An upper bound on the hot spots constant

Stefan Steinerberger

Abstract. Let $D \subset \mathbb{R}^d$ be a bounded, connected domain with smooth boundary, and let $-\Delta u = \mu_1 u$ be the first nontrivial eigenfunction of the Laplace operator with Neumann boundary conditions. We prove that

$$\max_{x \in D} u(x) \leq 60 \cdot \max_{x \in \partial D} u(x)$$

and we emphasize that this constant is uniform among all connected domains with smooth boundary in all dimensions. In particular, the hot spots conjecture cannot fail by an arbitrarily large factor. The inequality also holds for other (Neumann-) eigenfunctions (possibly with a different constant) provided their eigenvalue is smaller than the first Dirichlet eigenvalue. An example of Kleefeld shows that the optimal constant is at least $1 + 10^{-3}$.

1. Introduction

Let $D \subset \mathbb{R}^d$ be a bounded, connected domain with smooth boundary, and let u denote the first nontrivial Laplacian eigenfunction with Neumann boundary conditions on ∂D, i.e.,

$$-\Delta u = \mu_1 u \quad \text{in } D,$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D,$$

where $\mu_1 > 0$ and ν denotes the normal derivative on the boundary. The function u describes the asymptotic profile of the long-time behavior of generic solutions of the heat equation.

One might expect that if D is 'simple', then the maximum and minimum of u should be at the boundary; this is the 1974 *hot spots conjecture* of Rauch [6]. The conjecture is widely assumed to be true for convex domains, and possibly even for simply connected domains if $d = 2$. This is being actively investigated, and there are many settings where it is known to hold [3, 4, 6–9, 15, 18–20, 28–30, 34, 36].
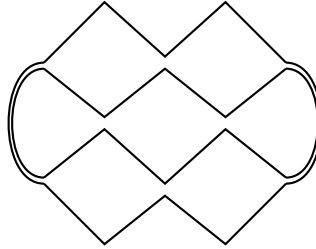
Figure 1. A type of domain on which hot spots fails (see [10, 11, 21]).

2. Results

2.1. Main result

However, there have to be at least some geometric restrictions on the domain D: Burdzy and Werner [11] constructed a planar domain with two holes where u assumes its maximum strictly inside the domain. Burdzy [10] later constructed a planar counterexample with one hole. Kleefeld [21] used high-precision numerics to numerically investigate examples of domains with one hole: Burdzy-type counterexamples [10] seem to be robust. Kleefeld also constructs an explicit (numerical) example of a domain for which

$$\|u\|_{L^\infty(D)} \geq (1 + 10^{-3}) \cdot \|u\|_{L^\infty(\partial D)}.$$

This leads to a natural question: how 'wrong' can the hot spots conjecture be? We show that it cannot fail by more than a fixed factor.

Theorem (Abridged version). *Let $D \subset \mathbb{R}^d$ be a bounded and connected domain with smooth boundary, and let $-\Delta u = \mu_1 u$ be the first nontrivial eigenfunction of the Laplace operator with Neumann boundary conditions. Then*

$$\max_{x \in D} u(x) \leq 60 \cdot \max_{x \in \partial D} u(x).$$

Remarks.

(1) The optimal constant might conceivably be quite close to 1 (at least for $d = 2$). An example of Kleefeld [21] shows that it is bigger than $1 + 10^{-3}$.

(2) The proof leads to better results in higher dimensions: the constant 60 can be replaced by 23 in $d = 3$ dimensions and by 15 in $d = 4$ dimensions. Asymptotically, as $d \to \infty$, the constant obtained by the argument converges to $\sqrt{e^e} \sim 3.89\ldots$

(3) Smoothness of the boundary can be replaced by weaker assumptions; we have not tried to optimize this condition.

(4) The proof also applies for other (Neumann-) eigenfunctions $-\Delta u = \mu u$ whose eigenvalue μ is smaller than the first Dirichlet eigenvalue. The strength of the result then depends on the ratio $0 < \mu/\lambda < 1$ (see §2.2.)

2.2. Other eigenfunctions

The proof shows a slightly stronger result: we never need $-\Delta u = \mu u$ to be the *first* eigenfunction of the Neumann-Laplacian, we only require that the associated eigenvalue satisfies

$$\mu < \lambda_1 = \inf_{\substack{f \in C_c^\infty(D) \\ f \not\equiv 0}} \frac{\int_D |\nabla f|^2 dx}{\int_D f^2 dx},$$

i.e., that the (Neumann-) eigenvalue is smaller than the first nontrivial Dirichlet eigenvalue. The inequality $\mu_1 < \lambda_1$ is elementary and follows from the variational characterization. Payne [31] showed that for convex $D \subset \mathbb{R}^2$ with C^2-boundary, we also have $\mu_2 < \lambda_1$. Levine and Weinberger [24] (see also Aviles [5]) show that for a bounded convex domain $D \subset \mathbb{R}^d$ with sufficiently smooth boundary $\mu_d < \lambda_1$ (see also Friedlander [17]). It is also not difficult to construct, in any dimension $d \geq 2$ and for any $n \in \mathbb{N}$, a connected domain $D_n \subset \mathbb{R}^d$ for which $\mu_n < 0.5 \cdot \lambda_1$ (think of many balls that are connected by very thin tubes).

Theorem (General version). *Let $D \subset \mathbb{R}^d$ be a bounded and connected domain with smooth boundary, and let $-\Delta u_k = \mu_k u_k$ be an eigenfunction of the Laplacian with Neumann boundary conditions. If $\mu_k < \lambda_1$, then there exists a constant $0 < c < \infty$ depending only on the ratio μ_k / λ_1 and the dimension such that*

$$\max_{x \in D} u_k(x) \leq c(\mu_k / \lambda_1, d) \cdot \max_{x \in \partial D} u_k(x).$$

We note that the constant $c(\mu_k / \lambda_1, d)$ can be explicitly computed for any choice of parameters (we refer to the proof for details). It could be interesting to understand whether and to what extent the limitation $\mu < \lambda_1$ is necessary or whether this could be further relaxed.

2.3. Other equations

There is a bigger picture here: to broad families of elliptic equations, it is possible to associate a drift-diffusion dynamical system. Eigenfunctions of the operator diagonalize the system and have a profile under the flow which changes multiplicatively – however, the profile also has a variational characterization from which we can infer that the first nontrivial Neumann eigenvalue is smaller than the first nontrivial Dirichlet eigenvalue: the associated drift-diffusion hits the boundary quickly. In order for this to make sense in terms of the slower decay of the Neumann eigenfunction, the solution at the boundary cannot be too small. This type of reasoning can be applied to more general elliptic equations and to the manifold setting – however, getting a uniform constant will require at least some assumptions on the PDE and/or the underlying geometry. We believe this to be an interesting avenue for further research.

3. Proof

We will only give a proof of the abridged version: getting a uniform constant requires more work, the proof of the general version follows easily from the proof of the abridged version. The proof decouples into the following steps.

- §3.1 deduces an elementary inequality for solutions of $-\Delta u = \mu u$ with Neumann conditions on domains $D \subset \mathbb{R}^d$ with smooth boundaries in terms of the (Dirichlet) heat kernel $p_t(\cdot, \cdot) : D \times D \to \mathbb{R}$.

- §3.2 discusses an estimate on the asymptotic behavior of the optimal constant in the inequality $\mu_1 \leq c \cdot \lambda_1$ and how it depends on the dimension. This section does not require any new ideas and follows from combining existing results: the Faber–Krahn inequality [13,23], the Szegő–Weinberger inequality [37,39] and an estimate of Lorch and P. Szegő [25] concerning the first nontrivial root of a special function.

- §3.3 quantifies the following phenomenon: the heat kernel $p_t(\cdot, \cdot)$ decays exponentially at rate $\exp(-\lambda_1 t)$. This implies that for $\alpha < \lambda_1$, it is possible to obtain decay estimates for $e^{\alpha t} p_t(\cdot, \cdot)$. §3.2 will imply that we can pick $\alpha = \mu_1$, which will be useful in terms of the inequality proved in §3.1.

- §3.4 combines all the arguments to finish the proof.

3.1. An upper bound on $\|u\|_{L^\infty(D)}$

The goal of this section is to deduce a general inequality for Neumann eigenfunctions. The argument is not sensitive to the geometry of D; however, we require the boundary to be sufficiently regular so that reflected Brownian motion exists (smoothness is more than sufficient).

Lemma 1. *Let D be a bounded, connected domain with smooth boundary and let u be a nontrivial solution of $-\Delta u = \mu u$ with Neumann boundary conditions assuming its maximum in $x_0 \in D$. Then, for all $t > 0$,*

$$1 \leq e^{\mu t} \int_D p_t(x_0, y) \, dy + e^{\mu t} \left(1 - \int_D p_t(x_0, y) \, dy \right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}},$$

where $p_t(x, y)$ denotes the Dirichlet heat kernel on D.

Proof. Let us assume, without loss of generality (after possibly replacing u by $-u$), that u assumes its maximum inside the domain D and

$$u(x_0) = \|u\|_{L^\infty(D)} > 0.$$

We will show that the maximum value on the boundary cannot be too small. Solving the heat equation with u as initial datum and Neumann boundary conditions is simple because u is an eigenfunction, the solution is $u(t, x) = e^{-\mu t} u(x)$. The probabilistic interpretation of the heat equation then allows us to write

$$u(t, x) = \mathbb{E}_x u(\omega(t)),$$

where the expectation $\mathbb{E}_x$ is taken with respect to Brownian motions started in x and reflected on the boundary. We now fix a time $t > 0$ and use the probabilistic interpretation to derive an upper bound on $u(x_0) = \|u\|_{L^\infty(D)}$. Let us consider a fixed Brownian path $\omega_{x_0}(s)$ started in x_0 and running for time $0 \le s \le t$. We distinguish two cases:

(1) $\omega_{x_0}(s)$ never touches the boundary for all $0 \le s \le t$,

(2) there exists a first time $0 \le t_0 \le t$ such that $\omega_{x_0}(t_0) \in \partial D$.

The first case is naturally related to the evolution of the heat equation with *Dirichlet* boundary conditions

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } D,$$
$$u = 0 \quad \text{on } \partial D.$$

We recall that the Dirichlet heat equation with initial data $u(0, x) = f(x)$ has an explicit solution

$$u(t, x) = \int_D p_t(x, y) f(y) \, dy,$$

where $p_t \colon D \times D \to \mathbb{R}$ is the heat kernel associated to the Dirichlet heat equation on D. Using $(\phi_n)_{n=1}^\infty$ to denote the sequence of L^2-normalized eigenfunctions of $-\Delta\phi_n = \lambda_n\phi_n$ with Dirichlet boundary conditions, we have an explicit expression for the heat kernel:

$$p_t(x, y) = \sum_{n=1}^\infty e^{-\lambda_n t} \, \phi_n(x) \, \phi_n(y).$$

At the same time, the Dirichlet heat equation has an explicit stochastic representation as

$$u(t, x) = \mathbb{E}_w u(\omega(t)),$$

where $\omega(t)$ is a Brownian motion started in x and running for time t with the additional rule that, once hitting the boundary, it remains there for all time (the boundary is 'sticky'). Since we are dealing with zero boundary conditions, the particles at the boundary do not contribute to the expectation and the expectation is only taken with respect to Brownian motion that never touches the boundary. This reproduces exactly the first case in our case distinction. Moreover, it implies that the density of the particles who have never touched the boundary is given by $p_t(x_0, \cdot)$, and the likelihood of Brownian motion starting in x_0, running for time t and never hitting the boundary, is given by

$$\mathbb{P}(\forall \, 0 \le s \le t : \omega_{x_0}(s) \notin \partial D) = \int_D p_t(x_0, y) \, dy.$$

This shows that we have a very precise understanding of the first case. As for the second case, we will not derive such a precise representation and instead bound it from above: suppose $\omega_{x_0}(t_0) \in \partial D$. We can use markovianity and the property that u is an eigenfunction to take an expectation over all Brownian paths started in t_0 and running for $t - t_0$ units of time:

$$\mathbb{E}_\omega \, \omega(t - t_0) = e^{-\mu(t - t_0)} \, u(\omega_{x_0}(t_0)) \le \|u\|_{L^\infty(\partial D)}.$$

Combining both estimates, we have

$$u(t, x_0) \leq \int_D p_t(x_0, y) u(y) \, dy + \left(1 - \int_D p_t(x_0, y) \, dy\right) \|u\|_{L^\infty(\partial D)}$$

$$\leq \|u\|_{L^\infty(D)} \int_D p_t(x_0, y) \, dy + \left(1 - \int_D p_t(x_0, y) \, dy\right) \|u\|_{L^\infty(\partial D)}.$$

Plugging in the explicit solution of the heat equation $u(t, x) = e^{-\mu t} u(x)$ and using that $u(x_0) = \|u\|_{L^\infty(D)}$, we obtain the desired inequality

$$\|u\|_{L^\infty(D)} \leq e^{\mu t} \|u\|_{L^\infty(D)} \int_D p_t(x_0, y) \, dy + e^{\mu t} \left(1 - \int_D p_t(x_0, y) \, dy\right) \|u\|_{L^\infty(\partial D)},$$

which can also be written as

$$1 \leq e^{\mu t} \int_D p_t(x_0, y) \, dy + e^{\mu t} \left(1 - \int_D p_t(x_0, y) \, dy\right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}}. \qquad \blacksquare$$

It is clear from the form of the inequality that if we can find a value of t such that

$$e^{\mu t} \int_D p_t(x_0, y) \, dy < 1,$$

then this inequality implies that the maximum at the boundary cannot be too small compared to the maximum inside the domain. Moreover, for each domain D,

$$p_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

decays asymptotically like $\sim e^{-\lambda_1 t}$, and since $\mu < \lambda_1$, there always exist a $t > 0$ such that the desired quantity is less than 1. It remains to show that all these quantities can be chosen uniformly among all domains D.

3.2. Inequalities for eigenvalues

We discuss an inequality for eigenvalues due to Szegő [37] (refining earlier work of Pólya [33]). These types of inequalities are now well-studied, we refer to [1, 2, 14, 16, 22, 24, 32, 35].

Lemma 2 (Szegő, 1954). *If D is a planar domain, then*

$$\mu_1 \leq 0.587 \cdot \lambda_1.$$

Proof. The Faber–Krahn inequality [13, 23] says that among all domains with fixed volume the ball minimizes the first eigenvalue of the Dirichlet-Laplacian and thus

$$\lambda_1 \geq \frac{\pi \cdot j_{0,1}^2}{|D|},$$

where $j_{0,1} \sim 2.404\ldots$ is the first positive zero of the Bessel function J_0. The case of equality is given when D is a disk. There is a corresponding upper bound for μ_1, the first nontrivial eigenvalue of Laplacian with Neumann boundary condition

$$\mu_1 \leq \frac{\pi \cdot \hat{j}_{1,1}^2}{|D|},$$

where $\hat{j}_{1,1} \sim 1.841$ is the first positive zero of J_1. This inequality was first proven by Szegő [37] for simply connected planar domains, and then more generally by Weinberger [39], who also established the corresponding analogue in higher dimensions. Equality is again attained on the ball. ∎

A corresponding inequality also exists in higher dimensions with an improved constant. The argument is essentially the same: the new ingredient is the need to control roots of certain Bessel functions.

Lemma 3. *Let $D \subset \mathbb{R}^d$ and $d \geq 3$. We have*

$$\mu_1 \leq \min\left\{0.587, \frac{d+2}{(d/2 - 2/d)^2}\right\} \cdot \lambda_1.$$

Proof. Using again the Faber–Krahn inequality [13, 23] and the Szegő–Weinberger inequality [37, 39], we have

$$\mu_1 \leq \frac{p_{d/2,1}^2}{j_{d/2-1,1}^2} \cdot \lambda_1,$$

where $j_{d/2-1,1}$ is the first positive root of the Bessel function $J_{d/2-1}(x)$ and $p_{d/2,1}$ is the first positive root of the derivative of $x^{1-d/2}J_{d/2}(x)$. We refer to a paper of Ashbaugh and Benguria [2], which gives a clear exposition of these inequalities. We have the known estimate (see e.g. Watson [38])

$$j_{d/2-1,1} \geq \sqrt{(d/2-1)(d/2+1)} \geq \frac{d}{2} - \frac{2}{d}.$$

It remains to obtain an upper bound on $p_{d/2,1}^2$. We could invoke bounds of Lorch and Szegő [25] which guarantee

$$d + \frac{8}{d+6} < p_{d/2,1}^2 < d + 2,$$

and this proves the desired inequality. However, we can also give an elementary and self-contained proof of the upper bound. The Szegő–Weinberger inequality

$$\mu_1(D) \leq \left(\frac{c_d}{|D|}\right)^{2/d} \cdot p_{d/2,1}^2,$$

where c_d is the volume of the unit ball, is attained on the unit ball $\mathbb{B}_1$. Invoking the variational characterization, we have for any $f : \mathbb{B}_1 \to \mathbb{R}$ with mean value 0,

$$\mu_1(\mathbb{B}_1) = p_{d/2,1}^2 \leq \frac{\int_{\mathbb{B}_1} |\nabla f(x)|^2 dx}{\int_{\mathbb{B}_1} f(x)^2 dx}.$$

This allows us to obtain upper bounds on μ_1 by plugging in test functions. Assuming $\mathbb{B}_1$ to be centered at the origin, we choose $f(x) = x_1$. Applying Fubini, we get, for a dimensional constant $\omega_d > 0$,

$$\int_{\mathbb{B}_1} |\nabla f(x)|^2 dx = \int_{\mathbb{B}_1} 1 \, dx = \omega_d \int_{-1}^1 (1 - x_1^2)^{(d-1)/2} \, dx_1$$

as well as

$$\int_{\mathbb{B}_1} f(x)^2 \, dx = \omega_d \int_{-1}^1 (1 - x_1^2)^{(d-1)/2} x_1^2 \, dx_1.$$

The constant ω_d cancels out and we arrive at $p_{d/2,1}^2 \leq d + 2$. $\blacksquare$

3.3. A decay estimate

The goal of this section is to derive an upper bound on the quantity

$$e^{\mu_1 t} \int_D p_t(x_0, y) \, dy,$$

where $x_0 \in D$ is the point in which u assumes its maximum. We will not use the fact that x_0 is the point in which u assumes its maximum and will instead bound the bigger quantity

$$e^{\mu_1 t} \int_D p_t(x_0, y) \, dy \leq \sup_{x \in D} e^{\mu_1 t} \int_D p_t(x, y) \, dy.$$

Moreover, the argument does not require that μ_1 is the first Neumann eigenvalue, it only requires that $\mu_1/\lambda_1 < 1$.

Lemma 4. *Let D be as above and let $\mu < \lambda_1$. Then*

$$\sup_{x \in D} \int_D e^{\mu t} p_t(x, y) \, dy \leq \frac{|D|}{(4(1 - \frac{\mu}{\lambda_1}) \pi t)^{d/2}}.$$

Proof. We only consider the special case $\mu = \mu_1$ (which then allows us to invoke uniform bounds on μ_1/λ_1 which depend only on the dimension), but it is easy to see that nothing changes in the proof in the general case. We recall that

$$p_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

and that therefore

$$\int_D e^{\mu_1 t} p_t(x, y) \, dy = \sum_{k=1}^{\infty} e^{(\mu_1 - \lambda_k)t} \phi_k(x) \int_D \phi_k(y) \, dy.$$

Using the L^2-normalization of the eigenfunctions,

$$\left(\int_D \phi_k(y) \right)^2 \leq |D| \cdot \int_D \phi_k(y)^2 \, dy = |D|,$$

in combination with Cauchy–Schwarz leads to

$$\sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x) \int_D \phi_k(y)\, dy \leq \Big(\sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x)^2 \Big)^{1/2} \Big(\sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \cdot |D| \Big)^{1/2}.$$

This requires us to bound

$$\sup_{x \in D} \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x)^2 \quad \text{and} \quad |D| \cdot \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t}.$$

For both these terms, the dimension of D will start playing a role. Recalling Lemma 3, we will denote the best constant in the inequality by

$$\mu_1 \leq \alpha_d \cdot \lambda_1.$$

We note that $\alpha_d \leq 0.587$ and that α_d converges to 0 as $d \to \infty$. Then

$$\frac{\mu_1 - \lambda_k}{\lambda_k} = \frac{\mu_1}{\lambda_k} - 1 \leq \frac{\mu_1}{\lambda_1} - 1 \leq -(1 - \alpha_d).$$

We can now bound the first term by

$$\sup_{x \in D} \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x)^2 \leq \sup_{x \in D} \sum_{k=1}^{\infty} e^{-\lambda_k(1-\alpha_d)t} \phi_k(x)^2$$

$$= \sup_{x \in D} p_{(1-\alpha_d)t}(t, x, x).$$

At this point we invoke domain monotonicity of the heat kernel: $p_t(\cdot, \cdot)$, the heat kernel associated to the domain D, is bounded from above by the heat kernel in $\mathbb{R}^d$:

$$p_t(x, y) \leq \frac{1}{(4\pi t)^{d/2}} \exp\Big(-\frac{\|x-y\|^2}{4t} \Big) \leq \frac{1}{(4\pi t)^{d/2}},$$

and therefore

$$\sup_{x \in D} \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x)^2 \leq \frac{1}{(4(1-\alpha_d)\pi t)^{d/2}}.$$

A similar argument can be used for the second term: using the L^2-normalization of the eigenfunctions shows

$$|D| \cdot \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} = |D| \cdot \int_D \sum_{k=1}^{\infty} e^{(\mu_1-\lambda_k)t} \phi_k(x)^2\, dx$$

$$\leq |D| \cdot \int_D p_{(1-\alpha_d)t}(t, x, x)\, dx \leq |D|^2 \cdot \sup_{x \in D} p_{(1-\alpha_d)t}(t, x, x),$$

which can be bounded by the same term as above. Combining all these estimates, we arrive at

$$\sup_{x \in D} \int_D e^{\mu_1 t} p_t(x, y)\, dy \leq \frac{|D|}{(4(1-\alpha_d)\pi t)^{d/2}}. \qquad \blacksquare$$

3.4. Finishing the argument

Recalling Lemma 1,

$$1 \leq e^{\mu_1 t} \int_D p_t(x_0, y) \, dy + \left(e^{\mu_1 t} - e^{\mu_1 t} \int_D p_t(x_0, y) \, dy \right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}},$$

we observe that in the case of interest to us, the quantity $\|u\|_{L^\infty(\partial D)}/\|u\|_{L^\infty(D)}$ is strictly smaller than 1 and we also note that

$$e^{\mu_1 t} \int_D p_t(x_0, y) \, dy + \left(e^{\mu_1 t} - e^{\mu_1 t} \int_D p_t(x_0, y) \, dy \right) = e^{\mu_1 t}.$$

Thus, replacing the first term by an upper bound on the first term (both times that it appears in the inequality), leads to an upper bound. Applying this together with Lemma 4, we end up with

$$1 \leq \frac{|D|}{(4(1 - \alpha_d)\pi t)^{d/2}} + \left(e^{\mu_1 t} - \frac{|D|}{(4(1 - \alpha_d)\pi t)^{d/2}} \right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}}.$$

We use the Szegő–Weinberger inequality once more in the form

$$\mu_1(D) \leq \frac{c_d^{2/d}}{|D|^{2/d}} \cdot p_{d/2,1}^2.$$

At this point, we can invoke a scaling symmetry which has not been used so far: we can exchange a dilation of time with a rescaling of time. This is reflected in the fact that powers of t and $|D|$ are not independent; we exploit this by introducing a rescaling of time, $\alpha = t|D|^{-2/d}$. The inequality then simplifies to, for all $\alpha > 0$,

$$1 \leq \frac{1}{(4(1 - \alpha_d)\pi \alpha)^{d/2}} + \left(\exp\left(c_d^{2/d} \cdot p_{d/2,1}^2 \alpha \right) - \frac{1}{(4(1 - \alpha_d)\pi \alpha)^{d/2}} \right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}}.$$

This inequality can now be optimized in α, and the results will depend on the values of α_d and $p_{d/2,1}^2$. We also recall that $c_d = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of the d-dimensional unit ball. We discuss the first few cases which will illustrate the general pattern. In the planar case, $d = 2$, we have $p_{d/2,1} \sim 1.841$ and $\alpha_d \sim 0.587$. Setting $\alpha = 0.258$, we obtain

$$\frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}} \geq \frac{1}{58.35}.$$

For $d = 3$, we have $p_{d/2,1} \sim 2.081$ and $\alpha_d \sim 0.439$. Setting $\alpha = 0.194$, we obtain

$$\frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}} \geq \frac{1}{22.03}.$$

For $d = 4$, we have $p_{d/2,1} \sim 2.299$ and $\alpha_d \sim 0.3602$. Setting $\alpha = 0.17$, we obtain

$$\frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}} \geq \frac{1}{14.71}.$$

The same computation can be carried out for $5 \leq d \leq 20$ and we observe that the constants are monotonically decreasing. It remains to prove the desired result for $d \geq 21$ as well as to derive the desired asymptotic behavior for $d \to \infty$.

3.5. Bounding the constants

We first observe, using standard asymptotics for the Gamma function, that for $d \geq 20$,

$$d \, \frac{\pi}{\Gamma(d/2+1)^{2/d}} \leq c_d^{2/d} \cdot p_{d/2,1}^2 \leq (d+2) \, \frac{\pi}{\Gamma(d/2+1)^{2/d}} \leq 18.$$

Taking a limit, this also implies that

$$\lim_{d \to \infty} c_d^{2/d} \cdot p_{d/2,1}^2 = 2\pi e.$$

Recalling Lemma 3, we have

$$\mu_1 \leq \min\left\{0.587, \frac{d+2}{(d/2-2/d)^2}\right\} \cdot \lambda_1$$

and thus, for $d \geq 20$, we have that $\alpha_d \leq 0.23$. We now set, in dimension d,

$$\alpha = \frac{1.1}{4(1-\alpha_d)\pi}$$

from which we infer that $\alpha \leq 0.12$ as well as

$$\frac{1}{(4(1-\alpha_d)\pi\alpha)^{d/2}} \leq \frac{1}{1.1^{20}} \leq 0.15,$$

and therefore,

$$1 \leq \frac{1}{(4(1-\alpha_d)\pi\alpha)^{d/2}} + \left(\exp\left(c_d^{2/d} \cdot p_{d/2,1}^2 \alpha\right) - \frac{1}{(4(1-\alpha_d)\pi\alpha)^{d/2}}\right) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}}$$

$$\leq 0.15 + (\exp(18 \cdot 0.12) - 0.15) \frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}}.$$

This implies that for $d \geq 20$,

$$\frac{\|u\|_{L^\infty(\partial D)}}{\|u\|_{L^\infty(D)}} \geq \frac{0.85}{\exp(18 \cdot 0.12) - 0.15} \geq \frac{1}{11}.$$

The pattern continues in the same manner for $d = 5, 6, \ldots$ and the constants keep improving. It remains to understand the asymptotic behavior. As $d \to \infty$, there is an easy way to see the asymptotic behavior. We have $\alpha_d \to 0$ and hence for any fixed $\alpha > 1/(4\pi)$, one of the terms become small:

$$\lim_{d \to \infty} \frac{1}{(4(1-\alpha_d)\pi\alpha)^{d/2}} = 0.$$

Ignoring these terms (which can be made precise since they become sufficiently small for d sufficiently large), we expect that the best lower bound in the limit converges to $\exp(-c_d^{2/d} \cdot p_{d/2,1}^2 \alpha)$ provided that $\alpha > 1/(4\pi)$. Thus, invoking the limit behavior, we see that for $\alpha = 1/(4\pi) + \varepsilon$,

$$\exp\left(-\frac{\pi}{\Gamma(d/2+1)^{2/d}} \cdot (d+2)\alpha\right) \geq \exp\left(-2e\pi \frac{1}{4\pi}\right) = \sqrt{e^{-e}} = \frac{1}{3.89\cdots}.$$

This is the best constant that can be achieved with this type of argument.

3.6. A final remark

There is one step in the proof that is guaranteed to be wasteful: when evaluating the second sum in §3.3, we have the inequalities

$$|D| \cdot \sum_{k=1}^{\infty} e^{(\mu_1 - \lambda_k)t} = |D| \cdot \int_D \sum_{k=1}^{\infty} e^{(\mu_1 - \lambda_k)t} \, \phi_k(x)^2 \, dx$$

$$\leq |D| \cdot \int_D p_{(1-\alpha_d)t}(x, x) \, dx \leq |D|^2 \cdot \sup_{x \in D} p_{(1-\alpha_d)t}(x, x).$$

It is clear that this estimate is wasteful since an integral is replaced by a pointwise supremum. Indeed, the integral

$$\int_D p_t(x, x) \, dx$$

is known as the heat trace. Luttinger [26, 27] proved that, among all domains with fixed volume, the heat trace is maximized by the ball; this would lead to a better estimate. However, we are not aware of any reasonably explicit estimates for the heat trace of the ball. Asymptotic expansions are available (see e.g. Davies [12]), however, our argument would require explicit bounds.

References

[1] Ashbaugh, M. S. and Benguria, R. D.: Proof of the Payne–Pólya–Weinberger conjecture. *Bull. Amer. Math. Soc. (N.S.)* **25** (1991), no. 1, 19–29.

[2] Ashbaugh, M. S. and Benguria, R. D.: Universal bounds for the low eigenvalues of Neumann Laplacians in *n* dimensions. *SIAM J. Math. Anal.* **24** (1993), no. 3, 557–570.

[3] Atar, R. and Burdzy, K.: On nodal lines of Neumann eigenfunctions. *Electron. Comm. Probab.* **7** (2002), 129–139.

[4] Atar, R. and Burdzy, K.: On Neumann eigenfunctions in lip domains. *J. Amer. Math. Soc.* **17** (2004), no. 2, 243–262.

[5] Aviles, P.: Symmetry theorems related to Pompeiu's problem. *Amer. J. Math.* **108** (1986), no. 5, 1023–1036.

[6] Bañuelos, R. and Burdzy, K.: On the "hot spots" conjecture of J. Rauch. *J. Func. Anal.* **164** (1999), no. 1, 1–33.

[7] Bañuelos, R. and Pang, M.: An inequality for potentials and the "hot-spots" conjecture. *Indiana Univ. Math. J.* **53** (2004), no. 1, 35–47.

[8] Bass, R. and Burdzy, K.: Fiber Brownian motion and the "hot spots" problem. *Duke Math. J.* **105** (2000), no. 1, 25–58.

[9] Beck, T.: Uniform level set estimates for ground state eigenfunctions. *SIAM J. Math. Anal.* **50** (2018), no. 4, 4483–4502.

[10] Burdzy, K.: The hot spots problem in planar domains with one hole. *Duke Math. J.* **129** (2005), no. 3, 481–502.

[11] Burdzy, K. and Werner, W.: A counterexample to the "hot spots" conjecture. *Ann. of Math. (2)* **149** (1999), no. 1, 309–317.

[12] Davies, E. B.: *Heat kernels and spectral theory*. Cambridge Tracts in Mathematics 92, Cambridge University Press, Cambridge, 1989.

[13] Faber, G.: Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisformige den tiefste Grundton gibt. *Sitzungsber. Bayer. Akad. Wiss. München, Math. Phys. Kl.* (1923), 169–172.

[14] Filonov, N.: On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator. *St. Petersburg Math. J.* **16** (2005), no. 2, 413–416.

[15] Freitas, P.: Closed nodal lines and interior hot spots of the second eigenfunctions of the Laplacian on surfaces. *Indiana Univ. Math. J.* **51** (2002), no. 2, 305–316.

[16] Friedlander, L.: Some inequalities between Dirichlet and Neumann eigenvalues. *Arch. Rational Mech. Anal.* **116** (1991), no. 2, 153–160.

[17] Friedlander, L.: Remarks on Dirichlet and Neumann eigenvalues. *Amer. J. Math.* **117** (1995), no. 1, 257–262.

[18] Jerison, D.: Locating the first nodal line in the Neumann problem. *Trans. Amer. Math. Soc.* **352** (2000), no. 5, 2301–2317.

[19] Jerison, D. and Nadirashvili, N.: The "hot spots" conjecture for domains with two axes of symmetry. *J. Amer. Math. Soc.* **13** (2000), no. 4, 741–772.

[20] Judge, C. and Mondal, S.: Euclidean triangles have no hot spots. *Ann. of Math. (2)* **191**, no. 1, 167–211.

[21] Kleefeld, A.: The hot spots conjecture can be false: Some numerical examples. *Adv. Comput. Math.* **47** (2021), no. 6, Paper no. 85, 31 pp.

[22] Kornhauser, E. and Stakgold, I.: A variational theorem for $\nabla^2 u + \lambda u = 0$ and its applications. *J. Math. Physics* **31** (1952), 45–54.

[23] Krahn, E.: Uber eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.* **94** (1925), no. 1, 97–100.

[24] Levine, H. and Weinberger, H.: Inequalities between Dirichlet and Neumann eigenvalues. *Arch. Rational Mech. Anal.* **94** (1986), no. 3, 193–208.

[25] Lorch, L. and Szegő, P.: Bounds and monotonicities for the zeros of derivatives of ultraspherical Bessel functions. *SIAM J. Math. Anal.* **25** (1994), no. 2, 549–554.

[26] Luttinger, J. M.: Generalized isoperimetric inequalities. *J. Mathematical Phys.* **14** (1973), 586–593.

[27] Luttinger, J. M.: Generalized isoperimetric inequalities. II, III. *J. Mathematical Phys.* **14** (1973), 1444–1447; ibid. **14** (1973), 1448–1450.

[28] Miyamoto, Y.: The "hot spots" conjecture for a certain class of planar convex domains. *J. Math. Phys.* **50** (2009), no. 10, 103530, 7 pp.

[29] Miyamoto, Y.: A planar convex domain with many isolated "hot spots" on the boundary. *Jpn. J. Ind. Appl. Math.* **30** (2013), no. 1, 145–164.

[30] Pascu, M.: Scaling coupling of reflecting Brownian motions and the hot spots problem. *Trans. Amer. Math. Soc.* **354** (2002), no. 11, 4681–4702.

[31] Payne, L. E.: Inequalities for eigenvalues of membranes and plates. *J. Rational Mech. Anal.* **4** (1955), 517–529.

[32] Payne, L. E. and Weinberger, H. F.: An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960), 286–292.

[33] Pólya, G.: Remarks on the foregoing paper. *J. Math. Physics* **31** (1952), 55–57.

[34] Siudeja, B.: Hot spots conjecture for a class of acute triangles. *Math. Z.* **280** (2015), no. 3-4, 783–806.

[35] Steinerberger, S.: Sharp L^1-Poincaré inequalities correspond to optimal hypersurface cuts. *Arch. Math. (Basel)* **105** (2015), no. 2, 179–188.

[36] Steinerberger, S.: Hot spots in convex domains are in the tips (up to an inradius). *Comm. Partial Differential Equations* **45** (2020), no. 6, 641–654.

[37] Szegő, G.: Inequalities for certain eigenvalues of a membrane of given area. *J. Rational Mech. Anal.* **3** (1954), 343–356.

[38] Watson, G.: *A treatise on the theory of Bessel functions*. Cambridge University Press, 1922.

[39] Weinberger, H. F.: An isoperimetric inequality for the N dimensional free membrane problem. *J. Rational Mech. Anal.* **5** (1956), 633–636.

Stefan Steinerberger
Department of Mathematics, University of Washington, Seattle, WA 98195, USA;
steinerb@uw.edu