq \text{dist}(x_{ik}, X \setminus \Omega)/10\lambda + r_{ik}$. From (4.7) we get

$$r_{jk} \leq \frac{1}{k} \quad \text{and} \quad \min\left\{\frac{1}{k}, \frac{\text{dist}(x_{ik}, X \setminus \Omega)}{80\lambda}\right\} < 2r_{ik}.$$

Combining these gives

$$7r_{jk} \leq \min\left\{\frac{7}{k}, \frac{\text{dist}(x_{ik}, X \setminus \Omega)}{10\lambda}\right\} + r_{ik} < 17r_{ik},$$

and, by construction, the quotient r_{jk}/r_{ik} can only take dyadic values.

For a fixed i , let $J_i = \{j : 10\lambda B_{ik} \cap 10\lambda B_{jk} \neq \emptyset\}$. If $j \in J_i$, then it follows from (iii) that $B_{jk} \subset 40\lambda B_{ik}$. The ball $40\lambda B_{ik}$ is a globally doubling metric space, by Björn–Björn [6, Proposition 3.4], with a doubling constant only depending on C_μ . As the balls $\{B(x_{jk}, \frac{1}{10}r_{ik})\}_{j \in J_i}$ are pairwise disjoint, property (ii) is satisfied with m only depending on λ and C_μ .

Finally, a Lipschitz partition of unity satisfying (iv) and (v) can now be constructed as in [38, pp. 104–105].

The following lemma is a special case of Hakkarainen–Kinnunen–Lahti–Lehtelä [33, Lemma 4.2 and Remark 4.3]; see also Franchi–Hajlasz–Koskela [25, Lemma 6] for an earlier very similar result. The last statement in the lemma is obtained using the Dunford–Pettis theorem (see e.g. Ambrosio–Fusco–Pallara [2, Theorem 1.38]).

Lemma 4.6. *Assume that μ is doubling within an open set Ω and for each $k = 1, 2, \dots$, let $\{B_{ik}\}_i$ be the above Whitney-type covering of Ω . For $g \in L^1(\Omega)$ define the functions*

$$g_k := \sum_{i=1}^{\infty} \chi_{B_{ik}} \int_{10\lambda B_{ik}} g \, d\mu, \quad k = 1, 2, \dots$$

Then the sequence $\{g_k\}_{k=1}^{\infty}$ is equi-integrable. Moreover, a subsequence of g_k converges weakly in $L^1(\Omega)$ to a function \tilde{g} satisfying $\tilde{g} \leq mg$ a.e. in Ω , where m is as in (4.6).

Proof of Theorem 1.2. We want to extend $u \in D^1(\Omega)$ and its minimal 1-weak upper gradient $g_u := g_{u,\Omega}$ to Ω^\wedge . Consider the above Whitney-type covering and Lipschitz partition of unity (extended continuously to Ω^\wedge).

As in the proofs of Heikkinen–Koskela–Tuominen [34, Lemma 5.3] and Björn–Björn [8, Theorem 4.1], it can be shown that for each $k = 1, 2, \dots$ and some C_0 , depending only on the doubling and Poincaré constants within Ω , the constant functions $C_0 \int_{10\lambda B_{ik}} g_u \, d\mu$ are upper gradients of

$$u_k := \sum_j u_{B_{jk}} \varphi_{jk} \quad \text{in } \hat{B}_{ik} := \hat{B}(x_{ik}, r_{ik}),$$

where x_{ik} are the centres of B_{ik} . Hence,

$$g_k := C_0 \sum_i \chi_{\hat{B}_{ik}} \int_{10\lambda B_{ik}} g_u \, d\mu$$

is an upper gradient of u_k in Ω^\wedge . Moreover, by (4.6) and the doubling property of μ , we have for every Lebesgue point $x \in \Omega$ of u that

$$|u_k(x) - u(x)| = \left| \sum_{2B_{ik} \ni x} (u_{B_{ik}} - u(x)) \varphi_{ik}(x) \right| \leq \sum_{2B_{ik} \ni x} \int_{B_{ik}} |u - u(x)| d\mu \rightarrow 0$$

as $k \rightarrow 0$. Since μ is doubling within Ω and $u \in L^1_{\text{loc}}(\Omega)$ (see Remark 4.8 below), u has Lebesgue points a.e., by e.g. Heinonen [35, Theorem 1.8]. We thus conclude that $u_k \rightarrow u$ a.e. in Ω .

Lemma 4.6 shows that the sequence $\{g_k\}_{k=1}^\infty \subset L^1(\Omega)$ is equi-integrable and there exists a subsequence (also denoted $\{g_k\}_{k=1}^\infty$) converging weakly in $L^1(\Omega)$ (and hence also in $L^1(\Omega^\wedge)$) to a function g such that $g \leq C_0 m g_u$ a.e. in Ω .

Mazur’s lemma, applied repeatedly to the subsequences $\{g_k\}_{k=j}^\infty$, for $j = 1, 2, \dots$, provides us with convex combinations of g_k converging to g in $L^1(\Omega^\wedge)$, such that the corresponding convex combinations of u_k converge a.e. to the function $\hat{u} := \limsup_{k \rightarrow \infty} u_k$ on Ω^\wedge , which has g as a 1-weak upper gradient (with respect to Ω^\wedge), see the proof of [4, Proposition 2.3]. In particular, $\hat{u} \in D^1(\Omega^\wedge)$. Since $\hat{u} = u$ a.e. in Ω and $u, \hat{u} \in D^1(\Omega)$, also $\hat{u} = u$ C^1_Ω -q.e. in Ω , and thus C^1_X -q.e. in Ω by Lemma 2.24 in [4].

If Ω is 1-path open in \hat{X} , then also the capacities C^1_Ω and $C^1_{\hat{X}}$ have the same zero sets in Ω , by Lemma 4.2. This shows that we may choose $\hat{u} = u$ in Ω . Lemma 4.1 then shows that $g_u = g_{\hat{u}}$ a.e. within Ω .

Finally, if \tilde{u} is defined to be the right-hand side of (1.1), then $\hat{u} = \tilde{u}$ at all Lebesgue points of \hat{u} , i.e., $C^1_{\hat{X}}$ -q.e. in Ω^\wedge , by the proof of Proposition 4.11 below with $\hat{G} = \Omega^\wedge$. Hence, \hat{u} may also be chosen so that it satisfies (1.1). ■

Remark 4.7. (a) The simple example $X = \Omega = \mathbb{R} \setminus \{0\}$ with $u(x) = \chi_{(0,\infty)}(x)(1 - |x|)_+$ demonstrates that under local assumptions on the measure μ , functions in $N^{1,p}(\Omega)$ may fail to have extensions even to $D^p_{\text{loc}}(\Omega^\wedge)$ when $p \geq 1$. A partial remedy for this situation is provided by Proposition 4.9 below.

(b) Under semilocal assumptions for μ (see Definition 2.5), the conclusion of Theorem 1.2 clearly holds for all bounded Ω . If Ω is unbounded, the semilocal assumptions do not imply the doubling property and the 1-Poincaré inequality within Ω , and so Theorem 1.2 is not directly available. However, if Ω is 1-path open in \hat{X} , then so is $\Omega \cap B(x, k)$ for every k and some fixed $x \in \Omega$. Since the doubling property and the 1-Poincaré inequality hold within each $\Omega \cap B(x, k)$, applying Theorem 1.2 for each k and letting $k \rightarrow \infty$ shows that the conclusion holds with $A_0 = 1$ also for Ω .

(c) Theorem 1.2 is formulated under assumptions holding within Ω . The corresponding result [8, Theorem 4.1] for $p > 1$ can also be formulated similarly, with the proof given in [8] still applying.

Remark 4.8. The extension result in Theorem 1.2 makes it possible to obtain quasicontinuity and Lebesgue points for local Newtonian functions on noncomplete spaces under local assumptions. If X supports a local 1-Poincaré inequality, then $N^{1,1}_{\text{loc}}(\Omega) = D^1_{\text{loc}}(\Omega)$ for every open $\Omega \subset X$; this follows as in [4, Proposition 4.14]. Moreover, since local assumptions are inherited by open subsets, the results in the rest of this section directly apply also to open $\Omega \subset X$. We therefore formulate them using $N^{1,1}(X)$ rather than $N^{1,1}_{\text{loc}}(\Omega) = D^1_{\text{loc}}(\Omega)$.

Proposition 4.9. *Assume that μ is locally doubling and supports a local 1-Poincaré inequality on X . Then for every $u \in N_{\text{loc}}^{1,1}(X)$, there is an open set $\hat{G} \supset X$ in \hat{X} and a function $\hat{u} \in N_{\text{loc}}^{1,1}(\hat{G})$ such that $u = \hat{u} C_1^X$ -q.e. on X . Moreover, \hat{G} is locally compact and $\mu|_{\hat{G}}$ is locally doubling and supports a local 1-Poincaré inequality. If X is 1-path open in \hat{X} , then one can choose $\hat{u} \equiv u$ and $g_{\hat{u}} \equiv g_u$ in X .*

Note that the set \hat{G} in general depends on u , cf. Björn–Björn [8, Example 4.7].

Proof. Since X is Lindelöf, we can find a countable cover of X by balls $B_j = B(x_j, r_j) \subset X$ such that $u \in N^{1,1}(B_j)$ and both the 1-Poincaré inequality and the doubling property for μ hold within each B_j , $j = 1, 2, \dots$. Let $\hat{B}_j = \hat{B}(x_j, r_j)$ and $\hat{G} = \bigcup_{j=1}^{\infty} \hat{B}_j$.

Using Theorem 1.2, we can extend $u|_{B_j}$ to $\hat{u}_j \in N^{1,1}(\hat{B}_j)$ so that $\hat{u}_j = u C_1^X$ -q.e. in B_j , $j = 1, 2, \dots$. Then $\hat{u}_i = \hat{u}_j$ a.e. (and hence $C_1^{\hat{X}}$ -q.e.) in $\hat{B}_i \cap \hat{B}_j$ for all i, j . We can thus construct $\hat{u} \in N_{\text{loc}}^{1,p}(\hat{G})$ so that $\hat{u} = u C_1^X$ -q.e. in X and $g_{\hat{u}} \leq A_j g_u$ a.e. in B_j , where A_j is the constant provided by Theorem 1.2 in B_j . Hence, $\hat{u} \in N_{\text{loc}}^{1,1}(\hat{G})$. If X is 1-path open in \hat{X} , then it follows from the last part of Theorem 1.2 that we can choose $\hat{u} \equiv u$ and $g_{\hat{u}} \equiv g_u$ in X .

The local doubling property and the local 1-Poincaré inequality for $\mu|_{\hat{G}}$ follow from Björn–Björn [8, Propositions 3.3 and 3.6]. Consequently, each \hat{B}_j (and thus also \hat{G}) is locally compact, by [8, Proposition 3.9]. ■

The following two results are now relatively easy consequences of the above extension to $\hat{G} \subset \hat{X}$ and the corresponding results in complete spaces. Recall the definition of quasicontinuity from Definition 2.4.

Corollary 4.10. *Assume that μ is locally doubling and supports a local 1-Poincaré inequality on X , and that X is 1-path open in \hat{X} . Then every $u \in N_{\text{loc}}^{1,1}(X)$ is C_1^X -quasicontinuous.*

Proof. Find a locally compact open set $\hat{G} \subset \hat{X}$ and a function $\hat{u} \in N_{\text{loc}}^{1,1}(\hat{G})$ as in Proposition 4.9 with $\hat{u} \equiv u$ in X and so that $\mu|_{\hat{G}}$ is locally doubling and supports a local 1-Poincaré inequality. It then follows from Theorem 9.1 in Björn–Björn [6] that \hat{u} is $C_1^{\hat{G}}$ -quasicontinuous on \hat{G} , which immediately yields that u is C_1^X -quasicontinuous on X , since C_1^X is dominated by $C_1^{\hat{G}}$. ■

Proposition 4.11. *Assume that μ is locally doubling and supports a local 1-Poincaré inequality on X . Then every $u \in N_{\text{loc}}^{1,1}(X)$ has Lebesgue points C_1^X -q.e., and moreover the extension \hat{u} in Proposition 4.9 can be given by*

$$(4.8) \quad \hat{u}(x) = \limsup_{r \rightarrow 0} \int_{\hat{B}(x,r) \cap X} u \, d\mu, \quad x \in \hat{G}.$$

The proof below shows that the limit

$$\lim_{r \rightarrow 0} \int_{B(x,r)} u \, d\mu$$

actually exists for $C_1^{\widehat{X}}$ -q.e. $x \in X$, even though it only equals $u(x)$ for C_1^X -q.e. x . In general, $C_1^X \leq C_1^{\widehat{X}}$, but it follows from Lemma 4.2 that they have the same zero sets if X is 1-path open in \widehat{X} .

Remark 4.12. Even when X is complete, the Lebesgue point result in Proposition 4.11 generalizes earlier results obtained under global assumptions, as in Kinnunen–Korte–Shanmugalingam–Tuominen [43, Theorem 4.1 and Remark 4.7]. Therein, $\mu(X) = \infty$ is assumed, but we shall now explain how the Lebesgue point result from [43, Theorem 4.1 and Remark 4.7] can be obtained also for a complete metric space X equipped with a globally doubling measure μ supporting a global 1-Poincaré inequality and satisfying $\mu(X) < \infty$. (Under these assumptions, $\mu(X) < \infty$ if and only if X is bounded.) We will use this fact when proving Proposition 4.11.

For this, let $\widetilde{X} = X \times \mathbb{R}$, equipped with the product metric

$$d_{\widetilde{X}}((x, t), (y, s)) = \max\{d(x, y), |t - s|\}$$

and the product measure

$$d\tilde{\mu}(x, t) = d\mu(x) dt.$$

Note that $\tilde{\mu}(\widetilde{X}) = \infty$. By Björn–Björn [7, Theorem 3 and Remark 4], $\tilde{\mu}$ is globally doubling and supports a global 1-Poincaré inequality. Let η be a Lipschitz cut-off function on \mathbb{R} such that $\eta = 1$ in $[-1, 1]$ and $\eta = 0$ outside $[-2, 2]$. If $u \in N^{1,1}(X)$, then

$$\tilde{u}(x, t) := u(x)\eta(t) \in N^{1,1}(\widetilde{X})$$

and [43, Theorem 4.1 and Remark 4.7] implies that \tilde{u} has Lebesgue points at $C_1^{\widetilde{X}}$ -q.e. $x \in \widetilde{X}$. Clearly, for $0 < r < 1$,

$$\int_{B(x,r) \times (-r,r)} \tilde{u} d\tilde{\mu} = 2r \int_{B(x,r)} u d\mu \quad \text{and} \quad \tilde{\mu}(B(x,r) \times (-r,r)) = 2r\mu(B(x,r)),$$

which implies that $x \in X$ is a Lebesgue point of u if and only if $(x, t) \in \widetilde{X}$ is a Lebesgue point of \tilde{u} for some (and equivalently all) $t \in (-1, 1)$. Hence, if $E \subset X$ is the set of non-Lebesgue points of u , then $C_1^{\widetilde{X}}(E \times (-1, 1)) = 0$ and for every $\varepsilon > 0$, there exists $\tilde{v} \in N^{1,1}(\widetilde{X})$, with an upper gradient g , such that $\tilde{v} \geq 1$ on $E \times (-1, 1)$ and

$$\int_{\widetilde{X}} (|\tilde{v}| + g) d\tilde{\mu} < 2\varepsilon.$$

Then there exists $t \in (-1, 1)$ such that

$$(4.9) \quad \int_X (|v(x, t)| + g(x, t)) d\mu(x) < \varepsilon.$$

Clearly, $g(\cdot, t)$ is an upper gradient of $v(\cdot, t)$ with respect to X and $v(\cdot, t)$ is admissible for $C_1^X(E)$. It therefore follows from (4.9) that $C_1^X(E) < \varepsilon$. Letting $\varepsilon \rightarrow 0$ now shows that $C_1^X(E) = 0$ and so u has Lebesgue points C_1^X -q.e. in X .

Proof of Proposition 4.11. Find \widehat{G} and $\widehat{u} \in N_{\text{loc}}^{1,1}(\widehat{G})$ as in Proposition 4.9. Let $x \in \widehat{G}$. As \widehat{G} is locally compact, it follows from Theorem 3.2 that there is a bounded uniform do-

main G_x in \hat{X} such that $x \in G_x \Subset \hat{G}$ and such that $\mu|_{\bar{G}_x}$ is globally doubling and supports a global p -Poincaré inequality on \bar{G}_x , where the closure is taken with respect to \hat{X} . In particular, $\hat{u} \in N^{1,1}(\bar{G}_x)$.

By [43, Theorem 4.1 and Remark 4.7] and the argument in Remark 4.12, \hat{u} has Lebesgue points $C_1^{G_x}$ -q.e. in G_x . By Lemma 4.2, the capacities $C_1^{G_x}$ and $C_1^{\hat{X}}$ have the same zero sets in G_x . Hence, as \hat{G} is Lindelöf, \hat{u} has Lebesgue points $C_1^{\hat{X}}$ -q.e. in \hat{G} , and so u has Lebesgue points C_1^X -q.e. in X .

Finally, if \tilde{u} is given by the right-hand side of (4.8), then $\hat{u} = \tilde{u}$ at all Lebesgue points of \hat{u} , i.e., $C_p^{\hat{X}}$ -q.e. in \hat{G} . Hence, \hat{u} may also be chosen so that it satisfies (4.8). ■

Even for $u \in N^{1,1}(X)$, (the proof of) Proposition 4.9 only guarantees an extension in the local Newtonian space $N_{loc}^{1,1}(\hat{G})$ (but with \hat{G} independent of u), unless X is 1-path open in \hat{X} . However, under slightly stronger uniform assumptions we can obtain the following partial nonlocal conclusion, which also includes $p > 1$, see [8, Remark 4.10].

Proposition 4.13. *Assume that there are constants C_μ , C_{PI} and λ such that for each $x \in X$, there is $r_x > 0$ such that μ is doubling within $B_x = B(x, r_x)$ with constant C_μ , and μ supports a p -Poincaré inequality within B_x with constants C_{PI} and λ . Then there is an open set $\hat{G} \supset X$ in \hat{X} such that for every $u \in N^{1,p}(X)$, the function \hat{u} given by (4.8) satisfies $\hat{u} = u$ C_p^X -q.e. on X and belongs to $N^{1,p}(\hat{G})$. If also r_x is independent of x , then we may choose $\hat{G} = \hat{X}$.*

Such assumptions are called *semiuniformly local*, and *uniformly local* in the case where r_x is independent of x , in [6, Definition 6.1]. Riemannian manifolds always support at least semiuniformly local assumptions and often uniformly local ones. Uniformly local assumptions are natural e.g. on Gromov hyperbolic spaces, see Björn–Björn–Shanmugalingam [14, 15] and Butler [18]. Semiuniformly local assumptions were also used by e.g. Holopainen–Shanmugalingam [39].

Proof. Let $\hat{B}_x = \hat{B}(x, r_x)$ and $\hat{G} = \bigcup_{x \in X} \hat{B}_x$. By [8, Proposition 4.8 and the proof of Lemma 4.6] (for $p > 1$) or Proposition 4.11 and the proof of Proposition 4.9 (for $p = 1$), we get that $\hat{u} \in N_{loc}^{1,p}(\hat{G})$. By [8, Theorem 4.1] (for $p > 1$) and Theorem 1.2 (for $p = 1$), we see that $g_{\hat{u}} \leq A_0 g_u$ a.e. in X , where A_0 only depends on p, C_μ, C_{PI} and λ . Thus,

$$\int_{\hat{G}} |\hat{u}|^p d\hat{\mu} = \int_X |u|^p d\mu < \infty \quad \text{and} \quad \int_{\hat{G}} g_{\hat{u}}^p d\hat{\mu} \leq A_0^p \int_X g_u^p d\mu < \infty,$$

i.e., $\hat{u} \in N^{1,p}(\hat{G})$. If r_x is independent of x , then clearly $\hat{G} = \hat{X}$. ■

5. Removable sets for Newtonian spaces

We assume in this section that $1 \leq p < \infty$ and that $Y = (Y, d, \mu_Y)$ is a metric measure space equipped with a metric d and a positive complete Borel measure μ_Y such that $0 < \mu_Y(B) < \infty$ for all balls $B \subset Y$. Moreover, $X \subset Y$ is such that $Y \subset \hat{X}$. We also let $E = Y \setminus X$ and assume that the inner measure satisfies

$$(5.1) \quad \mu_{Y,in}(E) := \sup\{\mu_Y(A) : A \subset E \text{ is } \mu_Y\text{-measurable}\} = 0.$$

Our main interest in this section is removability of sets with zero measure, i.e., when $X \subset Y$ are two metric spaces with $\mu_Y(Y \setminus X) = 0$. In order to be able (as before) to include the case when $Y = \widehat{X}$ and X is a nonmeasurable subset of Y , we merely impose the condition (5.1). This will only necessitate a little extra care in some of the formulations. At the end of this section we give examples of nonmeasurable removable sets with zero inner measure. Removability of sets with positive measure is a different topic, related to extension domains, see e.g. Hajłasz–Koskela–Tuominen [31] and Björn–Shanmugalingam [17, Section 5]. As in (4.1), it follows from (5.1) that

$$\mu_{Y,\text{in}}(E) = \sup\{\mu_Y(A) : A \subset E \text{ is a Borel set in } Y\}.$$

Since we want Y to satisfy our standing assumption that balls have positive measure, necessarily $Y \subset \widehat{X} = \widehat{Y}$. In the nonmeasurable case, we cannot just let $\mu_X = \mu_Y|_X$, but need to define μ_X by letting

$$(5.2) \quad \mu_X(A \cap X) = \mu_Y(A) \quad \text{for every } \mu_Y\text{-measurable set } A \subset Y.$$

This is well-defined since $\mu_{Y,\text{in}}(E) = 0$, and makes μ_X into a complete Borel regular measure on X , which coincides with the restriction $\mu_Y|_X$ when X is μ_Y -measurable.

We note that q.e. defined equivalence classes may depend on whether the capacity is C_p^X or C_p^Y , whereas the a.e. condition coincides in both spaces, due to (4.2). So for simplicity we restrict the discussion to removability with respect to the following spaces, where we implicitly assume that $u: X \rightarrow \overline{\mathbb{R}}$ is defined pointwise in X :

$$\begin{aligned} \widehat{N}^{1,p}(X) &= \{u : u = v \text{ a.e. for some } v \in N^{1,p}(X)\}, \\ \widehat{D}^p(X) &= \{u : u = v \text{ a.e. for some } v \in D^p(X)\}. \end{aligned}$$

In both cases we define $g_u = g_v$. This is well-defined a.e. and independent of the choice of v such that $v = u$ a.e. The spaces $\widehat{N}^{1,p}(Y)$ and $\widehat{D}^p(Y)$ are defined similarly.

Definition 5.1. The set $E = Y \setminus X$ is *removable* for $\widehat{N}^{1,p}(X)$ if $\widehat{N}^{1,p}(X) = \widehat{N}^{1,p}(Y)$ in the sense that $\widehat{N}^{1,p}(X) = \{u|_X : u \in \widehat{N}^{1,p}(Y)\}$. If moreover $g_{u,X} = g_{u,Y}$ a.e. in X for every $u \in \widehat{N}^{1,p}(Y)$, then E is *isometrically removable* for $\widehat{N}^{1,p}(X)$.

Removability and isometric removability for $\widehat{D}^p(X)$ are defined similarly.

It is easily seen that removability for $\widehat{N}^{1,p}$ is the same as for the corresponding spaces of a.e.-equivalence classes

$$(5.3) \quad \widehat{N}^{1,p}(X)/\overset{\text{ae}}{\sim} \quad \text{and} \quad \widehat{N}^{1,p}(Y)/\overset{\text{ae}}{\sim},$$

where $u \overset{\text{ae}}{\sim} v$ if $u - v = 0$ a.e. However, to make it clearer what exactly is meant, especially in the nonmeasurable case, we prefer to work with the spaces $\widehat{N}^{1,p}$ of pointwise defined functions. In fact, the proofs below show that when E is removable, then any μ_Y -measurable extension of u from X to Y will do the job.

Note also that the quotient spaces in (5.3) are Banach spaces. Since we have clearly $\|u\|_{N^{1,p}(X)} \leq \|u\|_{N^{1,p}(Y)}$, the bounded inverse theorem shows that the norms in these spaces are equivalent when E is removable for $\widehat{N}^{1,p}(X)$.

As a first observation we deduce the following result.

Proposition 5.2. *If $C_p^Y(E) = 0$, then $E = Y \setminus X$ is isometrically removable for $\widehat{N}^{1,p}(X)$ and $\widehat{D}^p(X)$.*

Note that no assumptions on Y are needed (other than the standing assumptions from the beginning of this section) and that X is automatically measurable in this case, since $\mu_Y(E) = 0$ (which follows directly from Definition 2.3).

Proof. Let $\hat{u} \in \widehat{D}^p(X)$ and let $u \in D^p(X)$ be such that $u = \hat{u}$ a.e. in X . Let g be any p -weak upper gradient of u in X . Extend u and g by 0 to $Y \setminus X$. Note that as X is measurable so are u and g . Since $C_p^Y(E) = 0$, it follows from [4, Proposition 1.48] that p -almost no curve in Y hits E . Hence, g is a p -weak upper gradient of u also on Y . Since $u = \hat{u}$ a.e. in X , any extension of \hat{u} to Y will coincide with u a.e. in Y and so belongs to $\widehat{D}^p(Y)$. Thus, E is isometrically removable both for $\widehat{N}^{1,p}(X)$ and $\widehat{D}^p(X)$. ■

Example 5.3. Let $Y = \mathbb{R}^n$, $n \geq 2$, $1 \leq p \leq n$ and let $E \subset \mathbb{R}^n$ be a countable or finite set. Then it is well known that $C_p^{\mathbb{R}^n}(E) = 0$, and thus E is isometrically removable for $\widehat{N}^{1,p}(\mathbb{R}^n \setminus E)$ and $\widehat{D}^p(\mathbb{R}^n \setminus E)$, by Proposition 5.2.

If $E \subset H$ is dense in a hyperplane H , then $\bar{E} = H$ is not removable for $\widehat{N}^{1,p}(\mathbb{R}^n \setminus \bar{E})$ nor for $\widehat{D}^p(\mathbb{R}^n \setminus \bar{E})$. This follows from Theorem 5.4 below since $\mathbb{R}^n \setminus H$ is disconnected and hence does not support any global Poincaré inequality.

This shows that removability for nonclosed sets cannot be achieved by only studying removability of their closures. In Proposition 6.4 we give a much more general result which includes this example as a special case.

The following is the main result in this section.

Theorem 5.4. *Assume that μ_Y is globally doubling and supports a global p -Poincaré inequality on Y . Consider the following statements:*

- (a) E is removable for $\widehat{N}^{1,p}(X)$.
- (b) E is removable for $\widehat{D}^p(X)$.
- (c) E is isometrically removable for $\widehat{N}^{1,p}(X)$.
- (d) E is isometrically removable for $\widehat{D}^p(X)$.
- (e) X supports a global p -Poincaré inequality with the same C and λ as on Y .
- (f) X supports a global p -Poincaré inequality.

Then (c) \Leftrightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (b) \Rightarrow (a).

If in addition X is p -path almost open in Y , then (a)–(f) are all equivalent.

As mentioned in the introduction, this generalizes Theorem C in Koskela [44], see also Koskela–Shanmugalingam–Tuominen [46, p. 335]. Koskela obtained such a characterization of removability for $W^{1,p}(\mathbb{R}^n \setminus E)$ on unweighted \mathbb{R}^n , with $p > 1$ and E closed (and thus $X = \mathbb{R}^n \setminus E$ open and hence p -path almost open). In the classical situation, on unweighted \mathbb{R}^n , our result thus extends Koskela’s result to $p = 1$. The classical Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n \setminus E)$, for E closed, coincide with $\widehat{N}^{1,p}(\mathbb{R}^n)/^{\text{ac}}$ and $\widehat{N}^{1,p}(\mathbb{R}^n \setminus E)/^{\text{ac}}$ (with the same norm), respectively, by Theorem 7.13 in Hajlasz [30] (or [4, Corollary A.4]). This is true also in weighted Euclidean spaces, for p -admissible weights when $p > 1$, see [4, Proposition A.12]. (A weight w is p -admissible if $d\mu = w \, dx$

is a globally doubling measure supporting a global p -Poincaré inequality.) For $p = 1$ and a 1-admissible weight, Proposition 4.26 in Cheeger [23], together with the arguments in [4, Propositions A.11 and A.12], implies that the norms are comparable, see also Eriksson-Bique–Soultanis [24].

Theorem 6.1 below shows that the assumptions in Theorem 5.4 can be fulfilled without X being p -path almost open in $Y = \widehat{X}$, and that even in this case, it is possible that (a)–(f) all hold. Some of the implications hold under weaker assumptions and we begin with deducing them.

Proposition 5.5. *If E is removable for $\widehat{D}^p(X)$, then it is removable for $\widehat{N}^{1,p}(X)$.*

Proof. Let $u \in \widehat{N}^{1,p}(X)$. Since $u \in \widehat{D}^p(X)$ and E is removable for $\widehat{D}^p(X)$, there exists $\hat{u} \in \widehat{D}^p(Y)$ such that $\hat{u} = u$ in X . As $\|\hat{u}\|_{L^p(Y)} = \|u\|_{L^p(X)} < \infty$ by (5.2), we see that $\hat{u} \in \widehat{N}^{1,p}(Y)$. Hence, E is removable for $\widehat{N}^{1,p}(X)$. ■

Proposition 5.6. *Assume that μ_X is doubling and supports a p -Poincaré inequality within an open set $\Omega \subset X$. Then $E \cap \Omega^\wedge$ is removable both for $\widehat{N}^{1,p}(\Omega)$ and $\widehat{D}^p(\Omega)$.*

Proof. By Proposition 5.5 (with Ω in place of X), it suffices to prove removability for $\widehat{D}^p(\Omega)$. Let $\hat{u} \in \widehat{D}^p(\Omega)$. Then there is $u \in D^p(\Omega)$ such that $u = \hat{u}$ a.e. in Ω . By Theorem 1.2 (when $p = 1$) and [8, Theorem 4.1] (when $p > 1$, see Remark 4.7 (c)), there exists $v \in D^p(\Omega^\wedge)$ such that $v = u$ C_p^X -q.e. in Ω . Since $v = \hat{u}$ a.e. in Ω , any μ_Y -measurable extension of \hat{u} to $Y \cap \Omega^\wedge$ will coincide with v a.e. in $Y \cap \Omega^\wedge$ and so belongs to $\widehat{D}^p(Y \cap \Omega^\wedge)$. Hence, $E \cap \Omega^\wedge$ is removable for $\widehat{D}^p(\Omega)$. ■

Theorem 5.7. *The set E is isometrically removable for $\widehat{N}^{1,p}(X)$ if and only if it is isometrically removable for $\widehat{D}^p(X)$.*

Proof. Assume first that E is isometrically removable for $\widehat{D}^p(X)$. By Proposition 5.5, the set E is removable for $\widehat{N}^{1,p}(X)$. As the removability for $\widehat{D}^p(X)$ is isometric, it follows directly from the definition that E is isometrically removable also for $\widehat{N}^{1,p}(X)$.

Conversely, assume that E is isometrically removable for $\widehat{N}^{1,p}(X)$. Let $u \in \widehat{D}^p(X)$ and let $v \in D^p(X)$ be such that $v = u$ a.e. in X . First consider the case when $u \geq 0$, so that we can assume also $v \geq 0$. Fix $x_0 \in X$ and let

$$u_k(x) = (1 - \text{dist}(x, B_X(x_0, k)))_+ \min\{v(x), k\}, \quad k = 1, 2, \dots$$

Then $u_k \in N^{1,p}(X)$ and there is $\hat{u}_k \in N^{1,p}(Y)$ such that $\hat{u}_k = u_k$ a.e. in X , and thus C_p^X -q.e. in X . As $\hat{u}_{k+1} \geq \hat{u}_k$ a.e., it follows from Corollary 1.60 in [4] that $\hat{u}_{k+1} \geq \hat{u}_k$ C_p^Y -q.e., and thus we can choose \hat{u}_{k+1} so that $\hat{u}_{k+1} \geq \hat{u}_k$ everywhere. Hence, $\hat{u} = \lim_{k \rightarrow \infty} \hat{u}_k$ is well-defined pointwise.

Next, let $\hat{g} = g_{u,X}$, extended measurably to $Y \setminus X$. By the isometric removability and truncation, $g_{\hat{u}_k,Y} = g_{u_k,X} \leq \hat{g}$ a.e. in $B_Y(x_0, k)$, and thus \hat{g} is a p -weak upper gradient of \hat{u}_k in $B_Y(x_0, k)$. Since, by (5.2),

$$\mu_Y(\{x \in Y : |\hat{u}(x)| = \infty\}) = \mu_X(\{x \in X : |v(x)| = \infty\}) = 0,$$

it follows from [4, Lemma 1.52] that \hat{g} is a p -weak upper gradient of \hat{u} in each $B_Y(x_0, k)$ and hence in Y . Therefore, $\hat{u} \in D^p(Y)$, and clearly $\hat{u} = u$ a.e. in X . Now any μ_Y -measurable extension of u will coincide with \hat{u} a.e. in Y and so belongs to $\hat{D}^p(Y)$. For general u we write $u = u_+ - u_-$, extend u_+ and u_- as above, and take their difference. Thus, E is isometrically removable for $\hat{D}^p(X)$. ■

Remark 5.8. In the proof of Theorem 5.7, we used the fact that functions in $D^p(X)$ are finite a.e. when applying [4, Lemma 1.52]. This is the reason why our definition of $D^p(X)$ slightly deviates from the one in [4], see Section 2. It may also be more natural to only consider functions that are finite a.e.

Proof of Theorem 5.4. (c) \Leftrightarrow (d) This follows from Theorem 5.7.

(c) \Rightarrow (e) Let $u \in N^{1,p}(X) \subset \hat{N}^{1,p}(X)$. As E is isometrically removable for $\hat{N}^{1,p}(X)$, there is $\hat{u} \in N^{1,p}(Y)$ such that $\hat{u} = u$ a.e. in X and $g_{\hat{u},Y} = g_{\hat{u},X} = g_{u,X}$ a.e. in X , see Section 2. Let $B_X = B_X(x, r)$ be a ball in X , and let $B_Y = B_Y(x, r)$ be the corresponding ball in Y . Then, in view of (5.2) and using the Poincaré inequality on Y ,

$$\begin{aligned} \int_{B_X} |u - u_{B_X}| d\mu_X &= \int_{B_Y} |\hat{u} - \hat{u}_{B_Y}| d\mu_Y \\ &\leq Cr \left(\int_{\lambda B_Y} g_{\hat{u},Y}^p d\mu_Y \right)^{1/p} = Cr \left(\int_{\lambda B_X} g_{u,X}^p d\mu_X \right)^{1/p}. \end{aligned}$$

Thus, X supports a global p -Poincaré inequality with the same constants C and λ as on Y , by Lemma 2.6.

(e) \Rightarrow (f) This is trivial.

(f) \Rightarrow (b) It follows directly from (5.2) that μ_X is globally doubling on X . Hence, this implication follows from Proposition 5.6.

(b) \Rightarrow (a) This follows from Proposition 5.5.

Finally, if X is p -path almost open in Y , then (a) \Rightarrow (c) by Lemma 4.1. ■

Under local assumptions we obtain the following result. Recall that local assumptions are inherited by open sets and thus X and Y in the following theorem can be replaced by $\Omega \cap X$ and Ω , respectively, for any open set $\Omega \subset Y$, cf. Remark 4.8.

Theorem 5.9 (Local version). *Assume that μ_Y is locally doubling and supports a local p -Poincaré inequality on Y . Consider the following statements:*

- (a) E is removable for $\hat{N}^{1,p}(X)$.
- (b) E is removable for $\hat{D}^p(X)$.
- (c) E is isometrically removable for $\hat{N}^{1,p}(X)$.
- (d) E is isometrically removable for $\hat{D}^p(X)$.
- (e) Whenever $x \in X$ and the Poincaré inequality (2.2) holds for a ball $B_Y(x, r)$ in Y , it holds for the ball $B_X(x, r)$ in X with the same constants C and λ .
- (f) There is a cover of Y by at most countably many balls $B_{Y,j} = B_Y(x_j, r_j)$, $x_j \in X$, such that the p -Poincaré inequality holds within each ball $B_{X,j} = B_X(x_j, r_j)$.

Then

$$(a) \Leftarrow (b) \Leftarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Rightarrow (f).$$

If in addition X is p -path almost open in Y , then (a)–(f) are all equivalent.

Note that $Y = \mathbb{R}$ with $E = \{0\}$ shows that in order for the equivalences in the last part to hold, it is not possible to replace (f) by the assumption that “ X supports a local p -Poincaré inequality”.

Proof. (c) \Leftrightarrow (d) This follows from Theorem 5.7.

(d) \Rightarrow (b) This is trivial.

(b) \Rightarrow (a) This follows from Proposition 5.5.

(c) \Rightarrow (e) The proof of this implication is similar to the proof of the corresponding implication in Theorem 5.4.

(e) \Rightarrow (f) Since Y is Lindelöf and supports a local p -Poincaré inequality, this is straightforward.

Now assume that X is p -path almost open in Y .

(a) \Rightarrow (c) This follows from Lemma 4.1.

(f) \Rightarrow (b) Since μ_Y is locally doubling on Y , we may assume that the cover $B_{Y,j}$ has been chosen so that μ_Y is doubling within each $B_{Y,j}$. It follows directly from (5.2) that μ_X is doubling within each $B_{X,j}$. Let $\hat{u} \in \hat{D}^p(X)$. Then there is $u \in D^p(X)$ such that $u = \hat{u}$ a.e. in X . Note that $Y \subset \hat{X}$.

Using Theorem 1.2 (when $p = 1$) and [8, Theorem 4.1] (when $p > 1$), we can find $u_j \in D^p(B_{Y,j})$ such that $u_j = u C_p^X$ -q.e. in $B_{X,j}$, $j = 1, 2, \dots$. As $u_i, u_j \in D^p(B_{Y,i} \cap B_{Y,j})$ and the set $\{y \in B_{Y,i} \cap B_{Y,j} : u_i(y) \neq u_j(y)\}$ has measure zero, it must be of zero C_p^Y -capacity for all i, j . We can thus construct $v \in D_{\text{loc}}^p(Y)$ such that $v = u_j C_p^Y$ -q.e. in $B_{Y,j}$, $j = 1, 2, \dots$, and hence $v = u C_p^X$ -q.e. in X .

Since X is p -path almost open in Y , we have $g_{u_j,Y} = g_{u,X}$ a.e. in $B_{X,j}$, by Lemma 4.1. As every curve γ in Y is compact, it can be covered by finitely many $B_{Y,j}$. From this it follows that $g_{u,X}$ (extended measurably to $Y \setminus X$) is a p -weak upper gradient also of v in Y , and thus $v \in D^p(Y)$. Since $v = \hat{u}$ a.e. in X , any μ_Y -measurable extension of \hat{u} will belong to $\hat{D}^p(Y)$. Hence, E is removable for $\hat{D}^p(X)$. ■

The following result, albeit a bit trivial, gives us plenty of examples of nonmeasurable removable sets with zero inner measure. Consider e.g. Y to be the von Koch snowflake curve (see e.g. [4, Example 1.23]) and $X \subset Y$ be any nonmeasurable subset with full outer measure.

Proposition 5.10. *Assume that there are no or p -almost no curves in Y , i.e., $\text{Mod}_{p,Y}(\Gamma) = 0$, where Γ is the collection of all nonconstant rectifiable curves in Y . Then any $E \subset Y$ satisfying (5.1) is isometrically removable for $\hat{N}^{1,p}(X)$ and $\hat{D}^p(X)$.*

Proof. In this case $g_u = 0$ a.e. for every measurable function u on X or Y , and so $\hat{N}^{1,p}(X) = N^{1,p}(X) = L^p(X)$ and $\hat{N}^{1,p}(Y) = N^{1,p}(Y) = L^p(Y)$. It thus follows directly from (5.2) that E is removable for $\hat{N}^{1,p}(X)$. Since $g_{u,X} = g_{u,Y}$ a.e. in X , the removability is isometric. By Theorem 5.7, E is isometrically removable for $\hat{D}^p(X)$. ■

6. Extension from a non- p -path almost open set

We are now going to construct a set $X \subset \mathbb{R}^2$ which satisfies the assumptions in Theorem 1.2 but is not p -path almost open in \mathbb{R}^2 . However, its complement is isometrically removable for $\hat{N}^{1,p}(X)$ and $\hat{D}^p(X)$.

We first construct a planar Cantor set $C \subset [0, 1] \times [0, 1]$ as follows. Let $H_0 = [0, 1]$ and for every $k = 0, 1, \dots$, let H_{k+1} be the set obtained by removing from the centre of every interval in H_k the open interval of length 2^{-2k-1} . Then let $C = \bigcap_{k=1}^{\infty} (H_k \times H_k)$, which is a planar Cantor set. This set projects (orthogonally) onto full intervals on the lines

$$(6.1) \quad y = \pm \frac{1}{2}x + c \quad \text{and} \quad y = \pm 2x + c,$$

but has zero length projections on all other lines. This is easy to check by sketching the set $H_1 \times H_1$ and then noting the self-similarity of the construction. In particular, C has 1-dimensional Hausdorff measure $0 < \mathcal{H}^1(C) < \infty$ (where the latter inequality is easy to show).

The Cantor set C is often called the four corners Cantor set, as well as the Garnett–Ivanov set in complex analysis, since Garnett [29] and Ivanov [40, footnote on p. 346] (independently) showed that it is removable for bounded analytic functions.¹

Let next $\{q_j\}_{j=1}^{\infty}$ be an enumeration of \mathbb{Q}^2 and define

$$(6.2) \quad A = \bigcup_{j=1}^{\infty} (q_j + C),$$

i.e., we shift C by all rational numbers and take the union. We are now going to show the following properties for $X = \mathbb{R}^2 \setminus A$.

Theorem 6.1. *Let $X = \mathbb{R}^2 \setminus A$, where $A \subset \mathbb{R}^2$ is as in (6.2). Also let Ω' be a nonempty open subset of \mathbb{R}^2 and $\Omega = \Omega' \cap X$, all sets being equipped with the Lebesgue measure \mathcal{L}^2 . Then the following are true:*

- (a) $A \cap \Omega'$ is isometrically removable for $\hat{N}^{1,p}(\Omega)$.
- (b) X supports a global 1-Poincaré inequality.
- (c) Ω is not p -path almost open in \mathbb{R}^2 .

This in particular shows that the assumptions and conclusions in Theorem 1.2, as well as in the corresponding Theorem 4.1 in Björn–Björn [8] for $1 < p < \infty$, can be fulfilled even if Ω is not p -path almost open in \hat{X} . Similarly, it shows that the assumptions in Theorem 5.4 can be fulfilled without X being p -path almost open in $Y = \hat{X}$, and that even in this case, it is possible that (a)–(f) all hold. Moreover, the conclusions in Theorem 1.1 hold.

Proof. (a) Let $u \in N^{1,p}(\Omega)$. Then u is absolutely continuous on p -almost every curve in Ω , by Proposition 3.1 in Shanmugalingam [53] (or [4, Theorem 1.56]). Let l be any line which is not among those in (6.1). The orthogonal projection of C , and thus of A , on l has zero length. Hence, almost every line in \mathbb{R}^2 , which is perpendicular to l , does not

¹For the historically interested reader it may be worth noting that Veltmann [56] considered planar Cantor sets in 1882 before Cantor [21, p. 590 (p. 407 in *Acta Math.*)] published his ternary set in 1883.

intersect A . Thus, by [4, Lemmas 2.14 and A.1], u is absolutely continuous along the intersection of almost every such line with Ω' and the corresponding directional derivative u'_d of u satisfies $|u'_d| \leq g_{u,\Omega}$ a.e. (Note that $\mathcal{L}^2(\Omega' \setminus \Omega) = 0$.)

In particular, $u \in \text{ACL}(\Omega')$, and thus $u \in W^{1,p}(\Omega') = \widehat{N}^{1,p}(\Omega')/\sim^{\text{ac}}$, by e.g. Theorem 2.1.4 in Ziemer [57]. Since we have only excluded four directions of lines for l , the distributional gradient of u satisfies $|\nabla u| \leq g_{u,\Omega}$ a.e. in Ω' . Thus,

$$g_{u,\Omega'} = |\nabla u| \leq g_{u,\Omega} \quad \text{a.e. in } \Omega',$$

by Theorem 7.13 in Hajlasz [30] (or [4, Corollary A.4]), while the reverse inequality is trivial. Hence, A is isometrically removable for $\widehat{N}^{1,p}(\Omega)$.

(b) This now follows directly from (a) and Theorem 5.4.

(c) Consider the family Γ_0 of all lines $\{\gamma(t) := (t/\sqrt{5}, (2t+c)/\sqrt{5}) : t \in \mathbb{R}\}_{c \in \mathbb{R}}$. The crucial property of the Cantor set C is that if any such line intersects $[0, 4^{-i}] \times [0, 4^{-i}]$, then it intersects $4^{-i}C \subset C$, $i = 0, 1, \dots$, though only in a set of zero 1-dimensional Lebesgue measure. Thus, if any line $\gamma \in \Gamma_0$ intersects $q_j + [0, 4^{-i}] \times [0, 4^{-i}]$ for some indices i, j , then it intersects $q_j + 4^{-i}C$.

Fix $\gamma \in \Gamma_0$, $t \in \mathbb{R}$ and $\varepsilon > 0$. We then find i, j such that $4^{-i} < \varepsilon/2$ and

$$\gamma(t) \in q_j + [0, 4^{-i}] \times [0, 4^{-i}].$$

As explained above, the line γ intersects $q_j + 4^{-i}C$ and so there is $s \in \mathbb{R}$ with $|s - t| < \varepsilon$ such that $\gamma(s) \in q_j + 4^{-i}C \subset A$. It follows that $\gamma^{-1}(A)$ is dense in \mathbb{R} but of zero 1-dimensional Lebesgue measure. The lines $\gamma \in \Gamma_0$ are not rectifiable curves since they are not of finite length, but we can define Γ as the collection of all compact line segments on these lines that also belong to Ω' . Let $\gamma: [0, l_\gamma] \rightarrow \Omega'$, $\gamma \in \Gamma$, be an arc-length parameterized curve. Then by the above argument, $\gamma^{-1}(A)$ is dense in $[0, l_\gamma]$ but of zero 1-dimensional Lebesgue measure, and so $\gamma^{-1}(\Omega) = [0, l_\gamma] \setminus \gamma^{-1}(A)$ is not the union of an open set and a set of zero 1-dimensional Lebesgue measure. By [4, Lemma A.1], we also have $\text{Mod}_{p,\mathbb{R}^2}(\Gamma) > 0$ for all $1 \leq p < \infty$. In conclusion, Ω is not p -path almost open in \mathbb{R}^2 . ■

In the rest of this section, we provide examples of removable sets E fulfilling the assumptions in Theorem 5.4, with $X = Y \setminus E$ that is p -path almost open but not p -path open.

Example 6.2. Let $p > 2$ and let Y be the so-called bow-tie

$$\begin{aligned} Y &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 0\}, \\ E &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0\} \setminus \{(0, 0)\}, \\ X &= Y \setminus E. \end{aligned}$$

We equip Y with the Lebesgue measure, which is globally doubling on Y . Then Y supports a global p -Poincaré inequality, by [4, Example A.23]. The same proof also shows that X supports a global p -Poincaré inequality. By Theorem 1.3, X is p -path almost open in Y . Thus, by Theorem 5.4, E is isometrically removable for $\widehat{N}^{1,p}(X)$. Note that the closure \overline{E} (taken in Y or, equivalently, \mathbb{R}^2) separates Y and thus is not removable for $\widehat{N}^{1,p}(Y \setminus \overline{E})$.

Since $p > 2$, it is well known that $C_p^{\mathbb{R}^2}(\{x\}) = C_p^{\mathbb{R}^2}(\{0\}) > 0$ for $x \in \mathbb{R}^2$. It is not difficult to see that $C_p^Y(\{x\}) \geq \frac{1}{4}C_p^{\mathbb{R}^2}(\{x\})$ for $x \in Y$. Thus, by definition, every p -quasiopen set in Y is open. By Theorem 3.7, every p -path open set in Y is open, and in particular X is not p -path open.

By adding a weight, we now modify the previous example to cover all $p \geq 1$.

Example 6.3. Let Y, E and X be as in Example 6.2, but this time we equip Y with the measure $d\mu = w dx$, where $w(x) = |x|^{-1}$, which is globally doubling on Y . Then Y supports a global 1-Poincaré inequality, by [4, Example A.24]. The same proof also shows that X supports a global 1-Poincaré inequality. By Theorem 1.3, X is p -path almost open in Y for every $p \geq 1$. Thus, by Theorem 5.4, E is isometrically removable for $\hat{N}^{1,p}(X)$. Note that \bar{E} separates Y and thus is not removable for $\hat{N}^{1,p}(Y \setminus \bar{E})$.

We shall see that X is not p -path open in Y for any $p \geq 1$. This will be done by showing that E is not 1-thin at $0 = (0, 0)$ and hence that X is not 1-finely open. Since $C_1^Y(\{0\}) > 0$, by [4, Example A.24 and Lemma 6.15], it then follows from Theorem 3.7 that X is not 1-path open in Y .

Let u be a function admissible for $\text{cap}_1^Y(E \cap B(0, b), B(0, 2b))$ and let g be an upper gradient of u . Then for each $0 < a < b$,

$$\int_0^{2b} g(a, t)w(a, t) dt \geq \frac{1}{2\sqrt{2}b} \int_0^{2b} g(a, t) dt \geq \frac{1}{2\sqrt{2}b},$$

since g is an upper gradient, $u(a, 0) = 1$ and $u(a, 2b) = 0$. It follows that

$$\int_{B(0,2b)} gw dx \geq \int_0^b \int_0^{2b} g(a, t)w(a, t) dt da \geq b \frac{1}{2\sqrt{2}b} = \frac{1}{2\sqrt{2}}.$$

Hence, by taking infimum over all such u and g , we see that

$$\text{cap}_1^Y(E \cap B(0, b), B(0, 2b)) \geq \frac{1}{2\sqrt{2}}.$$

Testing with $u(x) = \min\{(2b - |x|)/b, 1\}_+$ shows that

$$\text{cap}_1^Y(B(0, b), B(0, 2b)) \leq \pi,$$

and so E is not 1-thin at 0 by the definition (3.2), and thus X is not 1-path open in Y .

With a bit more work we can create similar examples of removable sets E with non- p -path open complements in unweighted \mathbb{R}^n . Moreover, it can be done so that any $E' \supset E$ with p -path open complement is not removable.

We start with the following result. As in Example 5.3 this gives a lot of examples of removable sets whose closure is not removable.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be open and equipped with the Lebesgue measure \mathcal{L}^n . Let $E \subset \Omega$ be a set with $(n - 1)$ -dimensional Hausdorff measure $\mathcal{H}^{n-1}(E) = 0$. Then E is isometrically removable for $\hat{N}^{1,p}(\Omega \setminus E)$ for every $p \geq 1$.*

Proof. The proof is essentially identical to the proof of Theorem 6.1 (a). However, this time we do not have any exceptional directions as given by (6.1). ■

Example 6.5. Let $Y = \mathbb{R}^n$, $n \geq 3$, equipped with the Lebesgue measure \mathcal{L}^n , and let $A \subset [0, 1]$ be a nonempty set of zero 1-dimensional Lebesgue measure. Let $\{q_j\}_{j=1}^\infty$ be an enumeration of \mathbb{Q} and let $X = \mathbb{R}^n \setminus E$, where

$$E = \left(\bigcup_{i,j=1}^\infty (q_j + 2^{-i}A) \right) \times \mathbb{R}^{n-2} \times \{0\}.$$

Let $p > 2 - d$, where $0 \leq d \leq 1$ is the Hausdorff dimension of A . Note that all $p > 1$ are included when $\dim_H A = 1$ and $\mathcal{L}^1(A) = 0$. It follows from Proposition 6.4 that E is removable for $\widehat{N}^{1,p}(X)$. As E is contained in the hyperplane $H := \mathbb{R}^{n-1} \times \{0\}$, X is a union of the open set $\mathbb{R}^n \setminus H$ and a set of measure zero, and thus p -path almost open in \mathbb{R}^n , by Theorem 1.3. We shall now show that X is not p -path open in \mathbb{R}^n .

By Theorem 3.7, this amounts to showing that $C_p^{\mathbb{R}^n}(X \setminus \text{fine-int } X) > 0$, with $\text{fine-int } X$ denoting the p -fine interior of X , which consists of all points $x \in X$ for which

$$(6.3) \quad \sum_{i=0}^\infty \left(\frac{\text{cap}_p^{\mathbb{R}^n}(B(x, 2^{-i}) \cap E, B(x, 2^{1-i}))}{\text{cap}_p^{\mathbb{R}^n}(B(x, 2^{-i}), B(x, 2^{1-i}))} \right)^{1/(p-1)} < \infty,$$

see Malý–Ziemer [49, Theorem 2.136]. We alert the reader that it is not enough to show that $C_p^{\mathbb{R}^n}(X \setminus \text{int } X) > 0$, since e.g. the complement of any countable dense set in \mathbb{R}^n , $n \geq p$, is p -path open but has empty interior.

By [36, Lemma 12.10], (6.3) is equivalent to the p -thinness condition (3.1). It is clear that (6.3) holds for all $x \in X \setminus H$. For $x = (x_1, \dots, x_n) \in X \cap H$ and $r = 2^{1-i}$, $i = 1, 2, \dots$, find

$$y = (y_1, \dots, y_n) \in H, \quad \text{with } y_1 \in \mathbb{Q} \text{ and } |x - y| < \frac{1}{2}r = 2^{-i}.$$

Let

$$A_i := 2^{-i}A \times \mathbb{R}^{n-2} \times \{0\}, \quad i = 0, 1, \dots$$

Then, by the scaling property and translation invariance of $\text{cap}_p^{\mathbb{R}^n}$ together with the construction of E ,

$$\begin{aligned} \text{cap}_p^{\mathbb{R}^n}(B(x, r) \cap E, B(x, 2r)) &\geq \text{cap}_p^{\mathbb{R}^n}(B(y, \frac{1}{2}r) \cap (y + A_i), B(y, \frac{5}{2}r)) \\ &= 2^{-i(n-p)} \text{cap}_p^{\mathbb{R}^n}(A_0 \cap B(0, 1), B(0, 5)) \\ &=: C_0 r^{n-p}. \end{aligned}$$

Since A_0 is $(d + n - 2)$ -dimensional and $p > 2 - d$, it follows from e.g. Heinonen–Kilpeläinen–Martio [36, Theorem 2.26] that $C_0 > 0$. It is crucial here that C_0 , by its definition above, only depends on the set A fixed at the beginning, and not on the ball $B(x, r)$. Similarly,

$$\text{cap}_p^{\mathbb{R}^n}(B(x, r), B(x, 2r)) = C r^{n-p} \quad \text{for some } C > 0.$$

Hence, for all $r = 2^{1-i}$, $i = 1, 2, \dots$,

$$\left(\frac{\text{cap}_p^{\mathbb{R}^n}(B(x, r) \cap E, B(x, 2r))}{\text{cap}_p^{\mathbb{R}^n}(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \geq \frac{C_0}{C} > 0,$$

and inserting this into the Wiener criterion (6.3) shows that $x \notin \text{fine-int } X$, and hence $\text{fine-int } X = \mathbb{R}^n \setminus H$. Moreover, $H \setminus E$ has infinite $(n - 1)$ -dimensional Hausdorff measure and thus by e.g. [36, Theorem 2.26] again,

$$C_p^{\mathbb{R}^n}(X \setminus \text{fine-int } X) = C_p^{\mathbb{R}^n}(H \setminus E) > 0,$$

i.e., X is not p -path open, by Theorem 3.7.

It also follows that if $E' \supset E$ is any set such that $X' = \mathbb{R}^n \setminus E'$ is p -path open (and $\mathcal{L}^n(E') = 0$), then $C_p^{\mathbb{R}^n}(H \cap X') = 0$. Since $H \cap X'$ separates X' , it follows that X' cannot support a p -Poincaré inequality and thus E' is not removable for $\widehat{N}^{1,p}(X')$, by Theorem 5.4. Thus, the removability of E cannot be achieved by considering larger sets with p -path open complements.

7. p -path almost open sets

Despite the example given in Theorem 6.1, p -path almost open sets played a rather central role in our studies of removable sets in Section 5. In this section, we therefore characterize p -path almost open sets, and in particular answer Open problem 3.4 in Björn–Björn [5], which asked whether every p -path almost open set can be written as a union of a p -path open set and a set of a measure zero. We give an affirmative answer for measurable sets, under natural assumptions. At the same time, we also answer it in the negative for nonmeasurable sets in unweighted \mathbb{R}^n , $n \geq 2$, and give a measurable counterexample with a nondoubling underlying measure on \mathbb{R} .

We call a set $N \subset X$ p -path negligible if for p -almost every arc-length parameterized curve γ , we have $\mathcal{L}^1(\gamma^{-1}(N)) = 0$, where \mathcal{L}^1 denotes the 1-dimensional Lebesgue measure. (Recall that we only consider rectifiable curves.) A p -path negligible set is obviously p -path almost open.

It is easy to check that a set of measure zero is p -path negligible, see Shanmugalingam [53, proof of Lemma 3.2] (or [4, Lemma 1.42]). Conversely, we have the following result.

Proposition 7.1. *Assume that μ is locally doubling and supports a local p -Poincaré inequality. Let $N \subset X$ be measurable and p -path negligible. Then $\mu(N) = 0$.*

Proposition 7.5 below shows that the measurability assumption cannot be dropped.

Proof. First we make the following observation: if $u \in N^{1,p}(X)$, then the minimal p -weak upper gradient satisfies $g_u = 0$ a.e. in N . To see this, note that for p -almost every curve γ , we have $\mathcal{L}^1(\gamma^{-1}(N)) = 0$ and so

$$\int_{\gamma} g_u ds = \int_{\gamma} g_u \chi_{X \setminus N} ds.$$

Thus, $g_u \chi_{X \setminus N}$ is also a p -weak upper gradient of u , and then by the minimality of g_u , we must have $g_u = 0$ a.e. in N .

In order to prove that $\mu(N) = 0$, suppose instead that $\mu(N) > 0$. Then there exists a point $x \in N$ of density one, see e.g. Heinonen [35, Theorem 1.8]. For each $i = 1, 2, \dots$, let $B_i = B(x, i^{-1})$ and $\eta_i(y) = (1 - i \text{dist}(y, B_i))_+$. Then $g_{\eta_i} \leq i \chi_{2B_i}$ and in fact $g_{\eta_i} \leq i \chi_{2B_i \setminus N}$, by the earlier observation.

By the local p -Poincaré inequality, we have for all sufficiently large i that the sphere $\partial \frac{5}{2} B_i$ is nonempty and

$$(7.1) \quad \int_{3B_i} |\eta_i - c_i| d\mu \leq \frac{C}{i} \left(\int_{3B_i} g_{\eta_i}^p d\mu \right)^{1/p},$$

where $c_i := \int_{3B_i} \eta_i d\mu$ is the integral average. Considering the cases $c_i \leq \frac{1}{2}$ and $c_i \geq \frac{1}{2}$ separately, we conclude that the left-hand side satisfies

$$\int_{3B_i} |\eta_i - c_i| d\mu \geq \frac{\min\{\mu(B_i), \mu(3B_i \setminus 2B_i)\}}{2\mu(3B_i)} \geq \frac{1}{C'},$$

by the local doubling property (and for large i). On the other hand, the right-hand side satisfies

$$\frac{1}{i} \left(\int_{3B_i} g_{\eta_i}^p d\mu \right)^{1/p} \leq \left(\frac{\mu(3B_i \setminus N)}{\mu(3B_i)} \right)^{1/p},$$

which tends to zero as $i \rightarrow \infty$, since x is a density point of N . This contradicts (7.1), and so we have the result. ■

Next we prove the following characterization of p -path almost open sets. Note that it applies also to nonmeasurable sets.

Theorem 7.2. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality. Then $U \subset X$ is p -path almost open if and only if it can be written as a union $U = V \cup N$, where V is p -path open and N is p -path negligible.*

Recall that under these assumptions, a set is p -path open if and only if it is p -quasi-open, by Theorem 3.7.

Proof. If $U = V \cup N$, where V is p -path open and N is p -path negligible, then it is easy to see that U is p -path almost open.

Conversely, suppose that U is p -path almost open. Now the family Γ of curves γ , for which $\gamma^{-1}(U)$ is not the union of an open set and a set of zero 1-dimensional Lebesgue measure, has zero p -modulus, i.e., there is a Borel function $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho ds = \infty$ for every $\gamma \in \Gamma$, see [4, Proposition 1.37].

Assume first that U is bounded and let B be a ball containing a 1-neighbourhood of U . Define

$$u(x) = \min \left\{ 1, \inf_\gamma \int_\gamma (\rho + \chi_B) ds \right\}, \quad x \in X,$$

where the infimum is taken over all rectifiable curves (including constant curves) from x to $X \setminus U$. Then $u = 0$ in $X \setminus U$, and $\rho + \chi_B$ is an upper gradient of u , by Björn–Björn–Shanmugalingam [12, Lemma 3.1] (or [4, Lemma 5.25]). By Corollary 1.10 in Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [41] (or Theorem 3.4), u is measurable. As u and U are bounded and $\rho \in L^p(X)$, it follows that $u \in N^{1,p}(X)$.

Let $V = \{x \in U : u(x) > 0\} = \{x \in X : u(x) > 0\}$ and $N = U \setminus V$. Then V is p -path open, since $u \in N^{1,p}(X)$ is (absolutely) continuous on p -almost every curve in X , by Proposition 3.1 in Shanmugalingam [53] (or [4, Theorem 1.56]). It remains to show that N is p -path negligible. Assume it is not. Then there necessarily is an arc-length parameterized curve $\hat{\gamma}$ for which $\mathcal{L}^1(D) > 0$, where $D := \hat{\gamma}^{-1}(N)$, but $\int_{\hat{\gamma}} \rho ds < \infty$.

Let $x \in N$ and $0 < \delta \leq 1$. As $u(x) = 0$, there are arc-length parameterized curves $\gamma_j: [0, l_{\gamma_j}] \rightarrow X$, $j = 1, 2, \dots$, such that $\gamma_j(0) = x$, $\gamma_j(l_{\gamma_j}) \in X \setminus U$ and

$$\int_{\gamma_j} (\rho + \chi_B) ds \leq 2^{-j-1}\delta.$$

Since B contains a 1-neighbourhood of U , necessarily $l_{\gamma_j} \leq 2^{-j-1}\delta$. We define a curve γ_x as follows. Let $L_0 = 0$,

$$L_i = 2 \sum_{j=1}^i l_{\gamma_j} \leq 2 \sum_{j=1}^{\infty} l_{\gamma_j} =: L \leq 2\delta \sum_{j=1}^{\infty} 2^{-j-1} = \delta \quad \text{for } i = 1, 2, \dots,$$

$$\gamma_x = \begin{cases} \gamma_j(t - L_{j-1}) & \text{for } L_{j-1} \leq t \leq L_{j-1} + l_{\gamma_j}, \\ \gamma_j(L_j - t) & \text{for } L_{j-1} + l_{\gamma_j} \leq t \leq L_j, \end{cases}$$

and $\gamma_x(L) := x$. Then $\gamma_x: [0, L] \rightarrow X$ is an arc-length parameterized curve with

$$\gamma_x(0) = x = \gamma_x(L_j) = \gamma_x(L) \quad \text{and} \quad \gamma_x(L_j + l_{\gamma_{j+1}}) \in X \setminus U \quad \text{for all } j = 1, 2, \dots,$$

with $L_j + l_{\gamma_{j+1}} \rightarrow L$ as $j \rightarrow \infty$. Also, $\text{length}(\gamma_x) = L \leq \delta$ and $\int_{\gamma_x} \rho ds \leq \delta$. In essence, γ_x is a short “zigzagging loop” at x which intersects $X \setminus U$ arbitrarily close to its end point.

Now take a dense set $\{s_k\}_{k=1}^{\infty} \subset D$. For every $k = 1, 2, \dots$, we find such a zigzagging loop $\hat{\gamma}_k := \gamma_{x_k}$ at $x_k = \hat{\gamma}(s_k)$, with $l_{\hat{\gamma}_k} \leq 2^{-k}$ and $\int_{\hat{\gamma}_k} \rho ds \leq 2^{-k}$. Next we define a curve γ that is obtained from $\hat{\gamma}$ by adding the “loops” $\hat{\gamma}_k$ at the points x_k , for $k = 1, 2, \dots$. More precisely, first let $l = \sum_{k=1}^{\infty} l_{\hat{\gamma}_k}$. Then define the function

$$f: [0, l_{\hat{\gamma}}] \rightarrow [0, l_{\hat{\gamma}} + l], \quad f(t) := v([0, t]) \quad \text{with } v = \mathcal{L}^1 + \sum_{k=1}^{\infty} l_{\hat{\gamma}_k} \delta_{s_k},$$

where δ_{s_k} are Dirac measures at the points s_k . Now f^{-1} is defined on a subset of $[0, l_{\hat{\gamma}} + l]$ and it is 1-Lipschitz. We define a curve γ on $[0, l_{\hat{\gamma}} + l]$ as follows. For $t \in f([0, l_{\hat{\gamma}}])$, let $\gamma(t) = \hat{\gamma}(f^{-1}(t))$. If $t \in [0, l_{\hat{\gamma}} + l] \setminus f([0, l_{\hat{\gamma}}])$, then for some $k = 1, 2, \dots$, the number t belongs to an interval of length $l_{\hat{\gamma}_k}$ which does not intersect $f([0, l_{\hat{\gamma}}])$ apart from the right end point $f(s_k)$. Define γ to be the curve $\hat{\gamma}_k$ on this interval. Note that γ is a 1-Lipschitz mapping and that $\text{length}(\gamma) = l_{\hat{\gamma}} + l$. Thus, γ is arc-length parameterized, and so it is indeed a “curve” in our sense.

Since $\hat{\gamma}(D) \subset N$, we also get $\gamma(f(D)) \subset N$. Moreover, since f^{-1} is 1-Lipschitz, $\mathcal{L}^1(f(D)) \geq \mathcal{L}^1(D) > 0$ and so γ travels a positive length in N . Let $t := f(\xi) \in f(D)$ and $\varepsilon > 0$. Then by the construction of f , together with the density of $\{s_k\}_{k=1}^{\infty}$ in D , we can find k and $j_0(k)$ such that $\lim_{k \rightarrow \infty} j_0(k) = \infty$ and

$$|f(s_k) - t| = |f(s_k) - f(\xi)| \leq |s_k - \xi| + \sum_{j \geq j_0(k)} l_{\gamma_j} \leq |s_k - \xi| + 2^{1-j_0(k)} < \varepsilon.$$

By the construction of the zigzagging loop $\hat{\gamma}_k$, there is a sequence $t_l \nearrow f(s_k)$ such that $\gamma(t_l) \in X \setminus U$ for $l = 1, 2, \dots$. Since $\varepsilon > 0$ was arbitrary, we conclude that t is not in the interior of $\gamma^{-1}(U)$. Thus, no $t \in f(D)$ is an interior point of $\gamma^{-1}(U)$, and since we had $\mathcal{L}^1(f(D)) > 0$, $\gamma^{-1}(U)$ is not the union of a relatively open set and a set of zero \mathcal{L}^1 -measure. This shows that $\gamma \in \Gamma$.

At the same time,

$$\int_{\gamma} \rho \, ds = \int_{\hat{\gamma}} \rho \, ds + \sum_{k=1}^{\infty} \int_{\hat{\gamma}_k} \rho \, ds \leq \int_{\hat{\gamma}} \rho \, ds + \sum_{k=1}^{\infty} 2^{-k} = \int_{\hat{\gamma}} \rho \, ds + 1 < \infty.$$

This contradicts the choice of ρ . Thus, N is in fact a p -path negligible set and we have the result for bounded sets U .

If U is p -path almost open and unbounded, we know that each $U \cap B(x_0, j)$ is a disjoint union of a p -path open set V_j and a p -path negligible set N_j , $j = 1, 2, \dots$, where $x_0 \in X$ is fixed. Now we can write U as the union

$$U = \bigcup_{j=1}^{\infty} V_j \cup \bigcup_{j=1}^{\infty} N_j,$$

where $\bigcup_{j=1}^{\infty} V_j$ is obviously p -path open and $\bigcup_{j=1}^{\infty} N_j$ is p -path negligible. ■

Finally, we obtain the following natural characterization of measurable p -path almost open sets. This answers Open problem 3.4 in Björn–Björn [5] in the affirmative for measurable sets, under natural assumptions.

Theorem 7.3. *Assume that X is locally compact and that μ is locally doubling and supports a local p -Poincaré inequality. Suppose that $U \subset X$ is measurable. Then $U \subset X$ is p -path almost open if and only if it can be written as $U = V \cup N$, where V is p -path open and $\mu(N) = 0$.*

Under these assumptions, it follows from Theorem 3.7 that every p -path open set is p -quasiopen and thus measurable. Hence, it follows from Proposition 7.5 below that the measurability assumption in Theorem 7.3 cannot be dropped.

Proof. If U is p -path almost open, then by Theorem 7.2 we know that it is a union $U = V \cup N'$, where V is p -path open and N' is p -path negligible. Then $U = V \cup N$, where $N = N' \setminus V$ is also p -path negligible. By Theorem 3.7, V is measurable. As U is measurable by assumption, so is $N = U \setminus V$. Thus, by Proposition 7.1, we have $\mu(N) = 0$.

Conversely, if $U = V \cup N$, where V is p -path open and $\mu(N) = 0$, then N is p -path negligible by [4, Lemma 1.42], and hence U is p -path almost open by Theorem 7.2. ■

A natural question is whether there exist nonmeasurable p -path almost open sets. If there are no nonconstant rectifiable curves in X , as e.g. on the von Koch snowflake curve, then all sets are p -path open as well as p -path almost open, and thus there are plenty of nonmeasurable p -path open and p -path almost open sets. But what can happen under natural assumptions, such as doubling and a Poincaré inequality?

First consider the 1-dimensional case.

Proposition 7.4. *Let $X = \mathbb{R}$ be equipped with a locally doubling measure μ supporting a local p -Poincaré inequality. Then every p -path almost open set G is a union of an open set and a set of measure zero, and is in particular measurable.*

Proof. By Björn–Björn–Shanmugalingam [13, Theorem 1.2], $d\mu = w \, dx$ and w is a local A_p -weight. Let $a > 0$ and $\gamma: [-a, a] \rightarrow \mathbb{R}$ with $\gamma(t) = t$. If $\rho \geq 0$ is a function admissible

in the definition of $\text{Mod}_{p,X}(\{\gamma\})$ and $p > 1$, then

$$1 \leq \int_{\gamma} \rho \, ds = \int_{-a}^a \rho w^{1/p} w^{-1/p} \, dx \leq \left(\int_{-a}^a \rho^p \, d\mu \right)^{1/p} \left(\int_{-a}^a w^{-1/(p-1)} \, dx \right)^{(p-1)/p}.$$

Taking infimum over all such ρ and in view of the local A_p -condition [13, equation (5.1)], we see that the single curve family $\{\gamma\}$ has positive p -modulus. (The calculation is similar when $p = 1$.) Thus, necessarily $\gamma^{-1}(G) = G \cap [-a, a]$ is a union of an open set and a set of measure zero. Hence, also $G = \bigcup_{k=1}^{\infty} (G \cap [-k, k])$ is a union of an open set and a set of measure zero. ■

The same argument applies to any connected metric graph X equipped with a locally doubling measure μ supporting a local p -Poincaré inequality, where each edge is considered to be a segment. To see this, first note that there are at most a countable number of vertices and edges, and that $\mu(\{x\}) = 0$ for each $x \in X$, see [4, Corollary 3.9]. It follows that the set of vertices has zero measure. On each open edge, μ is given by a locally p -admissible weight, by [13, Theorem 4.6], and we can apply the argument above.

On the contrary, in higher dimensions there always exist nonmeasurable p -path almost open sets, at least if we assume the continuum hypothesis.

Proposition 7.5. *Assume that the continuum hypothesis is true. Let $X = \mathbb{R}^n$, $n \geq 2$, be equipped with a measure $d\mu = w \, dx$ such that $0 < w \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then there is a nonmeasurable dense p -path negligible set S . In particular, S is a nonmeasurable dense p -path almost open set.*

In particular, Proposition 7.5 applies to p -admissible weights w , as studied extensively in Heinonen–Kilpeläinen–Martio [36] when $p > 1$. Note that μ and the Lebesgue measure \mathcal{L}^n have the same measurable sets.

We shall use Sierpiński sets to prove Proposition 7.5. A *Sierpiński set* S is an uncountable subset of \mathbb{R}^n such that $E \cap S$ is at most countable for every set E of Lebesgue measure $\mathcal{L}^n(E) = 0$. Such sets exist if we assume the continuum hypothesis, see Sierpiński [55] (Proposition C₂₆ in [55, p. 80] gives the existence for \mathbb{R} , while in the paragraph just before Proposition C_{26a} in [55, p. 81], it is explained how to deduce the existence for \mathbb{R}^2) and Morgan [50, Theorem 7, p. 86] (for \mathbb{R}^n). On the other hand, there are other models of set theory containing ZFC (Zermelo–Fraenkel’s system plus the axiom of choice) for which the existence of Sierpiński sets fails, e.g. if one adds Martin’s axiom for \aleph_1 , see Kunen [47, Exercise V.6.29].

Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a Sierpiński set and $A \subset S$. Then $A \cap H \subset S \cap H$ is at most countable for every hyperplane H . If A is measurable, then it follows from Fubini’s theorem that $\mathcal{L}^n(A) = 0$, but then $A = A \cap S$ is at most countable. Thus, every uncountable subset of S is nonmeasurable. In particular, S itself is nonmeasurable. Conversely it is easy to show that if $S \subset \mathbb{R}^n$, $n \geq 1$, is an uncountable set such that every uncountable subset is nonmeasurable, then S is a Sierpiński set.

In fact, there exist Sierpiński sets with additional, perhaps surprising, properties. For example, Bienias–Głab–Rałowski–Żeberski [3, Theorem 5.5] have shown that in \mathbb{R}^2 there is a Sierpiński set that intersects every line in at most two points. (This is again assuming the continuum hypothesis.)

When proving Proposition 7.5 we will need the following lemma, which is no doubt well known. As we have not found a good reference, we provide a short proof.

Lemma 7.6. *Let $\gamma: [0, l_\gamma] \rightarrow X$ be an arc-length parameterized curve. Then*

$$\mathcal{L}^1(\gamma^{-1}(x)) = 0 \quad \text{for every } x \in X.$$

Proof. The metric derivative

$$|\dot{\gamma}|(t) := \lim_{h \rightarrow 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|}$$

satisfies $|\dot{\gamma}|(t) = 1$ for a.e. $t \in [0, l_\gamma]$, see e.g. Hajłasz [30, Corollary 3.7]. At the same time, clearly $|\dot{\gamma}|(t) = 0$ at every point t of density one for the closed set $\gamma^{-1}(\{x\})$ (provided that the limit exists), and thus at a.e. $t \in \gamma^{-1}(\{x\})$. ■

Proof of Proposition 7.5. By the assumptions on the measure μ , it has the same zero sets and the same measurable sets as the Lebesgue measure \mathcal{L}^n . As mentioned above, there exists a Sierpiński set $S' \subset \mathbb{R}^n$. It is easy to see that a countable union of Sierpiński sets is a Sierpiński set, and hence $S = \bigcup_{q \in \mathbb{Q}^n} (S' + q)$ is a dense Sierpiński set.

If $\gamma: [0, l_\gamma] \rightarrow \mathbb{R}^n$ is an arc-length parameterized curve, then $\gamma([0, l_\gamma]) \cap S$ is at most countable, since $\mathcal{L}^n(\gamma([0, l_\gamma])) = 0$. Lemma 7.6 and the countable additivity of the Lebesgue measure \mathcal{L}^1 then imply that $\mathcal{L}^1(\gamma^{-1}(S)) = 0$. As this holds for every curve γ , the set S is p -path negligible for every p . However, S is nonmeasurable with respect to \mathcal{L}^n , and thus also with respect to μ . ■

We end the paper by constructing a measurable p -path almost open set which cannot be written as a union of a (quasi)open set and a set of measure zero. Note that the measure is not doubling and does not support a Poincaré inequality.

Example 7.7. Let $X = \mathbb{R}$, equipped with the measure $\mathcal{L}^1 + \delta_0$, where δ_0 is the Dirac measure at 0. Then $C_p(\{x\}) \geq 2$ for all $x \in X$ and hence all quasiopen sets in X are open. The interval $[0, 1)$ cannot therefore be written as a union of a quasiopen set and a set of measure zero. However, it is still p -path almost open for any $p \geq 1$, by Lemma 7.6.

For an example with a nonatomic measure, equip $\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})^{n-1}$ with the measure $(\mathcal{L}^1 + \delta_0) \times \mathcal{L}^{n-1}$ and consider $U = [0, 1) \times ((0, 1) \setminus \mathbb{Q})^{n-1}$, $n \geq 2$.

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Anders Björn

Department of Mathematics, Linköping University, 581 83 Linköping, Sweden;
anders.bjorn@liu.se

Jana Björn

Department of Mathematics, Linköping University, 581 83 Linköping, Sweden;
jana.bjorn@liu.se

Panu Lahti

Academy of Mathematics and Systems Science, Chinese Academy of Sciences,
Beijing 100190, P. R. China;
panulahti@amss.ac.cn