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On the rational approximation to p-adic Thue–Morse numbers

Yann Bugeaud

Abstract. Let p be a prime number and let ξ be an irrational p-adic number. Its multiplicative irrationality exponent $\mu^{\times}(\xi)$ is the supremum of the real numbers $\mu^{\times}$ for which the inequality $|b\xi - a|_p \le |ab|^{-\mu^{\times}/2}$ has infinitely many solutions in nonzero integers a and b. We show that $\mu^{\times}(\xi)$ can be expressed in terms of a new exponent of approximation attached to a sequence of rational numbers defined in terms of ξ. We establish that $\mu^{\times}(\xi_{\mathbf{t},p}) = 3$, where $\xi_{\mathbf{t},p}$ is the p-adic number $1 - p - p^2 + p^3 - p^4 + \cdots$, whose sequence of digits is given by the Thue–Morse sequence over $\{-1, 1\}$.

1. Introduction

Throughout this paper, we let p denote a prime number. Let ξ be an irrational p-adic number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ for which

$$0 < |b\xi - a|_p \le \max\{|a|, |b|\}^{-\mu} \tag{1.1}$$

has infinitely many solutions in nonzero integers a and b. As it has been already pointed out in [1, 10, 15], unlike in the real case, the integers $|a|$ and $|b|$ in (1.1) do not necessarily have the same order of magnitude, and one of them can be much larger than the other one. This has motivated the study in [10] of the following two exponents of p-adic multiplicative rational approximation.

Definition 1.1. Let ξ be an irrational p-adic number. The multiplicative irrationality exponent $\mu^{\times}(\xi)$ of ξ is the supremum of the real numbers $\mu^{\times}$ for which

$$0 < |b\xi - a|_p \le (|ab|^{1/2})^{-\mu^{\times}} \tag{1.2}$$

has infinitely many solutions in integers a, b. The uniform multiplicative irrationality exponent $\widehat{\mu}^{\times}(\xi)$ of ξ is the supremum of the real numbers $\widehat{\mu}^{\times}$ for which the system

$$0 < |ab|^{1/2} \le X, \quad |b\xi - a|_p \le X^{-\widehat{\mu}^{\times}} \tag{1.3}$$

has a solution in integers a, b for every sufficiently large real number X.

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We point out that a and b are not assumed to be coprime in (1.1), (1.2), nor in (1.3). Adding this assumption would not change the values of $\mu(\xi)$ and $\mu^\times(\xi)$, but would change the value of the uniform exponent $\widehat{\mu}^\times$ at some p-adic numbers ξ.

It follows from the Minkowski theorem (see [12, 13]) and the obvious inequality $\max\{|a|, |b|\} \le |ab|$, valid for all nonzero integers a and b, that we have

$$2 \le \mu(\xi) \le \mu^\times(\xi) \le 2\mu(\xi). \tag{1.4}$$

Inequalities (1.4) are best possible; see [10]. Furthermore, Theorem 3.1 in [10] asserts that

$$2 \le \widehat{\mu}^\times(\xi) \le \frac{5+\sqrt{5}}{2}, \tag{1.5}$$

for every irrational p-adic number ξ, while there exist p-adic numbers ξ with $\widehat{\mu}^\times(\xi) = 3$, an example being given by the p-adic Liouville number $\sum_{j\ge1} p^{j!}$; see [10] for additional results.

It is readily verified that $\mu^\times(\xi) = \mu^\times(u\xi/v)$ and $\widehat{\mu}^\times(\xi) = \widehat{\mu}^\times(u\xi/v)$ hold for every irrational p-adic number ξ and every nonzero rational number u/v. However, the exponents $\mu^\times$ and $\widehat{\mu}^\times$ are not invariant by rational translations. To see this, observe that $|b\xi - a|_p = |b(\xi+1) - (a+b)|_p$, while the product $|b(a+b)|$ is much larger than the product $|ab|$ when $|b|$ exceeds $|a|$. Consequently, $\mu^\times(\xi+1)$ may be strictly smaller than $\mu^\times(\xi)$.

The purpose of the present paper is to establish a somehow unexpected link between p-adic multiplicative approximation and sequences of continued fractions of rational numbers. The determination of $\mu^\times(\xi)$ and $\widehat{\mu}^\times(\xi)$ then boils down to the study of the size of the partial quotients of an infinite sequence of rational numbers. As an example of application, we determine the exact values of $\mu^\times(\xi_{\mathbf{t},p})$ and $\widehat{\mu}^\times(\xi_{\mathbf{t},p})$, where $\xi_{\mathbf{t},p}$ is the p-adic Thue–Morse number over $\{-1, 1\}$, and lower bounds for $\mu^\times(\xi_{\mathbf{f},p})$ and $\widehat{\mu}^\times(\xi_{\mathbf{f},p})$, where $\xi_{\mathbf{f},p}$ is the p-adic Fibonacci number over $\{0, 1\}$.

2. Approximation to p-adic Thue–Morse numbers

The Thue–Morse sequence $(t_n)_{n\ge0}$ over $\{-1, 1\}$ is defined by $t_0 = 1$ and the recursion $t_{2n} = t_n, t_{2n+1} = -t_n$ for $n \ge 0$. The first letters of the Thue–Morse infinite word $\mathbf{t} = t_0 t_1 t_2 \ldots$ are then $1, -1, -1, 1, -1, \ldots$ Said differently, $\mathbf{t}$ is the fixed point starting by 1 of the substitution τ defined by $\tau(1) = 1, -1$ and $\tau(-1) = -1, 1$ (here and in the beginning of the proof of Proposition 6.1, we use commas to separate the letters of a word, for sake of readability). Diophantine properties of real numbers whose g-ary expansion, for some integer $g \ge 2$, is given by the Thue–Morse sequence over some alphabet have been investigated in [2, 3, 7, 9]. In the present work, we study multiplicative rational approximations to the p-adic Thue–Morse number $\xi_{\mathbf{t},p}$ defined by

$$\xi_{\mathbf{t},p} = \sum_{n\ge0} t_n p^n = 1 - p - p^2 + p^3 - p^4 + p^5 + p^6 - p^7 + \cdots$$

Since $\mathbf{t}$ is not ultimately periodic, $\xi_{\mathbf{t},p}$ is an irrational number. It has been established in [11] that $\mu(\xi_{\mathbf{t},p}) = 2$. We complement this result as follows.

Theorem 2.1. *The p-adic Thue–Morse number $\xi_{\mathbf{t},p}$ satisfies*

$$\mu^\times(\xi_{\mathbf{t},p}) = 3 \quad \textit{and} \quad \widehat{\mu}^\times(\xi_{\mathbf{t},p}) = 2.$$

More precisely, there exist positive real numbers c_1 and c_2 with the following properties. There exist nonzero integers a and b with $|ab|$ arbitrarily large and

$$|b\xi_{\mathbf{t},p} - a|_p < c_1 |ab|^{-3/2}, \tag{2.1}$$

while, for every nonzero integers x and y, we have

$$|y\xi_{\mathbf{t},p} - x|_p > c_2 |xy|^{-3/2}. \tag{2.2}$$

Admittedly, it would be more natural to consider the p-adic number $\widetilde{\xi}_{\mathbf{t},p}$ whose Hensel expansion is given by the Thue–Morse sequence written over $\{0, 1\}$, namely the p-adic number

$$\widetilde{\xi}_{\mathbf{t},p} = \sum_{n \geq 0} t_n p^n = 1 + p^3 + p^5 + p^6 + \cdots = \frac{\xi_{\mathbf{t},p}}{2} + \frac{1}{2(1-p)} \cdot$$

It is very likely that the conclusion of Theorem 2.1 holds for $\widetilde{\xi}_{\mathbf{t},p}$ and, more generally, for every p-adic number of the form $u\xi_{\mathbf{t},p} + v$, where u and v are rational numbers with u nonzero. The difficulty lies in the control of $|b(u\xi_{\mathbf{t},p} + v) - a|_p$ when $|b|$ exceeds $|a|$; see at the end of Section 7 for a short discussion. In principle, the same method based on Hankel determinants and Padé approximants could be used, but new non-vanishing results for Hankel determinants are needed.

The strategy of the proof is the following. The easiest part, done in Section 4, consists in using repetitions in the Thue–Morse word to exhibit an infinite family of integer pairs (a, b) realizing (2.1). Then, to prove that there are no better approximations, we apply an idea of Mahler to associate with every solution to

$$|b\xi_{\mathbf{t},p} - a|_p < \frac{1}{2|ab|}$$

an integer m and a large partial quotient of the rational number

$$z_m = z_{m,p} = \frac{t_{m-1}}{p} + \frac{t_{m-2}}{p^2} + \cdots + \frac{t_0}{p^m} \cdot \tag{2.3}$$

This allows us to transform a Diophantine question on p-adic numbers into a Diophantine question on a sequence of rational numbers. As a consequence, in order to establish (2.2), it is sufficient to prove that no z_m has a partial quotient exceeding some absolute constant times $p^{m/3}$; see Proposition 3.5. Furthermore, we derive from an easy relation between the rational numbers z_m and z_n, for distinct integers m and n, that the quotient of the largest partial quotient of z_m by that of z_n is, roughly speaking, bounded from above by $p^{|m-n|}$ and from below by $p^{-|m-n|}$. Consequently, it is sufficient for our purpose to prove that, for every large integer k, neither z_{2^k}, nor $z_{3 \cdot 2^k}$ have very large partial quotients (apart, in the case of $z_{3 \cdot 2^k}$, from the one coming from a solution to (2.1)). Very good bounds for the partial quotients of z_{2^k} have been obtained in [8]; see Theorem 6.2 below. Regarding $z_{3.2^k}$, we use a similar argument as in [7] to get in Section 5 an upper bound for its partial quotients strong enough for our purpose. The proof of Theorem 2.1 is completed in Section 6. We discuss the case of the p-adic Fibonacci number in Section 7, while the last section is devoted to additional comments and alternative proofs of some results of [10].

Throughout this text, the constants implied by $\ll$ and $\asymp$ are positive and absolute, and those implied by $\ll_c$ and $\asymp_c$ are positive and depend at most on the parameter c.

3. Diophantine exponents associated with sequences of rational numbers

In this section, we use an idea of Mahler (see [14], pp. 64–67) to make a link between the multiplicative Diophantine exponents of a p-adic number ξ and new exponents of approximation associated with an infinite sequence of rational numbers defined by means of the Hensel expansion of ξ or some other expansion $c_0 + c_1 p + c_2 p^2 + \cdots$ of ξ, where $(c_k)_{k\geq 0}$ is a bounded sequence of integers.

Definition 3.1. Let $g \geq 2$ be an integer. Let $\mathbf{c} = (c_k)_{k\geq 0}$ be a bounded sequence of integers. For $m \geq 1$, define

$$x_m = \frac{c_0}{g^m} + \frac{c_1}{g^{m-1}} + \cdots + \frac{c_{m-1}}{g},$$

and let η_m denote the real number such that the largest partial quotient of x_m is equal to $g^{\eta_m m}$. Put

$$\eta_g(\mathbf{c}) = \limsup_{m\to+\infty} \eta_m \quad \text{and} \quad \widehat{\eta}_g(\mathbf{c}) = \liminf_{m\to+\infty} \eta_m.$$

Said differently, $\eta_g(\mathbf{c})$ (respectively, $\widehat{\eta}_g(\mathbf{c})$) denotes the supremum of the real numbers η such that, for arbitrarily large m (respectively, for every m large enough), the rational number x_m has a partial quotient greater than $g^{\eta m}$.

The exponents η_g and $\widehat{\eta}_g$ take their values in the interval [0, 1]. They can also be defined for unbounded sequences $\mathbf{c}$, but for simplicity we do not discuss this case. In the sequel, we only consider the case where g is a prime number and, when there is no ambiguity, we simply write η and $\widehat{\eta}$, without subscript.

Intuitively, there is no reason for $\eta(\mathbf{c})$ to be equal to $\eta(\mathbf{c}')$ when two sequences $\mathbf{c}$ and $\mathbf{c}'$ differ only by their first term, since x_m may well have a very large partial quotient, while $x_m + 1/g^m$ has none. Said differently, $\mu^\times(\xi)$ and $\mu^\times(\xi + 1)$ may well be different, as we have already noticed.

Throughout the rest of this section, p denotes a prime number and we let $\mathbf{c} = (c_k)_{k\geq 0}$ and $(x_m)_{m\geq 1}$ be as in Definition 3.1 with $g = p$.

We begin with an analysis of the evolution of the partial quotients of the rational numbers x_m. We refer to [6, 16] for an introduction to the theory of continued fractions and use classical results without further notice.

Definition 3.2. Let $[a_0; a_1, a_2, \ldots]$ be a real number. For a positive integer m, we say that the partial quotient a_m and the convergent $[a_0; a_1, a_2, \ldots, a_{m-1}]$ are associated.

Let $m \geq 2$ be an integer and let a/b be a convergent of x_m associated with the partial quotient r. Assume that $r \geq 2p$. It follows from the theory of continued fractions that

$$\frac{1}{(r+2)b^2} < \left| x_m - \frac{a}{b} \right| < \frac{1}{rb^2}.$$

Since $px_m = x_{m-1} + c_{m-1}$, we get

$$\frac{p}{(r+2)b^2} < \left| x_{m-1} - \frac{pa - bc_{m-1}}{b} \right| = p \left| x_m - \frac{a}{b} \right| < \frac{p}{rb^2} \leq \frac{1}{2b^2}. \tag{3.1}$$

By Legendre's theorem, the rational number $(pa - bc_{m-1})/b$ is a convergent of x_{m-1}. If p does not divide b, then it is written under its reduced form and is associated with a partial quotient r' satisfying

$$\frac{1}{(r'+2)b^2} < \left| x_{m-1} - \frac{pa - bc_{m-1}}{b} \right| < \frac{1}{r'b^2},$$

thus, $r/p - 2 < r' < (r+2)/p$, by (3.1). If p divides b, then $(a - bc_{m-1}/p)/(b/p)$ is a convergent of x_{m-1} written under its reduced form and associated with a partial quotient r' satisfying

$$\frac{1}{(r'+2)(b/p)^2} < \left| x_{m-1} - \frac{a - bc_{m-1}/p}{(b/p)} \right| < \frac{1}{r'(b/p)^2},$$

thus, $pr - 2 < r' < p(r+2)$, again by (3.1).

Likewise, since $x_{m+1} = (x_m + c_m)/p$, we get

$$\frac{1}{(r+2)pb^2} < \left| x_{m+1} - \frac{a + bc_m}{bp} \right| = \frac{1}{p} \left| x_m - \frac{a}{b} \right| < \frac{1}{rpb^2} \leq \frac{1}{2(pb)^2}.$$

By Legendre's theorem, the rational number $(a + bc_m)/(bp)$ is a convergent of x_{m+1}. If p does not divide $a + bc_m$, then it is written under its reduced form and is associated with a partial quotient r' satisfying $r/p - 2 < r' < (r+2)/p$. If p divides $a + bc_m$, then $((a + bc_m)/p)/b$ is a convergent of x_{m-1} written under its reduced form and associated with a partial quotient r' satisfying $pr - 2 < r' < p(r+2)$.

Since a and b are coprime, p cannot simultaneously divide b and $a + bc_m$. Consequently, when a/b is a convergent to x_m associated with a partial quotient r greater than $2p$, only three cases can occur, namely:

Case (i): x_{m-1} has a convergent of denominator b associated to a partial quotient in the interval $(r/p - 2, r/p + 1)$ and x_{m+1} has a convergent of denominator bp associated to a partial quotient in the interval $(r/p - 2, r/p + 1)$.

Case (ii): x_{m-1} has a convergent of denominator b associated to a partial quotient in the interval $(r/p - 2, r/p + 1)$ and x_{m+1} has a convergent of denominator b associated to a partial quotient in the interval $(pr - 2, p(r+2))$.

Case (iii): x_{m-1} has a convergent of denominator b/p associated to a partial quotient in the interval $(pr - 2, p(r+2))$ and x_{m+1} has a convergent of denominator bp associated to a partial quotient in $(r/p - 2, r/p + 1)$.

In Case (i), we say that r is a maximal partial quotient of x_m.

The same argument shows that the following holds. Let h be any positive integer such that $r \geq 2p^h$. If p does not divide b and $h < m$, then x_{m-h} has a convergent of denominator b associated to a partial quotient r_{m-h} with

$$rp^{-h} - 2 < r_{m-h} < (r+2)p^{-h}.$$

If p does not divide $a + bc_m$, then x_{m+h} has a convergent of denominator bp^h associated to a partial quotient r_{m+h} with

$$rp^{-h} - 2 < r_{m+h} < (r+2)p^{-h}.$$

Furthermore, if $x_m, x_{m+1}, \ldots, x_{m+h}$ have a convergent of denominator b, then x_{m+h} has a partial quotient r_{m+h} with

$$p^h r - 2 < r_{m+h} < p^h(r+2).$$

Set $r_m = r$ and, for $j > -m$, define $\widetilde{\eta}_{m+j}$ by

$$r_{m+j} = p^{\widetilde{\eta}_{m+j}(m+j)},$$

that is, by setting

$$\widetilde{\eta}_{m+j} = \frac{\log r_{m+j}}{(m+j)\log p}.$$

Note that $\widetilde{\eta}_{m+j}$ may differ from η_{m+j}. Assuming that r_m is a maximal partial quotient of x_m, we get

$$(3.2) \qquad \begin{aligned} (m-h)\widetilde{\eta}_{m-h} &= m\widetilde{\eta}_m - h + o(1), \\ (m+h)\widetilde{\eta}_{m+h} &= m\widetilde{\eta}_m - h + o(1), \end{aligned}$$

for any integer h with $0 \leq h < m\widetilde{\eta}_m$. Here and below, the notation $o(1)$ refers to a quantity that tends to 0 as the indices tend to infinity. This shows that the function $n \mapsto \widetilde{\eta}_n$ increases until $n = m$ and then decreases. In particular, if $\widetilde{\eta}_m > 1/2$, then we get

$$\widetilde{\eta}_{\lfloor m/2 \rfloor} = 2\widetilde{\eta}_m - 1 + o(1) \quad \text{and} \quad \widetilde{\eta}_{\lfloor 3m/2 \rfloor} = \frac{2\widetilde{\eta}_m - 1}{3} + o(1).$$

Let m and n be integers with $m < n$. Inequalities (3.2) imply that lower and upper bounds for the greatest partial quotient of x_u where $m < u < n$ can be expressed in terms of the greatest partial quotients of x_m and x_n. A precise statement is as follows.

Proposition 3.3. *Keep the notation of Definition 3.1. Let m, n be integers with $1 \leq m < n$. For any integer u with $m \leq u \leq n$, we have*

$$\eta_u \leq \frac{n(1+\eta_n) - m(1-\eta_m)}{n(1+\eta_n) + m(1-\eta_m)} + o(1).$$

Furthermore, denoting by $p^{\widetilde{\eta}_m m}$ and $p^{\widetilde{\eta}_n n}$ any partial quotients of x_m and x_n, respectively, we have, for any integer u with $m \leq u \leq n$, the lower bound

$$\eta_u \geq \frac{m(1+\widetilde{\eta}_m) - n(1-\widetilde{\eta}_n)}{m(1+\widetilde{\eta}_m) + n(1-\widetilde{\eta}_n)} + o(1).$$

Proof. We may assume that $p^{\eta_u u}$ is a maximal partial quotient of x_u. It then follows from (3.2) that

$$m\eta_m = u\eta_u - (u-m) + o(1) \quad \text{and} \quad n\eta_n = u\eta_u - (n-u) + o(1).$$

By eliminating u, this gives

$$\frac{m(1-\eta_m)}{1-\eta_u} = \frac{n(1+\eta_n)}{1+\eta_u} + o(1) \quad \text{and} \quad \eta_u \le \frac{n(1+\eta_n) - m(1-\eta_m)}{n(1+\eta_n) + m(1-\eta_m)} + o(1).$$

Likewise, for $u = m+1, \dots, n-1$, it follows from (3.2) that

$$u\eta_u \ge \max\{m\tilde{\eta}_m - (u-m), n\tilde{\eta}_n - (n-u)\} + o(1). \tag{3.3}$$

This maximum attains its minimal value when both quantities are equal, that is, when

$$2u = m(1+\tilde{\eta}_m) + n(1-\tilde{\eta}_n).$$

If this equation has no integer solution, we simply take the integer part, and derive from (3.3) that

$$\eta_u \ge \frac{2m\tilde{\eta}_m + m(1-\tilde{\eta}_m) - n(1-\tilde{\eta}_n)}{m(1+\tilde{\eta}_m) + n(1-\tilde{\eta}_n)} + o(1).$$

This implies the claimed lower bound. ■

The exponents $\mu^\times$ and η are closely related to each other. As usual, $1/0$ means infinity.

Theorem 3.4. *Let p be a prime number and let $\mathbf{c} = (c_k)_{k\ge0}$ be a bounded sequence of integers. The p-adic number $\xi = c_0 + c_1 p + c_2 p^2 + \cdots$ satisfies*

$$\mu^\times(\xi) = \frac{2}{1-\eta(\mathbf{c})} \quad \text{and} \quad \hat{\mu}^\times(\xi) = \frac{2}{1-\hat{\eta}(\mathbf{c})}.$$

We point out that, in the statement of Theorem 3.4, it is not assumed that $\mathbf{c}$ is the Hensel expansion of ξ. We only assume that $\mathbf{c}$ is bounded, but this is mostly by convenience.

Proof. Without any restriction, we assume that c_0 is nonzero. For $m \ge 1$, set

$$C_m = c_0 + c_1 p + c_2 p^2 + \cdots + c_{m-1} p^{m-1}.$$

Let a and b be coprime integers satisfying

$$|ab| \cdot \left|\xi - \frac{a}{b}\right|_p < \frac{1}{2}, \quad b \ge 1. \tag{3.4}$$

Since c_0 is nonzero, p does not divide the product ab. Let m be the positive integer such that $|b\xi - a|_p = p^{-m}$. Since $|\xi - C_m|_p \le p^{-m}$, we deduce that p^m divides $bC_m - a$. Thus, there exists an integer T, which may be divisible by p, such that

$$bC_m - a = p^m T.$$

Furthermore, it follows from (3.4) that $2|ab| < p^m$. Consequently, we get

$$\Big| b\,\frac{C_m}{p^m} - T \Big| = \frac{|a|}{p^m} < \frac{1}{2b},$$

and, by Legendre's theorem, T/b is a convergent of C_m/p^m. Note that T/b is written in its lowest form, since a and b are coprime. Also, we have $b < p^m$.

We can say a bit more. Write

$$\frac{C_m}{p^m} = [0; r_1, \ldots, r_k] \quad \text{and} \quad \frac{T}{b} = [0; r_1, \ldots, r_j],$$

with $j < k$ and $r_k = 1$ (recall that a rational number has two different continued fraction expansions, and only one of them terminates with 1, except for $1 = [0; 1]$). Then,

$$\frac{1}{3r_{j+1}b^2} \le \frac{|a|}{bp^m} = \Big| \frac{C_m}{p^m} - \frac{T}{b} \Big| \le \frac{1}{r_{j+1}b^2},$$

giving that

$$\frac{p^m}{3b|a|} \le r_{j+1} \le \frac{p^m}{b|a|}. \tag{3.5}$$

Define η by $r_{j+1} = p^{\eta m}$. Then, we get

$$|3ba|^{-1/(1-\eta)} \le |b\xi - a|_p = p^{-m} \le |ba|^{-1/(1-\eta)}.$$

Since, for every $\varepsilon > 0$ there are integers a, b and m as above with m arbitrarily large and $p^{-m} < |ab|^{-\mu^\times(\xi)+\varepsilon}$, this implies the inequality

$$\eta(\mathbf{c}) \ge 1 - \frac{2}{\mu^\times(\xi)}.$$

Let $m \ge 1$ be an integer. Let $T'/b' = [0; r_1, \ldots, r_h]$ denote any convergent to C_m/p^m with $b' < p^m$. Set

$$a' = b'C_m - p^m T'.$$

Then,

$$\begin{aligned} |b'\xi - a'|_p &= |b'\xi - b'C_m + p^m T'|_p \\ &= |b'(c_m p^m + c_{m+1} p^{m+1} + \cdots) + p^m T'|_p \le p^{-m}, \end{aligned}$$

with equality if and only if p does not divide $b'c_m + T'$. As above, we have

$$\frac{1}{3r_{h+1}b'^2} \le \frac{|a'|}{b'p^m} = \Big| \frac{C_m}{p^m} - \frac{T'}{b'} \Big| \le \frac{1}{r_{h+1}b'^2},$$

and, writing $r_{h+1} = p^{\eta' m}$, we get $|a'b'| \le p^{(1-\eta')m}$ and

$$|b'\xi - a'|_p \le p^{-m} \le |a'b'|^{-1/(1-\eta')}.$$

This implies the inequalities

$$\mu^{\times}(\xi) \geq \frac{2}{1-\eta(\mathbf{c})} \quad \text{and} \quad \widehat{\mu}^{\times}(\xi) \geq \frac{2}{1-\widehat{\eta}(\mathbf{c})}\cdot$$

The fourth inequality is slightly more delicate to establish. We introduce the sequence of multiplicative best approximation pairs $((a_k^{\times}, b_k^{\times}))_{k\geq 1}$ to ξ.

For a given p-adic number ξ with

$$\inf_{a,b\neq 0} |ab|\cdot|b\xi-a|_p = 0 \tag{3.6}$$

(this can be assumed, since otherwise $\mu^{\times}(\xi)=\widehat{\mu}^{\times}(\xi)=2$), we define the sequence of integer pairs $((\tilde{a}_k^{\times}, \tilde{b}_k^{\times}))_{k\geq 1}$ by taking a pair of *coprime* integers (a,b) minimizing $|b\xi-a|_p$ among all the integer pairs with $0<\sqrt{|ab|}\leq Q$, and letting the positive real number Q grow to infinity. Write

$$\tilde{Q}_k = \sqrt{|\tilde{a}_k^{\times}\,\tilde{b}_k^{\times}|} \quad \text{for } k\geq 1.$$

By construction, we have

$$\tilde{Q}_1 < \tilde{Q}_2 < \cdots \quad \text{and} \quad |\tilde{b}_1^{\times}\xi-\tilde{a}_1^{\times}|_p > |\tilde{b}_2^{\times}\xi-\tilde{a}_2^{\times}|_p > \cdots.$$

However, we cannot guarantee that

$$\tilde{Q}_k\,|\tilde{b}_k^{\times}\xi-\tilde{a}_k^{\times}|_p > \tilde{Q}_{k+1}\,|\tilde{b}_{k+1}^{\times}\xi-\tilde{a}_{k+1}^{\times}|_p \quad \text{for every } k\geq 1.$$

Therefore, we extract a subsequence $((\tilde{a}_{i_k}^{\times}, \tilde{b}_{i_k}^{\times}))_{k\geq 1}$ from $((\tilde{a}_k^{\times}, \tilde{b}_k^{\times}))_{k\geq 1}$, where $i_1=1$ and, for $k\geq 1$, the index i_{k+1} is the smallest index $j>i_k$ such that

$$\tilde{Q}_j\,|\tilde{b}_j^{\times}\xi-\tilde{a}_j^{\times}|_p < \tilde{Q}_{i_k}\,|\tilde{b}_{i_k}^{\times}\xi-\tilde{a}_{i_k}^{\times}|_p.$$

This gives an infinite subsequence since ξ satisfies (3.6).

To simplify the notation, put $a_k^{\times}=\tilde{a}_{i_k}^{\times}$, $b_k^{\times}=\tilde{b}_{i_k}^{\times}$, and $Q_k=\tilde{Q}_{i_k}$, for $k\geq 1$.

Observe that

$$\mu^{\times}(\xi) = \limsup_{k\to\infty} \frac{-\log|b_k^{\times}\xi-a_k^{\times}|_p}{\log Q_k}$$

and

$$\widehat{\mu}^{\times}(\xi) = \liminf_{k\to\infty} \frac{-\log|b_k^{\times}\xi-a_k^{\times}|_p+\log(Q_{k+1}/Q_k)}{\log Q_{k+1}}\cdot \tag{3.7}$$

Take two consecutive best approximation pairs $(a_k^{\times}, b_k^{\times})$ and $(a_{k+1}^{\times}, b_{k+1}^{\times})$, and define m_k and m_{k+1} by

$$|b_k^{\times}\xi-a_k^{\times}|_p = p^{-m_k} \quad \text{and} \quad |b_{k+1}^{\times}\xi-a_{k+1}^{\times}|_p = p^{-m_{k+1}}.$$

It follows from (3.5) that the largest partial quotients of x_{m_k} and $x_{m_{k+1}}$ are $\asymp p^{m_k}Q_k^{-2}$ and $\asymp p^{m_{k+1}}Q_{k+1}^{-2}$, respectively.

We apply the second inequality of Proposition 3.3 with

$$m = m_k, \quad \widetilde{\eta}_m = \frac{\log p^{m_k} Q_k^{-2}}{m_k \log p} = 1 - \frac{\log Q_k^2}{m_k \log p},$$
$$n = m_{k+1}, \quad \widetilde{\eta}_n = \frac{\log p^{m_{k+1}} Q_{k+1}^{-2}}{m_{k+1} \log p} = 1 - \frac{\log Q_{k+1}^2}{m_{k+1} \log p},$$

to derive that any x_u with $m_k < u < m_{k+1}$ has a partial quotient at least as large as $p^{\eta_u u}$ with

$$\eta_u \geq \frac{(2m_k \log p - 2\log Q_k) - 2\log Q_{k+1}}{(2m_k \log p - 2\log Q_k) + 2\log Q_{k+1}} + o(1)$$
$$= 1 - \frac{2\log Q_{k+1}}{m_k \log p + \log Q_{k+1}/Q_k} + o(1).$$

Recalling that $m_k \log p = -\log |b_k^\times \xi - a_k^\times|_p$, it then follows from (3.7) that

$$\widehat{\eta}(\mathbf{c}) \geq 1 - \frac{2}{\widehat{\mu}^\times(\xi)}.$$

This completes the proof of the theorem. ∎

In the course of the proof of Theorem 3.4, we have obtained the following statement.

Proposition 3.5. *If there exist positive real numbers c_1 and δ such that, for every $m \geq 1$, all the partial quotients of x_m are less than $c_1 p^{\delta m}$, then there exists $c_2 > 0$ such that*

$$|b\xi - a|_p > c_2 |ab|^{-1/(1-\delta)}, \quad \textit{for all } a \textit{ and } b.$$

If there exist positive real numbers c_3 and δ and arbitrarily large m such that x_m has a partial quotient greater than $c_3 p^{\delta m}$, then there exist $c_4 > 0$ and integers a and b with $|ab|$ arbitrarily large, such that

$$|b\xi - a|_p < c_4 |ab|^{-1/(1-\delta)}.$$

4. Very good rational approximations to the p-adic Thue–Morse number

In this section, we use combinatorial properties of the Thue–Morse word $\mathbf{t}$ to establish (2.1) and to exhibit an infinite family of rational numbers z_m (see (2.3) for their definition) having a very large partial quotient.

Proof of (2.1). Observe that

$$(1 + p^2)\, \xi_{\mathbf{t},p} = 1 - p - 2p^4 + 2p^5 + 2p^{12} - 2p^{13} + \cdots,$$

where the coefficients of p^6 up to p^{11} are 0. More generally, for $k \geq 1$, we get

$$(1 + p^{2^k})\, \xi_{\mathbf{t},p} = R_k(p) + 2(-1)^{k+1} p^{3\cdot 2^{k+1}} + p^{3\cdot 2^{k+1}+1} s_k,$$

where s_k is a nonzero element of $\mathbb{Z}_p$ and $R_k(X)$ is a polynomial with coefficients in $\{0, \pm 1, \pm 2\}$ and of degree $3 \cdot 2^k - 1$.

This can be checked either by using the substitution τ, or by a direct computation based on the recursion defining $\mathbf{t}$. Namely, we observe that

$$t_j + t_{j+2} = 0, \quad \text{for } j = 4, 5, \ldots, 9, \text{ that is, for } j = 2^2, \ldots, 2^3 + 2 - 1,$$

and

$$t_3 + t_5 = t_{10} + t_{12} = 2, \quad t_{11} + t_{13} = -2.$$

Furthermore, $t_j + t_{j+2} = 0$ implies that

$$t_{2j} + t_{2(j+2)} = t_{2j} + t_{2j+4} = 0$$

and

$$t_{2j+1} + t_{(2j+1)+4} = -t_j - t_{j+2} = 0.$$

Consequently, we derive that

$$t_j + t_{j+2^k} = 0, \quad \text{for } k \ge 1 \text{ and } j = 2^{k+1}, \ldots, 2^{k+2} + 2^k - 1.$$

In addition, we check that

$$t_{2^{k+1}-1} + t_{2^{k+1}+2^k-1} = 2 \cdot (-1)^{k+1}, \quad \text{for } k \ge 1,$$

and

$$t_{2^{k+2}+2^k} + t_{2^{k+2}+2^{k+1}} = 2, \quad t_{2^{k+2}+2^k+1} + t_{2^{k+2}+2^{k+1}+1} = -2, \quad \text{for } k \ge 1.$$

We get eventually

$$|(1 + p^{2^k})\xi_{\mathbf{t},p} - R_k(p)|_p = p^{-3 \cdot 2^{k+1}}, \quad \text{for } p \ge 3,$$

while

$$|(1 + 2^{2^k})\xi_{\mathbf{t},2} - R_k(2)|_2 = 2^{-3 \cdot 2^{k+1}-1}.$$

For $k \ge 1$, putting

$$b_{k,p} = 1 + p^{2^k} \quad \text{and} \quad a_{k,p} = R_k(p),$$

we check that

$$|b_{k,p}| \le p^{2^k+1}, \quad |a_{k,p}| \le 2p^{3 \cdot 2^k+1},$$

and

$$|b_{k,p}\xi_{\mathbf{t},p} - a_{k,p}|_p \le p^{-3 \cdot 2^{k+1}} \le \left(2p^2 |a_{k,p} b_{k,p}|^{-1}\right)^{3/2} \le 4p^3 |a_{k,p} b_{k,p}|^{-3/2}.$$

This establishes (2.1) and implies that $\mu^{\times}(\xi_{\mathbf{t},p}) \ge 3$. ∎

It follows from Proposition 3.5 that every rational number $z_{3 \cdot 2^k}$ has a large partial quotient.

Proposition 4.1. *There exist a positive real number c such that, for every $k \geq 1$, one among the rational numbers $z_{3 \cdot 2^k}$ and $z_{3 \cdot 2^k + 1}$ has a maximal partial quotient in the interval $[c^{-1} p^{2^k}, c p^{2^k}]$ associated with a convergent whose denominator is $2^{2^{k-1}} + 1$ if $p = 2$ and $(p^{2^{k-1}} + 1)/2$ if p is odd.*

Proof. Observe that

$$\Big(1 + \frac{1}{p^2}\Big) z_{12} = -\frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^9} - \frac{2}{p^{10}} - \frac{1}{p^{13}} + \frac{1}{p^{14}} \cdot$$

Since $t_0 = t_5$, for any $k \geq 1$, the prefix of $\mathbf{t}$ of length 2^k is equal to the suffix of length 2^k of the prefix of $\mathbf{t}$ of length $6 \cdot 2^k$. Consequently, we have

$$\Big(1 + \frac{1}{p^{2^k}}\Big) z_{3 \cdot 2^{k+1}} = \frac{T_k(p)}{p^{2^k}} + \frac{2}{p^{2^{k+2}+1}} - \frac{2}{p^{2^{k+2}+2}} + \cdots ,$$

where

$$T_k(X) = t_0 + t_1 X + \cdots + t_{2^k - 1} X^{2^k - 1} = \prod_{j=0}^{k-1} (1 - X^{2^j}).$$

This implies

$$\frac{1}{p^{3 \cdot 2^k}} \ll |(p^{2^k} + 1) z_{3 \cdot 2^{k+1}} - T_k(p)| \leq \frac{2}{p^{3 \cdot 2^k}} \cdot \tag{4.1}$$

Thus, there must be a very large partial quotient in the continued fraction expansion of $z_{3 \cdot 2^{k+1}}$. For $h = 0, \ldots, k-1$, since $p^{2^h} - 1$ divides $p^{2^k} - 1$, we see that

$$\gcd(p^{2^h} - 1, p^{2^k} + 1) \text{ divides } 2.$$

Furthermore, 4 does not divide $p^{2^k} + 1$. We conclude that $T_k(p)$ and $p^{2^k} + 1$ are coprime for $p = 2$, while their greatest common divisor is 2 for $p \geq 3$. This shows that $z_{3 \cdot 2^{k+1}}$ has a partial quotient $r_{3 \cdot 2^{k+1}}$ with $r_{3 \cdot 2^{k+1}} \asymp p^{2^k}$.

Note that $t_{3 \cdot 2^{k+1}} = t_3 = 1$. If $p \geq 3$, then p does not divide $T_k(p) + (p^{2^k} + 1)$ and we conclude that $r_{3 \cdot 2^{k+1}}$ is a maximal partial quotient. For $p = 2$ we check that $z_{3 \cdot 2^{k+1}+1}$ has a maximal partial quotient. This concludes the proof. ∎

A deeper study of the combinatorial properties of $\mathbf{t}$ shows that, for $j \geq 0$ and $k \geq 1$, there are polynomials $R_{k,j}(X)$ of degree at most equal to $2^{k-1}(6 + j 2^4) - 1$ such that

$$p^{-2^{k-1}(12 + j 2^4)} \ll |(1 + p^{2^k}) \xi_{\mathbf{t},p} - R_{k,j}(p)|_p \leq p^{-2^{k-1}(12 + j 2^4)}.$$

This shows that, for every $j \geq 0$, there are integers a and b with $|ab|$ arbitrarily large such that

$$|b \xi_{\mathbf{t},p} - a|_p \asymp |ab|^{-(3+4j)/(2+4j)}.$$

The exponents form the sequence of rational numbers $3/2, 7/6, 11/10, \ldots$ We suspect that, for any given $\varepsilon > 0$, all but finitely many solutions to

$$|b \xi_{\mathbf{t},p} - a|_p < |ab|^{-1-\varepsilon}$$

belong to the families described above.

5. Use of Hankel determinants

For $k \geq 1$, let

$$z_{3\cdot 2^k} = \frac{t_{3\cdot 2^k-1}}{p} + \frac{t_{3\cdot 2^k-2}}{p^2} + \cdots + \frac{t_0}{p^{3\cdot 2^k}} = [0; d_{1,k}, \ldots, d_{\ell(k),k}]$$

denote the continued fraction expansion of $z_{3\cdot 2^k}$ with $d_{\ell(k),k} \geq 2$. By Proposition 4.1, there exists $m(k)$ such that

$$\text{(5.1)} \qquad d_{m(k),k} \asymp p^{2^k} \quad \text{and} \quad \text{denominator}([0; d_{1,k}, \ldots, d_{m(k)-1,k}]) \asymp p^{2^{k-1}}.$$

To establish the upper bound $\mu^\times(\xi_{\mathbf{t},p}) \leq 3$, we need to ensure that the second largest partial quotient of $z_{3\cdot 2^k}$ is much smaller than $d_{m(k),k}$. To do this, we use Hankel determinants in a similar spirit as in [7, 8].

The purpose of this section is to establish the following statement.

Proposition 5.1. *There exists a positive constant c such that, for every sufficiently large k, every partial quotient of $z_{3\cdot 2^k}$ different from $d_{m(k),k}$ is at most equal to $cp^{2^{k-1}}$.*

By (5.1), the partial quotients $d_{1,k}, \ldots, d_{m(k)-1,k}$ are all $\ll p^{2^{k-1}}$. To establish Proposition 5.1, it thus remains to bound from above the partial quotients $d_{m(k)+1,k}, \ldots, d_{\ell(k),k}$. To this end, by (5.1), it is sufficient to consider only the convergents a/b of $z_{3\cdot 2^k}$ with $b \gg p^{3\cdot 2^{k-1}}$ and to show that none of them is associated with a partial quotient $\gg p^{2^{k-1}}$.

The method of the proof and some additional computation yields a stronger conclusion, with $p^{2^{k-1}}$ replaced by $p^{2^{k-h}}$, for some integer $h \geq 3$. It is even likely that the following statement holds:

For every $\varepsilon > 0$ and every sufficiently large k, every partial quotient of $z_{3\cdot 2^k}$ different from $d_{m(k),k}$ is at most equal to $p^{\varepsilon 2^k}$.

We follow very closely the argumentation of [7], where Padé approximants are used to construct a dense, in a suitable sense, sequence of good rational approximations to the real Thue–Morse–Mahler numbers. However, our problem is different, since we have to control the partial quotients of the *rational numbers* $z_{3\cdot 2^k}$.

As in [7, 8] we work with the Thue–Morse sequence written over the alphabet $\{-1, 1\}$. This is at this step of the proof that the choice of the alphabet does matter.

Proof of Proposition 5.1. As in [7], we briefly recall several basic facts on Padé approximants. We refer the reader to [4, 5] for the proofs and for additional results. Let

$$f(z) = \sum_{k\geq 0} c_k z^k, \quad c_k \in \mathbb{Q},$$

be a power series. Let u and v be non-negative integers. The Padé approximant $[u/v]_f(z)$ is any rational fraction $A(z)/B(z)$ in $\mathbb{Q}[[z]]$ such that

$$\deg(A) \leq u, \quad \deg(B) \leq v, \quad \text{and} \quad \text{ord}_{z=0}(B(z)f(z) - A(z)) \geq u + v + 1.$$

For $k \geq 1$, let

$$H_k(f) := \begin{vmatrix} c_0 & c_1 & \dots & c_{k-1} \\ c_1 & c_2 & \dots & c_k \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_k & \dots & c_{2k-2} \end{vmatrix}$$

denote the Hankel determinant of order k associated to $f(z)$. If $H_k(f)$ is non-zero, then the Padé approximant $[k-1/k]_f(z)$ exists and we have

$$f(z) - [k-1/k]_f(z) = \frac{H_{k+1}(f)}{H_k(f)}\, z^{2k} + O(z^{2k+1}).$$

For a positive integer k, set

$$\widetilde{g}_0(z) = 1 + z - z^2 = -(t_2 + t_1 z + t_0 z^2)$$

and

$$\widetilde{g}_k(z) = (1-z)(1-z^2)\cdots(1-z^{2^{k-1}})\, \widetilde{g}(z^{2^k}), \quad k \geq 1.$$

The definition of $\mathbf{t}$ implies that

$$\widetilde{g}_k(z) = (-1)^{k+1}(t_{3\cdot 2^k - 1} + t_{3\cdot 2^k - 2}\, z + \cdots + t_0\, z^{3\cdot 2^k - 1}), \quad k \geq 0,$$

thus

$$z_{3\cdot 2^k} = (-1)^{k+1}\, \frac{\widetilde{g}_k(1/p)}{p}, \quad k \geq 0.$$

It is sufficient for our purpose to show that the continued fraction expansion of the rational number $\widetilde{g}_k(1/p)$ has no 'too large' partial quotient associated with a convergent of denominator $\gg p^{3\cdot 2^{k-1}}$. Note that the fact that p is prime does not play any rôle in this section and all what follows also holds for the rational number $\widetilde{g}_k(1/b)$, where $b \geq 2$ is an integer.

Let $K \geq 2$ be an integer to be fixed later. Assume that we have checked that

$$H_j(\widetilde{g}_K) \neq 0, \quad j = 3\cdot 2^{K-1} + 1, \dots, 3\cdot 2^K.$$

Consequently, there exist integer polynomials $P_{j,0}(z)$ and $Q_{j,0}(z)$ of degree at most $j-1$ and j, respectively, and a non-zero rational number h_j such that

$$\widetilde{g}_K(z) - P_{j,0}(z)/Q_{j,0}(z) = h_j\, z^{2j} + O(z^{2j+1}), \quad 3\cdot 2^{K-1} + 1 \leq j \leq 3\cdot 2^K - 1.$$

The real numbers $c_1, c_2, \dots$ occurring below are all positive and depend only at most on K (note that the index j is bounded from above and from below in terms of K). There exists c_1 such that

$$\left| \widetilde{g}_K(z^{2^m}) - \frac{P_{j,0}(z^{2^m})}{Q_{j,0}(z^{2^m})} - h_j\, z^{2^{m+1} j} \right| \leq c_1\, z^{2^{m+1} j + 2^m}, \tag{5.2}$$

for $0 < z \leq 1/2$ and $3\cdot 2^{K-1} + 1 \leq j \leq 3\cdot 2^K - 1$. An immediate induction yields

$$(1-z)(1-z^2)\cdots(1-z^{2^{m-1}})\, \widetilde{g}_K(z^{2^m}) = \widetilde{g}_{K+m}(z).$$

Set

$$P_{j,m}(z) = \prod_{h=0}^{m-1}(1 - z^{2^h})\, P_{j,0}(z^{2^m}) \quad \text{and} \quad Q_{j,m}(z) = Q_{j,0}(z^{2^m}).$$

Note that $P_{j,m}(z)/Q_{j,m}(z)$ is the Padé approximant $[2^m j - 1/2^m j]_{\widetilde{g}_{K+m}}(z)$.

By multiplying both members of (5.2) by $(1 - z)(1 - z^2)\cdots(1 - z^{2^{m-1}})$, we obtain

$$\left|\widetilde{g}_{K+m}(z) - \frac{P_{j,m}(z)}{Q_{j,m}(z)} - h_j \prod_{h=0}^{m-1}(1 - z^{2^h})\, z^{2^{m+1} j}\right| \le c_2\, z^{2^{m+1} j + 2^m},$$

for $0 < z \le 1/2$ and $3 \cdot 2^{K-1} + 1 \le j \le 3 \cdot 2^K - 1$.

Evaluating at $z = 1/p$ and arguing as in [7], there exist an absolute, positive Ξ and an integer m_0, depending only on K, such that the inequalities

$$\frac{h_j\,\Xi}{2}\, p^{-2^{m+1} j} \le \left|\widetilde{g}_{K+m}(1/p) - \frac{P_{j,m}(1/p)}{Q_{j,m}(1/p)}\right| \le \frac{3h_j}{2}\, p^{-2^{m+1} j} \tag{5.3}$$

hold for $m > m_0$ and $3 \cdot 2^{K-1} + 1 \le j \le 3 \cdot 2^K - 1$.

Define the integers

$$p_{j,m} = p^{2^m j}\, P_{j,m}(1/p) \quad \text{and} \quad q_{j,m} = p^{2^m j}\, Q_{j,m}(1/p).$$

There exist $c_3, \dots, c_8$ such that

$$c_3\, p^{2^m j} \le q_{j,m} \le c_4\, p^{2^m j}, \tag{5.4}$$

$$\frac{c_5}{p^{2^{m+1} j}} \le \left|\widetilde{g}_{K+m}(1/p) - \frac{p_{j,m}}{q_{j,m}}\right| \le \frac{c_6}{p^{2^{m+1} j}}, \tag{5.5}$$

and, by combining (5.4) and (5.5),

$$\frac{c_7}{q_{j,m}^2} \le \left|\widetilde{g}_{K+m}(1/p) - \frac{p_{j,m}}{q_{j,m}}\right| \le \frac{c_8}{q_{j,m}^2}, \tag{5.6}$$

for $m > m_0$ and $3 \cdot 2^{K-1} + 1 \le j \le 3 \cdot 2^K - 1$.

Let r/s be a convergent to $\widetilde{g}_{K+m}(1/p)$ with $s > c_9\, p^{3 \cdot 2^{K+m-1}}$, for some absolute positive real number c_9. Assume that there is j with $3 \cdot 2^{K-1} + 1 \le j \le 3 \cdot 2^K - 2$ such that

$$q_{j,m} \le 2c_8\, s < q_{j+1,m}.$$

Then,

$$\begin{aligned}\left|\widetilde{g}_{K+m}(1/p) - \frac{r}{s}\right| &\ge \left|\frac{r}{s} - \frac{\tilde{p}_{j+1,m}}{q_{j+1,m}}\right| - \left|\widetilde{g}_{K+m}(1/p) - \frac{\tilde{p}_{j+1,m}}{q_{j+1,m}}\right| \\ &\ge \frac{1}{s\, q_{j+1,m}} - \frac{c_8}{q_{j+1,m}^2} \ge \frac{1}{2s\, q_{j+1,m}} \ge \frac{1}{c_{10}\, s^{2+1/j}},\end{aligned}$$

since $q_{j+1,m} \le c_{11}\, q_{j,m}^{1+1/j} \le c_{12}\, s^{1+1/j}$.

Since $s > c_9 p^{3\cdot 2^{K+m-1}}$, the case $2c_8 s \le q_{3\cdot 2^{K-1}+1,m}$ can be treated analogously. Furthermore, if $2c_8 s \ge q_{3\cdot 2^K-1,m}$, then the partial quotient A associated with the convergent r/s satisfies

$$\frac{1}{s\,p^{3\cdot 2^{K+m-1}}} \le \left|\widetilde{g}_{K+m}(1/p) - \frac{r}{s}\right| \le \frac{1}{A s^2},$$

thus

$$A \le \frac{p^{3\cdot 2^{K+m-1}}}{s} \le c_{13}\, p^{2^m} \le c_{14}\, s^{1/(3\cdot 2^K-1)}.$$

This shows that, for m large enough, the second largest partial quotient of the rational number $z_{3\cdot 2^{m+K}} = (-1)^{k+1}\widetilde{g}_{K+m}(1/p)/p$ is at most equal to $p^{2^{m+1}\eta}$, for $\eta = 1/(3\cdot 2^{K-1})$.

An easy calculation shows that, for

$$\widetilde{g}_2(z) = 1 - z - z^2 + z^3 + z^4 - z^5 - z^6 + z^7 - z^8 + z^9 + z^{10} - z^{11},$$

we have

$$H_2(\widetilde{g}_2) = -2, \quad H_3(\widetilde{g}_2) = \cdots = H_6(\widetilde{g}_2) = 0, \quad H_7(\widetilde{g}_2) = 64, \quad H_8(\widetilde{g}_2) = 128,$$
$$H_9(\widetilde{g}_2) = -64, \quad H_{10}(\widetilde{g}_2) = -56, \quad H_{11}(\widetilde{g}_2) = -14, \quad H_{12}(\widetilde{g}_2) = 1.$$

Consequently, we can take $K = 2$ in the above computation and we get $\eta = 1/6$, as announced.

A rapid check shows that the Hankel determinants $H_{25}(\widetilde{g}_4), \ldots, H_{48}(\widetilde{g}_4)$ do not vanish, thus we can take $K = 4$ and conclude that, for k large enough, every partial quotient $d_{j,k}$ of $z_{3\cdot 2^k}$ with $j > m(k)$ is at most equal to $cp^{3\cdot 2^k/24}$ for some positive constant c. ∎

6. Proof of Theorem 2.1

Recall that z_m is defined in (2.3). The following statement, which partly gathers results from Sections 4 and 5, is the key ingredient for the proof of Theorem 2.1.

Proposition 6.1. *For every positive real number ε, and for any sufficiently large integer k, all the partial quotients of z_{2^k} are less than $p^{\varepsilon 2^k}$. There exists a positive real number c such that, for every positive integer k, the rational number $z_{3\cdot 2^k}$ has a partial quotient in the interval $[c^{-1}p^{2^k}, cp^{2^k}]$, while all its other partial quotients are less than $cp^{2^{k-1}}$.*

The first statement of Proposition 6.1 is a direct consequence of Theorem 1.1 in [8], reproduced below.

Theorem 6.2. *There exists a positive real number K such that, for every integer $g \ge 2$ and every integer $\ell \ge 2$, the inequality*

$$\left|\prod_{h=0}^{\ell}(1 - g^{-2^h}) - \frac{p}{q}\right| > \frac{1}{q^2 \exp(K \log g\, \sqrt{\log q}\, \log\log q)}$$

holds for every rational number p/q different from $\prod_{h=0}^{\ell}(1 - g^{-2^h})$.

Proof of Proposition 6.1. Observe that the prefix of length 4 of $\mathbf{t}$ is a palindrome and that

$$\tau^2(1) = 1, -1, -1, 1 \quad \text{and} \quad \tau^2(-1) = -1, 1, 1, -1$$

are both palindromes. Consequently, for $k \geq 1$, the prefix of length 4^k of $\mathbf{t}$ is a palindrome and

$$\begin{aligned} z_{4^k} &= \frac{t_{4^k-1}}{p} + \frac{t_{4^k-2}}{p^2} + \cdots + \frac{t_0}{p^{4^k}} \\ &= \frac{t_0}{p} + \frac{t_1}{p^2} + \cdots + \frac{t_{4^k-1}}{p^{4^k}} = p^{-1} \prod_{h=0}^{2k-1} (1 - p^{-2^h}). \end{aligned}$$

Define the involution σ by $\sigma(1) = -1$ and $\sigma(-1) = 1$. It follows from the definition of $\mathbf{t}$ that its prefix of length $2 \cdot 4^k$ is equal to the concatenation of its prefix T_{4^k} of length 4^k with $\sigma(T_{4^k})$. Consequently,

$$t_{2\cdot 4^k-1} \ldots t_1 t_0 = \sigma(t_0 t_1 \ldots t_{2\cdot 4^k-1})$$

is the prefix of length $2 \cdot 4^k$ of the Thue–Morse word obtained from $\mathbf{t}$ after exchanging 1's and -1's, and we get

$$z_{2\cdot 4^k} = -p^{-1} \prod_{h=0}^{2k} (1 - p^{-2^h}), \quad k \geq 0.$$

The first assertion of Proposition 6.1 then follows from Theorem 6.2.

The second assertion has been established in Proposition 4.1, and the last one in Proposition 5.1. ■

Completion of the proof of Theorem 2.1. The first assertion of Proposition 6.1 implies that $\widehat{\eta}(\mathbf{t}) = 0$, which, by Theorem 3.4, gives that $\widehat{\mu}^\times(\xi_{\mathbf{t},p}) = 2$.

Since we have already established (2.1), it only remains for us to prove (2.2), that is, by the first assertion of Proposition 3.5, to show that there exists a positive real number c such that, for every $m \geq 1$, all the partial quotients of z_m are less than $cp^{m/3}$.

By Proposition 4.1, we know that, for $k \geq 1$, there is a maximal partial quotient of size $\asymp p^{2^k}$ attained at the rational number $z_{3\cdot 2^k}$ or $z_{3\cdot 2^k+1}$. It remains for us to control the partial quotients which do not derive from this maximal partial quotient. Therefore, it is sufficient to bound from above any other maximal partial quotient attained at some z_u and written under the form $p^{\eta_u u}$.

Let ε be a small positive real number. For k large enough, all the partial quotients of z_{2^k} are $< p^{\varepsilon 2^k}$ and all the partial quotients of $z_{3\cdot 2^k}$ except the largest one are $< p^{2^{k-1}+\varepsilon}$. By Proposition 3.3 applied with $m = 2^k$ and $n = 3 \cdot 2^{k-1}$, we get that every η_u with $2^k \leq u \leq 3 \cdot 2^{k-1}$ satisfies

$$\eta_u \leq \frac{3(1 + 1/6 + \varepsilon) - 2(1 - \varepsilon)}{3(1 + 1/6 + \varepsilon) + 2(1 - \varepsilon)} + o(1) \leq \frac{3}{11} + 5\varepsilon + o(1),$$

where, as below, $o(1)$ denotes some quantity which tends to 0 as k tends to infinity. Similarly, every η_u with $3 \cdot 2^{k-1} \le u \le 2^{k+1}$ satisfies

$$\eta_u \le \frac{4(1+\varepsilon) - 3(1 - 1/6 - \varepsilon)}{3(1 - 1/6 - \varepsilon) + 4(1+\varepsilon)} + o(1) \le \frac{3}{13} + 7\varepsilon + o(1).$$

Taking ε small enough and k large enough, both upper bounds are less than $3/10$. Consequently, we have proved that, for m sufficiently large, all the partial quotients of z_m are bounded from above by some positive constant times $p^{m/3}$. We then apply Proposition 3.5 to get (2.2) and Theorem 3.4 to derive that $\mu^\times(\xi_{\mathbf{t},p}) \le 3$. ■

As indicated above, the fact that $\widehat{\mu}^\times(\xi_{\mathbf{t},p}) = 2$ is a direct consequence of [8], which is also used in the proof that $\mu^\times(\xi_{\mathbf{t},p}) \le 3$. However, the latter proof can be made independent of [8]. Namely, by arguing as in Section 5 and checking that some Hankel determinants are nonzero, we can prove that, for k large enough, every partial quotient of z_{2^k} is at most equal to $cp^{2^k/24}$ for some positive constant c. Then, we use Proposition 3.3 in a similar way as above.

7. Rational approximation to the p-adic Fibonacci number

The Fibonacci word $\mathbf{f} = f_1 f_2 f_3 \dots$ over $\{0, 1\}$ is the limit of the sequence of finite words

$$0, 01, 010, 01001, \dots,$$

starting with $0, 01$, and such that, for $n \ge 1$, its $(n+2)$-th element is the concatenation of its $(n+1)$-th and its n-th elements. We then have

$$\mathbf{f} = 0100101001001010010100\dots$$

In other words, $\mathbf{f}$ is the fixed point of the substitution φ defined by $\varphi(0) = 01$ and $\varphi(1) = 0$. Note that we have

$$\varphi(f_1 \dots f_{F_n}) = f_1 \dots f_{F_{n+1}}, \quad n \ge 1, \tag{7.1}$$

where $(F_n)_{n \ge 0}$ denote the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and the recursion $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. Let

$$\xi_{\mathbf{f},p} = \sum_{n \ge 1} \frac{f_n}{p^n} = \frac{1}{p^2} + \frac{1}{p^5} + \frac{1}{p^7} + \frac{1}{p^{10}} + \cdots$$

denote the p-adic number whose Hensel expansion is given by the Fibonacci word $\mathbf{f}$ over $\{0, 1\}$.

The existence of long repetitions at and near the beginning of $\mathbf{f}$ allows us to establish the following result. Let $\gamma = (1 + \sqrt{5})/2$ denote the golden ratio.

Theorem 7.1. *The p-adic Fibonacci number $\xi_{\mathbf{f},p}$ satisfies*

$$\mu(\xi_{\mathbf{f},p}) = \gamma^2, \quad \mu^\times(\xi_{\mathbf{f},p}) \ge \gamma^2 + \frac{1}{1+\gamma^2}, \quad \textit{and} \quad \widehat{\mu}^\times(\xi_{\mathbf{f},p}) \ge 2 + \frac{1}{2\gamma}.$$

To establish the lower bounds in Theorem 7.1, we use the fact that there are many repetitions in $\mathbf{f}$, which yield very good rational approximations to $\xi_{\mathbf{f},p}$. Our key auxiliary result is the following proposition.

Proposition 7.2. *For $n \geq 4$, the word*

$$f_1 \dots f_{F_n} f_1 \dots f_{F_n} f_1 \dots f_{F_{n-1}-2},$$

that is of length $F_{n+2} - 2$, is the longest common prefix of the word $\mathbf{f}$ and the word $(f_1 \dots f_{F_n})^\infty$. For $n \geq 4$, the word

$$f_1 \dots f_{F_{n+1}} f_1 \dots f_{F_n} f_1 \dots f_{F_n} f_1 \dots f_{F_n} f_1 \dots f_{F_{n-1}-2},$$

that is of length $2F_{n+2} - 2$, is the longest common prefix of the word $\mathbf{f}$ and the word $f_1 \dots f_{F_{n+1}} (f_1 \dots f_{F_n})^\infty$.

Proof. This follows from (7.1) by induction, after having checked, also by induction, that, if $n \geq 4$ is odd,

$$f_{F_n - 2} f_{F_n - 1} f_{F_n} = 001,$$

and if $n \geq 4$ is even,

$$f_{F_n - 2} f_{F_n - 1} f_{F_n} = 010.$$

We omit the details. ∎

Proof of Theorem 7.1. It follows from Proposition 7.2 that, for $n \geq 1$, there exist integers u_n and x_n such that

$$|(p^{F_n} - 1)\xi_{\mathbf{f},p} - u_n|_p \asymp_p p^{-F_{n+2}}, \quad |u_n| \asymp_p p^{F_n},$$

and

$$|(p^{F_n} - 1)\xi_{\mathbf{f},p} - x_n|_p \asymp_p p^{-2F_{n+2}}, \quad |x_n| \asymp_p p^{F_{n+2}}.$$

Setting $y_n = p^{F_n} - 1$ for $n \geq 1$, we check that

$$|u_n y_n| \asymp_p p^{2F_n} \quad \text{and} \quad |x_n y_n| \asymp_p p^{F_{n+2}+F_n}, \quad n \geq 1.$$

Since $F_{n+1} = \gamma F_n + o(1)$, this gives

$$\mu(\xi_{\mathbf{f},p}) \geq \gamma^2 \quad \text{and} \quad \mu^\times(\xi_{\mathbf{f},p}) \geq \frac{4\gamma^2}{1+\gamma^2} = \gamma^2 + \frac{1}{1+\gamma^2}.$$

By using triangle inequalities, it is easy to show that

$$\mu(\xi_{\mathbf{f},p}) = \gamma^2;$$

we omit the details. However, triangle inequalities are useless when we consider the multiplicative exponent.

Let us now consider the point of view developed in Section 3 and work with the rational numbers

$$w_m = w_{m,p} = \frac{f_m}{p} + \frac{f_{m-1}}{p^2} + \cdots + \frac{f_1}{p^m}, \quad m \geq 1.$$

It follows from Proposition 7.2 that, for $n \geq 5$, there exist integers $\widetilde{u}_n$ and $\widetilde{x}_n$ such that

$$\left| w_{F_{n+2}-2} - \frac{\widetilde{u}_n}{p^{F_n} - 1} \right| \asymp_p p^{-2F_n - F_{n-1}}$$

and

$$\left| w_{2F_{n+2}-2} - \frac{\widetilde{x}_n}{p^{F_n} - 1} \right| \asymp_p p^{-3F_n - F_{n-1}}.$$

This implies, respectively, that $w_{F_{n+2}-2}$ has a partial quotient $\asymp_p p^{F_{n-1}}$ and that $w_{2F_{n+2}-2}$ has a partial quotient $\asymp_p p^{F_n + F_{n-1}} \asymp_p p^{F_{n+1}}$.

We apply Proposition 3.3 and use the fact that $F_n = \gamma^n/\sqrt{5} + o(1)$.

Let u be an integer such that $2F_n - 2 \leq u \leq 2F_{n+1} - 2$. By Proposition 3.3, if $u \leq F_{n+2} - 2$, then the rational number w_u has a partial quotient $p^{\eta_u u}$ with

$$\begin{aligned} \eta_u &\geq \frac{2\gamma^n(1 + 1/(2\gamma)) - \gamma^{n+2}(1 - \gamma^{-3})}{2\gamma^n(1 + 1/(2\gamma)) + \gamma^{n+2}(1 - \gamma^{-3})} + o(1) \\ &= \frac{2\gamma + 1 - (\gamma^3 - 1)}{2\gamma + 1 + (\gamma^3 - 1)} + o(1) = \frac{1}{4\gamma + 1} + o(1). \end{aligned}$$

By Proposition 3.3, if $u \geq F_{n+2} - 2$, then the rational number w_u has a partial quotient $p^{\eta_u u}$ with

$$\begin{aligned} \eta_u &\geq \frac{\gamma^{n+2}(1 + \gamma^{-3}) - 2\gamma^{n+1}(1 - 1/(2\gamma))}{\gamma^{n+2}(1 + \gamma^{-3}) + 2\gamma^{n+1}(1 - 1/(2\gamma))} + o(1) \\ &= \frac{\gamma^3 + 1 - (2\gamma^2 - \gamma)}{\gamma^3 + 1 + (2\gamma^2 - \gamma)} + o(1) = \frac{\gamma}{3\gamma^2 + 1} + o(1). \end{aligned}$$

Since the second lower bound is larger than the first one, we deduce from Theorem 3.4 that

$$\widehat{\mu}^{\times}(\xi_{\mathbf{f},p}) \geq \frac{2}{1 - 1/(4\gamma + 1)} = \frac{8\gamma + 2}{4\gamma} = 2 + \frac{1}{2\gamma}.$$

This completes the proof of the theorem. ■

Presumably, both inequalities in Theorem 7.1 are indeed equalities. This is the case if we restrict our attention to the approximations $|b\xi_{\mathbf{f},p} - a|_p$, with $|b| \leq |a|$. However, we do not see how to handle the approximations $|b\xi_{\mathbf{f},p} - a|_p$, with $|b| > |a|$. To establish the lower bounds in Theorem 7.1, we have used the very good rational approximations to the rational numbers w_m coming from the repetitions in the prefix of length m of $\mathbf{f}$. But this allows us only to find approximations of denominator at most $p^{m/2}$. A way to handle the approximations $|b\xi_{\mathbf{f},p} - a|_p$ with $|b| > |a|$ would be to look at the expansion of $1/\xi_{\mathbf{f},p}$ and hope to see repetitions in it. But we have no information on it.

Numerical experiments suggest that no w_m has a very large partial quotient associated with a convergent of denominator greater than $p^{m/2}$, but this seems difficult to prove.

8. Additional remarks

Let us briefly sketch how Theorem 3.4 can be used to reprove some of the results of [10].

For a prime p, let $\mathbf{c} = (c_k)_{k\geq 0}$ be a sequence of elements of $\{0, 1, \ldots, p-1\}$. As in Section 3, write

$$x_m = \frac{c_0}{p^m} + \frac{c_1}{p^{m-1}} + \cdots + \frac{c_{m-1}}{p}, \quad m \geq 1.$$

Assume that $\eta(\mathbf{c}) > 0$. Let m be such that x_m has a partial quotient $\asymp p^{\eta m}$, for some positive η close to $\eta(\mathbf{c})$. Then, the other partial quotients of x_m are $\ll p^{(1-\eta)m}$. By the analysis made in Section 3, for any positive integer h less than ηm, the rational number x_{m+h} has a partial quotient $\asymp p^{\eta m-h}$, while its other partial quotients are $\ll p^{(1-\eta)m+h}$. In particular, for $h = \lceil (2\eta - 1)m/2 \rceil$, all the partial quotients of x_{m+h} are $\ll p^{m/2}$, giving that

$$\widehat{\eta}(\mathbf{c}) \leq \frac{m}{2(m+h)} \leq \frac{1}{2\eta + 1}.$$

We conclude that

$$\widehat{\eta}(\mathbf{c}) \leq \frac{1}{2\eta(\mathbf{c}) + 1}. \tag{8.1}$$

By Theorem 3.4, this gives

$$1 - \frac{2}{\widehat{\mu}^{\times}(\xi)} \leq \frac{\mu^{\times}(\xi)}{3\mu^{\times}(\xi) - 4}, \quad \text{that is,} \quad \widehat{\mu}^{\times}(\xi) \leq 3 + \frac{2}{\mu^{\times}(\xi) - 2},$$

and we recover the first inequality of Theorem 3.1 in [10].

We can also recover the upper bound given in (1.5). In view of Theorem 3.4, it is equivalent to prove that $\widehat{\eta}(\mathbf{c}) \leq 1/\sqrt{5}$ always hold. The strategy is the following. For $m \geq 1$, recall that η_m is defined in such a way that $p^{\eta_m m}$ is the largest partial quotient of x_m. We take a local maximum α of the function $m \mapsto \eta_m$ and we consider the previous local maximum and the next one.

For simplicity, we ignore the integral parts. Let α, β, γ, δ, ε and m be such that

- x_m has partial quotients $r_m \asymp p^{\alpha m}$, $\asymp p^{\beta m}$, and $\asymp p^{\varepsilon m}$;
- $x_{(1+\gamma)m}$ has partial quotients $\asymp p^{(\alpha-\gamma)m}$, $\asymp p^{(\beta+\gamma)m}$;
- $x_{(1-\delta)m}$ has partial quotients $\asymp p^{(\alpha-\delta)m}$, $\asymp p^{(\beta-\delta)m}$, and $\asymp p^{(\delta+\varepsilon)m}$.

We assume that x_m has a maximal partial quotient $r_m \asymp p^{\alpha m}$ and that α is arbitrarily close to $\eta(\mathbf{c})$. We discuss the size of the largest partial quotients of $x_{(1+\gamma)m}$ and $x_{(1-\delta)m}$. Since r_m is assumed to be maximal, they have partial quotients $\asymp r_m p^{-\gamma m} \asymp p^{(\alpha-\gamma)m}$ and $\asymp r_m p^{-\delta m} \asymp p^{(\alpha-\delta)m}$, respectively.

Clearly, we have $0 \leq \varepsilon \leq 1 - \alpha - \beta$. To bound $\widehat{\eta}(\mathbf{c})$ from above, we choose γ in such a way that the two largest partial quotients of $x_{(1+\gamma)m}$ are equal, that is, such that $\alpha - \gamma = \beta + \gamma$ holds. Thus, we take $\gamma = (\alpha - \beta)/2$ and get the bound

$$\widehat{\eta}(\mathbf{c}) \leq \frac{\alpha + \beta}{2(1+\gamma)} = \frac{\alpha + \beta}{2 + \alpha - \beta}.$$

Likewise, we choose δ in such a way that the two largest partial quotients of $x_{(1-\delta)m}$ are equal, that is, such that $\alpha - \delta = \delta + \varepsilon$. Thus, we take $\delta = (\alpha - \varepsilon)/2$ and get the bound

$$\widehat{\eta}(\mathbf{c}) \le \frac{\alpha + \varepsilon}{2(1-\delta)} = \frac{\alpha + \varepsilon}{2 - \alpha + \varepsilon}.$$

Since $\beta \le 1 - \alpha - \varepsilon$, we have established that

$$\widehat{\eta}(\mathbf{c}) \le \min \left\{ \frac{\alpha + \varepsilon}{2 - \alpha + \varepsilon}, \frac{1 - \varepsilon}{2\alpha + 1 + \varepsilon} \right\}.$$

The right-hand side quantity in the minimum is at most equal to $1/\sqrt{5}$ when α is greater than or equal to $(\sqrt{5} - 1)/2$. The left-hand side quantity in the minimum is greater than $1/\sqrt{5}$ if

$$\varepsilon > \frac{2 - (1 + \sqrt{5})\alpha}{\sqrt{5} - 1}.$$

Under this assumption, the right-hand side quantity in the minimum is

$$< \frac{\sqrt{5} - 3 + (1 + \sqrt{5})\alpha}{(2\alpha + 1)(\sqrt{5} - 1) + 2 - (1 + \sqrt{5})\alpha} = \frac{\alpha + 2 - \sqrt{5}}{(2 - \sqrt{5})\alpha + 1} \le \frac{1}{\sqrt{5}},$$

for $\alpha \le (\sqrt{5} - 1)/2$.

This gives $\widehat{\eta}(\mathbf{c}) \le 1/\sqrt{5}$ in all cases and, by Theorem 3.4, we recover the upper bound

$$\widehat{\mu}^{\times}(\xi) \le \frac{5 + \sqrt{5}}{2},$$

established in Theorem 3.1 of [10]. Our approach also shows that if $\widehat{\eta}(\mathbf{c}) = 1/\sqrt{5}$, then $\eta(\mathbf{c}) = (\sqrt{5} - 1)/2$, which, by Theorem 3.4, corresponds to $\mu^{\times}(\xi) = 3 + \sqrt{5}$, as in [10].

We conclude with an open question.

Problem 8.1. *Let $g \ge 2$ be an integer. Do there exist a bounded non-zero sequence of integers $\mathbf{c} = (c_k)_{k \ge 0}$, an infinite set $\mathcal{M}$ of positive integers, and a positive integer C such that all the rational numbers*

$$x_m = \frac{c_0}{g^m} + \frac{c_1}{g^{m-1}} + \cdots + \frac{c_{m-1}}{g}, \quad m \in \mathcal{M},$$

have their partial quotients bounded from above by C?

For a given prime number p, this is a weaker question than the existence of a p-adic number ξ such that

$$\inf_{a,b \ne 0} |ab| \cdot |b\xi - a|_p > 0,$$

which, by Theorem 3 in [1], is equivalent to the existence of a sequence of integers $(c_k)_{k \ge 0}$ in $\{0, 1, \ldots, p-1\}$ and a positive integer C such that all the rational numbers

$$x_m = \frac{c_0}{p^m} + \frac{c_1}{p^{m-1}} + \cdots + \frac{c_{m-1}}{p}, \quad m \ge 1,$$

have their partial quotients bounded from above by C.

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Yann Bugeaud
Université de Strasbourg, IRMA, CNRS, UMR 7501,
7 rue René Descartes, 67084 Strasbourg, France;
Institut universitaire de France;
bugeaud@math.unistra.fr