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Growth of Sobolev norms for $2d$ NLS with harmonic potential

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Abstract. We prove polynomial upper bounds on the growth of solutions to the $2d$ cubic nonlinear Schrödinger equation where the Laplacian is confined by the harmonic potential. Due to better bilinear effects, our bounds improve on those available for the $2d$ cubic nonlinear Schrödinger equation in the periodic setting: our growth rate for a Sobolev norm of order s is $t^{2(s-1)/3+\varepsilon}$, for $s = 2k$ and $k \geq 1$ integer. In the appendix we provide a direct proof, based on integration by parts, of bilinear estimates associated with the harmonic oscillator.

Dedicated to Professor Vladimir Georgiev for his 65's birthday.

1. Introduction

In recent years, the growth of Sobolev norms for solutions to nonlinear dispersive equations generated a huge interest, in relation with weak turbulence phenomena. Concerning upper bounds, we quote the pioneering work of Bourgain [2] and its extension in a series of subsequent papers ([6,8,9,14,18,19,21], to quote only a few of them). On the other end, growth of Sobolev norm cannot occur in settings where the dispersive effect is too strong. For instance, consider the translation invariant cubic defocusing nonlinear Schrödinger equation (NLS) on $\mathbb{R}^2$. Then [10] proved the long standing conjecture that nonlinear solutions scatter to free waves when time goes to infinity and hence no growth phenomena is possible in such setting.

We are interested in the growth of solutions to the following nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + Au \pm u|u|^2 = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0,x) = \varphi(x) \in \mathcal{H}^s, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2)$, the operator A is the usual Laplacian with a harmonic potential,

$$A = -\Delta + |x|^2, \quad \text{where } \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \quad |x|^2 = x_1^2 + x_2^2,$$

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and $\|\varphi\|_{\mathcal{H}^s} = \|A^{s/2}\varphi\|_{L^2}$, where in general we use the notation $L^p = L^p(\mathbb{R}^2)$. We shall also denote $L^p_{t,x} = L^p(\mathbb{R}\times\mathbb{R}^2)$ to emphasize the Lebesgue space of space-time dependent functions, and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$ will denote the set of non-zero positive integers.

Let us first comment briefly about the local Cauchy theory associated with (1.1). By combining preservation of regularity for the linear flow $\|e^{itA}\varphi\|_{\mathcal{H}^s} = \|\varphi\|_{\mathcal{H}^s}$, and that $\mathcal{H}^s$ is an algebra for $s > 1$, one proves the existence of a local solution to (1.1) by fixed point methods; its local time of existence depends on the $\mathcal{H}^s$ norm of the initial datum. Moreover, the solution map is Lipschitz continuous. In order to globalize our solution, one can rely on the Brezis–Gallouët inequality (see [4]) provided that

$$\sup_{t\in(-T_{\min}(\varphi),T_{\max}(\varphi))} \|u(t,x)\|_{\mathcal{H}^1} < \infty, \tag{1.2}$$

where $(-T_{\min}(\varphi), T_{\max}(\varphi))$, with $T_{\min}(\varphi), T_{\max}(\varphi) > 0$, is the maximal time interval of existence of the solution associated with (1.1). In particular, assuming (1.2), $T_{\max}(\varphi) = T_{\min}(\varphi) = \infty$ and a double exponential bound holds:

$$\|u(t,x)\|_{\mathcal{H}^s} \le C\exp(C\exp(C|t|)). \tag{1.3}$$

Solutions to (1.1) satisfy the conservation of the Hamiltonian:

$$\frac{1}{2}\|u(t,x)\|^2_{\mathcal{H}^1} \pm \frac{1}{4}\|u(t,x)\|^4_{L^4} = \frac{1}{2}\|\varphi\|^2_{\mathcal{H}^1} \pm \frac{1}{4}\|\varphi\|^4_{L^4}.$$

Therefore, in the defocusing case, (1.2) is automatically satisfied, while in the focusing case it is not granted for free. Of course, by using more sophisticated tools, e.g., Bourgain's spaces $X^{s,b}$ associated with $i\partial_t + A$, one can deal with initial data at lower regularity than $\mathcal{H}^{1+\varepsilon}$. These $X^{s,b}$ spaces will play a key role in our analysis, as they allow us to exploit a bilinear effect associated with the propagator e^{itA}. They will be defined in Section 2, where we also provide more useful facts about the Cauchy theory.

Our main goal is to improve (1.3) and prove polynomial upper bounds for the quantity $\|u(t,x)\|_{\mathcal{H}^s}$ when $t\to\pm\infty$, with $s > 1$. Along the rest of the paper, the following equivalence of norms will be useful: for every $s \ge 0$, there exists $C > 0$ such that

$$\frac{1}{C}\left(\|D^s u\|^2_{L^2} + \|\langle x\rangle^s u\|^2_{L^2}\right) \le \|\varphi\|^2_{\mathcal{H}^s} \le C\left(\|D^s u\|^2_{L^2} + \|\langle x\rangle^s u\|^2_{L^2}\right), \tag{1.4}$$

where with D^s we denote the operator associated with the Fourier multiplier $|\xi|^s$, and $\langle x\rangle = (1+x_1^2+x_2^2)^{1/2}$. The proof of the equivalence (1.4) is a special case of a more general result proved in [1, 11]. In particular, establishing growth upper bounds on the $\mathcal{H}^s$ norm of the solution is equivalent to establish polynomial bounds on the classical Sobolev norms H^s and the corresponding moments of order s. We now state our main result.

Theorem 1.1. *Let $\varepsilon > 0$ and $k \in \mathbb{N}^*$. For every global solution u to (1.1) such that $u(t,x) \in \mathcal{C}(\mathbb{R}, \mathcal{H}^{2k})$ and*

$$\sup_{t\in\mathbb{R}} \|u(t,x)\|_{\mathcal{H}^1} < \infty, \tag{1.5}$$

there exists a constant C such that

$$\|D^{2k}u(t,x)\|_{L^2} + \|\langle x\rangle^{2k}u(t,x)\|_{L^2} \le C(1+|t|)^{2(2k-1)/3+\varepsilon}.$$

Our bound may be compared to the corresponding bound for solutions to NLS on a generic compact surface M^2, and more specifically on the torus $\mathbb{T}^2$. In fact, at the best of our knowledge, the best known upper bound available on the growth of the classical Sobolev norm $H^{2k}(\mathbb{T}^2)$ for solutions to cubic NLS on $\mathbb{T}^2$ is $(1+t)^{2k-1+\varepsilon}$, as proved in [14, 21]. Notice also that in our case we control the growth of the moments as well (see also [20] for a different perspective on the moments).

Theorem 1.1 may also be compared with Theorem 2 in [6], where the same bound on the growth of Sobolev norm was achieved for the translation invariant cubic NLS posed on $\mathbb{R}^2$, at a time where Dodson's definitive result was not available. As already mentioned, unlike the situation considered in Theorem 1.1, where in general scattering theory is not available, in the Euclidean setting one can deduce uniform boundedness of high order Sobolev norms, at least in the defocusing situation. Nevertheless, the bounds provided in Theorem 1.1 are still meaningful and non-trivial in the flat case either, if one considers solutions to the focusing NLS such that the H^1 norm is uniformly bounded. In fact, under this assumption, it is not true in general that the solutions scatter to a free wave and hence the uniform boundedness of Sobolev norms is not granted.

It would be very interesting to construct solutions to the defocusing equation (1.1) such that the H^k norms do not remain bounded in time for some $k > 1$. Unfortunately, such results are rare in the context of canonical dispersive models (with the notable exception of [12]).

2. $X^{s,b}$ framework and linear estimates

We first define the $X^{s,b}$ spaces associated with the harmonic oscillator in dimension two: the spectrum of the $2d$ harmonic oscillator is given by the following set: $\{2n+2 \,|\, n \in \mathbb{N}\}$. We shall denote by Π_n the orthogonal projector on the eigenspace associated with the eigenvalue $2n+2$. Then the $X^{s,b}$ norm associated with the $2d$ harmonic oscillator A is given by the expression

$$\|u\|^2_{X^{s,b}} = \sum_{n\in\mathbb{N}} (2n+2)^s \left\| \langle \tau + 2n + 2\rangle^b \mathcal{F}_{t\to\tau}(\Pi_n u(t,x)) \right\|^2_{L^2_{\tau,x}},$$

where $u(t,x)$ is a function globally defined on space-time and $\mathcal{F}_{t\to\tau}$ denotes the Fourier transform with respect to the time variable. Along with the $X^{s,b}$ spaces, which are defined for global space-time functions, we also introduce its localized version for every $T > 0$. More precisely, for functions $v(t,x)$ on the strip $(-T,T)\times\mathbb{R}^2$, we define

$$\|v\|_{X^{s,b}_T} = \inf_{\substack{\tilde{v}\in X^{s,b} \\ v(t,x)=\tilde{v}(t,x)_{|(-T,T)\times\mathbb{R}^2}}} \|\tilde{v}\|_{X^{s,b}}.$$

The main result of this section is the continuity of some suitable linear operators in the Bourgain spaces $X^{s,b}_T$.

Proposition 2.1. *For every $\delta \in (0,1/2)$ and $b \in (0,1)$, there exists $C > 0$ such that we have the following estimates for every $T > 0$:*

$$\|Lu\|_{X_T^{-1/2+\delta,1/2-\delta+2\delta b}} \le C\|u\|_{X_T^{1/2+\delta,1/2-\delta+2\delta b}} \tag{2.1}$$

and

$$\|Lu\|_{X_T^{\delta,(1-\delta)b}} \leq C\|u\|_{X_T^{1+\delta,(1-\delta)b}}, \tag{2.2}$$

where L can be either ∂_{x_i} for $i = 1, 2$, or multiplication by $\langle x\rangle$.

Proof. We prove Proposition 2.1 without the time localization. The corresponding version in localized Bourgain spaces is straightforward. We will prove the following bounds:

$$\|Lu\|_{X^{0,b}} \leq C\|u\|_{X^{1,b}}, \quad b \in [0, 1], \tag{2.3}$$

$$\|Lu\|_{X^{1,0}} \leq C\|u\|_{X^{2,0}}. \tag{2.4}$$

Notice that (2.2) follows by interpolation between (2.3) and (2.4). Moreover, we get

$$\|Lu\|_{X^{-1,0}} \leq C\|u\|_{X^{0,0}} \tag{2.5}$$

by duality from (2.3) for $b = 0$, and we also get

$$\|Lu\|_{X^{-1/2,1/2}} \leq C\|u\|_{X^{1/2,1/2}} \tag{2.6}$$

by interpolation between (2.5) and (2.3) for $b = 1$. Then (2.1) follows interpolating (2.3) and (2.6). Hence we focus on (2.3) and (2.4). Since the proof is slightly different depending from the operator L that we consider, we distinguish two cases.

First case: proof of (2.3) and (2.4) for $Lu = \partial_{x_i}u$.

First we prove that, for space-time dependent functions $u(t, x)$, we have

$$\|\partial_{x_i}u\|_{X^{0,0}} \leq C\|u\|_{X^{1,0}}. \tag{2.7}$$

This estimate is a consequence of the following one for time independent functions $v(x)$:

$$\|\partial_{x_i}v\|_{L^2} \leq C\|\sqrt{A}v\|_{L^2},$$

that in turn follows by $\|\sqrt{A}v\|_{L^2} = \|v\|_{\mathcal{H}^1}$ and by recalling (1.4) for $s = 1$. Next we prove

$$\|\partial_{x_i}u\|_{X^{0,1}} \leq C\|u\|_{X^{1,1}}, \tag{2.8}$$

and by interpolation with (2.7), (2.3) will follow for $L = \partial_{x_i}$. As $\|w(t,x)\|_{X^{0,1}}$ is equivalent to $\|(i\partial_t + A)w\|_{L^2_{t,x}} + \|w\|_{L^2_{t,x}}$, in order to get (2.8) we estimate

$$\begin{aligned}
&\|(i\partial_t + A)\,\partial_{x_i}u\|_{L^2_{t,x}} + \|\partial_{x_i}u\|_{L^2_{t,x}} \\
&\quad = \|\partial_{x_i}(i\partial_t + A)u + [|x|^2, \partial_{x_i}]u\|_{L^2_{t,x}} + \|\partial_{x_i}u\|_{L^2_{t,x}} \\
&\quad \leq \|\partial_{x_i}(i\partial_t + A)u\|_{L^2_{t,x}} + 2\||x|u\|_{L^2_{t,x}} + \|\partial_{x_i}u\|_{L^2_{t,x}}.
\end{aligned} \tag{2.9}$$

By combining (2.7) with the identity

$$\|\sqrt{A}v\|_{L^2}^2 = (Av, v) = \|\nabla_x v\|_{L^2}^2 + \||x|v\|_{L^2}^2,$$

we can continue (2.9) as follows:

$$(\cdots) \leq \|(i\partial_t + A)u\|_{X^{1,0}} + 3\|\sqrt{A}u\|_{L^2_{t,x}} \leq \|u\|_{X^{1,1}} + 3\|u\|_{X^{1,0}} \leq 4\|u\|_{X^{1,1}}.$$

and (2.8) for $L = \partial_{x_i}$ follows. Next we prove (2.4) (where $L = \partial_{x_i}$), namely

$$\|\partial_{x_i} u\|_{X^{1,0}} \leq C\|u\|_{X^{2,0}}.$$

This estimate is a consequence of the following one for time independent functions $v(x)$:

$$\|\sqrt{A}\partial_{x_i} v\|_{L^2} \leq C\|Av\|_{L^2},$$

that in turn is equivalent to

$$(A\partial_{x_i} v, \partial_{x_i} v) \leq C(Av, Av).$$

As on the right-hand side we get $\|v\|^2_{\mathcal{H}^2}$, by (1.4) and elementary considerations it is sufficient to prove

$$\int |x|^2 |\partial_{x_i} v|^2 \leq C(\|D^2 v\|^2_{L^2} + \|\langle x\rangle^2 v\|^2_{L^2}). \tag{2.10}$$

In turn, this last inequality follows by combining integration by parts and the Cauchy–Schwarz inequality:

$$\begin{aligned}\int |x|^2|\partial_{x_i} v|^2 &= -\int |x|^2\, \partial^2_{x_i} v\, \bar{v} - 2\int x_i \partial_{x_i} v\, \bar{v} \\ &\leq \|\partial^2_{x_i} v\|_{L^2}\, \||x|^2 v\|_{L^2} + 2\||x|\,\partial_{x_i} v\|_{L^2}\, \|v\|_{L^2} \\ &\leq \frac{1}{2}\|D^2 u\|^2_{L^2} + \frac{1}{2}\|\langle x\rangle^2 v\|^2_{L^2} + \frac{1}{2}\||x|\partial_{x_i} v\|^2_{L^2} + 2\|\langle x\rangle^2 v\|^2_{L^2},\end{aligned}$$

from which we easily conclude moving $\frac{1}{2}\||x|\partial_{x_i} v\|^2_{L^2}$ to the left-hand side.

Second case: proof of (2.3) *and* (2.4) *for* $Lu = \langle x\rangle u$.

The proof follows the same steps as in the case $L = \partial_{x_i}$, with minor modifications. First notice that, for space-time dependent functions $u(t,x)$, we have the following:

$$\|\langle x\rangle u\|_{X^{0,0}} \leq C\|u\|_{X^{1,0}}. \tag{2.11}$$

This is a consequence of the following estimate for time independent functions $v(x)$:

$$\|\langle x\rangle v\|_{L^2} \leq C\|\sqrt{A}v\|_{L^2},$$

that in turn follows by noticing that $\|\sqrt{A}v\|_{L^2} = \|v\|_{\mathcal{H}^1}$ and recalling (1.4) for $s = 1$. Moreover, we have

$$\|\langle x\rangle u\|_{X^{0,1}} \leq C\|u\|_{X^{1,1}}.$$

that by interpolation with (2.11) implies (2.3) for $L = \langle x \rangle$. In order to prove this estimate, recall again that $\|w(t,x)\|_{X^{0,1}}$ is equivalent to $\|i\partial_t w + Aw\|_{L^2_{t,x}} + \|w\|_{L^2_{t,x}}$ and hence we compute

$$\begin{aligned}\|(i\partial_t + A)(\langle x\rangle u)\|_{L^2_{t,x}} + \|\langle x\rangle u\|_{L^2_{t,x}} &= \|\langle x\rangle(i\partial_t + A)u + [\Delta, \langle x\rangle]u\|_{L^2_{t,x}} + \|\langle x\rangle u\|_{L^2_{t,x}}\\ &\le \|\langle x\rangle(i\partial_t + A)u\|_{L^2_{t,x}} + \|2\nabla(\langle x\rangle)\cdot\nabla u + \Delta(\langle x\rangle)u\|_{L^2_{t,x}} + \|\langle x\rangle u\|_{L^2_{t,x}}\\ &\le C\big(\|\langle x\rangle(i\partial_t + A)u\|_{L^2_{t,x}} + \|\nabla u\|_{L^2_{t,x}} + \|\langle x\rangle u\|_{L^2_{t,x}}\big).\end{aligned}$$

By combining (2.11) with the identity $\|\sqrt{A}u\|_{L^2_{t,x}} = \|u\|_{\mathcal{H}^1}$ and by recalling (1.4) for $s = 1$, we can proceed with our estimate above:

$$(\cdots) \le C\big(\|(i\partial_t + A)u\|_{X^{1,0}} + \|\sqrt{A}u\|_{L^2_{t,x}}\big) = C(\|u\|_{X^{1,1}} + \|u\|_{X^{1,0}}) \le C\|u\|_{X^{1,1}}.$$

Next we prove (2.4) (where $L = \langle x \rangle$), namely

$$\|\langle x\rangle u\|_{X^{1,0}} \le C\|u\|_{X^{2,0}}.$$

This estimate is a consequence of the following one for time independent functions $v(x)$:

$$\|\sqrt{A}(\langle x\rangle v)\|_{L^2} \le C\|Av\|_{L^2},$$

that in turn is equivalent to

$$(A(\langle x\rangle v), \langle x\rangle v) \le C\|v\|_{\mathcal{H}^2}.$$

By (1.4), this is equivalent to

$$\|\nabla(\langle x\rangle v)\|_{L^2}^2 + \|\langle x\rangle\,|x|\,v\|_{L^2}^2 \le C(\|D^2 v\|_{L^2}^2 + \|\langle x\rangle^2 v\|_{L^2}^2).$$

In turn, developing the gradient on the left-hand side, the estimate above follows from

$$\int \langle x\rangle^2\,|\nabla v|^2 \le C(\|D^2 v\|_{L^2}^2 + \|\langle x\rangle^2 v\|_{L^2}^2),$$

whose proof proceeds by integration by parts and Cauchy–Schwarz inequality as we did for (2.10). ∎

3. The Cauchy theory in $X^{s,b}$ and consequences

We first obtain a trilinear estimate, whose proof heavily relies on the analysis of [16] (also available as [17]); for the sake of completeness, we provide a relatively elementary proof of the crucial bilinear estimate from [16] in the appendix, using the bilinear virial techniques from [15]. The only novelty in our trilinear estimate is that we prove a tame estimate, while such an estimate was not needed for the low regularity analysis of [16]. We first recall the following key bilinear estimate (see Theorem 2.3.13 in [16]). There exists $\delta_0 \in (0, 1/2]$ such that for every $\delta \in (0, \delta_0]$ there exist $b' < 1/2$ and $C > 0$ such that

$$\begin{aligned}(3.1)\quad &\|\Delta_N(u)\,\Delta_M(v)\|_{L^2((0,T);L^2)} \\ &\leq C(\min(M,N))^{\delta}\Big(\frac{\min(M,N)}{\max(M,N)}\Big)^{1/2-\delta}\|\Delta_N(u)\|_{X_T^{0,b'}}\,\|\Delta_M(v)\|_{X_T^{0,b'}},\end{aligned}$$

where Δ_N and Δ_M are the Littlewood–Paley localization operators associated with A, and N and M are dyadic integers.

Proposition 3.1. *Let $0 < T < 1$ and $\varepsilon > 0$ be fixed. Then there exist $C > 0$, $b > 1/2$ and $\gamma > 0$ such that, for $s \geq \varepsilon$,*

$$\begin{aligned}(3.2)\quad &\Big\|\int_0^t e^{i(t-\tau)A}\,(u_1(\tau)u_2(\tau)\bar{u}_3(\tau))\,d\tau\Big\|_{X_T^{s,b}} \\ &\leq CT^{\gamma}\sum_{\sigma\in\mathcal{S}_3}\|u_{\sigma(1)}\|_{X_T^{s,b}}\,\|u_{\sigma(2)}\|_{X_T^{\varepsilon,b}}\,\|u_{\sigma(3)}\|_{X_T^{\varepsilon,b}}.\end{aligned}$$

Proof. Using standard arguments (see for instance Proposition 3.3 in [5]), it suffices to prove that

$$\|u_1u_2\bar{u}_3\|_{X_T^{s,-b'}} \leq C\sum_{\sigma\in\mathcal{S}_3}\|u_{\sigma(1)}\|_{X_T^{s,b}}\,\|u_{\sigma(2)}\|_{X_T^{\varepsilon,b}}\,\|u_{\sigma(3)}\|_{X_T^{\varepsilon,b}}$$

for some $b > 1/2$ and $b' < 1/2$ such that $b + b' < 1$. Using duality, the last estimate is equivalent to

$$\Big|\iint u_1u_2\bar{u}_3\bar{u}_0\Big| \leq C\|u_0\|_{X_T^{-s,b'}}\sum_{\sigma\in\mathcal{S}_3}\|u_{\sigma(1)}\|_{X_T^{s,b}}\|u_{\sigma(2)}\|_{X_T^{\varepsilon,b}}\|u_{\sigma(3)}\|_{X_T^{\varepsilon,b}}.$$

where $\iint$ denotes a space-time integral on $(-T,T)\times\mathbb{R}^2$ with respect to the Lebesgue measure $dx\,dt$. We now perform a Littlewood–Paley decomposition in the left-hand side of the last inequality, and using a symmetry argument, we are reduced to obtaining a bound on

$$(3.3)\qquad \Big|\sum_{N_1\geq N_2\geq N_3}\sum_{N_0}\iint \Delta_{N_0}(\bar{u}_0)\,\Delta_{N_1}(u_1)\,\Delta_{N_2}(u_2)\,\Delta_{N_3}(\bar{u}_3)\Big|,$$

where the summation is meant over dyadic values of N_1, N_2, N_3 and N_0. The other possible orders of magnitudes of N_1, N_2 and N_3 provide all permutations involved in the sum of the right hand-side of (3.2). Next we split the analysis of the terms in the expression (3.3) depending on the relation between N_0, N_1, N_2 and N_3.

First case: $N_0 \geq N_1^{1+\rho}$ *for some* $\rho > 0$.

In this case, we can apply the $2d$ version of Lemme 2.1.23 in [16] to obtain that for every K, there is C_K such that

$$\begin{aligned}&\Big|\iint \Delta_{N_0}(\bar{u}_0)\,\Delta_{N_1}(u_1)\,\Delta_{N_2}(u_2)\,\Delta_{N_3}(\bar{u}_3)\Big| \\ &\leq C_K N_0^{-K}\,\|\Delta_{N_0}u_0\|_{X_T^{0,b'}}\,\|\Delta_{N_1}u_1\|_{X_T^{0,b'}}\,\|\Delta_{N_2}u_2\|_{X_T^{0,b'}}\|\,\Delta_{N_3}u_3\|_{X_T^{0,b'}},\end{aligned}$$

where $b' < 1/2$. And hence we get

$$\Big| \sum_{\substack{N_1 \geq N_2 \geq N_3 \\ N_0 \geq N_1^{1+\rho}}} \iint \Delta_{N_0}(\bar{u}_0)\, \Delta_{N_1}(u_1)\, \Delta_{N_2}(u_2)\, \Delta_{N_3}(\bar{u}_3) \Big|$$
$$\leq \sum_{\substack{N_1 \geq N_2 \geq N_3 \\ N_0 \geq N_1^{1+\rho}}} \frac{C_K}{N_0^{K-s} N_1^s} \, \|\Delta_{N_0} u_0\|_{X_T^{-s,b'}} \, \|\Delta_{N_1} u_1\|_{X_T^{s,b'}} \, \|\Delta_{N_2} u_2\|_{X_T^{0,b'}} \| \, \Delta_{N_3} u_3\|_{X_T^{0,b'}},$$

and thus, by choosing $K > s$, we can successively sum in N_3, N_2, N_1 and N_0 to get

$$(\cdots) \leq C \, \|u_0\|_{X_T^{-s,b'}} \, \|u_1\|_{X_T^{s,b'}} \, \|u_2\|_{X_T^{\varepsilon,b'}} \, \|u_3\|_{X_T^{\varepsilon,b'}}.$$

Second case: $N_1 \leq N_0 \leq N_1^{1+\rho}$, *with* $0 < \rho s < \varepsilon$.

We have, by combining Cauchy–Schwarz and (3.1),

$$\Big| \sum_{\substack{N_1^{1+\rho} \geq N_0 \\ N_0 \geq N_1 \geq N_2 \geq N_3}} \iint \Delta_{N_0}(\bar{u}_0)\, \Delta_{N_1}(u_1)\, \Delta_{N_2}(u_2)\, \Delta_{N_3}(\bar{u}_3) \Big|$$
$$\leq C (N_2 N_3)^{\varepsilon/2} \left(\frac{N_2 N_3}{N_0 N_1} \right)^{1/2-\varepsilon/2} \prod_{j=0}^{3} \|\Delta_{N_j}(u_j)\|_{X_T^{0,b'}}$$
$$\leq C \, \frac{(N_2 N_3)^{1/2-\varepsilon}}{(N_0 N_1)^{1/2-\varepsilon/2}} \left(\frac{N_0}{N_1} \right)^s \|\Delta_{N_0}(u_0)\|_{X_T^{-s,b'}} \, \|\Delta_{N_1}(u_1)\|_{X_T^{s,b'}}$$
$$\times \|\Delta_{N_2}(u_2)\|_{X_T^{\varepsilon,b'}} \, \|\Delta_{N_3}(u_3)\|_{X_T^{\varepsilon,b'}}.$$

Summing over $N_2, N_3 \leq N_1$ and using $N_0 \leq N_1^{1+\rho}$, we get

$$(\cdots) \leq C \, \frac{N_1^{1-2\varepsilon}}{(N_0 N_1)^{1/2-\varepsilon/2}} \, N_1^{\rho s} \, \|\Delta_{N_0}(u_0)\|_{X_T^{-s,b'}} \, \|\Delta_{N_1}(u_1)\|_{X_T^{s,b'}} \, \|u_2\|_{X_T^{\varepsilon,b'}} \, \|u_3\|_{X_T^{\varepsilon,b'}}$$
$$\leq \|u_0\|_{X_T^{-s,b'}} \, \|u_1\|_{X_T^{s,b'}} \, \|u_2\|_{X_T^{\varepsilon,b'}} \, \|u_3\|_{X_T^{\varepsilon,b'}},$$

where at the last step we summed over $N_0 \geq N_1$ and then used that $\rho s - \varepsilon < 0$ in order to sum on N_1.

Third case: $N_2 \leq N_0 \leq N_1$.

Again by combining Cauchy–Schwarz and (3.1), we get

$$\Big| \sum_{N_1 \geq N_0 \geq N_2 \geq N_3} \iint \Delta_{N_0}(\bar{u}_0)\, \Delta_{N_1}(u_1)\, \Delta_{N_2}(u_2)\, \Delta_{N_3}(\bar{u}_3) \Big|$$
$$\leq C \, (N_2 N_3)^{\varepsilon/2} \left(\frac{N_2 N_3}{N_0 N_1} \right)^{1/2-\varepsilon/2} \prod_{j=0}^{3} \|\Delta_{N_j}(u_j)\|_{X_T^{0,b'}},$$

and from $N_2 N_3 \le N_0^2$, we get

$$(\cdots) \le \frac{C}{(N_2 N_3)^{\varepsilon/2}} \Big(\frac{N_0}{N_1}\Big)^{s+1/2-\varepsilon/2} \|\Delta_{N_0}(u_0)\|_{X_T^{-s,b'}} \|\Delta_{N_1}(u_1)\|_{X_T^{s,b'}} \|u_2\|_{X_T^{\varepsilon,b'}} \|u_3\|_{X_T^{\varepsilon,b'}},$$

where we can sum over N_2 and N_3, and then sum on $N_0 \le N_1$ by Schur's lemma, as $s + 1/2 - \varepsilon/2 > 0$.

Fourth case: $N_0 \le N_2$.

We have, by combining Cauchy–Schwarz and (3.1),

$$\begin{aligned}
\Big| \sum_{\substack{N_1 \ge N_2 \ge N_3 \\ N_2 \ge N_0}} \iint \Delta_{N_0}(\bar u_0)\, \Delta_{N_1}(u_1)\, \Delta_{N_2}(u_2)\, \Delta_{N_3}(\bar u_3) \Big| \\
&\le C (N_0 N_3)^{\varepsilon/2} \Big(\frac{N_0 N_3}{N_1 N_2}\Big)^{1/2-\varepsilon/2} \prod_{j=0}^{3} \|\Delta_{N_j}(u_j)\|_{X_T^{0,b'}} \\
&\le C\, (N_2 N_3)^{\varepsilon/2} \Big(\frac{N_0}{N_1}\Big)^{1/2+s-\varepsilon/2} \|\Delta_{N_0}(u_0)\|_{X_T^{-s,b'}} \\
&\quad \times \|\Delta_{N_1}(u_1)\|_{X_T^{s,b'}} \|\Delta_{N_2}(u_2)\|_{X_T^{0,b'}} \|\Delta_{N_3}(u_3)\|_{X_T^{0,b'}}
\end{aligned}$$

where we used that $N_0 \le N_2$ and $N_3 \le N_2$. We can then sum as in the third case to get

$$(\cdots) \le C \|u_0\|_{X_T^{-s,b'}} \|u_1\|_{X_T^{s,b'}} \|u_2\|_{X_T^{\varepsilon,b'}} \|u_3\|_{X_T^{\varepsilon,b'}},$$

which concludes our proof. ∎

As a standard consequence of Proposition 3.1 (see e.g. Proposition 3.3 in [5]), we can obtain the following well-posedness result.

Proposition 3.2. *For $R > 0$, there exist $T = T(R) > 0$ and $b > 1/2$ such that (1.1) has a unique local solution $u \in X_T^{s_0,b}$ for every $\varphi \in \mathcal{H}^{s_0}$, $s_0 \ge 1$, with $\|\varphi\|_{\mathcal{H}^1} < R$. Moreover, for every $s \in (0, s_0]$ there exists $C = C(R, s)$ such that*

$$\|u(t,x)\|_{X_T^{s,b}} \le C \|\varphi\|_{\mathcal{H}^s}. \tag{3.4}$$

Our next proposition reduces studying the growth of the $\mathcal{H}^{2k}$ norm of the solution $u(t,x)$ to the analysis of the growth of $\|\partial_t^k u(t,x)\|_{L^2}$. In fact, this last quantity is easier to handle, as ∂_t has better commutation properties with the nonlinear Schrödinger flow than the operator A.

Proposition 3.3. *Let $s, k \in \mathbb{N}$ and $R > 0$. Set $T = T(R)$ and $s_0 = 2k + s$ in Proposition 3.2, and let $u(t,x) \in X_T^{s_0,b}$ be the unique local solution to (1.1) with $\varphi \in \mathcal{H}^{s_0}$. Assume moreover that $\sup_{t\in(-T,T)} \|u(t,x)\|_{\mathcal{H}^1} < R$. Then there exists $C = C(R, s_0)$ such that*

$$\forall t \in (-T,T), \quad \|\partial_t^k u(t) - i^k A^k u(t)\|_{\mathcal{H}^s} \le C \|u(t)\|_{\mathcal{H}^{s_0-1}}. \tag{3.5}$$

Proof. We temporarily drop the dependence on t, since the estimates we prove are pointwise in time. We start from the identity

$$\partial_t^h u = i^h A^h u + \sum_{j=0}^{h-1} c_j\, \partial_t^j A^{h-j-1}(u|u|^2), \tag{3.6}$$

that holds for every integer $h \geq 1$ and for suitable coefficients $c_j \in \mathbb{C}$. Its elementary proof follows by induction on h, using the equation solved by $u(t,x)$.

Next we argue by induction on k in order to establish (3.5). For $k = 0$ and all $s \in \mathbb{N}$, there is nothing to prove, as the left-hand side vanishes: $\partial_t^0 u - i^0 A^0 u = 0$. Assuming that (3.5) holds for all $s \geq 0$ integers and for all integers up to rank k, we shall prove that the same estimate is true for $k+1$ and for every integer $s \geq 0$. Indeed, by (3.6), where we choose $h = k+1$, the estimate (3.5) for $k+1$ reduces to

$$\|\partial_t^j (u|u|^2)\|_{\mathcal{H}^{2k-2j+s}} \leq C\|u\|_{\mathcal{H}^{s+2k+1}}, \quad j = 0,\dots,k,\ s \in \mathbb{N}.$$

Recalling (1.4), we have to prove

$$\|D^{2k-2j+s}\partial_t^j (u|u|^2)\|_{L^2} \leq C\|u\|_{\mathcal{H}^{s+2k+1}}, \quad j = 0,\dots,k,\ s \in \mathbb{N}, \tag{3.7}$$

$$\|\langle x\rangle^{2k-2j+s}\partial_t^j (u|u|^2)\|_{L^2} \leq C\|u\|_{\mathcal{H}^{s+2k+1}}, \quad j = 0,\dots,k,\ s \in \mathbb{N}. \tag{3.8}$$

We may now replace the operator D by the usual gradient operator ∇, as we are operating on L^2; then, to prove (3.7), we expand the time and space derivatives on the new left-hand side, using the Leibniz rule. Hence, by expanding the space-time derivatives and by using Hölder, we can estimate as follows the left-hand side in (3.7):

$$\begin{aligned}\sum_{\substack{j_1+j_2+j_3=j\\ s_1+s_2+s_3=2k-2j+s}} \prod_{l=1,2,3} \|\nabla^{s_l}\partial_t^{j_l} u\|_{L^6} &\leq C \sum_{\substack{j_1+j_2+j_3=j\\ s_1+s_2+s_3=2k-2j+s}} \prod_{l=1,2,3} \|\partial_t^{j_l} u\|_{\mathcal{H}^{s_l+1}}\\ &\leq C \sum_{\substack{j_1+j_2+j_3=j\\ s_1+s_2+s_3=2k-2j+s}} \prod_{l=1,2,3} \|u\|_{\mathcal{H}^{2j_l+s_l+1}},\end{aligned}$$

where we used the (non-sharp) Sobolev embedding and the induction hypothesis.

We proceed with a trivial interpolation argument, with $\theta_l(s+2k+1) + (1-\theta_l) = 2j_l + s_l + 1$, to get

$$\prod_{l=1,2,3} \|u\|_{\mathcal{H}^{s_l+2j_l+1}} \leq \prod_{l=1,2,3} \|u\|_{\mathcal{H}^{s+2k+1}}^{\theta_l} \|u\|_{\mathcal{H}^1}^{(1-\theta_l)} \leq \|u\|_{\mathcal{H}^{s+2k+1}} \|u\|_{\mathcal{H}^1}^2.$$

As $\theta_l = (2j_l + s_l)/(s+2k)$, we observe that $\sum_l \theta_l = 1$. This closes the induction argument.

We can deal with (3.8) in a similar way: by using the Leibniz rule with respect to the time variable, and the Sobolev embedding, we estimate the left-hand side in (3.8) as follows:

$$\sum_{\substack{j_1+j_2+j_3=j\\ s_1+s_2+s_3=2k-2j+s}} \prod_{l=1,2,3} \|\langle x\rangle^{s_l}\partial_t^{j_l} u\|_{L^6} \leq C \sum_{\substack{j_1+j_2+j_3=j\\ s_1+s_2+s_3=2k-2j+s}} \prod_{l=1,2,3} \|\partial_t^{j_l} u\|_{\mathcal{H}^{s_l+1}}.$$

Then we are reduced at the same situation as above in order to get (3.7), which concludes the proof of the proposition. ■

The next proposition will be crucial in the sequel. It allows to estimate the norm of time derivatives of the solution in the localized $X_T^{s,b}$ spaces, by using suitable Sobolev norms of the initial datum.

Proposition 3.4. *Let $l \in \mathbb{N}$, $R > 0$ and $s \in (0,2)$. Set $T = T(R)$ and $s_0 = 2l+2$ in Proposition 3.2, let b satisfy Proposition 3.1 and let $u(t,x) \in X_T^{2l+2,b}$ be the unique local solution to (1.1) with initial condition $\varphi \in \mathcal{H}^{2l+2}$, $\|\varphi\|_{\mathcal{H}^1} < R$.*

Assume moreover that

$$\sup_{t\in(-T,T)} \|u(t,x)\|_{\mathcal{H}^1} < R.$$

Then there exists $C > 0$ such that

$$\|\partial_t^l u\|_{X_T^{s,b}} \le C\|\varphi\|_{\mathcal{H}^{2l}}^{1-s} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s}, \quad \text{if } s \in (0,1] \tag{3.9}$$

and

$$\|\partial_t^l u\|_{X_T^{s,b}} \le C\|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s} \|\varphi\|_{\mathcal{H}^{2l+2}}^{s-1}, \quad \text{if } s \in (1,2). \tag{3.10}$$

Proof. We shall prove separately (3.9) and (3.10) by induction on l. In the case $l = 0$, the estimates (3.9) and (3.10) follow from (3.4). Consider the integral formulation of the equation solved by $\partial_t^l u$:

$$\partial_t^l u(t) = e^{itA}\,\partial_t^l u(0) + \int_0^t e^{i(t-\tau)A}\,\partial_\tau^l(u(\tau)|u(\tau)|^2)\,d\tau.$$

Then, by standard properties of the $X^{s,b}$ spaces,

$$\|\partial_t^l u\|_{X_T^{s,b}} \le C\Big(\|\partial_t^l u(0)\|_{\mathcal{H}^s} + \Big\|\int_0^t e^{i(t-\tau)A}\,\partial_\tau^l(u(\tau)|u(\tau)|^2)\,d\tau\Big\|_{X_T^{s,b}}\Big). \tag{3.11}$$

Next we argue by induction to show that (3.9) is true for l provided that it is satisfied for all integers up to $l-1$. Expanding the time derivative in (3.11) and using Proposition 3.1, we get

$$\|\partial_t^l u(t)\|_{X_T^{s,b}} \le C\Big(\|\partial_t^l u(0)\|_{\mathcal{H}^s} + T^\gamma \sum_{l_1+l_2+l_3=l} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{s,b}} \|\partial_t^{l_3} u\|_{X_T^{s,b}}\Big). \tag{3.12}$$

By interpolation and Proposition 3.3, we also have

$$\begin{aligned}\|\partial_t^l u(0)\|_{\mathcal{H}^s} &\le \|\partial_t^l u(0)\|_{\mathcal{H}^1}^s \|\partial_t^l u(0)\|_{L^2}^{1-s} \le C\|u(0)\|_{\mathcal{H}^{2l+1}}^s \|u(0)\|_{\mathcal{H}^{2l}}^{1-s} \\ &\le C\|\varphi\|_{\mathcal{H}^{2l+1}}^s \|\varphi\|_{\mathcal{H}^{2l}}^{1-s}.\end{aligned} \tag{3.13}$$

Therefore, estimating the second term on the right-hand side in (3.12) is sufficient. We deal with three cases: the first two are lower order terms and use the induction hypothesis, the third one leads to a bootstrap argument to close the estimate.

First case: $0 < \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l$.

We use our induction hypothesis on l_1, l_2 and l_3 and estimate as follows:

$$\begin{aligned}
&\sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l\\ \min\{l_1,l_2,l_3\}>0}} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{s,b}} \|\partial_t^{l_3} u\|_{X_T^{s,b}} \\
&\quad \leq C \sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l\\ \min\{l_1,l_2,l_3\}>0}} \|\varphi\|_{\mathcal{H}^{2l_1}}^{1-s} \|\varphi\|_{\mathcal{H}^{2l_2}}^{1-s} \|\varphi\|_{\mathcal{H}^{2l_3}}^{1-s} \|\varphi\|_{\mathcal{H}^{2l_1+1}}^{s} \|\varphi\|_{\mathcal{H}^{2l_2+1}}^{s} \|\varphi\|_{\mathcal{H}^{2l_3+1}}^{s} \\
&\quad \leq C \|\varphi\|_{\mathcal{H}^{2l}}^{(1-s)(\eta_1+\eta_2+\eta_3)} \|\varphi\|_{\mathcal{H}^{1}}^{(1-s)(3-\eta_1-\eta_2-\eta_3)} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s(\theta_1+\theta_2+\theta_3)} \|\varphi\|_{\mathcal{H}^{1}}^{s(3-\theta_1-\theta_2-\theta_3)},
\end{aligned}$$

where

$$\begin{cases} 2l\eta_1 + 1 - \eta_1 = 2l_1, \\ 2l\eta_2 + 1 - \eta_2 = 2l_2, \\ 2l\eta_3 + 1 - \eta_3 = 2l_3, \end{cases} \quad \text{and} \quad \begin{cases} \theta_1(2l+1) + 1 - \theta_1 = 2l_1 + 1, \\ \theta_2(2l+1) + 1 - \theta_2 = 2l_2 + 1, \\ \theta_3(2l+1) + 1 - \theta_3 = 2l_3 + 1. \end{cases}$$

Noting that $\theta_1 + \theta_2 + \theta_3 = 1$ and $\eta_1 + \eta_2 + \eta_3 = \eta < 1$, we get

$$(\cdots) \leq C \|\varphi\|_{\mathcal{H}^{2l}}^{(1-s)\eta} \|\varphi\|_{\mathcal{H}^{1}}^{(1-s)(3-\eta)+2s} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s} \leq C \|\varphi\|_{\mathcal{H}^{2l}}^{(1-s)\eta} \|\varphi\|_{\mathcal{H}^{1}}^{(1-s)(1-\eta)} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s},$$

where we used our bound on $\mathcal{H}^1$ to subsume a factor $\|\varphi\|_{\mathcal{H}^1}^2$ into the constant. Using that $\mathcal{H}^{2l} \subset \mathcal{H}^1$, we conclude that

$$\sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l\\ \min\{l_1,l_2,l_3\}>0}} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{s,b}} \|\partial_t^{l_3} u\|_{X_T^{s,b}} \leq C \|\varphi\|_{\mathcal{H}^{2l}}^{1-s} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s}.$$

Second case: $0 = \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l$.

We can assume $l_1 = 0$. Then we argue exactly as above except that, since $\|\varphi\|_{\mathcal{H}^{2l_1}} = \|\varphi\|_{L^2}$ is bounded (since we assume a control on the $\mathcal{H}^1$ norm of the initial datum), it is not necessary to introduce the parameter η_1. Hence we need only η_2, η_3, θ_1, θ_2 and θ_3. The conclusion is the same as above.

Third case: $\max\{l_1, l_2, l_3\} = l$.

We estimate the terms in the sum at the right-hand side of (3.12) as follows:

$$\|u\|_{X_T^{s,b}}^2 \|\partial_t^l u\|_{X_T^{s,b}} \leq \|u\|_{X_T^{1,b}}^2 \|\partial_t^l u\|_{X_T^{s,b}} \leq C \|\varphi\|_{\mathcal{H}^1}^2 \|\partial_t^l u\|_{X_T^{s,b}},$$

where we have used (3.4) for $s_0 = 1$.

Collecting (3.12), (3.13) and the estimates above in the three cases, we get

$$\|\partial_t^l u(t)\|_{X_T^{s,b}} \leq C \big(\|\varphi\|_{\mathcal{H}^{2l+1}}^{s} \|\varphi\|_{\mathcal{H}^{2l}}^{1-s} + T^{\gamma} \|\partial_t^l u\|_{X_T^{s,b}} \big).$$

We conclude by choosing a time $\bar{T} > 0$ small enough in such a way that the second term on the right-hand side can be absorbed by the left-hand side. Notice that the bound that we get on the short time $\bar{T}$ can be iterated since the constants depends only from the $\mathcal{H}^1$ norm of the solution, and hence we get the desired bound (3.9) up to our chosen time T after a finite iteration of the previous argument.

Next we proceed with proving (3.10) again by induction on l. By developing as above the time derivative in (3.11) and by using Proposition 3.1, we get

$$\|\partial_t^l u(t)\|_{X_T^{s,b}} \le C\Big(\|\partial_t^l u(0)\|_{\mathcal{H}^s} + T^\gamma \sum_{l_1+l_2+l_3=l} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{2-s,b}} \|\partial_t^{l_3} u\|_{X_T^{2-s,b}}\Big). \tag{3.14}$$

We first notice that by interpolation and Proposition 3.3 (see the proof of (3.13)),

$$\|\partial_t^l u(0)\|_{\mathcal{H}^s} \le C\|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s}\|\varphi\|_{\mathcal{H}^{2l+2}}^{s-1}.$$

Next we estimate the sum on the right-hand side of (3.14) by again considering three cases.

First case: $0 < \min\{l_1, l_2, l_3\} \le \max\{l_1, l_2, l_3\} < l$.

Using our induction claim on l_1, l_2, l_3 and (3.9), we get

$$\begin{aligned}
&\sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l\\ \min\{l_1,l_2,l_3\}>0}} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{2-s,b}} \|\partial_t^{l_3} u\|_{X_T^{2-s,b}}\\
&\le C \sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l\\ \min\{l_1,l_2,l_3\}>0}} \|\varphi\|_{\mathcal{H}^{2l_1+1}}^{2-s}\|\varphi\|_{\mathcal{H}^{2l_2}}^{s-1}\|\varphi\|_{\mathcal{H}^{2l_3}}^{s-1}\|\varphi\|_{\mathcal{H}^{2l_1+2}}^{s-1}\|\varphi\|_{\mathcal{H}^{2l_2+1}}^{2-s}\|\varphi\|_{\mathcal{H}^{2l_3+1}}^{2-s}\\
&\le C\|\varphi\|_{\mathcal{H}^{2l+1}}^{(2-s)(\eta_1+\theta_2+\theta_3)}\|\varphi\|_{\mathcal{H}^{2l+2}}^{(s-1)(\theta_1+\eta_2+\eta_3)}\|\varphi\|_{\mathcal{H}^1}^{(s-1)(3-\theta_1-\eta_2-\eta_3)}\|\varphi\|_{\mathcal{H}^1}^{(2-s)(3-\eta_1-\theta_2-\theta_3)},
\end{aligned}$$

where we used the convexity of Sobolev norms, with

$$\begin{cases} \eta_1(2l+1)+1-\eta_1 = 2l_1+1,\\ \eta_2(2l+2)+1-\eta_2 = 2l_2,\\ \eta_3(2l+2)+1-\eta_3 = 2l_3, \end{cases} \quad\text{and}\quad \begin{cases} \theta_1(2l+2)+1-\theta_1 = 2l_1+2,\\ \theta_2(2l+1)+1-\theta_2 = 2l_2+1,\\ \theta_3(2l+1)+1-\theta_3 = 2l_3+1. \end{cases}$$

By direct computation, $\eta_1+\theta_2+\theta_3 = 1$ and $\theta_1+\eta_2+\eta_3 = \mu < 1$. Hence, using $\mathcal{H}^{2l+2} \subset \mathcal{H}^1$ on the next to last factor and discarding the last one, we deduce that

$$\begin{aligned}
&\sum_{\substack{l_1+l_2+l_3=l\\ \max\{l_1,l_2,l_3\}<l}} \|\partial_t^{l_1} u\|_{X_T^{s,b}} \|\partial_t^{l_2} u\|_{X_T^{2-s,b}} \|\partial_t^{l_3} u\|_{X_T^{2-s,b}}\\
&\qquad\le C\|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s}\|\varphi\|_{\mathcal{H}^{2l+2}}^{(s-1)\mu}\|\varphi\|_{\mathcal{H}^1}^{(s-1)(1-\mu)}\|\varphi\|_{\mathcal{H}^1}^{2(s-1)+2(2-s)} \le C\|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s}\|\varphi\|_{\mathcal{H}^{2l+2}}^{s-1}.
\end{aligned}$$

Second case: $0 = \min\{l_1, l_2, l_3\} \le \max\{l_1, l_2, l_3\} < l$.

If $l_1 = 0$, then our previous proof is valid since we have to deal with the norm $\|\varphi\|_{\mathcal{H}^{2l_1+1}}$ and hence we have regularity $\mathcal{H}^1$ and the interpolation argument above can be applied. However, in the cases $l_2 = 0$ or $l_3 = 0$, the proof needs to be slightly modified.

We can assume $l_2 = 0$. Then in this case $\|\varphi\|_{\mathcal{H}^{2l_2}} = \|\varphi\|_{L^2}$ is bounded, since we assume a control on the $\mathcal{H}^1$ norm of the initial datum, hence it is not necessary to introduce the parameter η_2 in the interpolation step. The conclusion is the same as above.

Third case: $\max\{l_1, l_2, l_3\} = l$.

We have to consider three cases: $(l_1, l_2, l_3) = (l, 0, 0)$, $(l_1, l_2, l_3) = (0, l, 0)$ and $(l_1, l_2, l_3) = (0, 0, l)$ (the last two cases are similar). Notice that we have, as a consequence of (3.4) (where we choose $s_0 = 1$),

$$\|u\|_{X_T^{2-s,b}} \le C\|\varphi\|_{\mathcal{H}^1}. \tag{3.15}$$

Start with one of the later two: for $(l_1, l_2, l_3) = (0, l, 0)$ by Proposition 3.2, (3.15) and (3.9),

$$\begin{aligned}
&\|u\|_{X_T^{s,b}} \|\partial_t^l u\|_{X_T^{2-s,b}} \|u\|_{X_T^{2-s,b}} \\
&\quad \le C\|\varphi\|_{\mathcal{H}^s} \|\partial_t^l u\|_{X_T^{2-s,b}} \le C\|\varphi\|_{\mathcal{H}^s} \|\varphi\|_{\mathcal{H}^{2l}}^{s-1} \|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s} \\
&\quad \le C\|\varphi\|_{\mathcal{H}^{2l+2}}^{\frac{s-1}{2l+1}} \|\varphi\|_{\mathcal{H}^{2l+1}}^{s-1} \|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s} \le C\|\varphi\|_{\mathcal{H}^{2l+2}}^{\frac{s-1}{2l+1}} \|\varphi\|_{\mathcal{H}^{2l+2}}^{\frac{2l(s-1)}{2l+1}} \|\varphi\|_{\mathcal{H}^{2l+1}}^{2-s},
\end{aligned}$$

where we used, again, the convexity of Sobolev norms, the fact that $\mathcal{H}^{2l+1} \subset \mathcal{H}^{2l}$, and the a priori bound on the $\mathcal{H}^1$ norm of φ.

In the first case we have, using again (3.15),

$$\|\partial_t^l u\|_{X_T^{s,b}} \|u\|_{X_T^{2-s,b}} \|u\|_{X_T^{2-s,b}} \le C\|\partial_t^l u\|_{X_T^{s,b}}.$$

We then conclude by choosing T small enough, exactly as we did along the proof of (3.9). This concludes the proof of (3.10). ∎

4. Modified energies and proof of Theorem 1.1

The aim of this section is to introduce suitable energies and to measure how far they are from being exact conservation laws. Those energies are the key tool in order to achieve the growth estimate provided in Theorem 1.1. Along this section we denote by $\int$ the integral on $\mathbb{R}^2$ with respect to the Lebesgue measure dx, and $\iint$ the integral on $\mathbb{R}^2 \times \mathbb{R}$ with respect to the Lebesgue measure $dxdt$.

Proposition 4.1. *Let $u(t,x) \in \mathcal{C}((-T,T); \mathcal{H}^{2k+2})$ be a local solution to (1.1) with initial datum $\varphi \in \mathcal{H}^{2k+2}$. Then we have*

$$\frac{d}{dt}\Big(\frac{1}{2}\|\partial_t^k A u(t,x)\|_{L^2}^2 + \mathcal{S}_{2k+2}(u(t,x))\Big) = \mathcal{R}_{2k+2}(u(t,x)), \tag{4.1}$$

where $\mathcal{S}_{2k+2}(u(t,x))$ is a linear combination of terms of the following type:

$$\int \partial_t^k L u_0 \partial_t^{m_1} L u_1\, \partial_t^{m_2} u_2\, \partial_t^{m_3} u_3, \quad m_1 + m_2 + m_3 = k, \tag{4.2}$$

and $\mathcal{R}_{2k+2}(u(t,x))$ *is a linear combination of terms of the following type*:

$$\int \partial_t^k Lu_0\, \partial_t^{l_1} Lu_1\, \partial_t^{l_2} u_2\, \partial_t^{l_3} u_3, \quad l_1+l_2+l_3=k+1, \quad l_1 \le k, \tag{4.3}$$

where in (4.2) *and* (4.3) *we have* $u_0, u_1, u_2, u_3 \in \{u, \bar u\}$ *and* L *can be any of the following operators*:

$$Lu = \partial_{x_i} u \quad i=1,2, \quad Lu = \langle x\rangle u, \quad \text{or} \quad Lu = u.$$

Proof. We have

$$i\partial_t(\partial_t^k \sqrt{A} u) + A(\partial_t^k \sqrt{A} u) \pm \partial_t^k \sqrt{A}(u|u|^2) = 0.$$

Next we multiply the equation above by $\partial_t^{k+1}\sqrt{A}\,\bar u$ and we take the real part:

$$\frac{1}{2}\frac{d}{dt}(\|\partial_t^k A u\|_{L^2}^2) = \mp \mathrm{Re} \int \partial_t^k \sqrt{A}(u|u|^2)\, \partial_t^{k+1}\sqrt{A}\,\bar u.$$

By the symmetry of the operator $\sqrt{A}$, we have

$$\begin{aligned}\mathrm{Re}\int \partial_t^k \sqrt{A}(u|u|^2)\,\partial_t^{k+1}\sqrt{A}\,\bar u &= \mathrm{Re}\int \partial_t^k(u|u|^2)\,\partial_t^{k+1} A\bar u\\ &= -\mathrm{Re}\int \partial_t^k(u|u|^2)\,\partial_t^{k+1}\Delta \bar u + \mathrm{Re}\int \partial_t^k(u|u|^2)\,\partial_t^{k+1}(|x|^2\bar u),\end{aligned}$$

and we proceed by integration by parts:

$$(\cdots) = \sum_{i=1}^2 \mathrm{Re}\int \partial_t^k \partial_{x_i}(u|u|^2)\,\partial_t^{k+1}\partial_{x_i}\bar u + \mathrm{Re}\int |x|^2\,\partial_t^k(u|u|^2)\,\partial_t^{k+1}\bar u. \tag{4.4}$$

Next notice that the first term on the right-hand side in (4.4) can be written as follows:

$$\begin{aligned}&\sum_{i=1}^2 \mathrm{Re}\int \partial_t^k \partial_{x_i}(u|u|^2)\,\partial_t^{k+1}\partial_{x_i}\bar u\\ &\quad= \sum_{i=1}^2 \Big(2\mathrm{Re}\int |u|^2\,\partial_t^k\partial_{x_i} u\,\partial_t^{k+1}\partial_{x_i}\bar u + \mathrm{Re}\int u^2\,\partial_t^k \partial_{x_i}\bar u\,\partial_t^{k+1}\partial_{x_i}\bar u\Big)\\ &\qquad+ \sum_{\substack{l_1+l_2+l_3=k\\ \max\{l_1,l_2,l_3\}<k}} \mathrm{Re}\,\big(a_{l_1,l_2,l_3}\,\partial_t^{l_1}\partial_{x_i} u\,\partial_t^{l_2} u\,\partial_t^{l_3}\bar u\,\partial_t^{k+1}\partial_{x_i}\bar u\\ &\qquad\qquad+ b_{l_1,l_2,l_3}\,\partial_t^{l_1}\partial_{x_i} u\,\partial_t^{l_2} u\,\partial_t^{l_3}\partial_{x_i}\bar u\,\partial_t^{k+1}\partial_{x_i}\bar u\big)\end{aligned}$$

where a_{l_1,l_2,l_3} and b_{l_1,l_2,l_3} are suitable real numbers.

Rewriting,

$$
\begin{aligned}
(\cdots) = \frac{d}{dt}\sum_{i=1}^{2}\Big(\int |\partial_t^k\partial_{x_i}u|^2\,|u|^2 + \frac{1}{2}\,\mathrm{Re}\int(\partial_t^k\partial_{x_i}\bar u)^2\,u^2\Big)\\
-\sum_{i=1}^{2}\Big(\int |\partial_t^k\partial_{x_i}u|^2\,\partial_t(|u|^2) + \frac{1}{2}\,\mathrm{Re}\int(\partial_t^k\partial_{x_i}\bar u)^2\,\partial_t(u^2)\Big)\\
+\sum_{\substack{l_1+l_2+l_3=k\\ l_1<k}}\mathrm{Re}\,\big(a_{l_1,l_2,l_3}\,\partial_t^{l_1}\partial_{x_i}u\,\partial_t^{l_2}u\,\partial_t^{l_3}\bar u\,\partial_t^{k+1}\partial_{x_i}\bar u\\
+\,b_{l_1,l_2,l_3}\,\partial_t^{l_1}\partial_{x_i}u\,\partial_t^{l_2}u\,\partial_t^{l_3}\partial_{x_i}\bar u\,\partial_t^{k+1}\partial_{x_i}\bar u\big).
\end{aligned}
$$

By elementary manipulations on the last line, we get

$$
\begin{aligned}
(\cdots) = \frac{d}{dt}\sum_{i=1}^{2}\Big(\int |\partial_t^k\partial_{x_i}u|^2\,|u|^2 + \frac{1}{2}\,\mathrm{Re}\int(\partial_t^k\partial_{x_i}\bar u)^2\,u^2\Big)\\
-\sum_{i=1}^{2}\Big(\int |\partial_t^k\partial_{x_i}u|^2\,\partial_t(|u|^2) + \frac{1}{2}\,\mathrm{Re}\int(\partial_t^k\partial_{x_i}\bar u)^2\,\partial_t(u^2)\Big)\\
+\frac{d}{dt}\sum_{\substack{l_1+l_2+l_3=k\\ l_1<k}}\mathrm{Re}\,\big(a_{l_1,l_2,l_3}\,\partial_t^{l_1}\partial_{x_i}u\,\partial_t^{l_2}u\,\partial_t^{l_3}\bar u\,\partial_t^{k}\partial_{x_i}\bar u+b_{l_1,l_2,l_3}\,\partial_t^{l_1}u\,\partial_t^{l_2}u\,\partial_t^{l_3}\partial_{x_i}\bar u\,\partial_t^{k}\partial_{x_i}\bar u\big)\\
+\sum_{\substack{l_1+l_2+l_3=k+1\\ l_1\leq k}}\mathrm{Re}\,\big(\tilde a_{l_1,l_2,l_3}\partial_t^{l_1}\partial_{x_i}u\partial_t^{l_2}u\partial_t^{l_3}\bar u\partial_t^{k}\partial_{x_i}\bar u+\tilde b_{l_1,l_2,l_3}\partial_t^{l_1}u\partial_t^{l_2}u\partial_t^{l_3}\partial_{x_i}\bar u\partial_t^{k}\partial_{x_i}\bar u\big).
\end{aligned}
$$

This last expression is a sum of two terms: a linear combination of terms with structure (4.2) with $Lu=\partial_{x_i}u$, and a time derivative of a linear combination of terms of type (4.3) with $Lu=\partial_{x_i}u$. We proceed with the second term on the right-hand side in (4.4): it can be rewritten as

$$
\mathrm{Re}\int |x|^2\partial_t^k(u|u|^2)\,\partial_t^{k+1}\bar u = \mathrm{Re}\int \partial_t^k(\langle x\rangle u|u|^2)\,\partial_t^{k+1}(\langle x\rangle\bar u) - \mathrm{Re}\int \partial_t^k(u|u|^2)\,\partial_t^{k+1}\bar u,
$$

and arguing as above one checks, by first expanding the derivative of order k with respect to time, that we get again a sum of two terms: a time derivative of terms of type (4.2), where $Lu=\langle x\rangle u$ or $Lu=u$, and a linear combination of terms with structure (4.3) where $Lu=\langle x\rangle u$ or $Lu=u$. This concludes the proof of Proposition 4.1. ■

Next we estimate the energy $\mathcal{R}_{2k+2}$ we just introduced.

Proposition 4.2. *Let $k\in\mathbb{N}$ and $R>0$ be given, and let $u(t,x)\in X_T^{2k+2,b}$ be the unique local solution to (1.1) with initial condition $\varphi\in\mathcal{H}^{2k+2}$ and $\|\varphi\|_{\mathcal{H}^1}<R$, with $T=T(R)$ and $s_0=2k+2$ in Proposition 3.2. Assume moreover that $\sup_{t\in(-T,T)}\|u(t,x)\|_{\mathcal{H}^1}<R$. Then for every $\delta>0$ there exists $C=C(\delta,R)>0$ such that:*

$$
\Big|\int_0^T \mathcal{R}_{2k+2}(u(\tau,x))\,d\tau\Big| \leq C\,\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta}.
$$

Proof. We have to estimate integrals like (4.3), namely $\iint (\partial_t^k Lu_0)(\partial_t^{l_1} Lu_1)\, \partial_t^{l_2} u_2\, \partial_t^{l_3} u_3$, under the condition

$$l_1 + l_2 + l_3 = k + 1, \quad l_1 \le k.$$

Here and below, abusing notation, we denote by $\iint$ a space-time integral on $(-T, T) \times \mathbb{R}^2$. Using the equation solved by u, and noticing that with the imposed conditions on l_1, l_2 and l_3 we may assume $l_2 \ge 1$ (as l_2 and l_3 play symmetric roles), we get

$$\begin{aligned} (4.5) \qquad & \Big| \iint (\partial_t^k Lu_0)\, (\partial_t^{l_1} Lu_1) \partial_t^{l_2} u_2\, \partial_t^{l_3} u_3 \Big| \\ & \le C \Big(\Big| \iint (\partial_t^k Lu_0)\, (\partial_t^{l_1} Lu_1)\, (\partial_t^{l_2-1} Au_2)\, \partial_t^{l_3} u_3 \Big| \\ & \quad + \iint |(\partial_t^k Lu_0)|\, |(\partial_t^{l_1} Lu_1)|\, |(\partial_t^{l_2-1}(u|u|^2)|\, |\partial_t^{l_3} u| \Big). \end{aligned}$$

The second term on the right-hand side is estimated, by Cauchy–Schwarz, as

$$\begin{aligned} & \Big(\int_0^T \|\partial_t^k Lu_0\|_{L^2} \|\partial_t^{l_1} Lu_1\|_{L^2} d\tau \Big) \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);L^\infty)} \|\partial_t^{l_3} u\|_{L^\infty((0,T);L^\infty)} \\ & \le \Big(\int_0^T \|\partial_t^k Lu_0\|_{L^2} \|\partial_t^{l_1} Lu_1\|_{L^2} d\tau \Big) \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})} \|\partial_t^{l_3} u\|_{L^\infty((0,T);\mathcal{H}^{1+\delta})} \end{aligned}$$

where we used the Sobolev embedding. Using that $X_T^{s,b} \subset \mathcal{C}(-T, T); \mathcal{H}^s)$, (1.4), (3.9) and (3.10), we proceed with

$$\begin{aligned} (\cdots) & \le C \|\varphi\|_{\mathcal{H}^{2k+1}} \|\varphi\|_{\mathcal{H}^{2l_1+1}} \|\varphi\|_{\mathcal{H}^{2l_3+1}}^{1-\delta} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta} \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})} \\ & \le C \|\varphi\|_{\mathcal{H}^{2k+2}}^{\eta+\eta_1+(1-\delta)\eta_3} \|\varphi\|_{\mathcal{H}^1}^{3-\delta-(\eta+\eta_1+\eta_3)+\delta\eta_3} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta} \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})}, \end{aligned}$$

where

$$\begin{cases} \eta(2k+2) + (1-\eta) = 2k+1, \\ \eta_i(2k+2) + (1-\eta_i) = 2l_i + 1, \quad i = 1, 2, 3, \end{cases}$$

and hence, by using that $\mathcal{H}^{2k+2} \subset \mathcal{H}^1$, we can continue the estimate above as follows:

$$(\cdots) \le C \|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{2k+2l_1+2l_3}{2k+1}} \|\varphi\|_{\mathcal{H}^1}^{3-\delta-\frac{2k+2l_1+2l_3}{2k+1}} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta} \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})}.$$

Expanding $\partial_t^{l_2-1}(u|u|^2)$ and using that $\mathcal{H}^{1+\delta/3}$ is an algebra, we get

$$\begin{aligned} \|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})} \le C & \sum_{j_1+j_2+j_3=l_2-1} \|\partial_t^{j_1} u\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})} \\ & \times \|\partial_t^{j_2} u\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})} \|\partial_t^{j_3} u\|_{L^\infty((0,T);\mathcal{H}^{1+\delta/3})}, \end{aligned}$$

and using the convexity of Sobolev norms and (3.5), we proceed with

$$\begin{aligned} (\cdots) & \le C \sum_{j_1+j_2+j_3=l_2-1} \|\varphi\|_{\mathcal{H}^{2j_1+1}}^{1-\delta/3} \|\varphi\|_{\mathcal{H}^{2j_2+1}}^{1-\delta/3} \|\varphi\|_{\mathcal{H}^{2j_3+1}}^{1-\delta/3} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta} \\ & \le C \sum_{j_1+j_2+j_3=l_2-1} \|\varphi\|_{\mathcal{H}^{2k+2}}^{(1-\delta/3)(\theta_1+\theta_2+\theta_3)} \|\varphi\|_{\mathcal{H}^1}^{(1-\delta/3)(3-\theta_1-\theta_2-\theta_3)} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta}, \end{aligned}$$

where

$$\theta_i(2k+2)+(1-\theta_i)=2j_i+1, \quad i=1,2,3.$$

Using that $\mathcal{H}^{2k+2}\subset\mathcal{H}^1$ and writing

$$(1-\delta/3)(3-\theta_1-\theta_2-\theta_3)=(\delta/3)(\theta_1+\theta_2+\theta_3)+3-\delta-\theta_1-\theta_2-\theta_3$$

to compensate the $(1-\delta/3)$ on the $\mathcal{H}^{2k+2}$ factor with part of the $\mathcal{H}^1$ factor, we conclude that

$$\|(\partial_t^{l_2-1}(u|u|^2)\|_{L^\infty((0,T);\mathcal{H}^{1+\delta})}\leq C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{2l_2-2}{2k+1}+\delta}.$$

By combining the estimates above, we get that the second term on the right-hand side in (4.5) can be estimated by

$$C\|\varphi\|_{\mathcal{H}^1}^{3-\delta-\frac{2k+2l_1+2l_3}{2k+1}}\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{2k+2l_1+2l_2+2l_3-2}{2k+1}+\delta}\leq C\|\varphi\|_{\mathcal{H}^1}^{\frac{1}{4k+2}+5\delta}\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{4k}{2k+1}+\delta}\leq C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+6\delta},$$

where we have used the uniform bound on $\mathcal{H}^1$ along with the inclusion $\mathcal{H}^{2k+2}\subset\mathcal{H}^1$, and

$$3-\delta-\frac{2k+2l_1+2l_3}{2k+1}\geq 3-\delta-\frac{4k}{2k+1}=1+\frac{4}{4k+2}-\delta\geq\frac{1}{4k+2}+5\delta.$$

We now focus on the first term on the right-hand side in (4.5). In this case, we shall use a Littlewood–Paley decomposition, and we are reduced to estimating

$$\sum_{N_0,N_1,N_2,N_3}\iint \Delta_{N_0}(\partial_t^k Lu_0)\,\Delta_{N_1}(\partial_t^{l_1}Lu_1)\,\Delta_{N_2}(\partial_t^{l_2-1}Au_2)\,\Delta_{N_3}(\partial_t^{l_3}u_3).$$

Here Δ_N denote the localization operator associated with the operator A at dyadic frequency N. We split the sum in several pieces, depending on the frequencies N_0, N_1, N_2 and N_3, and we shall make extensively use of the following bilinear estimate (see Proposition 2.3.15 in [16]): for every $\delta\in(0,1/2]$ and $b>1/2$, there exists $C>0$ such that

$$\|(\Delta_N u)(\Delta_M v)\|_{L^2((0,T);L^2)}\leq C\Big(\frac{\min\{N,M\}}{\max\{N,M\}}\Big)^{1/2-\delta}\|\Delta_N u\|_{X_T^{0,b}}\|\Delta_M v\|_{X_T^{0,b}}. \tag{4.6}$$

We point out that an alternative proof of (4.6) can be obtained following Section 5, where a key bilinear estimate is established via integration by parts.

Next we consider several subcases.

First subcase: $\min\{N_0,N_2\}\geq\max\{N_1,N_3\}$.

By Cauchy–Schwarz,

$$\Big|\iint \Delta_{N_0}(\partial_t^k Lu_0)\,\Delta_{N_1}(\partial_t^{l_1}Lu_1)\,\Delta_{N_2}(\partial_t^{l_2-1}Au_2)\,\Delta_{N_3}(\partial_t^{l_3}u_3)\Big|$$
$$\leq\|\Delta_{N_0}(\partial_t^k Lu_0)\,\Delta_{N_1}(\partial_t^{l_1}Lu_1)\|_{L^2((0,T);L^2)}\|\Delta_{N_2}(\partial_t^{l_2-1}Au)\,\Delta_{N_3}(\partial_t^{l_3}u)\|_{L^2((0,T);L^2)},$$

and by (4.6) we can continue as follows:

$$
\begin{aligned}
(\cdots) \leq C\, & \frac{(N_1 N_3)^{1/2-\delta}}{(N_0 N_2)^{1/2-\delta}} \|\Delta_{N_0}(\partial_t^k Lu)\|_{X_T^{0,b}} \|\Delta_{N_1}(\partial_t^{l_1} Lu)\|_{X_T^{0,b}} \\
& \times \|\Delta_{N_2}(\partial_t^{l_2-1} Au_2)\|_{X_T^{0,b}} \|\Delta_{N_3}(\partial_t^{l_3} u)\|_{X_T^{0,b}} \\
\leq C\, & \frac{N_3^{1/2-\delta}}{N_2^{1/2-\delta}} \|\Delta_{N_0}(\partial_t^k Lu)\|_{X_T^{0,b}} \|\Delta_{N_1}(\partial_t^{l_1} Lu)\|_{X_T^{0,b}} \\
& \times \|\Delta_{N_2}(\partial_t^{l_2-1} Au_2)\|_{X_T^{0,b}} \|\Delta_{N_3}(\partial_t^{l_3} u)\|_{X_T^{0,b}} \\
\leq C\, & \|\Delta_{N_0}(\partial_t^k Lu)\|_{X_T^{0,b}} \|\Delta_{N_1}(\partial_t^{l_1} Lu)\|_{X_T^{0,b}} \\
& \times \|\Delta_{N_2}(\partial_t^{l_2-1} Au)\|_{X_T^{-1/2+\delta,b}} \|\Delta_{N_3}(\partial_t^{l_3} u)\|_{X_T^{1/2-\delta,b}} .
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
\sum_{\substack{N_0,N_1,N_2,N_3 \\ \min\{N_0,N_2\}\geq\max\{N_1,N_3\}}} & \Big| \iint \Delta_{N_0}(\partial_t^k Lu_0)\, \Delta_{N_1}(\partial_t^{l_1} Lu_1)\, \Delta_{N_2}(\partial_t^{l_2-1} Au_2)\, \Delta_{N_3}(\partial_t^{l_3} u_3) \Big| \\
& \leq C\, \|L\partial_t^k u\|_{X_T^{\delta,b}} \|L\partial_t^{l_1} u\|_{X_T^{\delta,b}} \|A\partial_t^{l_2-1} u\|_{X_T^{-1/2+\delta,b}} \|\partial_t^{l_3} u\,\|_{X_T^{1/2-\delta,b}} \\
& \leq C\, \|\partial_t^k u\|_{X_T^{1+\delta,b}} \|\partial_t^{l_1} u\|_{X_T^{1+\delta,b}} \|\partial_t^{l_2-1} u\|_{X_T^{3/2+\delta,b}} \|\partial_t^{l_3} u\,\|_{X_T^{1/2+\delta,b}} ,
\end{aligned}
$$

where we used Proposition 2.1 at the last step, assuming we chose $b > 1/2$ in such a way the estimate at the last line fits with Proposition 2.1.

Second subcase: $\min\{N_1, N_2\} \geq \max\{N_0, N_3\}$.

We can argue as above and we are reduced to the previous case by noticing that $N_0 N_3/(N_1 N_2) \leq N_3/N_2$, since in this subcase $N_0 \leq N_1$.

Third subcase: $\min\{N_3, N_2\} \geq \max\{N_0, N_1\}$.

Arguing as above, we are reduced to the first subcase by noticing that $N_0 N_1 \leq N_3^2$ and hence $N_0 N_1/(N_2 N_3) \leq N_3/N_2$.

Fourth subcase: $\min\{N_1, N_3\} \geq \max\{N_0, N_2\}$.

Again, we can argue as above and we are reduced to the first subcase by noticing that $N_0 N_2/(N_1 N_3) \leq N_3/N_2$, as $N_0 N_2^2 \leq N_1 N_3^2$, which clearly holds in that subcase.

Fifth subcase: $\min\{N_0, N_3\} \geq \max\{N_1, N_2\}$.

Once more, we can argue as above and we are reduced to the first subcase by noticing that $N_1 N_2/(N_0 N_3) \leq N_3/N_2$, as $N_1 N_2^2 \leq N_0 N_3^2$, which clearly holds in this subcase.

Sixth subcase: $\min\{N_0, N_1\} \geq \max\{N_2, N_3\}$.

One more time, we can argue as above and we are reduced to the first subcase by noticing that $N_2 N_3/(N_0 N_1) \leq N_3/N_2$ which in turn follows from $N_2^2 \leq N_0 N_1$ which holds in that subcase.

We are therefore left with proving

$$\|\partial_t^k u\|_{X_T^{1+\delta,b}} \|\partial_t^{l_1} u\|_{X_T^{1+\delta,b}} \|\partial_t^{l_2-1} u\|_{X_T^{3/2+\delta,b}} \|\partial_t^{l_3} u\|_{X_T^{1/2+\delta,b}} \le C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta}.$$

Using (3.9) and (3.10), we can control the left-hand side with

$$\|\varphi\|_{\mathcal{H}^{2k+1}}^{(1-\delta)} \|\varphi\|_{\mathcal{H}^{2l_1+1}}^{(1-\delta)} \|\varphi\|_{\mathcal{H}^{2l_2-1}}^{1/2-\delta} \|\varphi\|_{\mathcal{H}^{2l_2}}^{1/2+\delta} \|\varphi\|_{\mathcal{H}^{2l_3}}^{1/2-\delta} \|\varphi\|_{\mathcal{H}^{2l_3+1}}^{1/2+\delta} \|\varphi\|_{\mathcal{H}^{2l_1+2}}^{\delta} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\delta},$$

which in turn, using that $\mathcal{H}^{2k+2} \subset \mathcal{H}^{2l_1+1}$ for the next to last term, is bounded by

$$\Big(\|\varphi\|_{\mathcal{H}^{2k+1}}^{\frac{1-\delta}{1-2\delta}} \|\varphi\|_{\mathcal{H}^{2l_1+1}}^{\frac{1-\delta}{1-2\delta}} \|\varphi\|_{\mathcal{H}^{2l_2-1}}^{1/2} \|\varphi\|_{\mathcal{H}^{2l_2}}^{\frac{1/2+\delta}{1-2\delta}} \|\varphi\|_{\mathcal{H}^{2l_3}}^{1/2} \|\varphi\|_{\mathcal{H}^{2l_3+1}}^{\frac{1/2+\delta}{1-2\delta}}\Big)^{1-2\delta} \|\varphi\|_{\mathcal{H}^{2k+2}}^{2\delta}.$$

Again by embeddings in the $\mathcal{H}$ scale, we get an upper bound

$$\big(\|\varphi\|_{\mathcal{H}^{2k+1}} \|\varphi\|_{\mathcal{H}^{2l_1+1}} \|\varphi\|_{\mathcal{H}^{2l_2-1}}^{1/2} \|\varphi\|_{\mathcal{H}^{2l_2}}^{1/2} \|\varphi\|_{\mathcal{H}^{2l_3}}^{1/2} \|\varphi\|_{\mathcal{H}^{2l_3+1}}^{1/2}\big)^{1-2\delta} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\mu}$$

and

$$\mu = 2\delta + \delta + \delta + 2\delta + 2\delta = 8\delta.$$

We will have to deal differently with $l_3 = 0$ and $l_3 \ge 1$. By interpolation and recalling the a priori bound on $\mathcal{H}^1$ norm, the quantity inside $(\cdots)^{1-2\delta}$ can be estimated as

$$C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\theta} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\theta_1} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\gamma_2/2} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\eta_2/2} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\eta_3/2} \|\varphi\|_{\mathcal{H}^{2k+2}}^{\theta_3/2},$$

where C contains some power of $\|\varphi\|_{\mathcal{H}^1}$ (recall our quantity has four factors of u) and

$$\begin{cases} \theta(2k+2) + (1-\theta) = 2k+1, \\ \theta_1(2k+2) + (1-\theta_1) = 2l_1+1, \\ \theta_3(2k+2) + (1-\theta_3) = 2l_3+1, \\ \gamma_2(2k+2) + (1-\gamma_2) = 2l_2-1, \\ \eta_2(2k+2) + (1-\eta_2) = 2l_2, \\ \eta_3(2k+2) + (1-\eta_3) = 2l_3, \end{cases}$$

except when $l_3 = 0$, where no interpolation takes place and therefore one sets $\eta_3 = 0$. We conclude by computing θ, θ_1, θ_3, γ_2, η_2 and η_3, and noticing that, for $l_3 \ge 1$,

$$\theta + \theta_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\eta_2 + \frac{1}{2}\eta_3 + \frac{1}{2}\theta_3 = \frac{4k}{2k+1},$$

while for $l_3 = 0$,

$$\theta + \theta_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\eta_2 + \frac{1}{2}\theta_3 = \frac{4k}{2k+1} + \frac{1}{2(2k+1)} = \frac{8k+1}{4k+2}.$$

For $l_3 \ge 1$, we may trade a bit of $\mathcal{H}^1$ norm to get the same exponent as for $l_3 = 0$. Then,

$$\frac{8k+1}{4k+2}(1-2\delta) + 8\delta = \frac{8k+1}{4k+2} + \Big(4 - \frac{8k+1}{4k+2}\Big)2\delta = \frac{8k+1}{4k+2} + \frac{8k+7}{2k+1}\delta,$$

and we get our final bound $C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta}$, up to relabeling $\frac{8k+3}{2k+1}\delta$ to be δ. ■

Next we estimate $\mathcal{S}_{2k+2}$ using its expansion as introduced in Proposition 4.1.

Proposition 4.3. *Let $k \in \mathbb{N}$, $R > 0$ be given, and let $u(t,x) \in X_T^{2k+2,b}$ be the unique local solution to* (1.1) *with initial condition $\varphi \in \mathcal{H}^{2k+2}$ and $\|\varphi\|_{\mathcal{H}^1} < R$, where $T = T(R)$ and $s_0 = 2k+2$ as in Proposition* 3.2*. Assume moreover that* $\sup_{t\in(-T,T)} \|u(t,x)\|_{\mathcal{H}^1} < R$*. Then for every $\delta > 0$, there exists $C > 0$ such that*

$$\sup_{t\in(-T,T)} |\mathcal{S}_{2k+2}(u(t,x))| \le C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{4k}{2k+1}+\delta}.$$

Proof. We prove the desired estimate for every expression with type (4.2). Indeed by Hölder we have for every fixed $t \in (-T,T)$,

$$\Big|\int \partial_t^k Lu_0\, \partial_t^{m_1} Lu_1\, \partial_t^{m_2} u_2\, \partial_t^{m_3} u_3\Big| \le \|\partial_t^k Lu_0\|_{L^2}\, \|\partial_t^{m_1} Lu_1\|_{L^2}\, \|\partial_t^{m_2} u_2\|_{L^\infty}\, \|\partial_t^{m_3} u_3\|_{L^\infty},$$

and by the Sobolev embedding and (1.4), we proceed with

$$\begin{aligned}(\cdots) &\le C\|\partial_t^k Lu\|_{L^2}\, \|\partial_t^{m_1} Lu\|_{L^2}\, \|\partial_t^{m_2} u\|_{\mathcal{H}^1}^{1-\delta}\, \|\partial_t^{m_2} u\|_{\mathcal{H}^2}^{\delta}\, \|\partial_t^{m_3} u\|_{\mathcal{H}^1}^{1-\delta}\, \|\partial_t^{m_3} u\|_{\mathcal{H}^2}^{\delta}\\ &\le C\|\partial_t^k u\|_{\mathcal{H}^1}\, \|\partial_t^{m_1} u\|_{\mathcal{H}^1}\, \|\partial_t^{m_2} u\|_{\mathcal{H}^1}^{1-\delta}\, \|\partial_t^{m_2} u\|_{\mathcal{H}^2}^{\delta}\, \|\partial_t^{m_3} u\|_{\mathcal{H}^1}^{1-\delta}\, \|\partial_t^{m_3} u\|_{\mathcal{H}^2}^{\delta}.\end{aligned}$$

Then, using (3.5), we get

$$\begin{aligned}(\cdots) &\le C\|u\|_{\mathcal{H}^{2k+1}}\, \|u\|_{\mathcal{H}^{2m_1+1}}\, \|u\|_{\mathcal{H}^{2m_2+1}}^{1-\delta}\, \|u\|_{\mathcal{H}^{2m_3+1}}^{(1-\delta)}\, \|u\|_{\mathcal{H}^{2m_2+2}}^{\delta}\, \|u\|_{\mathcal{H}^{2m_3+2}}^{\delta}\\ &\le C\|\varphi\|_{\mathcal{H}^{2k+1}}\, \|\varphi\|_{\mathcal{H}^{2m_1+1}}\, \|\varphi\|_{\mathcal{H}^{2m_2+1}}^{(1-\delta)}\, \|\varphi\|_{\mathcal{H}^{2m_3+1}}^{(1-\delta)}\, \|\varphi\|_{\mathcal{H}^{2m_2+2}}^{\delta}\, \|\varphi\|_{\mathcal{H}^{2m_3+2}}^{\delta},\end{aligned}$$

where we remark that the bound holds irrespective of the value of m_2 and m_3 (which may be zero). The last step follows from the embedding $X_T^{s,b} \subset \mathcal{C}((-T,T);\mathcal{H}^s)$ for $b > 1/2$ and (3.4). Next we choose $\theta, \theta_1, \theta_2, \theta_3 \in [0,1]$ such that

$$\begin{cases}\theta(2k+2) + (1-\theta) = 2k+1,\\ \theta_i(2k+2) + (1-\theta_i) = 2m_i + 1, \quad i = 1,2,3\end{cases}$$

(again, $m_2 = 0$ or $m_3 = 0$ are admissible, as then $\theta_2 = 0$ or $\theta_3 = 0$), and by further interpolation,

$$\Big|\int \partial_t^k Lu_0\, \partial_t^{m_1} Lu_1\, \partial_t^{m_2} u_2\, \partial_t^{m_3} u_3\Big| \le C\|\varphi\|_{\mathcal{H}^{2k+2}}^{\theta+\theta_1+(1-\delta)(\theta_2+\theta_3)+\delta(\theta_2+\theta_3+\frac{2}{2k+1})}.$$

The exponent turns out to be

$$\theta + \theta_1 + \theta_2 + \theta_3 + \frac{2\delta}{2k+1} = \frac{4k}{2k+1} + \frac{2\delta}{2k+1},$$

and this concludes the proof by relabeling δ.

Note that, since the constant C has a factor $\|\varphi\|_{\mathcal{H}^1}$ from the interpolation steps, we may trade part of it to increase the power $4k/(2k+1)$ to $(8k+1)/(4k+2)$, to have matching exponents in both Propositions 4.2 and 4.3. ∎

Proof of Theorem 1.1. It will follow as a consequence of Propositions 4.1, 4.2 and 4.3. Let

$$\sup_{t\in(-\infty,\infty)} \|u(t,x)\|_{\mathcal{H}^1} = R. \tag{4.7}$$

Then $R<\infty$, by (1.5). By integration of the identity (4.1) on the strip $(0,T)$, we get

$$\frac{1}{2}\|\partial_t^k Au(T,x)\|_{L^2}^2 + \mathcal{S}_{2k+2}(u(T,x)) = \frac{1}{2}\|\partial_t^k Au(0,x)\|_{L^2}^2 + \mathcal{S}_{2k+2}(u(0,x)) + \int_0^T \mathcal{R}_{2k+2}(u(\tau,x))\,d\tau,$$

where $T=T(R)$ is the local time of existence as defined in Propositions 4.2 and 4.3. Then we get, taking advantage of the remark at the end of the proof of Proposition 4.3 to match both exponents on the right-hand side,

$$\frac{1}{2}\|\partial_t^k Au(T,x)\|_{L^2}^2 - \frac{1}{2}\|\partial_t^k Au(0,x)\|_{L^2}^2 \le C\|u(0,x)\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta}.$$

One easily checks that by (4.7) the bound above can be iterated with the same constants, giving

$$\frac{1}{2}\|\partial_t^k Au((n+1)T,x)\|_{L^2}^2 - \frac{1}{2}\|\partial_t^k Au(nT,x)\|_{L^2}^2 \le C\|u(nT,x)\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta}$$

for every $n\in\mathbb{N}$. By summing up for $n\in[0,N-1]$ (subsuming the data at $n=0$ on the left-hand side into the constant and $n=0$ term on the right-hand side), we obtain

$$\|\partial_t^k Au(NT,x)\|_{L^2}^2 \le C\sum_{n\in\{0,\dots,N-1\}} \|u(nT,x)\|_{\mathcal{H}^{2k+2}}^{\frac{8k+1}{4k+2}+\delta},$$

and then,

$$\sup_{n\in[0,N]} \|\partial_t^k Au(nT,x)\|_{L^2}^2 \le CN\Big(\sup_{n\in[0,N]}\|u(nT,x)\|_{\mathcal{H}^{2k+2}}\Big)^{\frac{8k+1}{4k+2}+\delta}.$$

By (3.5) (δ may change from line to line but can always be chosen arbitrary small),

$$\sup_{n\in[0,N]} \|u(nT,x)\|_{\mathcal{H}^{2k+2}} \le CN^{\frac{2}{3}(2k+1)+\delta},$$

and therefore

$$\|u(NT,x)\|_{\mathcal{H}^{2k+2}} \le CN^{\frac{2}{3}(2k+1)+\delta}, \quad \forall N\in\mathbb{N}^*.$$

Using (3.4), we easily obtain

$$\sup_{t\in[NT,(N+1)T]} \|u(t,x)\|_{\mathcal{H}^{2k+2}} \le CN^{\frac{2}{3}(2k+1)+\delta}$$

provided that we suitably modify the multiplicative constant C. Summarizing, we get that, for all $t>0$,

$$\|u(t,x)\|_{\mathcal{H}^{2k+2}} \le C(1+|t|)^{\frac{2}{3}(2k+1)+\delta}.$$

The same argument works for $t<0$, concluding the proof of Theorem 1.1. ■

5. Appendix

We intend to provide a direct proof, based on integration by parts, of the crucial bilinear estimate from [16], for solutions to

$$i\partial_t u - \Delta u + |x|^2 u = 0. \tag{5.1}$$

Theorem 5.1. *Let $1 \leq M \leq N$ be dyadic numbers. For $T \in (0,\infty)$, there exists C_T such that*

$$\|u_N v_M\|^2_{L^2((0,T);L^2)} \leq C_T M N^{-1} \|u_N(0)\|^2_{L^2} \|v_N(0)\|^2_{L^2}, \tag{5.2}$$

where u_N and v_N are spectrally localized solutions to (5.1) (namely, $\Delta_N u_N = u_N$ and $\Delta_M v_M = v_M$) with initial datum $u_N(0)$ and $v_M(0)$, respectively.

Such bilinear estimates were first obtained for solutions to the classical linear Schrödinger equation in [3], using direct computations in Fourier variables. In [7], the so-called interaction Morawetz estimates were introduced for the 3D nonlinear Schrödinger equation, relying on a bilinear version of the classical Morawetz estimate. Here, we rely on the bilinear computation from [15], that non only extended such bilinear virial estimates to low dimensions, but also allowed to recover Bourgain's estimates from [3]. We will follow the strategy from [13], where bilinear estimates on bounded domains were obtained, bypassing the need for Fourier localization. We split the proof in several steps.

First we prove the following: for a given $T \in (0,\infty)$,

$$\int_0^T \Big(\iint_{|x-y|<1/M} M|u_N(x)\nabla_y \bar{v}_M(y) + \bar{v}_M(y)\nabla_x u_N(x)|^2\, dx\, dy\Big)\, dt \\ \leq C_T N \|u_N(0)\|^2_{L^2} \|v_M(0)\|^2_{L^2}. \tag{5.3}$$

Next we deduce from (5.3) that

$$\int_0^T \Big(\int |\nabla_x (v_M u_N)|^2\, dx\Big)\, dt \leq C_T M N \|u_N(0)\|^2_{L^2} \|v_M(0)\|^2_{L^2}. \tag{5.4}$$

Estimate (5.4), along with a companion easier estimate for $\int_0^T \big(\int |x|^2 |v_M u_N|^2\, dx\big)\, dt$, implies

$$\int_0^T \|v_M \bar{u}_N\|^2_{\mathcal{H}^1}\, dt \leq C_T M N \|u_N(0)\|^2_{L^2} \|v_M(0)\|^2_{L^2}. \tag{5.5}$$

Finally, by a spectral localization argument, we prove that (5.5) implies (5.2).

Proof of (5.3)

We first remark for later use that once (5.3) will be established, then we are allowed to replace v_N by Av_N (which is still a localized solution to (5.1)), and we get

$$\int_0^T \Big(\iint_{|x-y|<1/M} M|u_N(x)\nabla_y (A\bar{v}_M)(y) + (A\bar{v}_M)(y)\nabla_x u_N(x)|^2\, dx\, dy\Big) dt \\ \leq C_T N M^2 \|u_N(0)\|^2_{L^2} \|v_M(0)\|^2_{L^2}. \tag{5.6}$$

Next we focus on the proof of (5.3). From now on, T is fixed in $(0,+\infty)$. Let $\rho\colon\mathbb{R}^2\to\mathbb{R}$ be a C^1 function whose derivative is piecewise differentiable, with H_ρ denoting the bilinear form associated to its Hessian (as a distribution), $H_\rho(a,b)=\sum_{k,l}(\partial^2_{kl}\rho)a_k b_l$; all $\partial^2_{kl}\rho$ are actually piecewise continuous functions, and under such assumptions, all subsequent integrations by parts are fully justified in the classical sense. We claim that, for any couple of solutions u,v of (5.1),

$$\int_0^T\Big(\iint H_\rho(x-y)(\bar v(y)\nabla_x u(x)+u(x)\nabla_y\bar v(y),v(y)\nabla_x\bar u(x)+\bar u(x)\nabla_y v(y))\,dx\,dy\Big)\,dt$$

$$\leq C_T\|\nabla\rho\|_{L^\infty}(\|v(0)\|^2_{L^2}\|u(0)\|_{L^2}\|u(0)\|_{\mathcal{H}^1}+\|u(0)\|^2_{L^2}\|v(0)\|_{L^2}\|v(0)\|_{\mathcal{H}^1}),\tag{5.7}$$

where we dropped time dependence for notational simplicity. Following [13], we define a convex function $\rho_M\colon\mathbb{R}\to\mathbb{R}$,

$$\rho_M(z)=\begin{cases}\frac{M}{2}z^2+\frac{1}{2M}, & |z|\leq 1/M,\\ |z|, & |z|>1/M,\end{cases}$$

and we use (5.7) with $\rho(x-y)=\rho_M(x_1-y_1)$: we get, by direct computation of the Hessian H_ρ,

$$\int_0^T\Big(\iint_{|x_1-y_1|<1/M} M|(\bar v(y)\,\partial_{x_1}u(x)+u(x)\,\partial_{y_1}\bar v(y)|^2\,dx\,dy\Big)\,dt$$
$$\leq C_T(\|v(0)\|^2_{L^2}\|u(0)\|_{L^2}\|u(0)\|_{\mathcal{H}^1}+\|u(0)\|^2_{L^2}\|v(0)\|_{L^2}\|v(0)\|_{\mathcal{H}^1}),$$

where there is no contribution in the region $|x_1-y_1|>1/M$ as $H_\rho=0$ there, and we used that $\|\rho'_M(z)\|_{L^\infty}\leq 1$. Of course, by choosing $\rho(x-y)=\rho_M(x_2-y_2)$, we get a similar estimate where x_1,y_1 are replaced by x_2,y_2, and by combining the two estimates we get (5.3), where we noticed that $|x-y|<1/M\subset\max\{|x_1-y_1|,|x_2-y_2|\}<1/M$. Replacing u and v by u_N and v_M and using spectral localization, we get (5.3).

We now go back to the proof of (5.7), with a generic weight ρ. We compute the second derivative with respect to time of the functional

$$I_\rho(t)=\iint|u(x)|^2\,\rho(x-y)\,|v(y)|^2\,dx\,dy,$$

where for simplicity we have dropped the time dependence of u,v. In order to do so, recall that by the classical virial computation we get for a solution $w(t,x)$ to (5.1) (we drop again time-dependence of w and set $\rho_y(x)=\rho(x-y)$ to emphasize that y is a fixed base point here), the following:

$$\frac{d}{dt}\int\rho_y(x)\,|w(x)|^2\,dx=2\int\nabla\rho_y(x)\cdot\mathrm{Im}\,(\nabla\bar w(x)w(x))\,dx\tag{5.8}$$

$$\frac{d^2}{dt^2}\int\rho_y(x)\,|w|^2\,dx=4\int H_{\rho_y}(\nabla w(x),\nabla\bar w(x))-\int\Delta\rho_y(x)\,\Delta(|w(x)|^2)\,dx\tag{5.9}$$
$$-4\int x\cdot\nabla\rho_y(x)|w(x)|^2dx,$$

where we emphasize that we will not be using more than two derivatives on ρ_y. Next, using (5.8) we get

$$\frac{d}{dt} I_\rho(t) = 2 \iint \nabla\rho(x-y) \cdot \mathrm{Im}\,(\nabla_x \bar{u}(x) u(x))\, |v(y)|^2\, dx\, dy \\ - 2 \iint \nabla\rho(x-y) \cdot \mathrm{Im}\,(\nabla_y \bar{v}(y) v(y))\, |u(x)|^2\, dx\, dy\,.$$

Using that $\|\nabla w\|_{L^2} \le C\|w\|_{\mathcal{H}^1}$ at fixed time, followed by the conservation of mass and energy for (5.1), we get

$$\left|\frac{d}{dt} I_\rho(t)\right| \le 2\|\nabla\rho\|_{L^\infty} \big(\|v\|_{L^2}^2\, \|u\|_{L^2}\, \|\nabla u\|_{L^2} + \|u\|_{L^2}^2 \|v\|_{L^2}\, \|\nabla v\|_{L^2}\big) \\ \le C\|\nabla\rho\|_{L^\infty} \big(\|v(0)\|_{L^2}^2\, \|u(0)\|_{L^2}\, \|u(0)\|_{\mathcal{H}^1} + \|u(0)\|_{L^2}^2\, \|v(0)\|_{L^2}\, \|v(0)\|_{\mathcal{H}^1}\big). \tag{5.10}$$

For later use, notice also that using (5.8) on both mass densities,

$$\int \Big(\int \rho(x-y)\, \frac{d}{dt}|u(x)|^2\, dx \Big)\, \frac{d}{dt}|v(y)|^2\, dy \\ = 2\int \Big(\int \nabla\rho(x-y) \cdot \mathrm{Im}\,(\nabla_x \bar{u}(x) u(x))\, dx \Big)\, \frac{d}{dt}|v(y)|^2\, dy \\ = -4 \iint H_\rho(\mathrm{Im}\,(\nabla_x \bar{u}(x) u(x)), \mathrm{Im}\,(\nabla_y \bar{v}(y) v(y))\, dx\, dy\,. \tag{5.11}$$

On the other hand, by combining (5.9) and (5.11), we get

$$\frac{d^2}{dt^2} I_\rho(t) = 4 \iint H_\rho(x-y)(\nabla u(x), \nabla\bar{u}(x))\, |v(y)|^2 \\ + 4 \iint H_\rho(x-y)(\nabla v(y), \nabla\bar{v}(y))\, |u(x)|^2\, dx\, dy \\ - \iint \Delta\rho(x-y) \Delta(|u(x)|^2) |v(y)|^2\, dx\, dy \\ - \iint \Delta\rho(x-y) |u(x)|^2 \Delta(|u(y)|^2)\, dx\, dy \\ - 8 \iint H_\rho(x-y)(\mathrm{Im}\,(\nabla\bar{u}(x) u(x)), \mathrm{Im}\,(\nabla\bar{v}(y) v(y)))\, dx\, dy \\ - 4\mathrm{Re} \iint (x-y)\cdot\nabla\rho(x-y) |u(x)|^2 |v(y)|^2\, dx dy \\ = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}.$$

Following [15], we rewrite $\Delta\rho(x-y) = -\nabla_x \cdot \nabla_y \rho(x-y)$ and integrate by parts with respect to to x and y to obtain

$$\mathrm{III} + \mathrm{IV} = 8 \iint H_\rho(x-y)(\mathrm{Re}\,(\bar{u}(x)\nabla u(x)), \mathrm{Re}\,(\bar{v}(y)\nabla v(y)))\, dx\, dy\,.$$

Now, thinking about just one direction of derivation, we have the following identity:

$$\begin{aligned}
&4|v|^2(y)|\partial u|^2(x) + 4|u|^2(x)\,|\partial v|^2(y) \\
&+ 8\,\frac{(v\partial\bar v + \bar v\partial v)(y)}{2}\,\frac{(u\partial\bar u + \bar u\partial u)(x)}{2} - 8\,\frac{(v\partial\bar v - \bar v\partial v)(y)}{2}\,\frac{(\bar u\partial u - u\partial\bar u)(x)}{2} \\
&= 4|v|^2(y)\,|\partial u|^2(x) + 4|u|^2(x)\,|\partial v|^2(y) + 4v\,\partial\bar v(y) u\,\partial\bar u(x) + 4\bar v\,\partial v(y)\bar u\,\partial u(x) \\
&= 4|\bar v(y)\,\partial u(x) + u(x)\,\partial\bar v(y)|^2,
\end{aligned}$$

which allows to recombine I + II + III + IV + V, to get

$$\begin{aligned}
\frac{d^2}{dt^2} I_\rho(t) = 4 \iint H_\rho(x-y)(\bar v(y)\nabla u(x) + u(x)\nabla\bar v(y), v(y)\nabla\bar u(x) + \bar u(x)\nabla v(y))\,dxdy \\
- 4\mathrm{Re} \iint \nabla\rho(x-y)\cdot(x-y)|v(y)|^2|u(x)|^2\,dxdy.
\end{aligned}$$

After integration in time of the identity above, and by recalling (5.10), we get

$$\begin{aligned}
&4\int_0^T\Big(\iint H_\rho(x-y)(\bar v(y)\nabla_x u(x) + u(x)\nabla_y\bar v(y), v(y)\nabla_x\bar u(x) + \bar u(x)\nabla_y v(y))\,dxdy\Big)dt \\
&\le C\,\|\nabla\rho\|_{L^\infty}\big(\|v(0)\|_{L^2}^2\,\|u(0)\|_{L^2}\,\|u(0)\|_{\mathcal{H}^1} + \|u(0)\|_{L^2}^2\,\|v(0)\|_{L^2}\,\|v(0)\|_{\mathcal{H}^1}\big) \\
&\quad + 4\int_0^T\Big(\iint |\nabla\rho(x-y)||y-x||v(y)|^2|u(x)|^2\,dxdy\Big)dt \\
&\le C\,\|\nabla\rho\|_{L^\infty}\big(\|v(0)\|_{L^2}^2\,\|u(0)\|_{L^2}\,\|u(0)\|_{\mathcal{H}^1} + \|u(0)\|_{L^2}^2\,\|v(0)\|_{L^2}\,\|v(0)\|_{\mathcal{H}^1}\big) \\
&\quad + C_T\,\|\nabla\rho\|_{L^\infty}\sup_{t\in(0,T)}\big(\|u(t)\|_{L^2}^2\,\|v(t)\|_{L^2}\,\|yv(t)\|_{L^2} + \|v(t)\|_{L^2}^2\,\|u(t)\|_{L^2}\,\|xu(t)\|_{L^2}^2\big).
\end{aligned}$$

Using that $\|yw\|_{L^2} \le \|w\|_{\mathcal{H}^1}$ and, again, the conservation of mass and energy for (5.1), this estimate implies (5.7).

Proof of (5.3) ⇒ (5.4)

We need a suitable local elliptic estimate for our operator $A = -\Delta + |x|^2$ to reproduce the computation from [13]. The next lemma is a modification of Lemma 4.2 in [13].

Lemma 5.1. *There exist $C > 0$ and $\lambda_0 \ge 1$ such that, for any smooth function ϕ in $\mathbb{R}^2$ and $\lambda \ge \lambda_0$, the following pointwise estimate holds*:

$$|\phi(x)|^2 \le C\lambda^{-2}\int_{|x-y|<\lambda^{-1}} |A\phi|^2\,dy + C\lambda^2\int_{|x-y|<\lambda^{-1}} |\phi|^2\,dy, \quad \forall x\in\mathbb{R}^2.$$

Proof. Without loss of generality, we may restrict to real-valued ϕ. The lemma is proved in [13] if we replace in the right-hand side the operator A by $-\Delta$, and the domain of integration by the smaller domain $|x-y| < (4\lambda)^{-1}$ (this fact follows from classical elliptic theory and the Sobolev embedding for $\lambda = 1$, and then any $\lambda > 0$ by rescaling). Thus we conclude provided that we prove

$$\lambda^{-2}\int_{|x-y|<(4\lambda)^{-1}} |\Delta\phi|^2\,dy \le C\lambda^{-2}\int_{|x-y|<\lambda^{-1}} |A\phi|^2\,dy + C\lambda^2\int_{|x-y|<\lambda^{-1}} |\phi|^2\,dy. \tag{5.12}$$

In order to prove this estimate, we expand the square $\int |\Delta f|^2 = \int |Af - |y|^2 f|^2$, and after integrations by parts we get

$$\int (|\Delta f|^2 + |y|^4 |f|^2 + 2|y|^2 |\nabla f|^2)\, dy = \int (|Af|^2 + 4|f|^2)\, dy \tag{5.13}$$

for any real-valued function $f \in C_0^\infty(\mathbb{R}^2)$. Next we pick $f(y) = \chi_\lambda(y)\phi(y)$, where $\chi_\lambda(y) = \chi(\lambda(y-x))$, with $\chi(|z|) = 1$ on $|z| < 1/4$ and $\chi(|z|) = 0$ on $|z| > 1/2$. All subsequent cutoffs with respect to y will be centered at x. Expanding $\int |\Delta(\chi_\lambda \phi)|^2$ and $\int |A(\chi_\lambda \phi)|^2$ and replacing in the previous identity, we get

$$\begin{aligned}&\int (|\chi_\lambda|^2 |\Delta\phi|^2 + |y|^4 |\chi_\lambda|^2 |\phi|^2 + 2|y|^2 |\nabla(\chi_\lambda\phi)|^2)\, dy \\ &= \int (|\chi_\lambda|^2 |Av|^2 + 4|\chi_\lambda|^2 |\phi|^2)\, dy - 2\int \chi_\lambda |y|^2 \phi (2\nabla\chi_\lambda \cdot \nabla\phi + \Delta\chi_\lambda \phi)\, dy.\end{aligned} \tag{5.14}$$

By Cauchy–Schwarz and elementary manipulations, we estimate the last term on the right-hand side as follows: for every $\mu > 0$ and with a universal constant $C > 0$,

$$\begin{aligned}&\Big| \int \chi_\lambda |y|^2 \phi (2\nabla\chi_\lambda \cdot \nabla\phi + \phi\Delta\chi_\lambda)\, dy \Big|^2 \\ &\qquad \leq C\mu \int |y|^4 |\chi_\lambda|^2 |\phi|^2\, dy + \frac{C}{\mu} \int (|\nabla\chi_\lambda \cdot \nabla\phi|^2 + |\Delta\chi_\lambda|^2 |\phi|^2)\, dy.\end{aligned}$$

If we choose the constant μ small enough, then we can absorb $\int |y|^4 |\chi_\lambda v|^2 dx$ on the left-hand side in (5.14), and by neglecting some positive terms, we get, abusing notation for the constant C,

$$\int |\chi_\lambda|^2 |\Delta\phi|^2\, dy \leq C \int (|\chi_\lambda|^2 |A\phi|^2 + 4|\chi_\lambda|^2 |\phi|^2 + |\nabla\chi_\lambda \cdot \nabla\phi|^2 + |\Delta\chi_\lambda|^2) |\phi|^2\, dy,$$

and by elementary considerations,

$$\begin{aligned}\int_{|x-y|<(4\lambda)^{-1}} |\Delta\phi|^2\, dy &\leq C \int_{|x-y|<(2\lambda)^{-1}} |A\phi|^2\, dy \\ &\quad + C\lambda^2 \int \tilde{\chi}_\lambda |\nabla\phi|^2\, dy + C(1+\lambda^4) \int_{|x-y|<(2\lambda)^{-1}} |\phi|^2\, dy,\end{aligned}$$

where $\tilde{\chi}_\lambda$ is a suitable enlargement of χ_λ, namely $\tilde{\chi}_\lambda(y) = \tilde{\chi}(\frac{y-x}{\lambda})$, with $\tilde{\chi}(|z|) = 1$ on $|z| < 1/2$ and $\tilde{\chi}(|z|) = 0$ on $|z| > 1$. Then (5.12) follows provided that

$$\int \tilde{\chi}_\lambda |\nabla\phi|^2 dy \leq C\lambda^{-2} \int_{|x-y|<\lambda^{-1}} |A\phi|^2\, dy + C\lambda^2 \int_{|x-y|<\lambda^{-1}} |\phi|^2\, dy.$$

In order to do that, we write (either integrating by parts or replacing $-\Delta$ by $A - |x|^2$)

$$\begin{aligned}-2\int \tilde{\chi}_\lambda \phi\, \Delta\phi\, dy &= 2\int \tilde{\chi}_\lambda |\nabla\phi|^2\, dy - \int \Delta\tilde{\chi}_\lambda |\phi|^2\, dy \\ &= 2\int \tilde{\chi}_\lambda \phi\, A\phi\, dy - 2\int |y|^2 \tilde{\chi}_\lambda |\phi|^2\, dy,\end{aligned}$$

and hence

$$2\int \tilde{\chi}_\lambda |\nabla\phi|^2\,dy + 2\int |y|^2\,\tilde{\chi}_\lambda |\phi|^2\,dy = 2\int \tilde{\chi}_\lambda\,\phi\,A\phi\,dy + \int \Delta\tilde{\chi}_\lambda |\phi|^2\,dy$$
$$\le C\lambda^{-2}\int_{|x-y|<\lambda^{-1}} |A\phi|^2\,dy + C(1+\lambda^2)\int_{|x-y|<\lambda^{-1}} |\phi|^2\,dy,$$

where we used Cauchy–Schwarz at the last step. ∎

We now proceed to prove that (5.3) ⇒ (5.4). The first term in the square at the left-hand side of (5.3) turns out to be of lower order: we compute, by change of variable, the Cauchy–Schwarz inequality and the Strichartz estimate,

$$\int_0^T \Big(\iint_{|x-y|<1/M} |u_N(x)\nabla_y \bar{v}_M(y)|^2 dx dy\Big) dt$$
$$= \int_0^T \int_{|z|<1/M} |u_N(x)\nabla_x \bar{v}_M(x-z)|^2\,dx dz dt$$
$$\le \int_{|z|<1/M} \|u_N\|^2_{L^4((0,T);L^4)}\,\|\nabla v_M\|^2_{L^4((0,T);L^4)}\,dz \le C_T\|u_N(0)\|_2^2\,\|v_M(0)\|^2_{L^2}, \tag{5.15}$$

where at the last step we used that $\|\nabla v_M\|_{L^4((0,T);L^4)} \le C_T M\|v_M(0)\|_{L^2}$. In turn, this bound follows by noticing that ∇v_M is solution to the inhomogeneous equation associated with (5.1) with forcing term $2xv_M$. Hence by the inhomogeneous Strichartz estimate, placing the forcing term in $L^1((0,T);L^2)$,

$$\|\nabla v_M\|_{L^4((0,T);L^4)} \le C\|\nabla v_M(0)\|_{L^2} + C\||x|v_M\|_{L^1((0,T);L^2)}$$
$$\le C_T\|v_M(0)\|_{\mathcal{H}^1} \le C_T M\|v_M(0)\|_{L^2}, \tag{5.16}$$

where we used the conservation of energy for (5.1) and the bound $\||x|w\|_{L^2} \le C\|w\|_{\mathcal{H}^1}$ for every time independent function.

Recall that (5.15) holds with v_M replaced by Av_M (it is still a solution to (5.1)). Hence we get

$$\int_0^T\Big(\iint_{|x-y|<1/M} |u_N(x)\nabla_y(A\bar{v}_M)(y)|^2\,dx dy\Big)\,dt \le C_T M^4\|u_N(0)\|^2_{L^2}\|v_M(0)\|^2_{L^2}. \tag{5.17}$$

We now proceed using the Lemma 5.1, and we get

$$\int_0^T \Big(\int |\bar{v}_M(x)\nabla_x u_N(x)|^2\,dx\Big)\,dt \le C\int_0^T \Big(\iint_{|x-y|<1/M} M^2|\bar{v}_M(y)\nabla_x u_N(x)|^2$$
$$+ \frac{1}{M^2}|A\bar{v}_M(y)\nabla_x u_N(x)|^2\,dx\,dy\Big)\,dt,$$

that by (5.3) and (5.6) implies

$$(\cdots) \le C_T NM\|u_N(0)\|^2_{L^2}\,\|v_M(0)\|^2_{L^2}$$
$$+ C\int_0^T\Big(\iint_{|x-y|<1/M} M^2|u_N(x)\nabla_y\bar{v}_M(y)|^2 + \frac{1}{M^2}|u_N(x)\nabla_y(A\bar{v}_M)(y)|^2\,dx\,dy\Big)dt.$$

Combining the above estimate with (5.15) and (5.17), we obtain

$$\int_0^T \Big(\int |\bar{v}_M(x) \nabla_x u_N(x)|^2 \, dx \Big) \, dt \le C_T (M^2 + NM) \|u_N(0)\|_{L^2}^2 \, \|v_M(0)\|_{L^2}^2$$
$$\le C_T NM \|u_N(0)\|_{L^2}^2 \, \|v_M(0)\|_{L^2}^2. \tag{5.18}$$

On the other hand, by Cauchy–Schwarz, the Strichartz estimate and (5.16), we have

$$\int_0^T \Big(\int |u_N(x) \nabla_x \bar{v}_M(x)|^2 \, dx \Big) \, dt$$
$$\le \|u_N\|_{L^4((0,T);L^4)}^2 \, \|\nabla v_M\|_{L^4((0,T);L^4)}^2 \le C_T M^2 \|v_M(0)\|_{L^2}^2 \, \|u_N(0)\|_{L^2}^2.$$

Therefore, combining this last estimate with (5.18), we get (5.4).

Proof of (5.5)

Due to (5.4), it suffices to prove

$$\int_0^T \Big(\int |x|^2 |v_M \bar{u}_N|^2 \, dx \Big) \, dt \le C_T MN \, \|u_N(0)\|_{L^2}^2 \, \|v_M(0)\|_{L^2}^2. \tag{5.19}$$

By Hölder's inequality, we have

$$\int_0^T \Big(\int |x|^2 |v_M \bar{u}_N|^2 \, dx \Big) \, dt \le \| |x| v_M \|_{L^4((0,T);L^4)}^2 \, \|u_N\|_{L^4((0,T);L^4)}^2. \tag{5.20}$$

Next notice that $|x|^2 v_M$ is solution to the inhomogeneous equation associated with (5.1), with source term $-4v_M - 2x \cdot \nabla v_M$. Again, using Strichartz and placing the source term in $L^1((0,T);L^2)$,

$$\| |x|^2 v_M \|_{L^4((0,T);L^4)} \le C \| |x|^2 v_M(0) \|_{L^2} + C \|v_M\|_{L^1((0,T);L^2)}$$
$$+ C \|x \cdot \nabla v_M\|_{L^1((0,T);L^2)} \le C_T \|v_M(0)\|_{\mathcal{H}^2} \le C_T M^2 \|v_M(0)\|_{L^2}, \tag{5.21}$$

where we used the time independent estimate $\|x \cdot \nabla w\|_{L^2} \le C \|w\|_{\mathcal{H}^2}$ (see (5.13)) and the conservation of the $\mathcal{H}^2$ norm for (5.1). Interpolation between (5.21) and the Strichartz estimate $\|v_M\|_{L^4((0,T);L^4)} \le C \|v_M(0)\|_{L^2}$ implies $\| |x| v_M \|_{L^4((0,T);L^4)} \le CM \|v_M(0)\|_{L^2}$. Combining this estimate, Strichartz for u_N and (5.20), we obtain (5.19) (in fact, a stronger version of (5.19), as on the right-hand side we get M^2).

Proof of the implication (5.5) ⇒ (5.2)

We can write

$$\|v_M u_N\|_{L^2((0,T);L^2)}^2 = \sum_{K \in 2^{\mathbb{N}}} \|\Delta_K (v_M u_N)\|_{L^2((0,T);L^2)}^2.$$

If $K > N$, we may forget about Δ_K and use (5.5) in order to get

$$\begin{aligned}&\sum_{K>N} \|\Delta_K(v_M u_N)\|^2_{L^2((0,T);L^2)} \\ &\quad \le C \sum_{K>N} (1+K)^{-2} \|v_M u_N\|^2_{L^2((0,T);\mathcal{H}^1)} \le C_T M N^{-1} \|u_N(0)\|^2_{L^2} \|v_M(0)\|^2_{L^2}.\end{aligned}$$

For $K \le N$, denote

$$S_N = \sum_{K \le N} \Delta_K$$

and write directly

$$S_N(v_M u_N) = S_N(v_M N^{-2} A \tilde{u}_N), \tag{5.22}$$

where $\tilde{u}_N = \tilde{\Delta}_N u_N$, and the localization operator $\tilde{\Delta}_N$ was chosen so that $N^{-2} A \tilde{\Delta}_N$ is the identity on the support of Δ_N. We may now write

$$v_M A \tilde{u}_N = A(v_M \tilde{u}_N) + \tilde{u}_N \Delta v_M + 2 \nabla v_M \cdot \nabla \tilde{u}_N,$$

and hence by (5.22), the uniform boundedness of S_N and $N^{-1}\sqrt{A} S_N$ on L^2, we get

$$\begin{aligned}&\|S_N(v_M u_N)\|^2_{L^2} \\ &\le C N^{-2} \|N^{-1}\sqrt{A} S_N(\sqrt{A}(v_M \tilde{u}_N)\|^2_{L^2} + C N^{-4}(\|\tilde{u}_N \Delta v_M\|^2_{L^2} + \|\nabla v_M \cdot \nabla \tilde{u}_N\|^2_{L^2}) \\ &\le C N^{-2} \|\sqrt{A}(v_M \tilde{u}_N)\|^2_{L^2} + C N^{-4} \|\Delta v_M\|^2_{L^4} \|\tilde{u}_N\|^2_{L^4} + C N^{-4} \|\nabla v_M\|^2_{L^4} \|\nabla \tilde{u}_N\|^2_{L^4}.\end{aligned}$$

After integration in time, using the Strichartz estimates to control L^4 norms (use (5.16) to control $\|\nabla v_M\|_{L^4_{t,x}}$ and a similar argument to control $\|\nabla \tilde{u}_N\|_{L^4_{t,x}}$) and (5.5), we get

$$\begin{aligned}&\int_0^T \|S_N(v_M u_N)\|^2_{L^2}\, dt \\ &\quad \le C N^{-2} \int_0^T \|v_M \tilde{u}_N\|^2_{\mathcal{H}^1}\, dt + C_T M^2 N^{-4}(M^2 + N^2) \|v_M(0)\|^2_{L^2} \|\tilde{u}_N(0)\|^2_{L^2} \\ &\quad \le C_T M N^{-1} \|v_M(0)\|^2_{L^2} \|\tilde{u}_N(0)\|^2_{L^2} + C_T M^2 N^{-2} \|v_M(0)\|^2_{L^2} \|\tilde{u}_N(0)\|^2_{L^2},\end{aligned}$$

and we complete the proof with $\|\tilde{u}_N(0)\|_{L^2} \le C \|u_N(0)\|_{L^2}$. ∎

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