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On the uniqueness of multi-breathers of the modified Korteweg–de Vries equation

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Abstract. We consider the modified Korteweg–de Vries equation, and prove that given any sum P of solitons and breathers (with distinct velocities), there exists a solution p such that $p(t) - P(t) \to 0$ when $t \to +\infty$, which we call multi-breather. In order to do this, we work at the H^2 level (even if usually solitons are considered at the H^1 level). We will show that this convergence takes place in any H^s space and that this convergence is exponentially fast in time. We also show that the constructed multi-breather is unique in two cases: in the class of solutions which converge to the profile P faster than the inverse of a polynomial of a large enough degree in time (we will call this a super polynomial convergence), or when all the velocities are positive (without any hypothesis on the convergence rate).

1. Introduction

1.1. Setting of the problem

We consider the modified Korteweg–de Vries equation (mKdV) on $\mathbb{R}$:

$$\begin{cases} u_t + (u_{xx} + u^3)_x = 0, & (t,x) \in \mathbb{R}^2, \\ u(0) = u_0, & u(t,x) \in \mathbb{R}. \end{cases} \tag{1.1}$$

The mKdV equation appears as a model in a variety of physical studies, such as plasma physics [9, 39], electrodynamics [38], fluid mechanics [22], ferromagnetic vortices [46], and more.

In [24], Kenig, Ponce and Vega established local well-posedness in H^s, for $s \geq 1/4$, of the Cauchy problem for (1.1), by a fixed point argument in $L^p_x L^q_t$ type spaces. Moreover, if $s > 1/4$, the Cauchy problem is globally well posed [12]. Recently, Harrop-Griffiths, Killip and Visan [21] proved local well-posedness in H^s for $s > -1/2$. However, in this paper, we will only use the global well-posedness in H^2.

Equation (1.1) is an integrable equation (like the original Korteweg–de Vries equation) and thus it has an infinity of conservation laws, see [1, 37]. We will use three of them (the

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first two of them are called *mass* and *energy*; the third is sometimes called *second energy*):

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)\,dx,$$

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{4}\int_{\mathbb{R}} u^4(t,x)\,dx,$$

$$F[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_{xx}^2(t,x)\,dx - \frac{5}{2}\int_{\mathbb{R}} u^2(t,x)u_x^2(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}} u^6(t,x)\,dx.$$

Observe that if u is a solution of (1.1), then $-u$ and $(t,x) \mapsto u(t, x-x_0)$, for any $x_0 \in \mathbb{R}$, are solutions of (1.1) too.

Equation (1.1) is a dispersive nonlinear equation, which is a special case of a more general class of equations: the general Korteweg–de Vries equation (gKdV), where the nonlinearity u^3 is replaced by $f(u)$ for some real-valued function f. The particularity of (1.1) in comparison to other gKdV equations is that it admits special nonlinear solutions, namely, breather solutions.

The most simple nonlinear solutions of (1.1) are solitons, i.e., a bump of a constant shape that translates with a constant velocity without deformation, that is, solutions of the form $u(t,x) = Q_c(x-ct)$, where c is the velocity and Q_c is the profile function that depends only on one variable. The function $Q_c \in H^1(\mathbb{R})$ should solve the elliptic equation

$$Q_c'' = cQ_c - Q_c^3. \tag{1.2}$$

We can show that necessarily $c > 0$ and that, if $c > 0$, equation (1.2) has a unique solution in $H^1(\mathbb{R})$, up to translations and reflection with respect to the x-axis. Actually, one has the explicit formula

$$Q_c(x) := \Big(\frac{2c}{\cosh^2(c^{1/2}x)}\Big)^{1/2}.$$

Observe that we chose Q_c so that it is even and positive.

A *soliton* is a solution of (1.1), parametrized by a velocity parameter $c > 0$, a sign parameter $\kappa \in \{-1, 1\}$ and a translation parameter $x_0 \in \mathbb{R}$ (it corresponds to the initial position of the soliton) that has the following expression:

$$R_{c,\kappa}(t,x;x_0) := \kappa Q_c(x - x_0 - ct).$$

When $\kappa = -1$, this object is sometimes called *antisoliton*. Notice that solitons are smooth and decaying. The generalized Korteweg–de Vries equation (gKdV) also admits soliton type solutions, and so does the focusing nonlinear Schrödinger equation (NLS). Solitons have been extensively studied, in particular, their stability. Cazenave, Lions and Weinstein in [7, 8, 44, 45] were interested in orbital stability of gKdV and NLS solitons in H^1. A soliton of (1.1) is indeed orbitally stable, i.e., if a solution is initially close to a soliton in $H^1(\mathbb{R})$, then it stays close to the soliton, up to a space translation defined for any time in $H^1(\mathbb{R})$. General results about orbital stability of nonlinear dispersive solitons are presented by Grillakis, Shatah and Strauss in [20]. The result about orbital stability of a soliton can be improved in a result of asymptotic stability, as it was done in the works by Martel and Merle [29, 31, 33], see also [17].

A *breather* is a solution of (1.1), parametrized by $\alpha, \beta > 0$, $x_1, x_2 \in \mathbb{R}$ that has the following expression:

$$B_{\alpha,\beta}(t,x;x_1,x_2) := 2\sqrt{2}\,\partial_x\Big[\arctan\Big(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\Big)\Big],$$

where

$$y_1 := x + \delta t + x_1, \quad y_2 := x + \gamma t + x_2, \quad \text{with } \delta := \alpha^2 - 3\beta^2 \text{ and } \gamma := 3\alpha^2 - \beta^2.$$

It corresponds to a localized periodic in time function (with frequency α, and exponential localization with decay rate β) that propagates at a constant velocity $-\gamma$ in time. Like solitons, breathers are smooth and decaying in space. Unlike solitons, breather's velocities can be positive, zero or negative. The parameters α, β are the shape parameters and x_1, x_2 are the translation parameters of a breather. Note that if we replace the parameter x_1 by $x_1 + \pi/\alpha$, we transform $B_{\alpha,\beta}(\cdot,\cdot;x_1,x_2)$ in $-B_{\alpha,\beta}(\cdot,\cdot;x_1,x_2)$ (therefore, we do not need to talk about "antibreathers").

Breathers were first introduced by Wadati in [42], and they were used by Kenig, Ponce and Vega in [25] to prove that the flow map associated to (1.1) is *not* uniformly continuous in H^s for $s < 1/4$: the point is that two breathers with close velocities can be very close at $t = 0$ and can separate as fast as we want in H^s with $s < 1/4$, if α is taken large enough.

Breathers for (1.1) and their properties, as well as breathers for other equations, are well studied by Alejo and Muñoz and co-authors in [2–6].

Let us singularize a result of H^2 orbital stability for breathers established in [3], and improved to H^1 orbital stability in [4]. In this last paper, a partial result of asymptotic stability is also given, for breathers travelling to the right only, with positive velocity $-\gamma > 0$; asymptotic stability for breathers in full generality is still an open problem.

When $\alpha \to 0$, $B_{\alpha,\beta}$ tends to a solution of (1.1) called *double-pole solution* [43]; the methods employed in this article as well as the proof of orbital stability made by Alejo and Muñoz seem not to apply for this limit, which is expected to be unstable according to the numerical computations in [18].

An important result regarding the long time dynamics of (1.1) is the soliton-breather resolution [10]. It asserts that any generic solution can be approached by a sum of solitons and breathers when $t \to +\infty$ (up to a dispersive and a self-similar term). Together with their stability properties, the soliton-breather resolution shows why solitons and breathers are essential objects to study. This resolution was established for initial conditions in a weighted Sobolev space in [10] (see also Schuur [40]) by an inverse scattering method; see also [40] for the soliton resolution for KdV. Observe that (1.1) breathers do not decouple into simple solitons for large time (it is a *fully bounded state* as it is called in [3]); therefore, they must appear in the resolution. The soliton-breather resolution is one of the motivations of the study of multi-breathers, which we define below.

There are works in the literature about a more complicated object obtained from several solitons, that is, a *multi-soliton*. A multi-soliton is a solution $r(t)$ of (1.1) with the following property: there exist $0 < c_1 < c_2 < \cdots < c_N$, $\kappa_1, \ldots, \kappa_N \in \{-1, 1\}$ and $x_1, \ldots, x_N \in \mathbb{R}$ such that

$$\lim_{t\to+\infty}\Big\|r(t) - \sum_{j=1}^{N} R_{c_j,\kappa_j}(t,\cdot;x_j)\Big\|_{H^1(\mathbb{R})} = 0.$$

This definition is not specific to (1.1) and makes sense for many other nonlinear dispersive PDEs as soon as they admit solitons. This object is introduced by Schuur [40] and Lamb [26], see also Miura [36], where explicit formulas are given; these were obtained by an inverse scattering method thanks to the integrability of the equation. Multi-solitons were first constructed in a non-integrable context by Merle [34] for the mass critical NLS. Martel [28] constructed multi-solitons for mass-subcritical and critical gKdV equations and proved that they are unique in $H^1(\mathbb{R})$, smooth and converge exponentially fast to their profile in any Sobolev space H^s. Similar studies were done for other nonlinear dispersive PDEs. Martel and Merle [32] have proved the existence of multi-solitons for the NLS in H^1, Côte, Martel and Merle extended this construction to the mass supercritical gKdV and NLS in [15]. Friederich and Côte in [14] proved smoothness, and uniqueness in a class of algebraic convergence. Côte and Muñoz constructed in [16] multi-solitons for the nonlinear Klein–Gordon equation. Ming, Rousset and Tzvetkov have constructed multi-solitons for the water-waves systems in [35]. Valet has proved in [41] the existence and uniqueness of multi-solitons in H^1 for the Zakharov–Kuznetsov equation, which generalizes gKdV to higher dimensions.

1.2. Main results

We prove in this article that given any sum of solitons and breathers with distinct velocities, there exists a solution of (1.1) whose difference with this sum tends to zero when time goes to infinity. This solution will be called a multi-breather. Let us make the definition more precise.

Let $J \in \mathbb{N}$ and $K, L \in \mathbb{N}$ be such that $J = K + L$. We will consider a set of L solitons and K breathers:

- the breather parameters are $\alpha_k > 0$, $\beta_k > 0$, $x^0_{1,k} \in \mathbb{R}$ and $x^0_{2,k} \in \mathbb{R}$ for $1 \le k \le K$.
- the solitons parameters are $c_l > 0$, $\kappa_l \in \{-1, 1\}$ and $x^0_{0,l} \in \mathbb{R}$ for $1 \le l \le L$.

We define for $1 \le k \le K$, the kth breather

$$B_k(t,x) := B_{\alpha_k,\beta_k}(t,x; x^0_{1,k}, x^0_{2,k}); \tag{1.3}$$

and for $1 \le l \le L$, the lth soliton

$$R_l(t,x) := R_{c_l,\kappa_l}(t,x; x^0_{0,l}). \tag{1.4}$$

We now define the *velocity* of our objects. Recall that for $1 \le k \le K$, the velocity of B_k is

$$v^b_k := -\gamma_k = \beta_k^2 - 3\alpha_k^2, \tag{1.5}$$

and for $1 \le l \le L$, the velocity of R_l is

$$v^s_l := c_l. \tag{1.6}$$

The most important assumption we make is that all these velocities are distinct:

$$v^b_k \neq v^b_{k'} \quad \text{for all } k \neq k', \qquad v^s_l \neq v^s_{l'} \quad \text{for all } l \neq l', \qquad v^b_k \neq v^s_l \quad \text{for all } k, l. \tag{1.7}$$

This implies for any two of these objects to be far from each other when time is large, and this assumption is essential in our analysis.

It will be useful to order our breathers and solitons by increasing velocities. As these are distinct, we can define an increasing function

$$\underline{v} : \{1, \dots, J\} \to \{v_k^b, 1 \le k \le K\} \cup \{v_l^s, 1 \le l \le L\}.$$

The set $\{v_1, \dots, v_J\}$ is thus the (ordered) set of all possible velocities of our objects. We define P_j, for $1 \le j \le J$, as the object (either a soliton R_l or a breather B_k) that corresponds to the velocity v_j. Hence, $P_1, \dots, P_J$ are the considered objects ordered by increasing velocity.

We will need both notations: the indexation by k and l, and the indexation by j, and we will keep these notations to avoid ambiguity.

We will denote by x_j the centre of mass of P_j, that is,

- if $P_j = B_k$ is a breather, we set $x_j(t) := -x_{2,k}^0 + v_j t$;
- if $P_j = R_l$ is a soliton, we set $x_j(t) := x_{0,l}^0 + v_j t$.

We denote

$$R = \sum_{l=1}^{L} R_l, \quad B = \sum_{k=1}^{K} B_k, \quad P = R + B = \sum_{j=1}^{J} P_j. \tag{1.8}$$

We can now define a multi-breather: as solitons are objects which can be studied naturally in $H^1(\mathbb{R})$, it turns out that breathers are best studied in $H^2(\mathbb{R})$; therefore, it is in this latter space that we develop our analysis.

Definition 1.1. A *multi-breather* associated to the sum P of solitons and breathers, given in (1.8), is a solution $p \in \mathcal{C}([T^*, +\infty), H^2(\mathbb{R}))$, for a constant $T^* > 0$, of (1.1) such that

$$\lim_{t \to +\infty} \|p(t) - P(t)\|_{H^2} = 0.$$

We will prove two results in this article. The first one is the existence and the regularity of a multi-breather, the second one is the uniqueness of a multi-breather. The uniqueness is established in two settings: in the case when all velocities are positive, and without any assumption on the sign of the considered velocities. However, in the last case, the uniqueness is obtained in a narrower class of functions.

Theorem 1.2. *Given solitons and breathers* (1.3), (1.4) *whose velocities* (1.5) *and* (1.6) *satisfy* (1.7)*, there exists a multi-breather p associated to P given in* (1.8)*. Moreover,*

$$p \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R}, H^s(\mathbb{R})) \quad \text{for any } s \ge 0,$$

and there is $\theta > 0$ such that for any $s \ge 0$, there exist $A_s \ge 1$ and $T^* > 0$ such that

$$\|p(t) - P(t)\|_{H^s} \le A_s e^{-\theta t} \quad \text{for all } t \ge T^*. \tag{1.9}$$

Remark 1.3. We will also show that θ does only depend on the shape parameters of our objects: α_k, β_k and c_l. Moreover, if there exists $D > 0$ such that for all $j \ge 2$, we have $x_j(0) \ge x_{j-1}(0) + D$, then A_s and T^* do not depend on $x_{1,k}^0, x_{2,k}^0, x_{0,l}^0$ but only on α_k, β_k, c_l and D. Finally, if $D > 0$ is large enough with respect to the problem data, then (1.9) is true for $T^* = 0$. See Section 3.2 for further details.

Theorem 1.4. *Given the same set of solitons and breathers as in Theorem 1.2 whose velocities satisfy (1.7) and $v_1 > 0$ (so that all the velocities are positive), the multi-breather p associated to P by Theorem 1.2, in the sense of Definition 1.1, is unique.*

Proposition 1.5. *Given the same set of solitons and breathers as in Theorem 1.2 whose velocities satisfy (1.7), there exists $N > 0$ large enough such that the multi-breather p associated to P by Theorem 1.2 is the unique solution $u \in \mathcal{C}([T_0, +\infty), H^2(\mathbb{R}))$ of (1.1) such that*

$$\|u(t) - P(t)\|_{H^2} = O\Big(\frac{1}{t^N}\Big) \quad \text{as } t \to +\infty.$$

In [43], there exists a formula for a multi-breather, obtained by an inverse scattering method, that in some sense already gives the existence of a multi-breather. However, the proof of the Theorem 1.2 from this formula is rather involved.

In this paper, we give a different approach to prove the existence of a multi-breather, and we clearly show that we have convergence of the constructed multi-breather to the corresponding sum of solitons and breathers in H^s, that this convergence is exponentially fast in time and that the constructed multi-breather is smooth. To do this, we use the variational structure of solitons and breathers. This is why we give a proof that is potentially generalizable to non-integrable equations, and that uses similar type of techniques as in the proof of the uniqueness (the latter cannot be deduced from the formula). In any case, uniqueness of multi-breathers is new.

In this paper, we adapt the arguments given by Martel and Merle [32], by Martel [28] and by Côte and Friederich [14] to the context of breathers. To do so, one needs to understand the variational structure of breathers, in the same fashion as Weinstein did in [45] for NLS solitons. Such results were obtained by Alejo and Muñoz in [3]: a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to several (but finitely many) orthogonal conditions, see Section 2 for details. As we see from [3], the H^2 regularity level is the most natural setting to study breathers, and the H^1 regularity level is natural for the study of solitons (as we see in [28, 32]). One important issue we face is therefore to understand the soliton variational structure at H^2 level, and to adapt the Lyapunov functional in [3] to accommodate for a sum of breathers (and solitons). Notice that arguments based on monotonicity may be adapted only if we suppose that all the considered velocities are positive. Because [14, 32] are not based on monotonicity (these are results for the NLS equation, which is not well suited for monotonicity), we can adapt their arguments to obtain existence and uniqueness results for our case without any condition on the sign of velocities. The uniqueness result obtained in this setting is however weaker than the one that is obtained with monotonicity arguments.

1.3. Outline of the proof

The proof of Theorem 1.2 (the existence of multi-breathers) is split into two main parts: the construction of an H^2 multi-breather and the proof that this multi-breather is smooth.

1.3.1. An H^2 multi-breather. Let us start with the first part. We consider an increasing sequence (T_n) of $\mathbb{R}_+$ with $T_n \to +\infty$, and for $n \in \mathbb{N}$, let p_n be the unique global H^2 solution of (1.1) such that $p_n(T_n) = P(T_n)$ (recall that the Cauchy problem for (1.1) is globally well-posed in H^2).

We will prove the following *uniform estimate*.

Proposition 1.6. *There exist $T^* > 0$, $A > 0$ and $\theta > 0$ such that, for any $n \in \mathbb{N}$ satisfying $T_n \geq T^*$,*

$$\|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \quad \textit{for all } t \in [T^*, T_n],$$

With this proposition in hand, we can construct an H^2 multi-breather which converges exponentially fast to its profile, which is the first part of Theorem 1.2, as stated below.

Proposition 1.7. *There exist $T^* \in \mathbb{R}$, $A > 0$, $\theta > 0$ and a solution $p \in C([T^*, +\infty), H^2(\mathbb{R}))$ of (1.1) such that*

$$\|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \quad \textit{for all } t \geq T^*,$$

Proof of Proposition 1.7 assuming Proposition 1.6. We show that the sequence $(p_n(T^*))$ is L^2-compact, in the following sense.

Lemma 1.8. *For any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} p_n^2(T^*, x)\, dx < \varepsilon \quad \textit{for all } n \in \mathbb{N}.$$

An analogous lemma has already been proved on p. 1111 of [28], which is the proof of formula (14) (and can also be found in [32]). The same proof works here. We need to use Proposition 1.6 for T_n large enough and then make a time variation to obtain the result in T^*. We can first find R that works for $P^2(t_0)$ instead of $p_n^2(T^*)$ for a fixed $t_0 > T^*$ large enough. From Proposition 1.6, we see that if we take t_0 large enough, we obtain the desired lemma for $p_n^2(t_0)$ instead of $p_n^2(T^*)$. To finish, with the help of a cut-off function, we control time variations of $\int_{|x|>R} p_n^2(t)\, dx$, where R is taken larger if needed. This is why we obtain the result at $t = T^*$.

As a consequence of Proposition 1.6 above, $(\|p_n(T^*)\|_{H^2})$ is a bounded sequence. Thus, there exists $p^* \in H^2(\mathbb{R})$ such that, up to a subsequence,

$$p_n(T^*) \rightharpoonup p^* \quad \text{in } H^2.$$

Thus, from Lemma 1.8, we have the strong convergence

$$p_n(T^*) \to p^* \quad \text{in } L^2.$$

Therefore, we obtain, by interpolation,

$$p_n(T^*) \to p^* \quad \text{in } H^1.$$

Now, let us consider the global H^1 (even H^2) solution p of (1.1) such that $p(T^*) = p^*$. As shown in [28], the Cauchy problem for (1.1) has a continuous dependence in H^1 on compact sets of time. Let $t \geq T^*$. By continuous dependence, we deduce that $p_n(t) \to p(t)$ in H^1. The sequence $(p_n(t) - P(t))$ is bounded in H^2, which admits a unique weak limit, and so

$$p_n(t) - P(t) \rightharpoonup p(t) - P(t) \quad \text{in } H^2.$$

By weak convergence and from Proposition 1.6, we obtain

$$\|p(t) - P(t)\|_{H^2} \leq \liminf_{n\to+\infty} \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t}.$$

As this is true for any $t \geq T^*$, this completes the proof of Proposition 1.7. ∎

It remains to prove Proposition 1.6, for which we rest on a bootstrap argument. More precisely, we will reduce the proof to the following proposition.

Proposition 1.9. *There exist $T^* > 0$, $A > 0$ and $\theta > 0$ such that, for any $n \in \mathbb{N}$ satisfying $T_n \geq T^*$, and any $t^* \in [T^*, T_n]$, if*

$$\|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \quad \text{for all } t \in [t^*, T_n],$$

then

$$\|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2} e^{-\theta t} \quad \text{for all } t \in [t^*, T_n].$$

The proof of Proposition 1.6 then follows from a simple continuity argument.

Proof of Proposition 1.6 assuming Proposition 1.9. We define t_n^* in the following way:

$$t_n^* := \inf\{t^* \in [T^*, T_n) : \|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \text{ for all } t \in [t^*, T_n]\}.$$

The map $t \mapsto \|p_n(t) - P(t)\|_{H^2}$ is a continuous function and $\|p_n(T_n) - P(T_n)\|_{H^2} = 0$. This means that there exists $T^* \leq t^* < T_n$ such that

$$\|p_n(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \quad \text{for all } t \in [t^*, T_n],$$

Therefore, we have that

$$T^* \leq t_n^* < T_n.$$

We would like to prove that $t_n^* = T^*$. Let us argue by contradiction and assume that $t_n^* > T^*$. Proposition 1.9 allows us to deduce that

$$\|p_n(t) - P(t)\|_{H^2} \leq \frac{A}{2} e^{-\theta t} \quad \text{for all } t \in [t_n^*, T_n].$$

This means that

$$\|p_n(t_n^*) - P(t_n^*)\|_{H^2} \leq \frac{A}{2} e^{-\theta t_n^*},$$

which means that t_n^* could be chosen smaller, by continuity. This is a contradiction. ∎

Hence, we are left to prove Proposition 1.9, which will be done in Section 2.

1.3.2. The H^2 multi-breather is smooth. We now turn to the second part of Theorem 1.2, which is strongly adapted from [28]. The heart of this part is to prove uniform estimates in H^s for $p_n - P$, for any $s \geq 0$.

Proposition 1.10. *There exist $T^* > 0$, $\theta > 0$, and $A_s \geq 1$ for any $s \geq 0$, such that for any $n \in \mathbb{N}$ satisfying $T_n \geq T^*$,*

$$\|p_n(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t} \quad \text{for all } t \in [T^*, T_n].$$

With this improved version of Proposition 1.6, one can prove, by the same reasoning as in the proof of the Proposition 1.7, that for any $s \geq 0$, p actually belongs to $L^\infty([T^*, +\infty), H^s(\mathbb{R}))$ and that the convergence of $p(t) - P(t)$ occurs in H^s with an exponential decay rate. More precisely, the following holds.

Theorem 1.11. *For any $s \geq 2$, we have that $p \in \mathcal{C}([T^*, +\infty), H^s(\mathbb{R}))$ and, furthermore,*

$$\|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t} \quad \text{for all } t \geq T^*.$$

It remains to prove Proposition 1.10, which will be done in Section 3.

1.3.3. The uniqueness result. We denote p the multi-breather constructed in the previous sections, the existence of which is established. Let u be a solution of (1.1) such that

$$\|u - P\|_{H^2} \to 0 \quad \text{as } t \to +\infty. \tag{1.10}$$

Equivalently, we have

$$\|u - p\|_{H^2} \to 0 \quad \text{as } t \to +\infty.$$

We denote

$$z := u - p.$$

The goal is to prove that $z = 0$. We prove it in two configurations: when all the velocities are positive (Theorem 1.4), and without any assumption on velocities (Proposition 1.5), but in this last case we need to assume a stronger convergence than given in (1.10).

The proof of Theorem 1.4 will be carried out in two steps.

We start with Proposition 1.5, which is adapted from [14]. For this, we do not study $u - P$ anymore, we deal only with $z = u - p$, the difference of two solutions of (1.1), which is much more precise than $u - P$. Thus, we do not modulate parameters of the solitons, as it is needed in other parts of the proof, in order to deal with the soliton part of the linear part of the Lyapunov functional, and we avoid some difficulty. In order to prove our inequalities, we need again to use coercivity of the same type of quadratic forms. In order to do this, we replace z by $\tilde{z} = z + \sum_{j=1}^{J} c_j K_j$, where K_j, $j = 1, \dots, J$, is a well chosen basis of the kernel of the quadratic form, in order to have $\tilde{z}$ orthogonal to any K_j. An important idea is to use slow variations of localized functionals with adapted cut-off functions of the form $\varphi(\frac{x-vt}{\delta t})$, which provides an extra $O(1/t)$ decay when derivatives fall on the cut-off, and ultimately explain why algebraic decay comes into play.

In the context of Theorem 1.4, we actually prove that

$$v := u - P$$

converges exponentially fast to 0. This is the purpose of Proposition 4.10, which uses some ideas of [28]. Due to Proposition 1.5, we deduce immediately from there that an exponential convergence is trivial, that is, $z = 0$.

To prove Proposition 4.10, we use monotonicity properties combined with the coercivity of an energy type functional very similar to that used for the existence result. This is why we also need to modulate, and the choice of the orthogonality condition is essential: it allows to bound linear terms in w that appear in the computations. An issue of the mixed breathers/solitons context is that one cannot build a functional adapted to all the nonlinear objects at once, as it is done in [28]. Instead, we carry out an induction and we argue successively around each object, soliton or breather, separately.

1.3.4. Organisation of the paper. Sections 2 and 3 are devoted to the proof of the existence of a multi-breather. Proposition 1.9 is proved in Section 2 and Proposition 1.10 is proved in Section 3. Section 4 gathers the proofs of the uniqueness results. Section 4.1 is devoted to the proof of Proposition 1.5, and Sections 4.2 and 4.3 are devoted to the proof of Theorem 1.4.

2. Construction of a multi-breather in $H^2(\mathbb{R})$

We set

$$\beta := \min\{\beta_k, 1 \le k \le K\} \cup \{\sqrt{c_l}, 1 \le l \le L\},$$
$$\tau := \min\{v_{j+1} - v_j, 1 \le j \le J-1\}. \tag{2.1}$$

Our goal in this section is to prove Proposition 1.9.

2.1. Elementary results

Let us first collect a few basic facts that will be used throughout the article. One may check an exponential decay result for any of our objects.

Proposition 2.1. *Let $j = 1, \dots, J$, $n, m \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \partial_t^m P_j(t,x)| \le Ce^{-\beta|x - v_j t|}.$$

Corollary 2.2. *Let $r > 0$. For t and x such that $v_j t + r < x < v_{j+1} t - r$, we have that*

$$|P(t,x)| \le Ce^{-\beta r}.$$

The same is true for any space or time derivative of P.

We will also use the following cross-product result.

Proposition 2.3. *Let $i \ne j \in \{1, \dots, J\}$ and $m, n \in \mathbb{N}$. There exists a constant C, that depends only on P, such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m P_i \, \partial_x^n P_j \right| \le Ce^{-\beta\tau t/2}.$$

There is also an orthogonality result for breathers that will be useful.

Lemma 2.4. *Let $B := B_{\alpha,\beta}$ be a breather. We denote $B_1 := \partial_{x_1} B$ and $B_2 := \partial_{x_2} B$. Then*

$$\int BB_1 = \int BB_2 = 0.$$

Proof. Note that $\mathrm{Span}(B_1, B_2) = \mathrm{Span}(B_x, B_t)$. Therefore, it is enough to prove that

$$\int BB_x = \int BB_t = 0.$$

Firstly,

$$\int BB_x = \frac{1}{2} \int (B^2)_x = 0.$$

Secondly,

$$\int BB_t = \frac{1}{2} \int (B^2)_t = \frac{1}{2} \frac{d}{dt} \int B^2 = 0,$$

by mass conservation and because a breather is a solution of (1.1). ∎

2.2. Almost-conservation of localized conservation laws

From now on, we will fix $n \in \mathbb{N}$. This is why, for the simplicity of notations, we can write T for T_n, and p for p_n. The goal will be to find constants T^*, $A > 1$ and θ that do not depend on n, nor on the translation parameters of the given objects, and that will be chosen later (T^* will depend on A and θ), such that Proposition 1.9 is verified. We will take $t^* \in [T^*, T]$, and we will make the following bootstrap assumption for the remaining of the article:

$$\|p(t) - P(t)\|_{H^2} \leq Ae^{-\theta t} \quad \text{for all } t \in [t^*, T], \tag{2.2}$$

where $p(T) = P(T)$.

Remark 2.5. We have the following property for solutions of (1.1): there exists $C_0 > 0$ such that for any solution w of (1.1), w is global and

$$\|w(t)\|_{H^2} \leq C_0 \|w(T)\|_{H^2} \quad \text{for all } t \in \mathbb{R},$$

Therefore,

$$\|p(t)\|_{H^2} \leq C_0 \|P(T)\|_{H^2} \leq C_0 \sum_{j=1}^{J} \|P_j(T)\|_{H^2} \leq C_0 C \quad \text{for all } t \in \mathbb{R},$$

where C is a constant that depends only on the problem data (because the H^s-norm of solitons or breathers can be easily bounded).

Let $\theta := \beta\tau/32$. Let $\min(1, \tau/4) > \delta > 0$ be a constant to be chosen later. This part of the proof is adapted from [32]. Let $\psi(x)$ be a C^3 function such that

$$0 \leq \psi \leq 1 \quad \text{and} \quad \psi' \geq 0 \quad \text{on } \mathbb{R}, \quad \psi(x) = \begin{cases} 0 & \text{for } x \leq -1, \\ 1 & \text{for } x \geq 1, \end{cases}$$

and satisfying, for a constant $C > 0$, for any $x \in \mathbb{R}$,

$$(\psi'(x))^{4/3} \leq C\psi(x), \quad (\psi'(x))^{4/3} \leq C(1 - \psi(x)) \quad \text{and} \quad |\psi''(x)|^{3/2} \leq C\psi'(x).$$

Note that it is enough to take ψ equal to $(1+x)^4$ on a neighbourhood of -1 and equal to $1 - (-1+x)^4$ on a neighbourhood of 1. These conditions on ψ will be needed for the proof of Proposition 2.19.

For any $j = 2, \dots, J$, let

$$\sigma_j := \frac{1}{2}(v_{j-1} + v_j).$$

For any $j = 2, \dots, J-1$, let

$$\varphi_j(t,x) := \psi\Big(\frac{x - \sigma_j t}{\delta t}\Big) - \psi\Big(\frac{x - \sigma_{j+1} t}{\delta t}\Big), \tag{2.3}$$

and let

$$\varphi_1(t,x) := 1 - \psi\Big(\frac{x - \sigma_2 t}{\delta t}\Big), \quad \varphi_J(t,x) := \psi\Big(\frac{x - \sigma_J t}{\delta t}\Big), \tag{2.4}$$

so that the function φ_j corresponds obviously to the object P_j. We will also use the notations φ_l^s and φ_k^b, which represent the same functions, and where φ_l^s corresponds to the soliton R_l and φ_k^b corresponds to the breather B_k.

We will also denote, for $j = 2, \dots, J-1$,

$$\varphi_{1,j}(t,x) := \psi'\Big(\frac{x-\sigma_j t}{\delta t}\Big) - \psi'\Big(\frac{x-\sigma_{j+1} t}{\delta t}\Big), \tag{2.5}$$

and

$$\varphi_{1,1}(t,x) := -\psi'\Big(\frac{x-\sigma_2 t}{\delta t}\Big), \quad \varphi_{1,J}(t,x) := \psi'\Big(\frac{x-\sigma_J t}{\delta t}\Big). \tag{2.6}$$

Of course, the notations $\varphi_{1,k}^b$, $\varphi_{1,l}^s$ or $\varphi_{2,j}$ will be used with similar obvious definitions.

We have that, for $j = 1, \dots, J$,

$$|\varphi_{1,j}| \leq C\varphi_j^{3/4}.$$

Remark 2.6. If $\delta \leq \tau/4$,

$$\int_{-\infty}^{\sigma_j t+\delta t} e^{-2\beta|x-v_j t|}\,dx = e^{-2\beta v_j t}\int_{-\infty}^{\sigma_j t+\delta t} e^{2\beta x}\,dx = \frac{1}{2\beta}e^{-2\beta v_j t}e^{\beta(v_j+v_{j-1})t}e^{2\beta\delta t}$$
$$\leq Ce^{-\beta\tau t}e^{2\beta\delta t} \leq Ce^{-\beta\tau t/2},$$

and

$$\int_{\sigma_{j+1}t-\delta t}^{+\infty} e^{-2\beta|x-v_j t|}\,dx \leq Ce^{-\beta\tau t/2}$$

for the same reason, and if $i \neq j$, e.g., $j > i$,

$$\int_{\sigma_j t-\delta t}^{\sigma_{j+1}t+\delta t} e^{-2\beta|x-v_i t|}\,dx = e^{2\beta v_i t}\int_{\sigma_j t-\delta t}^{\sigma_{j+1}t+\delta t} e^{-2\beta x}\,dx$$
$$\leq \frac{1}{2\beta}e^{2\beta v_i t}e^{-\beta(v_j+v_{j-1})t}e^{2\beta\delta t} \leq Ce^{-\beta\tau t}e^{2\beta\delta t} \leq Ce^{-\beta\tau t/2}.$$

Finally, we set, for all $j = 1, \dots, J$,

$$\begin{aligned} M_j(t) &:= \int \frac{1}{2}p^2(t,x)\varphi_j(t,x)\,dx =: M_j[p](t), \\ E_j(t) &:= \int \Big(\frac{1}{2}p_x^2(t,x) - \frac{1}{4}p^4(t,x)\Big)\varphi_j(t,x)\,dx =: E_j[p](t). \end{aligned} \tag{2.7}$$

The notations $M_l^s, M_k^b, E_l^s, E_k^b$ will also be used.

These are local versions of the mass and the energy of the solution p considered (localized around each breather or soliton). We will prove the following result for the localized mass and energy.

Lemma 2.7. *There exist $C > 0$ and $T_1^* := T_1^*(A)$ such that if $T^* \geq T_1^*$, then for any $j = 1, \dots, J$, and any $t \in [t^*, T]$,*

$$|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \leq \frac{C}{\delta^2 t}A^2 e^{-2\theta t}.$$

Proof. We will use the results of the computations made on the bottom of p. 1115 and p. 1116 of [28] to claim the following facts:

$$\frac{d}{dt}\frac{1}{2}\int p^2 f = \int\Big(-\frac{3}{2}p_x^2+\frac{3}{4}p^4\Big)f' - \int p_x p f'',$$
$$\frac{d}{dt}\int\Big[\frac{1}{2}p_x^2-\frac{1}{4}p^4\Big]f = \int\Big[-\frac{1}{2}(p_{xx}+p^3)^2-p_{xx}^2+3p_x^2p^2\Big]f'-\int p_{xx}p_x f'',$$

where f is a C^2 function that does not depend on time.

For $M_j(t)$, which is a sum of quantities of the form $\frac{1}{2}\int p^2\psi(\frac{x-\sigma_j t}{\delta t})$, see (2.7), (2.3), we compute

$$\frac{d}{dt}\frac{1}{2}\int p^2\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big) = \frac{1}{\delta t}\int\Big(-\frac{3}{2}p_x^2+\frac{3}{4}p^4\Big)\psi'\Big(\frac{x-\sigma_j t}{\delta t}\Big) - \frac{1}{(\delta t)^2}\int p_x p\psi''\Big(\frac{x-\sigma_j t}{\delta t}\Big) - \frac{1}{2}\int p^2\frac{x}{\delta t^2}\psi'\Big(\frac{x-\sigma_j t}{\delta t}\Big).$$

The function $\psi'(\frac{x-\sigma_j t}{\delta t})$ is zero outside of $\Omega_j(t) := (-\delta t+\sigma_j t, \delta t+\sigma_j t)$. Thus, for $x\in\Omega_j(t)$, $|x/t|\le|\sigma_j|+|\delta|\le|\sigma_j|+1$, this means that $|x/t|$ is bounded by a constant (that depends only on the given parameters). We can deduce that

$$\Big|\frac{d}{dt}\frac{1}{2}\int p^2\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big)\Big| \le \frac{C}{\delta^2 t}\Big(\int_{\Omega_j(t)}p_x^2+\int_{\Omega_j(t)}p^4+\int_{\Omega_j(t)}p^2\Big).$$

We bound $\int_{\Omega_j(t)}p^4$, using the Sobolev embedding and Remark 2.5, as follows:

$$\int_{\Omega_j(t)}p^4 \le \|p\|_{L^\infty}^2\int_{\Omega_j(t)}p^2 \le C\|p\|_{H^1}^2\int_{\Omega_j(t)}p^2 \le C\int_{\Omega_j(t)}p^2.$$

Thus, for any $t\in[t^*,T]$, we have

$$\Big|\frac{d}{dt}\frac{1}{2}\int p^2\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big)\Big| \le \frac{C}{\delta^2 t}\Big(\int_{\Omega_j(t)}p_x^2+\int_{\Omega_j(t)}p^2\Big).$$

For $E_j(t)$, which is a sum of quantities of the form $\int[\frac{1}{2}p_x^2-\frac{1}{4}p^4]\psi(\frac{x-\sigma_j t}{\delta t})$, see (2.7) and (2.3), we compute

$$\begin{aligned}\frac{d}{dt}\int\Big[\frac{1}{2}p_x^2-\frac{1}{4}p^4\Big]\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big) &= \frac{1}{\delta t}\int\Big[-\frac{1}{2}(p_{xx}+p^3)^2-p_{xx}^2+3p_x^2p\Big]\psi'\Big(\frac{x-\sigma_j t}{\delta t}\Big)\\ &\quad-\frac{1}{(\delta t)^2}\int p_{xx}p_x\psi''\Big(\frac{x-\sigma_j t}{\delta t}\Big)-\int\Big[\frac{1}{2}p_x^2-\frac{1}{4}p^4\Big]\frac{x}{\delta t^2}\psi'\Big(\frac{x-\sigma_j t}{\delta t}\Big).\end{aligned}$$

We deduce from this, by using similar arguments as for the mass, that for any $t\in[t^*,T]$,

$$\Big|\frac{d}{dt}\int\Big[\frac{1}{2}p_x^2-\frac{1}{4}p^4\Big]\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big)\Big| \le \frac{C}{\delta^2 t}\Big(\int_{\Omega_j(t)}p^2+\int_{\Omega_j(t)}p_x^2+\int_{\Omega_j(t)}p_{xx}^2\Big).$$

Now, we write $p(t) = P(t) + (p(t) - P(t))$ and use the triangular inequality to obtain

$$\int_{\Omega_j(t)} (p^2 + p_x^2 + p_{xx}^2) \leq 2\int_{\Omega_j(t)} (P^2 + P_x^2 + P_{xx}^2) + 2\|p - P\|_{H^2}^2.$$

We have assumed that $\|p - P\|_{H^2}^2 \leq A^2 e^{-2\theta t}$, so we need to study P on $\Omega_j(t)$. The following computations work also for the derivatives of P:

$$\begin{aligned}\int_{\Omega_j(t)} P^2 = \int_{\Omega_j(t)} \Big(\sum_{m=1}^{J} P_m(t,x)\Big)^2 dx = \sum_{1\leq m,l\leq J} \int_{\Omega_j(t)} P_m(t,x)P_l(t,x)\,dx \\ \leq C \sum_{1\leq m,l\leq J} \int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|}\,dx,\end{aligned}$$

where we used Proposition 2.1.

We assume that $m \geq j$ (we argue similarly if $m \leq j-1$). Then

$$\begin{aligned} x \in \Omega_j(t) &\iff -\delta t + \sigma_j t \leq x \leq \delta t + \sigma_j t \\ &\iff -\delta t + (\sigma_j - v_m)t \leq x - v_m t \leq \delta t + (\sigma_j - v_m)t. \end{aligned}$$

We note that $\sigma_j - v_m \leq -\tau/2 < 0$. We can thus deduce from the condition on δ that $\sigma_j - v_m + \delta \leq -\tau/4 < 0$. We deduce that $x - v_m t$ is negative for $x \in \Omega_j(t)$. Similarly, if $m \leq j-1$, then $x - v_m t$ is positive for $x \in \Omega_j(t)$. We will now make calculations for different cases. If $m, l \leq j-1$,

$$\begin{aligned}\int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|}\,dx &\leq \int_{\Omega_j(t)} e^{-\beta(x-v_m t)} e^{-\beta(x-v_l t)}\,dx \\ &= \frac{1}{2\beta} e^{\beta t(-v_j - v_{j-1} + v_m + v_l)} (e^{2\beta\delta t} - e^{-2\beta\delta t}) \\ &\leq C e^{\beta t(-v_j - v_{j-1} + v_m + v_l + 2\delta)} \leq C e^{-\beta\tau t/2}.\end{aligned}$$

Similarly, if $m, l \geq j$,

$$\int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|}\,dx \leq C e^{-\beta\tau t/2}.$$

And, if $m \leq j-1$ and $l \geq j$,

$$\begin{aligned}\int_{\Omega_j(t)} e^{-\beta|x-v_m t|} e^{-\beta|x-v_l t|}\,dx &\leq \int_{\Omega_j(t)} e^{-\beta(x-v_m t)} e^{\beta(x-v_l t)}\,dx \\ &\leq 2\delta t e^{\beta t(v_m - v_l)} \leq C e^{-\beta\tau t/2}.\end{aligned}$$

Thus,

$$\int_{\Omega_j(t)} P^2 \leq C e^{-\beta\tau t/2},$$

and the same is valid for the derivatives of P.

Thus, for $t \in [t^*, T]$,

$$\sum_{j=1}^{J}\Big|\frac{d}{dt}\frac{1}{2}\int p^2\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big)\Big| + \Big|\frac{d}{dt}\int\Big[\frac{1}{2}p_x^2 - \frac{1}{4}p^4\Big]\psi\Big(\frac{x-\sigma_j t}{\delta t}\Big)\Big|$$
$$\leq \frac{C}{\delta^2 t}A^2 e^{-2\theta t} + \frac{C}{\delta^2 t}e^{-\beta\tau t/2} \leq \frac{C}{\delta^2 t}(A^2 + e^{-2\theta t})e^{-2\theta t} \leq \frac{C}{\delta^2 t}A^2 e^{-2\theta t}.$$

Thus, for $j = 1, \dots, J$, $t \in [t^*, T]$,

$$|M_j(T) - M_j(t)| + |E_j(T) - E_j(t)| \leq \int_t^T \frac{C}{\delta^2 s}A^2 e^{-2\theta s}\,ds \leq \frac{C}{\delta^2 t}A^2\int_t^T e^{-2\theta s}\,ds$$
$$= \frac{C}{\delta^2 t}A^2\frac{1}{2\theta}(e^{-2\theta t} - e^{-2\theta T}) \leq \frac{C}{\delta^2 t}A^2 e^{-2\theta t}. \quad \blacksquare$$

2.3. Modulation

Lemma 2.8. *There exist $C > 0$ and $T_2^* = T_2^*(A)$ such that if $T^* > T_2^*$, then there exist unique C^1 functions $x_{1,k}\colon [t^*, T] \to \mathbb{R}$, $x_{2,k}\colon [t^*, T] \to \mathbb{R}$, for $1 \leq k \leq K$, and $x_{0,l}, c_{0,l}\colon [t^*, T] \to \mathbb{R}$, for $1 \leq l \leq L$, such that if we set*

$$\varepsilon(t, x) = p(t, x) - \widetilde{B}(t, x) - \widetilde{R}(t, x) = p(t, x) - \widetilde{P}(t, x),$$

with

$$\widetilde{B}(t, x) := \sum_{k=1}^{K}\widetilde{B}_k(t, x), \quad \widetilde{B}_k(t, x) = B_{\alpha_k,\beta_k}(t, x; x_{1,k}^0 + x_{1,k}(t), x_{2,k}^0 + x_{2,k}(t)),$$
$$\widetilde{R}(t, x) := \sum_{l=1}^{L}\widetilde{R}_l(t, x), \quad \widetilde{R}_l(t, x) = \kappa_l Q_{c_l + c_{0,l}(t)}(x - x_{0,l}^0 + x_{0,l}(t) - c_l t),$$

and

$$\widetilde{P}(t, x) := \widetilde{R}(t, x) + \widetilde{B}(t, x) = \sum_{j=1}^{J}\widetilde{P}_j(t, x),$$

where there is the usual correspondence between $\widetilde{P}_j$ and $\widetilde{B}_k$ or $\widetilde{R}_l$, then $\varepsilon(t)$ satisfies, for any $k = 1, \dots, K$, any $l = 1, \dots, L$ and any $t \in [t^*, T]$,

$$\int \widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} = \int \partial_x\widetilde{R}_l(t)\varepsilon(t)\sqrt{\varphi_l^s(t)} = 0,$$
$$\int \partial_{x_1}\widetilde{B}_k(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} = \int \partial_{x_2}\widetilde{B}_k(t)\varepsilon(t)\sqrt{\varphi_k^b(t)} = 0. \tag{2.8}$$

Moreover, for any $t \in [t^*, T]$,

$$\|\varepsilon(t)\|_{H^2} + \sum_{k=1}^{K}(|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^{L}(|x_{0,l}(t)| + |c_{0,l}(t)|) \leq CAe^{-\theta t} \tag{2.9}$$

and

$$(2.10)\quad \sum_{k=1}^{K}(|x'_{1,k}(t)|+|x'_{2,k}(t)|)+\sum_{l=1}^{L}(|x'_{0,l}(t)|+|c'_{0,l}(t)|)\le C\,\|\varepsilon(t)\|_{L^2}+Ce^{-\theta t}.$$

Finally,

$$p(T)=P(T)=\widetilde{P}(T)$$

and

$$\varepsilon(T)=x_{0,l}(T)=x_{1,k}(T)=x_{2,k}(T)=c_{0,l}(T)=0.$$

Proof (see, for example, [13]). Let

$$F_t : L^2(\mathbb{R})\times\mathbb{R}^{2K}\times\mathbb{R}^{2L}\to\mathbb{R}^{2K+2L}\quad \text{for } t\in[t^*,T]$$

be such that

$$\begin{aligned}(w,x_{1,k},x_{2,k},x_{0,l},c_{0,l})\mapsto \Big(&\int\sqrt{\varphi_k^b(t,x)}\,\partial_{x_1}B_{\alpha_k,\beta_k}(t,x;x_{1,k}^0+x_{1,k},x_{2,k}^0+x_{2,k})\epsilon,\\ &\int\sqrt{\varphi_k^b(t,x)}\,\partial_{x_2}B_{\alpha_k,\beta_k}(t,x;x_{1,k}^0+x_{1,k},x_{2,k}^0+x_{2,k})\epsilon,\\ &\int\sqrt{\varphi_l^s(t,x)}\,\partial_x\kappa_l Q_{c_l+c_{0,l}}(x-x_{0,l}^0+x_{0,l}-c_lt)\epsilon,\\ &\int\sqrt{\varphi_l^s(t,x)}\,\kappa_l Q_{c_l+c_{0,l}}(x-x_{0,l}^0+x_{0,l}-c_lt)\epsilon\Big),\end{aligned}$$

where

$$\begin{aligned}\epsilon := w&-\sum_{m=1}^{K}B_{\alpha_m,\beta_m}(t,x;x_{1,m}^0+x_{1,m},x_{2,m}^0+x_{2,m})\\ &-\sum_{n=1}^{L}\kappa_n Q_{c_n+c_{0,n}}(x-x_{0,n}^0+x_{0,n}-c_nt).\end{aligned}$$

We observe that F_t is a C^1 function and that $F_t(P(t),0,0,0,0)=0$. Now, let us consider the matrix which gives the differential of F_t (with respect to $x_{1,k},x_{2,k},x_{0,l},c_{0,l}$) in $(P(t),0,0,0,0)$ (we consider diagonal and extra-diagonal terms for each block):

$$DF_t=\begin{pmatrix} B^1_{k,k} & B^3_{k,k} & \times & \times & \times & \times & \times & \times\\ B^3_{k,k} & B^2_{k,k} & \times & \times & \times & \times & \times & \times\\ \times & \times & B^1_{k',k'} & B^3_{k',k'} & \times & \times & \times & \times\\ \times & \times & B^3_{k',k'} & B^2_{k',k'} & \times & \times & \times & \times\\ \times & \times & \times & \times & R^1_{l,l} & R^4_{l,l} & \times & \times\\ \times & \times & \times & \times & R^3_{l,l} & R^2_{l,l} & \times & \times\\ \times & \times & \times & \times & \times & \times & R^1_{l',l'} & R^4_{l',l'}\\ \times & \times & \times & \times & \times & \times & R^3_{l',l'} & R^2_{l',l'} \end{pmatrix},$$

where

$$\begin{cases} B^1_{k,k} := -\int (\partial_{x_1} B_{\alpha_k,\beta_k})^2 \sqrt{\varphi^b_k}, \quad B^2_{k,k} := -\int (\partial_{x_2} B_{\alpha_k,\beta_k})^2 \sqrt{\varphi^b_k}, \\ B^3_{k,k} := -\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \sqrt{\varphi^b_k}, \end{cases}$$

$$\begin{cases} R^1_{l,l} := -\int (\partial_x Q_{c_l}(y^0_{0,l}))^2 \sqrt{\varphi^s_l}, \quad R^3_{l,l} := -\int Q_{c_l}(y^0_{0,l})\partial_x Q_{c_l}(y^0_{0,l}) \sqrt{\varphi^s_l}, \\ R^2_{l,l} := -\frac{1}{2c_l} \int Q_{c_l}(y^0_{0,l})\big(Q_{c_l}(y^0_{0,l}) + y^0_{0,l}\partial_x Q_{c_l}(y^0_{0,l})\big) \sqrt{\varphi^s_l}, \\ R^4_{l,l} := -\frac{1}{2c_l} \int \partial_x Q_{c_l}(y^0_{0,l})\big(Q_{c_l}(y^0_{0,l}) + y^0_{0,l}\partial_x Q_{c_l}(y^0_{0,l})\big) \sqrt{\varphi^s_l}, \end{cases}$$

denoting $y^0_{0,l} := x - x^0_{0,l} - c_l t$, and sign "×" indicates exponentially decaying terms when $t \to +\infty$; and where we consider variables in the following order: $x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, \ldots, x_{1,K}, x_{2,K}, x_{0,1}, c_{0,1}, \ldots, x_{0,L}, c_{0,L}$, and we order the coefficients of the function in a similar way. This is a matrix with dominant diagonal blocks.

Note that $B^1_{k,k}$ is exponentially close to $-\int (\partial_{x_1} B_{\alpha_k,\beta_k})^2$, because if $P_j = B_k$ is a breather, then

$$\begin{aligned} \int (\partial_{x_1} B_{\alpha_k,\beta_k})^2 (1 - \sqrt{\varphi^b_k}) &\leq \int_{-\infty}^{\sigma_j t + \delta t} (\partial_{x_1} B_{\alpha_k,\beta_k})^2 + \int_{\sigma_{j+1} t + \delta t}^{+\infty} (\partial_{x_1} B_{\alpha_k,\beta_k})^2 \\ &\leq C \int_{-\infty}^{\sigma_j t + \delta t} e^{-2\beta |x - v_j t|} + \int_{\sigma_{j+1} t + \delta t}^{+\infty} e^{-2\beta |x - v_j t|} \\ &\leq C e^{-\beta \tau t/2}, \end{aligned}$$

and the same is true for the other dominant diagonal terms of the matrix (we can get rid of φs).

Therefore, the determinant of the matrix is exponentially close to

$$\begin{aligned} \det(DF_t) = \prod_{k=1}^{K} \Big(&\int (\partial_{x_1} B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k}))^2 \int (\partial_{x_2} B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k}))^2 \\ &- \Big(\int \partial_{x_1} B_{\alpha_k,\beta_k} \partial_{x_2} B_{\alpha_k,\beta_k} \Big)^2 \Big) \\ &\times \prod_{l=1}^{L} \Big(\frac{1}{2c_l} \int Q_{c_l}(y^0_{0,l})(Q_{c_l}(y^0_{0,l}) + y^0_{0,l} \partial_x Q_{c_l}(y^0_{0,l})) \int (\partial_x Q_{c_l}(y^0_{0,l}))^2 \Big), \end{aligned}$$

because $\int Q_{c_l}(y^0_{0,l}) \partial_x Q_{c_l}(y^0_{0,l})\, dx = 0$.

By the Cauchy–Schwarz inequality and the fact that

$$\partial_{x_1} B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k}) \quad \text{and} \quad \partial_{x_2} B_{\alpha_k,\beta_k}(t, x; x^0_{1,k}, x^0_{2,k})$$

are linearly independent as functions of the x variable, for any time t fixed, we see that the first product is positive. Since each member of the product is periodic in time, the first product is bounded below by a positive constant independent from time and translation parameters.

For the second product, by translation of the variable in the integrations, for any time t fixed, we see that we can replace $y^0_{0,l}$ by x. Then, by integration by parts,

$$\int x Q_{c_l}(x)\,\partial_x Q_{c_l}(x)\,dx = -\frac{1}{2}\int Q_{c_l}(x)^2\,dx.$$

By scaling, if q denotes the soliton with $c = 1$, i.e., $q = Q_1$, then

$$\int Q^2_{c_l} = \sqrt{c_l}\int q^2, \quad \int \partial_x Q^2_{c_l} = c_l^{3/2}\int q^2_x.$$

Therefore,

$$\begin{aligned}(2.11)\qquad &\frac{1}{2c_l}\int Q_{c_l}(y^0_{0,l})\big(Q_{c_l}(y^0_{0,l}) + y^0_{0,l}\partial_x Q_{c_l}(y^0_{0,l})\big)\int(\partial_x Q_{c_l}(y^0_{0,l}))^2\\ &= \frac{1}{4}c_l\int q^2\int (q_x)^2 \geq \frac{1}{4}\min\{c_n, 1\leq n\leq L\}\int q^2\int q^2_x.\end{aligned}$$

This means that the second product is bounded below by a positive constant independent from time and translation parameters.

Therefore, if T^*_2 is large enough, the considered matrix is invertible.

Now, we may use the implicit function theorem (actually, we use a quantitative version of the implicit function theorem, see Section 2.2 in [11] for a precise statement). If w is close enough to $P(t)$, then there exists

$$(2.12)\qquad (x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l})$$

such that

$$F_t(w, x_{1,k}, x_{2,k}, x_{0,l}, c_{0,l}) = 0,$$

where (2.12) depends in a regular C^1 way on w. It is possible to show that the "close enough" in the previous sentence does not depend on t; for this, it is required to use a uniform implicit function theorem. This means that for T^*_2 large enough (depending on A), $Ae^{-\theta t}$ is small enough for $t \in [t^*, T]$, thus for $t \in [t^*, T]$, $p(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have, for $t \in [t^*, T]$, the existence of $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{0,l}(t)$ and $c_{0,l}(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $p(t)$ and that $p(t)$ has a similar regularity in time (see [13] for more details).

Now, we prove inequalities (2.9) and (2.10). We can take the differential of the implicit functions with respect to $p(t)$ for $t \in [t^*, T]$. For this, we differentiate the following equation with respect to $p(t)$:

$$F_t\big(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t)), c_{0,l}(p(t))\big) = 0.$$

We know that the matrix that gives the differential of F_t (with respect to $x_{1,k}$, $x_{2,k}$, $x_{0,l}$ and $c_{0,l}$) in

$$\big(p(t), x_{1,k}(p(t)), x_{2,k}(p(t)), x_{0,l}(p(t)), c_{0,l}(p(t))\big)$$

is invertible and its inverse is bounded in time (from the formula giving the inverse of a matrix from the comatrix and the determinant). The differential of F_t with respect to the first variable is also bounded. Thus, by the mean value theorem,

$$|x_{1,k}| \leq C\|p - P\| \leq CAe^{-\theta t}.$$

The same is true for $x_{2,k}$, $x_{0,l}$ and $c_{0,l}$.

By applying the mean value theorem (inequality) for Q_{c_l} with respect to $x_{0,l}$ and $c_{0,l}$ or for B_{α_k,β_k} with respect to $x_{1,k}$ and $x_{2,k}$, we deduce that

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C(|x_{1,k}(t)| + |x_{2,k}(t)|)$$

if $P_j = B_k$ is a breather, and

$$\|P_j(t) - \widetilde{P}_j(t)\|_{H^2} \leq C(|x_{0,l}(t)| + |c_{0,l}(t)|)$$

if $P_j = R_l$ is a soliton.

Finally, by the triangular inequality,

$$\begin{aligned}\|\varepsilon(t)\|_{H^2} &\leq \|p(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2} \\ &\leq \|p(t)-P(t)\|_{H^2} + C\left(\sum_{k=1}^{K}(|x_{1,k}(t)|+|x_{2,k}(t)|)+\sum_{l=1}^{L}(|x_{0,l}(t)| + |c_{0,l}(t)|)\right) \\ &\leq C\|p(t) - P(t)\|_{H^2} \leq CAe^{-\theta t}.\end{aligned}$$

This completes the proof of (2.9).

For (2.10), we will take time derivatives of the equations (2.8). From now on, we write $\widetilde{B}_{k1}$ for $\partial_{x_1}\widetilde{B}_k$ and $\widetilde{B}_{k2}$ for $\partial_{x_2}\widetilde{B}_k$. Firstly, we write the PDE satisfied by ε (knowing that $p, B_1, \ldots, B_K, R_1, \ldots, R_L$ are solutions of (1.1)):

$$\begin{aligned}\partial_t\varepsilon = -\varepsilon_{xxx} - \left[\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{j=1}^{J}\widetilde{P}_j + 3\sum_{i,j=1}^{J}\widetilde{P}_i\widetilde{P}_j\right)\right]_x - \sum_{k=1}^{K}x'_{1,k}(t)\widetilde{B}_{k1} - \sum_{k=1}^{K}x'_{2,k}(t)\widetilde{B}_{k2} \\ - \sum_{l=1}^{L}x'_{0,l}(t)\widetilde{R}_{lx} - \sum_{l=1}^{L}\frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))}(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{lx}) - \sum_{h\neq i \text{ or } i\neq j}(\widetilde{P}_h\widetilde{P}_i\widetilde{P}_j)_x,\end{aligned}$$

where $y_{0,l}(t) := x - x^0_{0,l} + x_{0,l}(t) - c_l t$. Now, we will take the time derivative of the equation $\int \widetilde{B}_{k1}\,\varepsilon\sqrt{\varphi^b_k} = 0$ (and perform integration by parts):

$$\begin{aligned}(2.13)\quad & -\int(\widetilde{B}^3_k)_{1x}\,\varepsilon\sqrt{\varphi^b_k} - \int\widetilde{B}_{k1}\sum_{h\neq i \text{ or } g\neq h}(\widetilde{P}_h\widetilde{P}_i\widetilde{P}_g)_x\sqrt{\varphi^b_k} + x'_{2,k}(t)\int\widetilde{B}_{k12}\,\varepsilon\sqrt{\varphi^b_k} \\ &+ \frac{1}{2\delta t}\int\widetilde{B}_{k1}\,\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P}_h\widetilde{P}_i\right)\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\ &+ \int\widetilde{B}_{k1x}\,\varepsilon\left(\varepsilon^2 + 3\varepsilon\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P}_h\widetilde{P}_i\right)\sqrt{\varphi^b_k} \\ &- \frac{1}{2\delta t^2}\int\widetilde{B}_{k1}\,\varepsilon x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} + \frac{1}{2\delta t}\int\widetilde{B}_{k1}\,\varepsilon_{xx}\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} \\ &- \frac{1}{2\delta t}\int\widetilde{B}_{k1x}\,\varepsilon_x\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} + \frac{1}{2\delta t}\int\widetilde{B}_{k1xx}\,\varepsilon\frac{\varphi^b_{1,k}}{\sqrt{\varphi^b_k}} + x'_{1,k}(t)\int\widetilde{B}_{k11}\,\varepsilon\sqrt{\varphi^b_k}\end{aligned}$$

$$= \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{B}_{k1} \widetilde{B}_{m1} \sqrt{\varphi_k^b} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{B}_{k1} \widetilde{B}_{m2} \sqrt{\varphi_k^b}$$

$$+ \sum_{n=1}^{L} x'_{0,n}(t) \int \widetilde{B}_{k1} \widetilde{R}_{nx} \sqrt{\varphi_k^b} + \sum_{n=1}^{L} \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{B}_{k1} (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_k^b}.$$

Similarly, taking the time derivative of $\int \widetilde{B}_{k2} \varepsilon \sqrt{\varphi_k^b} = 0$,

$$(2.14) \quad -\int (\widetilde{B}_k^3)_{2x}\, \varepsilon \sqrt{\varphi_k^b} - \int \widetilde{B}_{k2} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \sqrt{\varphi_k^b} + x'_{2,k}(t) \int \widetilde{B}_{k22}\, \varepsilon \sqrt{\varphi_k^b}$$

$$+ \frac{1}{2\delta t} \int \widetilde{B}_{k2}\, \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}$$

$$+ \int \widetilde{B}_{k2x}\, \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \sqrt{\varphi_k^b}$$

$$+ \frac{1}{2\delta t} \int \widetilde{B}_{k2}\, \varepsilon_{xx} \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} - \frac{1}{2\delta t} \int \widetilde{B}_{k2x}\, \varepsilon_x \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} + \frac{1}{2\delta t} \int \widetilde{B}_{k2xx}\, \varepsilon \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}}$$

$$- \frac{1}{2\delta t^2} \int \widetilde{B}_{k2}\, \varepsilon x \frac{\varphi_{1,k}^b}{\sqrt{\varphi_k^b}} + x'_{1,k}(t) \int \widetilde{B}_{k12}\, \varepsilon \sqrt{\varphi_k^b}$$

$$= \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{B}_{k2} \widetilde{B}_{m1} \sqrt{\varphi_k^b} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{B}_{k2} \widetilde{B}_{m2} \sqrt{\varphi_k^b}$$

$$+ \sum_{n=1}^{L} x'_{0,n}(t) \int \widetilde{B}_{k2} \widetilde{R}_{nx} \sqrt{\varphi_k^b} + \sum_{n=1}^{L} \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{B}_{k2} (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_k^b}.$$

Similarly, taking the time derivative of $\int \widetilde{R}_{lx}(t)\, \varepsilon(t) \sqrt{\varphi_l^s} = 0$,

$$(2.15) \quad -\int (\widetilde{R}_l^3)_{xx}\, \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))} \int (\widetilde{R}_{lx} + y_{0,l}(t) \widetilde{R}_{lxx})\, \varepsilon \sqrt{\varphi_l^s}$$

$$+ x'_{0,l}(t) \int \widetilde{R}_{lxx}\, \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R}_{lx}\, \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}}$$

$$+ \int \widetilde{R}_{lxx}\, \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \sqrt{\varphi_l^s}$$

$$- \frac{1}{2\delta t^2} \int \widetilde{R}_{lx}\, \varepsilon x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t} \int \widetilde{R}_{lx}\, \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R}_{lxx}\, \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}}$$

$$+ \frac{1}{2\delta t} \int \widetilde{R}_{lxxx}\, \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \int \widetilde{R}_{lx} \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \sqrt{\varphi_l^s}$$

$$= \sum_{n=1}^{L} x'_{0,n}(t) \int \widetilde{R}_{lx} \widetilde{R}_{nx} \sqrt{\varphi_l^s} + \sum_{n=1}^{L} \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{R}_{lx} (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_l^s}$$
$$+ \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{R}_{lx} \widetilde{B}_{m1} \sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{R}_{lx} \widetilde{B}_{m2} \sqrt{\varphi_l^s}.$$

Finally, taking the time derivative of $\int \widetilde{R}_l \varepsilon \sqrt{\varphi_l^s} = 0$,

$$(2.16) \quad -\int (\widetilde{R}_l^3)_x \varepsilon \sqrt{\varphi_l^s} + \frac{c'_{0,l}(t)}{2(c_l + c_{0,l}(t))} \int (\widetilde{R}_l + y_{0,l}(t) \widetilde{R}_{lx}) \varepsilon \sqrt{\varphi_l^s}$$
$$+ x'_{0,l}(t) \int \widetilde{R}_{lx} \varepsilon \sqrt{\varphi_l^s} + \frac{1}{2\delta t} \int \widetilde{R}_l \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}}$$
$$+ \int \widetilde{R}_{lx} \varepsilon \Big(\varepsilon^2 + 3\varepsilon \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i \Big) \sqrt{\varphi_l^s}$$
$$- \frac{1}{2\delta t^2} \int \widetilde{R}_l \varepsilon x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} + \frac{1}{2\delta t} \int \widetilde{R}_l \varepsilon_{xx} \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \frac{1}{2\delta t} \int \widetilde{R}_{lx} \varepsilon_x \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}}$$
$$+ \frac{1}{2\delta t} \int \widetilde{R}_{lxx} \varepsilon \frac{\varphi_{1,l}^s}{\sqrt{\varphi_l^s}} - \int \widetilde{R}_l \sum_{h \neq i \text{ or } g \neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x \sqrt{\varphi_l^s}$$
$$= \sum_{n=1}^{L} x'_{0,n}(t) \int \widetilde{R}_l \widetilde{R}_{nx} \sqrt{\varphi_l^s} + \sum_{n=1}^{L} \frac{c'_{0,n}(t)}{2(c_n + c_{0,n}(t))} \int \widetilde{R}_l (\widetilde{R}_n + y_{0,n}(t) \widetilde{R}_{nx}) \sqrt{\varphi_l^s}$$
$$+ \sum_{m=1}^{K} x'_{1,m}(t) \int \widetilde{R}_l \widetilde{B}_{m1} \sqrt{\varphi_l^s} + \sum_{m=1}^{K} x'_{2,m}(t) \int \widetilde{R}_l \widetilde{B}_{m2} \sqrt{\varphi_l^s}.$$

By Proposition 2.10 below (that follows from the first part of the lemma we prove) and its corollary, several terms of equalities (2.13), (2.14), (2.15) and (2.16) are bounded by $Ce^{-\theta t}$; other terms are $O(\|\varepsilon\|_{L^2})$. We recall that $O(\|\varepsilon\|_{L^2}) \leq CAe^{-\theta t}$. From the basic properties of φ_j (see Section 2.2), $\varphi_{1,j}/\sqrt{\varphi_j}$ is bounded. Because of the compact support of φ_j, $\frac{x}{t}\varphi_{1,j}/\sqrt{\varphi_j}$ is bounded independently of x and t. Using these bounds, and after several linear combinations, we obtain the desired inequalities. ∎

Remark 2.9. As a consequence of Lemma 2.8, there exists a constant $C > 0$ such that, for all $t \in [t^*, T]$,

$$\sum_{k=1}^{K} (|x_{1,k}(t)| + |x_{2,k}(t)|) + \sum_{l=1}^{L} (|x_{0,l}(t)| + |c_{0,l}(t)|) \leq CAe^{-\theta T^*}.$$

This means that if we take T_2^* eventually larger (which we will assume in the following of the article), we may extend Proposition 2.1 to $\widetilde{P}_j$ in the following way, by integration of the bounds given by modulation (the constant C is a bit larger in a controlled way, we write $\beta/2$ because the shape of the solitons is a bit modified in a controlled way).

Proposition 2.10. *Let $j = 1, \dots, J$, $n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant $C > 0$ such that for any $t, x \in \mathbb{R}$,*

$$|\partial_x^n \widetilde{P}_j(t,x)| \le C e^{-\frac{\beta}{2}|x - v_j t|}.$$

We will also use that any $\|\partial_x^n \widetilde{P}_j\|_{H^2}$ is bounded by C.

Corollary 2.11. *Let $i \neq j \in \{1, \dots, J\}$ and $m, n \in \mathbb{N}$. If $T^* > T_2^*$, then there exists a constant C that depends only on P such that for any $t \in \mathbb{R}$,*

$$\left| \int \partial_x^m \widetilde{P}_i \, \partial_x^n \widetilde{P}_j \right| \le C e^{-\beta \tau t/8}.$$

2.4. Study of coercivity

In [3], the Lyapunov functional that was introduced to study the orbital stability of a breather is the following conserved-in-time functional:

$$F[p](t) + 2(\beta^2 - \alpha^2) E[p](t) + (\alpha^2 + \beta^2)^2 M[p](t).$$

The functional that we will consider here is adapted from the latter. For $t \in [t^*, T]$, we set

$$\begin{aligned} \mathcal{H}[p](t) := F[p](t) &+ \sum_{k=1}^{K} \big(2(\beta_k^2 - \alpha_k^2) E_k^b[p](t) + (\alpha_k^2 + \beta_k^2)^2 M_k^b[p](t) \big) \\ &+ \sum_{l=1}^{L} \big(2c_l E_l^s[p](t) + c_l^2 M_l^s[p](t) \big). \end{aligned}$$

For simplicity of notations, for $j \in \{1, \dots, J\}$, a_j will denote α_k if P_j is the breather B_k or 0 if P_j is a soliton, and b_j will denote β_k if P_j is the breather B_k or $c_l^{1/2}$ if P_j is the soliton R_l. With these notations, we may write

$$\mathcal{H}[p](t) = F[p](t) + \sum_{j=1}^{J} \big(2(b_j^2 - a_j^2) E_j[p](t) + (a_j^2 + b_j^2)^2 M_j[p](t) \big).$$

We would like to study locally this functional around the considered sum of breathers and solitons. The aim of this section will be to prove the two following propositions.

Proposition 2.12 (Expansion of H^2 conserved quantity). *There exists $T_4^* > 0$ such that if $T^* \ge T_4^*$, for all $t \in [t^*, T]$, we have that*

$$\begin{aligned} \mathcal{H}[p](t) = &\sum_{j=1}^{J} \big(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t) \big) \\ &+ H_2[\varepsilon](t) + O(\|\varepsilon(t)\|_{H^2}^3) + O(e^{-2\theta t} \|\varepsilon(t)\|_{H^2}) + O(e^{-2\theta t}), \end{aligned}$$

where

$$\begin{aligned} H_2[\varepsilon](t) := &\frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int \widetilde{P}^2 \varepsilon_x^2 + \frac{5}{2} \int \widetilde{P}_x^2 \varepsilon^2 + 5 \int \widetilde{P} \widetilde{P}_{xx} \varepsilon^2 + \frac{15}{4} \int \widetilde{P}^4 \varepsilon^2 \\ &+ \sum_{j=1}^{J} (b_j^2 - a_j^2) \Big(\int \varepsilon_x^2 \varphi_j - 3 \int \widetilde{P}^2 \varepsilon^2 \varphi_j \Big) + \sum_{j=1}^{J} (a_j^2 + b_j^2)^2 \frac{1}{2} \int \varepsilon^2 \varphi_j. \end{aligned}$$

Proposition 2.13 (Coercivity of H_2). *There exist $\mu > 0$ and $T_3^* = T_3^*(A)$ such that if $T^* \geq T_3^*$, then we have, for any $t \in [t^*, T]$,*

$$H_2[\varepsilon](t) \geq \mu \|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu} \sum_{k=1}^{K} \Big(\int \varepsilon \widetilde{B}_k \sqrt{\varphi_k^b} \Big)^2.$$

Propositions 2.12 and 2.13 will be used in the next concluding subsection to prove Proposition 1.9.

Firstly, let us prove Proposition 2.12.

Proof of Proposition 2.12. We would like to compare $\mathcal{H}[\widetilde{P} + \varepsilon](t)$ and $\mathcal{H}[\widetilde{P}](t)$ (recall that $p = \widetilde{P} + \varepsilon$) by studying the difference asymptotically when ε is small. First, let us see how we could simplify the expression of $\mathcal{H}[\widetilde{P}](t)$.

Claim 2.14. If T^* is large enough, for all $t \in [t^*, T]$, we have that

$$\mathcal{H}[\widetilde{P}](t) = \sum_{j=1}^{J} \big(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t) \big) + O(e^{-2\theta t}).$$

Proof. We prove that, for $t \in [t^*, T]$,

$$\Big| \mathcal{H}[\widetilde{P}] - \sum_{j=1}^{J} \big(F[\widetilde{P}_j] + 2(b_j^2 - a_j^2) E[\widetilde{P}_j] + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j] \big) \Big| \leq C e^{-2\theta t}.$$

Let us compare $F_j[\widetilde{P}]$ and $F[\widetilde{P}_j]$:

$$F_j[\widetilde{P}] = \int \Big(\frac{1}{2} \widetilde{P}_{xx}^2 - \frac{5}{2} \widetilde{P}^2 \widetilde{P}_x^2 + \frac{1}{4} \widetilde{P}^6 \Big) \varphi_j(t,x)\, dx,$$
$$F[\widetilde{P}_j] = \int \Big(\frac{1}{2} \widetilde{P}_{jxx}^2 - \frac{5}{2} \widetilde{P}_j^2 \widetilde{P}_{jx}^2 + \frac{1}{4} \widetilde{P}_j^6 \Big)\, dx.$$

We compare the corresponding terms of these equalities. Let us start with the first one:

$$\begin{aligned}
&\Big| \int (\widetilde{P}_{xx}^2 \varphi_j(t,x) - \widetilde{P}_{jxx}^2) \Big| \\
&\leq \int \widetilde{P}_{jxx}^2 |1 - \varphi_j(t,x)| + \sum_{(r,s) \neq (j,j)} \int |\widetilde{P}_{rxx} \widetilde{P}_{sxx}| \varphi_j(t,x) \\
&\leq C \int e^{-\frac{\beta}{2}|x - v_j t|} e^{\beta \tau t/32} \big| 1 - \varphi_j(t,x) \big|\, dx + C \sum_{i \neq j} \int e^{-\frac{\beta}{2}|x - v_i t|} e^{\beta \tau t/32} \varphi_j(t,x)\, dx \\
&\leq C e^{\beta \tau t/32} \Big[\Big(\int_{-\infty}^{\sigma_j t + \delta t} + \int_{\sigma_{j+1} t - \delta t}^{+\infty} \Big) e^{-\frac{\beta}{2}|x - v_j t|}\, dx + \sum_{i \neq j} \int_{\sigma_j t - \delta t}^{\sigma_{j+1} t + \delta t} e^{-\frac{\beta}{2}|x - v_i t|}\, dx \Big] \\
&\leq C e^{-\beta \tau t/16},
\end{aligned}$$

by Proposition 2.10 and Remark 2.6. For the other terms of the difference to be bounded, we reason in a similar way. This completes the proof of the claim. ∎

Therefore, when we will be able to compare $\mathcal{H}[p](t)$ and $\mathcal{H}[\widetilde{P}](t)$, we will also be able to compare $\mathcal{H}[p](t)$ and

$$\sum_{j=1}^{J}\big(F[\widetilde{P}_j](t)+2(b_j^2-a_j^2)E[\widetilde{P}_j](t)+(a_j^2+b_j^2)^2M[\widetilde{P}_j](t)\big).$$

We compute the Taylor expansion of $\mathcal{H}[p]=\mathcal{H}[\widetilde{P}+\varepsilon]$:

$$\begin{aligned}\mathcal{H}[\widetilde{P}+\varepsilon] &= \frac{1}{2}\int(\widetilde{P}+\varepsilon)_{xx}^2-\frac{5}{2}\int(\widetilde{P}+\varepsilon)^2(\widetilde{P}+\varepsilon)_x^2+\frac{1}{4}\int(\widetilde{P}+\varepsilon)^6\\
&\quad+\sum_{j=1}^{J}\Big[(b_j^2-a_j^2)\Big(\int(\widetilde{P}+\varepsilon)_x^2\varphi_j-\frac{1}{2}\int(\widetilde{P}+\varepsilon)^4\varphi_j\Big)\Big]\\
&\quad+\sum_{j=1}^{J}\Big[(a_j^2+b_j^2)^2\frac{1}{2}\int(\widetilde{P}+\varepsilon)^2\varphi_j\Big]\\
&=\frac{1}{2}\int\widetilde{P}_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\widetilde{P}_x^2+\frac{1}{4}\int\widetilde{P}^6+\int\widetilde{P}_{(4x)}\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon\\
&\quad+5\int\widetilde{P}^2\widetilde{P}_{xx}\,\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon+\frac{1}{2}\int\varepsilon_{xx}^2-\frac{5}{2}\int\widetilde{P}^2\varepsilon_x^2\\
&\quad+\frac{5}{2}\int\widetilde{P}_x^2\,\varepsilon^2+5\int\widetilde{P}\widetilde{P}_{xx}\,\varepsilon^2+\frac{15}{4}\int\widetilde{P}^4\varepsilon^2+O(\|\varepsilon(t)\|_{H^2}^3)\\
&\quad+\sum_{j=1}^{J}(b_j^2-a_j^2)\Big(\int\widetilde{P}_x^2\varphi_j-\frac{1}{2}\int\widetilde{P}^4\varphi_j-2\int\widetilde{P}_{xx}\,\varepsilon\varphi_j\\
&\quad-2\int\widetilde{P}_x\,\varepsilon\varphi_{j,x}-2\int\widetilde{P}^3\varepsilon\varphi_j+\int\varepsilon_x^2\varphi_j-3\int\widetilde{P}^2\varepsilon^2\varphi_j\Big)\\
&\quad+\sum_{j=1}^{J}(a_j^2+b_j^2)^2\frac{1}{2}\Big(\int\widetilde{P}^2\varphi_j+2\int\widetilde{P}\varepsilon\varphi_j+\int\varepsilon^2\varphi_j\Big).\end{aligned}\tag{2.17}$$

We can observe that the sum (2.17) is composed of 0th-order terms in ε, of 1st-order terms in ε, of 2nd-order terms in ε; 3rd and larger-order terms in ε are contained in $O(\|\varepsilon(t)\|_{H^2}^3)$. The sum of the 0th-order terms is actually $\mathcal{H}[\widetilde{P}]$. The sum of 2nd-order terms in ε is $H_2[\varepsilon](t)$.

Let us study more closely the 1st-order terms:

$$\begin{aligned}H_1&=\int\widetilde{P}_{(4x)}\,\varepsilon+5\int\widetilde{P}\widetilde{P}_x^2\varepsilon+5\int\widetilde{P}^2\widetilde{P}_{xx}\,\varepsilon+\frac{3}{2}\int\widetilde{P}^5\varepsilon\\
&\quad+\sum_{j=1}^{J}(b_j^2-a_j^2)\Big(2\int\widetilde{P}_x\,\varepsilon_x\,\varphi_j-2\int\widetilde{P}^3\varepsilon\varphi_j\Big)+\sum_{j=1}^{J}(a_j^2+b_j^2)^2\int\widetilde{P}\varepsilon\varphi_j.\end{aligned}$$

From [3], we know that a breather $A=A_{\alpha,\beta}$ satisfies for any fixed $t\in\mathbb{R}$, the following nonlinear equation:

$$A_{(4x)}-2(\beta^2-\alpha^2)(A_{xx}+A^3)+(\alpha^2+\beta^2)^2A+5AA_x^2+5A^2A_{xx}+\frac{3}{2}A^5=0.$$

This equation is also satisfied for $A = \widetilde{B}_k$ with $\alpha = \alpha_k$ and $\beta = \beta_k$ for any $k = 1, \dots, K$ (the shape parameters of a breather are not changed by modulation).

For a soliton $Q = R_{c,\kappa}$, we know from $Q_{xx} = cQ - Q^3$ that Q satisfies, for any fixed $t \in \mathbb{R}$, the following nonlinear equation (see Appendix A.1):

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 = 0.$$

This equation is not exactly satisfied for $Q = \widetilde{R}_l$ for any $l = 1, \dots, L$ (the shape parameters of a soliton are changed by modulation). The exact equation satisfied by $Q = \widetilde{R}_l$ is

$$\begin{aligned} Q_{(4x)} - 2c_l(Q_{xx} + Q^3) + c_l^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 \\ = 2c_{0,l}(t)(Q_{xx} + Q^3) - 2c_l c_{0,l}(t) Q - c_{0,l}(t)^2 Q. \end{aligned}$$

We will compare H_1 and

$$\begin{aligned} H_1' := \int \widetilde{P}_{(4x)}\varepsilon + 5\sum_{j=1}^{J}\int \widetilde{P}_j \widetilde{P}_{jx}^2\varepsilon + 5\sum_{j=1}^{J}\int \widetilde{P}_{jx}^2 \widetilde{P}_{jxx}\varepsilon + \frac{3}{2}\sum_{j=1}^{J}\int \widetilde{P}_j^5\varepsilon \\ - 2\sum_{j=1}^{J}(b_j^2 - a_j^2)\left(\int \widetilde{P}_{jxx}\varepsilon + \int \widetilde{P}_j^3\varepsilon\right) + \sum_{j=1}^{J}(a_j^2 + b_j^2)^2 \int \widetilde{P}_j\varepsilon. \end{aligned}$$

Firstly, let us compare $\int \widetilde{P}\widetilde{P}_x^2\varepsilon$ and $\sum_{j=1}^{J}\int \widetilde{P}_j \widetilde{P}_{jx}^2\varepsilon$:

$$\int \widetilde{P}\widetilde{P}_x^2\varepsilon = \int \left(\sum_{j=1}^{J}\widetilde{P}_j\right)\left(\sum_{j=1}^{J}\widetilde{P}_{jx}\right)^2\varepsilon = \sum_{j=1}^{J}\int \widetilde{P}_j \widetilde{P}_{jx}^2\varepsilon + \sum_{h\neq i \text{ or } i\neq j}\int \widetilde{P}_h \widetilde{P}_{ix}\widetilde{P}_{jx}\varepsilon.$$

To succeed, we need to find a bound for a term of the type $\int \widetilde{P}_h \widetilde{P}_{ix}\widetilde{P}_{jx}\varepsilon$, where $h \neq i$ or $i \neq j$. We can perform the following upper bounding (where, without loss of generality, we suppose that $i \neq j$):

$$\begin{aligned} \left|\int \widetilde{P}_h \widetilde{P}_{ix}\widetilde{P}_{jx}\varepsilon\right| &\leq Ce^{\beta\tau t/16}\int e^{-\frac{\beta}{2}|x-v_i t|}e^{-\frac{\beta}{2}|x-v_j t|}|\varepsilon| \\ &\leq C\|\varepsilon\|_{L^\infty}e^{\beta\tau t/16}\int e^{-\frac{\beta}{2}|x-v_i t|}e^{-\frac{\beta}{2}|x-v_j t|} \leq C\|\varepsilon\|_{H^2}e^{-\beta\tau t/8}, \end{aligned}$$

by Sobolev embeddings and Proposition 2.3.

The bounding is quite similar for $\int \widetilde{P}^2\widetilde{P}_{xx}\varepsilon$ and $\int \widetilde{P}^5\varepsilon$. We observe that $-\int \widetilde{P}_{jxx}\varepsilon = \int \widetilde{P}_{jx}\varepsilon_x$. To compare $\int \widetilde{P}_x\varepsilon_x\varphi_j$ and $\int_{\mathbb{R}}\widetilde{P}_{jx}\varepsilon_x$, and for similar terms, we can use computations that we have already performed at the beginning of this proof. Therefore,

$$\left|\int \widetilde{P}_x\varepsilon_x\varphi_j - \int_{\mathbb{R}}\widetilde{P}_{jx}\varepsilon_x\right| \leq C\|\varepsilon\|_{H^2}e^{-\beta\tau t/16}.$$

This enables us to bound the difference between H_1 and H_1':

$$|H_1 - H_1'| \leq C\|\varepsilon(t)\|_{H^2}e^{-\beta\tau t/16}.$$

Now, because our objects are not only breathers, H_1' is not equal to 0. Actually, we have

$$H_1' = 2\sum_{l=1}^{L} c_{0,l}(t)\Big(\int \widetilde{R}_{l\,xx}\,\varepsilon + \int \widetilde{R}_l^3\,\varepsilon\Big) - 2\sum_{l=1}^{L} c_l\, c_{0,l}(t)\int \widetilde{R}_l\,\varepsilon - \sum_{l=1}^{L} c_{0,l}(t)^2 \int \widetilde{R}_l\,\varepsilon.$$

Now, we introduce

$$\begin{aligned} H_1'' = {} & 2\sum_{l=1}^{L} c_{0,l}(t)\Big(\int \widetilde{R}_{l\,xx}\,\varepsilon\sqrt{\varphi_l^s} + \int \widetilde{R}_l^3\,\varepsilon\sqrt{\varphi_l^s}\Big) \\ & - 2\sum_{l=1}^{L} c_l\, c_{0,l}(t)\int \widetilde{R}_l\,\varepsilon\sqrt{\varphi_l^s} - \sum_{l=1}^{L} c_{0,l}(t)^2\int \widetilde{R}_l\,\varepsilon\sqrt{\varphi_l^s}. \end{aligned}$$

By reasoning the same way as for H_1 and H_1', we see that

$$|H_1' - H_1''| \leq C\|\varepsilon(t)\|_{H^2}\, e^{-2\theta t}.$$

Because of (2.8) and because of the elliptic equation satisfied by a soliton, we have that $H_1'' = 0$. Thus,

$$|H_1| = |H_1 - H_1'| + |H_1' - H_1''| + |H_1''| \leq C\|\varepsilon(t)\|_{H^2}\, e^{-2\theta t}.$$

The proof of Proposition 2.12 is now completed. ∎

Now, we would like to study the quadratic terms in ε of the development of $\mathcal{H}[\widetilde{P} + \varepsilon]$. They are contained in $H_2[\varepsilon](t)$.

Let $A = B_{\alpha,\beta}$ be a breather (we denote $A_1 := \partial_{x_1} A$ and $A_2 := \partial_{x_2} A$). We define a quadratic form associated to this breather:

$$\begin{aligned} \mathcal{Q}^b_{\alpha,\beta}[\epsilon] := {} & \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int A^2\epsilon_x^2 + \frac{5}{2}\int A_x^2\epsilon^2 + 5\int AA_{xx}\,\epsilon^2 + \frac{15}{4}\int A^4\epsilon^2 \\ & + (\beta^2-\alpha^2)\Big(\int \epsilon_x^2 - 3\int A^2\epsilon^2\Big) + (\alpha^2+\beta^2)^2\,\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{\alpha,\beta}[\epsilon]. \end{aligned}$$

From [3], we know that the kernel of this quadratic form is of dimension 2 and is spanned by $\partial_{x_1} B_{\alpha,\beta}$ and $\partial_{x_2} B_{\alpha,\beta}$, and that this quadratic form has only one negative eigenvalue that is of multiplicity 1.

Proposition 2.15 (Proposition 4.11, [32]). *There exists $\mu^b_{\alpha,\beta} > 0$ that depends only on α and β (and does not depend on time), such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int A_1\epsilon = \int A_2\epsilon = 0,$$

then

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon] \geq \mu^b_{\alpha,\beta}\|\epsilon\|_{H^2}^2 - \frac{1}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon A\Big)^2.$$

Remark 2.16. $\mu^b_{\alpha,\beta}$ is continuous in α, β. Note that the translation parameters are implicit in $\mathcal{Q}^b_{\alpha,\beta}$.

Let $Q = R_{c,\kappa}$ be a soliton. We define a quadratic form associated to this soliton:

$$\mathcal{Q}_c^s[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 + c\Big(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\Big) + c^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\epsilon].$$

By the same techniques, such as those presented in [3], adapted to the quadratic form of a soliton, we may establish that the kernel of this quadratic form is of dimension 2, and it is spanned by $\partial_x Q$ and $\partial_c Q$, and that this quadratic form does not have any negative eigenvalue (see Appendix A.2). After that, from Appendix A.3, we deduce that the coercivity still works when ϵ is orthogonal to Q and $\partial_x Q$. More precisely, we have the following.

Proposition 2.17. *There exists $\mu_c^s > 0$ that depends only on c (and does not depend on time) such that if $\epsilon \in H^2(\mathbb{R})$ is such that*

$$\int Q\epsilon = \int Q_x\epsilon = 0,$$

then

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c^s\|\epsilon\|_{H^2}^2.$$

Remark 2.18. μ_c^s is continuous in c. Note that the translation and sign parameters are implicit in the notation $\mathcal{Q}_c^s$.

We would like to find a similar minoration for H_2 (which is a generalization of $\mathcal{Q}$). For $j = 1,\dots,J$, let us define, for $\epsilon \in H^2$,

$$\begin{aligned}\mathcal{Q}_j[\epsilon] := {}&\frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P}_j^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P}_{jx}^2\epsilon^2\varphi_j \\ &+ 5\int \widetilde{P}_j\widetilde{P}_{jxx}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P}_j^4\epsilon^2\varphi_j \\ &+ (b_j^2 - a_j^2)\Big(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P}_j^2\epsilon^2\varphi_j\Big) + (a_j^2+b_j^2)^2\frac{1}{2}\int \epsilon^2\varphi_j,\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}'_j[\epsilon] := {}&\frac{1}{2}\int \epsilon_{xx}^2\varphi_j - \frac{5}{2}\int \widetilde{P}^2\epsilon_x^2\varphi_j + \frac{5}{2}\int \widetilde{P}_x^2\epsilon^2\varphi_j \\ &+ 5\int \widetilde{P}\widetilde{P}_{xx}\epsilon^2\varphi_j + \frac{15}{4}\int \widetilde{P}^4\epsilon^2\varphi_j \\ &+ (b_j^2 - a_j^2)\Big(\int \epsilon_x^2\varphi_j - 3\int \widetilde{P}^2\epsilon^2\varphi_j\Big) + (a_j^2+b_j^2)^2\frac{1}{2}\int \epsilon^2\varphi_j.\end{aligned}$$

We have that

$$H_2[\varepsilon(t)] = \sum_{j=1}^J \mathcal{Q}'_j[\varepsilon(t)].$$

The notations $\mathcal{Q}_k^b$, $(\mathcal{Q}_k^b)'$, $\mathcal{Q}_l^s$ and $(\mathcal{Q}_l^s)'$ will also be used.

We note that the support of φ_j increases with time, so that $\mathcal{Q}_j$ is near a $\mathcal{Q}^b_{\alpha_k,\beta_k}$ or a $\mathcal{Q}^s_{c_l}$ when time is large (note that $\mathcal{Q}^b_{\alpha_k,\beta_k}$ is the canonical quadratic form associated to the breather $\widetilde{B}_k$, but the canonical quadratic form associated to the soliton $\widetilde{R}_c$ is $\mathcal{Q}^s_{c_l+c_{0,l}(t)}$). However, firstly, let us study the difference between $\mathcal{Q}_j$ and $\mathcal{Q}'_j$. Using the computations carried out at the beginning of this part (those done for the linear part) and Sobolev inequalities, we obtain

$$|\mathcal{Q}_j[\epsilon] - \mathcal{Q}'_j[\epsilon]| \le Ce^{-2\theta t}\|\epsilon\|^2_{H^2(\mathbb{R})}.$$

Lemma 2.19. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that for $T^* \ge T_3^*$, any $\epsilon \in H^2(\mathbb{R})$, and any $t \in [t^*, T]$, if*

$$\int \widetilde{B}_{k1}(t)\epsilon\sqrt{\varphi^b_k(t)} = \int \widetilde{B}_{k2}(t)\epsilon\sqrt{\varphi^b_k(t)} = 0,$$

then

$$\mathcal{Q}^b_k[\epsilon] \ge \mu\int(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2)\varphi^b_k(t) - \frac{1}{\mu}\Big(\int \epsilon\widetilde{B}_k(t)\sqrt{\varphi^b_k(t)}\Big)^2 - \rho\|\epsilon\|^2_{H^2}.$$

Proof of Lemma 2.19. The idea is to write $\mathcal{Q}^b_k[\epsilon]$ as $\mathcal{Q}_{\alpha_k,\beta_k}[\epsilon\sqrt{\varphi^b_k}]$ plus several error terms. Let j be such that $\widetilde{P}_j = \widetilde{B}_k$. We will denote $\varphi_{1,j} := \psi'(\frac{x-\sigma_j t}{\delta t}) - \psi'(\frac{x-\sigma_{j+1}t}{\delta t})$ and $\varphi_{2,j} := \psi''(\frac{x-\sigma_j t}{\delta t}) - \psi''(\frac{x-\sigma_{j+1}t}{\delta t})$, as defined by (2.5) and (2.6), which will be useful to write the derivatives of φ_j. We recall that they have the same support and bounding properties as φ_j. We have that

$$\begin{aligned}\int(\epsilon\sqrt{\varphi_j})^2_{xx} &= \int\epsilon_{xx}^2\varphi_j + \int\frac{\epsilon_x^2}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \frac{1}{4}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}^2}{\varphi_j} + \frac{1}{16}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{1,j}^4}{\varphi_j^3}\\ &\quad - \frac{1}{4}\int\frac{\epsilon^2}{(\delta t)^4}\frac{\varphi_{2,j}\varphi_{1,j}^2}{\varphi_j^2} + 2\int\frac{\epsilon_{xx}\epsilon_x}{\delta t}\varphi_{1,j} + \int\frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\varphi_{2,j}\\ &\quad - \frac{1}{2}\int\frac{\epsilon_{xx}\epsilon}{(\delta t)^2}\frac{\varphi_{1,j}^2}{\varphi_j} + \int\frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}\varphi_{2,j}}{\varphi_j} - \frac{1}{2}\int\frac{\epsilon_x\epsilon}{(\delta t)^3}\frac{\varphi_{1,j}^3}{\varphi_j^2}.\end{aligned} \tag{2.18}$$

We observe that, for T_3^* large enough, and by using the inequalities that define ψ, the error terms can be bounded by $\frac{C}{\delta t}\|\epsilon\|^2_{H^2} \le \frac{\rho}{100}\|\epsilon\|^2_{H^2}$. The computation for the other terms is similar and the same bound can be used for the error terms.

Because $\epsilon\sqrt{\varphi^b_k}$ satisfies the orthogonality conditions, we can apply Proposition 2.15 and obtain that

$$\mathcal{Q}_{\alpha_k,\beta_k}[\epsilon\sqrt{\varphi^b_k}] \ge \mu^b_k\|\epsilon\sqrt{\varphi^b_k}\|^2_{H^2} - \frac{1}{\mu^b_k}\Big(\int\epsilon\sqrt{\varphi^b_k}\widetilde{B}_k\Big)^2.$$

To complete the proof, we note that $\|\epsilon\sqrt{\varphi^b_k}\|^2_{H^2}$ is $\int(\epsilon^2+\epsilon_x^2+\epsilon_{xx}^2)\varphi^b_k(t)$ plus several error terms as in (2.18). ■

Lemma 2.20. *There exists $\mu > 0$ such that for $\rho > 0$, there exists T_3^* such that, for $T^* \geq T_3^*$, any $\epsilon \in H^2(\mathbb{R})$, and any $t \in [t^*, T]$, if*

$$\int \widetilde{R}_l(t)\epsilon\sqrt{\varphi_l^s(t)} = \int \widetilde{R}_{l\,x}(t)\epsilon\sqrt{\varphi_l^s(t)} = 0,$$

then

$$\mathcal{Q}_l^s[\epsilon] \geq \mu \int (\epsilon^2 + \epsilon_x^2 + \epsilon_{xx}^2)\varphi_l^s(t) - \rho\|\epsilon\|_{H^2}^2.$$

Proof. As in the previous proof, we write $\mathcal{Q}_l^s[\epsilon]$ as $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ (with $Q = \widetilde{R}_l$) plus several error terms that are all bounded by $\rho\|\epsilon\|_{H^2}^2$ if T_3^* is chosen large enough. However, $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ is not appropriate in order to have coercivity, the appropriate quadratic form is $\mathcal{Q}_{c_l+c_{0,l}(t)}^s[\epsilon\sqrt{\varphi_l^s}]$. This is why we need to bound the difference between $\mathcal{Q}_{c_l}^s[\epsilon\sqrt{\varphi_l^s}]$ and $\mathcal{Q}_{c_l+c_{0,l}(t)}^s[\epsilon\sqrt{\varphi_l^s}]$. This difference is

$$c_{0,l}(t)\Big(\int \epsilon_x^2\varphi_j - 3\int \widetilde{R}_l^2\epsilon^2\varphi_j\Big) + c_l c_{0,l}(t)\int \epsilon^2\varphi_j + c_{0,l}(t)^2\frac{1}{2}\int \epsilon^2\varphi_j,$$

which, because of the bound for $c_{0,l}(t)$, for T_3^* large enough (depending on A), can be bounded by $\rho\|\epsilon\|_{H^2}^2$.

Now, $\epsilon\sqrt{\varphi_l^s}$ satisfies the orthogonality conditions we need, and as in the previous proof, we may apply coercivity. ■

Proof of Proposition 2.13. We will now use Lemma 2.19 and its version for solitons (Lemma 2.20) for $\epsilon = \varepsilon(t)$. From this, we deduce that for $\rho > 0$ small enough, we have that

$$\sum_{j=1}^{J}\mathcal{Q}_j[\varepsilon(t)] \geq \mu\|\varepsilon(t)\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\Big(\int \varepsilon(t)\widetilde{B}_k\sqrt{\varphi_k^b}\Big)^2,$$

for a suitable constant $\mu > 0$. This means that for T_3^* large enough, by taking, if needed, a smaller constant μ,

$$H_2[\varepsilon(t)] \geq \mu\|\varepsilon\|_{H^2}^2 - \frac{1}{\mu}\sum_{k=1}^{K}\Big(\int \varepsilon\widetilde{B}_k\sqrt{\varphi_k^b}\Big)^2.$$

The proof of Proposition 2.13 is now completed. ■

2.5. Proof of Proposition 1.9 (Bootstrap)

We recall that p_n from Proposition 1.9 is denoted by p and T_n is denoted by T in what follows in order to simplify the notations. We do the proof that follows under the assumption (2.2), so that the propositions proved above are true for $t \in [t^*, T]$.

The aim of this subsection is to complete the proof of Proposition 1.9 by using Propositions 2.12 and 2.13.

We note that by Lemma 2.7, the conservation of $F[p](t)$ and the definition of $\mathcal{H}[p]$, we have, for any $t \in [t^*, T]$, that

$$|\mathcal{H}[p](T) - \mathcal{H}[p](t)| \leq \frac{CA^2}{\delta^2 t}e^{-2\theta t}.$$

Thus, for any $t \in [t^*, T]$,

$$\mathcal{H}[p](t) \le \mathcal{H}[p](T) + \frac{CA^2}{\delta^2 t} e^{-2\theta t}. \tag{2.19}$$

From Proposition 2.12,

$$\begin{aligned} &\Big| \mathcal{H}[\widetilde{P} + \varepsilon](t) - H_2[\varepsilon](t) \\ &\quad - \sum_{j=1}^{J} \big(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2) E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t) \big) \Big| \\ &\quad \le Ce^{-2\theta t} + C\|\varepsilon\|_{H^2} e^{-2\theta t} + C\|\varepsilon\|_{H^2}^3 \le Ce^{-2\theta t} + \frac{\mu}{100}\|\varepsilon\|_{H^2}^2. \end{aligned} \tag{2.20}$$

In order to obtain the last inequality, we use the fact that $\|\varepsilon(t)\|_{H^2} \le CAe^{-\theta t}$, and we take $T^* \ge T_5^*$ for T_5^* large enough (depending on A) so that $\|\varepsilon\|_{H^2} \le C$ and $C\|\varepsilon(t)\|_{H^2} \le \mu/100$, and thus $C\|\varepsilon(t)\|_{H^2}^3 \le \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2$.

We remark that if $P_j = B_k$ is a breather, then $F[\widetilde{P}_j]$, $E[\widetilde{P}_j]$ and $M[\widetilde{P}_j]$ are all constants in time. If $P_j = R_l$ is a soliton and we denote q the basic ground state (i.e., the ground state for $c = 1$), we have the following:

$$\begin{aligned} M[\widetilde{R}_l](t) &= (c_l + c_{0,l}(t))^{1/2} M[q], \\ E[\widetilde{R}_l](t) &= (c_l + c_{0,l}(t))^{3/2} E[q], \\ F[\widetilde{R}_l](t) &= (c_l + c_{0,l}(t))^{5/2} F[q]. \end{aligned}$$

Using that, we can simplify $\mathcal{R}_l(t) := F[\widetilde{R}_l](t) + 2c_l E[\widetilde{R}_l](t) + c_l^2 M[\widetilde{R}_l](t)$ as follows:

$$\begin{aligned} \mathcal{R}_l(t) &= (c_l + c_{0,l}(t))^{5/2} F[q] + 2c_l(c_l + c_{0,l}(t))^{3/2} E[q] \\ &\quad + c_l^2 (c_l + c_{0,l}(t))^{1/2} M[q] \\ &= c_l^{5/2}\Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{5/2} F[q] + 2c_l^{5/2}\Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{3/2} E[q] \\ &\quad + c_l^{5/2}\Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{1/2} M[q]. \end{aligned} \tag{2.21}$$

Note that from Lemma 2.8, $|c_{0,l}(t)|^3 \le CA^3 e^{-\theta t} e^{-2\theta t}$. That is why, if we take T_5^* eventually larger, $|c_{0,l}(t)|^3 \le Ce^{-2\theta t}$. For this reason, we will do Taylor expansions of order 2 of (2.21):

$$\begin{aligned} \Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{5/2} &= 1 + \frac{5}{2}\frac{c_{0,l}(t)}{c_l} + \frac{15}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}), \\ \Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{3/2} &= 1 + \frac{3}{2}\frac{c_{0,l}(t)}{c_l} + \frac{3}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}), \\ \Big(1 + \frac{c_{0,l}(t)}{c_l}\Big)^{1/2} &= 1 + \frac{1}{2}\frac{c_{0,l}(t)}{c_l} - \frac{1}{8}\frac{c_{0,l}(t)^2}{c_l^2} + O(e^{-2\theta t}). \end{aligned}$$

This allows us to write

$$\mathcal{R}_l(t) = c_l^{5/2}(F[q]+2E[q]+M[q]) + c_l^{3/2}c_{0,l}(t)\Big(\frac{5}{2}F[q]+3E[q]+\frac{1}{2}M[q]\Big) + c_l^{1/2}c_{0,l}(t)^2\Big(\frac{15}{8}F[q] + \frac{3}{4}E[q] - \frac{1}{8}M[q]\Big) + O(e^{-2\theta t}).$$

Now, $c_l^{5/2}(F[q] + 2E[q] + M[q])$ is constant in time. For both other terms, we use that $M[q] = 2$, $E[q] = -2/3$ and $F[q] = 2/5$, and we see that $\frac{5}{2}F[q] + 3E[q] + \frac{1}{2}M[q] = 0$ and $\frac{15}{8}F[q] + \frac{3}{4}E[q] - \frac{1}{8}M[q] = 0$. This allows us to write

$$\mathcal{R}_l(t) = \frac{16}{15}c_l^{5/2} + O(e^{-2\theta t}).$$

From this, we deduce that

$$\mathcal{R}_l(t) - \mathcal{R}_l(T) = O(e^{-2\theta t}).$$

By using $\mathcal{H}[p](T) = \mathcal{H}[P](T) = \mathcal{H}[\widetilde{P}](T)$, equations (2.20) and (2.19), Claim 2.14, and the fact that for $t \geq T_4^*$, $O(\|\varepsilon(t)\|_{H^2}^3) \leq \frac{\mu}{100}\|\varepsilon\|_{H^2}^2$, we have that

$$\begin{aligned}
H_2[\varepsilon](t) &\leq \mathcal{H}[p](t) + Ce^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2 \\
&\quad - \sum_{j=1}^{J}\big(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t)\big) \\
&\leq \mathcal{H}[\widetilde{P}](T) + C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2 \\
&\quad - \sum_{j=1}^{J}\big(F[\widetilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](t)\big) \\
&\leq \mathcal{H}[\widetilde{P}](T) + C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2 + \sum_{l=1}^{L}(\mathcal{R}_l(T) - \mathcal{R}_l(t)) \\
&\quad - \sum_{j=1}^{J}\big(F[\widetilde{P}_j](T) + 2(b_j^2 - a_j^2)E[\widetilde{P}_j](T) + (a_j^2 + b_j^2)^2 M[\widetilde{P}_j](T)\big) \\
&\leq C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-2\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2.
\end{aligned}$$

From Proposition 2.13, we deduce (by taking a smaller constant μ) that

$$\mu\|\varepsilon\|_{H^2}^2 \leq C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-2\theta t} + \frac{1}{\mu}\sum_{k=1}^{K}\Big(\int \varepsilon \widetilde{B}_k\sqrt{\varphi_k^b}\Big)^2.$$

We now need to establish a result close to Lemma 2.7. We set, for any $j = 1, \dots, J$,

$$m_j(t) := \int \frac{1}{2}p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx =: m_j[p](t).$$

Lemma 2.21. *There exist $C > 0$ and $T_6^* = T_6^*(A)$ such that if $T^* \geq T_6^*$, then for any $j = 1, \dots, J$, and any $t \in [t^*, T]$,*

$$|m_j(T) - m_j(t)| \leq \frac{C}{\delta^2 t} A^2 e^{-2\theta t}.$$

Proof. We compute

$$\frac{d}{dt}\int \frac{1}{2} p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx = \frac{1}{2\delta t}\int \Big(-\frac{3}{2}p_x^2 + \frac{3}{4}p^4\Big)\frac{\varphi_{1,j}}{\sqrt{\varphi_j}} - \frac{1}{2(\delta t)^2}\int p_x p \frac{\varphi_{2,j}}{\sqrt{\varphi_j}} + \frac{1}{4(\delta t)^2}\int p_x p \frac{\varphi_{1,j}^2}{\varphi_j^{3/2}} - \frac{1}{4}\int p^2 \frac{x}{\delta t^2}\frac{\varphi_{1,j}}{\sqrt{\varphi_j}}.$$

From the inequalities that define ψ, we find that

$$\Big|\frac{d}{dt}\int \frac{1}{2} p^2(t,x)\sqrt{\varphi_j(t,x)}\,dx\Big| \leq \frac{C}{\delta^2 t}\int_{\Omega_j(t)\cup\Omega_{j+1}(t)} (p_x^2 + p^2 + p^4).$$

From now on, we can follow the proof of Lemma 2.7. ∎

Now, we observe the following:

$$\int (\widetilde{P} + \varepsilon)^2 \sqrt{\varphi_k^b} = \int \widetilde{B}_k^2 + 2\int \widetilde{B}_k \varepsilon \sqrt{\varphi_k^b} + \int \varepsilon^2 \sqrt{\varphi_k^b} + \mathrm{Err}, \tag{2.22}$$

where Err stands for the other terms of the sum, which we consider as error terms, and we will show that they are bounded by $Ce^{-\theta t}$.

For $i \neq j$ and any h (if $P_j = B_k$ is a breather),

$$\Big|\int \widetilde{P}_i \widetilde{P}_h \sqrt{\varphi_j}\Big| \leq C\int_{-\delta t + \sigma_j t}^{\delta t + \sigma_{j+1} t} e^{-\frac{\beta}{2}|x - v_i t|}\,dx \leq Ce^{-\theta t}$$

and

$$\Big|\int \widetilde{P}_i \varepsilon \sqrt{\varphi_j}\Big| \leq \sqrt{\Big(\int \widetilde{P}_i^2 \varphi_j\Big)\Big(\int \varepsilon^2\Big)} \leq Ce^{-\theta t/2}\|\varepsilon\|_{H^2} \leq CAe^{-\theta t}e^{-\theta t/2} \leq Ce^{-\theta t},$$

where $T^* \geq T_7^*$, with T_7^* being large enough depending on A. If we use the calculations we have made in the proof of Claim 2.14, we see that

$$\Big|\int \widetilde{P}_j^2 - \int \widetilde{P}_j^2 \sqrt{\varphi_j}\Big| \leq Ce^{-\theta t}.$$

This proves the bound for the error terms.

Now, we study the variations of (2.22). We know that $\int \widetilde{P}_j^2 = \int \widetilde{B}_k^2$ has no variations. We can apply Lemma 2.21 for $\int (\widetilde{P} + \varepsilon)^2 \sqrt{\varphi_j}$. By writing the difference of equation (2.22) between t and T, and using that $\varepsilon(T) = 0$, we deduce, for $T^* \geq \max(T_6^*, T_7^*)$,

$$\Big|\int \widetilde{P}_j \varepsilon \sqrt{\varphi_j}(t)\Big| \leq C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-\theta t} + \|\varepsilon\|_{H^2}^2 \leq C\Big(\frac{A^2}{\delta^2 t} + 1\Big)e^{-\theta t} + \frac{\mu}{100}\|\varepsilon(t)\|_{H^2}.$$

Thus,

$$\mu\|\varepsilon\|_{H^2}^2 \le C\Big(\frac{A^2}{\delta^2 t}+1\Big)e^{-2\theta t}+\frac{1}{\mu}\sum_{j=1}^{J}\Big(\int \varepsilon\widetilde{P}_j\sqrt{\varphi_j}\Big)^2$$
$$\le C\Big(\frac{A^4}{\delta^4 t}+1\Big)e^{-2\theta t}+\frac{\mu}{100}\|\varepsilon(t)\|_{H^2}^2.$$

Therefore,

$$\|\varepsilon(t)\|_{H^2}^2 \le C\Big(\frac{A^4}{\delta^4 t}+1\Big)e^{-2\theta t}. \tag{2.23}$$

By using (2.23), the mean value theorem and Lemma 2.8, we deduce, for $t\in[t^*,T]$,

$$\|p(t)-P(t)\|_{H^2}\le\|\varepsilon(t)\|_{H^2}+\|\widetilde{P}(t)-P(t)\|_{H^2}$$
$$\le C\Big(\sqrt{\frac{A^4}{\delta^4 t}+1}\Big)e^{-\theta t}+C\Big(\sum_{k=1}^{K}(|x_{1,k}(t)|+|x_{2,k}(t)|)+\sum_{l=1}^{L}(|x_{0,l}(t)|+|c_{0,l}(t)|)\Big)$$
$$\le C\Big(\sqrt{\frac{A^4}{\delta^4 t}+1}\Big)e^{-\theta t}+C\sum_{k=1}^{K}\Big(\Big|\int_t^T x_{1,k}'(s)\,ds\Big|+\Big|\int_t^T x_{2,k}'(s)\,ds\Big|\Big)$$
$$+C\sum_{l=1}^{L}\Big(\Big|\int_t^T x_{0,l}'(s)\,ds\Big|+\Big|\int_t^T c_{0,l}'(s)\,ds\Big|\Big)$$
$$\le C\Big(\frac{A^4}{\delta^4 t}+1\Big)e^{-\theta t}+C\Big(\int_t^T\|\varepsilon(s)\|_{H^2}\,ds+\int_t^T e^{-\theta s}\,ds\Big)\le C\Big(\frac{A^4}{\delta^4 t}+1\Big)e^{-\theta t}.$$

We take $A=4C$ (where C is a constant that can be used anywhere in the proof above) and

$$T^*:=\max\big(T_1^*,T_2^*,T_3^*,T_4^*,T_5^*,T_6^*,T_7^*,T_8^*\big)$$

(depending on A), where $T_8^*:=T_8^*(A)$ is such that for $t\ge T_8^*$, we have $\frac{A^4}{\delta^4 t}\le 1$. Thus, for any $t\in[t^*,T]$,

$$C\Big(\frac{A^4}{\delta^4 t}+1\Big)\le 2C=\frac{A}{2},$$

which is exactly what we wanted to prove.

3. *p* is a smooth multi-breather

Our goal here is to prove Proposition 1.10.

3.1. Estimates in higher order Sobolev norms

Firstly, we notice that the proposition is already established for $s=2$. We note also that if this proposition is proved for an $s\ge 2$ with a corresponding constant A_s, then it is also valid for any $s' \leq s$ with the same constant A_s. This means that A_s can possibly increase with s and that this proposition is already established for $0 \leq s \leq 2$. From now on, we will denote (as before) p_n by p, T_n by T and $p_n - P$ by v, and make sure that the constant A_s that we will obtain in the proof does not depend on n (although it will depend on s). For the constant θ, we will take the usual value: $\theta := \beta\tau/32$. For the constant T^*, we will also take the value that works for Proposition 1.6.

We will prove the proposition by induction on s (it is sufficient to prove it for any integer s). Let $s \geq 3$. We will prove the proposition for s, assuming that it is true for any $0 \leq s' \leq s - 1$.

Let us deduce from (1.1) the equation satisfied by v:

$$\begin{aligned} v_t = p_t - \sum_{j=1}^{J} P_{jt} &= -\Big(p_{xx} + p^3 - \sum_{j=1}^{J} P_{jxx} - \sum_{j=1}^{J} P_j^3\Big)_x \\ &= -\Big(v_{xx} + (v+P)^3 - \sum_{j=1}^{J} P_j^3\Big)_x \\ &= -\Big(v_{xx} + v^3 + 3v^2 P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3\Big)_x. \end{aligned}$$

Firstly, we compute $\frac{d}{dt}\int (\partial_x^s v)^2$ by integration by parts:

$$\begin{aligned} \frac{d}{dt}\int (\partial_x^s v)^2 &= 2\int (\partial_x^s v_t)(\partial_x^s v) \\ &= -2\int \partial_x^{s+1}\Big(v_{xx} + v^3 + 3v^2 P + 3vP^2 + P^3 - \sum_{j=1}^{J} P_j^3\Big)(\partial_x^s v) \\ &= 2(-1)^{s+1}\int \partial_x^{2s+1}\Big(P^3 - \sum_{j=1}^{J} P_j^3\Big)v - 2\int \partial_x^{s+1}(v^3)(\partial_x^s v) \\ &\quad - 6\int \partial_x^{s+1}(v^2 P)(\partial_x^s v) - 6\int \partial_x^{s+1}(vP^2)(\partial_x^s v), \end{aligned}$$

because $\int (\partial_x^{s+3} v)(\partial_x^s v) = -\int (\partial_x^{s+2} v)(\partial_x^{s+1} v) = 0$.

We will now bound above each of the terms of the obtained sum. By the Sobolev embedding, Proposition 2.3 and Proposition 1.6,

$$\begin{aligned} \left|\int \partial_x^{2s+1}\Big(P^3 - \sum_{j=1}^{J} P_j^3\Big)v\right| &\leq \|v\|_{L^\infty}\int \left|\partial_x^{2s+1}\Big(P^3 - \sum_{j=1}^{J} P_j^3\Big)\right| \\ &\leq C\|v\|_{H^1} e^{-\beta\tau t/2} \leq CAe^{-\theta t} e^{-\beta\tau t/2} \\ &\leq CAe^{-2\theta t} \leq CA_{s-1}^2 e^{-2\theta t}, \end{aligned}$$

where $C \geq 0$ is a constant that depends only on s.

We observe that

$$\partial_x^{s+1}(v^3) = 3(\partial_x^{s+1}v)v^2 + 6(s+1)(\partial_x^s v)v_x v + Z_1(v, v_x, \dots, \partial_x^{s-1}v),$$
$$\partial_x^{s+1}(v^2 P) = 2(\partial_x^{s+1}v)vP + 2(s+1)(\partial_x^s v)(vP)_x + Z_2(v, v_x, \dots, \partial_x^{s-1}v, P, P_x, \dots, \partial_x^{s+1}P),$$

where Z_1 and Z_2 are homogeneous polynomials of degree 3 with constant coefficients.

Now, let us look for a bound for $\int \partial_x^{s+1}(v^3)(\partial_x^s v)$. Firstly, by integration by parts,

$$\begin{aligned}\int \partial_x^{s+1}(v^3)(\partial_x^s v) &= \frac{3}{2}\int \big((\partial_x^s v)^2\big)_x v^2 + 3(s+1)\int (\partial_x^s v)^2 (v^2)_x + \int (\partial_x^s v) Z_1 \\ &= \frac{6(s+1)-3}{2}\int (\partial_x^s v)^2 (v^2)_x + \int (\partial_x^s v) Z_1.\end{aligned}$$

Then we bound above each of the terms of the obtained sum:

$$\begin{aligned}\Big|\int (\partial_x^s v)^2 (v^2)_x\Big| &\le C\|v\|_{L^\infty}\|v_x\|_{L^\infty}\int (\partial_x^s v)^2 \\ &\le C\|v\|_{H^2}^2 \int (\partial_x^s v)^2 \le C(\|p\|_{H^2} + \|P\|_{H^2})Ae^{-\theta t}\int (\partial_x^s v)^2 \\ &\le CC_0 Ae^{-\theta t}\int (\partial_x^s v)^2 \le CA_{s-1}e^{-\theta t}\int (\partial_x^s v)^2.\end{aligned}$$

We have actually shown in the computation above that $\|v\|_{H^2}^2$ can be bounded above by $\|v\|_{H^2}$ (with a constant that depends only on the problem data), and therefore the degree of $\|v\|_{H^2}$ can be lowered without harm in the upper bound. We will use this fact again for the rest of the proof. In fact, all what it means is that, for several terms, what we have is more than what we need.

By the Cauchy–Schwarz and Gagliardo–Nirenberg–Sobolev inequalities,

$$\begin{aligned}\Big|\int (\partial_x^s v) Z_1\Big| &\le C\int |\partial_x^s v|\Big(\sum_{s'=0}^{s-1}|\partial_x^{s'}v|^3\Big) \le C\Big(\int |\partial_x^s v|^2\Big)^{1/2}\sum_{s'=0}^{s-1}\Big(\int |\partial_x^{s'}v|^6\Big)^{1/2} \\ &\le C\Big(\int |\partial_x^s v|^2\Big)^{1/2}\sum_{s'=0}^{s-1}\Big(\int |\partial_x^{s'}v|^2\Big)\Big(\int |\partial_x^{s'+1}v|^2\Big)^{1/2} \\ &\le C\sum_{s'=0}^{s-1}\Big(\int |\partial_x^{s'}v|^2\Big)\Big(\int |\partial_x^s v|^2 + \int |\partial_x^{s'+1}v|^2\Big) \\ &\le CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}e^{-\theta t}\int |\partial_x^s v|^2.\end{aligned} \tag{3.1}$$

Similarly, we bound $\int \partial_x^{s+1}(v^2 P)(\partial_x^s v)$. By integration by parts,

$$\begin{aligned}\int \partial_x^{s+1}(v^2 P)(\partial_x^s v) &= \int \big((\partial_x^s v)^2\big)_x vP + 2(s+1)\int (\partial_x^s v)^2 (vP)_x + \int (\partial_x^s v) Z_2 \\ &= (2s+1)\int (\partial_x^s v)^2 (vP)_x + \int (\partial_x^s v) Z_2.\end{aligned}$$

We bound above each of the terms of the obtained sum, starting by

$$\Big|\int (\partial_x^s v)^2 (vP)_x\Big| \le C(\|v\|_{L^\infty} + \|v_x\|_{L^\infty}) \int (\partial_x^s v)^2 \le CAe^{-\theta t} \int (\partial_x^s v)^2.$$

The upper bound of $|\int (\partial_x^s v) Z_2|$ is similar to (3.1):

$$\Big|\int (\partial_x^s v) Z_2\Big| \le CA_{s-1}^2 e^{-2\theta t} + CA_{s-1} e^{-\theta t} \int |\partial_x^s v|^2.$$

The term $\int \partial_x^{s+1}(vP^2)(\partial_x^s v)$ remains to be bounded. By integration by parts,

$$\begin{aligned}
\int \partial_x^{s+1}(vP^2)(\partial_x^s v) &= -\int \partial_x^{s+2}(vP^2)(\partial_x^{s-1} v) \\
&= -\int (\partial_x^{s+2} v)(\partial_x^{s-1} v) P^2 - (s+2) \int (\partial_x^{s+1} v)(\partial_x^{s-1} v)(P^2)_x \\
&\quad - \frac{(s+2)(s+1)}{2} \int (\partial_x^s v)(\partial_x^{s-1} v)(P^2)_{xx} + \int (\partial_x^{s-1} v) Z_3^0(v, v_x, \dots, \partial_x^{s-1} v) \\
&= \frac{1}{2} \int \big((\partial_x^s v)^2\big)_x P^2 + (s+1) \int (\partial_x^s v)^2 (P^2)_x \\
&\quad - \frac{s(s+1)}{4} \int \big((\partial_x^{s-1} v)^2\big)_x (P^2)_{xx} + \int (\partial_x^{s-1} v) Z_3^0(v, v_x, \dots, \partial_x^{s-1} v) \\
&= \frac{2s+1}{2} \int (\partial_x^s v)^2 (P^2)_x + \int (\partial_x^{s-1} v) Z_3(v, v_x, \dots, \partial_x^{s-1} v),
\end{aligned}$$

where Z_3^0 and Z_3 are homogeneous polynomials of degree 1 whose coefficients are polynomials in P and its space derivatives. We have that $|Z_3| \le C(\sum_{s'=0}^{s-1} |\partial_x^{s'} v|)$. Therefore,

$$\Big|\int (\partial_x^{s-1} v) Z_3\Big| \le CA_{s-1}^2 e^{-2\theta t}.$$

Thus, by taking the sum of all those inequalities, we obtain

$$\Big|\frac{d}{dt}\int (\partial_x^s v)^2 + 3(2s+1) \int (\partial_x^s v)^2 (P^2)_x\Big| \le CA_{s-1}^2 e^{-2\theta t} + CA_{s-1} e^{-\theta t} \int |\partial_x^s v|^2.$$

Next, we perform similar computations for $\frac{d}{dt}\int (\partial_x^{s-1} v)^2 P^2$:

$$\begin{aligned}
\frac{d}{dt}\int (\partial_x^{s-1} v)^2 P^2 &= 2\int (\partial_x^{s-1} v_t)(\partial_x^{s-1} v) P^2 + 2\int (\partial_x^{s-1} v)^2 P_t P \\
&= -2\int \partial_x^s\Big(v_{xx} + v^3 + 3v^2 P + 3vP^2 + P^3 - \sum_{j=1}^J P_j^3\Big)(\partial_x^{s-1} v) P^2 \\
&\quad - 2\int (\partial_x^{s-1} v)^2 \Big(P_{xx} + \sum_{j=1}^J P_j^3\Big)_x P.
\end{aligned}$$

Let us study each of the obtained terms.

Firstly,

$$
\begin{aligned}
-2\int (\partial_x^{s+2}v)(\partial_x^{s-1}v)P^2 &= 2\int (\partial_x^{s+1}v)(\partial_x^{s}v)P^2 + 2\int (\partial_x^{s+1}v)(\partial_x^{s-1}v)(P^2)_x \\
&= -3\int (\partial_x^{s}v)^2(P^2)_x - 2\int (\partial_x^{s}v)(\partial_x^{s-1}v)(P^2)_{xx} \\
&= -3\int (\partial_x^{s}v)^2(P^2)_x + \int (\partial_x^{s-1}v)^2(P^2)_{xxx}.
\end{aligned}
$$

Indeed,

$$
\Big|\int (\partial_x^{s-1}v)^2(P^2)_{xxx}\Big| \le CA^2e^{-2\theta t}.
$$

Secondly,

$$
\Big|\int \partial_x^s\Big(P^3 - \sum_{j=1}^{J} P_j^3\Big)(\partial_x^{s-1}v)P^2\Big| \le CA_{s-1}^2e^{-2\theta t},
$$

which can be obtained similarly to the first part of the proof (starting by an integration by parts with $\partial_x^{s-2}v$ in the place of $\partial_x^{s-1}v$).

Thirdly,

$$
\begin{aligned}
\int \partial_x^s(v^3)(\partial_x^{s-1}v)P^2 &= 3\int (\partial_x^{s}v)(\partial_x^{s-1}v)v^2P^2 + \int Z_4(v, v_x, \ldots, \partial_x^{s-1}v)P^2 \\
&= -\frac{3}{2}\int (\partial_x^{s-1}v)^2(v^2P^2)_x + \int Z_4P^2,
\end{aligned}
$$

where Z_4 is a homogeneous polynomial of degree 4 with constant coefficients. Both terms are easily bounded by $CA_{s-1}^2e^{-2\theta t}$.

Fourthly, for $\int \partial_x^s(v^2P)(\partial_x^{s-1}v)P^2$ and $\int \partial_x^s(vP^2)(\partial_x^{s-1}v)P^2$, we reason similarly.

Fifthly, it is clear that

$$
\Big|\int (\partial_x^{s-1}v)^2\Big(P_{xx} + \sum_{j=1}^{J} P_j^3\Big)_x P\Big| \le CA_{s-1}^2e^{-2\theta t}.
$$

Therefore,

$$
\Big|\frac{d}{dt}\int (\partial_x^{s-1}v)^2P^2 + 3\int (\partial_x^{s}v)^2(P^2)_x\Big| \le CA_{s-1}^2e^{-2\theta t}.
$$

We set

$$
F(t) := \int (\partial_x^{s}v)^2 - (2s+1)\int (\partial_x^{s-1}v)^2P^2.
$$

By putting both parts of the proof together,

$$
\Big|\frac{d}{dt}F(t)\Big| \le CA_{s-1}^2e^{-2\theta t} + CA_{s-1}e^{-\theta t}\int |\partial_x^s v|^2.
$$

Because $|\int (\partial_x^{s-1}v)^2P^2| \le CA^2e^{-2\theta t}$, we can write the following upper bound:

$$
\int (\partial_x^{s}v)^2 \le |F(t)| + CA_{s-1}^2e^{-2\theta t}.
$$

Therefore, we have, for a suitable constant $C > 0$ that depends only on s,

$$\left|\frac{d}{dt}F(t)\right| \leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}e^{-\theta t}|F(t)|.$$

For $t \in [T^*, T]$, by integration between t and T (we recall that $F(T) = 0$),

$$\begin{aligned}|F(t)| = |F(T) - F(t)| = \left|\int_t^T \frac{d}{dt}F(\sigma)\,d\sigma\right| &\leq \int_t^T \left|\frac{d}{dt}F(\sigma)\right| d\sigma \\ &\leq CA_{s-1}^2 \int_t^T e^{-2\theta\sigma}\,d\sigma + CA_{s-1}\int_t^T e^{-\theta\sigma}|F(\sigma)|\,d\sigma \\ &\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}\int_t^T e^{-\theta\sigma}|F(\sigma)|\,d\sigma.\end{aligned}$$

By Gronwall's lemma, for all $t \in [T^*, T]$,

$$\begin{aligned}|F(t)| &\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}\int_t^T e^{-\theta\sigma}CA_{s-1}^2 e^{-2\theta\sigma}\exp\Big(\int_t^\sigma CA_{s-1}e^{-\theta u}\,du\Big)\,d\sigma \\ &\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}^3 \exp\Big(\frac{CA_{s-1}}{\theta}e^{-\theta t}\Big)\int_t^T e^{-3\theta\sigma}\exp\Big(-\frac{CA_{s-1}}{\theta}e^{-\theta\sigma}\Big)\,d\sigma \\ &\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}^3 \exp\Big(\frac{CA_{s-1}}{\theta}\Big)\int_t^T e^{-3\theta\sigma}\,d\sigma \\ &\leq CA_{s-1}^2 e^{-2\theta t} + CA_{s-1}^3 \exp\Big(\frac{CA_{s-1}}{\theta}\Big)e^{-3\theta t} \\ &\leq CA_{s-1}^3 \exp\Big(\frac{CA_{s-1}}{\theta}\Big)e^{-2\theta t}.\end{aligned}$$

Therefore,

$$\int (\partial_x^s v)^2 \leq A_s e^{-2\theta t},$$

where $A_s := CA_{s-1}^3 \exp(CA_{s-1}/\theta)$ and C is a constant large enough that depends only on s. This conclude the proof of Proposition 1.10, and so of Theorem 1.2.

3.2. Uniformity of constants

We conclude this section with an explanation regarding Remark 1.3.

In the proof above, the constants that we obtain A, T^* and θ do depend on $P_j(0)$ ($1 \leq j \leq J$). Actually, we may characterize this dependence. In fact, they do not depend on the initial positions of our objects in the case where our objects are initially ordered in the right order and sufficiently far from each other.

Theorem 3.1. *Given parameters (1.3), (1.4), (1.5) and (1.6) which satisfy (1.7), there exists $D > 0$ large enough that depends only on α_k, β_k, c_l such that if*

$$x_j(0) \geq x_{j-1}(0) + D \quad \textit{for all } j \geq 2, \tag{3.2}$$

then the following holds. We set $\theta := \beta\tau/32$, with β and τ given by (2.1), and let $p(t)$ be the multi-breather associated to P by Proposition 1.7. There exists $A_s \geq 1$ for any $s \geq 2$ that depends only on α_k, β_k, c_l and D such that

$$\|p(t) - P(t)\|_{H^s} \leq A_s e^{-\theta t} \quad \text{for all } t \geq 0,$$

Initially, we will prove that for any $D > 0$, if (3.2) is satisfied, then the constants A_s and T^* do only depend on α_k, β_k, c_l and D. Then we will prove that if $D > 0$ is large enough with respect to the given parameters, then we can take $T^* = 0$.

To establish the validity of this theorem, it is enough to read again the whole article and to make sure that on any step of the proof, there is no dependence on initial positions of our objects when our objects are initially far from each other for the constant C. This will allow to claim the same for the constants A and T^* (but these constants may depend on D). This works, but we should change a bit the way we write our results.

For Proposition 2.1, we should write

$$|\partial_x^n \partial_t^m P_j(t,x)| \leq Ce^{-\beta|x-v_jt-x_j(0)|}.$$

Therefore, in Proposition 2.3, we have nothing to change, but the constant C depends on D. This will also be the case in the following propositions and lemmas of this proof.

We should replace $\sigma_j t$ by $\sigma_j t + (x_{j-1}(0) + x_j(0))/2$ in the definition of φ_j in (2.3) and (2.4) to take into account initial positions. More precisely, for any $j = 2, \dots, J-1$, we will have

$$\varphi_j(t,x) := \psi\left(\frac{x - \sigma_j t - \frac{x_{j-1}(0)+x_j(0)}{2}}{\delta t}\right) - \psi\left(\frac{x - \sigma_{j+1} t - \frac{x_j(0)+x_{j+1}(0)}{2}}{\delta t}\right),$$

and similarly for other definitions.

After having done the modulation with C and T^* depending on D, for Proposition 2.10, we should write

$$|\partial_x^n \widetilde{P}_j(t,x)| \leq Ce^{-\frac{\beta}{2}|x-v_jt-x_j(0)|}e^{\beta\tau t/32}.$$

Therefore, with these adaptations, the same proof works for proving that A_s and T^* do depend only on α_k, β_k, c_l and D.

Now, given α_k, β_k, c_l, we choose $D_0 > 0$ in an arbitrary manner. Therefore, we get $A_s(D_0)$ and $T^*(D_0)$ associated to D_0. Let $\Lambda := v_J - v_1$ be the maximal difference between two velocities. We set $D := D_0 + \Lambda \cdot T^*(D_0)$. Therefore, if we suppose (3.2) in $t = 0$ for D, then we have (3.2) in $t = -T^*(D_0)$ for D_0. Therefore, by applying the intermediate result for D_0, we obtain the desired conclusion with D and A_s depending on D_0.

4. Uniqueness

Let p be the multi-breather constructed in the existence part. The goal here is to prove that if a solution u converges to p when $t \to +\infty$ (in some sense), then $u = p$ (under well chosen assumptions).

We prove here two propositions. For both of them, we assume that the velocities of all our objects are distinct (this was also an assumption for the existence). The first proposition does not make more assumptions on the velocities of our objects, but it is a partial uniqueness result as we restrict ourselves to the class of super polynomial convergence to the multi-breather. The second proposition assumes that the velocities of all our objects are positive (this is a new assumption and it is needed because this proof uses monotonicity arguments).

4.1. A super polynomial convergence of a solution to a multi-breather is trivial

The goal of this subsection is to prove Proposition 1.5.

Remark 4.1. Note that in Proposition 1.5, we do not make any assumptions on the sign of v_1 or v_2. This uniqueness proposition has the same degree of generality as Theorem 1.2.

Proof of Proposition 1.5. Let $p(t)$ be the multi-breather associated to P by Theorem 1.2. Recall that for any s,

$$\|p(t)-P(t)\|_{H^s}=O(e^{-\theta t}), \tag{4.1}$$

for a suitable $\theta>0$.

Let $N>2$ to be chosen later. We take $u(t)$ an H^2 solution of (1.1), such that for a constant $C_0>0$ large enough, for any t large enough,

$$\|u(t)-P(t)\|_{H^2}\leq\frac{C_0}{t^N}.$$

From that, we may deduce that for t large enough (namely, $t\geq 2C_0$ along with the previous condition),

$$\|u(t)-P(t)\|_{H^2}\leq\frac{1}{2}\frac{1}{t^{N-1}}. \tag{4.2}$$

Our goal is to find a condition on N that does not depend on u, such that condition (4.2) on u for t large enough implies that $u\equiv p$.

Because of (4.1), the condition (4.2), for t large enough, is equivalent to

$$\|u(t)-p(t)\|_{H^2}\leq\frac{1}{t^{N-1}}.$$

We denote $z(t):=u(t)-p(t)$. Our goal is to find N large enough that does not depend on z, for which we will be able to prove that $z\equiv 0$, given

$$\|z(t)\|_{H^2}\leq\frac{1}{t^{N-1}}, \tag{4.3}$$

for t large enough. Because z is a difference of two solutions of (1.1), we may derive the following equation for z:

$$z_t+(z_{xx}+(z+p)^3-p^3)_x=0. \tag{4.4}$$

We divide our proof in several steps.

Step 1. *Modulation on z.* For $j = 1, \dots, J$, if $P_j = B_k$ is a breather, we denote

$$K_j := \begin{pmatrix} \partial_{x_1} B_k \\ \partial_{x_2} B_k \end{pmatrix},$$

and if $P_j = R_l$ is a soliton, we denote

$$K_j = \partial_x R_l.$$

We may derive the following equation for K_j:

$$(K_j)_t + ((K_j)_{xx} + 3P_j^2 K_j)_x = 0.$$

For $j = 1, \dots, J$, if $P_j = B_k$ is a breather, let $c_j(t) \in \mathbb{R}^2$ defined for t large enough and if $P_j = R_l$ is a soliton, let $c_j(t) \in \mathbb{R}$ defined for t large enough such that for

$$\widetilde{z}(t) := z(t) + \sum_{j=1}^{J} c_j(t) K_j(t), \tag{4.5}$$

where $c_j K_j$ is either a product of two numbers of $\mathbb{R}$ or a scalar product of two vectors of $\mathbb{R}^2$, the following condition is satisfied: for any $j = 1, \dots, J$, for t large enough,

$$\int \widetilde{z}(t) K_j(t) \sqrt{\varphi_j(t)} = 0, \tag{4.6}$$

where φ_j is defined in Section 2.2 (in this proof, we can take $\delta = 1$). It is possible to do so in a unique way, because the Gram matrix associated to $K_j(t)\sqrt{\varphi_j(t)}$, $1 \le j \le J$, is invertible; which is the case because $K_j(t)\sqrt{\varphi_j(t)}$, $1 \le j \le J$, are linearly independent. This is why $c_j(t)$, $1 \le j \le J$, are defined in a unique way. For the same reason, $c_j(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, with coefficients that depend only on K_k, $1 \le k \le J$. This is why, from Cauchy–Schwarz, we may deduce the following lemma.

Lemma 4.2. *For any $j = 1, \dots, J$ and for t large enough, there exists $C > 0$, which does not depend on z, such that*

$$|c_j(t)| \le C \|z(t)\|_{L^2} \quad \textit{and} \quad \|\widetilde{z}(t)\|_{H^2} \le C \|z(t)\|_{H^2}.$$

The Gram matrix is C^1 in time and invertible. This is why its inverse is C^1 in time. Because $\int K_j z \sqrt{\varphi_j}$ are C^1 in time, we deduce by multiplication that $c_j(t)$ are C^1 in time.

By differentiating in time the linear relation that defines $c_j(t)$, we see that $c_j'(t)$ is obtained linearly from $\int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, and from $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$, $1 \le k \le J$, with coefficients that depend on K_k, $1 \le k \le J$ (and their derivatives). Because it is easy to see that $\frac{d}{dt} \int K_k(t) z(t) \sqrt{\varphi_k(t)}$ may still be bounded by $C \|z(t)\|_{L^2}$, we deduce that for any $j = 1, \dots, J$ and for t large enough, there exists $C > 0$, which does not depend on z, such that

$$|c_j'(t)| \le C \|z(t)\|_{L^2}. \tag{4.7}$$

We may derive the following equation for $\tilde{z}$:

$$\tilde{z}_t + (\tilde{z}_{xx} + 3\tilde{z}p^2)_x = -(3z^2 p + z^3)_x + \sum_{k=1}^{J} c_k'(t) K_k - 3\sum_{k=1}^{J} c_k(t)((P_k^2 - p^2)K_k)_x. \tag{4.8}$$

Step 2. *A bound for* $|c_j'(t)|$. The goal here is to improve (4.7).

Lemma 4.3. *For any* $j = 1, \dots, J$*, and for* t *large enough, there exist* $C > 0$ *and* $\theta > 0$*, which do not depend on* z*, such that*

$$|c_j'(t)| \le C\|\tilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2.$$

Proof. We may differentiate (4.6):

$$\begin{aligned}
0 &= \frac{d}{dt}\int \tilde{z} K_j \sqrt{\varphi_j} \\
&= \int \tilde{z}_t K_j \sqrt{\varphi_j} + \int \tilde{z}(K_j)_t \sqrt{\varphi_j} + \int \tilde{z} K_j (\sqrt{\varphi_j})_t \\
&= -\int (\tilde{z}_{xx} + 3\tilde{z}p^2)_x K_j \sqrt{\varphi_j} - \int (3z^2 p + z^3)_x K_j \sqrt{\varphi_j} \\
&\quad + \sum_{k=1}^{J} \int (c_k'(t) \cdot K_k) K_j \sqrt{\varphi_j} - 3\sum_{k=1}^{J} c_k(t) \int \big(c_k(t) \cdot ((P_k^2 - p^2)K_k)_x\big) K_j \sqrt{\varphi_j} \\
&\quad - \int \tilde{z}((K_j)_{xx} + 3K_j P_j^2)_x \sqrt{\varphi_j} + \int \tilde{z} K_j (\sqrt{\varphi_j})_t.
\end{aligned}$$

We know that $(\sqrt{\varphi_j})_x$ and $(\sqrt{\varphi_j})_t$ are bounded (from inequalities established in Section 2.2). This is why, for any t large enough,

$$\left|\int \tilde{z} K_j (\sqrt{\varphi_j})_t\right| \le C\|\tilde{z}(t)\|_{H^2}.$$

For the same reason, after eventually doing an integration by parts, for any t large enough,

$$\left|\int (\tilde{z}_{xx} + 3\tilde{z}p^2)_x K_j \sqrt{\varphi_j}\right| + \left|\int \tilde{z}((K_j)_{xx} + 3K_j P_j^2)_x \sqrt{\varphi_j}\right| \le C\|\tilde{z}(t)\|_{H^2}.$$

The term $\int (3z^2 p + z^3)_x K_j \sqrt{\varphi_j}$ is clearly bounded by $C\|z(t)\|_{H^2}^2$. Finally, we see that $(P_k^2 - p^2)K_k$ is exponentially bounded in time (in Sobolev or L^∞ norm), and using Lemma 4.2, we deduce that

$$\int \big(c_k(t) \cdot ((P_k^2 - p^2)K_k)_x\big) K_j \sqrt{\varphi_j} \le Ce^{-\theta t}\|z(t)\|_{H^2},$$

for a suitable $\theta > 0$ that does not depend on z. This is why we deduce, for any $j = 1, \dots, J$ and for t large enough, that there exist $C > 0$ and $\theta > 0$, which do not depend on z, such that

$$\left|\sum_{k=1}^{J} \int (c_k'(t) \cdot K_k) K_j \sqrt{\varphi_j}\right| \le C\|\tilde{z}(t)\|_{H^2} + Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2.$$

We recall that for any $(e_1, e_2) \in (\mathbb{R})^2$ or $(\mathbb{R}^2)^2$, $e_3 \in \mathbb{R}$ or $\mathbb{R}^2$, we have the following equality between two elements of $\mathbb{R}$ or $\mathbb{R}^2$ (where vectors are columns)

$$(e_1 \cdot e_2) e_3 = \big(e_1^T (e_2 \, e_3^T)\big)^T,$$

where T denotes the transpose.

First of all, because $\int K_k K_j^T \sqrt{\varphi_j}$ converges exponentially to $\int K_k K_j^T$, for $k \neq j$, we conclude that $\int K_k K_j^T$ is exponentially decreasing, and from (4.7), we may write that for any $j = 1, \dots, J$ and t large enough, there exist $C > 0$ and $\theta > 0$, which do not depend on z, such that

$$\left| \left(c_j'(t)^T \int K_j K_j^T \right)^T \right| \leq C \|\tilde{z}(t)\|_{H^2} + C e^{-\theta t} \|z(t)\|_{H^2} + C \|z(t)\|_{H^2}^2.$$

Now, when $K_j \in \mathbb{R}^2$, using the fact that its components are linearly independent and the Cauchy–Schwarz inequality, we deduce the desired lemma. ■

Step 3. *Coercivity.* We define the following functional, which is quadratic in $\tilde{z}$:

$$\begin{aligned} H(t) &= \frac{1}{2} \int \tilde{z}_{xx}^2 - \frac{5}{2} \int p^2 \tilde{z}_x^2 + \frac{5}{2} \int p_x^2 \tilde{z}^2 + 5 \int p p_{xx} \tilde{z}^2 + \frac{15}{4} \int p^4 \tilde{z}^2 \\ &\quad + \sum_{j=1}^{J} (b_j^2 - a_j^2) \Big(\int \tilde{z}_x^2 \varphi_j - 3 \int p^2 \tilde{z}^2 \varphi_j \Big) + \sum_{j=1}^{J} (a_j^2 + b_j^2)^2 \, \frac{1}{2} \int \tilde{z}^2 \varphi_j . \end{aligned}$$

We will prove the following lemma.

Lemma 4.4. *There exists $C > 0$, which does not depend on z, such that for t large enough,*

$$\|\tilde{z}(t)\|_{H^2}^2 \leq C H(t) + C \sum_{j=1}^{J} \Big(\int \tilde{z} P_j \Big)^2 .$$

Proof. We denote $\mathcal{Q}_j$ the quadratic form associated with P_j. We recall that

$$\begin{aligned} \mathcal{Q}_j[\varepsilon] &:= \frac{1}{2} \int \varepsilon_{xx}^2 - \frac{5}{2} \int P_j^2 \varepsilon_x^2 + \frac{5}{2} \int (P_j)_x^2 \varepsilon^2 + 5 \int P_j (P_j)_{xx} \varepsilon^2 \\ &\quad + \frac{15}{4} \int P_j^4 \varepsilon^2 + (b_j^2 - a_j^2) \Big(\int \varepsilon_x^2 - 3 \int P_j^2 \varepsilon^2 \Big) + (a_j^2 + b_j^2)^2 \, \frac{1}{2} \int \varepsilon^2 . \end{aligned}$$

In any case, we have that for any $j = 1, \dots, J$, there exists $\mu_j > 0$ such that if $\varepsilon \in H^2$ satisfies $\int K_j \varepsilon = 0$, then we have

$$\mathcal{Q}_j[\varepsilon] \geq \mu_j \|\varepsilon\|_{H^2}^2 - \frac{1}{\mu_j} \Big(\int \varepsilon P_j \Big)^2 .$$

Here, we apply this coercivity result with $\varepsilon = \tilde{z} \sqrt{\varphi_j}$, for which the orthogonality conditions (4.6) are satisfied. Thus,

$$\|\tilde{z} \sqrt{\varphi_j}\|_{H^2}^2 \leq C \mathcal{Q}_j[\tilde{z} \sqrt{\varphi_j}] + C \Big(\int \tilde{z} P_j \sqrt{\varphi_j} \Big)^2 .$$

We denote

$$\mathcal{Q}_j'[\varepsilon] := \frac{1}{2}\int \varepsilon_{xx}^2\varphi_j - \frac{5}{2}\int p^2\varepsilon_x^2\varphi_j + \frac{5}{2}\int p_x^2\varepsilon^2\varphi_j + 5\int pp_{xx}\varepsilon^2\varphi_j$$
$$+ \frac{15}{4}\int p^4\varepsilon^2\varphi_j + (b_j^2-a_j^2)\Big(\int \varepsilon_x^2\varphi_j - 3\int p^2\varepsilon^2\varphi_j\Big) + (a_j^2+b_j^2)^2\,\frac{1}{2}\int \varepsilon^2\varphi_j,$$

and we observe that

$$H(t) = \sum_{j=1}^{J} \mathcal{Q}_j'[\tilde{z}(t)].$$

In $\mathcal{Q}_j'[\tilde{z}(t)]$, we may replace p by P_j with an error bounded by $Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}^2$, because of (4.1) mainly. After that, the expression obtained may be replaced by $\mathcal{Q}_j[\tilde{z}(t)\sqrt{\varphi_j(t)}]$ with an error bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$ (cf. calculations done in the proof of Lemma 2.19). For the same reason, $\|\tilde{z}\sqrt{\varphi_j}\|_{H^2}^2$ may be replaced by $\int(\tilde{z}^2 + \tilde{z}_x^2 + \tilde{z}_{xx}^2)\varphi_j$ with an error bounded by $\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2$. Therefore, because of

$$\|\tilde{z}\|_{H^2}^2 = \sum_{j=1}^{J}\int(\tilde{z}^2 + \tilde{z}_x^2 + \tilde{z}_{xx}^2)\varphi_j,$$

the fact that $P_j\sqrt{\varphi_j}$ converges exponentially to P_j, and the fact that C/t may be as small as we want, if we take t large enough, we deduce the desired lemma. ■

Step 4. *Modification of H for the sake of simplification.* We define

$$\widetilde{H}(t) := \int\Big[\frac{1}{2}\tilde{z}_{xx}^2 - \frac{5}{2}\big((\tilde{z}+p)^2(\tilde{z}+p)_x^2 - p^2p_x^2 - 2\tilde{z}pp_x^2 - 2\tilde{z}_xp^2p_x\big)$$
$$+ \frac{1}{4}\big((\tilde{z}+p)^6 - p^6 - 6\tilde{z}p^5\big)\Big] + \frac{1}{2}\sum_{j=1}^{J}(a_j^2+b_j^2)^2\int \tilde{z}^2\varphi_j$$
$$+ 2\sum_{j=1}^{J}(b_j^2-a_j^2)\int\Big[\frac{1}{2}\tilde{z}_x^2 - \frac{1}{4}\big((\tilde{z}+p)^4 - p^4 - 4\tilde{z}p^3\big)\Big]\varphi_j.$$

We observe that the difference between H and $\widetilde{H}$ is bounded by $O(\|\tilde{z}(t)\|_{H^2}^3)$. We can thus claim the following.

Lemma 4.5. *There exists $C>0$, which does not depend on z, such that for t large enough,*

$$\|\tilde{z}(t)\|_{H^2}^2 \le C\widetilde{H}(t) + C\sum_{j=1}^{J}\Big(\int \tilde{z}P_j\Big)^2.$$

Step 5. *A bound for $d\widetilde{H}/dt$.*

Lemma 4.6. *There exist $C>0$ and $\theta>0$, which do not depend on z, such that for t large enough,*

$$\Big|\frac{d\widetilde{H}}{dt}\Big| \le \frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2.$$

Proof. We develop the expression of $\widetilde{H}(t)$, we differentiate each term obtained, we use equation (4.8), the fact that p is a solution of (1.1) and the fact that $(\varphi_j)_t = -\frac{x}{t}(\varphi_j)_x$, where x/t is bounded independently from z because of the compact support of φ_j. We obtain several sorts of terms after doing several integrations by parts and several obvious simplifications.

Several terms are clearly bounded by one of the bounds of the lemma, because in these terms, the accumulated degree of z and $\widetilde{z}$ is larger than 2. As an example, we show how to deal with $\int z_{xxx} z \widetilde{z}_{xx} p$. We use the fact that $z = \widetilde{z} - \sum_{j=1}^{J} c_j K_j$, and we obtain the following:

$$\int z_{xxx} z \widetilde{z}_{xx} p = \int \widetilde{z}_{xxx} \widetilde{z} \widetilde{z}_{xx} p - \int \widetilde{z}_{xxx} \Big(\sum_{j=1}^{J} c_j K_j \Big) \widetilde{z}_{xx} p$$
$$- \int \Big(\sum_{j=1}^{J} c_j (K_j)_{xxx} \Big) \widetilde{z} \widetilde{z}_{xx} p + \int \Big(\sum_{j=1}^{J} c_j (K_j)_{xxx} \Big) \Big(\sum_{j=1}^{J} c_j K_j \Big) \widetilde{z}_{xx} p.$$

It is easy to see that any of these terms is bounded as we want in the lemma (several of them are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$, the last one is bounded by $C\|\widetilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2$), because of Lemma 4.2 and (4.3).

Other terms contain $\widetilde{z}$ quadratically and contain $(\varphi_j)_x$. In addition, $(\varphi_j)_x$ is bounded by C/t. This is why such terms are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Several other terms can be, by doing suitable integrations by parts transformed in one of the two following expressions:

$$6\sum_{j=1}^{J} \int \widetilde{z}\widetilde{z}_x p \Big[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5 \Big] \varphi_j,$$
$$3\sum_{j=1}^{J} \int \widetilde{z}^2 p_x \Big[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5 \Big] \varphi_j.$$

To deal with these two expressions, we use the elliptic equation satisfied by P_j:

$$(P_j)_{xxxx} - 2(b_j^2 - a_j^2)((P_j)_{xx} + P_j^3) + (a_j^2 + b_j^2)^2 P_j + 5P_j (P_j)_x^2 + 5P_j^2 (P_j)_{xx} + \frac{3}{2} P_j^5 = 0, \tag{4.9}$$

and the fact that

$$\Big[p_{xxxx} - 2(b_j^2 - a_j^2)(p_{xx} + p^3) + (a_j^2 + b_j^2)^2 p + 5pp_x^2 + 5p^2 p_{xx} + \frac{3}{2}p^5 \Big] \varphi_j$$

converges exponentially to

$$(P_j)_{xxxx} - 2(b_j^2 - a_j^2)((P_j)_{xx} + P_j^3) + (a_j^2 + b_j^2)^2 P_j + 5P_j (P_j)_x^2 + 5P_j^2 (P_j)_{xx} + \frac{3}{2} P_j^5,$$

which is a direct consequence of (4.1). This is why such terms are bounded by $\frac{C}{t}\|\widetilde{z}(t)\|_{H^2}^2$.

Other terms contain $(P_j^2 - p^2)K_j$, which is bounded exponentially, with c_j bounded by $\|z\|_{H^2}$. Those terms are obviously bounded by $Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$.

Other terms contain K_k (or a derivative) and φ_j with $j \neq k$. In this case, this product gives an exponential decreasing, and such a term is bounded by $Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}$, using (4.7).

Therefore, we are left with the following terms:

$$\sum_{j=1}^{J} c_j'(t) \int \Big[(K_j)_{xx}\tilde{z}_{xx} - 10K_j\tilde{z}_x p p_x - 5K_j\tilde{z}p_x^2 - 10(K_j)_x\tilde{z}pp_x - 5(K_j)_x\tilde{z}_x p^2 \\ + \frac{15}{4}K_j\tilde{z}p^4 + 2(b_j^2 - a_j^2)(K_j)_x\tilde{z}_x - 6(b_j^2 - a_j^2)K_j\tilde{z}p^2 + (a_j^2 + b_j^2)^2 K_j\tilde{z}\Big]\varphi_j.$$

We may replace p by P_j in the preceding expression with an error bounded by

$$Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2},$$

because of (4.7) and (4.1). This is acceptable, knowing the result we want to prove. By integration by parts, we obtain several terms of the form $c_j'(t)\int (K_j)_{xx}\tilde{z}_x(\varphi_j)_x$, which are bounded by $\frac{C}{t}|c_j'(t)|\|\tilde{z}(t)\|_{H^2}$. Now, from Lemma 4.3, we deduce that they are bounded by

$$\frac{C}{t}\|\tilde{z}(t)\|_{H^2}^2 + Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2,$$

which is exactly the bound that we want. We are left with the following terms:

$$\sum_{j=1}^{J} c_j'(t) \int \Big[(K_j)_{xxxx} + 10(K_j)_x P_j(P_j)_x + 5K_j(P_j)_x^2 + 10K_jP_j(P_j)_{xx} + 5(K_j)_{xx}P_j^2 \\ + \frac{15}{2}K_jP_j^4 - 2(b_j^2 - a_j^2)(K_j)_{xx} - 6(b_j^2 - a_j^2)K_jP_j^2 + (a_j^2 + b_j^2)^2K_j\Big]\tilde{z}\varphi_j.$$

The last expression equals zero, because of the elliptic equation satisfied by K_j, which we may derive by differentiating (4.9). ■

Step 6. *A bound for* $\frac{d}{dt}\int \tilde{z}P_j$.

Lemma 4.7. *There exist $C > 0$ and $\theta > 0$, which do not depend on z, such that for t large enough, for any $j = 1, \dots, J$,*

$$\left|\frac{d}{dt}\int \tilde{z}P_j\right| \le Ce^{-\theta t}\|z(t)\|_{H^2} + C\|z(t)\|_{H^2}^2.$$

Proof. We observe that

$$\int \tilde{z}P_j = \int zP_j + \sum_{k=1}^{J} c_k(t)\int K_kP_j.$$

Firstly, for $k = j$,

$$\int K_jP_j = 0,$$

and for $k \neq j$,

$$\frac{d}{dt}\Big[c_k(t)\int K_k P_j\Big] = c_k'(t)\int K_k P_j + c_k(t)\int (K_k)_t P_j + c_k(t)\int K_k (P_j)_t,$$

and it is obvious, from Lemma 4.2 and (4.7), that the latter is bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$.

It is left to bound $\frac{d}{dt}\int zP_j$. We use (4.4) and we obtain

$$\frac{d}{dt}\int zP_j = -\int (z_{xx} + (z+p)^3 - p^3)_x P_j - \int z((P_j)_{xx} + P_j^3)_x.$$

Several terms are immediately boundable by $C\|z(t)\|_{H^2}^2$, we kill several others by integration by parts and we are left with

$$\int z(p^2 - P_j^2)(P_j)_x,$$

which is obviously bounded by $Ce^{-\theta t}\|z(t)\|_{H^2}$, because of (4.1). ■

By differentiation of a square, we obtain the following.

Lemma 4.8. *There exist $C > 0$ and $\theta > 0$, which do not depend on z, such that for t large enough, and for any $j = 1, \dots, J$,*

$$\left|\frac{d}{dt}\Big(\int \tilde{z} P_j\Big)^2\right| \leq Ce^{-\theta t}\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2} + C\|\tilde{z}(t)\|_{H^2}\|z(t)\|_{H^2}^2.$$

Step 7. *A bound for $\|z(t)\|_{H^2}$ in terms of $\tilde{z}(t)$.* Because we have chosen $N > 2$ and because of (4.3), we may claim that for t large enough, the following integral is finite:

$$\int_t^{+\infty} \|z(s)\|_{H^2}\, ds.$$

Because of Lemma 4.2 and (4.3), we deduce that

$$c_j(t) \to 0 \quad \text{as } t \to +\infty.$$

From this and Lemma 4.3, we deduce by integration that

$$\begin{aligned}
|c_j(t)| &\leq \int_t^{+\infty} |c_j'(s)|\, ds \\
&\leq C\int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2}\, ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2\, ds.
\end{aligned}$$

Knowing this and using (4.5), we may deduce that

$$\begin{aligned}
\|z(t)\|_{H^2} &\leq C\|\tilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds \\
&\qquad + C\int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2}\, ds + \int_t^{+\infty} \|z(s)\|_{H^2}^2\, ds \\
&\leq C\|\tilde{z}(t)\|_{H^2} + C\int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds \\
&\qquad + C\sup_{s\geq t}\|z(s)\|_{H^2} e^{-\theta t} + C\sup_{s\geq t}\|z(s)\|_{H^2}\int_t^{+\infty} \|z(s)\|_{H^2}\, ds,
\end{aligned}$$

which implies, because

$$\int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds, \quad \sup_{s\geq t}\|z(s)\|_{H^2}\, e^{-\theta t}, \quad \sup_{s\geq t}\|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2}\, ds$$

are decreasing in time, that

$$\sup_{s\geq t}\|z(s)\|_{H^2} \leq C \sup_{s\geq t}\|\tilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds + C \sup_{s\geq t}\|z(s)\|_{H^2}\, e^{-\theta t} + C \sup_{s\geq t}\|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2}\, ds.$$

Since $e^{-\theta t}$ and $\int_t^{+\infty}\|z(s)\|_{H^2}\, ds$ may be as small as we want for t large enough (dependent on z), we may deduce the following.

Lemma 4.9. *There exists $C > 0$, which does not depend on z, such that for t large enough,*

$$\|z(t)\|_{H^2} \leq \sup_{s\geq t}\|z(s)\|_{H^2} \leq C \sup_{s\geq t}\|\tilde{z}(s)\|_{H^2} + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\, ds.$$

Step 8. *Conclusion.* By integration, from Lemmas 4.5, 4.6 and 4.8, for t large enough (depending on z), with constants C and θ that do not depend on z,

$$\begin{aligned}\|\tilde{z}(t)\|_{H^2}^2 &\leq C \int_t^{+\infty} \frac{1}{s}\|\tilde{z}(s)\|_{H^2}^2\, ds + C \int_t^{+\infty} e^{-\theta s}\|\tilde{z}(s)\|_{H^2}\|z(s)\|_{H^2}\, ds \\ &\quad + C \int_t^{+\infty} \|\tilde{z}(s)\|_{H^2}\|z(s)\|_{H^2}^2\, ds \\ &\leq C \sup_{s\geq t}\|\tilde{z}(s)\|_{H^2} \int_t^{+\infty} \Big(\frac{1}{s}\|\tilde{z}(s)\|_{H^2} + e^{-\theta s}\|z(s)\|_{H^2} + \|z(s)\|_{H^2}^2\Big)\, ds.\end{aligned}$$

Because the right-hand side of the inequality above is decreasing in time, we deduce, after taking the supremum of the previous inequality and after simplification, that for t large enough,

$$\begin{aligned}\sup_{s\geq t}\|\tilde{z}(s)\|_{H^2} &\leq C \int_t^{+\infty} \frac{1}{s}\|\tilde{z}(s)\|_{H^2}\, ds + C \int_t^{+\infty} e^{-\theta s}\|z(s)\|_{H^2}\, ds \\ &\quad + C \int_t^{+\infty} \|z(s)\|_{H^2}^2\, ds \\ &\leq C \int_t^{+\infty} \frac{1}{s}\|\tilde{z}(s)\|_{H^2}\, ds + C \sup_{s\geq t}\|z(s)\|_{H^2} e^{-\theta t} \\ &\quad + C \sup_{s\geq t}\|z(s)\|_{H^2} \int_t^{+\infty} \|z(s)\|_{H^2}\, ds.\end{aligned}$$

Using (4.3), the fact that $N-1 > 1$ and the fact that $e^{-\theta t}$ is decreasing faster than $1/t^{N-2}$, we deduce that for t large enough,

$$\sup_{s\geq t}\|\tilde{z}(s)\|_{H^2} \leq C \int_t^{+\infty} \frac{1}{s}\|\tilde{z}(s)\|_{H^2}\, ds + C \frac{1}{t^{N-2}} \sup_{s\geq t}\|z(s)\|_{H^2}.$$

Using Lemma 4.9, we deduce that

$$\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} \leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\,\frac{1}{t^{N-2}}\sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} + C\,\frac{1}{t^{N-2}}\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds.$$

And because $1/t^{N-2}$ can be as small as we want for t large enough, we deduce that for t large enough and for a constant $C>0$ that does not depend on z or on N,

$$\|\widetilde{z}(t)\|_{H^2} \leq \sup_{s\geq t}\|\widetilde{z}(s)\|_{H^2} \leq C\int_t^{+\infty}\frac{1}{s}\|\widetilde{z}(s)\|_{H^2}\,ds + C\,\frac{1}{t^{N-2}}\int_t^{+\infty}\|\widetilde{z}(s)\|_{H^2}\,ds. \tag{4.10}$$

Let us pick $T>0$ large enough such that for $t\geq T$, inequality (4.10) works (i.e., T is large enough so that every part of the preceding proof works). From (4.5) and Lemma 4.2, we know that for $t\geq T$ (by taking T larger if needed),

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{C}{t^{N-1}}. \tag{4.11}$$

This is why the following quantity is well defined:

$$A := \sup_{t\geq T}\{t^{N-1}\|\widetilde{z}(t)\|_{H^2}\}, \tag{4.12}$$

which means that for $t\geq T$,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{A}{t^{N-1}}. \tag{4.13}$$

Now, using (4.11) and (4.13), we deduce from (4.10) that for $t\geq T$, with $C>0$ that does not depend on z, N or A,

$$\|\widetilde{z}(t)\|_{H^2} \leq \frac{CA}{N-1}\frac{1}{t^{N-1}} + \frac{CA}{N-2}\frac{1}{t^{2N-4}} \leq \frac{CA}{N-2}\frac{1}{t^{N-1}}, \tag{4.14}$$

if we assume that $N>3$. Now, from (4.12), we deduce that there exists $T^*>T$ such that

$$(T^*)^{N-1}\|\widetilde{z}(T^*)\|_{H^2} \geq \frac{A}{2}.$$

This is why, by evaluating (4.14) in $t=T^*$, we find that

$$\frac{A}{2(T^*)^{N-1}} \leq \frac{CA}{N-2}\frac{1}{(T^*)^{N-1}},$$

which, if we assume that $A>0$, after simplification yields

$$N-2\leq 2C.$$

This means that if we assume that $N>2C+2$ and $N>3$, the assumption $A>0$ leads to a contradiction. Therefore, $A=0$ under that assumption on N, which implies $\|\widetilde{z}(t)\|_{H^2}=0$, and from Lemma 4.9, this implies that $z\equiv 0$. This means that the condition that we have established for N, namely,

$$N>\max(2C+2,3),$$

does not depend on z and allows us to deduce that under (4.3), we may establish that $z\equiv 0$. Proposition 1.5 is now proved. ∎

4.2. A convergence of a solution to a multi-breather is always exponentially fast, in the case when all the velocities are positive

Proposition 4.10. *Let $u(t)$ be an H^2 solution of (1.1) on $[T,+\infty)$, for $T \in \mathbb{R}$. We assume that*

$$\|u(t) - p(t)\|_{H^2} \to 0 \quad \text{as } t \to +\infty,$$

where p is the multi-breather constructed in Section 2. If $v_1 > 0$, then there exist $\varpi > 0$, $T_0 \geq T$ and $C > 0$ such that for any $t \geq T_0$,

$$\|u(t) - p(t)\|_{H^2} \leq Ce^{-\varpi t}.$$

Note that in the formulation of the proposition above, we may replace p by P without changing its content (this is a consequence from (1.9)).

Proof. We set $v(t) := u(t) - P(t)$, such that $\|v(t)\|_{H^2} \to 0$ as $t \to +\infty$.

We denote

$$\Psi(x) := \frac{2}{\pi} \arctan\big(\exp(-\sqrt{\sigma}x/2)\big),$$

where $\sigma > 0$ is small enough (with precise conditions that will be mentioned throughout the proof). By direct calculations,

$$\Psi'(x) = \frac{-\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma}x/2)}.$$

Thus,

$$|\Psi'(x)| \leq C \exp(-\sqrt{\sigma}|x|/2).$$

We have the following properties: $\lim_{+\infty} \Psi = 0$, $\lim_{-\infty} \Psi = 1$, and $\Psi(-x) = 1 - \Psi(x)$, $\Psi'(x) < 0$, $|\Psi''(x)| \leq \sqrt{\sigma}|\Psi'(x)|/2$, $|\Psi'''(x)| \leq \sqrt{\sigma}|\Psi''(x)|/2$, $|\Psi'(x)| \leq \sqrt{\sigma}\Psi/2$ and $|\Psi'(x)| \leq \sqrt{\sigma}(1-\Psi)/2$ for all $x \in \mathbb{R}$.

For $j = 2, \dots, J$, let

$$m_j = \frac{v_{j-1} + v_j}{2}.$$

Let us denote $\tau_0 > 0$ the minimal distance between a v_j and an m_j.

We define for $j = 2, \dots, J$,

$$\Phi_j(t,x) := \Psi(x - m_j t).$$

We may extend this definition to $j = 1$ and $j = J+1$ in the following way: $\Phi_1 := 0$ and $\Phi_{J+1} := 1$. Thus, the function that allows us to study the properties around each object P_j (for $j = 1, \dots, J$) is $\chi_j := \Phi_{j+1} - \Phi_j$.

The goal is to prove that, for t large enough,

$$\|v(t)\|_{H^2} \leq Ce^{-\varpi t}, \tag{4.15}$$

where $\varpi > 0$ is a constant to be deduced from the constants of the problem. Proposition 4.10 follows from this, because of Theorem 1.2.

We will prove (4.15) by induction. In particular, we will prove, for $j = 2, \dots, J+1$ and for t large enough, that $\int (v^2 + v_x^2 + v_{xx}^2)\Phi_j \leq Ce^{-2\varpi t}$ holds, in the knowledge of

$\int(v^2+v_x^2+v_{xx}^2)\Phi_{j-1}\leq Ce^{-2\varpi t}$ (note that this assumption is empty when $j=2$). This implies the desired inequality. (Note that it is OK if ϖ becomes smaller after a step of this induction, as long as it stays positive.)

Let us write the j-th step of our reasoning by induction (where $j\in\{2,\dots,J+1\}$). Thus, j is fixed in the rest of the proof. We assume that

$$\int(v^2+v_x^2+v_{xx}^2)\Phi_{j-1}\leq Ce^{-2\varpi t}. \tag{4.16}$$

We divide our proof in several steps.

Step 1. *Almost-conservation of localized conservation laws.* We define quantities that are similar to quantities defined in Section 2.2. We note that we localize around the first $j-1$ objects, not only around the $(j-1)$-th object. The notations given in Section 2.2 should not be considered in the proof and should be replaced by the following notations:

$$M_j(t):=\frac12\int u^2(t)\Phi_j(t),\quad E_j(t):=\int\Big[\frac12 u_x^2-\frac14 u^4\Big]\Phi_j(t),$$
$$F_j(t):=\int\Big[\frac12 u_{xx}^2-\frac52 u^2u_x^2+\frac14 u^6\Big]\Phi_j(t).$$

Lemma 4.11. *Let $\omega_2,\omega_6>0$ be as small as desired. There exist $T_1\geq T$ and $C>0$ such that for $t\geq T_1$,*

$$\sum_{i=1}^{j-1}M[P_i]-M_j(t)\geq -Ce^{-2\varpi t},$$
$$\sum_{i=1}^{j-1}(E[P_i]+\omega_2M[P_i])-(E_j(t)+\omega_2M_j(t))\geq -Ce^{-2\varpi t},$$
$$\sum_{i=1}^{j-1}(F[P_i]+\omega_6M[P_i])-(F_j(t)+\omega_6M_j(t))\geq -Ce^{-2\varpi t}.$$

Proof. We will use the results of the computations made at the bottom of p. 1115 and at the bottom of p. 1116 of [28], as well as in Appendix A.5 to claim the three following facts:

$$\frac{d}{dt}\frac12\int u^2f=\int\Big(-\frac32u_x^2+\frac34u^4\Big)f'+\frac12\int u^2f''',$$
$$\frac{d}{dt}\int\Big[\frac12 u_x^2-\frac14u^4\Big]f=\int\Big[-\frac12(u_{xx}+u^3)^2-u_{xx}^2+3u_x^2u^2\Big]f'+\frac12\int u_x^2f''',$$

and

$$\frac{d}{dt}\int\Big(\frac12u_{xx}^2-\frac52u^2u_x^2+\frac14u^6\Big)f$$
$$=\int\Big(-\frac32u_{xxx}^2+9u_{xx}^2u^2+15u_x^2uu_{xx}+\frac{9}{16}u^8+\frac14u_x^4+\frac32u_{xx}u^5-\frac{45}{4}u^4u_x^2\Big)f'$$
$$+5\int u^2u_xu_{xx}f''+\frac12\int u_{xx}^2f'''.$$

where f is a C^3 function that does not depend on time.

For the mass: If $j \leq J$,

$$2\,\frac{d}{dt}\,M_j(t) = -\int \Big(3u_x^2 + m_j u^2 - \frac{3}{2}\,u^4\Big)\Phi_{jx}(t) + \int u^2\,\Phi_{jxxx}(t).$$

We recall that

$$|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2}|\Phi_{jx}|, \quad |\Phi_{jxxx}| \leq \frac{\sigma}{4}|\Phi_{jx}|, \quad \Phi_{jx} \leq 0,$$

where we can choose σ as small as desired. For this proof, we assume

$$0 < \sigma \leq m_2 \leq m_j.$$

Thus,

$$2\,\frac{d}{dt}\,M_j(t) \geq \int \Big(3u_x^2 + \frac{3\sigma}{4}\,u^2 - \frac{3}{2}\,u^4\Big)|\Phi_{jx}(t)|.$$

By Corollary 2.2, for $r > 0$, if t, x satisfy $v_{j-1}\,t + r < x < v_j\,t - r$, then

$$|u(t,x)| \leq |P(t,x)| + \|v(t)\|_{L^\infty} \leq Ce^{-\beta r} + C\|v(t)\|_{H^2},$$

and the same could be said for u_x.

We can thus deduce, for r and T_1 large enough, and for $x \in (v_{j-1}\,t + r, v_j\,t - r)$, that $|u|$ is bounded by any fixed constant, that can be taken as small as desired. Here, we will use the latter to bound $3u^2/2$ by $\sigma/4$.

For $t \geq T_1$ and $x \leq v_{j-1}\,t + r$ or $x \geq v_j\,t - r$, we have $|x - m_j t| \leq \tau_0\,t - r$, and therefore for such t, x,

$$|\Phi_{jx}(t,x)| \leq C\,\exp(-\sqrt{\sigma}|x - m_j t|/2) \leq C\,\exp(-\sqrt{\sigma}\tau_0\,t/2)\exp(\sqrt{\sigma}r/2).$$

Because $\int u^4$ is bounded by a constant for any time and $\exp(\sqrt{\sigma}r/2)$ is a fixed constant (r is already chosen), we have, for $t \geq T_1$,

$$\frac{d}{dt}\,M_j(t) \geq \int \Big(\frac{3}{2}\,u_x^2 + \frac{\sigma}{4}\,u^2\Big)|\Phi_{jx}(t)| - Ce^{-2\varpi t} \geq -Ce^{-2\varpi t},$$

where ϖ is chosen as a suitable function of σ and τ_0.

By integration, we deduce that for any $t_1 \geq t$, with a constant $C > 0$ that does not depend on t_1, we have

$$M_j(t_1) - M_j(t) \geq -Ce^{-2\varpi t}. \tag{4.17}$$

We note that this conclusion is immediate when $j = J + 1$, because we have exactly the conserved quantity.

We have that

$$\begin{aligned}\Big|\sum_{i=1}^{j-1} M[P_i] - M_j(t_1)\Big| &\leq \Big|\sum_{i=1}^{j-1}\frac{1}{2}\int P_i^2 - \frac{1}{2}\int P^2\,\Phi_j(t_1)\Big| + \frac{1}{2}\Big|\int P^2\,\Phi_j(t_1) - \int u^2\,\Phi_j(t_1)\Big| \\ &\leq Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + \frac{1}{2}\int |P^2 - u^2|\Phi_j(t_1) \\ &\leq Ce^{-\kappa(\beta,\sigma,\tau_0)t_1} + C\int |P^2 - u^2| \to 0 \quad \text{as } t_1 \to +\infty.\end{aligned}$$

This means that when we take the limit of (4.17) when $t_1 \to +\infty$, we obtain, for $t \geq T_1$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) \geq -Ce^{-2\varpi t},$$

which is exactly what we wished to prove.

For the energy: If $j \leq J$,

$$\begin{aligned}
2\frac{d}{dt}E_j(t) &= \int \big[-(u_{xx}+u^3)^2 - 2u_{xx}^2 + 6u_x^2u^2\big]\Phi_{jx}(t) \\
&\quad - m_j\int\Big(u_x^2 - \frac{1}{2}u^4\Big)\Phi_{jx}(t) + \frac{1}{2}\int u_x^2\,\Phi_{jxxx}(t) \\
&\geq \int\Big[(u_{xx}+u^3)^2 + 2u_{xx}^2 - 6u_x^2u^2 + \frac{3\sigma}{4}u_x^2 - \frac{m_j}{2}u^4\Big]|\Phi_{jx}(t)|.
\end{aligned}$$

We can do the same reasoning as for the mass to bound above $m_ju^2/2$ by ω_1, a constant that we can choose as small as desired, and to bound above $6u^2$ by $\sigma/4$. We obtain that if T_1 is large enough (dependently on the chosen constant ω_1),

$$2\frac{d}{dt}E_j(t) \geq \int\Big[(u_{xx}+u^3)^2 + 2u_{xx}^2 + \frac{\sigma}{2}u_x^2 - \omega_1u^2\Big]|\Phi_{jx}(t)| - Ce^{-2\varpi t}.$$

By using what we have performed for the mass, if we take ω_1 small enough with respect to $\omega_2\sigma/2$, we have that

$$\frac{d}{dt}(E_j + \omega_2 M_j)(t) \geq -Ce^{-2\varpi t}.$$

Then, by integration, in a similar manner as for the mass, we obtain that the desired conclusion is true for any j.

For F: If $j \leq J$,

$$\begin{aligned}
&2\frac{d}{dt}F_j(t) \\
&= \int\Big(-3u_{xxx}^2 + 18u_{xx}^2u^2 + 30u_x^2uu_{xx} + \frac{9}{8}u^8 + \frac{1}{2}u_x^4 + 3u_{xx}u^5 - \frac{45}{2}u^4u_x^2\Big)\Phi_{jx}(t) \\
&\quad - m_j\int\Big(u_{xx}^2 - 5u^2u_x^2 + \frac{1}{2}u^6\Big)\Phi_{jx}(t) + 10\int u^2u_{xx}u_x\Phi_{jxx}(t) + \int u_{xx}^2\,\Phi_{jxxx}(t) \\
&\geq \int\Big(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 - 18u_{xx}^2u^2 - 15u_x^2u^2 - 15u_x^2u_{xx}^2 - \frac{9}{8}u^8 \\
&\quad - \frac{1}{2}u_x^4 - \frac{3}{2}u_{xx}^2u^4 - \frac{3}{2}u^6\Big)|\Phi_{jx}(t)| + \int\Big(\sigma u_{xx}^2 + \frac{\sigma}{2}u^6 - 5m_ju^2u_x^2\Big)|\Phi_{jx}(t)| \\
&\quad - 5\int u^2u_x^2|\Phi_{jxx}(t)| - 5\int u^2u_{xx}^2|\Phi_{jxx}(t)| - \int u_{xx}^2|\Phi_{jxxx}(t)|.
\end{aligned}$$

By the same reasoning as for the energy and the mass, if we set $\omega_3, \omega_4, \omega_5 > 0$, constants that we can take as small as desired, and if T_1 is large enough depending on these constants, for $t \geq T_1$, we have that

$$2\frac{d}{dt}F_j(t) \geq \int \Big(3u_{xxx}^2 + \frac{45}{2}u^4u_x^2 + \frac{3\sigma}{4}u_{xx}^2 + \frac{\sigma}{2}u^6 \\ - \omega_3 u_{xx}^2 - \omega_4 u_x^2 - \omega_5 u^2\Big)|\Phi_{jx}(t)| - Ce^{-2\varpi t}.$$

By using what we have done for the mass, if we take $\omega_3, \omega_4, \omega_5$ small enough (with respect to ω_6), we have that

$$\frac{d}{dt}(F_j + \omega_6 M_j)(t) \geq -Ce^{-2\varpi t}.$$

Then, by integration, similarly, as before, we obtain that the desired conclusion is true for any j. ∎

Remark 4.12. If $j = J+1$, we have that

$$\sum_{i=1}^{J} M[P_i] - M_{J+1}(t) = 0, \quad \sum_{i=1}^{J} E[P_i] - E_{J+1}(t) = 0, \quad \sum_{i=1}^{J} F[P_i] - F_{J+1}(t) = 0.$$

Step 2. *Modulation.* The notations that were given in Section 2.3 should not be taken into consideration in the following proof and should be replaced by new ones provided below.

Lemma 4.13. *There exist $C > 0$, $T_2 \geq T$ and unique C^1 functions $y_1, y_2 \colon [T_2, +\infty) \to \mathbb{R}$ such that if we set*

$$w(t,x) := u - \widetilde{P},$$

where

$$\widetilde{P}(t,x) := \sum_{i=1}^{J} \widetilde{P}_i(t,x),$$

with

$$\widetilde{P}_i(t,x) := P_i(t,x) \quad \textit{for } i \neq j-1,$$

and either

$$\widetilde{P}_{j-1}(t,x) := \kappa_l Q_{c_l + y_1(t)}(x - x_{0,l}^0 + y_2(t) - c_l t) \quad \textit{if } P_{j-1} = R_l \textit{ is a soliton},$$

or

$$\widetilde{P}_{j-1}(t,x) := B_{\alpha_k,\beta_k}(t, x; x_{1,k} + y_1(t), x_{2,k} + y_2(t)) \quad \textit{if } P_{j-1} = B_k \textit{ is a breather},$$

then, $w(t)$ satisfies, for any $t \in [T_2, +\infty)$, either

$$\int \widetilde{P}_{j-11}(t)w(t) = \int \widetilde{P}_{j-12}(t)w(t) = 0 \quad \textit{if } P_{j-1} \textit{ is a breather}, \tag{4.18}$$

or

$$\int \widetilde{P}_{j-1}(t)w(t) = \int \widetilde{P}_{j-1x}(t)w(t) = 0 \quad \textit{if } P_{j-1} \textit{ is a soliton}, \tag{4.19}$$

where in the case P_{j-1} is a breather, we denote

$$\widetilde{P}_{j-11}(t,x) := \partial_{x_1}\widetilde{P}_{j-1}, \quad \widetilde{P}_{j-12}(t,x) := \partial_{x_2}\widetilde{P}_{j-1}.$$

Moreover, for any $t \in [T_2, +\infty)$,

$$\|w(t)\|_{H^2} + |y_1(t)| + |y_2(t)| \le C\|v(t)\|_{H^2}, \tag{4.20}$$

and, if ϖ *is small enough,*

$$|y_1'(t)| + |y_2'(t)| \le C\Big(\int w(t)^2\,\Phi_j\Big)^{1/2} + Ce^{-\varpi t}. \tag{4.21}$$

Proof. The proof that has to be performed is similar to the proof of Lemma 2.8, which is a consequence of a quantitative version of the implicit function theorem. See Section 2.2 in [11] for a precise statement. The proof of (4.21) is also similar: as in the proof of Lemma 2.8, we take the time derivative of $\int \widetilde{P}_{j-1\,1}(t)\,w(t) = \int \widetilde{P}_{j-1\,2}(t)\,w(t) = 0$. For completeness, let us perform this proof.

For $t \in [T_2, +\infty)$, let

$$F_t\colon L^2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$$

be such that if $P_{j-1} = B_k$ is a breather,

$$(U, y_1, y_2) \mapsto \Big(\int \partial_{x_1} B_{\alpha_k,\beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2)\epsilon\,dx,$$
$$\int \partial_{x_2} B_{\alpha_k,\beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2)\epsilon\,dx\Big),$$

where

$$\epsilon := U - P + P_{j-1} - B_{\alpha_k,\beta_k}(t, x; x_{1,k}^0 + y_1, x_{2,k}^0 + y_2),$$

and if $P_{j-1} = R_l$ is a soliton,

$$(U, y_1, y_2) \mapsto \Big(\int \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t)\epsilon\,dx,$$
$$\int \partial_x \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t)\epsilon\,dx\Big),$$

where

$$\epsilon := U - P + P_{j-1} - \kappa_l Q_{c_l+y_1}(x - x_{0,l}^0 + y_2 - c_l t).$$

We observe that F_t is a C^1 function and that $F_t(P(t), 0, 0) = 0$. Now, let us consider the matrix which gives the differential of F_t (with respect to y_1, y_2) in $(P(t), 0, 0)$.

In the case when $P_{j-1} = B_k$ is a breather, this matrix is

$$DF_t = \begin{pmatrix} -\int(\partial_{x_1} B_k)^2\,dx & -\int \partial_{x_1} B_k\,\partial_{x_2} B_k\,dx \\ -\int \partial_{x_1} B_k\,\partial_{x_2} B_k\,dx & -\int(\partial_{x_2} B_k)^2\,dx \end{pmatrix},$$

whose determinant is

$$\det(DF_t) = \int(\partial_{x_1} B_k)^2\,dx \int(\partial_{x_2} B_k)^2\,dx - \Big(\int \partial_{x_1} B_k\,\partial_{x_2} B_k\,dx\Big)^2.$$

By the Cauchy–Schwarz inequality and the fact that $\partial_{x_1} B_k$ and $\partial_{x_2} B_k$ are linearly independent as functions of the x variable, for any time t fixed, we conclude that $\det(DF_t)$

is positive. Since each member of its expression is periodic in time, $\det(DF_t)$ is bounded below by a positive constant independent of time and the translation parameters of B_k.

In the case when $P_{j-1} = R_l$ is a soliton, denoting $y_{0,l} := x - x^0_{0,l} + y_2 - c_l t$, let us recall that

$$\partial_{y_1} Q_{c_l+y_1}(y_{0,l}) = \frac{1}{2c_l}\big(Q_{c_l+y_1}(y_{0,l}) + y_{0,l}\partial_x Q_{c_l+y_1}(y_{0,l})\big).$$

Thus, denoting $Q_{c_l}(x - x^0_{0,l} - c_l t)$ by Q_{c_l} and $x - x^0_{0,l} - c_l t$ by $y^0_{0,l}$,

$$DF_t = \begin{pmatrix} -\frac{1}{2c_l}\int Q_{c_l}(Q_{c_l} + y^0_{0,l}\partial_x Q_{c_l})\,dx & -\int Q_{c_l}\partial_x Q_{c_l}\,dx \\ -\frac{1}{2c_l}\int \partial_x Q_{c_l}(Q_{c_l} + y^0_{0,l}\partial_x Q_{c_l})\,dx & -\int(\partial_x Q_{c_l})^2\,dx \end{pmatrix},$$

whose determinant is

$$\det(DF_t) = \frac{1}{2c_l}\int Q_{c_l}(Q_{c_l} + y^0_{0,l}\partial_x Q_{c_l})\,dx \int(\partial_x Q_{c_l})^2\,dx,$$

because $\int Q_{c_l}\partial_x Q_{c_l}\,dx = 0$. And, from the computations made to obtain (2.11), we have

$$\det(DF_t) = \frac{1}{4}c_l \int q^2 \int q_x^2,$$

where q denotes the soliton with $c = 1$, i.e., $q = Q_1$.

This means that $\det(DF_t)$ is bounded below by a positive constant independent of time and the translation parameters of R_l. Thus, in any case, DF_t is invertible.

Now, we may use the implicit function theorem. If U is close enough to $P(t)$, then there exists (y_1, y_2) such that $F_t(U, y_1, y_2) = 0$, where (y_1, y_2) depends on a regular C^1 way on U. It is possible to show that the "close enough" in the previous sentence does not depend on t; for this, it is required to use a uniform implicit function theorem. This means that for T_2 large enough, $\|v(t)\|_{H^2}$ is small enough for $t \in [T_2, +\infty)$, thus for $t \geq T_2$, $u(t)$ is close enough to $P(t)$ in order to apply the implicit function theorem. Therefore, we have, for $t \in [T_2, +\infty)$, the existence of $y_1(t)$ and $y_2(t)$. It is possible to show that these functions are C^1 in time. Basically, this comes from the fact that they are C^1 in $u(t)$ and that $u(t)$ has a similar regularity in time (see [13] for more details).

Now, we prove inequalities (4.20) and (4.21). We can take the differential of the implicit functions with respect to $u(t)$ for $t \in [T_2, +\infty)$. For this, we differentiate the following equation with respect to $u(t)$:

$$F_t(u(t), y_1(u(t)), y_2(u(t))) = 0.$$

We know that the matrix that gives the differential of F_t (with respect to y_1, y_2) in

$$(u(t), y_1(u(t)), y_2(u(t)))$$

is invertible and that its inverse is bounded in time. The differential of F_t with respect to the first variable is also bounded (from its expression, F_t is linear in U). Thus, by the mean value theorem (given $(y_1, y_2)(P(t)) = (0, 0)$),

$$|y_1(u(t))| + |y_2(u(t))| \leq C\|u(t) - P(t)\| \leq C\|v(t)\|_{H^2}.$$

By applying the mean value theorem (inequality) for Q_{c_l} or B_{α_k,β_k} with respect to y_1 and y_2, we deduce that

$$\|P_{j-1}(t) - \widetilde{P}_{j-1}(t)\|_{H^2} \leq C(|y_1(t)| + |y_2(t)|).$$

Finally, by the triangular inequality,

$$\begin{aligned}\|w(t)\|_{H^2} &\leq \|u(t) - P(t)\|_{H^2} + \|P(t) - \widetilde{P}(t)\|_{H^2}\\ &\leq \|u(t) - P(t)\|_{H^2} + C(|y_1(t)| + |y_2(t)|)\\ &\leq C\|v(t)\|_{H^2}.\end{aligned}$$

This completes the proof of (4.20).

For (4.21), we will take time derivatives of equations (4.18) and (4.19). Firstly, we may write the PDE satisfied by w:

$$\partial_t w = -w_{xxx} - \Big[w\Big(w^2 + 3w\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{i,m=1}^{J}\widetilde{P}_i\widetilde{P}_m\Big)\Big]_x - \sum_{h\neq i \text{ or } i\neq m}(\widetilde{P}_h\widetilde{P}_i\widetilde{P}_m)_x - E,$$

where, if $P_{j-1} = B_k$ is a breather,

$$E := y_1'(t)\widetilde{B}_{k1} + y_2'(t)\widetilde{B}_{k2},$$

and if $P_{j-1} = R_l$ is a soliton, denoting $y_{0,l}(t) := x - x_{0,l}^0 + y_2(t) - c_l t$,

$$E := \frac{y_1'(t)}{2(c_l + y_1(t))}(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{lx}) + y_2'(t)\widetilde{R}_{lx}.$$

If $P_{j-1} = B_k$, we start by taking the time derivative of $\int \widetilde{B}_{k1}w = 0$ and perform some integrations by parts to obtain

$$\begin{aligned}&-\int(\widetilde{B}_k^3)_{1x}w + y_1'(t)\int\widetilde{B}_{k11}w + y_2'(t)\int\widetilde{B}_{k12}w\\ &\quad+\int\widetilde{B}_{k1x}w\Big(w^2 + 3w\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P}_h\widetilde{P}_i\Big) - \int\widetilde{B}_{k1}\sum_{h\neq i \text{ or } g\neq h}(\widetilde{P}_h\widetilde{P}_i\widetilde{P}_g)_x\\ &\quad= y_1'(t)\int\widetilde{B}_{k1}^2 + y_2'(t)\int\widetilde{B}_{k1}\widetilde{B}_{k2},\end{aligned}$$

then, we take the time derivative of $\int \widetilde{B}_{k2}w = 0$:

$$\begin{aligned}&-\int(\widetilde{B}_k^3)_{2x}w + y_1'(t)\int\widetilde{B}_{k12}w + y_2'(t)\int\widetilde{B}_{k22}w\\ &\quad+\int\widetilde{B}_{k2x}w\Big(w^2 + 3w\sum_{i=1}^{J}\widetilde{P}_i + 3\sum_{h,i=1}^{J}\widetilde{P}_h\widetilde{P}_i\Big) - \int\widetilde{B}_{k2}\sum_{h\neq i \text{ or } g\neq h}(\widetilde{P}_h\widetilde{P}_i\widetilde{P}_g)_x\\ &\quad= y_1'(t)\int\widetilde{B}_{k1}\widetilde{B}_{k2} + y_2'(t)\int\widetilde{B}_{k2}^2.\end{aligned}$$

If $P_{j-1} = R_l$, we start by taking the time derivative of $\int \widetilde{R}_l w = 0$ and perform some integrations by parts to obtain

$$-\int (\widetilde{R}_l^3)_x w + \frac{y_1'(t)}{2c_l} \int (\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{l\,x})w + y_2'(t) \int \widetilde{R}_{l\,x} w$$
$$+ \int \widetilde{R}_{l\,x} w\Big(w^2 + 3w \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i\Big) - \int \widetilde{R}_l \sum_{h\neq i \text{ or } g\neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x$$
$$= \frac{y_1'(t)}{2(c_l + y_1(t))} \int \widetilde{R}_l(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{l\,x}) + y_2'(t) \int \widetilde{R}_l \widetilde{R}_{l\,x},$$

then, we take the time derivative of $\int \widetilde{R}_{l\,x} w = 0$:

$$-\int (\widetilde{R}_l^3)_{xx} w + \frac{y_1'(t)}{2c_l} \int (\widetilde{R}_{l\,x} + y_{0,l}(t)\widetilde{R}_{l\,xx})w + y_2'(t) \int \widetilde{R}_{l\,xx} w$$
$$+ \int \widetilde{R}_{l\,xx} w\Big(w^2 + 3w \sum_{i=1}^{J} \widetilde{P}_i + 3 \sum_{h,i=1}^{J} \widetilde{P}_h \widetilde{P}_i\Big) - \int \widetilde{R}_{l\,x} \sum_{h\neq i \text{ or } g\neq h} (\widetilde{P}_h \widetilde{P}_i \widetilde{P}_g)_x$$
$$= \frac{y_1'(t)}{2(c_l + y_1(t))} \int \widetilde{R}_{l\,x}(\widetilde{R}_l + y_{0,l}(t)\widetilde{R}_{l\,x}) + y_2'(t) \int (\widetilde{R}_{l\,x})^2.$$

As a consequence of (4.20), we see that $|y_1(t)| + |y_2(t)|$ tends to 0 when $t \to +\infty$. This is why we may use Proposition 2.10 and Corollary 2.11 here, if T_2 is large enough. So, several terms of the four equalities above are obviously bounded by $(w(t)^2\Phi_j)^{1/2}$ or $e^{-\varpi t}$ for $\varpi > 0$, a constant chosen small enough. Using these bounds, and after several linear combinations, we obtain (4.21). ∎

Step 3. *Quadratic approximations of localized conservation laws.*

Lemma 4.14. *Let $\omega > 0$ be as small as we want. There exist $C > 0$, $T_3 \geq T$ such that the following holds for $t \geq T_3$:*

$$\Big|M_j(t) - \sum_{i=1}^{j-1} M[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int \widetilde{P}_i w - \frac{1}{2} \int w^2 \Phi_j\Big| \leq Ce^{-2\varpi t},$$
$$\Big|E_j(t) - \sum_{i=1}^{j-1} E[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int [\widetilde{P}_{i\,x} w_x - \widetilde{P}_i^3 w] - \int \Big[\frac{1}{2} w_x^2 - \frac{3}{2} \widetilde{P}^2 w^2\Big] \Phi_j\Big|$$
$$\leq Ce^{-2\varpi t} + \omega \int w^2 \Phi_j,$$
$$\Big|F_j(t) - \sum_{i=1}^{j-1} F[\widetilde{P}_i] - \sum_{i=1}^{j-1} \int \Big[\widetilde{P}_{i\,xx} w_{xx} - 5\widetilde{P}_i \widetilde{P}_{i\,x}^2 w - 5\widetilde{P}_i^2 \widetilde{P}_{i\,x} w_x + \frac{3}{2} \widetilde{P}_i^5 w\Big]$$
$$- \int \Big[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \widetilde{P}_x^2 - 10\widetilde{P} w \widetilde{P}_x w_x - \frac{5}{2} \widetilde{P}^2 w_x^2 + \frac{15}{4} \widetilde{P}^4 w^2\Big] \Phi_j(t)\Big|$$
$$\leq Ce^{-2\varpi t} + \omega \int (w^2 + w_x^2)\Phi_j.$$

Proof. For the mass, we compute

$$M_j(t) = \frac{1}{2}\int (\widetilde{P} + w)^2\,\Phi_j = \frac{1}{2}\int \widetilde{P}^2\,\Phi_j + \int \widetilde{P} w\,\Phi_j + \frac{1}{2}\int w^2\,\Phi_j.$$

As in step 1, we can show that $\frac{1}{2}\int \widetilde{P}^2\Phi_j$ converges exponentially (we choose ϖ with respect to this exponential convergence) to $\sum_{i=1}^{j-1} M[\widetilde{P}_i]$. Similarly, the difference between $\int \widetilde{P} w \Phi_j$ and $\sum_{i=1}^{j-1} \widetilde{P}_i w$ converges exponentially to 0 (the velocity of a soliton is not modified a lot by modulation, this is why it works in any cases).

For E and F, we perform similar basic computations with the only difference being that there will also be terms of degree 3 or more in w. We know that $\|w(t)\|_{H^2} \to 0$ as $t \to +\infty$, this is the reason why for t large enough, such terms are boundable by $\omega \int w^2\,\Phi_j$ or $\omega \int w_x^2\,\Phi_j$. ∎

Step 4. *Approximation of the Lyapunov functional.* By analogy with the existence part, we introduce the following Lyapunov functional:

$$\mathcal{H}_j(t) := F_j(t) + 2(b_{j-1}^2 - a_{j-1}^2)E_j(t) + (a_{j-1}^2 + b_{j-1}^2)^2 M_j(t).$$

We will use the previous steps to approximate $\mathcal{H}_j(t)$.

Lemma 4.15. *There exists $T_4 \geq T$ such that the following holds for $t \geq T_4$:*

$$\begin{aligned}\mathcal{H}_j(t) = &\sum_{i=1}^{j-1} F[\widetilde{P}_i] + 2(b_{j-1}^2 - a_{j-1}^2)\sum_{i=1}^{j-1} E[\widetilde{P}_i] + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} M[\widetilde{P}_i] \\ &+ H_j(t) + O(e^{-2\varpi t}) + o\Big(\int (w^2 + w_x^2)\Phi_j\Big),\end{aligned}$$

where

$$\begin{aligned}H_j(t) := &\int \Big[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w_x^2 \widetilde{P}_{j-1}^2 + \frac{5}{2} w^2 \widetilde{P}_{j-1x}^2 + 5w^2 \widetilde{P}_{j-1}\widetilde{P}_{j-1xx} + \frac{15}{4} w^2 \widetilde{P}_{j-1}^4\Big]\Phi_j(t) \\ &+ (b_{j-1}^2 - a_{j-1}^2)\int [w_x^2 - 3w^2\widetilde{P}_{j-1}^2]\Phi_j(t) + \frac{1}{2}\big(a_{j-1}^2 + b_{j-1}^2\big)^2 \int w^2\,\Phi_j(t).\end{aligned}$$

Proof. This lemma is obtained from the summation of the facts established in the previous lemma. We get rid of the linear terms in the following way, by integrations by parts:

$$\begin{aligned}&\sum_{i=1}^{j-1}\int \Big(\widetilde{P}_{ixx} w_{xx} - 5\widetilde{P}_i \widetilde{P}_{ix}^2 w - 5\widetilde{P}_i^2 \widetilde{P}_{ix} w_x + \frac{3}{2}\widetilde{P}_i^5 w\Big) \\ &\quad + 2(b_{j-1}^2 - a_{j-1}^2)\sum_{i=1}^{j-1}\int (\widetilde{P}_{ix} w_x - \widetilde{P}_i^3 w) + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \widetilde{P}_i w \\ &= \sum_{i=1}^{j-1}\int \Big(\widetilde{P}_{ixxxx} + 5\widetilde{P}_i \widetilde{P}_{ix}^2 + 5\int \widetilde{P}_i^2 \widetilde{P}_{ixx} + \frac{3}{2}\widetilde{P}_i^5\Big) w \\ &\quad + 2(b_{j-1}^2 - a_{j-1}^2)\sum_{i=1}^{j-1}\int (-\widetilde{P}_{ixx} - \widetilde{P}_i^3) w + (a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1}\int \widetilde{P}_i w.\end{aligned}$$

If we consider that this sum goes from $i = 1$ to $j-2$, we see that for $1 \le i \le j-2$, this sum is exponentially bounded by the induction assumption (we use that for $i \le j-2$, a polynomial in $\widetilde{P}_i$ and its derivatives are bounded by $C\Phi_{j-1}$, and $w = v + (P_{j-1} - \widetilde{P}_{j-1})$). It is left to consider the sum of the terms with $i = j-1$.

For $i = j-1$, we have nearly the elliptic equation satisfied by $\widetilde{P}_{j-1}$. It is actually exactly this equation in the case when $\widetilde{P}_{j-1}$ is a breather. When $\widetilde{P}_{j-1}$ is a soliton, its shape parameter is modified by modulation. This is why, in this case, the sum of the terms with $i = j-1$ is equal to

$$2y_1(t)\int(-\widetilde{P}_{j-1xx} - \widetilde{P}_{j-1}^3)w + 2b_{j-1}^2 y_1(t)\int \widetilde{P}_{j-1}w + y_1(t)^2\int \widetilde{P}_{j-1}w,$$

which vanishes because of the orthogonality condition from the modulation (Lemma 4.13) and the elliptic equation (1.2) satisfied by a soliton.

The term H_j is obtained as the sum of the quadratic parts of the previous lemma on which we have performed some integrations by parts, and some simplifications based on the fact that for $i \ge j$, $\widetilde{P}_i\Phi_j(t)$ is exponentially decreasing, and the fact that for $i \le j-2$, $\int \widetilde{P}_i w^2$ is exponentially decreasing by the induction assumption (4.16). Therefore, H_j corresponds to the sum of the quadratic parts of the previous lemma to which we have to add $5\int w^2\widetilde{P}\widetilde{P}_x\Phi_{jx}$, which is bounded exponentially. ■

Step 5. *Bound from above for $H_j(t)$.* Because $v_1 > 0$, we have that $b_{j-1}^2 - a_{j-1}^2 \ge 0$. By taking ω_2 and ω_6 small enough (with respect to $(a_{j-1}^2 + b_{j-1}^2)^2$), we obtain, by the facts of Lemma 4.11, the following inequality:

$$\mathcal{H}_j(t) - \sum_{i=1}^{j-1} F[P_i] - 2(b_{j-1}^2 - a_{j-1}^2)\sum_{i=1}^{j-1} E[P_i] - (a_{j-1}^2 + b_{j-1}^2)^2\sum_{i=1}^{j-1} M[P_i] \le Ce^{-2\varpi t}.$$

From Lemma 4.15, for $t \ge T_3$,

$$\begin{aligned} H_j(t) \le\ & F[P_{j-1}] - F[\widetilde{P}_{j-1}] + 2(b_{j-1}^2 - a_{j-1}^2)(E[P_{j-1}] - E[\widetilde{P}_{j-1}]) \\ & + (a_{j-1}^2 + b_{j-1}^2)^2(M[P_{j-1}] - M[\widetilde{P}_{j-1}]) + Ce^{-2\varpi t} + \omega\int(w^2 + w_x^2)\Phi_j. \end{aligned}$$

If P_{j-1} is a breather, we obtain immediately that

$$H_j(t) \le Ce^{-2\varpi t} + \omega\int(w^2 + w_x^2)\Phi_j.$$

The case when P_{j-1} is a soliton needs more inspection. As in the existence part, we have the following relations:

$$\begin{aligned} M[\widetilde{P}_{j-1}](t) &= (b_{j-1}^2 + y_1(t))^{1/2}M[q], \\ E[\widetilde{P}_{j-1}](t) &= (b_{j-1}^2 + y_1(t))^{3/2}E[q], \\ F[\widetilde{P}_{j-1}](t) &= (b_{j-1}^2 + y_1(t))^{5/2}F[q]. \end{aligned}$$

We set

$$\mathcal{R}_{j-1}(t) := F[\widetilde{P}_{j-1}](t) + 2b_{j-1}^2 E[\widetilde{P}_{j-1}](t) + b_{j-1}^4 M[\widetilde{P}_{j-1}](t),$$

and we simplify it as follows:

$$\mathcal{R}_{j-1}(t) = b_{j-1}^5\Big(1 + \frac{y_1(t)}{b_{j-1}^2}\Big)^{5/2} F[q] + 2b_{j-1}^5\Big(1 + \frac{y_1(t)}{b_{j-1}^2}\Big)^{3/2} E[q] + b_{j-1}^5\Big(1 + \frac{y_1(t)}{b_{j-1}^2}\Big)^{1/2} M[q].$$

After making a Taylor expansion as in Section 2.5,

$$\mathcal{R}_{j-1}(t) - F[P_{j-1}] - 2b_{j-1}^2 E[P_{j-1}] - b_{j-1}^4 M[P_{j-1}] = O(y_1(t)^3). \tag{4.22}$$

Therefore, if T_4 is large enough, $\|v(t)\|_{H^2}$ can be as small as we want, and for $t \geq T_4$, if P_{j-1} a soliton, we may write

$$H_j(t) \leq Ce^{-2\varpi t} + \omega \int (w^2 + w_x^2)\Phi_j + \omega y_1(t)^2.$$

Step 6. *Coercivity.* The term H_j can be seen as the quadratic form associated to $\widetilde{P}_{j-1}$ and evaluated in $w\sqrt{\Phi_j}$, modulo several terms that can be bounded by $C\sqrt{\sigma}\int(w^2 + w_x^2 + w_{xx}^2)\Phi_j$ (because these terms depend on derivatives of Φ_j). Let us prove that we can apply Appendix A.4 for $w\sqrt{\Phi_j}$.

More precisely, we need to prove that, for $\nu > 0$ small enough (from Appendix A.4),

$$\Big|\int w\sqrt{\Phi_j}\,\widetilde{P}_{j-11}\Big| + \Big|\int w\sqrt{\Phi_j}\,\widetilde{P}_{j-12}\Big| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}$$

if P_{j-1} is a breather, or that

$$\Big|\int w\sqrt{\Phi_j}\,\widetilde{P}_{j-1}\Big| + \Big|\int w\sqrt{\Phi_j}\,\widetilde{P}_{j-1x}\Big| \leq \nu\|w\sqrt{\Phi_j}\|_{H^2}$$

if P_{j-1} is a soliton. In any case, the proof is the same and let us write K at the place of $\widetilde{P}_{j-11}$, $\widetilde{P}_{j-12}$, $\widetilde{P}_{j-1}$ or $\widetilde{P}_{j-1x}$. This means that we want to bound $\int w\sqrt{\Phi_j}K$.

From (4.18) and (4.19), we can see that it is enough to bound $\int w(1-\sqrt{\Phi_j})K$ by $\nu\|w\sqrt{\Phi_j}\|_{H^2}$. The reasoning that follows works for $j \leq J$; for $j = J+1$, the result is immediate because $\Phi_{J+1} = 1$. Since Φ_j is a translate of Ψ, using the fact that

$$\sqrt{1+v} = 1 + O(v) \quad \text{as } v \to 0,$$

we have

$$1 - \sqrt{\Psi} = 1 - \sqrt{1 + \Psi - 1} = 1 - \sqrt{1 - \Psi(-x)} = O(\Psi(-x)),$$

which means that

$$1 - \sqrt{\Phi_j} \leq C\min\Big(1, \exp\Big(\frac{\sqrt{\sigma}(x - m_j t)}{2}\Big)\Big).$$

We may deduce now that

$$\Big|\int w(1-\sqrt{\Phi_j})K\Big| = \Big|\int w\sqrt{\Phi_j}\,\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}\,K\Big|$$
$$\leq \Big\|\frac{1-\sqrt{\Phi_j}}{\sqrt{\Phi_j}}\,K\Big\|_{L^2}\|w\sqrt{\Phi_j}\|_{L^2} \leq Ce^{\sqrt{\sigma}(m_j-v_{j-1})t}\|w\sqrt{\Phi_j}\|_{L^2}$$

if $\sqrt{\sigma}/4 < \beta/2$. And so, if t is large enough, we get the bound we want.

Thus, there exists $\mu > 0$ such that for $t \geq T_5$ (where T_5 is large enough and depends on σ),

$$\begin{aligned}\mu\|w\sqrt{\Phi_j}\|_{H^2}^2 &\leq H_j(t) + C\sqrt{\sigma}\int(w^2+w_x^2+w_{xx}^2)\Phi_j + \frac{1}{\mu}\Big(\int \widetilde{P}_{j-1}w\sqrt{\Phi_j}\Big)^2 \\ &\leq Ce^{-2\varpi t} + \omega\int(w^2+w_x^2)\Phi_j + C\sqrt{\sigma}\int(w^2+w_x^2+w_{xx}^2)\Phi_j \\ &\quad + \omega y_1(t)^2 + \frac{1}{\mu}\Big(\int \widetilde{P}_{j-1}w\sqrt{\Phi_j}\Big)^2,\end{aligned}$$

where the term $\frac{1}{\mu}(\int \widetilde{P}_{j-1}w\sqrt{\Phi_j})^2$ is present only if $\widetilde{P}_{j-1}$ is a breather and the term $\omega y_1(t)^2$ is present only if $\widetilde{P}_{j-1}$ is a soliton.

For σ and ω small enough, we deduce that

$$\int(w^2+w_x^2+w_{xx}^2)\Phi_j \leq Ce^{-2\varpi t} + \omega y_1(t)^2 + C\Big(\int \widetilde{P}_{j-1}w\sqrt{\Phi_j}\Big)^2. \tag{4.23}$$

We set $T_0 := \max(T_1, T_2, T_3, T_4, T_5)$.

Step 7. *Bound for* $|\int \widetilde{P}_{j-1}w\sqrt{\Phi_j}|$ (*in case* $\widetilde{P}_{j-1}$ *is a breather*). We would like to prove that $\int \widetilde{P}_{j-1}w\sqrt{\Phi_j}$ is exponentially decreasing. To do so, we would like to get rid of $\sqrt{\Phi_j}$. It is clear that $\int \widetilde{P}_{j-1}w(1-\sqrt{\Phi_j})$ is exponentially decreasing. Thus, it is enough to prove that $\int \widetilde{P}_{j-1}w$ is exponentially decreasing.

If $i \leq j-2$, we know that $\int \widetilde{P}_i w$ is exponentially decreasing by the induction assumption (4.16). Thus, it is enough to prove that $\sum_{i=1}^{j-1}\int \widetilde{P}_i w$ is exponentially decreasing.

From the mass approximation of Lemma 4.14 and Lemma 4.11, we have, for $t \geq T_0$,

$$\begin{aligned}\sum_{i=1}^{j-1}\int \widetilde{P}_i w &= O(e^{-2\varpi t}) + M_j(t) - \sum_{i=1}^{j-1} M[P_i] - \frac{1}{2}\int w^2\Phi_j \\ &\leq Ce^{-2\varpi t} - \frac{1}{2}\int w^2\Phi_j \leq Ce^{-2\varpi t}.\end{aligned}$$

Now, we use the fact that the sum of the linear parts of our localized conservation laws is exponentially decreasing, which we have established in the proof of Lemma 4.15. Therefore, the linear terms of $F_j + 2(b_{j-1}^2 - a_{j-1}^2)E_j$ are equal to

$$O(e^{-2\varpi t}) - (a_{j-1}^2 + b_{j-1}^2)^2\sum_{i=1}^{j-1}\int \widetilde{P}_i w.$$

Now, from the energy and F approximations of Lemma 4.14 and Lemma 4.11, and from (4.22), we observe that (we recall that $b_{j-1}^2 - a_{j-1}^2 \geq 0$), for $t \geq T_0$,

$$
\begin{aligned}
&-(a_{j-1}^2 + b_{j-1}^2)^2 \sum_{i=1}^{j-1} \int \tilde{P}_i w \\
&= O(e^{-2\varpi t}) + o\Big(\int (w^2 + w_x^2)\Phi_j\Big) \\
&\quad + F_j(t) + 2(b_{j-1}^2 - a_{j-1}^2)E_j(t) - \sum_{i=1}^{j-1} F[P_i] - 2(b_{j-1}^2 - a_{j-1}^2) \sum_{i=1}^{j-1} E[P_i] \\
&\quad - \int \Big[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \tilde{P}_x^2 - 10 \tilde{P} w \tilde{P}_x w_x - \frac{5}{2} \tilde{P}^2 w_x^2 + \frac{15}{4} \tilde{P}^4 w^2\Big]\Phi_j \\
&\quad - 2(b_{j-1}^2 - a_{j-1}^2) \int \Big[\frac{1}{2} w_x^2 - \frac{3}{2} \tilde{P}^2 w^2\Big]\Phi_j + o(y_1(t)^2) \\
&= O(e^{-2\varpi t}) + o\Big(\int (w^2 + w_x^2)\Phi_j\Big) \\
&\quad + F_j(t) + \omega_6 M_j(t) - \sum_{i=1}^{j-1} F[P_i] - \omega_6 \sum_{i=1}^{j-1} M[P_i] \\
&\quad + 2(b_{j-1}^2 - a_{j-1}^2)\Big[E_j(t) + \omega_2 M_j(t) - \sum_{i=1}^{j-1} E[P_i] - \omega_2 \sum_{i=1}^{j-1} M[P_i]\Big] \\
&\quad + \big(\omega_6 + 2\omega_2(b_{j-1}^2 - a_{j-1}^2)\big)\Big(\sum_{i=1}^{j-1} M[P_i] - M_j(t)\Big) \\
&\quad - \int \Big[\frac{1}{2} w_{xx}^2 - \frac{5}{2} w^2 \tilde{P}_x^2 - 10 \tilde{P} w \tilde{P}_x w_x - \frac{5}{2} \tilde{P}^2 w_x^2 + \frac{15}{4} \tilde{P}^4 w^2\Big]\Phi_j \\
&\quad - 2(b_{j-1}^2 - a_{j-1}^2) \int \Big[\frac{1}{2} w_x^2 - \frac{3}{2} \tilde{P}^2 w^2\Big]\Phi_j + o(y_1(t)^2) \\
&\leq C e^{-2\varpi t} + C \int (w^2 + w_x^2)\Phi_j + o(y_1(t)^2) \\
&\quad - \big(\omega_6 + 2\omega_2(b_{j-1}^2 - a_{j-1}^2)\big)\Big(\sum_{i=1}^{j-1} \int \tilde{P}_i w + \frac{1}{2} \int w^2 \Phi_j\Big),
\end{aligned}
$$

where the term $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton. And therefore, for ω_2 and ω_6 small enough,

$$
-\sum_{i=1}^{j-1} \int \tilde{P}_i w \leq C e^{-2\varpi t} + C \int (w^2 + w_x^2)\Phi_j + o(y_1(t)^2).
$$

Thus, we deduce the following bound:

$$
\Big| \int \tilde{P}_{j-1} w \sqrt{\Phi_j} \Big| \leq C e^{-2\varpi t} + C \int (w^2 + w_x^2)\Phi_j + o(y_1(t)^2).
$$

Because $\|w(t)\|_{H^2} \to 0$ as $t \to +\infty$, we deduce that

$$\left(\int \widetilde{P}_{j-1} w \sqrt{\Phi_j}\right)^2 = o(e^{-2\varpi t}) + o\Big(\int (w^2 + w_x^2)\,\Phi_j\Big) + o(y_1(t)^2). \tag{4.24}$$

Step 8. *Conclusion.* From (4.23) and (4.24), we deduce, for $t \geq T_0$, that

$$\int (w^2 + w_x^2 + w_{xx}^2)\,\Phi_j = O(e^{-2\varpi t}) + o(y_1(t)^2) + o\Big(\int (w^2 + w_x^2)\,\Phi_j\Big).$$

This means that if we take T_0 large enough, we have

$$\int (w^2 + w_x^2 + w_{xx}^2)\,\Phi_j = o(y_1(t)^2) + O(e^{-2\varpi t}), \tag{4.25}$$

where the term $o(y_1(t)^2)$ is present only if P_{j-1} is a soliton.

Before completing the proof, we need to find a better bound for $y_1(t)$ than just a convergence to 0 given by the modulation (in the case when P_{j-1} is a soliton). For this, we study $M_j(t)$:

$$\begin{aligned} M_j(t) &= \frac{1}{2}\int u^2(t)\,\Phi_j(t) = \frac{1}{2}\int (\widetilde{P}(t) + w(t))^2\,\Phi_j(t) \\ &= \frac{1}{2}\int \widetilde{P}(t)^2\,\Phi_j(t) + \int \widetilde{P}(t)\,w(t)\,\Phi_j(t) + \frac{1}{2}\int w(t)^2\,\Phi_j(t) \\ &= \frac{1}{2}\sum_{i=1}^{j-1}\int \widetilde{P}_i(t)^2 + \sum_{i=1}^{j-1}\int \widetilde{P}_i(t)\,w(t) + O(e^{-2\varpi t}) + \frac{1}{2}\int w(t)^2\,\Phi_j(t) \\ &= \frac{1}{2}\int \widetilde{P}_{j-1}(t)^2 + \int \widetilde{P}_{j-1}(t)\,w(t) + O(e^{-2\varpi t}) \\ &\quad + \frac{1}{2}\int w(t)^2\,\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2, \end{aligned}$$

by the induction assumption (4.16). Then

$$M_j(t) = \frac{1}{2}\int \widetilde{P}_{j-1}(t)^2 + O(e^{-2\varpi t}) + \frac{1}{2}\int w(t)^2\,\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2,$$

by the orthogonality condition from the modulation (Lemma 4.13). Therefore,

$$M_j(t) = (b_{j-1}^2 + y_1(t))^{1/2} M[q] + O(e^{-2\varpi t}) + \frac{1}{2}\int w(t)^2\,\Phi_j(t) + \frac{1}{2}\sum_{i=1}^{j-2}\int P_i(t)^2.$$

Now, if we take $t_1 \geq t$, we obtain from (4.25) that

$$\begin{aligned} M_j(t_1) - M_j(t) = {} & \big[(b_{j-1}^2 + y_1(t_1))^{1/2} - (b_{j-1}^2 + y_1(t))^{1/2}\big] M[q] \\ & + O(e^{-2\varpi t}) + o(y_1(t)^2) + o(y_1(t_1)^2). \end{aligned} \tag{4.26}$$

By doing a Taylor expansion of order 1, as in the existence part, we obtain

$$(b_{j-1}^2 + y_1(t_1))^{1/2} = b_{j-1}\Big(1 + \frac{1}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O(y_1(t_1)^2)\Big).$$

Therefore,

$$\begin{aligned}(b_{j-1}^2 &+ y_1(t_1))^{1/2} - (b_{j-1}^2 + y_1(t))^{1/2} \\ &= \frac{1}{2b_{j-1}}(y_1(t_1) - y_1(t)) + O(y_1(t_1)^2) + O(y_1(t)^2).\end{aligned}$$

Now, we recall that when $t_1 \to +\infty$, we have $y_1(t_1) \to 0$. Therefore, by taking the limit of the previous formula when $t_1 \to +\infty$, we obtain

$$b_{j-1} - (b_{j-1}^2 + y_1(t))^{1/2} = -\frac{y_1(t)}{2b_{j-1}} + O(y_1(t)^2).$$

Therefore, from (4.26), with $t_1 \to +\infty$,

$$\sum_{i=1}^{j-1} M[P_i] - M_j(t) = -\frac{y_1(t)}{2b_{j-1}} M[q] + O(e^{-2\varpi t}) + O(y_1(t)^2). \tag{4.27}$$

The second step is to study $E_j(t)$ (we do the same reasoning as for M_j):

$$\begin{aligned}E_j(t) &= \int \Big[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\Big]\Phi_j(t) \\ &= \int \Big[\frac{1}{2}\widetilde{P}_x^2 - \frac{1}{4}\widetilde{P}^4\Big]\Phi_j(t) + \int [\widetilde{P}_x w_x - \widetilde{P}^3 w]\,\Phi_j(t) + O\Big(\int w^2\,\Phi_j(t)\Big),\end{aligned}$$

and after simplifications by Φ_j due to exponential convergences, the induction assumption (4.16) and orthogonality conditions (Lemma 4.13),

$$\begin{aligned}E_j(t) &= E[\widetilde{P}_{j-1}(t)] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + O\Big(\int w^2\,\Phi_j(t)\Big) \\ &= (b_{j-1}^2 + y_1(t))^{3/2} E[q] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + O\Big(\int w^2\,\Phi_j(t)\Big) \\ &= (b_{j-1}^2 + y_1(t))^{3/2} E[q] + \sum_{i=1}^{j-2} E[P_i] + O(e^{-2\varpi t}) + o(y_1(t)^2),\end{aligned}$$

by (4.25). And then, by taking the difference for $t_1 \geq t$,

$$\begin{aligned}E_j(t_1) - E_j(t) = \big[&(b_{j-1}^2 + y_1(t_1))^{3/2} - (b_{j-1}^2 + y_1(t))^{3/2}\big]E[q] \\ &+ O(e^{-2\varpi t}) + o(y_1(t_1)^2) + o(y_1(t)^2).\end{aligned}$$

By taking a Taylor expansion of order 1, we obtain

$$(b_{j-1}^2 + y_1(t_1))^{3/2} = b_{j-1}^3\Big(1 + \frac{3}{2}\frac{y_1(t_1)}{b_{j-1}^2} + O(y_1(t_1)^2)\Big).$$

Therefore, after taking $t_1 \to +\infty$, we obtain

$$\sum_{i=1}^{j-1} E[P_i] - E_j(t) = -\frac{3}{2} b_{j-1} y_1(t) E[q] + O(e^{-2\varpi t}) + O(y_1(t)^2). \tag{4.28}$$

This is why, from (4.27), (4.28) and Lemma 4.11, we obtain

$$-\frac{y_1(t)}{2b_{j-1}} M[q] + O(e^{-2\varpi t}) + O(y_1(t)^2) \geq -Ce^{-2\varpi t}, \tag{4.29}$$

and

$$-\frac{3}{2} b_{j-1} y_1(t) E[q] + O(e^{-2\varpi t}) + O(y_1(t)^2) \geq -Ce^{-2\varpi t}. \tag{4.30}$$

Because $M[q] = 2$ and $E[q] = -2/3$, we rewrite the previous inequalities (4.29) and (4.30) in the following way (and we pass $O(e^{-2\varpi t})$ on the other side of each inequality):

$$-\frac{y_1(t)}{b_{j-1}} + O(y_1(t)^2) \geq -Ce^{-2\varpi t}, \tag{4.31}$$

and

$$b_{j-1} y_1(t) + O(y_1(t)^2) \geq -Ce^{-2\varpi t}. \tag{4.32}$$

Because $y_1(t) \to +\infty$, by taking T_0 larger if needed, $O(y_1(t)^2)$ can be bounded above by any positive constant multiplied by $|y_1(t)|$, so by taking this constant small enough (by taking T_0 large enough) and combining both previous inequalities (4.31) and (4.32), we obtain

$$|y_1(t)| \leq Ce^{-2\varpi t}.$$

Therefore, we have obtained a better bound for $y_1(t)$ in the case when P_{j-1} is a soliton. Thus, we may conclude that in any case, for $t \geq T_0$, and T_0 large enough,

$$\int (w^2 + w_x^2 + w_{xx}^2)\Phi_j(t) = O(e^{-2\varpi t}).$$

Then we deduce from (4.21) that

$$|y_1'(t)| + |y_2'(t)| = O(e^{-\varpi t}).$$

Because $|y_1(t)| + |y_2(t)| \to 0$ as $t \to +\infty$, we obtain by integration

$$|y_1(t)| + |y_2(t)| = O(e^{-\varpi t}).$$

And so, by the mean value theorem,

$$\|\widetilde{P}_{j-1} - P_{j-1}\|_{H^2} \le C(|y_1(t)| + |y_2(t)|) \le Ce^{-\varpi t}.$$

From $v = w + \widetilde{P}_{j-1} - P_{j-1}$, we deduce

$$\int (v^2 + v_x^2 + v_{xx}^2)\Phi_j \le C \int (w^2 + w_x^2 + w_{xx}^2)\Phi_j + C \int [(\widetilde{P}_{j-1} - P_{j-1})^2 + (\widetilde{P}_{j-1} - P_{j-1})_x^2 + (\widetilde{P}_{j-1} - P_{j-1})_{xx}^2]\Phi_j \le Ce^{-2\varpi t},$$

and this completes the induction. ∎

4.3. Proof of Theorem 1.4

Proof of Theorem 1.4. We suppose that $v_1 > 0$. Let p be the associated multi-breather given by Theorem 1.2. Let u be a solution of (1.1) such that

$$\|u(t) - p(t)\|_{H^2} \to 0 \quad \text{as } t \to +\infty.$$

From Proposition 4.10, we deduce that there exist constants $C > 0$ and $\varpi > 0$ such that for t large enough,

$$\|u(t) - p(t)\|_{H^2} \le Ce^{-\varpi t}.$$

This implies that u satisfies the assumptions of Proposition 1.5. Thus, $u = p$ and Theorem 1.4 is proved. ∎

A. Appendix

The first two subsections of the appendix show that a soliton has similar properties as a "limit breather" of parameter $\alpha = 0$. Firstly, the corresponding elliptic equation is satisfied by a soliton. Secondly, the corresponding quadratic form is coercive for a soliton, and we see that its kernel is spanned by $\partial_x Q$ and $\partial_c Q$. In the third subsection, we prove that it is possible for ϵ to be orthogonal to Q and $\partial_x Q$ (instead of $\partial_x Q$ and $\partial_c Q$) in order to satisfy a coercivity for the quadratic form. We will use this fact for the proof of the existence, as well as for the first part of the proof of the uniqueness. In the fourth subsection, we prove that we can have coercivity for quadratic forms when the orthogonality condition is not exactly satisfied. We will use this result for the proof of the uniqueness. The last subsection is about computations for the third conservation law. It will be useful for the monotonicity property for the localized F that we need in the proof of the uniqueness.

A.1. Elliptic equation satisfied by a soliton

Lemma A.1. *A soliton $Q = R_{c,\kappa}$ satisfies, for any time $t \in \mathbb{R}$, the following nonlinear elliptic equation*:

$$Q_{(4x)} - 2c(Q_{xx} + Q^3) + c^2 Q + 5QQ_x^2 + 5Q^2 Q_{xx} + \frac{3}{2}Q^5 = 0. \tag{A.1}$$

Proof. In order to derive this equation, we will use the equation that defines a soliton (and that is satisfied by Q at any time):

$$Q_{xx} = cQ - Q^3.$$

We will also need the following equation:

$$Q_x^2 = cQ^2 - \frac{1}{2}Q^4,$$

that can be derived by taking the space derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$, and by showing that this derivative is zero. From this, we deduce that $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is constant, and by taking its limit when $x \to \pm\infty$, we see that this constant is zero. More precisely, the derivative of $Q_x^2 - cQ^2 + \frac{1}{2}Q^4$ is

$$2Q_xQ_{xx} - 2cQQ_x + 2Q^3Q_x = 2Q_x(Q_{xx} - cQ + Q^3) = 0.$$

From now on, the derivation of (A.1) is straightforward. It is sufficient to take space derivatives of $Q_{xx} = cQ - Q^3$ and to inject them into the right-hand side of equation (A.1), which we want to prove that is equal to zero. By doing this, we make the maximal order of a derivative of Q present in the right-hand side of the equation lower. In the end, we have only zero and first order derivatives. To have only a polynomial in Q, we have to use $Q_x^2 = cQ^2 - \frac{1}{2}Q^4$, and the calculations show that this polynomial is zero. ∎

A.2. Study of coercivity of the quadratic form associated to a soliton

In this article, we adapt the argument for the breathers in [3] to the soliton case. We consider

$$\begin{aligned}\mathcal{Q}_c^s[\epsilon] &:= \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ &\quad + c\Big(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\Big) + c^2\frac{1}{2}\int \epsilon^2 =: \mathcal{Q}_{0,\sqrt{c}}[\epsilon].\end{aligned}$$

Firstly, we prove, by simple calculations, as in the previous section, that Q_x and $Q + xQ_x$ are in the kernel of this quadratic form. It is easy to see, by asymptotic study, that these two functions are linearly independent.

The self-adjoint linear operator associated to this quadratic form is

$$\begin{aligned}\mathcal{L}_c^s[\epsilon] &:= \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon + 5Q^2\epsilon_{xx} + 10QQ_x\epsilon_x \\ &\quad + \Big(5Q_x^2 + 10QQ_{xx} + \frac{15}{2}Q^4 - 6cQ^2\Big)\epsilon,\end{aligned}$$

so that $\mathcal{Q}_c^s[\epsilon] = \int \epsilon\mathcal{L}_c^s[\epsilon]$, where $\mathcal{L}_c^s$ is a compact perturbation of the constant coefficients operator:

$$\mathcal{M}[\epsilon] := \epsilon_{(4x)} - 2c\epsilon_{xx} + c^2\epsilon.$$

A direct analysis involving ODEs shows that the null space of $\mathcal{M}$ is spawned by four linearly independent functions:

$$e^{\pm\sqrt{c}x}, \quad xe^{\pm\sqrt{c}x}.$$

Among these four functions, there are only two L^2-integrable ones in the semi-infinite line $[0, +\infty)$. Therefore, the null space of $\mathcal{L}_c^s|_{H^4(\mathbb{R})}$ is spanned by at most two L^2-functions. Thus,

$$\ker(\mathcal{L}_c^s) = \mathrm{Span}(\partial_x Q, Q + x\partial_x Q).$$

Lemma A.2. *The operator $\mathcal{L}_c^s$ does not have any negative eigenvalue.*

Proof. The operator $\mathcal{L}_c^s$ has

$$\sum_{x\in\mathbb{R}} \dim\ker W[Q_x, Q + xQ_x](t,x)$$

negative eigenvalues, counting multiplicities, where W is the Wronskian matrix:

$$W[Q_x, Q + xQ_x](t,x) := \begin{bmatrix} Q_x & Q + xQ_x \\ Q_{xx} & (Q + xQ_x)_x \end{bmatrix}.$$

For this result, see [19], where the finite interval case was considered. As shown in several articles [23, 27], the extension to the real line is direct.

Thus, it is sufficient to see that $\det W[Q_x, Q + xQ_x](t,x)$ is never zero. For this, let us simply calculate this determinant:

$$\begin{aligned} Q_x(2Q_x + xQ_{xx}) - (Q + xQ_x)Q_{xx} &= 2Q_x^2 - QQ_{xx} \\ &= 2cQ^2 - Q^4 - Q(cQ - Q^3) = cQ^2 > 0. \quad \blacksquare \end{aligned}$$

A.3. Coercivity of the quadratic form associated to a soliton

For $Q = R_{c,\kappa}$, let

$$\begin{aligned} \mathcal{Q}_c^s[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int Q^2\epsilon_x^2 + \frac{5}{2}\int Q_x^2\epsilon^2 + 5\int QQ_{xx}\epsilon^2 + \frac{15}{4}\int Q^4\epsilon^2 \\ + c\Big(\int \epsilon_x^2 - 3\int Q^2\epsilon^2\Big) + c^2\frac{1}{2}\int \epsilon^2. \end{aligned}$$

Lemma A.3. *There exists $\mu_c > 0$ such that for any $\epsilon \in H^2$ satisfying $\int \epsilon Q = \int \epsilon Q_x = 0$, we have that*

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2.$$

Proof. From Section A.2, we know that if $\int \epsilon\partial_x Q = \int \epsilon\partial_c Q = 0$, then, for a constant $\nu_c > 0$, we have that

$$\mathcal{Q}_c^s[\epsilon] \geq \nu_c \|\epsilon\|_{H^2}^2.$$

Let $\epsilon \in H^2$ be such that $\int \epsilon Q = \int \epsilon\partial_x Q = 0$. There exist $a \in \mathbb{R}$ and $\epsilon_\perp$ belonging to $\mathrm{Span}(\partial_x Q, \partial_c Q)^\perp$ such that

$$\epsilon = a\partial_c Q + \epsilon_\perp.$$

From $\int \epsilon Q = 0$, we have that

$$a\int \partial_c Q \cdot Q + \int \epsilon_\perp Q = 0,$$

thus,

$$\frac{a}{2}\int Q^2 + \int \epsilon_\perp Q = 0,$$

which allows us to derive

$$a = -2\,\frac{\int \epsilon_\perp Q}{\int Q^2}.$$

Because $\partial_c Q$ is in the kernel of $\mathcal{Q}_c^s$, we have that

$$\mathcal{Q}_c^s[\epsilon] = \mathcal{Q}_c^s[\epsilon_\perp] \geq \nu_c \|\epsilon_\perp\|_{H^2}^2.$$

Now, from

$$\epsilon = -2\,\frac{\int \epsilon_\perp Q}{\int Q^2}\,\partial_c Q + \epsilon_\perp,$$

we have, by the triangular and Cauchy–Schwarz inequalities, that

$$\begin{aligned}\|\epsilon\|_{H^2} &\leq \|\epsilon_\perp\|_{H^2} + 2\,\frac{|\int \epsilon_\perp Q|}{\|Q\|_{L^2}^2}\|\partial_c Q\|_{H^2}\\ &\leq \|\epsilon_\perp\|_{H^2} + 2\,\frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\|\epsilon_\perp\|_{L^2} \leq \Big(1 + 2\,\frac{\|\partial_c Q\|_{H^2}}{\|Q\|_{L^2}}\Big)\|\epsilon_\perp\|_{H^2}.\end{aligned}$$

Therefore, we may derive a constant μ_c (independent of ϵ) such that

$$\mathcal{Q}_c^s[\epsilon] \geq \mu_c \|\epsilon\|_{H^2}^2.$$ ■

A.4. Coercivity with almost orthogonality conditions (to be used for the uniqueness)

For $B := B_{\alpha,\beta}$ or any of its translations, we define the canonical quadratic form associated to B:

$$\begin{aligned}\mathcal{Q}_{\alpha,\beta}^b[\epsilon] := \frac{1}{2}\int \epsilon_{xx}^2 - \frac{5}{2}\int B^2\epsilon_x^2 + \frac{5}{2}\int B_x^2\epsilon^2 + 5\int BB_{xx}\epsilon^2 + \frac{15}{4}\int B^4\epsilon^2\\ + (\beta^2-\alpha^2)\Big(\int \epsilon_x^2 - 3\int B^2\epsilon^2\Big) + (\alpha^2+\beta^2)^2\,\frac{1}{2}\int \epsilon^2,\end{aligned}$$

and we know that $\partial_{x_1} B$ and $\partial_{x_2} B$ span the kernel of $\mathcal{Q}_{\alpha,\beta}^b$. More precisely, there exists $\mu_{\alpha,\beta}^b > 0$ such that if ϵ is orthogonal to $\partial_{x_1} B$ and $\partial_{x_2} B$, we have that

$$\mathcal{Q}_{\alpha,\beta}^b[\epsilon] \geq \mu_{\alpha,\beta}^b\,\|\epsilon\|_{H^2}^2 - \frac{1}{\mu_{\alpha,\beta}^b}\Big(\int \epsilon B\Big)^2.$$

We would like to prove the following lemma (adapted from Appendix A of [30]).

Lemma A.4. *There exists* $\nu := \nu_{\alpha,\beta}^b > 0$ *such that, for* $\epsilon \in H^2(\mathbb{R})$, *if*

$$\Big|\int (\partial_{x_1} B_{\alpha,\beta})\epsilon\Big| + \Big|\int (\partial_{x_2} B_{\alpha,\beta})\epsilon\Big| < \nu\|\epsilon\|_{H^2},$$

then

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon] \geq \frac{\mu^b_{\alpha,\beta}}{4}\|\epsilon\|^2_{H^2} - \frac{4}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon B_{\alpha,\beta}\Big)^2,$$

where $B_{\alpha,\beta}$ denotes the breather of parameters α and β or any of its translations (in space or in time).

Proof. Take $\nu > 0$ (we will find a condition on ν later in the proof) and take ϵ satisfying the assumption of the lemma. Then (denoting $B = B_{\alpha,\beta}$)

$$\epsilon = \epsilon_1 + aB_1 + bB_2 = \epsilon_1 + \epsilon_2, \tag{A.2}$$

where $\int \epsilon_1 B_1 = \int \epsilon_1 B_2 = \int \epsilon_1\epsilon_2 = 0$.

By performing a L^2-scalar product of (A.2) with B_1 and B_2, we obtain, by assumption, that

$$\Big|a\int B_1^2 + b\int B_1B_2\Big| \leq \nu\|\epsilon\|_{H^2} \quad \text{and} \quad \Big|a\int B_1B_2 + b\int B_2^2\Big| \leq \nu\|\epsilon\|_{H^2}.$$

Therefore, by making linear combinations of these two inequalities, using the triangular and Cauchy–Schwarz inequalities, we obtain that

$$|a| + |b| \leq C\nu\|\epsilon\|_{H^2}.$$

We can take space derivatives of (A.2). And thus, we obtain, for ν small enough, that

$$\frac{1}{2}\|\epsilon\|^2_{H^2} \leq \|\epsilon_1\|^2_{H^2} \leq 2\|\epsilon\|^2_{H^2}.$$

Because of $\int BB_1 = \int BB_2 = 0$,

$$\int \epsilon B = \int \epsilon_1 B.$$

By bilinearity,

$$\begin{aligned}\mathcal{Q}^b_{\alpha,\beta}[\epsilon] &= \mathcal{Q}^b_{\alpha,\beta}[\epsilon_1] + \mathcal{Q}^b_{\alpha,\beta}[\epsilon_2] + \int \epsilon_{1,xx}\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\epsilon_{2,x}\\ &\quad + 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\epsilon_1\epsilon_2 + \frac{15}{2}\int B^4\epsilon_1\epsilon_2\\ &\quad + (\beta^2-\alpha^2)\Big(2\int \epsilon_{1,x}\epsilon_{2,x} - 6\int B^2\epsilon_1\epsilon_2\Big) + (\alpha^2+\beta^2)^2\int \epsilon_1\epsilon_2.\end{aligned}$$

We know from the coercivity of $\mathcal{Q}^b_{\alpha,\beta}$ that

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon_1] \geq \mu^b_{\alpha,\beta}\|\epsilon_1\|^2_{H^2} - \frac{1}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon_1 B\Big)^2 \geq \frac{\mu^b_{\alpha,\beta}}{2}\|\epsilon\|^2_{H^2} - \frac{2}{\mu^b_{\alpha,\beta}}\Big(\int \epsilon B\Big)^2.$$

Moreover, if we denote by $\mathcal{L}^b_{\alpha,\beta}$ the self-adjoint operator associated to the quadratic form $\mathcal{Q}^b_{\alpha,\beta}$,

$$\mathcal{Q}^b_{\alpha,\beta}[\epsilon_2] = a^2\,\mathcal{Q}^b_{\alpha,\beta}[B_1] + b^2\,\mathcal{Q}^b_{\alpha,\beta}[B_2] + 2ab\int \mathcal{L}^b_{\alpha,\beta}[B_1]B_2 \le C\nu^2\|\epsilon\|^2_{H^2}.$$

Actually, in this case, $\mathcal{Q}^b_{\alpha,\beta}[\epsilon_2] = 0$, because ϵ_2 is in the kernel of $\mathcal{Q}^b_{\alpha,\beta}$ (however, when we adapt this proof for solitons, we can only write the bound).

Now, we recall that $\int \epsilon_1\epsilon_2 = 0$, and study the other terms by using Cauchy–Schwarz:

$$\begin{aligned}&\Big|\int \epsilon_{1,xx}\,\epsilon_{2,xx} - 5\int B^2\epsilon_{1,x}\,\epsilon_{2,x} + 5\int B_x^2\epsilon_1\epsilon_2 + 10\int BB_{xx}\,\epsilon_1\epsilon_2\\ &\quad + \frac{15}{2}\int B^4\epsilon_1\epsilon_2 + (\beta^2-\alpha^2)\Big(2\int \epsilon_{1,x}\,\epsilon_{2,x} - 6\int B^2\epsilon_1\,\epsilon_2\Big)\Big|\\ &\quad \le C(|a|+|b|)\|\epsilon_1\|_{H^2} \le C\nu\|\epsilon\|^2_{H^2(\mathbb{R})}.\end{aligned}$$

We observe that if we take ν small enough, the claim of the lemma is proved. ■

We prove in the same way that we have similar lemmas for solitons.

Lemma A.5. *There exists $\nu := \nu^s_c > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\Big|\int (\partial_c R_{c,\kappa})\epsilon\Big| + \Big|\int (\partial_x R_{c,\kappa})\epsilon\Big| \le \nu\|\epsilon\|_{H^2},$$

then

$$\mathcal{Q}^s_c[\epsilon] \ge \frac{\mu^s_c}{4}\,\|\epsilon\|^2_{H^2},$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

Lemma A.6. *There exists $\nu := \nu^s_c > 0$, such that, for $\epsilon \in H^2(\mathbb{R})$, if*

$$\Big|\int R_{c,\kappa}\epsilon\Big| + \Big|\int (\partial_x R_{c,\kappa})\epsilon\Big| \le \nu\|\epsilon\|_{H^2},$$

then

$$\mathcal{Q}^s_c[\epsilon] \ge \frac{\mu^s_c}{4}\,\|\epsilon\|^2_{H^2},$$

where $R_{c,\kappa}$ denotes the soliton of parameter c and sign κ or any of its translations.

A.5. Computations for the third localized integral (to be used for the uniqueness)

Lemma A.7. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^3 function that does not depend on time and u a solution of* (1.1). *Then*

$$\begin{aligned}&\frac{d}{dt}\int\Big(\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\Big)f\\ &\quad = \int\Big(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2\,u\,u_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4u_x^2\Big)f'\\ &\qquad + 5\int u^2u_xu_{xx}f'' + \frac{1}{2}\int u_{xx}^2 f'''.\end{aligned}$$

Proof. We perform by doing integrations by parts when needed and basic calculations:

$$
\begin{aligned}
&\tfrac{d}{dt}\int\big(\tfrac12 u_{xx}^2-\tfrac52 u^2u_x^2+\tfrac14 u^6\big)f\\
&=\int u_{txx}u_{xx}f-5\int u_tuu_x^2f-5\int u^2u_{tx}u_xf+\tfrac32\int u_tu^5f\\
&=-\int(u_{xx}+u^3)_{xxx}u_{xx}f+5\int(u_{xx}+u^3)_xuu_x^2f\\
&\quad+5\int u^2(u_{xx}+u^3)_{xx}u_xf-\tfrac32\int(u_{xx}+u^3)_xu^5f\\
&=\int(u_{xx}+u^3)_{xx}u_{xxx}f+\int(u_{xx}+u^3)_{xx}u_{xx}f'+5\int(u_{xx}+u^3)_xuu_x^2f\\
&\quad+5\int u^2(u_{xx}+u^3)_{xx}u_xf-\tfrac32\int(u_{xx}+u^3)_xu^5f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u^3)_{xx}u_{xxx}f+\int(u_{xx}+u^3)_{xx}u_{xx}f'\\
&\quad+5\int u_{xxx}uu_x^2f+5\int(u^3)_xuu_x^2f+5\int u^2u_{xxxx}u_xf\\
&\quad+5\int u^2(u^3)_{xx}u_xf-\tfrac32\int u_{xxx}u^5f-\tfrac32\int(u^3)_xu^5f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+\int(3u_{xx}u^2+6u_x^2u)u_{xxx}f\\
&\quad+5\int u_{xxx}uu_x^2f+15\int u_x^3u^3f+5\int u^2u_{xxxx}u_xf\\
&\quad+5\int u^2(3u_{xx}u^2+6u_x^2u)u_xf-\tfrac32\int u_{xxx}u^5f-\tfrac92\int u_xu^7f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+3\int u^2u_{xx}u_{xxx}f+5\int u^2u_{xxxx}u_xf\\
&\quad+11\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\tfrac32\int u_{xxx}u^5f+\tfrac{9}{16}\int u^8f'\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+\tfrac{9}{16}\int u^8f'-2\int u^2u_{xx}u_{xxx}f\\
&\quad+\int uu_x^2u_{xxx}f-5\int u^2u_xu_{xxx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\tfrac32\int u^5u_{xxx}f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+\tfrac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'\\
&\quad-\int u^2(u_{xx}^2)_xf+\int uu_x^2u_{xxx}f+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\tfrac32\int u^5u_{xxx}f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+\tfrac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'\\
&\quad+\int u^2u_{xx}^2f'+2\int uu_xu_{xx}^2f-\int u_x^3u_{xx}f-2\int uu_xu_{xx}^2f\\
&\quad-\int uu_x^2u_{xx}f'+45\int u^3u_x^3f+15\int u^4u_xu_{xx}f-\tfrac32\int u^5u_{xxx}f\\
&=-\tfrac12\int u_{xxx}^2f'+\int(u_{xx}+u^3)_{xx}u_{xx}f'+\tfrac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'+\int u^2u_{xx}^2f'\\
&\quad-\int uu_x^2u_{xx}f'-\tfrac14\int(u_x^4)_xf+45\int u^3u_x^3f+\tfrac{45}{4}\int u^4(u_x^2)_xf+\tfrac32\int u^5u_{xx}f'\\
&=-\tfrac12\int u_{xxx}^2f'+\int\big(u_{xx}+u^3\big)_{xx}u_{xx}f'+\tfrac{9}{16}\int u^8f'-5\int u^2u_xu_{xxx}f'\\
&\quad+\int u^2u_{xx}^2f'-\int uu_x^2u_{xx}f'+\tfrac14\int u_x^4f'+\tfrac32\int u^5u_{xx}f'+45\int u^3u_x^3f\\
&\quad-45\int u^3u_x^3f-\tfrac{45}{4}\int u^4u_x^2f'\\
&=-\tfrac32\int u_{xxx}^2f'-\int u_{xxx}u_{xx}f''+4\int u_{xx}^2u^2f'+5\int u_x^2uu_{xx}f'-5\int u^2u_xu_{xxx}f'\\
&\quad+\tfrac{9}{16}\int u^8f'+\tfrac14\int u_x^4f'+\tfrac32\int u^5u_{xx}f'-\tfrac{45}{4}\int u^4u_x^2f'\\
&=-\tfrac32\int u_{xxx}^2f'+9\int u_{xx}^2u^2f'+15\int u_x^2uu_{xx}f'+\tfrac{9}{16}\int u^8f'+\tfrac14\int u_x^4f'\\
&\quad+\tfrac32\int u^5u_{xx}f'-\tfrac{45}{4}\int u^4u_x^2f'-\int u_{xxx}u_{xx}f''+5\int u^2u_xu_{xx}f''\\
&=\int\big(-\tfrac32u_{xxx}^2+9u_{xx}^2u^2+15u_x^2uu_{xx}+\tfrac{9}{16}u^8+\tfrac14u_x^4+\tfrac32u_{xx}u^5-\tfrac{45}{4}u^4u_x^2\big)f'\\
&\quad+5\int u^2u_xu_{xx}f''+\tfrac12\int u_{xx}^2f'''.
\end{aligned}
$$

which is exactly the desired expression. ■

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