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Constant mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ with small planar boundary

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Abstract. We show that constant mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, with small and pinched boundary contained in a horizontal slice P, are topological disks provided they are contained in one of the two halfspaces determined by P. This is the analogous in $\mathbb{H}^n \times \mathbb{R}$ of a result in $\mathbb{R}^3$ by A. Ros and H. Rosenberg [J. Differential Geom. 44 (1996), 807–817].

1. Introduction

There is little known about the topological and geometrical structure of constant mean curvature hypersurfaces with convex boundary.

For example, it is unknown if a surface embedded in $\mathbb{R}^3$, with boundary a circle and constant mean curvature, is isometric to a spherical cap. During the years, there have been partial results concerning this problem. Let us recall those we consider the deepest.

The result by Brito, W. Meeks, H. Rosenberg and R. Sa Earp in [3] yields that an embedded constant mean curvature surface, with boundary a circle in a plane P, is a spherical cap provided it is transverse to P along the boundary. In fact, the authors are able to prove that the transversality condition forces the surface to stay in one of the two halfspaces determined by P. Then one can use the Alexandrov reflection method to get the result.

Notice that Alexandrov's theorem states that a closed, embedded surface with constant mean curvature in $\mathbb{R}^3$ is a round sphere [5] (see also [7] for a survey about the subject). In light of Alexandrov's theorem, it is reasonable to expect that, if the boundary curve of a constant mean curvature surface M is small, then M is a topological disk. By rescaling, this is analogous to expect that M is topologically a disk if the mean curvature of M is small when compared with the curvature of its boundary.

Indeed, A. Ros and H. Rosenberg showed in [11] that if Γ is a convex curve contained in the plane P, the mean curvature H of an embedded constant mean curvature surface M is small when compared with the curvature of Γ, and if M is contained in the halfspace bounded by P, then M is a topological disk. Their result was extended to the

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hyperbolic 3-space $\mathbb{H}^3$ by B. Semmler [13], and to $\mathbb{R}^n$, for all symmetric functions of the principal curvatures, by B. Nelli and B. Semmler [9].

In this paper we extend the Ros–Rosenberg result to constant mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. Namely, we prove the following theorem (see Theorem 4.1). The notion of r_{ext} (exterior radius) and r_{int} (interior radius) in the statement are intuitively clear. For a precise definition, as well as for the definition of horoconvexity, see the beginning of Section 4.

Theorem. *Let M be a n-dimensional compact hypersurface of $\mathbb{H}^n \times \mathbb{R}$ embedded in $\mathbb{H}^n \times [0, \infty[$ with constant mean curvature $H > (n-1)/n$ and boundary $\partial M = \Gamma$. Assume that Γ is a closed $(n-1)$-dimensional horoconvex hypersurface of the slice $P = \mathbb{H}^n \times \{0\}$ satisfying the pinching $2r_{\text{int}} > r_{\text{ext}}$. Then there is a constant $\delta(n, H) > 0$, depending only on n and H, such that if $r_{\text{ext}} \le \delta(n, H)$, then M is topologically a disk. Moreover, either M is a graph over the domain Ω bounded by Γ, or $N = M \cap (\Omega \times]0, \infty[)$ is a graph over Ω and $M \setminus N$ is a graph over $\Gamma \times]0, \infty[$ with respect to the lines orthogonal to $\Gamma \times]0, \infty[$.*

We emphasize that the assumption $H > (n-1)/n$ in the previous theorem is not restrictive because the following theorem holds.

Theorem. *Let Ω be a connected domain in a horizontal section P of $\mathbb{H}^n \times \mathbb{R}$, with horoconvex boundary $\Gamma = \partial\Omega$. If M is an embedded compact constant mean curvature hypersurface with boundary Γ and mean curvature $H \le (n-1)/n$, then M is a graph on Ω. In particular, M is a topological disk.*

The previous theorem is proved for $n = 2$ by B. Nelli, R. Sa Earp, W. Santos and E. Toubiana in Theorem 2.2 of [8], and for $n > 2$, by P. Bérard and R. Sa Earp in Theorem 3.3 of [2].

Ros and Rosenberg's proof relies on a crucial rescaling theorem (see Theorem 1 in [11]). In trying to adapt their proof to $\mathbb{H}^n \times \mathbb{R}$, however, one is faced with the first obstacle that rescalings are not available in $\mathbb{H}^n$. This obstacle was overcome by B. Semmler in [13], where she uses constant mean curvature horizontal half-cylinders as barriers to give a different proof of Theorem 2 in [11]. In order to be able to use Semmler's type proof, we need *horizontal cylinders*, i.e., constant mean curvature hypersurfaces invariant by hyperbolic translations in $\mathbb{H}^n \times \mathbb{R}$. Such cylinders for $n = 2$ are described by J. M. Manzano and I. Onnis [6,10], and by P. Bérard and R. Sa Earp [2] for $n > 2$. Then, one is faced with the second obstacle that, when $H \to (n-1)/n$, compact constant mean curvature and invariant by rotations hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ converge to a complete non-compact graph different from a slice (see Remark 3.1). In order to overcome this second obstacle, we need two ingredients:

- a precise estimate of how much we can go beyond the boundary of M, when doing an Alexandrov reflection: limaçon construction, see Section 2;
- the use as barriers of immersed constant mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by rotation and horizontal cylinders.

The paper is organized as follows. In Section 2, we describe the construction of the hyperbolic limaçon and study its geometry. In Section 3, we describe all the comparison hypersurfaces that we need in the following. In Section 4, we prove our main theorem. To simplify the reading, in Appendix A we summarize the principal notations introduced in the article. Throughout the article (except in a part of Section 3.2), we will use the Poincaré disk model for the hyperbolic space. Moreover, we will use extensively the Alexandrov reflection method, which is explained in details in Chapter 7 of [5] and in [7].

2. The limaçon in the hyperbolic space

In this section we describe a family of hypersurfaces of the hyperbolic space $\mathbb{H}^n$, analogous to the classical curve in the Euclidean plane known as *limaçon of Pascal*. Such hypersurfaces will be convenient to estimate radii of disks appearing in the proof of the main theorem.

Definition 2.1. We call *hyperbolic limaçon* the hypersurface $\mathcal{L}$ of $\mathbb{H}^n$ defined in the following way. Fix two points $A \neq C \in \mathbb{H}^n$ and a positive constant c, and let $\mathcal{C}$ be the geodesic sphere of center C and radius c. For any $P \in \mathcal{C}$, let A_P be the reflection of A across the totally geodesic hyperplane of $\mathbb{H}^n$ tangent to $\mathcal{C}$ at P. Then $\mathcal{L} = \{A_P \in \mathbb{H}^2 \mid P \in \mathcal{C}\}$. See Figure 1 below.

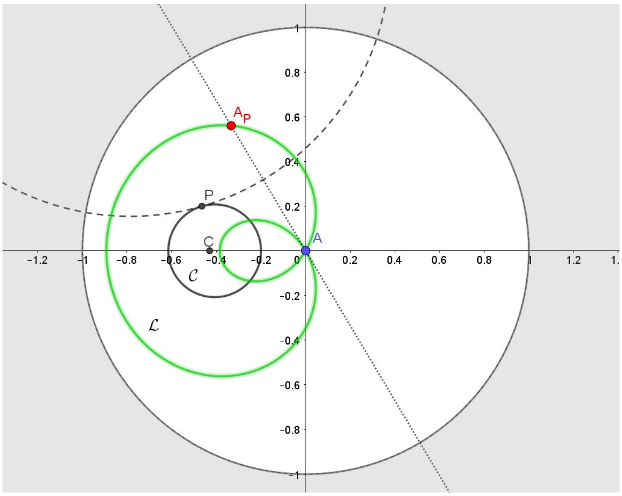

Figure 1. Hyperbolic limaçon for $n = 2$, $a > c$.

In what follows, we want to describe the main proprieties of $\mathcal{L}$. It is evident by definition that $\mathcal{L}$ is invariant under rotations around the geodesic passing through A and C, therefore it is enough to study only the case $n = 2$.

Remark 2.2. Since $\mathbb{H}^2$ is homogeneous, up to isometries of the ambient space, $\mathcal{L}$ depends only on two positive parameters: $a := d(A, C)$ and c.

In the following, a and c will be called the *parameters* of the limaçon, and A the *base point* of it.

Remark 2.3. The name of this curve is justified by the fact that the same construction in the Euclidean plane produces the classical limaçon parametrized, up to isometries, by

$$\mathcal{L}(\vartheta) = (-a\cos(2\vartheta) + 2c\cos(\vartheta), -a\sin(2\vartheta) + 2c\sin(\vartheta)).$$

It is well known that the Euclidean limaçon can be defined in many equivalent ways. For example, it is the pedal curve of a circumference. The pedal curve of a circumference in the hyperbolic plane is the subject of [14].

The following result says that the shape of the hyperbolic limaçon is qualitatively analogous to the shape of the Euclidean one.

Lemma 2.4. *Let $\mathcal{L}$ be the hyperbolic limaçon with parameters a and c, and base point A. Let $\mathcal{C}$ be the sphere defining $\mathcal{L}$, and let $C \neq A$ be its center. Then $\mathcal{L}$ is a closed continuous curve, which continuously depends on a and c. It is symmetric with respect to the geodesic through A and C. If $a < c$, $\mathcal{L}$ is a simple curve. If $a = c$, $\mathcal{L}$ has a cusp in A. If $a > c$, $\mathcal{L}$ has two loops, one inside the other, and it crosses itself only in A.*

Proof. From the very definition, it is evident that $\mathcal{L}$ is an immersion of a $\mathbb{S}^1$ in $\mathbb{H}^2$, that continuously depends on the parameters a and c, and that is symmetric with respect to the complete geodesic joining A and C.

Note that, for any choice of a and c, $\mathcal{L}$ has no multiple points except, possibly, A. In fact, take any point $B \in \mathcal{L}$, $B \neq A$, let $\overline{AB}$ the geodesic segment joining A and B, and let M be its middle point. Reversing the construction of Definition 2.1, the geodesic passing through M and orthogonal to $\overline{AB}$ has to be tangent to $\mathcal{C}$. Let X be the tangency point. Then $B = A_X$, and X is uniquely determined by this procedure. Moreover, by Definition 2.1, we have that $A \in \mathcal{L}$ if and only if there is a geodesic tangent to $\mathcal{C}$ containing A. After these general facts, we need to distinguish the three cases.

If $a < c$, the point A lies in the disk bounded by $\mathcal{C}$. In this case, $A \notin \mathcal{L}$, otherwise there would be a geodesic passing through A and tangent to $\mathcal{C}$. Therefore, $\mathcal{L}$ is a simple curve.

When $a > c$, A is clearly outside the disk bounded by $\mathcal{C}$; then there are two distinct geodesic passing though A and tangent to $\mathcal{C}$. Hence A is a double point for $\mathcal{L}$ and, as showed above, it is its only multiple point. Now let γ be the complete geodesic joining A and C: γ intersects $\mathcal{C}$ in exactly two points. Denote by P the closest to A, by Q the second one, and let A_P and A_Q be the corresponding points on $\mathcal{L}$. Since γ and $\mathcal{C}$ are orthogonal in P and Q, we have that $\gamma \cap \mathcal{L} = \{A, A_P, A_Q\}$. Moreover, walking along γ from A in the direction of C, we will meet A, A_P and A_Q exactly in that order. Suppose now that we walk along $\mathcal{C}$ clockwise starting from P; then in $\mathcal{L}$ we will meet A_P, A, A_Q and A, again exactly in that order. Since $\mathcal{L}$ is continuous, and by the order of the points A, A_P and A_Q on γ, it follows that $\mathcal{L}$ has two loops branching in A, one inside the other. By construction, $d(A, A_P) = 2d(A, P)$ and $d(A, A_Q) = 2d(A, Q)$, therefore the smaller loop is the one passing through A_P.

The case $a = c$ can be thought as the limit case when $a > c$ and a converges to c: in this case, P converges to A, and then the smaller loops shrinks to A producing a cusp. ∎

For applications, we would like to estimate the size of $\mathcal{L}$, with particular attention to the case of small c.

Lemma 2.5. *Let $\mathcal{L}$ be the hyperbolic limaçon with parameters $a > c$ and base point A. Let $\mathcal{C}$ be the sphere defining $\mathcal{L}$, let C be its center, and let P be the point of $\mathcal{C}$ closest to A. Then we have the following.*

(1) *The smaller (respectively, the larger) loop of $\mathcal{L}$ is contained in (respectively, contains) the disk with center P and radius $a - c$.*

(2) *If $a > 2c$, then the smaller loop of $\mathcal{L}$ bounds the disk with center C and radius $a - 2c$.*

(3) *The whole $\mathcal{L}$ is contained in the disk with center C and radius $a + 2c$.*

Moreover, for any fixed a, C and A, when c converges to zero, $\mathcal{L}$ converges to twice the sphere with center C and radius a.

Proof. Since $a > c$, by Lemma 2.4, $\mathcal{L}$ has two loops.

(1) Let A_P be as in the proof of Lemma 2.4, and let $\mathcal{C}_1$ be the geodesic sphere of center P and radius $a - c$. We claim that $\mathcal{C}_1 \cap \mathcal{L} = \{A,\ A_P\}$. In fact, $A, A_P \in \mathcal{C}_1 \cap \mathcal{L}$ trivially. For any point $B \in \mathcal{C}_1$ with $B \neq A, A_P$, let $\overline{AB}$ be the geodesic segment joining A and B, and let M be its middle point. See Figure 2 for a picture of the construction. For the choice of B, we have that $M \neq P$. Let γ be the geodesic through M and P. The geodesic triangles AMP and BMP are congruent, hence γ is orthogonal to $\overline{AB}$ in M. Now suppose that there exists a $B \in \mathcal{L} \cap \mathcal{C}_1$ with $B \neq A, A_P$; then, by the construction of Definition 2.1, γ should be tangent to $\mathcal{C}$, but we know that $P \in \mathcal{C} \cap \gamma$, hence we would get $B = A_P$, having a contradiction. From this claim it follows that the smaller (respectively, the larger) loop of $\mathcal{L}$ is inside (respectively, outside) $\mathcal{C}_1$.

(2) Now suppose that $a > 2c$. Let $\mathcal{C}_2$ be the geodesic sphere with center C and radius $a - 2c$. We want to prove that the smaller loop of $\mathcal{L}$ bounds $\mathcal{C}_2$. By construction, we have $A_P \in \mathcal{C}_2 \cap \mathcal{L}$. We claim that $\mathcal{C}_2 \cap \mathcal{L} = \{A_P\}$. In fact, fix $X \in \mathcal{C}_2$, $X \neq A_P$, let γ be the unique geodesic tangent to $\mathcal{C}$ and orthogonal to the geodesic α containing the segment $\overline{XA}$, and let $Y = \gamma \cap \mathcal{C}$ be the tangency point and let $Z = \gamma \cap \alpha$. See Figure 2 for a picture of the construction. In order to prove that $X \notin \mathcal{L}$, by Definition 2.1, we need to prove that Z is not the middle point of $\overline{XA}$. If $Z \notin \overline{XA}$, we are done, hence we assume $Z \in \overline{XA}$. Since $X \neq A_P$, we have that the following triangle inequality is strict:

$$d(X, Y) < d(X, C) + d(C, Y) = a - c.$$

On the other hand, since $Y \in \mathcal{C}$ and by definition of P, we have

$$d(A, Y) \geq d(A, P) = a - c.$$

Therefore

$$d(X, Y) < d(A, Y). \tag{2.1}$$

Now let us consider the right-angled hyperbolic triangles XYZ and AYZ. Let β (respectively, β') be the angle of XYZ (respectively, AYZ) with vertex X (respectively, A). By (2.1) and some hyperbolic trigonometry (see for instance Theorem 7.11.2 of [1]), we get

$$\tanh^2 d(X,Z) = \tanh^2 d(X,Y)\cos^2\beta = \tanh^2 d(X,Y)\Big(1 - \frac{\sinh^2 d(A,Y)}{\sinh^2 d(X,Y)}\sin^2\beta'\Big)$$
$$< \tanh^2 d(A,Y)\cos^2\beta' \;=\; \tanh^2 d(A,Z).$$

It follows that

$$d(X,Z) < d(Z,A),$$

hence Z cannot be the middle point of $\overline{XA}$ and therefore $X \notin \mathcal{L}$. The result follows noticing that $\mathcal{C}_2$ is a continuous curve and that the antipodal point of A_P in $\mathcal{C}_2$ is in the compact domain bounded by the smaller loop of $\mathcal{L}$, therefore the closed disk bounded by $\mathcal{C}_2$ is inside this domain as well.

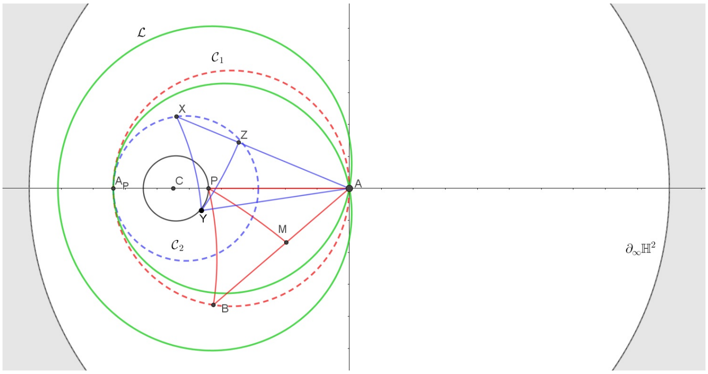

Figure 2. Estimate of the diameter of the smaller loop. Green: hyperbolic limaçon with $a > 2c$. Red: the construction in the proof of Lemma 2.5 (1). Blue: the construction in the proof of Lemma 2.5 (2).

(3) The proof is similar to that of the previous case, but this time the inequalities are reversed. Let $\mathcal{C}_3$ be the geodesic sphere with center C and radius $a + 2c$, let X a point of $\mathcal{C}_3$, and define Y and Z analogously to the previous case. We have that $d(X,Y) > a + c > d(A,Y)$. Therefore either $Z \notin \overline{XA}$ or $d(X,Z) > d(Z,A)$. In any case, Z is not the middle point of $\overline{XA}$, hence $X \notin \mathcal{L}$.

Finally, when c converges to 0, we have that P converges to C, and then, for any $i = 1,2,3$, we get that $\mathcal{C}_i$ converges to the geodesic sphere with center C and radius a. ∎

3. Constant mean curvature comparison hypersurfaces

There are many examples of constant mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by some ambient isometry. In this section we describe some of them, that will be mainly used as barriers in the proof of our main theorem.

3.1. Rotationally invariant hypersurfaces

P. Bérard and R. Sa Earp [2] classify the rotationally invariant hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ with constant mean curvature. Following their notations, for any $m \in \mathbb{N}$, for any $H > 0$

and for suitable choice of the parameter d, we define

$$I_m(t) = \int_0^t \sinh^m(\tau)\,d\tau,$$

$$\lambda_{H,d}(\rho) = \int_{\rho_0}^{\rho} \frac{nHI_{n-1}(t)+d}{\sqrt{\sinh^{2n-2}(t)-(nHI_{n-1}(t)+d)^2}}\,dt, \tag{3.1}$$

where $\rho_0 \geq 0$ is the infimum of the interval where the integrand function exists. The rotation around the axis $\{0\} \times \mathbb{R}$ of the graph of the function $\lambda_{H,d}$ produces, up to isometries of the ambient space, all the rotationally invariant hypersurfaces with constant mean curvature H. The case of $n = 2$ has been studied in detail in [8, 10, 12].

In this section we describe only the three types of hypersurfaces in this class that are most relevant for our purposes. We refer to the bibliography cited above for an exhaustive discussion about the topic.

Case 1. *The spheres* $\mathcal{S}_H$.

When $H > (n-1)/n$ and $d = 0$, $\lambda_{H,0}$ is defined for $\rho \in [0, R_{\mathcal{S}}]$, where $R_{\mathcal{S}}$ is the solution of the equation

$$\sinh^{n-1}(\rho) - nHI_{n-1}(\rho) = 0. \tag{3.2}$$

The graph of the function $\lambda_{H,0}$ is tangent to the plane $t = 0$ when $\rho = 0$, and it has a vertical tangent at $\rho = R_{\mathcal{S}}$. Let $h_{\mathcal{S}} = \lambda_{H,0}(R_{\mathcal{S}})$ be the maximal height of such curve. Let $\mathcal{S}_H$ be the hypersurface generated by rotating the graph of $\lambda_{H,0}$ and the graph of the function $-\lambda_{H,0} + 2h_{\mathcal{S}}$ around the t-axis. According to Theorem 2.3 in [2], $\mathcal{S}_H$ is a compact embedded smooth hypersurface with the topology of the sphere. When $n = 2$ (and hence $H > 1/2$), we have the explicit expression

$$\lambda_{H,0}(\rho) = \frac{4H}{\sqrt{4H^2-1}} \arcsin\frac{1}{2H} - \frac{4H}{\sqrt{4H^2-1}} \arctan\sqrt{\frac{1-4H^2\tanh^2\frac{\rho}{2}}{4H^2-1}},$$
$$\text{for } \rho \in \left[0, \cosh^{-1}\left(\frac{4H^2+1}{4H^2-1}\right)\right].$$

Case 2. *The complete graphs* $\mathcal{S}_{(n-1)/n}$.

When $H = (n-1)/n$ and $d = 0$, the curve $\lambda_{H,0}$ is defined for any $\rho \geq 0$. Let $\mathcal{S}_{(n-1)/n}$ be the hypersurface generated by rotating $\lambda_{(n-1)/n,0}$ around the t-axis. According to Theorem 2.1 in [2], $\mathcal{S}_{(n-1)/n}$ is a simply connected entire vertical graph, contained in a half-space and tangent to the hyperplane $t = 0$ at $\rho = 0$. Moreover, when $n = 2$, (3.1) can be solved explicitly and one has

$$\lambda_{1/2,0}(\rho) = 2\left(\cosh\frac{\rho}{2} - 1\right).$$

Case 3. *Immersed annuli* $\mathcal{A}_d$.

When $H = (n-1)/n$ and $d < 0$, the curve $\lambda_{(n-1)/n,d}$ is defined for any $\rho \geq r_{0,d}$, where $r_{0,d}$ is the unique solution of the equation

$$\sinh^{n-1}(\rho) + (n-1)I_{n-1}(\rho) + d = 0. \tag{3.3}$$

Moreover, $\lambda_{(n-1)/n,d}$ has a vertical tangent at $r_{0,d}$, it is negative for ρ close to $r_{0,d}$, and $\lim_{\rho\to+\infty} \lambda_{(n-1)/n,d}(\rho) = +\infty$. From (3.1), it easy to see that $\lambda_{(n-1)/n,d}$ has only one critical point at $\rho = r_{1,d}$, where $r_{1,d}$ is the unique solution of the equation

$$(n-1)I_{n-1}(\rho) + d = 0. \tag{3.4}$$

Let $\mathcal{A}_d$ be the hypersurface generated by rotating the graphs of $\lambda_{(n-1)/n,d}$ and $-\lambda_{(n-1)/n,d}$ around the t-axis. According to Theorem 2.1 in [2], $\mathcal{A}_d$ is a complete hypersurface, symmetric with respect to the hyperplane $t = 0$ with self-intersection along a sphere of this hyperplane. Moreover, $\mathcal{A}_d \cap \mathbb{R}^{\pm}$ are vertical graphs outside a disk of $\mathbb{H}^n \times \{0\}$ with center in the origin and radius $r_{0,d}$.

Remark 3.1. We sum up some important relations between the hypersurfaces described above.

- When $d \to 0$, then the hypersurfaces $\mathcal{A}_d$ tend to the union of $\mathcal{S}_{(n-1)/n}$ with its reflection with respect to the slice $\{t = 0\}$.
- When $H \to (n-1)/n$, the spheres $\mathcal{S}_H$ tend to the complete graph $\mathcal{S}_{(n-1)/n}$. Notice that, differently from the Euclidean case, here the spheres do not converge to a horizontal slice.

For future use, we need to estimate various quantities associated to the hypersurfaces $\mathcal{S}_H$, $\mathcal{S}_{(n-1)/n}$ and $\mathcal{A}_d$. In particular, we would like to compare $r_{0,d}$ and $r_{1,d}$ defined above with ρ_H^* defined in the following way: for fixed d and $H \geq (n-1)/n$, ρ_H^* is the radius of $\mathcal{S}_H$ at height h^*, where h^* is defined in (3.7) below and it is a suitable approximation of the height of the portion of $\mathcal{A}_d$ between $r_{0,d}$ and $r_{1,d}$. In particular, ρ_H^* satisfies the equation

$$\lambda_{H,0}(\rho_H^*) = h^*. \tag{3.5}$$

Lemma 3.2. *With the notations introduced so far, we have the following limits for any $H \geq (n-1)/n$:*

$$\lim_{d\to 0} r_{0,d} = 0, \quad \lim_{d\to 0} r_{1,d} = 0 \quad \textit{and} \quad \lim_{d\to 0} \rho_H^* = 0.$$

Moreover,

$$\lim_{d\to 0} \frac{r_{0,d}}{r_{1,d}} = 0 \quad \textit{and} \quad \lim_{d\to 0} \frac{r_{1,d}}{\rho_H^*} = 0.$$

Proof. The first two limits follow directly by the definition of $r_{0,d}$ and $r_{1,d}$. Now note that, by standard computations, we have that for any $m \in \mathbb{N}$,

$$\lim_{t\to 0} \frac{(m+1)I_m(t)}{t^{m+1}} = 1. \tag{3.6}$$

It follows that

$$\lim_{d\to 0} \frac{r_{1,d}}{|d|^{1/n}} = \left(\frac{n}{n-1}\right)^{1/n} \quad \text{and} \quad \lim_{d\to 0} \frac{r_{0,d}}{|d|^{1/(n-1)}} = 1.$$

Therefore we get $\lim_{d\to 0} r_{0,d}/r_{1,d} = 0$.

Using again (3.6), $\lambda_{(n-1)/n,d}(r_{1,d})$ can be estimated for $|d|$ very small as follows:

$$\lambda_{(n-1)/n,d}(r_{1,d}) \le -h^* := \int_{r_{0,d}}^{r_{1,d}} \frac{(n-1)I_{n-1}(t)+d}{\sinh^{n-1}(t)}\,dt \approx \int_{r_{0,d}}^{r_{1,d}} \frac{\frac{n-1}{n}t^n+d}{t^{n-1}}\,dt \tag{3.7}$$

$$= \frac{n-1}{n}(r_{1,d}^2 - r_{0,d}^2) + \frac{d}{2-n}(r_{1,d}^{2-n} - r_{0,d}^{2-n})$$

$$\approx -\frac{1}{n-2}\left(\frac{n-1}{n}\right)^{1/(n-1)} r_{1,d}^{n/(n-1)}, \tag{3.8}$$

By using (3.6) and the definition of h^* in (3.7), when $|d|$ is close to zero we have

$$h^* = \int_0^{\rho_H^*} \frac{nHI_{n-1}(t)}{\sqrt{\sinh^{2n-2}(t) - n^2H^2I_{n-1}^2(t)}}\,dt \approx H\int_0^{\rho_H^*} t\,dt = \frac{H}{2}(\rho_H^*)^2,$$

Together with (3.8) this implies that, as $d \to 0$, $\rho_H^* \approx c_{n,H}\, r_{1,d}^{\frac{n}{2(n-1)}}$, where

$$c_{n,H} = \sqrt{\frac{2}{H(n-2)}}\left(\frac{n-1}{n}\right)^{\frac{1}{2(n-1)}}.$$

Hence the results involving ρ_H^* follow easily. ∎

3.2. Horizontal cylinders

In [2], P. Bérard and R. Sa Earp describe constant mean curvature hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ which are invariant under hyperbolic translations of $\mathbb{H}^n$. Following their notations, for any $m \in \mathbb{N}$, for any $H > 0$ and for suitable choice of the parameter d, we define

$$J_m(t) = \int_0^t \cosh^m(\tau)\,d\tau,$$

$$\mu_{H,d}(\rho) = \int_{\rho_0}^{\rho} \frac{nHJ_{n-1}(t)+d}{\sqrt{\cosh^{2n-2}(t) - (nHJ_{n-1}(t)+d)^2}}\,dt, \tag{3.9}$$

where $\rho_0 \ge 0$ is the infimum of the interval where the integrand function exists.

The graphs of the functions $\mu_{H,d}$ are the generating curves of hypersurfaces constructed as follows. Let γ be a complete geodesic through the origin of the hyperbolic space $\mathbb{H}^n \times \{0\}$, parametrized by the signed distance ρ to the origin. Let π be the hyperplane of $\mathbb{H}^n \times \{0\}$ orthogonal to γ at the origin. Consider the curve $(\rho, \mu_{H,d}(\rho))$ embedded in the plane $\gamma \times \mathbb{R}$. For any ρ, let π_ρ be the vertical translation of π in the slice $\mathbb{H}^n \times \{\mu_{H,d}(\rho)\}$. The desired hypersurface is obtained translating each point $(\rho, \mu_{H,d}(\rho))$ along any geodesic of π_ρ passing through the origin of π_ρ. By [2], such hypersurface has constant mean curvature H.

The case of $n = 2$ has been studied in detail in [6, 8, 10, 12].

In this paper we are only interest in the case $H > (n-1)/n$ and $d = 0$. For any fixed H, we denote with $\mathcal{C}_H$ such hypersurface. The value of n will be clear from the context. The function $\mu_{H,0}$ is defined for $\rho \in [0, R_{\mathcal{C}}]$, where $R_{\mathcal{C}}$ is the unique solution of the equation

$$\cosh^{n-1}(\rho) - nHJ_{n-1}(\rho) = 0. \tag{3.10}$$

When $n = 2$, $\mathcal{C}_H$ was explicitly parametrized by J.M. Manzano in [6]: using the half-space model for $\mathbb{H}^2$, we have that, up to isometries of the ambient space,

$$\mathcal{C}_H(u, v) = \Big(\frac{e^v \sin u}{\sqrt{4H^2-1}}, e^v, \frac{2H}{\sqrt{4H^2-1}} \tan^{-1} \frac{\cos u}{\sqrt{4H^2-1+\sin^2 u}}\Big),$$

where $u, v \in \mathbb{R}$. In this case it is evident that $\mathcal{C}_H$ has the topology of the cylinder and it is a bi-graph on the non-compact domain of $\mathbb{H}^2$ bounded by the two equidistant curves of constant geodesic curvature $-1/(2H)$:

$$\mathcal{C}_H \cap \{t = 0\} = \Big\{\Big(\frac{\pm e^v}{\sqrt{4H^2-1}}, e^v, 0\Big) \mid v \in \mathbb{R}\Big\}.$$

The distance between these two curves is $2R_{\mathcal{C}}$, hence for fixed $v = 0$, we can compute

$$R_{\mathcal{C}} = \frac{1}{2} d_{\mathbb{H}^2}\Big(\Big(\frac{-1}{\sqrt{4H^2-1}}, 1\Big), \Big(\frac{1}{\sqrt{4H^2-1}}, 1\Big)\Big) = \frac{1}{2} \ln\Big(\frac{2H+1}{2H-1}\Big). \tag{3.11}$$

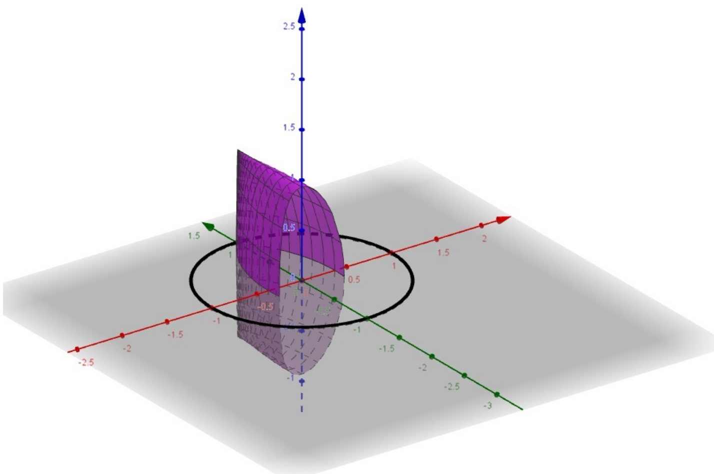

Figure 3. Horizontal cylinder $\mathcal{C}_H$ in $\mathbb{H}^2 \times \mathbb{R}$ in the disk model, $H = 0.77$.

When $n \geq 3$, we do not have an explicit parametrization for $\mu_{H,0}$, but part 1 of Theorem 2.4 in [2] can be easily extended to the case $H > (n-1)/n$ proving that $h_{\mathcal{C}} := \lim_{\rho \to R_{\mathcal{C}}} \mu_{H,0}(\rho)$ is finite, and that $\mu_{H,0}$ is a strictly increasing and convex function with a vertical tangent at $\rho = R_{\mathcal{C}}$. Therefore the reflections of $\mu_{H,0}$ with respect to $\rho = 0$ and $t = h_{\mathcal{C}}$ produce a compact strictly convex simple curve. It follows that the hypersurface $\mathcal{C}_H$ is embedded and it is symmetric with respect to a horizontal hyperplane, and the parts above and below this hyperplane are vertical graphs. Moreover, $\mathcal{C}_H$ has the topology of $\mathbb{S}^1 \times \mathbb{R}^{n-1}$. Note that $\lim_{H\to(n-1)/n} R_{\mathcal{C}}$ is infinite for $n = 2$ and finite otherwise.

We conclude this section comparing the radius of the cylinder $\mathcal{C}_H$ and of the compact sphere $\mathcal{S}_H$ with the same mean curvature.

Lemma 3.3. *For any $n \geq 2$ and for any $H > (n-1)/n$, we have*

$$R_{\mathcal{S}} > R_{\mathcal{C}},$$

where $R_{\mathcal{S}}$ is the solution of equation (3.2) *and $R_{\mathcal{C}}$ is the solution of equation* (3.10).

Proof. We start by introducing the following iteration formula taken from [2]:

$$\begin{aligned} J_0(x) &= x; \quad J_1(x) = \sinh(x); \\ mJ_m(x) &= \sinh(x)\cosh(x)^{m-1} + (m-1)\,J_{m-2}(x), \quad \forall m \geq 2; \end{aligned} \tag{3.12}$$

Notice that $J_m(x) > 0$ for any m and any $x > 0$. Fix n and H as in the statement. Let us define the real functions

$$\varphi(x) = \sinh^{n-1}(x) - nHI_{n-1}(x) \quad \text{and} \quad \psi(x) = \cosh^{n-1}(x) - nHJ_{n-1}(x).$$

By trivial arguments, we have that the unique strictly positive critical point of φ is

$$\tilde{R} := \tanh^{-1}\left(\frac{n-1}{nH}\right),$$

and that $\tilde{R} < R_{\mathcal{S}}$.

When $n = 2$, by (3.11), we have that $\tilde{R} = R_{\mathcal{C}}$, and the result holds.

Now let $n \geq 3$. By (3.12) and definition of $\tilde{R}$, we have that

$$\begin{aligned} \psi(\tilde{R}) &= \cosh^{n-1}(\tilde{R}) - \frac{nH}{n-1}\left(\sinh(\tilde{R})\cosh^{n-2}(\tilde{R}) + (n-2)J_{n-3}(\tilde{R})\right) \\ &< \cosh^{n-2}(\tilde{R})\left(\cosh(\tilde{R}) - \frac{nH}{n-1}\sinh(\tilde{R})\right) = 0 = \psi(R_{\mathcal{C}}), \end{aligned}$$

where, in the inequality, we use that $J_{n-3} > 0$. Therefore $R_{\mathcal{C}} < \tilde{R}$ holds because ψ is strictly decreasing. ∎

4. Main theorem

In this section, we prove the generalization of the Ros–Rosenberg theorem, stated in the introduction.

A closed hypersurface Γ in the hyperbolic space $\mathbb{H}^n$ is said *horoconvex* if all the principal curvatures of Γ are strictly larger than 1. Given a horoconvex hypersurface Γ, we call the exterior (respectively, interior) radius of Γ the minimum (respectively, maximum) of the radii ρ such that for any $p \in \Gamma$ there exists a geodesic sphere S with radius ρ tangent to Γ in p and such that Γ lies in (respectively, encloses) the closed disk bounded by S. We denote by r_{ext} the exterior radius and by r_{int} the interior radius. It is clear that, for any Γ, we have $r_{\text{ext}} \geq r_{\text{int}}$ and that the equality holds if and only if Γ is a geodesic sphere of radius r_{ext}.

The main result of the paper is the following.

Theorem 4.1. *Let M be a n-dimensional compact hypersurface of $\mathbb{H}^n \times \mathbb{R}$ embedded in $\mathbb{H}^n \times [0, \infty[$ with constant mean curvature $H > (n-1)/n$ and boundary $\partial M = \Gamma$. Assume that Γ is a closed $(n-1)$-dimensional horoconvex hypersurface of the slice $P = \mathbb{H}^n \times \{0\}$ satisfying the pinching $2r_{\text{int}} > r_{\text{ext}}$. Then there is a constant $\delta(n, H) > 0$, depending only on n and H, such that if $r_{\text{ext}} \leq \delta(n, H)$, then M is topologically a disk. Moreover, either M is a graph over the domain Ω bounded by Γ, or $N = M \cap (\Omega \times]0, \infty[)$ is a graph over Ω and $M \setminus N$ is a graph over $\Gamma \times]0, \infty[$ with respect to the lines orthogonal to $\Gamma \times]0, \infty[$.*

Remark 4.2. (1) In the case Γ is a geodesic sphere and M satisfies the assumption of Theorem 4.1, it is easy to prove that M is rotationally symmetric, by using Alexandrov reflections with respect to vertical hyperplanes. Hence M is a portion of a vertical translation of the hypersurface $\mathcal{S}_H$, defined in Section 3.1.

(2) Notice that a pinching assumption for the boundary of constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is also considered in [4].

From now on, we suppose that Γ is not a geodesic sphere, hence $r_{\text{ext}} > r_{\text{int}}$. The strategy of the proof of Theorem 4.1 is inspired by the one in [9,13], and it is divided into two cases according to the following definition.

Definition 4.3. A hypersurface M of $\mathbb{H}^n \times \mathbb{R}$ with constant mean curvature $H > (n-1)/n$ is called *short* if it is contained between two slices at distance smaller than the height of the cylinder $\mathcal{C}_H$ with the same mean curvature, defined in Section 3.2. M is called *tall* otherwise.

When M is short, the proof of Theorem 4.1 is more direct because shortness yields that M is a graph over Ω. When M is tall, we will prove that M is a union of pieces, each one graph in some system of coordinates.

In [11], the authors introduced the notion of small surface with constant mean curvature H: a hypersurface contained in a ball of mean curvature larger than H. Here, the vertical and the horizontal directions are not homogeneous, hence the suitable notion is that of short hypersurface.

The first step, common to both cases short and tall, is to show that in a small vertical cylinder, M is a graph. This will be proved in the following lemma, that is the analogous of Lemma 3 of [11]. Differently from [11], we need a quantitative estimate of the radius of the cylinder, that will be evaluated using the limaçon described in Section 2 .

In the following, we let W be the domain in $\mathbb{H}^n \times \mathbb{R}$ bounded by M and Ω, and we will denote by $D(\rho)$ any disk with radius ρ in a slice $\mathbb{H}^n \times \{t\}$, where the value of t and the center of $D(\rho)$ will be clear from the context.

Lemma 4.4. *Let M and Γ be as in the statement of Theorem 4.1. Then, there exists a disk $D(r_{\min})$ in $\mathbb{H}^n \times \{0\}$, where $r_{\min}$ depends only of the principal curvatures of $\Gamma = \partial M$, such that $M \cap (D(r_{\min}) \times \mathbb{R})$ is a graph. In particular, since Γ satisfies the pinching $2r_{\text{int}} > r_{\text{ext}}$, then $2r_{\text{int}} - r_{\text{ext}} < r_{\min} < r_{\text{int}}$ holds.*

Proof. Consider the Alexandrov reflection with horizontal hyperplanes coming down. If we can arrive to $P := \mathbb{H}^n \times \{0\}$ without having a contact point between M and its reflection, then M is a graph over Ω and the result holds. Otherwise there is a height $t_0 > 0$ where the reflected hypersurface touches Γ for the first time. Let $q \in \Gamma$ be the first touching point. So $\{q\}\times]0,\infty)$ intersects M exactly once and $\{q\} \times (0, 2t_0) \subset \text{int}(W)$. Notice that the part of M above $\mathbb{H}^n \times \{t_0\}$ is a vertical graph.

Now let v be a unit horizontal vector and Q_v be a vertical hyperplane orthogonal to v such that $Q_v \cap M = \emptyset$. Fix any $x \in Q_v$ and let γ_v be the geodesic passing through x with tangent vector v. We will do an Alexandrov reflection with the family of hyperplanes orthogonal to γ_v. Note that these hyperplanes are parallel by construction. We can move Q_v parallel to itself along γ_v until it touches M for the first time.

Keep moving Q_v (by abuse of notation we call Q_v any parallel translated of it) and let M^* be the reflection, across Q_v, of the part of M behind Q_v. Continue moving Q_v until there is a first touching point between M and M^*. Since for any $p \in \Gamma$ the domain Ω bounds a disk of radius r_{int} tangent at p to Γ, then we could do Alexandrov reflections at least until Q_v is at distance r_{int} from $p \in \Gamma$. In fact, let $p' \in \Gamma$ behind a Q_v, where Q_v is at distance less than r_{int} from $p \in \Gamma$. Then the disk of radius r_{int} tangent to Γ at p' is contained in Ω and its center is on the side of Q_v not containing p. Hence, the reflection of p' is inside the disk, and then it is inside Ω.

In order to avoid the dependence on the point q, we stop to do reflection earlier, precisely when Q_v becomes tangent to $\mathcal{C}$, where $\mathcal{C}$ is defined in the following way: let $\mathcal{C}_{\text{ext}}$ be the geodesic sphere of P with radius r_{ext} and tangent to Γ in q which encloses Γ; then $\mathcal{C}$ is the geodesic sphere with the same center of $\mathcal{C}_{\text{ext}}$ and radius $r_{\text{ext}} - r_{\text{int}}$.

For any unit horizontal vector v, let q_v be the reflection of the point $q \in P$ with respect to Q_v tangent to $\mathcal{C}$. Let $\mathcal{L}$ be set of all such points q_v. Since Q_v is vertical and $Q_v \cap P$ is a totally geodesic hyperplane of the hyperbolic space P, then $q_v \in P$ and $\mathcal{L}$ is a hyperbolic limaçon as in Definition 2.1 with base point q and parameters $a = r_{\text{ext}}$ and $c = r_{\text{ext}} - r_{\text{int}}$. Since $a > c$, $\mathcal{L}$ has two loops by Lemma 2.4. Moreover, as shown in the proof of Lemma 2.5, the smaller loop of $\mathcal{L}$ is bounded by the sphere of radius $a - c = r_{\text{int}}$, tangent to Γ in q, and such sphere is contained in Ω.

Furthermore, for any p in the smaller loop of $\mathcal{L}$, the vertical rectangle $\overline{qp} \times (0, 2t_0) \subset \text{int}(W)$. The result will follow taking $r_{\min}$ as the largest radius of a disk bounded by the smaller loop of $\mathcal{L}$. Note that $r_{\min}$ depends only on a and c, i.e., only on the curvature of Γ, and not on q.

Finally, since Γ satisfies the pinching $2r_{\text{int}} > r_{\text{ext}}$, by Lemma 2.5 it holds that

$$2r_{\text{int}} - r_{\text{ext}} < r_{\min} < r_{\text{int}}. \qquad \blacksquare$$

Remark 4.5. By the construction described in the proof above of Lemma 4.4, and by Lemma 2.5 (2), the centers of $\mathcal{C}_{\text{ext}}$, $\mathcal{C}$ and $D(r_{\min})$ coincide. Up to isometries, we can suppose that it is $(\sigma, 0) \in \mathbb{H}^n \times \mathbb{R}$, where σ is the origin of $\mathbb{H}^n$.

Proof of Theorem 4.1. The strategy is to prove that M is the union of components, each graphical above some domain. At the end of the proof, it will be clear that this determines the fact that M is topologically a disk. Denote by h_M the maximal height of M above the plane $P := \mathbb{H}^n \times \{0\}$. Fix any point $A \in \Gamma$ and consider a disk $D(r_{\text{ext}})$ of radius r_{ext} tangent to Γ at A. We divide the proof into two cases, depending on whether M is tall or short, as defined in Definition 4.3.

Case 1 (*M* short): $h_M < 2h_{\mathcal{C}}$.

Claim 1A. *M is contained in $D(r_{\text{ext}} + R_{\mathcal{C}}) \times [0, 2h_{\mathcal{C}}[$.*

Consider the horizontal cylinder $\mathcal{C}_H$ and let π be hyperplane defined in Section 3.2.

Denote by Π the vertical hyperplane containing π. Denote by $\mathcal{C}_H^+$ the intersection of $\mathcal{C}_H$ with one of the two halfspaces determined by Π. Notice that $\partial\mathcal{C}_H^+$ is the union of two hyperplanes, each contained in a slice. Up to a vertical translation, we may assume that the lower boundary is on the slice $t = 0$ and the upper boundary is in the slice $t = 2h_{\mathcal{C}}$. By abuse of notation, we will call $\mathcal{C}_H^+$ any horizontal translation and any horizontal rotation of $\mathcal{C}_H^+$. Since M is compact, we can translate horizontally $\mathcal{C}_H^+$ in such a way that $M \cap \mathcal{C}_H^+ = \emptyset$ and M lies in the side of $\mathcal{C}_H^+ \cap (\mathbb{H}^n \times [0, 2h_{\mathcal{C}}])$ which contains the axis of $\mathcal{C}_H$. Then, we translate $\mathcal{C}_H^+$ towards M and by the maximum principle, $\mathcal{C}_H^+$ and M cannot meet at an interior point. As $h_M < 2h_{\mathcal{C}}$, one can translate $\mathcal{C}_H^+$ till its lower boundary on $t = 0$ touches the boundary of $D(r_{\text{ext}})$. The same can be done for $\mathcal{C}_H^+$ with any horizontal axis in the slice $t = h_{\mathcal{C}}$, hence M is contained in $D(r_{\text{ext}} + R_{\mathcal{C}}) \times [0, 2h_{\mathcal{C}}[$.

Claim 1B. *For Γ sufficiently small, M is contained in the vertical cylinder above Ω.*

By Lemma 3.3, we know that, for any $n \geq 2$ and any $H > (n-1)/n$, $R_{\mathcal{S}} > R_{\mathcal{C}}$ holds. Choose Γ small enough such that $R_{\mathcal{S}} > r_{\text{ext}} + R_{\mathcal{C}}$. Let $\mathcal{S}_H$ be the sphere with constant mean curvature H. Translate it vertically such that $\mathcal{S}_H \subset \mathbb{H}^n \times [-h_{\mathcal{S}}, h_{\mathcal{S}}]$ and denote by $\mathcal{S}_H^+$ the part contained in $t > 0$. Translate $\mathcal{S}_H^+$ vertically so that it is above M. Then translate $\mathcal{S}_H^+$ down. By the maximum principle, there can not be an interior contact point between M and the translation of $\mathcal{S}_H^+$. Moreover, as $M \subset D(r_{\text{ext}} + R_{\mathcal{C}}) \times [0, 2h_{\mathcal{C}}[$ and $R_{\mathcal{S}} > r_{\text{ext}} + R_{\mathcal{C}}$, the boundary of $\mathcal{S}_H^+$ will not meet M before coming back to $t = 0$. Hence M is below $\mathcal{S}_H^+$. As $R_{\mathcal{S}} > r_{\text{ext}}$, by translating horizontally $\mathcal{S}_H^+$ one can touch all the points of Γ with $\mathcal{S}_H^+ \cap \{t = 0\}$. By the maximum principle, M stays below all such translations of $\mathcal{S}_H^+$. That is, M is contained in the vertical cylinder above Ω.

This concludes the proof of Theorem 4.1 in the case M short. In fact, since Claim 1B holds, using Alexandrov reflections with horizontal hyperplanes, it is easy to prove that M is a graph over Ω, hence M is topologically a disk. Moreover, we have that $\delta(n, H) = R_{\mathcal{S}} - R_{\mathcal{C}}$. Notice that, in the case M short, we do not use the assumption on the pinching for Γ.

Case 2 (*M* tall): $h_M \geq 2h_{\mathcal{C}}$.

Using Alexandrov reflections with horizontal and vertical hyperplanes, one gets that the part of M above the plane $t = h_M/2$ is a graph as well as the part of M outside the cylinder over Ω. Then, we have only to understand the topology of $M \cap \Omega \times [0, h_M/2]$. In what follows we will show that there is no point of M in $\Omega \times [h^*, h_M/2]$. Actually, we get event more: there is no point of M in $D(R) \times [h^*, h_M/2]$, where R will be fixed later and $\Omega \subset D(R)$. Then, we use the latter to prove that there is no interior point of M in $\Omega \times [0, h^*]$.

The bound $\delta(n, H)$ on the size of Γ will be determined by a careful choice of the parameter d of the family of immersed annuli $\mathcal{A}_d$ described in Section 3.1. Let us explain first how the choice of d affects the other quantities involved in the proof. Fix any $d < 0$ such that

$$2r_{\text{int}} - r_{\text{ext}} < r_{0,d} < r_{\min}, \tag{4.1}$$

where $r_{0,d}$ is the solution of the equation (3.3) and $r_{\min}$ is the radius found in Lemma 4.4. The reasons of the bounds in (4.1) will be clear in the following. The choice of d determines the hypersurface $\mathcal{A}_d$ discussed in Section 3.1, together with the radius $r_{1,d}$, i.e., the solution of equation (3.4). Consequently, the height h^* defined in (3.7), and ρ_H^*, the radius of the *spherical cap* of $\mathcal{S}_H$ of height h^*, i.e., the solution of equation (3.5), depend on the choice of d as well.

We point out that the choice of d is determined by Γ through the inequality (4.1), in particular if $r_{\text{ext}} \to 0$, which means that Γ shrinks to a point, then $d \to 0$. Moreover, when $d \to 0$, then all the radii and height h^* tend to zero by (3.8) and Lemma 3.2. It follows that if r_{ext} is small enough, we have $h^* \ll 2h_{\mathcal{C}}$. Furthermore, since Γ is compact and pinched, then there exists an $\varepsilon > 0$ such that $2r_{\text{int}} \geq (1+\varepsilon)r_{\text{ext}}$. By Lemma 4.4, we have

$$\varepsilon r_{\text{ext}} \leq 2r_{\text{int}} - r_{\text{ext}} < r_{0,d}$$

hence $\rho_H^*/r_{\text{ext}} > \varepsilon \rho_H^*/r_{0,d}$. Finally, taking d, and hence Γ, smaller if necessary, by Lemma 3.2 one has

$$\rho_H^* > 3r_{\text{ext}}. \tag{4.2}$$

Claim 2A. *The compact domain bounded by $M \cap \{t = h_M - h^*\}$ contains a geodesic segment of length at least ρ_H^*.*

Using Alexandrov reflections with respect to horizontal hyperplanes, we can prove that the reflections with respect to $\{t = h_M/2\}$ of the points at maximal height of M belong to the closure of Ω. Up to horizontal isometries, we can suppose that one of these points belongs to the t-axis, that is, it is of the form (σ, h_M), where σ was defined in Remark 4.5. The claim is proved once we show that there is a point $p \in M$ on the hyperplane $\{t = h_M - h^*\}$, at distance at least ρ_H^* from the t-axis. Denote by M' the part of M above the plane $\{t = h_M - h^*\}$. M' is a graph of height h^*. Assume, by contradiction, that the distance between $\partial M'$ and the t-axis is smaller than ρ_H^*. Cut $\mathcal{S}_H$ with a suitable horizontal hyperplane such that $\mathcal{S}_H'$, the spherical cap above this plane, has height h^*. Translate $\mathcal{S}_H'$ up until $\mathcal{S}_H' \cap M = \emptyset$, then move it down. By the maximum principle, there is no interior contact point between the spherical cap and M' till the boundary of $\mathcal{S}_H'$ reaches the height $t = h_M - h^*$. Then, M' has height less than h^*. This is a contradiction.

Claim 2B. *The compact domain bounded by $M \cap \{t = h_M - h^*\}$ bounds a disk $D(R)$ with $R > \rho_H^* - 3r_{\text{ext}}$ and center $(\sigma, h_M - h^*)$.*

Enclose Γ with the geodesic sphere $\mathcal{C}_{\text{ext}}$ defined in Lemma 4.4, and reflect the point p found in Claim 2A with respect to any vertical hyperplane tangent to $\mathcal{C}_{\text{ext}}$ in $\mathbb{H}^n \times \mathbb{R}$. In this way we have a hyperbolic limaçon $\mathcal{L}$ in $\{t = h_M - h^*\}$ defined by the base point p and the vertical translation of $\mathcal{C}_{\text{ext}}$ at height $h_M - h^*$. Therefore, the parameters of $\mathcal{L}$ are $a > \rho_H^* - r_{\text{ext}}$ and $c = r_{\text{ext}}$. In fact, the distance between p and $(\sigma, h_M - h^*)$ is larger than ρ_H^*, where σ was defined in Remark 4.5. Estimate (4.2) implies that $a > 2c > c$, hence, by Lemma 2.4, $\mathcal{L}$ has two loops. Arguing as in the proof of Lemma 4.4, we have that the smaller loop is contained in W. The claim follows by Lemma 2.5 and Remark 4.5.

Claim 2C. $M \cap \{D(R) \times [h^*, h_M - h^*]\} = \emptyset$.

By Claim 2B, $D(R) \subset W$, and by our choice of $h^* \ll h_{\mathcal{S}}$, the plane $\{t = h_M - h^*\}$ is above the plane $\{t = h_M/2\}$. By doing Alexandrov reflections with horizontal planes, the reflection $D^*(R)$ of $D(R)$ with respect to $\{t = \tau\}$ will be contained in W, for all $\tau \in [h_M/2, h_M - h^*]$. Therefore, $M \cap \{D(R) \times [h^*, h_M - h^*]\} = \emptyset$.

Claim 2D. $M \cap \{0 \le t \le h^*\}$ *is outside the cylinder* $\Omega \times \{0 \le t \le h^*\}$.

Denote by Σ the embedded part of the annulus $\mathcal{A}_d$ contained in $(D(r_{1,d}) \setminus D(r_{0,d})) \times [0, h^*]$. By construction, the hypersurface Σ has two boundary components, denoted by C_0 and C_1, both geodesic hyperspheres in the hyperbolic space. In particular, $C_0 \subset P$ and $C_1 \subset \{t = h^*\}$. By inequality (3.7), we have that the radius of C_1 is smaller than $r_{1,d}$. Moreover, up to horizontal translations, we can suppose that the center of C_0 is $(\sigma, 0)$. By the pinching of Γ and Lemma 3.2, we can take Γ small enough such that $R > r_{1,d}$ holds, where R is the radius found in Claim 2B. Therefore, by Claim 2C, we can translate vertically Σ such that $\Sigma \subset W$. We recall that our choice in (4.1) yields $r_{0,d} < r_{\min} \le r_{\text{int}}$ and that the mean curvature of Σ is strictly smaller than that of M. By the maximum principle, it follows that we can translate Σ down, until C_0 reaches P again, without having an interior contact point between Σ and M. Moreover, by translating C_0 in Ω, we can touch every point of Γ, while C_0 remains in Ω. Furthermore, by Claim 2B, the pinching of Γ and Lemma 3.2, we can take $|d|$ small enough such that

$$R > r_{\text{ext}} + r_{1,d} - r_{0,d}. \tag{4.3}$$

In fact, by Claim 2B we have that $R > \rho_H^* - 3r_{\text{ext}}$ and $\rho_H^* - 3r_{\text{ext}} > r_{\text{ext}} + r_{1,d} - r_{0,d}$ holds because, when $d \to 0$, the left-hand side tends to zero slower than the right-hand side.

Claim 2C and (4.3) imply that when translating C_0 inside Ω, the circle C_1 remains inside the disk $D^*(R) \subset \{t = h^*\}$, hence the upper boundary of Σ will not touch M. Notice also that Σ is a vertical graph over the exterior of $D(r_{0,d})$, hence, translating horizontally C_0, Σ and M cannot meet at an interior point otherwise we would have a contradiction with the maximum principle. Therefore $M \cap (\Omega \times [0, h^*]) \neq \emptyset$.

This completes the proof of Theorem 4.1. ■

A. Appendix: list of notations

To simplify the reading we summarize the principal notations introduced.

(1) Hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$:

$\mathcal{S}_H$: compact rotationally symmetric hypersurface with constant mean curvature $H > (n-1)/n$, see Section 3.1;

$\mathcal{S}_{(n-1)/n}$: complete rotationally symmetric entire graph with constant mean curvature $H = (n-1)/n$, see Section 3.1;

$\mathcal{A}_d$: rotationally symmetric annulus with self-intersection and constant mean curvature $H = (n-1)/n$, see Section 3.1;

$\mathcal{C}_H$: horizontal cylinder with constant mean curvature $H > (n-1)/n$, see Section 3.2.

(2) Heights:

$h_{\mathcal{S}}$: height of the half sphere $\mathcal{S}_H$, see Section 3.1;

h^*: approximated value of the height of the portion of $\mathcal{A}_d$ between $r_{0,d}$ and $r_{1,d}$, see (3.7);

$h_{\mathcal{C}}$: height of half of the cylinder $\mathcal{C}_H$, see Section 3.2;

h_M: height of M.

(3) Radii:

$R_{\mathcal{S}}$: radius of $\mathcal{S}_H$, it is the unique solution of (3.2);

$r_{0,d}$: the minimum radius for which $\mathcal{A}_d$ is defined, it is the unique solution of (3.3);

$r_{1,d}$: the radius for which $\mathcal{A}_d$ has horizontal tangent plane, it is the unique solution of (3.4);

ρ_H^*: radius of the spherical cap of $\mathcal{S}_H$ with height h^*, it is the unique solution of equation (3.5);

$R_{\mathcal{C}}$: radius of $\mathcal{C}_H$, it is the unique solution of (3.10);

r_{int}: interior radius of Γ;

r_{ext} : exterior radius of Γ;

$r_{\min}$: the minimum radius on which M is a graph, it is determined by Lemma 4.4.

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