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Factorization of the normalization of the Nash blowup of order n of $\mathcal{A}_n$ by the minimal resolution

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Abstract. We show that the normalization of the Nash blowup of order n of the toric surface singularity $\mathcal{A}_n$ can be factorized by the minimal resolution of $\mathcal{A}_n$. The result is obtained using the combinatorial description of these objects.

Introduction

The Nash blowup of an algebraic variety is a modification that replaces singular points by limits of tangent spaces at non-singular points. It was proposed to achieve a resolution of singularities by iterating this process [16, 20]. This question has been treated in [1, 5, 12, 13, 15, 16, 18, 21, 22]. The particular case of toric varieties is treated in [7, 10–12, 14] using their combinatorial structure.

There is a generalization of Nash blowups, called higher Nash blowups or Nash blowups of order n, that was proposed by Takehiko Yasuda. This modification replaces singular points by limits of infinitesimal neighborhoods of certain order at non-singular points. In particular, the higher Nash blowup looks for resolution of singularities in one step [23]. Yasuda proves that this is true for curves in characteristic zero, but conjectures that is false in general, proposing as a counterexample the toric surface $\mathcal{A}_3$.

There are several papers that deal with higher Nash blowups in the special case of toric varieties. The usual strategy for this special case is to translate the original geometric problem into a combinatorial one and then try to solve the latter. So far, the combinatorial description of higher Nash blowups of toric varieties has been obtained using Gröebner fans or higher-order Jacobian matrices.

The use of Gröebner fans for higher Nash blowups of toric varieties was initiated in [6]. Later, this tool was further developed in [22] to show that the Nash blowup of order n of the toric surface singularity $\mathcal{A}_3$ is singular for any $n > 0$, over the complex numbers. This problem was later revisited to show that it also holds in prime characteristic [9].

The techniques from [22] can be used to compute the Gröebner fan of the normalization of higher Nash blowup of $\mathcal{A}_n$ for some n's. Those computations suggest that the essential divisors of the minimal resolution of $\mathcal{A}_n$ appear in the normalization of the Nash

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blowup of order n of $\mathcal{A}_n$ for some n's. The main goal of this paper is to show that this happens for all n. In particular, this implies that the normalization of the Nash blowup of order n of $\mathcal{A}_n$ factors through its minimal resolution (see Corollary 1.9).

The approach to study higher Nash blowups of toric varieties using a higher order Jacobian matrix was initiated in [4]. That paper deals with a conjecture proposed by Yasuda concerning the semigroup associated to the higher Nash blowup of formal curves. There it was proved that the conjecture is true in the toric case but false in general. This was achieved by studying the properties of the higher order Jacobian matrix of monomial morphisms. In this paper we follow a similar but more general approach.

The normalization of the higher Nash blowup of $\mathcal{A}_n$ is a toric variety associated to a fan that subdivides the cone determining $\mathcal{A}_n$, see [4,11]. An explicit description of this fan could be obtained by effectively computing all minors of the corresponding higher order Jacobian matrix. This is a difficult task given the complexity of the matrix for large n. However, for the problem we are interested in, we do not require an explicit description of the entire fan.

The rays that subdivide the cone of $\mathcal{A}_n$ to obtain its minimal resolution can be explicitly specified. Thus, in order to show that these rays appear in the fan associated to the normalization of the higher Nash blowup, we need to be able to control only certain minors of the matrix. A great deal of this paper is devoted to construct combinatorial tools that allow us to accomplish that goal.

1. The main result

In this section we state the main result of this work. First, we introduce some notation that will be constantly used throughout this paper. *From now on, n will always denote a fixed positive natural number.*

Notation 1.1. Let $\gamma, \beta \in \mathbb{N}^t$ and $v \in \mathbb{N}^2$.

(1) We denote by $\pi_i(\beta)$ the projection to the i-th coordinate of β.

(2) $\gamma \leq \beta$ if and only if $\pi_i(\gamma) \leq \pi_i(\beta)$ for all $i \in \{1, \dots, t\}$. In particular, $\gamma < \beta$ if and only if $\gamma \leq \beta$ and $\pi_i(\gamma) < \pi_i(\beta)$ for some $i \in \{1, \dots, t\}$.

(3) $\binom{\beta}{\gamma} := \prod_{i=1}^{t} \binom{\pi_i(\beta)}{\pi_i(\gamma)}$.

(4) $|\beta| = \sum_{i=1}^{t} \pi_i(\beta)$.

(5) $\Lambda_{t,n} := \{\beta \in \mathbb{N}^t \mid 1 \leq |\beta| \leq n\}$. In addition, $\lambda_{t,n} := |\Lambda_{t,n}| = \binom{n+t}{n} - 1$.

(6) $\bar{v} := \left(\binom{v}{\alpha}\right)_{\alpha \in \Lambda_{2,n}} \in \mathbb{N}^{\lambda_{2,n}}$. We order the entries of this vector increasingly using graded lexicographical order on $\mathbb{N}^{\lambda_{2,n}}$.

(7) Let $A_n := \left(\begin{smallmatrix} 1 & 1 & n \\ 0 & 1 & n+1 \end{smallmatrix}\right)$.

(8) Given $J \subset \Lambda_{3,n}$, let $m_J := \sum_{\beta \in J} A_n \beta \in \mathbb{N}^2$.

Let $X \subset \mathbb{C}^s$ be an irreducible algebraic variety of dimension d. For a non-singular point $x \in X$, the $\mathbb{C}$-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee$ has dimension $\lambda_{d,n}$, where $\mathfrak{m}_x$ denotes the maximal ideal of x.

Definition 1.2 ([16,17,23]). With the previous notation, consider the morphism of Gauss:

$$G_n : X \setminus \mathrm{Sing}(X) \to \mathrm{Gr}(\lambda_{d,n}, \mathbb{C}^{\lambda_{s,n}}), \quad x \mapsto (\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee,$$

where $\mathrm{Sing}(X)$ denotes the singular locus of X and $\mathrm{Gr}(\lambda_{d,s}, \mathbb{C}^{\lambda_{s,n}})$ is the Grassmanian of vector subspaces of dimension $\lambda_{d,n}$ in $\mathbb{C}^{\lambda_{s,n}}$.

Denote by $\mathrm{Nash}_n(X)$ the Zariski closure of the graph of G_n. Call π_n the restriction to $\mathrm{Nash}_n(X)$ of the projection of $X \times \mathrm{Gr}(\lambda_{d,n}, \mathbb{C}^{\lambda_{s,n}})$ to X. The pair $(\mathrm{Nash}_n(X), \pi_n)$ is called the Nash blowup of X of order n.

This entire paper is devoted to study some aspects of the higher Nash blowup of the $\mathcal{A}_n$ singularity. Let us recall its definition and the notation we will use.

Definition 1.3. Consider the cone $\sigma_n = \mathbb{R}_{\geq 0}\{(0,1), (n+1, -n)\} \subset (\mathbb{R}^2)^\vee$. We denote by $\mathcal{A}_n$ the normal toric surface corresponding to σ_n, i.e., $\mathcal{A}_n = V(xz - y^{n+1})$.

In [4], the higher Nash blowup is studied through a higher-order Jacobian matrix. It is worth mentioning that there are other versions of higher order Jacobian matrices [2,3,8]. In the context of toric varieties, that matrix gave place to the following definition.

Definition 1.4 ([4, Proposition 2.4]). Let $J \subset \Lambda_{3,n}$ be such that $|J| = \lambda_{2,n}$. We define the matrix

$$L_J^c := \left(c_\beta\right)_{\beta \in J},$$

where

$$c_\beta = \sum_{\gamma \leq \beta} (-1)^{|\beta - \gamma|} \binom{\beta}{\gamma} \overline{A_n \gamma} \in \mathbb{N}^{\lambda_{2,n}}.$$

We order the rows of this matrix increasingly using graded lexicographical order on $J \subset \Lambda_{3,n}$. In addition, we denote

$$S_{A_n} := \left\{J \subset \Lambda_{3,n} \mid |J| = \lambda_{2,n} \text{ and } \det L_J^c \neq 0\right\}.$$

Proposition 1.5 ([4, Proposition 3.15]). *Let $I_n = \langle x^{m_J} \mid J \in S_{A_n}\rangle$. Then* $\mathrm{Nash}_n(\mathcal{A}_n) \cong \mathrm{Bl}_{I_n}(\mathcal{A}_n)$*, where $\mathrm{Bl}_{I_n}(\mathcal{A}_n)$ is the blowup of $\mathcal{A}_n$ centered on I_n.*

Abusing the notation, let $I_n = \{m_J \in \mathbb{R}^2 \mid J \in S_{A_n}\}$. The set I_n defines an order function:

$$\mathrm{ord}_{I_n} : \sigma_n \to \mathbb{R}, \quad v \mapsto \min_{m_J \in I_n} \langle v, m_J\rangle.$$

This function induces the following cones:

$$\sigma_{m_J} := \left\{v \in \sigma_n \mid \mathrm{ord}_{I_n}(v) = \langle v, m_J\rangle\right\}.$$

These cones form a fan $\Sigma(I_n) := \bigcup_{m_J \in I_n} \sigma_{m_J}$. This fan is a refinement of σ.

Proposition 1.6. *With the previous notation, we have*

$$\overline{\mathrm{Nash}_n(\mathcal{A}_n)} \cong X_{\Sigma(I_n)},$$

where $\overline{\mathrm{Nash}_n(\mathcal{A}_n)}$ is the normalization of the Nash blowup of $\mathcal{A}_n$ of order n and $X_{\Sigma(I_n)}$ is the normal variety corresponding to $\Sigma(I_n)$.

Proof. By the previous proposition we have that $\mathrm{Nash}_n(\mathcal{A}_n)$ is a monomial blowup. The result follows from Proposition 5.1 and Remark 4.6 of [11]. ∎

Definition 1.7. Let Σ_n' be the subdivision of σ_n given by the rays generated by $(k, 1-k)$, for each $k \in \{1, \dots, n\}$. We denote by $\mathcal{A}_n'$ the normal toric surface corresponding to Σ_n'. It is well known that $\mathcal{A}_n'$ is the minimal resolution of $\mathcal{A}_n$.

Moreover, for each $k \in \{1, \dots, n\}$, consider the function

$$f_k : \mathbb{N}^2 \to \mathbb{Z}, \quad v \mapsto \langle (k, 1-k), v \rangle.$$

The goal of this paper is to prove the following result about the shape of the fan $\Sigma(I_n)$.

Theorem 1.8. *For each $k \in \{1, \dots, n\}$, there exists $J_k, J_k' \in S_{\mathcal{A}_n}$ such that $f_k(m_{J_k}) = f_k(m_{J_k'}) \leq f_k(m_J)$ for all $J \in S_{\mathcal{A}_n}$. In particular, the rays generated by $(k, 1-k)$ appear in the fan $\Sigma(I_n)$.*

Corollary 1.9. *Let $\mathcal{A}_n'$ be the minimal resolution of $\mathcal{A}_n$ and let $\overline{\mathrm{Nash}_n(\mathcal{A}_n)}$ be the normalization of the higher Nash blowup of $\mathcal{A}_n$ of order n. Then there exists a proper birational morphism $\phi\colon \overline{\mathrm{Nash}_n(\mathcal{A}_n)} \to \mathcal{A}_n'$ such that the following diagram commutes:*

$$\begin{array}{ccc} \overline{\mathrm{Nash}_n(\mathcal{A}_n)} & \overset{\phi}{\dashrightarrow} & \mathcal{A}_n' \\ & \searrow & \downarrow \\ & & \mathcal{A}_n. \end{array}$$

Proof. The result follows by Theorem 1.8. ∎

2. A particular basis for the vector space $\mathbb{C}^{\lambda_{2,n}}$

As stated in Theorem 1.8, we need to find some subsets $J \subset \Lambda_{3,n}$ such that the determinant of L_J^c is non-zero. This will be achieved by reducing the matrix L_J^c to another matrix given by vectors formed by certain binomial coefficients. In this section, we prove that those vectors are linearly independent. We will see that this is equivalent to finding some basis of the vector space $\mathbb{C}^{\lambda_{2,n}}$.

The following results are stated for the field $\mathbb{C}$, but can be generalized for any field of characteristic 0.

Definition 2.1. Consider a sequence $\eta = (z, d_0, d_1, d_2, \dots, d_r)$, where $z \in \mathbb{Z}/2\mathbb{Z}$, $d_0 = 0$, $\{d_i\}_{i=1}^r \subset \mathbb{N} \setminus \{0\}$ and $\sum_{i=0}^r d_i = n$. We denote by Ω the set of all these possible sequences.

With this set let us define a subset of vectors of $\mathbb{N}^2$.

Definition 2.2. Let $\eta = (z, d_0, d_1, \dots, d_r) \in \Omega$. We construct a set of vectors $\{v_{j,\eta}\}_{j=1}^n \subset \mathbb{N}^2$ as follows. For each $j \in \{1, \dots, n\}$, there exists an unique $t \in \{1, \dots, r\}$ such that $\sum_{i=0}^{t-1} d_i < j \leq \sum_{i=0}^{t} d_i$. This implies that $j = \sum_{i=0}^{t-1} d_i + c$, where $0 < c \leq d_t$. Then we define

$$v_{j,\eta} = \begin{cases} \Big(\sum_{\substack{i \text{ odd} \\ i<t}} d_i + c, 0\Big) & \text{if } z = 1 \text{ and } t \text{ odd}, \\ \Big(0, \sum_{\substack{i \text{ even} \\ i<t}} d_i + c\Big) & \text{if } z = 1 \text{ and } t \text{ even}, \\ \Big(0, \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c\Big) & \text{if } z = 0 \text{ and } t \text{ odd}, \\ \Big(\sum_{\substack{i \text{ even} \\ i<t}} d_i + c, 0\Big) & \text{if } z = 0 \text{ and } t \text{ even}. \end{cases}$$

In addition, for each $j \in \{1, \ldots, n\}$, we let

$$T_{j,\eta} := \big\{v_{j,\eta}, v_{j,\eta} + (1,1), \ldots, v_{j,\eta} + (n-j)(1,1)\big\}.$$

Furthermore, we set $v_{0,\eta} := (1,1)$ and $T_{0,\eta} := \{(1,1), \ldots, (n,n)\}$. We define

$$T_\eta := \bigcup_{j=0}^{n} T_{j,\eta}.$$

Finally, recalling Notation 1.1, we define

$$\overline{T}_\eta = \{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_\eta\}.$$

Remark 2.3. Notice that this construction depends only on η. Moreover, geometrically, this construction is equivalent to taking vectors in an ordered way on the axes of $\mathbb{N}^2$.

Example 2.4. Let $n = 6$ and $\eta = (1,0,1,1,1,1,2)$. For $j = 3$ we have that $d_0 + d_1 + d_2 < 3 = d_0 + d_1 + d_2 + d_3$. Then $t = 3$, $v_{3,\eta} = (2,0)$ and $T_{3,\eta} = \{(2,0), (3,1), (4,2), (5,3)\}$. T_η is computed similarly and can be seen in Figure 1.

Now we give some basic properties of Definition 2.2.

Lemma 2.5. *Let $\eta \in \Omega$ and $u, v \in \mathbb{N}^2$. Then we have the following properties*:

(1) *If $u \neq v$, then $\bar{u} \neq \bar{v}$.*

(2) $|\overline{T}_\eta| = \lambda_{2,n}$.

(3) *If $v_{j,\eta} = (l,0)$ or $v_{j,\eta} = (0,l)$, then $l \leq j$.*

(4) *$\pi_i(v) \leq n$ for all $v \in T_\eta$ and $i \in \{1,2\}$.*

(5) *If $v_{j,\eta} = (0,p)$, then for all $q < p$, there exists $l < j$ such that $v_{l,\eta} = (0,q)$. If $v_{j,\eta} = (p,0)$, then for all $q < p$, there exists $l < j$ such that $v_{l,\eta} = (q,0)$.*

(6) *If $v_{j,\eta} = (0,l)$, then $\{v_{i,\eta}\}_{i=1}^{j} = \{(0,t)\}_{t=1}^{l} \cup \{(s,0)\}_{s=1}^{j-l}$ If $v_{j,\eta} = (l,0)$, then $\{v_{i,\eta}\}_{i=1}^{j} = \{(0,t)\}_{t=1}^{j-l} \cup \{(s,0)\}_{s=1}^{l}$.*

Proof. (1) Since $u \neq v$, $\pi_1(u) \neq \pi_1(v)$ or $\pi_2(u) \neq \pi_2(v)$. Suppose the first case; the other is analogous. By definition, $\bar{u} = \big(\binom{u}{\alpha}\big)_{\alpha \in \Lambda_{2,n}}$. Notice that $(1,0) \in \Lambda_{2,n}$. Then

$$\bar{u} = \left(\binom{u}{\alpha}\right) = \big(\pi_1(u), \ldots\big) \neq \big(\pi_1(v), \ldots\big) = \left(\binom{v}{\alpha}\right) = \bar{v}.$$

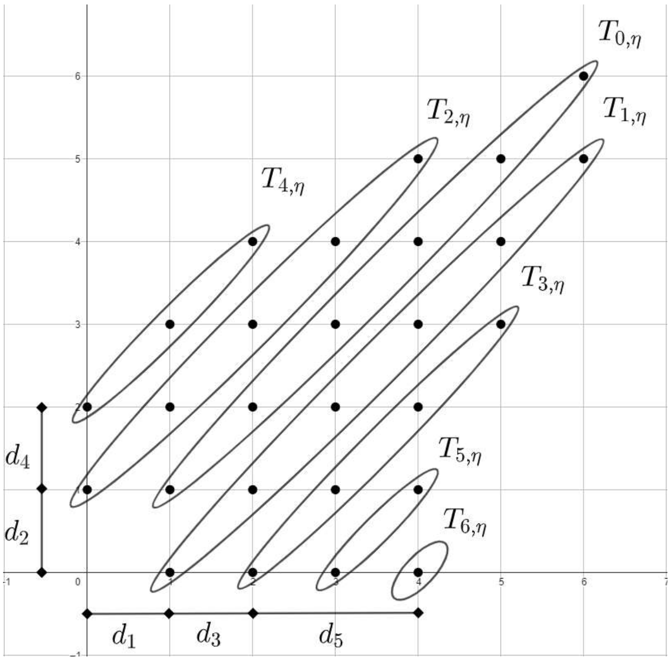

Figure 1. Example of T_η for $\eta = (1, 0, 1, 1, 1, 1, 2)$

(2) Notice that for each $j \in \{1, \dots, n\}$, $|T_{j,\eta}| = n - j + 1$ and $|T_{0,\eta}| = n$. This implies $|T_\eta| = (n+1)(n+2)/2 - 1 = \lambda_{2,n}$. By the previous item, we have that $|\overline{T}_\eta| = \lambda_{2,n}$.

(3) Let $t \leq r$ be such that $j = \sum_{i=0}^{t-1} d_i + c$. By definition, we have $l = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c$ or $l = \sum_{\substack{i \text{ even} \\ i<t}} d_i + c$. In any case, $l \leq j$.

(4) Let $v \in T_\eta$. If $v \in T_{0,\eta}$, then $v = (p, p)$, with $p \leq n$. If $v \notin T_{0,\eta}$, by Definition 2.2, we have that $v = v_{j,\eta} + p(1, 1)$, with $p \leq n - j$. Then

$$\pi_i(v) = \pi_i(v_{j,\eta}) + \pi_i(p(1,1)) = \pi_i(v_{j,\eta}) + p \leq j + p \leq n.$$

(5) Let $t \leq r$ be such that $j = \sum_{i=0}^{t-1} d_i + c$. Consider the case $v_{j,\eta} = (0, p)$. Suppose that t is odd. By Definition 2.2, $p = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c$ and $z = 0$. Let $q < p$. Then $q = \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c'$, where t' is odd, $t' < t$ and $c' \leq d_{t'}$ or $t' = t$ and $c' < c$. In any case, consider $l = \sum_{i=0}^{t'-1} d_i + c'$. Since t' is odd and $z = 0$, then we get $v_{l,\eta} = (0, \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c') = (0, q)$. If t is even, we have that $p = \sum_{\substack{i \text{ even} \\ i<t}} d_i + c$ and $z = 1$. In this case the proof is identical. If $v_{j,\eta} = (p, 0)$, the argument is analogous.

(6) If $v_{j,\eta} = (0, l)$, by the previous point, we have that $\{(0,t)\}_{t=1}^{l} \subset \{v_{i,\eta}\}_{i=1}^{j}$. On the other hand, we have that there exists $\{i_1, \dots, i_{j-l}\}$ such that $i_p \leq j$ and $v_{i_p,\eta} \notin \{(0,t)\}_{t=1}^{j}$

for all $p \in \{1, \dots, j-l\}$. Since $i_p < j$ for all p, using the previous point, we obtain that $v_{i_p,\eta} = (s_p, 0)$ for some $s_p \in \mathbb{N}$, and also $\{v_{i_p,\eta}\}_{p=1}^{j-l} = \{(s,0)\}_{s=1}^{j-l}$. This implies that $\{v_{i,\eta}\}_{i=1}^{n} = \{(0,t)\}_{t=1}^{l} \cup \{(s,0)\}_{s=1}^{n-l}$. ∎

2.1. Linear independence of $\overline{T}_\eta$

By Lemma 2.5 (2), we know that the cardinality of $\overline{T}_\eta$ is $\lambda_{2,n}$. In order to prove that it is a basis of $\mathbb{C}^{\lambda_{2,n}}$, we only have to see that it is linearly independent. For that, we need some preliminary lemmas.

Lemma 2.6. *Let $0 < c_0 < c_1 < \cdots < c_l$ be natural numbers. Then*

$$\det\left(\binom{c_i}{j}\right)_{\substack{0\le i\le l\\ 0\le j\le l}} \neq 0.$$

In particular, the set of vectors $\left\{\left(\binom{c_i}{j}\right)_{0\le j\le l} \in \mathbb{C}^{l+1} \mid 0 \le i \le l\right\}$ is linearly independent.

Proof. For each $j \le l$, consider the polynomial $b_j(x) = x(x-1)\cdots(x-j+1)/j!$ and $b_0 = 1$. Notice that for $x \in \mathbb{N}$, we have $b_j(x) = \binom{x}{j}$ and $\deg b_j(x) = j$ for all $j \in \{0, \dots, l\}$. Thus,

$$\left(\binom{c_i}{j}\right)_{\substack{0\le i\le l\\ 0\le j\le l}} = \left(b_j(c_i)\right)_{\substack{0\le i\le l\\ 0\le j\le l}}.$$

We show that the columns of this matrix are linearly independent. Let $\alpha_0, \dots, \alpha_l \in \mathbb{C}$ be such that $\sum_{j=0}^{l} \alpha_j b_j(c_i) = 0$ for each $i \in \{0, \dots, l\}$. Consider $f(x) = \sum_{j=0}^{l} \alpha_j b_j(x)$. Then $\{c_0, \dots, c_l\}$ are roots of $f(x)$. Since $\deg f(x) \le l$, we obtain that $f(x) = 0$. Since $\deg b_j(x) = j$, we conclude $\alpha_j = 0$ for all j. ∎

As we mentioned before, the goal is to prove that given $\eta \in \Omega$, the set of vectors $\overline{T}_\eta$ is linearly independent on $\mathbb{C}^{\lambda_{2,n}}$. Consider

$$\sum_{\bar{v}\in\overline{T}_\eta} a_{\bar{v}}\, \bar{v} = \bar{0} \in \mathbb{C}^{\lambda_{2,n}}. \tag{2.1}$$

Fix this notation for the next results.

Lemma 2.7. *Let $l, m, n \in \mathbb{N}$ be such that $1 \le l \le n$ and $m \le n - l + 1$. Let $\eta \in \Omega$. Suppose that $E = \{(c_1, l), \dots, (c_m, l)\}$ (resp. $\{(l, c_1), \dots, (l, c_m)\}$) is contained in T_η, for some $0 < c_1 < \cdots < c_m$. Moreover, suppose that for each $u \in T_\eta \setminus E$ such that $\pi_2(u) \ge l$ (resp. $\pi_1(u) \ge l$), we have that $a_{\bar{u}} = 0$. Then for all $v \in E$, we obtain that $a_{\bar{v}} = 0$.*

Proof. Consider the set of vectors $D = \{(0,l), (1,l), \dots, (n-l, l)\} \subset \Lambda_{2,n}$ (respectively $\{(l,0), (l,1), \dots, (l, n-l)\}$). Let $u \in T_\eta \setminus E$. If $\pi_2(u) < l$ (resp. $\pi_1(u) < l$), then $\binom{u}{\alpha} = 0$ for all $\alpha \in D$. If $\pi_2(u) \ge l$ (respectively $\pi_1(u) \ge l$), by hypothesis $a_{\bar{u}} = 0$. Consider $\pi_\alpha \colon \mathbb{C}^{\lambda_{2,n}} \to \mathbb{C}$ the projection on the α-th coordinate. Therefore, $\pi_\alpha(a_{\bar{u}}\bar{u}) = 0$ for all $u \in T_\eta \setminus E$ and $\alpha \in D$. This implies

$$\sum_{v\in E} \pi_\alpha(a_{\bar{v}}\bar{v}) = \sum_{\bar{v}\in\overline{T}_\eta} \pi_\alpha(a_{\bar{v}}\bar{v}) = 0 \quad \text{for all } \alpha \in D.$$

Since $\alpha = (j, l)$ (resp. (l, j)) with $0 \le j \le n - l$ and $v = (c_i, l)$ (resp. (l, c_i)), with $1 \le i \le m$, we obtain that $\pi_\alpha(\bar{v}) = \binom{c_i}{j}$. Thus,

$$\sum_{i=1}^{m} a_{\bar{v}} \binom{c_i}{j} = \sum_{v \in E} \pi_\alpha(a_{\bar{v}} \bar{v}) = 0,$$

for all $0 \le j \le n - l$. By Lemma 2.6, we obtain that $a_{\bar{v}} = 0$ for all $v \in E$. ■

Lemma 2.8. *Let $\eta = (z, d_0, d_1, \ldots, d_r) \in \Omega$ and $1 \le l < j \le n$.*

- *If $v_{l,\eta} = (p_l, 0)$ and $v_{j,\eta} = (p_j, 0)$, then*

$$\pi_1(v_{j,\eta} + (n - j)(1, 1)) \le \pi_1(v_{l,\eta} + (n - l)(1, 1)).$$

  The equality holds if and only if there exists $1 \le t \le r$ such that

$$\sum_{i=0}^{t-1} d_i < l < j \le \sum_{i=0}^{t} d_i.$$

- *If $v_{l,\eta} = (0, p_l)$ and $v_{j,\eta} = (0, p_j)$, then*

$$\pi_2(v_{j,\eta} + (n - j)(1, 1)) \le \pi_2(v_{l,\eta} + (n - l)(1, 1)).$$

  The equality holds if and only if there exists $1 \le t \le r$ such that

$$\sum_{i=0}^{t-1} d_i < l < j \le \sum_{i=0}^{t} d_i.$$

Proof. Suppose that $z = 1$. By Definition 2.2 and the fact $l < j$, $p_l = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c_t$ and $p_j = \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c_{t'}$, for some odd numbers $t \le t' \le r$, where $c_t \le d_t$ and $c_{t'} \le d_{t'}$. Moreover, by definition, $l = \sum_{i=0}^{t-1} d_i + c_t$ and $j = \sum_{i=0}^{t'-1} d_i + c_{t'}$. Then

$$\begin{aligned}\pi_1(v_{j,\eta} + (n - j)(1, 1)) = p_j + (n - j) = n - \sum_{\substack{i \text{ even} \\ i<t'}} d_i \le n - \sum_{\substack{i \text{ even} \\ i<t}} d_i \\ = p_l + (n - l) = \pi_1(v_{l,\eta} + (n - l)(1, 1)).\end{aligned}$$

Notice that the equality holds if and only if $t' = t$. For the other three cases ($z = 1$, $v_{l,\eta} = (0, p_l)$, $v_{j,\eta} = (0, p_j)$; $z = 0$, $v_{l,\eta} = (p_l, 0)$, $v_{j,\eta} = (p_j, 0)$; $z = 0$, $v_{l,\eta} = (0, p_l)$, $v_{j,\eta} = (0, p_j)$) the proof is analogous. ■

Now we are ready to prove the first important result of the section.

Proposition 2.9. *Let $\eta \in \Omega$. Then $\overline{T}_\eta$ is linearly independent.*

Proof. Let $\eta = (z, d_0, d_1, \ldots, d_r)$ and suppose that $z = 1$. Define the numbers

$$d_{+,r} = \sum_{\substack{i \le r \\ i \text{ odd}}} d_i, \quad d_{-,r} = \sum_{\substack{i \le r \\ i \text{ even}}} d_i.$$

Notice that, by Definition 2.1, we have that $n = d_{+,r} + d_{-,r}$.

Claim 1. For all $v \in T_\eta$ such that $\pi_2(v) > n - d_{+,r}$ or $\pi_1(v) > n - d_{-,r}$, we obtain that $a_{\bar{v}} = 0$ in (2.1).

Assume Claim 1 for the moment. For each $0 \le s \le d_{+,r}$, define the set $E_s = \{v \in T_\eta \mid \pi_1(v) = s \text{ and } \pi_2(v) \le d_{-,r}\}$. Notice that

$$|E_s| \le d_{-,r} + 1 = n - d_{+,r} + 1 \le n - s + 1.$$

Using Claim 1 and taking $s = d_{+,r}$ we obtain the conditions of Lemma 2.7. Therefore, $a_{\bar{v}} = 0$ for all $v \in E_{d_{+,r}}$. Now we can repeat the same argument for $s = d_{+,r} - 1$. Applying this process in a decreasing way for each $s \in \{0, \dots, d_{r,+}\}$, we obtain that $a_{\bar{v}} = 0$ for all $v \in \bigcup_{s=0}^{d_{+,r}} E_s$. Then for $v \in T_\eta$, we have three possibilities: $v \in \bigcup_{s=0}^{d_{+,r}} E_s$, $\pi_1(v) > d_{+,r}$, or $\pi_2(v) > d_{-,r}$. In any case, we obtain that $a_{\bar{v}} = 0$ by the previous argument or Claim 1. This implies that $\overline{T}_\eta$ is linearly independent.

Proof of Claim 1. For each $1 \le l \le r$, define

$$d_{+,l} = \sum_{\substack{i \le l \\ i \text{ odd}}} d_i, \quad d_{-,l} = \sum_{\substack{i \le l \\ i \text{ even}}} d_i.$$

We prove Claim 1 by induction on l. By definition, we have that $d_{+,1} = d_1$ and $d_{-,1} = 0$. Therefore, we only have to prove that if $\pi_2(v) > n - d_1$, then $a_{\bar{v}} = 0$. ∎

Claim 2. For all $v \in T_\eta$ such that $\pi_2(v) > n - d_1$, we have that $\pi_1(v) \ge \pi_2(v)$.

Proof of Claim 2. We proceed to prove Claim 2 by contrapositive. Let $v \in T_\eta$ be such that $\pi_2(v) > \pi_1(v)$. This implies that $v = (0, \pi_2(v) - \pi_1(v)) + \pi_1(v)(1,1) = v_{j,\eta} + \pi_1(v)(1,1)$ for some $j \le n$, where $\pi_1(v) \le n - j$, by Definition 2.2. By Lemma 2.5 (5), there exists $i < j$ such that $v_{i,\eta} = (0,1)$. Moreover, by Definition 2.2, $i = d_1 + 1$. By Lemma 2.8, we obtain

$$\begin{aligned} \pi_2(v) = \pi_2(v_{j,\eta} + \pi_1(v)(1,1)) &\le \pi_2(v_{j,\eta} + (n-j)(1,1)) \\ &\le \pi_2(v_{d_1+1,\eta} + (n - d_1 - 1)(1,1)) \\ &= n - d_1, \end{aligned}$$

concluding the proof of Claim 2. ∎

For each $s \in \{n - d_1 + 1, \dots, n\}$, we define the set $E(s) = \{v \in T_\eta \mid \pi_2(v) = s\}$. By Lemma 2.5 (4) and Claim 2, we have that for each $s \in \{n - d_1 + 1, \dots, n\}$, we have $|E(s)| \le n - s + 1$. Now we are in the conditions of Lemma 2.7. Applying the lemma for each s in a descendant way, we obtain the result for $l = 1$.

Now suppose that Claim 1 is true for l, i.e., for all $v \in T_\eta$ such that $\pi_2(v) > n - d_{+,l}$ or $\pi_1(v) > n - d_{-,l}$ for some $l \ge 1$, we have that $a_{\bar{v}} = 0$ and we prove for $l + 1$. We have two cases: l odd or l even. We prove the case l odd, the other case is analogous. Since l is odd, we obtain that $d_{+,l} = d_{+,l+1}$ and $d_{-,l} + d_{l+1} = d_{-,l+1}$. Then, by the induction hypothesis, we only need to check that for all $v \in T_\eta$ such that $n - d_{-,l+1} < \pi_1(v) \le n - d_{-,l}$ and $\pi_2(v) \le n - d_{+,l}$, we have $a_{\bar{v}} = 0$. For this, we are going to apply Lemma 2.8 in an iterative way. By definition, $v_{\sum_{i=0}^{l} d_i, \eta} = (d_{+,l}, 0)$.

Claim 3. For all $v \in T_\eta$ such that $\pi_1(v) \geq n - d_{-,l+1} + 1$, we have that $\pi_2(v) > \pi_1(v) - d_{+,l} - 1$.

Proof of Claim 3. We proceed to prove Claim 3 by contrapositive. Let $v \in T_\eta$ be such that $\pi_2(v) \leq \pi_1(v) - d_{+,l} - 1$. This implies that $v = (\pi_1(v) - \pi_2(v), 0) + \pi_2(v)(1,1) = v_{j,\eta} + \pi_2(v)(1,1)$, where $\pi_2(v) \leq n - j$, by Definition 2.2. Since $\pi_1(v) - \pi_2(v) \geq d_{+,l} + 1$, by Lemma 2.5 (5), there exists $i < j$ such that $v_{i,\eta} = (d_{+,l} + 1, 0)$. Moreover, by Definition 2.2, $i = \sum_{p=0}^{l+1} d_p + 1$. By Lemma 2.8, we obtain

$$\begin{aligned}\pi_1(v) &= \pi_1(v_{j,\eta} + \pi_2(v)(1,1)) \leq \pi_1(v_{j,\eta} + (n-j)(1,1)) \\ &\leq \pi_1\Big(v_{\sum_{p=0}^{l+1} d_p + 1,\eta} + \Big(n - \sum_{p=0}^{l+1} d_p - 1\Big)(1,1)\Big) \\ &= d_{+,l} + 1 + n - \sum_{p=0}^{l+1} d_p - 1 < n - d_{-,l+1} + 1,\end{aligned}$$

concluding the proof of Claim 3. ∎

For each $s \in \{n - d_{-,l+1} + 1, \ldots, n - d_{-,l}\}$, we define the set $E(s) = \{v \in T_\eta \mid \pi_1(v) = s \text{ and } \pi_2(v) \leq n - d_{+,l}\}$. Notice that, by Claim 3, we have that for each $s \in \{n - d_{-,l+1} + 1, \ldots, n - d_{-,l}\}$, $|E(s)| \leq (n - d_{+,l}) - (s - d_{+,l} - 1) = n - s + 1$. By the induction hypothesis, we are in the conditions of Lemma 2.7 for $s = n - d_{-,l}$. Applying the lemma for each s in a descendant way, we conclude the proof of Claim 1.

In the case $z = 0$, Claim 1 becomes: for each $v \in T_\eta$ such that $\pi_2(v) > n - d_{-,r}$ or $\pi_1(v) > n - d_{+,r}$, we have $a_{\bar{v}} = 0$. The proof of this case is analogous. ∎

2.2. Moving $T_{j,\eta}$ along a diagonal preserves linear independence

Proposition 2.9 shows that $\overline{T}_\eta$ is a basis of $\mathbb{C}^{\lambda_{2,n}}$ for all $\eta \in \Omega$. Our following goal is to show that we can move the set $T_{j,\eta}$ along a diagonal without losing the linear independence for all $j \in \{1, \ldots, n\}$. First we need the following combinatorial identities.

Lemma 2.10 ([19, Chapter 1]). *Given $n, m, p \in \mathbb{N}$, we have the following identities*:

(1) $\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{m-p}$.
(2) $\sum_j (-1)^j \binom{n-j}{m}\binom{p}{j} = \binom{n-p}{m-p} = \binom{n-p}{n-m}$.
(3) $\sum_j \binom{n}{m-j}\binom{p}{j} = \binom{n+p}{m}$.
(4) $\sum_j \binom{n-p}{m-j}\binom{p}{j} = \binom{n}{m}$.

Lemma 2.11. *For all $m \in \mathbb{N}$, we have that $\overline{(m,m)} \in \operatorname{span}_{\mathbb{C}}\{\overline{(1,1)}, \ldots, \overline{(n,n)}\}$.*

Proof. Recalling Notation 1.1, for each $j \in \{1, \ldots, n\}$, consider the vector

$$v_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \overline{(i,i)}.$$

Notice that for all $j \in \{1, \ldots, n\}$, we have $v_j \in \operatorname{span}_{\mathbb{C}}\{\overline{(1,1)}, \ldots, \overline{(n,n)}\}$. We claim that $\overline{(m,m)} = \sum_{j=1}^{n} \binom{m}{j} v_j$. Recall that $\overline{(i,i)} := \left(\binom{(i,i)}{\alpha}\right)_{\alpha \in \Lambda_{2,n}}$. Therefore, we have to prove the identity:

$$\binom{m}{p-q}\binom{m}{q} = \sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m}{j}\binom{j}{i}\binom{i}{q}\binom{i}{p-q},$$

for all $1 \leq p \leq n$ and $0 \leq q \leq p$. By Lemma 2.10 (1), we obtain the identities:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m}{j}\binom{j}{i}\binom{i}{q}\binom{i}{p-q} \\
&\quad = \sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m}{j}\binom{j}{q}\binom{j-q}{i-q}\binom{i}{p-q} \\
&\quad = \sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m}{q}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q} \\
&\quad = \binom{m}{q}\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q}.
\end{aligned}$$

With this, the claim is reduced to prove that

$$\binom{m}{p-q} = \sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q}.$$

Now we have the following identities, where the second identity comes from the reindexing $i \mapsto l - i$, and the fourth identity by Lemma 2.10 (2):

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q} \\
&\quad = \sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}\left(\sum_{i=1}^{j}(-1)^{i}\binom{j-q}{i-q}\binom{i}{p-q}\right) \\
&\quad = \sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}\left(\sum_{i}(-1)^{j-i}\binom{j-i}{p-q}\binom{j-q}{i}\right) \\
&\quad = \sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}(-1)^{j}\left(\sum_{i}(-1)^{i}\binom{j-i}{p-q}\binom{j-q}{i}\right) \\
&\quad = \sum_{j=1}^{n}\binom{m-q}{j-q}\binom{q}{p-j}.
\end{aligned}$$

Finally, we have the following identities, where the first identity comes from replacing j with $j+q$ (we can make this change since for values of l less than q, $\binom{m-q}{j-q}=0$, on the other hand, if $j+q>n>p$, then $\binom{q}{p-j}=0$). The second identity comes from Lemma 2.10 (3):

$$\sum_{j=1}^{n}\binom{m-q}{j-q}\binom{q}{p-j}=\sum_{j=1}^{n}\binom{m-q}{j}\binom{q}{(p-q)-j}=\binom{m}{p-q},$$

proving the claim. ■

Lemma 2.12. *For all $a,r\in\mathbb{N}$ and $l\leq n$, we have that*

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\overline{(a+r+j,r+i+j)}=\bar{0}.$$

Proof. The proof is by induction on r. First, consider $r=0$. We need to show that for all $(p-q,q)\in\Lambda_{2,n}$,

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q}=0.$$

Notice that

$$\begin{aligned}(2.2)\quad &\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q}\\ &=\sum_{i=0}^{l}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j+i}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q}\binom{l}{i}\\ &=\sum_{j=1}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\left(\sum_{i=0}^{l}(-1)^i\binom{i+j}{q}\binom{l}{i}\right).\end{aligned}$$

Now we have the following identity, where the first identity comes from the reindexing $i\mapsto l-i$ and the second comes from Lemma 2.10 (2):

$$\sum_{i=0}^{l}(-1)^i\binom{i+j}{q}\binom{l}{i}=(-1)^l\sum_{i=0}^{l}(-1)^i\binom{l+j-i}{q}\binom{l}{i}=(-1)^l\binom{j}{q-l}.$$

Replacing this identity in the sum (2.2) and using Lemma 2.10 (1), we obtain

$$\begin{aligned}(2.3)\quad &\sum_{j=0}^{n-l+1}(-1)^{n+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{j}{q-l}\\ &=\sum_{j=0}^{n-l+1}(-1)^{n+1+j}\binom{n-l+1}{q-l}\binom{n-q+1}{j-q+l}\binom{a+j}{p-q}\\ &=(-1)^{n+1}\binom{n-l+1}{q-l}\sum_{j=0}^{n-l+1}(-1)^j\binom{n-q+1}{(n-l+1)-j}\binom{a+j}{p-q}.\end{aligned}$$

Replacing j by $n-l+1-j$ on the sum (2.3) and using Lemma 2.10 (2), we obtain that

$$\begin{aligned}&\sum_{j=0}^{n-l+1}(-1)^j\binom{n-q+1}{(n-l+1)-j}\binom{a+j}{p-q}\\&=\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{a+(n-l+1)-j}{p-q}\binom{n-q+1}{j}=\binom{a+q-l}{p-n-1}.\end{aligned}$$

Since $p\leq n$, the claim is true for $r=0$.

Now suppose that the statement is true for $r-1$, i.e.,

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q}=0,$$

and we have to show that

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+r+j}{p-q}\binom{r+i+j}{q}=0,$$

for all $(p-q,q)\in\Lambda_{2,n}$.

Using basic properties of binomial coefficients, we have

$$\begin{aligned}&\binom{a+r+j}{p-q}\binom{r+i+j}{q}\\&=\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q}+\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q-1}\\&\quad+\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q}+\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q-1}.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+r+j}{p-q}\binom{r+i+j}{q}\\&=\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q}\\&+\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q-1}\\&+\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q}\\&+\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1-j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q-1}\\&=0.\end{aligned}$$

Notice that each element of $\{(p-q,q),(p-q,q-1),(p-q-1,q),(p-q-1,q-1)\}$ belongs to $\Lambda_{2,n}$ or has a negative entry. In any case, by the induction hypothesis, each of the four sums are zero, obtaining the result. ■

Corollary 2.13. *For all $a, r \in \mathbb{N}$ and $l \leq n$, we have that*

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\overline{(r+i+j,a+r+j)} = \bar{0}.$$

Proof. We need to prove that

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{r+i+j}{p-q}\binom{a+r+j}{q} = 0,$$

for all $(p-q,q) \in \Lambda_{2,n}$. Notice that, by the definition of $\Lambda_{2,n}$, if $(p-q,q) \in \Lambda_{2,n}$, then $(q, p-q) \in \Lambda_{2,n}$. With this and the previous lemma, we obtain the result. ■

Definition 2.14. Let $\eta \in \Omega$, $\{v_{i,\eta}\}_{i=1}^n$ and $\{T_{i,\eta}\}_{i=1}^n \subset \mathbb{N}^2$ as in Definition 2.2. We define the set

$$T_{i,\eta} + r\vec{1} := \{v + (r,r) \mid v \in T_{i,\eta}\},$$

for all $i \in \{1,\ldots,n\}$ and $r \in \mathbb{N}$.

Now we are ready to show the other important result of this section. As we mentioned before, the goal is to show that we can move the sets $T_{j,\eta}$ along a diagonal without losing the linear independence. We are going to prove this with some additional properties.

Proposition 2.15. *Let $\eta \in \Omega$, $l \in \{1,\ldots,n\}$ and $(r_1,\ldots,r_l) \in \mathbb{N}^l$. Then we have*

$$\operatorname{span}_{\mathbb{C}}\left\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \;\middle|\; v \in T_{0,\eta} \cup \bigcup_{i=1}^{l}(T_{i,\eta} + r_i\vec{1})\right\} = \operatorname{span}_{\mathbb{C}}\left\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \;\middle|\; v \in \bigcup_{i=0}^{l} T_{i,\eta}\right\}.$$

In particular, $\overline{v_{l,\eta} + (r,r)} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \bigcup_{i=0}^{l} T_{i,\eta}\}$, for all $r \in \mathbb{N}$.

Proof. Consider the notation $\varsigma(T) = \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T\}$, where $T \subset \mathbb{C}^2$.

Let $\eta \in \Omega$. The proof is by induction on l. Consider $l = 1$. There are two cases, $v_{1,\eta} = (1,0)$ or $v_{1,\eta} = (0,1)$. Suppose that $v_{1,\eta} = (1,0)$. Consider the sums

$$f_{0,r} = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\overline{(1+r+j,r+j)},$$
$$f_{1,r} = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\overline{(1+r+j,1+r+j)}.$$

Applying Lemma 2.12 for $a = 1$ and $l = 1$, we obtain that

$$f_{1,r} - f_{0,r} = \bar{0} \quad \text{for all } r \in \mathbb{N}. \tag{2.4}$$

By Lemma 2.11, we have that

$$\{\overline{(1+r+j, 1+r+j)}\}_{j=0}^{n} \subset \varsigma(T_{0,\eta}),$$

for all $r, j \in \mathbb{N}$. In particular, $f_{1,r} \in \varsigma(T_{0,\eta})$. Moreover, since $v_{1,\eta} = (1,0)$, for $r = 0$, we have that $f_{0,0} - \overline{(1+n, n)} \in \varsigma(T_{1,\eta})$. Then

$$\overline{(1+n, n)} = f_{1,0} - f_{0,0} + \overline{(1+n, n)} \in \varsigma(T_{1,\eta}).$$

Notice that the coefficient of $\overline{(1,0)}$ is not zero. By elementary results of linear algebra, we have that

$$\varsigma(T_{0,\eta} \cup T_{1,\eta}) = \varsigma(\{T_{0,\eta} \cup T_{1,\eta}\} \setminus \{\overline{(1,0)}\} \cup \{\overline{(1+n, n)}\}) = \varsigma(T_{0,\eta} \cup (T_{1,\eta} + 1\vec{1})).$$

Applying the same argument for $r = 1$ in (2.4), we obtain that

$$\begin{aligned}\varsigma(T_{0,\eta} \cup (T_{1,\eta} + 1\vec{1})) &= \varsigma\big(\{T_{0,\eta} \cup (T_{1,\eta} + 1\vec{1})\} \setminus \{\overline{(2,1)}\} \cup \{\overline{(2+n, 1+n)}\}\big) \\ &= \varsigma(T_{0,\eta} \cup (T_{1,\eta} + 2\vec{1})).\end{aligned}$$

Repeating the argument r_1 times for each r and putting together all the identities, we obtain that

$$\varsigma(T_{0,\eta} \cup T_{1,\eta}) = \varsigma(T_{0,\eta} \cup (T_{1,\eta} + r_1\vec{1})).$$

This completes the proof for $l = 1$ and $v_{1,\eta} = (1,0)$. For $v_{1,\eta} = (0,1)$, the proof is analogous using Corollary 2.13.

Now suppose that the statement is true for $l-1$ and let $(r_1, \dots, r_l) \in \mathbb{N}^l$. We claim that

$$\varsigma\Big(\bigcup_{i=0}^{l-1} T_{i,\eta} \cup (T_{l,\eta} + r_l\vec{1})\Big) = \varsigma\Big(\bigcup_{i=0}^{l} T_{i,\eta}\Big).$$

Assume this claim for the moment. By the induction hypothesis, we have that

$$\varsigma\Big(T_{0,\eta} \cup \bigcup_{i=1}^{l-1} (T_{i,\eta} + r_i\vec{1})\Big) = \varsigma\Big(\bigcup_{i=0}^{l-1} T_{i,\eta}\Big).$$

This implies that

$$\varsigma\Big(T_{0,\eta} \cup \bigcup_{i=1}^{l} (T_{i,\eta} + r_i\vec{1})\Big) = \varsigma\Big(\bigcup_{i=0}^{l} T_{i,\eta}\Big).$$

Now we proceed to prove the claim. There are two cases, $v_{l,\eta} = (a,0)$ or $v_{l,\eta} = (0,a)$, where $0 < a \le l$ by Lemma 2.5 (3). Suppose that $v_{l,\eta} = (a,0)$. For each $i \in \{0, \dots, l\}$ and $r \in \mathbb{N}$, consider the sum

$$f_{i,r} = \sum_{j=0}^{n-l+1} (-1)^{n-l+1+j} \binom{n-l+1}{j} \overline{(a+r+j, i+r+j)}.$$

Applying Lemma 2.12 for l and a, we have that

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}f_{i,r}=\bar{0}.$$

By Lemma 2.5 (6), we have that

$$\{(a-1,0),\dots,(1,0),(0,1),\dots,(0,l-a)\}=\{v_{i,\eta}\}_{i=1}^{l-1}.$$

Notice that if $i=a$, then $\overline{(a+r+j)(1,1)}\in\operatorname{span}_{\mathbb{C}}\{\bar{v}\subset\mathbb{C}^{\lambda_{2,n}}\mid v\in T_{0,\eta}\}$ by Lemma 2.11. If $1\leq i<a$, then

$$(a-i,0)+(i+r+j,i+r+j)=(a+r+j,i+r+j),$$

and if $a<i\leq l$, then

$$(0,i-a)+(a-i,a-i)+(i+r+j,i+r+j)=(a+r+j,i+r+j).$$

By the induction hypothesis, we obtain that

$$\{\overline{(a+r+j,i+r+j)}\}_{j=0}^{n-l+1}\subset\varsigma\Big(\bigcup_{i=0}^{l-1}T_{i,\eta}\Big),$$

for all $i\in\{1,\dots,l\}$, $r\in\mathbb{N}$. In particular, $f_{i,r}\in\varsigma(\bigcup_{i=0}^{l-1}T_{i,\eta})$ for all $i\in\{1,\dots,l\}$. Moreover, since $v_{l,\eta}=(a,0)$, for $r=0$, we have that $f_{0,0}-\overline{(a+n-l+1,n-l+1)}\in\varsigma(T_{l,\eta})$. Then

$$\begin{aligned}&\overline{(a+n-l+1,n-l+1)}\\&=-\Big(\sum_{i=0}^{l}(-1)^i\binom{l}{i}f_{i,0}\Big)+\overline{(a+n-l+1,n-l+1)}\in\varsigma\Big(\bigcup_{i=0}^{l}T_{i,\eta}\Big).\end{aligned}$$

Applying the same argument as in the case $l=1$, we obtain

$$\begin{aligned}\varsigma\Big(\bigcup_{i=0}^{l}T_{i,\eta}\Big)&=\varsigma\Big(\bigcup_{i=0}^{l-1}T_{i,\eta}\cup(T_{l,\eta}+1\vec{1})\Big)\\&=\varsigma\Big(\bigcup_{i=0}^{l-1}T_{i,\eta}\cup(T_{l,\eta}+2\vec{1})\Big)\\&\ \vdots\\&=\varsigma\Big(\bigcup_{i=0}^{l-1}T_{i,\eta}\cup(T_{l,\eta}+r_l\vec{1})\Big).\end{aligned}$$

Now suppose that $v_{l,\eta}=(0,a)$. In this case we have that

$$\{(0,a-1),\dots,(0,1),(1,1),(1,0),\dots,(l-a,0)\}=\{v_{i,\eta}\}_{i=0}^{l-1},$$

obtaining

$$\{\overline{(i+r+j,a+r+j)}\}_{j=0}^{n-l+1}\in\varsigma(T_{i,\eta}).$$

The proof is analogous using Corollary 2.13. ∎

3. Proof of Theorem 1.8

In this section we give the proof of the main theorem. We first associate to each $\eta \in \Omega$ a unique $J_\eta \in S_{A_n}$ with certain properties. Secondly, we construct a distinguished element J_{η_k} for each $k \in \{1, \ldots, n\}$ and prove that there exists another element $J_\eta \in S_{A_n}$ with the same value with respect to an order function. Finally, we prove that J_{η_k} is minimal in S_{A_n} with respect to the previous function.

Definition 3.1. Let $\eta \in \Omega$, $\{v_{i,\eta}\}_{i=1}^n$ and $\{T_{i,\eta}\}_{i=1}^n \subset \mathbb{N}^2$ as in Definition 2.2. Consider $r_{i,\eta} := n \cdot \pi_2(v_{i,\eta})$ for all $i \in \{1, \ldots, n\}$. We define the set

$$T'_\eta := T_{0,\eta} \cup \bigcup_{i=1}^{n} (T_{i,\eta} + r_{i,\eta}\vec{1}),$$

where $T_{i,\eta} + r_{i,\eta}\vec{1}$ comes from Definition 2.14.

Example 3.2. Let $n = 6$ and $\eta = (1, 0, 1, 1, 1, 1, 2)$. By Definition 2.2, we have that $v_{1,\eta} = (1, 0)$, $v_{2,\eta} = (0, 1)$, $v_{3,\eta} = (2, 0)$, $v_{4,\eta} = (0, 2)$, $v_{5,\eta} = (3, 0)$ and $v_{6,\eta} = (4, 0)$. By definition, we obtain that $r_{1,\eta} = 0$, $r_{2,\eta} = 6$, $r_{3,\eta} = 0$, $r_{4,\eta} = 12$, $r_{5,\eta} = 0$ and $r_{6,\eta} = 0$. Thus,

$$T'_\eta = T_{0,\eta} \cup T_{1,\eta} \cup (T_{2\eta} + 6\vec{1}) \cup T_{3,\eta} \cup (T_{4,\eta} + 12\vec{1}) \cup T_{5,\eta} \cup T_{6,\eta},$$

where

$$T_{2,\eta} + 6 = \{(6, 7), (7, 8), (8, 9), (9, 10), (10, 11)\},$$
$$T_{4,\eta} + 12 = \{(12, 14), (13, 15), (14, 16)\}.$$

Remark 3.3. Recall Notation 1.1. Let $\beta, \beta' \in \Lambda_{3,n}$ be such that $\beta \neq \beta'$. Then $A_n\beta \neq A_n\beta'$. This is a consequence of $|\beta| \leq n$ and $|\beta'| \leq n$ and the fact that the smallest relation among $(1, 0)$, $(1, 1)$ and $(n, n + 1)$ is $(n + 1)(1, 1) = 1(1, 0) + 1(n, n + 1)$.

Proposition 3.4. *For each $\eta \in \Omega$, there exists a unique $J_\eta \subset \Lambda_{3,n}$ such that*

$$A_n \cdot J_\eta := \{A_n \cdot \beta \in \mathbb{N}^2 \mid \beta \in J_\eta\} = T'_\eta.$$

Moreover, $J_\eta \in S_{A_n}$.

Proof. We need to show that for each $v \in T'_\eta$, there exists a unique element $\beta \in \Lambda_{3,n}$ such that $A_n\beta = v$. The uniqueness comes from Remark 3.3.

Now, let $v \in T'_\eta$. Then $v \in T_{\eta,0}$ or $v \in \bigcup_{i=1}^n T_{\eta,i} + r_{\eta,i}\vec{1}$. For the first case we have that $v = (t, t)$ with $t \leq n$. In this case we take $\beta = (0, t, 0)$. For the second case we have

$$v = v_{i,\eta} + (s, s) + r_{i,\eta}(1, 1) = v_{i,\eta} + (s, s) + n\pi_2(v_{i,\eta})(1, 1),$$

where $s \leq n - i$. By Definition 2.2, $v_{i,\eta} = (q, 0)$ or $v_{i,\eta} = (0, q)$, where $q \leq i$. Then

$$v = (q + s, s) \quad \text{or} \quad v = (nq + s, (n + 1)q + s).$$

For these, we take $\beta = (q, s, 0)$ and $\beta = (0, s, q)$, respectively. Using the previous inequalities, we obtain that $\beta \in \Lambda_{3,n}$.

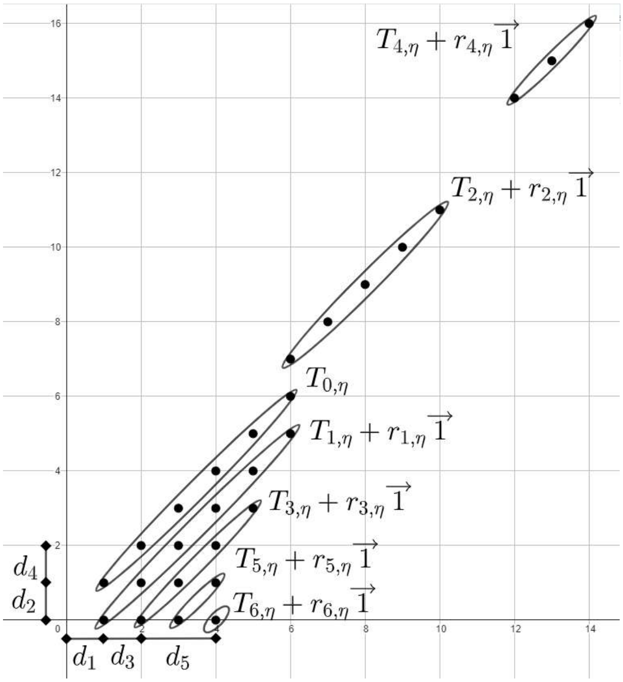

Figure 2. Example of T'_η, with $\eta = (1, 0, 1, 1, 1, 1, 2)$.

Now we have to see that $J_\eta \in S_{A_n}$. Since $\lambda_{2,n} = |T_\eta| = |T'_\eta| = |J_\eta|$, we only have to see that $\det L^c_{J_\eta} \neq 0$. Let $\{\beta_1, \beta_2, \dots, \beta_{\lambda_{2,n}}\} = J_\eta$ be such that $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_{\lambda_{2,n}}$, where $\prec$ denotes the lexicographic order. Notice that if $\beta' < \beta$ (see Notation 1.1), then $\beta' \prec \beta$. By the definition of $L^c_{J_\eta}$, we need to check that

$$\det\left(\sum_{\gamma \leq \beta_i} (-1)^{|\beta_i - \gamma|}\binom{\beta_i}{\gamma}\overline{A_n\gamma}\right)_{1\leq i\leq \lambda_{2,n}} \neq 0.$$

For this, we will show below that we can transform the previous matrix into $(\overline{A\beta_i})_{1\leq i\leq\lambda_{2,n}}$ using elementary row operations. This implies the result since we have that $A_nJ_\eta = T'_\eta$ and $\{\bar{v} \in \mathbb{Z}^{\lambda_{2,n}} \mid v \in T'_\eta\}$ is linearly independent by Propositions 2.9 and 2.15.

Fix the $\lambda_{2,n}$-row. Consider $\beta_{\lambda_{2,n}} \succ \gamma_{1,\lambda_{2,n}} \succ \cdots \succ \gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}$, where $\{\gamma_{i,\lambda_{2,n}}\}_{i=1}^{\lambda_{2,n}} = \{\gamma \in \Lambda_{3,n} \mid \gamma < \beta_{\lambda_{2,n}}\}$. We can write this row as the sum

$$\overline{A_n\beta_{\lambda_{2,n}}} + (-1)^{|\beta_{\lambda_{2,n}} - \gamma_{1,\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{1,\lambda_{2,n}}}\overline{A_n\gamma_{1,\lambda_{2,n}}} + \cdots + (-1)^{|\beta_{\lambda_{2,n}} - \gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}}\overline{A_n\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}}.$$

Since $A_n\beta_{\lambda_{2,n}} \in T'_\eta$, we have that $\beta_{\lambda_{2,n}}$ have the shape $(q,s,0)$ or $(0,s,q)$, with $s+q \le n$ and $A_n\beta_{\lambda_{2,n}}$ equals one of $(q+s,s)$ or $(nq+s,(n+1)q+s)$. Since $\gamma_{1,\lambda_{2,n}} < \beta_{\lambda_{2,n}}$, we obtain that $\gamma_{1,\lambda_{2,n}}$ have the shape $(q',s',0)$ or $(0,s',q')$, with $s' < s$ or $q' < q$. Thus, $A_n\gamma_{1,\lambda_{2,n}}$ have the shape $(q'+s',s')$ or $(nq'+s',(n+1)q'+s')$. In any case, we have that $A_n\gamma_{1,\lambda_{2,n}} \in T'_\eta$.

By the first part of the proposition, we have that $\gamma_{1,\lambda_{2,n}} = \beta_i$, for some $i < \lambda_{2,n}$. Then we subtract $(-1)^{|\beta_{\lambda_{2,n}}-\gamma_{1,\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{1,\lambda_{2,n}}}$-times the row i to the row $\lambda_{2,n}$ in the matrix $L^c_{J_\eta}$. Notice that if $\gamma < \beta_i$, we have $\gamma < \beta_{\lambda_{2,n}}$. Thus, we obtain that

$$\overline{A_n\beta_{\lambda_{2,n}}} + c_2\overline{A_n\gamma_{2,\lambda_{2,n}}} + \cdots + c_{r_{\lambda_{2,n}}}\overline{A_n\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}}$$

is the new $\lambda_{2,n}$-row, for some constants $\{c_2,\dots,c_{r_{\lambda_{2,n}}}\} \subset \mathbb{Z}$. Applying the same argument for each $\gamma_{i,\lambda_{2,n}}$ in a increasing way, we turn the $\lambda_{2,n}$-th row into $\overline{A_n\beta_{\lambda_{2,n}}}$.

Applying this process to the other rows of L^c_J in an ascending way, we obtain the matrix $(\overline{A\beta_i})_{1\le i\le\lambda_{2,n}}$. ∎

3.1. A distinguished element of S_{A_n}

Definition 3.5. Let $n \in \mathbb{N}\setminus\{0\}$, $1 \le k \le n$, $d_{k,0} = 0$ and consider $f_k\colon \mathbb{N}^2 \to \mathbb{Z}$ from Definition 1.7. If $f_k((1,0)) \le f_k((n,n+1))$, we take $z_k = 1$. If $f_k((n,n+1)) < f_k((1,0))$, we take $z_k = 0$.

Now, we define $d_{k,l}$ for $l > 0$ in an iterative way. Let

$$d_{k,l} = \min\Big\{n - \sum_{j=0}^{l-1} d_{k,j},\, t_l - s_l\Big\},$$

where

$$t_l = \begin{cases} \max\Big\{m \in \mathbb{N} \mid m\cdot f_k((1,0)) \le f_k\Big(\Big(\sum_{j\text{ even}}^{l-1} d_{k,j}+1\Big)(n,n+1)\Big)\Big\} & \text{if } z_k = 1 \text{ and } l \text{ odd},\\ \max\Big\{m \in \mathbb{N} \mid m\cdot f_k((n,n+1)) \le f_k\Big(\Big(\sum_{j\text{ odd}}^{l-1} d_{k,j}+1\Big)(1,0)\Big)\Big\} & \text{if } z_k = 1 \text{ and } l \text{ even},\\ \max\Big\{m \in \mathbb{N} \mid m\cdot f_k((n,n+1)) \le f_k\Big(\Big(\sum_{j\text{ even}}^{l-1} d_{k,j}+1\Big)(1,0)\Big)\Big\} & \text{if } z_k = 0 \text{ and } l \text{ odd},\\ \max\Big\{m \in \mathbb{N} \mid m\cdot f_k((1,0)) \le f_k\Big(\Big(\sum_{j\text{ odd}}^{l-1} d_{k,j}+1\Big)(n,n+1)\Big)\Big\} & \text{if } z_k = 0 \text{ and } l \text{ even}, \end{cases}$$

and

$$s_l = \begin{cases} 0 & \text{if } l = 1,\\ \sum_{j\text{ odd}}^{l-1} d_{k,j} & \text{if } l \text{ odd and } l > 1,\\ \sum_{j\text{ even}}^{l-1} d_{k,j} & \text{if } l \text{ even.} \end{cases}$$

If $\sum_{j=1}^{l} d_{k,j} < n$, we define $d_{k,l+1}$. Otherwise, we finish the process and we define $\eta_k = (z_k, d_{k,0}, \ldots, d_{k,r})$.

Example 3.6. Let $n = 6$ and $k = 3$. We have that $d_{3,0} = 0$. On the other hand, we have

$$f_3((1,0)) = 3 < 4 = f_3((6,7)).$$

Then $z_3 = 1$. For $l = 1$, we have that

$$t_1 = \max\{m \in \mathbb{N} \mid m \cdot 3 = m \cdot f_3((1,0)) \leq f_3((6,7)) = 4\} = 1,$$

and $s_1 = 0$. Then

$$d_{3,1} = \min\{6, 1 - 0\} = 1.$$

Now we compute $d_{3,2}$. By definition,

$$t_2 = \max\{m \in \mathbb{N} \mid m \cdot 4 = m \cdot f_3((6,7)) \leq f_5(2(1,0)) = 6\} = 1,$$

and $s_2 = 0$. This implies that

$$d_{3,2} = \min\{6 - 1 = 5, 1 - 0\} = 1.$$

In an analogous way we obtain that $d_{3,3} = 1$ and $d_{3,4} = 1$. Now we compute $d_{3,5}$. We have that

$$t_5 = \max\{m \in \mathbb{N} \mid m \cdot 3 = m \cdot f_3((1,0)) \leq f_5(3(6,7)) = 12\} = 4,$$

and $s_3 = d_{3,1} + d_{3,3} = 2$. Then

$$d_{3,5} = \min\{6 - 1 - 1 - 1 - 1 = 2, 4 - 2 = 2\} = 2.$$

Since $n - \sum_{j=0}^{3} d_{k,j} = 0$, we completed the process. Thus, $\eta_3 = (1,0,1,1,1,1,2)$.

Lemma 3.7. *Let $1 \leq k \leq n$ and η_k be as in Definition 3.5. Then we have the following properties*:

(1) $d_{k,l} > 0$ *for all* $l \in \{1, \ldots, r\}$. *In particular,* $\eta_k \in \Omega$.

(2) *For each* $i \in \{1, \ldots, n\}$, *we let* $l \in \{1, \ldots, r\}$ *be the unique element such that* $\sum_{j=0}^{l-1} d_{k,j} < i \leq \sum_{j=1}^{l} d_{k,j}$. *Then we have the following inequalities*:

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k\Big(\Big(\sum_{j \text{ even}}^{l} d_{k,j} + 1\Big)(n, n+1)\Big) \quad \text{if } z_k = 1 \text{ and } l \text{ odd},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k\Big(\Big(\sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big)(1,0)\Big) \quad \text{if } z_k = 1 \text{ and } l \text{ even},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k\Big(\Big(\sum_{j \text{ even}}^{l} d_{k,j} + 1\Big)(1,0)\Big) \quad \text{if } z_k = 0 \text{ and } l \text{ odd},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k\Big(\Big(\sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big)(n, n+1)\Big) \quad \text{if } z_k = 0 \text{ and } l \text{ even}.$$

(3) *Let* $i', i \in \mathbb{N} \setminus \{0\}$ *be such that* $\sum_{j=0}^{l-1} d_{k,j} < i < i' \leq \sum_{j=0}^{l} d_{k,j}$*, for some* $l \in \{1, \ldots, r\}$*. Then*

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k(v_{i',\eta_k}) + r_{i',\eta_k}.$$

(4) *For all* $1 \leq i < i' \leq n$*, we have that*

$$f_k(v_{i,\eta_k} + r_{i,\eta_k}(1,1)) \leq f_k(v_{i',\eta_k} + r_{i',\eta_k}(1,1)).$$

(5) *If* $l > 2$ *and* $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1))$*, then we have* $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) \geq f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1,1)) + 2$.

Proof. (1) By construction $n - \sum_{j=0}^{l-1} d_{k,j} > 0$. Then, by Definition 3.5, we only have to check that $t_l - s_l > 0$. Notice that, by definition, $t_1 > 0$ and $s_1 = 0$. This implies that the statement is true for $l = 1$. Now suppose that $l > 1$. We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Consider $z_k = 1$ and l odd. By the definition of t_{l-1},

$$f_k((t_{l-1} + 1)(n, n+1)) > f_k\Big(\Big(\sum_{j \text{ odd}}^{l-2} d_{k,j} + 1\Big)(1,0)\Big).$$

Since l is odd, $\sum_{j \text{ odd}}^{l-2} d_{k,j} = \sum_{j \text{ odd}}^{l-1} d_{k,j}$. It follows that

$$f_k\Big(\Big(\sum_{j \text{ odd}}^{l-2} d_{k,j} + 1\Big)(1,0)\Big) = f_k\Big(\Big(\sum_{j \text{ odd}}^{l-1} d_{k,j} + 1\Big)(1,0)\Big) = f_k((s_l + 1)(1,0)).$$

On the other hand, notice that if $d_{k,l-1} = n - \sum_{j=0}^{l-2} d_{k.j}$, then $n = \sum_{j=0}^{l-1} d_{k,j}$, and so there is no $d_{k,l}$, which is a contradiction. This implies that $d_{k,l-1} = t_{l-1} - s_{l-1}$. Thus,

$$f_k((t_{l-1} + 1)(n, n+1)) = f_k((d_{k,l-1} + s_{l-1} + 1)(n, n+1)).$$

Since $l - 1$ is even and $s_{l-1} = \sum_{j \text{ even}}^{l-2} d_{k,j}$, we have that $d_{k,l-1} + s_{l-1} = \sum_{j \text{ even}}^{l-1} d_{k,j}$. Then

$$f_k\Big(\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + 1\Big)(n, n+1)\Big) > f_k((s_l + 1)(1,0)).$$

By the definition of t_l, we obtain that $t_l \geq s_l + 1$, and so $t_l - s_l > 0$. The other three cases are analogous.

(2) Let $i \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, r\}$. We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Suppose that $z_k = 1$ and l odd. In this case, by definition, $v_{i,\eta_k} = (\sum_{j \text{ odd}}^{l-1} d_{k,j} + c)(1,0)$ with $c \leq d_{k,l}$, $r_{i,\eta_k} = 0$ and $s_l = \sum_{j \text{ odd}}^{l-1} d_{k,j}$. Then

$$\sum_{j \text{ odd}}^{l-1} d_{k,j} + c \leq \sum_{j \text{ odd}}^{l-1} d_{k,j} + d_{k,l} = s_l + d_{k,l} \leq t_l.$$

By the definition of t_l, we have that

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} = f_k\Big(\Big(\sum_{j \text{ odd}}^{l-1} d_{k,j} + c\Big)(1,0)\Big) \le f_k\Big(\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + 1\Big)(n, n+1)\Big).$$

Since l is odd, we have that $\sum_{j \text{ even}}^{l-1} d_{k,j} = \sum_{j \text{ even}}^{l} d_{k,j}$. This implies the inequality that we need.

Now suppose that $z_k = 1$ and l even. In this case, by the definition of η_k, we have that $v_{i,\eta_k} = (\sum_{j \text{ even}}^{l-1} d_{k,j} + c)(0,1)$, with $c \le d_{k,l}$, $r_{i,\eta} = n(\sum_{j \text{ even}}^{l-1} d_{k,j} + c)$ and $s_l = \sum_{j \text{ even}}^{l-1} d_{k,j}$. Using the above and the linearity of f_k, we obtain

$$\begin{aligned} f_k(v_{i,\eta_k}) + r_{i,\eta_k} &= f_k(v_{i,\eta_k}) + f_k(r_{i,\eta_k}(1,1)) = f_k(v_{i,\eta_k} + r_{i,\eta_k}(1,1)) \\ &= f_k\Big(\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c\Big)(n, n+1)\Big). \end{aligned}$$

Since

$$\sum_{j \text{ even}}^{l-1} d_{k,j} + c \le \sum_{j \text{ even}}^{l-1} d_{k,j} + d_{k,l} \le t_l,$$

by the definition of t_l, we obtain the inequality

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} = f_k\Big(\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c\Big)(n, n+1)\Big) \le f_k\Big(\Big(\sum_{j \text{ odd}}^{l-1} d_{k,j} + 1\Big)(1,0)\Big).$$

Since l is even, we have that $\sum_{j \text{ odd}}^{l-1} d_{k,j} = \sum_{j \text{ odd}}^{l} d_{k,j}$, obtaining the result.

The other two cases are analogous.

(3) The hypothesis implies that $i = \sum_{j=0}^{l-1} d_{k,j} + c_i$ and $i' = \sum_{j=0}^{l-1} d_{k,j} + c_{i'}$, where $0 < c_i < c_{i'} \le d_{k,l}$. We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Consider $z_k = 1$ and l even. By Definition 2.2, $v_{i,\eta_k} = (0, \sum_{j \text{ even}}^{l-1} d_{k,j} + c_i)$ and $v_{i',\eta_k} = (0, \sum_{j \text{ even}}^{l-1} d_{k,j} + c_{i'})$. Then

$$\begin{aligned} f_k(v_{i,\eta_k}) + r_{i,\eta_k} &= (1-k)\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c_i\Big) + n\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c_i\Big) \\ &= (n-k+1)\Big(\sum_{j \text{ even}}^{l-1} d_{k,j}\Big) + (n-k+1)c_i \\ &\le (n-k+1)\Big(\sum_{j \text{ even}}^{l-1} d_{k,j}\Big) + (n-k+1)c_{i'} \\ &= (1-k)\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c_{i'}\Big) + n\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + c_{i'}\Big) \\ &= f_k(v_{i',\eta_k}) + r_{i',\eta_k}. \end{aligned}$$

The other three cases are analogous.

(4) We fix in this proof the auxiliary notation $\delta_l = \sum_{j=0}^{l} d_{k,j}$, for all $l \in \mathbb{N}$.

Let $1 \le i < i' \le n$. Let $1 \le l \le l' \le r$ be such that $i = \delta_{l-1} + c_i$ and $i' = \delta_{l'-1} + c_{i'}$, where $0 < c_i \le d_{k,l}$ and $0 < c_{i'} \le d_{k,l'}$. By hypothesis, we have that $l \le l'$. If $l = l'$, the result follows from (3). Suppose that $l < l'$, this implies that $l' = l + c$ with $c > 0$.

We have four cases ($z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 0$ and l even). Consider $z_k = 0$ and l even. By Definition 2.2, we have that $v_{i,\eta_k} = (\sum_{j \text{ even}}^{l-1} d_{k,j} + c_i, 0)$. By (2), we have that

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \le f_k\Big(\Big(\sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big)(n, n+1)\Big).$$

On the other hand, by Definition 2.2, we obtain that $v_{\delta_l+1,\eta_k} = (0, \sum_{j \text{ odd}}^{l} d_{k,j} + 1)$. Then

$$\begin{aligned} f_k(v_{\delta_l+1,\eta_k}) + r_{\delta_l+1,\eta_k} &= f_k\Big(\Big(0, \sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big) + n \cdot \Big(\sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big)(1,1)\Big) \\ &= f_k\Big(\Big(\sum_{j \text{ odd}}^{l} d_{k,j} + 1\Big)(n, n+1)\Big) \ge f_k(v_{i,\eta_k}) + r_{i,\eta_k}. \end{aligned}$$

Now, if $c > 1$, by (2) and knowing that $l+1$ is odd, we obtain that

$$f_k(v_{\delta_l+1,\eta_k}) + r_{\delta_l+1,\eta_k} \le f_k\Big(\Big(\sum_{j \text{ even}}^{l+1} d_{k,j} + 1\Big)(1,0)\Big).$$

Using Definition 2.2, we have the vector $v_{\delta_{l+1}+1,\eta_k} = (\sum_{j \text{ even}}^{l+1} d_{k,j} + 1, 0)$. Then

$$\begin{aligned} f_k(v_{\delta_l+1,\eta_k}) + r_{\delta_l+1,\eta_k} &\le f_k\Big(\Big(\sum_{j \text{ even}}^{l+1} d_{k,j} + 1\Big)(1,0)\Big) \\ &= f_k(v_{\delta_{l+1}+1,\eta_k}) = f_k(v_{\delta_{l+1}+1,\eta_k}) + r_{\delta_{l+1}+1,\eta_k}. \end{aligned}$$

Repeating this argument c times, we obtain

$$\begin{aligned} f_k(v_{i,\eta_k}) + r_{i,\eta_k} &\le f_k(v_{\delta_l+1,\eta_k}) + r_{\delta_l+1,\eta_k} \\ &\le f_k(v_{\delta_{l+1}+1,\eta_k}) + r_{\delta_{l+1}+1,\eta_k} \\ &\vdots \\ &\le f_k(v_{\delta_{l'-1}+1,\eta_k}) + r_{\delta_{l'-1}+1,\eta_k} \\ &\le f_k(v_{i',\eta_k}) + r_{i',\eta_k}, \end{aligned}$$

where the last inequality comes from (3). The other cases are analogous.

(5) Notice that if $k = n$, $f_n(t(n, n+1)) = t$ for all $t \in \{1, \dots, n\}$ and $f_n((1,0)) = n$. Then, by Definition 3.5, $\eta_n = (0, 0, n)$, i.e., $v_{j,\eta_n} = (0, j)$ for all $j \in \{1, \dots, n\}$. In particular, $f_n(v_{i,\eta_n} + r_{i,\eta_n}(1,1)) < f_n(v_{j,\eta_n} + r_{j,\eta_n}(1,1))$ if $1 \le i < j \le n$. Then we

cannot have the conditions of the lemma. Analogously, if $k = 1$, $\eta_1 = (1, 0, n)$, and $f_1(v_{i,\eta_1} + r_{i,\eta_1}(1, 1)) < f_1(v_{j,\eta_1} + r_{j,\eta_1}(1, 1))$, for all $1 \leq i < j \leq n$. This implies that if there exists $l \in \{1, \dots, n\}$ such that $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1, 1))$, then we have that $k \in \{2, \dots, n-1\}$.

Now, we suppose that there exists $l \in \{1, \dots, n\}$ such that $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1, 1))$. If $v_{l,\eta_k} = (0, s)$ and $v_{l-1,\eta_k} = (0, s-1)$, then

$$\begin{aligned} f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) &= s(n-k+1) \\ &> (s-1)(n-k+1) \\ &= f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1, 1)). \end{aligned}$$

In an analogous way, we obtain a contradiction if $v_{l,\eta_k} = (t, 0)$ and $v_{l-1,\eta_k} = (t-1, 0)$. This implies that $v_{l,\eta_k} = (t, 0)$ and $v_{l-1,\eta_k} = (0, s)$ or $v_{l,\eta_k} = (0, s)$ and $v_{l-1,\eta_k} = (t, 0)$. Consider the first case, the other case is analogous. By definition,

$$f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) = f_k((t, 0)) = f_k((0, s) + (ns, ns)), \tag{3.1}$$

By Lemma 2.5 (5), we deduce that $v_{l-2,\eta_k} = (0, s-1)$ or $v_{l-2,\eta_k} = (t-1, 0)$. Suppose that $v_{l-2,\eta_k} = (0, s-1)$. Then we have

$$\begin{aligned} f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) &= f_k((0, s) + (ns, ns)) \\ &= f_k(s(n, n+1)) \\ &= s(n-k+1) \\ &= (s-1)(n-k+1) + n - k + 1 \\ &= f_k((s-1)(n, n+1)) + n - k + 1 \\ &\geq f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1, 1)) + 2, \end{aligned}$$

where the first equality comes from equation (3.1) and the last inequality comes from $k \leq n-1$.

Now suppose that $v_{l-2,\eta_k} = (t-1, 0)$. In an analogous way, we obtain that

$$\begin{aligned} f_k(v_{l,\eta_k} + r_{l,\eta_k}(1, 1)) = f_k((t, 0)) &= k(t-1) + k \\ &\geq f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1, 1)) + 2, \end{aligned}$$

where the first equality comes from equation (3.1) and the last inequality comes from $k \geq 2$. Obtaining the result. ∎

The previous lemma will be constantly used in the rest of the section.

Proposition 3.8. *Let $k \in \{1, \dots, n\}$. Let $\eta_k \in \Omega$ be from Definition 3.5. Then $v_{n,\eta_k} = (0, k)$ or $v_{n,\eta_k} = (n-k+1, 0)$.*

Proof. Suppose the statement is not true. By definition, we have that $\sum_{j=0}^{r} d_{k,j} = n$. Using Lemma 2.5 and since v_{n,η_k} is not $(n-k+1, 0)$ or $(0, k)$, we have that there exists $m < n$ such that $v_{m,\eta_k} = (n-k+1, 0)$ or $v_{m,\eta_k} = (0, k)$. Let $l \leq r$ be such that $m = \sum_{j=0}^{l-1} d_{k,j} + c$ and $0 < c \leq d_{k,l}$.

We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 0$ and l even. Consider $z = 1$ and l odd. In this case, by Definition 2.2, we have that $v_{m,\eta_k} = (\sum_{j \text{ odd}}^{l-1} d_{k,j} + c, 0) = (n-k+1, 0)$. Hence, $\sum_{j \text{ odd}}^{l-1} d_{k,j} + c = n-k+1$. Then

$$n-k+1 = m - \sum_{j \text{ even}}^{l-1} d_{k,j} < n - \sum_{j \text{ even}}^{l-1} d_{k,j}.$$

This implies that $\sum_{j \text{ even}}^{l-1} d_{k,j} + 1 < k$. Thus,

$$\begin{aligned} f_k\Big(\Big(\sum_{j \text{ even}}^{l-1} d_{k,j} + 1\Big)(n, n+1)\Big) &< f_k(k(n,n+1)) = k(n-k+1) \\ &= f_k((n-k+1, 0)) = f_k(v_{m,\eta_k}) + r_{m,\eta_k}. \end{aligned}$$

This is a contradiction to Lemma 3.7 (2).

Now, suppose that $z_k = 1$ and l even. For this case, by Definition 2.2, we have that $v_{m,\eta_k} = (0, k)$ and $k = \sum_{j \text{ even}}^{l-1} d_{k,j} + c$. Then

$$k = \sum_{j \text{ even}}^{l-1} d_{k,j} + c = m - \sum_{j \text{ odd}}^{l-1} d_{k,j} < n - \sum_{j \text{ odd}}^{l-1} d_{k,j}.$$

This implies that $\sum_{j \text{ odd}}^{l-1} d_{k,j} + 1 < n-k+1$. Thus,

$$\begin{aligned} f_k\Big(\Big(\sum_{j \text{ odd}}^{l-1} d_{k,j} + 1\Big)(1, 0)\Big) &< f_k((n-k+1)(1,0)) = k(n-k+1) \\ &= f_k(k(n,n+1)) = f_k((0,k) + k \cdot n(1,1)) \\ &= f_k(v_{m,\eta_k}) + r_{m,\eta_k}. \end{aligned}$$

This is a contradiction to Lemma 3.7 (2). The other two cases are analogous. ∎

Recall that for $J \in S_{A_n}$, we denote $m_J = \sum_{\beta \in J} A_n \beta$.

Corollary 3.9. *Let $n \in \mathbb{N} \setminus \{0\}$ and $1 \le k \le n$. Then there exists $\eta \in \Omega$ such that $\eta \ne \eta_k$ and $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$.*

Proof. By the previous proposition, we have that $v_{n,\eta_k} = (n-k+1, 0)$ or $v_{n,\eta_k} = (0,k)$. By Lemma 2.5 (6), we obtain that $\{v_{i,\eta_k}\}_{i=1}^{n-1} = \{(t,0)\}_{t=1}^{n-k} \cup \{(0,s)\}_{s=1}^{k-1}$. Moreover, we can deduce that v_{n-1,η_k} is $(n-k, 0)$ or $(0, k-1)$.

Suppose that $v_{n-1,\eta_k} = (n-k, 0)$. This implies that there exists $p < n-1$ such that $v_{p,\eta_k} = (0, k-1)$. Moreover, by Definition 2.2, we obtain that $p = \sum_{j=0}^{r-1} d_{k,j}$ and for all $p < q \le n-1$, we have that $v_{q,\eta_k} = (n-k-(n-1-q), 0) = (q+1-k, 0)$. In particular, $s_r = p+1-k$ (recall Definition 3.5). Since

$$\begin{aligned} (n-k+1) \cdot f_k((1,0)) = f_k((n-k+1, 0)) &= f_k(k(n,n+1)) \\ &= k \cdot f_k((n, n+1)), \end{aligned}$$

we obtain that $t_r = n - k + 1$. By Definition 3.5, we have that

$$d_{k,r} = \min\Big\{n - \sum_{j=0}^{r-1} d_{k,j}, t_r - s_r\Big\} = \min\{n - p, n - k + 1 - (p + 1 - k)\} = n - p.$$

This implies that $\sum_{j=0}^{r} d_{k,j} = n$, i.e., the process completes, and $d_{k,r} \geq 2$. In the case of $v_{n-1,\eta_k} = (0, k-1)$, we obtain that $d_{k,r} \geq 2$ using the same argument. In any case, we obtain that $d_{k,r} \geq 2$. Then we define $\eta = (z', d'_0, d'_1, \ldots, d'_r, d'_{r+1})$, where $z' = z_k$, $d'_i = d_{k,i}$ for all $i < r$, $d'_r = d_{k,r} - 1$ and $d'_{r+1} = 1$.

By construction, $\sum_{j=0}^{n} d'_j = n$ and $d'_j > 0$ for all $j \in \{1, \ldots, n\}$. This implies that $\eta \in \Omega$. On the other hand, we have that $v_{j,\eta_k} = v_{j,\eta}$ for all $j \leq n - 1$ and $v_{n,\eta} = (n - k + 1, 0)$ if $v_{n,\eta_k} = (0, k)$ or $v_{n,\eta} = (0, k)$ if $v_{n,\eta_k} = (n - k + 1, 0)$. Since $f_k(k(n, n + 1)) = f_k((n - k + 1, 0))$, we obtain that $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$. ■

Example 3.10. Let $n = 6$ and $k = 3$. By Example 3.6, we obtain that $\eta_3 = (1, 0, 1, 1, 1, 1, 2)$. Using the construction of the proof of Corollary 3.9, we obtain that $\eta = (1, 0, 1, 1, 1, 1, 1, 1)$.

3.2. $J_{\eta_k} \in S_{A_n}$ is minimal with respect to f_k

Lemma 3.11. *Let $\beta', \beta \in \mathbb{N}^3$ be such that $\beta' \leq \beta$ (Recall Notation 1.1). Then*

$$f_k(A_n\beta') \leq f_k(A_n\beta).$$

Proof. This is a straightforward computation. ■

Lemma 3.12. *Let $\beta \in \mathbb{N}^3$ be such that $A_n\beta \neq v + q(1,1)$ for all $v \in T'_{\eta_k}$ and $q \in \mathbb{N}$. Then $f_k(A_n\beta) \geq f_k(v)$ for all $v \in T'_{\eta_k}$.*

Proof. We claim that $f_k(v) \leq k(n - k + 1) \leq f_k(A_n\beta)$ for all $v \in T'_{\eta_k}$ and for all $\beta \in \mathbb{N}^3$ with the conditions of the lemma.

We are going to prove the first inequality of the claim. By Definition 3.1, we have that $T'_{\eta_k} = T_{0,\eta_k} \cup \bigcup_{j=1}^{n} T_{j,\eta_k} + r_{j,\eta_k}\vec{1}$, where $T_{0,\eta_k} = \{(q,q)\}_{q=1}^{n}$, $r_{j,\eta_k} = n \cdot \pi_2(v_{j,\eta_k})$ and $T_{j,\eta_k} + r_{j,\eta_k}\vec{1} = \{v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)\}_{p=0}^{n-j}$. By Proposition 3.8, we have that $v_{n,\eta_k} = (0, k)$ or $v_{n,\eta_k} = (n - k + 1, 0)$. Moreover, by Lemma 2.5 (5), we have that $\{v_{j,\eta_k}\}_{j=1}^{n-1} = \{(t, 0)\}_{t=1}^{n-k} \cup \{(0, s)\}_{s=1}^{k-1}$.

By definition, $T_{n,\eta_k} + r_{n,\eta_k}\vec{1} = \{v_{n,\eta_k} + r_{n,\eta_k}(1,1)\}$. Since we know the two possibilities for v_{n,η_k}, we obtain that $f_k(v_{n,\eta_k} + r_{n,\eta_k}(1,1)) = k(n - k + 1)$. On the other hand, if $v \in T_{0,\eta_k}$, we have that $v = (q,q)$ with $q \leq n$. Since $1 \leq k \leq n$, we obtain that $f_k(v) = q \leq n \leq k(n - k + 1)$. With this, we only have to check the desired inequality for $v \in \bigcup_{j=1}^{n-1}\{v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)\}_{p=0}^{n-j}$. This implies that $v = v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)$, for $1 \leq j \leq n - 1$ and $0 \leq p \leq n - j$.

Suppose that $v_{j,\eta_k} = (t, 0)$ for some $t \leq j$, and recall that $t \leq n - k$. Then

$$\begin{aligned} f_k(v) &= f_k(v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)) = f_k((t,0)) + p + r_{j,\eta_k} \\ &\leq kt + n - j \leq kt + n - t = (k-1)t + n \\ &\leq (k-1)(n-k) + n = nk - k^2 + k. \end{aligned}$$

Now suppose that $v_{j,\eta_k} = (0,s)$ for some $s \leq j$ and recall that $s < k$. Then

$$\begin{aligned} f_k(v) = f_k(v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)) &= f_k((0,s) + p(1,1) + n \cdot s(1,1)) \\ &= f_k(s(n,n+1)) + p \leq s(n-k+1) + (n-j) \\ &\leq s(n-k+1) + (n-s) \leq s(n-k+1) + (n-k) + (k-s) \\ &\leq s(n-k+1) + (k-s)(n-k) + (k-s) = nk - k^2 + k. \end{aligned}$$

This proves the first inequality of the claim. For the second inequality, notice that

$$\begin{aligned} A_n\beta &= \pi_1(\beta)(1,0) + \pi_2(\beta)(1,1) + \pi_3(\beta)(n,n+1) \\ &= (\pi_1(\beta),0) + (\pi_2(\beta),\pi_2(\beta)) + (n\pi_3(\beta), n\pi_3(\beta) + \pi_3(\beta)) \\ &= (\pi_1(\beta),\pi_3(\beta)) + (\pi_2(\beta) + n\pi_3(\beta))(1,1) \\ &= (\pi_1(\beta) - \pi_3(\beta),0) + (\pi_2(\beta) + (n+1)\pi_3(\beta))(1,1). \end{aligned}$$

Similarly, we obtain the expression

$$A_n\beta = (0, \pi_3(\beta) - \pi_1(\beta)) + (\pi_1(\beta) + \pi_2(\beta) + n\pi_3(\beta))(1,1)).$$

Working with the first expression of $A_n\beta$ and applying f_k to this vector, we obtain that

$$\begin{aligned} f_k(A_n\beta) &= f_k((\pi_1(\beta) - \pi_3(\beta),0) + (\pi_2(\beta) + (n+1)\pi_3(\beta))(1,1)) \\ &= f_k((\pi_1(\beta) - \pi_3(\beta),0)) + \pi_2(\beta) + (n+1)\pi_3(\beta) \\ &= k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta). \end{aligned}$$

By the hypothesis over β and recalling that $\{v_{j,\eta_k}\}_{j=1}^{n-1} = \{(t,0)\}_{t=1}^{n-k} \cup \{(0,s)\}_{s=1}^{k-1}$, we obtain two cases, $\pi_1(\beta) - \pi_3(\beta) \geq n-k+1$ or $\pi_3(\beta) - \pi_1(\beta) \geq k$. If $\pi_1(\beta) - \pi_3(\beta) \geq n-k+1$, then

$$\begin{aligned} f_k(A_n\beta) &= k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta) \\ &\geq k(n-k+1) + \pi_2(\beta) + (n+1)\pi_3(\beta) \\ &\geq k(n-k+1). \end{aligned}$$

If $\pi_3(\beta) - \pi_1(\beta) \geq k$, in particular, $\pi_3(\beta) \geq k$, then

$$\begin{aligned} f_k(A_n\beta) &= k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta) \\ &= (n+1)\pi_3(\beta) - k\pi_3(\beta) + k\pi_1(\beta) + \pi_2(\beta) \\ &= (n-k+1)\pi_3(\beta) + k\pi_1(\beta) + \pi_2(\beta) \\ &\geq (n-k+1)k + k\pi_1(\beta) + \pi_2(\beta) \\ &\geq nk - k^2 + k. \end{aligned}$$

In any case, we obtain that $f_k(A_n\beta) \geq k(n-k+1)$ for all $\beta \in \mathbb{N}^3$ with the conditions of the lemma, as claimed. ■

Lemma 3.13. *Let $v = v_{l,\eta_k} + q(1,1) \in \mathbb{N}^2$ with $l \leq n$ and $q \geq n-l+1+r_{l,\eta_k}$. Then $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}\vec{1}$.*

Proof. We proceed by induction over l. Consider $l = 1$. Then $v = v_{1,\eta_k} + q(1,1)$, with $q \geq n + r_{1,\eta_k}$ and we need to prove that $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \cup T_{1,\eta_k} + r_{1,\eta_k}\overrightarrow{1}$. If $u \in T_{1,\eta_k} + r_{1,\eta_k}\overrightarrow{1}$, then $u = v_{1,\eta_k} + (p + r_{1,\eta_k})(1,1)$, with $p \leq n-1$. It follows that

$$\begin{aligned} f_k(v) &= f_k(v_{1,\eta_k} + q(1,1)) = f_k(v_{1,\eta_k}) + q \\ &\geq f_k(v_{1,\eta_k}) + n + r_{1,\eta_k} \geq f_k(v_{1,\eta_k}) + p + r_{1,\eta_k} \\ &= f_k(v_{1,\eta_k} + p + r_{1,\eta_k}(1,1)) = f_k(u). \end{aligned}$$

If $u \in T_{0,\eta_k}$, then $u = p(1,1)$, with $p \leq n$. By Definition 2.2, $v_{1,\eta_k} = (1,0)$ or $v_{1,\eta_k} = (0,1)$. Thus, $f_k(v) = k + q \geq k + n$ or $f_k(v) = (1-k) + q \geq n + (n-k+1)$. Since $k \in \{1, \ldots, n\}$, in any case we have that

$$f_k(v) > n \geq p = f_k(u). \tag{3.2}$$

We conclude that it is true for $l = 1$.

Now, suppose that it is true for all $l' < l$, i.e., $f_k(v_{l',\eta_k} + q'(1,1)) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l'} T_{j,\eta_k} + r_{j,\eta_k}\overrightarrow{1}$ and $q' \geq n - l' + 1 + r_{l',\eta_k}$. Let $v = v_{l,\eta_k} + q(1,1)$ with $q \geq n - l + 1 + r_{l,\eta_k}$. If $u \in T_{l,\eta_k} + r_{l,\eta_k}\overrightarrow{1}$, we have that $u = v_{l,\eta_k} + (p + r_{l,\eta_k})(1,1)$, with $p \leq n - l$. Then

$$\begin{aligned} f_k(v) &= f_k(v_{l,\eta_k}) + q \geq f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k} \\ &\geq f_k(v_{l,\eta_k}) + p + r_{l,\eta_k} = f_k(v_{l,\eta_k} + (p + r_{l,\eta_k})(1,1)) = f_k(u). \end{aligned}$$

Thus $f_k(v) \geq f_k(u)$ for all $u \in T_{l,\eta_k} + r_{l,\eta_k}\overrightarrow{1}$. Consider $u \in T_{l-1,\eta_k} + r_{\eta_k,l-1}\overrightarrow{1}$. By definition, $u = v_{l-1,\eta_k} + (p + r_{l-1,\eta_k})(1,1)$ with $p \leq n - l + 1$. By Lemma 3.7 (4), $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) \geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1))$. Then

$$\begin{aligned} f_k(v) &= f_k(v_{l,\eta_k}) + q \geq f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k} \\ &= f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1)) + p \\ &= f_k(v_{l-1,\eta_k} + (p + r_{l-1,\eta_k})(1,1)) = f_k(u). \end{aligned}$$

Obtaining that the statement is true for all $u \in T_{l-1,\eta_k} + r_{l-1,\eta_k}\overrightarrow{1}$.

Suppose $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) \geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1)) + 1$. Obtaining that

$$\begin{aligned} f_k(v) &\geq f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \\ &\geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1)) + n - l + 2. \end{aligned}$$

Then, by the induction hypothesis over $l-1$, we have that $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l-1} T_{j,\eta_k} + r_{j,\eta_k}\overrightarrow{1}$, obtaining the result.

Now suppose that $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1))$. For this, we have two cases, $l = 2$ or $l > 2$. If $l = 2$, by (3.2), we have that

$$\begin{aligned} f_k(v) &= f_k(v_{2,\eta_k}) + q \geq f_k(v_{2,\eta_k}) + n - 1 + r_{2,\eta_k} \\ &= f_k(v_{2,\eta_k} + r_{2,\eta_k}(1,1)) + n - 1 = f_k(v_{1,\eta_k} + r_{1,\eta_k}(1,1)) + n - 1 \\ &= f_k(v_{1,\eta_k} + (n - 1 + r_{1,\eta_k})(1,1)) \geq n \geq f_k(u), \end{aligned}$$

for all $u \in T_{0,\eta_k}$. If $l > 2$, then for all $u \in T_{0,\eta_k} \bigcup \bigcup_{j=1}^{l-2} T_{j,\eta_k} + r_{j,\eta_k} \overrightarrow{1}$, we have that

$$\begin{aligned} f_k(v) &\geq f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \\ &\geq f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1,1)) + n - l + 3 \geq f_k(u), \end{aligned}$$

where the second inequality comes from Lemma 3.7 (5) and the last inequality comes from the induction hypothesis over $l-2$, obtaining the result. ■

Now we are ready to prove the other important result of this section.

Proposition 3.14. *Let $\eta_k \in \Omega$ and its respective $J_{\eta_k} \in S_{A_n}$. Then for all $J \in S_{A_n}$, we have that $f_k(m_{J_{\eta_k}}) \leq f_k(m_J)$.*

Proof. Let $J = \{\beta_1, \ldots, \beta_{\lambda_{2,n}}\} \in S_{A_n}$. By the definition of S_{A_n}, $\det(c_{\beta_i})_{1 \leq i \leq \lambda_{2,n}} \neq 0$, where $c_{\beta_i} := \sum_{\gamma \leq \beta_i} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \overline{A_n \gamma}$ (recall Notation 1.1).

Fixing the β_1-th row of this matrix and using basic properties of determinants, we obtain that

$$0 \neq \det(c_{\beta_i})_{1 \leq i \leq \lambda_{2,n}} = \sum_{\gamma \leq \beta_1} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \det \begin{pmatrix} \overline{A_n \gamma} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{2,n}}} \end{pmatrix}.$$

Since the determinant is not zero, this implies that there exists $\beta_1' \leq \beta_1$ such that

$$\det \begin{pmatrix} \overline{A_n \beta_1'} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

Applying this process for each row, we obtain a set of vectors $B = \{\beta_i'\}_{i=1}^{\lambda_{2,n}} \subset \Lambda_{3,n}$ such that $\beta_i' \leq \beta_i$ for all $i \in \{1, \ldots, \lambda_{2,n}\}$ and with the property $\det(\overline{A_n \beta_i'})_{1 \leq i \leq \lambda_{2,n}} \neq 0$.

The goal is to construct a bijective correspondence $\varphi \colon B \to T'_{\eta_k}$, such that $f_k(A_n \beta_i') \geq f_k(\varphi(\beta_i'))$. Consider the set $v_{j,\eta_k} + L := \{v_{j\eta_k} + p(1,1) \mid p \in \mathbb{N}\}$. Now, consider the following partition of B:

$$\begin{aligned} B_0 &= \{\beta_i' \in B \mid A_n \beta_i' \in T'_{\eta_k}\}, \\ B_1 &= \{\beta_i' \in B \mid A_n \beta_i' \in (v_{j,\eta_k} + L) \setminus T_{j,\eta_k} + r_{j,\eta_k} \overrightarrow{1} \text{ for some } j \in \{1, \ldots, n\}\}, \\ B_2 &= \{\beta_i' \in B \mid A_n \beta_i' = q(1,1) \text{ for some } q > n\}, \\ B_3 &= \{\beta_i' \in B \mid A_n \beta_i' \notin (v_{j,\eta_k} + L) \text{ for all } j \in \{0, \ldots, n\}\}, \end{aligned}$$

For all $\beta_i' \in B_0$, we define $\varphi(\beta_i') = A_n \beta_i'$. Since $\det(\overline{A_n \beta_i'})_{1 \leq i \leq \lambda_{2,n}} \neq 0$, we have that $\varphi(\beta_i') \neq \varphi(\beta_j')$ for all $\beta_i', \beta_j' \in B_0$.

Now, if $B_1 \neq \emptyset$, we rearrange B in such a way that $\{\beta'_1, \beta'_2, \ldots, \beta'_m\} = B_1$. Consider $\beta'_1 \in B_1$. By the construction of B_1, there exist $l \leq n$ and $q \in \mathbb{N}$ such that $A_n\beta'_1 = v_{l,\eta_k} + q(1,1)$. By Proposition 2.15, we have that

$$\begin{aligned}\overline{A_n\beta'_1} &\in \operatorname{span}_{\mathbb{C}}\left\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \bigcup_{j=0}^{l} T_{j,\eta_k}\right\}\\ &= \operatorname{span}_{\mathbb{C}}\left\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}\vec{1}\right\}.\end{aligned}$$

This implies that $\overline{A_n\beta'_1} = \sum_{v \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}\vec{1}} a_v \bar{v}$, for some constants $a_v \in \mathbb{C}$. Using again basic properties of the determinant, we obtain that there exists $u_{\beta'_1} \in T_{0,\eta_k} \cup \bigcup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}\vec{1}$ such that

$$\det\begin{pmatrix} \bar{u}_{\beta'_1} \\ \overline{A_n\beta'_2} \\ \vdots \\ \overline{A_n\beta'_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

Applying this process for each element of B_1, we obtain the vectors $\{u_{\beta'_j}\}_{j=1}^{m}$. We define $\varphi(\beta'_j) = u_{\beta'_j}$ for all $j \in \{1, \ldots, m\}$. Now, we need to check that φ is injective on $B_0 \cup B_1$ and $f_k(A_n\beta'_i) \geq f_k(\varphi(\beta'_i))$.

Notice that, by construction,

$$\det\begin{pmatrix} \bar{u}_{\beta'_1} \\ \vdots \\ \bar{u}_{\beta'_m} \\ \overline{A_n\beta'_{m+1}} \\ \vdots \\ \overline{A_n\beta'_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

This implies that $\bar{u}_{\beta'_i} \neq \bar{u}_{\beta'_j}$ for all $1 \leq i < j \leq m$. In particular, we have that $u_{\beta'_i} \neq u_{\beta'_j}$. Moreover, using the same argument, we have that $u_{\beta'_i} \neq A_n\beta'_j = \varphi(\beta'_j)$ for all $\beta'_j \in B_0$. Thus, φ is injective on $B_0 \cup B_1$.

On the other hand, $A_n\beta'_i = v_{l,\eta_k} + q(1,1) \notin T_{l,\eta_k} + r_{l,\eta_k}\vec{1}$ for some $l \leq n$. This implies that $q \geq n - l + 1 + r_{l,\eta_k}$. Then

$$\begin{aligned}f_k(A_n\beta'_i) &= f_k(v_{l,\eta_k} + q(1,1)) = f_k(v_{l,\eta_k}) + q\\ &\geq f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k}\\ &= f_k(v_{l,\eta_k} + (n - l + 1 + r_{l,\eta_k})(1,1)) \geq f_k(u_{\beta'_i}),\end{aligned}$$

where the last inequality comes from Lemma 3.13, obtaining the desired inequality.

For all $\beta'_i \in B_2$, we have that $A_n\beta'_i = q(1,1)$, for some $q > n$. By Lemma 2.11, we have that $\overline{A_n\beta'_i} = \sum_{v\in T_{0,\eta_k}} a_v\bar{v}$. Applying the same method for the elements of B_1, we can define φ with the properties that we need.

Since $|T'_{\eta_k}| = |B| = \lambda_{2,n}$, we have that $|B_3| = |T_{\eta_k} \setminus \{\varphi(\beta'_j) \mid \beta'_j \in B_0 \cup B_1 \cup B_2\}|$. Then we take $\varphi(\beta'_i) = v$, with $v \in T_{\eta_k} \setminus \{\varphi(\beta'_j) \mid \beta'_j \in B_0 \cup B_1 \cup B_2\}$, in such a way that $\varphi(\beta'_i) \neq \varphi(\beta'_j)$ for all $\beta'_i, \beta'_j \in B_3$ and $\beta'_i \neq \beta'_j$.

By construction, we obtain that φ is a bijective correspondence and by the definition of B_3 and Lemma 3.12, we have that $f_k(A_n\beta'_i) \geq f_k(\varphi(\beta'_i))$ for all $\beta'_i \in B_3$. Then

$$\begin{aligned} f_k(m_J) &= \sum_{\beta_i \in J} f_k(A_n\beta_i) \geq \sum_{\beta'_i \in B} f_k(A_n\beta'_i) \\ &\geq \sum_{\beta'_i \in B} f_k(\varphi(\beta'_i)) = \sum_{b \in T'_{\eta_k}} f_k(v) = f_k(m_{J_{\eta_k}}), \end{aligned}$$

where the first inequality comes from Lemma 3.11 and the second comes from the construction of φ. ■

Now we are ready to prove Theorem 1.8.

Proof. By Proposition 3.4, $J_{\eta_k} \in S_{A_n}$. By corollary 3.9, there exists $J_\eta \in S_{A_n}$ such that $J_\eta \neq J_{\eta_k}$ and $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$. From Proposition 3.14, we obtain $\mathrm{ord}_{I_n}((k, 1-k)) = f_k(m_{J_{\eta_k}})$. This implies that $(k, 1-k) \in \sigma_{m_{J_\eta}} \cap \sigma_{m_{J_{\eta_k}}}$. ■

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