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On complete hypersurfaces with negative Ricci curvature in Euclidean spaces

Alexandre Paiva Barreto and Francisco Fontenele

Abstract. In this paper, we prove that if M^n, $n \geq 3$, is a complete Riemannian manifold with negative Ricci curvature and $f: M^n \to \mathbb{R}^{n+1}$ is an isometric immersion such that $\mathbb{R}^{n+1} \setminus f(M)$ is an open set that contains balls of arbitrarily large radius, then $\inf_M |A| = 0$, where $|A|$ is the norm of the second fundamental form of the immersion. In particular, an n-dimensional complete Riemannian manifold with negative Ricci curvature bounded away from zero cannot be properly isometrically immersed in a half-space of $\mathbb{R}^{n+1}$. This gives a partial answer to a question raised by Reilly and Yau.

1. Introduction

A classical theorem by Hilbert states that the hyperbolic plane cannot be isometrically immersed in the 3-dimensional Euclidean space $\mathbb{R}^3$. Efimov [4] extended Hilbert's theorem by proving that there is no immersed complete surface in $\mathbb{R}^3$ with negative Gaussian curvature bounded away from zero. Independently, Reilly [8] and Yau [11] (see also [12], problem 56, p. 682) proposed the following extension of Efimov's theorem:

Question 1.1. *There are no complete hypersurfaces in $\mathbb{R}^{n+1}$ with negative Ricci curvature bounded away from zero.*

In a well-known work, Smyth and Xavier [9] proved that the question above has an affirmative answer for $n = 3$, with the stronger conclusion that the infimum of the length $|A|$ of the second fundamental form A is actually zero, and provided a partial answer for $n > 3$. In this paper, we give the following partial answer to that question:

Theorem 1.2. *A complete n-dimensional Riemannian manifold with negative Ricci curvature bounded away from zero cannot be properly isometrically immersed in a half-space of $\mathbb{R}^{n+1}$.*

Theorem 1.2 was obtained as a consequence of the stronger result below. As usual, the inradius of an open subset V of $\mathbb{R}^{n+1}$, denoted by $\mathrm{Inrad}(V)$, is the supremum of the radii of the open balls contained in V.

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Theorem 1.3. *Let M^n, $n \geq 3$, be a complete Riemannian manifold with negative Ricci curvature and let $f: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. If $f(M)$ is a closed subset of $\mathbb{R}^{n+1}$ and* $\mathrm{Inrad}(\mathbb{R}^{n+1} \backslash f(M)) = \infty$, *then* $\inf_M |A| = 0$.

Question 1.1 has an affirmative answer for entire graphs $M^n \subset \mathbb{R}^{n+1}$, $n \geq 3$, with negative Ricci curvature, since Chern [2] showed that $\inf |\mathrm{Ric}| = 0$ for any of these graphs. As a consequence of Theorem 1.3, one has the following improvement of this result of Chern (see Corollary 1.7 in [3] for a generalization).

Corollary 1.4 (Corollary 1.6 in [5]). *Let M^n, $n \geq 3$, be an entire graph in $\mathbb{R}^{n+1}$, i.e., the graph of a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. If the Ricci curvature of M^n is negative, then* $\inf_M |A| = 0$.

A regular algebraic hypersurface in $\mathbb{R}^{n+1}$ is the zero set $M = P^{-1}(0)$ of a polynomial function $P: \mathbb{R}^{n+1} \to \mathbb{R}$ with the property that the gradient vector field ∇P of P has no zeros on M. Since $\mathbb{R}^{n+1} \backslash M$ contains balls of arbitrarily large radius (see e.g. [1]), as an immediate consequence of Theorem 1.3 one has the following result.

Corollary 1.5. *Let M^n, $n \geq 3$, be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$. If the Ricci curvature of M^n is negative, then* $\inf_M |A| = 0$.

In dimension 2, Milnor [7] conjectured (see also [12], Problem 62, p. 684) the following: *if M^2 is a complete non-flat surface in $\mathbb{R}^3$ whose Gaussian curvature K does not change sign, then* $\inf(H^2 - K) = 0$, *where H is the mean curvature of M*. If true, Milnor's conjecture would be an improvement of Efimov's theorem. In fact, for surfaces in $\mathbb{R}^3$ with negative Gaussian curvature, one has $\inf(H^2 - K) = 0$ if and only if $\inf |A| = 0$. For partial answers to Milnor's conjecture, see [6, 10]. Smyth and Xavier [9] proposed a similar question in higher dimensions:

Question 1.6. *If M^n is a complete hypersurface in $\mathbb{R}^{n+1}$ with negative Ricci curvature, then* $\inf_M |A| = 0$.

Clearly, if Question 1.6 has an affirmative answer, then so will Question 1.1. As shown by Theorem 1.3, the answer to Question 1.6 is yes for a wide class of hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$, namely the class of hypersurfaces whose complement in $\mathbb{R}^{n+1}$ is an open subset with infinite inradius.

It follows from Theorem 1.3 that if M^n, $n \geq 3$, is a complete hypersurface of non-positive Ricci curvature in $\mathbb{R}^{n+1}$ whose complement is an open subset with infinite inradius, then $\sup \mathrm{Ric} = 0$. On the other hand, one may ask if a stronger conclusion could be obtained in the finite inradius case: does $\mathrm{Inrad}(\mathbb{R}^{n+1} \backslash M) < \infty$ implies $\sup \mathrm{Ric} = \max \mathrm{Ric} = 0$?

2. The arguments

Before presenting the proof of Theorem 1.3, let us introduce the notation and recall some basic facts about isometric immersions.

Given an isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ of an n-dimensional orientable Riemannian manifold into the $(n+1)$-dimensional Euclidean space, denote by σ the vector valued second fundamental form of the immersion, and by A the shape operator of M^n with respect to a global unit normal vector field ξ. The squared norm $|A|^2$ of A is defined as the trace of A^2. It is easy to see that

$$|A|^2 = \sum_{i=1}^{n} \lambda_i^2, \tag{2.1}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the principal curvatures functions of M^n with respect to ξ.

Denote by $\mathrm{Ric}_p(v)$ the Ricci curvature of M^n at a point p in the direction of a unity vector $v \in T_pM$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM such that $Ae_i = \lambda_i e_i$, for $i = 1, \ldots, n$, it follows from the Gauss equation that

$$\mathrm{Ric}_p(e_i) = \sum_{j \neq i} \lambda_i \lambda_j = \lambda_i(\mathrm{tr}\, A(p) - \lambda_i), \quad i = 1, \ldots, n, \tag{2.2}$$

where $\mathrm{tr} A(p)$ is the trace of $A(p)$.

In the proof of Theorem 1.3, we will use the following result from [9].

Theorem 2.1 (Principal curvature theorem). *Let M^n be a complete immersed orientable hypersurface in $\mathbb{R}^{n+1}$, which is not a hyperplane, and let A denote the shape operator with respect to a global unit normal vector field. Let $\Lambda \subset \mathbb{R}$ be the set of nonzero values assumed by the eigenvalues of A and let $\Lambda^{\pm} = \Lambda \cap \mathbb{R}^{\pm}$.*

(i) *If Λ^+ and Λ^- are both nonempty, then* $\inf \Lambda^+ = \sup \Lambda^- = 0$.

(ii) *If Λ^+ or Λ^- is empty, then the closure $\overline{\Lambda}$ of Λ is connected.*

Proof of Theorem 1.3. Since M^n has negative Ricci curvature by hypothesis, it follows from (2.2) that $\lambda_i(p) \neq 0$ for all $p \in M$ and all $i = 1, \ldots, n$. Moreover, there are principal curvatures of both signs at each point of M^n. By the continuity of the principal curvature functions, the number of negative principal curvatures (and so also the number of positive principal curvatures) is constant along M. Therefore, there are two cases to consider:

Case 1. At each point of M, we have $n-1$ negative principal curvatures and one positive principal curvature, or vice versa.

This case was treated in [9]. We will present the argument here for the convenience of the reader. We may choose the unit normal vector field ξ so that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} < 0 < \lambda_n,$$

along M. Since $n \geq 3$ and the Ricci curvature of M is negative, by (2.2) we have

$$\lambda_n > -\sum_{\substack{j=1 \\ j \neq i}}^{n-1} \lambda_j = \sum_{\substack{j=1 \\ j \neq i}}^{n-1} |\lambda_j|, \quad i = 1, \ldots, n-1,$$

which implies

$$\lambda_n(p) > |\lambda_i(p)|, \quad i = 1, \ldots, n-1, \; p \in M. \tag{2.3}$$

By the principal curvature theorem, Theorem 2.1, there is a sequence (p_k) in M such that $\lambda_n(p_k) \to 0$ when $k \to \infty$. Using this information in (2.3), we obtain that $\lambda_i(p_k) \to 0$ when $k \to \infty$, for all $i = 1, \dots, n$. It now follows from (2.1) that $|A|(p_k) \to 0$. This concludes the proof of the theorem in this case.

Case 2. At each point of M, we have at least two negative principal curvatures and two positive principal curvatures.

Given $r > 0$, there exists, by hypothesis, a closed ball $\overline{B}_r(p_0)$ of center p_0 and radius r contained in $\mathbb{R}^{n+1} - f(M)$. Let $h: M \to \mathbb{R}$ be the function defined by

$$h(x) = \frac{1}{2}\,\|f(x) - p_0\|^2.$$

It is easy to see that

$$df_x(\nabla h(x)) = \{f(x) - p_0\}^T, \tag{2.4}$$

and

$$\operatorname{Hess} h_x(v, w) = \langle v, w\rangle + \langle \sigma_x(v, w), f(x) - p_0\rangle, \tag{2.5}$$

for all $x \in M$ and $v, w \in T_xM$, where $\{f(x) - p_0\}^T$ is the component of $f(x) - p_0$ that is tangent to $df_x(T_xM)$.

Since $f(M)$ is a closed subset of $\mathbb{R}^{n+1}$ by hypothesis, there exists a point $q \in M$ such that $h(q) = \inf_M h$. Then $\nabla h(q) = 0$, and from (2.4) we obtain that $f(q) - p_0$ is orthogonal to $df_q(T_qM)$. Replacing ξ by $-\xi$ if necessary, we can assume that

$$\xi(q) = \frac{p_0 - f(q)}{\|p_0 - f(q)\|}. \tag{2.6}$$

Since h attains a minimum at q, it follows from (2.5) and (2.6) that

$$\begin{aligned} 0 \le \operatorname{Hess} h_q(v, v) &= 1 + \langle \sigma_q(v, v), \xi(q)\rangle\langle \xi(q), f(q) - p_0\rangle \\ &= 1 - \|f(q) - p_0\|\langle Av, v\rangle, \end{aligned}$$

for any unit vector $v \in T_qM$. Since $\|f(q) - p_0\| > r$ (for $\overline{B}_r(p_0)$ does not intersects $f(M)$), from the inequality above we obtain

$$\lambda_j(q) \le \frac{1}{\|f(q) - p_0\|} < \frac{1}{r}, \quad j = 1, \dots, n. \tag{2.7}$$

Let l be the number of negative principal curvatures. Notice that $l \ge 2$ in the case we are considering. Since the Ricci curvature of M is negative, by (2.2) we have

$$\lambda_{l+1}(q) + \cdots + \lambda_n(q) > \sum_{\substack{j=1 \\ j \ne i}}^{l} |\lambda_j(q)| \quad i = 1, \dots, l,$$

which implies

$$2\{\lambda_{l+1}(q) + \cdots + \lambda_n(q)\} > |\lambda_1(q)| + \cdots + |\lambda_l(q)|. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$|\lambda_1(q)| + \cdots + |\lambda_l(q)| < \frac{2(n-l)}{r}.$$

Then, by (2.1), (2.7) and the inequality above,

$$\begin{aligned} |A|^2(q) &= \sum_{i=1}^{l} \lambda_i^2(q) + \sum_{i=l+1}^{n} \lambda_i^2(q) < \Big(\sum_{i=1}^{l} |\lambda_i(q)|\Big)^2 + \frac{n-l}{r^2} \\ &< \frac{4(n-l)^2}{r^2} + \frac{n-l}{r^2} < \frac{5(n-l)^2}{r^2}, \end{aligned} \tag{2.9}$$

and so

$$\inf_M |A| \le |A|(q) < \frac{3(n-l)}{r} < \frac{3(n-1)}{r}.$$

Since the last inequality holds for every $r > 0$, we conclude that $\inf_M |A| = 0$ by letting $r \to \infty$. The proof of Theorem 1.3 is now complete. ∎

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Alexandre Paiva Barreto
Departamento de Matemática, Universidade Federal de São Carlos,
Rodovia Washington Luís, Km 235, 13565-905 São Carlos, SP, Brazil;
alexandre.paiva@ufscar.br

Francisco Fontenele
Departamento de Geometria, Universidade Federal Fluminense, Campus do Gragoatá,
Rua Professor Marcos Waldemar de Freitas Reis, s/n, 24210-201 Niterói, RJ, Brazil;
fontenele@mat.uff.br